

# Streaming Algorithms for Planar Convex Hulls

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## Abstract

Many classical algorithms are known for computing the convex hull of a set of  $n$  point in  $\mathbb{R}^2$  using  $O(n)$  space. For large point sets, whose size exceeds the size of the working space, these algorithms cannot be directly used. The current best streaming algorithm for computing the convex hull is computationally expensive, because it needs to solve a set of linear programs.

In this paper, we propose simpler and faster streaming and W-stream algorithms for computing the convex hull. Our streaming algorithm has small pass complexity, which is roughly a square root of the current best bound, and it is simpler in the sense that our algorithm mainly relies on computing the convex hulls of smaller point sets. Our W-stream algorithms, one of which is deterministic and the other of which is randomized, have nearly-optimal tradeoff between the pass complexity and space usage, as we established by a new unconditional lower bound.

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## 1 Introduction

The *convex hull* of a set  $P$  of points in  $\mathbb{R}^2$  is the smallest convex set that contains  $P$ . We denote the convex hull of  $P$  by  $\text{conv}(P)$  and denote the set of extreme points in  $\text{conv}(P)$  by  $\text{ext}(P)$ . Let  $n = |P|$  and  $h = |\text{ext}(P)|$ . Note that  $h \leq n$  because  $\text{ext}(P)$  is a subset of  $P$ . By computing the convex hull of  $P$ , we mean outputting the points in  $\text{ext}(P)$  in clockwise order.

There is a long line of research on computing the convex hull using  $O(n)$  space. In the RAM model, Graham [20] gave the first algorithm, called the *Graham Scan*, with running time  $O(n \log n)$ . Subsequently, several algorithms were devised with the same running time, but with different approaches [2, 6, 26, 34]. In the output-sensitive model, where the running time depends on  $n$  and  $h$ , Jarvis [25] proposed the *Gift Wrapping* algorithm, which has

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running time  $O(nh)$ . This algorithm was later improved by Kirkpatrick and Seidel [28] and Chan [12], both of which achieve running time of  $O(n \log h)$ . In the online model, where input points are given one by one and algorithms need to compute the convex hull of points seen so far, Overmars and van Leeuwen's algorithm [33] can update the convex hull in  $O(\log^2 n)$  time per incoming point. Brodal and Jacob [9] reduced the update time to  $O(\log n)$ .

**Streaming Model.** The algorithms mentioned above all require  $s = \Omega(n)$  working space (memory) in the worst case. Therefore, none of these can handle the case when  $s \ll n$ , that is, when either  $n$  is very large (a massive data set) or  $s$  is very small (such as in embedded systems). In order to explore the convex hull problem with such a memory restriction, we consider the standard streaming models [5, 15, 16, 32, 36], where the  $n$  given points are stored on a read-only or writable tape in an arbitrary order. If the tape is read-only, then the model is simply called the *streaming model* [5, 32]. Otherwise the tape is writable, and the model is called the *W-stream model* [15, 16, 36]. We refer to algorithms in the streaming model as *streaming algorithms* and algorithms in the W-stream model as *W-stream algorithms*. In both models, algorithms can manipulate the working space while reading the points sequentially from the beginning of the tape to the end; however, only algorithms in the W-stream model can modify the tape, detailed in Section 4. Hence, algorithms in this model cannot access the input randomly, which is different from the model for in-place algorithms [8, 10]. The extreme points are written to a write-only stream. The *pass complexity* of an algorithm refers to the number of times the algorithms scans the tape from the beginning to the end. The goal is to devise streaming and W-stream algorithms that have small pass and space complexities.

No single-pass streaming algorithm can compute the convex hull using  $o(n)$  space because it is no easier than sorting  $n$  positive numbers in  $\mathbb{R}$ . Since sorting  $n$  numbers using  $s$  spaces requires  $\Omega(n/s)$  passes [31], computing the convex hull in a single pass requires linear space. However, Chan and Chen [13] showed that the space requirement can be significantly reduced if multi-pass algorithms are allowed. Specifically, their streaming algorithm uses  $O(\delta^{-2})$  passes,  $O(\delta^{-2}hn^\delta)$  space, and  $O(\delta^{-2}n \log n)$  time for any constant  $\delta \in (0, 1)$ . On the other hand, to have small space complexity, one can appeal to a general scheme to convert PRAM algorithms to W-stream algorithms established by Demetrescu et al. [15], summarized in Section 4. Using this technique yields a W-stream algorithm that uses  $O((n/s) \log h)$  passes and  $O(s)$  space where  $s$  can be as small as constant.

**Our Contribution.** We devise a new  $O(n \log h)$ -time RAM algorithm to compute the convex hull (Section 2). Then, we adapt the RAM algorithm to both models.

In the streaming model, the pass complexity of our algorithm is roughly a square root of that of Chan and Chen's algorithm [13] if both have the same space usage. We have:

► **Theorem 1.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}^2$  on a read-only tape where  $|ext(P)| = h$ , there exists a deterministic streaming algorithm to compute the convex hull of  $P$  in  $O(\delta^{-1})$  passes using  $O(\min\{\delta^{-1}hn^\delta \log n, n\})$  space and  $O(\delta^{-2}n \log n)$  time for every constant  $\delta \in (0, 1)$ .*

In the W-stream model, we adapt the RAM algorithm to two W-stream algorithms. One uses  $O(s)$  space for any  $s = \Omega(\log n)$  and the other uses  $O(s)$  space for any  $s = \Omega(1)$ . The pass complexity of our W-stream algorithms are  $O(\lceil h/s \rceil \log n)$  and  $O(h/s + \log n)$ , which are smaller than  $O((n/s) \log h)$ , the best pass complexity among those W-stream algorithms that are converted from PRAM algorithms in algebraic decision tree model [15], when  $s \leq h$ .

The first  $W$ -stream algorithm is deterministic, and we get:

► **Theorem 2.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}^2$  where  $|ext(P)| = h$ , there exists a deterministic  $W$ -stream algorithm to compute the convex hull of  $P$  in  $O(\lceil h/s \rceil \log n)$  passes using  $O(s)$  space and  $O(n \log^2 n)$  time for any  $s = \Omega(\log n)$ .*

Next, we randomize the above  $W$ -stream algorithm. A logarithmic factor can be shaved off from the pass complexity with probability  $1 - 1/n^{\Omega(1)}$ , abbreviated as w.h.p. We have:

► **Theorem 3.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}^2$  where  $|ext(P)| = h$ , there exists a randomized  $W$ -stream algorithm to compute the convex hull of  $P$  in  $p$  passes using  $O(s)$  space and  $O(n \log^2 n)$  time for any  $s = \Omega(1)$ , where  $p = O(h/s + \log n)$  w.h.p.*

We prove that our  $W$ -stream algorithms have nearly-optimal tradeoff between pass and space complexities by showing Theorem 4, which generalizes Guha and McGregor's lower bound (Theorem 8 in [22]). We remark that this lower bound is sharp because it matches the bounds of our randomized  $W$ -stream algorithm when  $h = \Omega(s \log n)$ .

► **Theorem 4.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}^2$  where  $|ext(P)| = h = \Omega(1)$ , any streaming (or  $W$ -stream) algorithm that computes the convex hull of  $P$  with success rate  $\geq 2/3$ , and uses  $s$  bits requires  $\Omega(\lceil h/s \rceil)$  passes.*

We note here that space is measured in terms of bits for lower bounds and in terms of points for upper bounds. This asymmetry is a common issue for geometric problems because most geometric problems are analyzed under the RealRAM model, where precision of points (or other geometric objects) is unbounded.

**Applications.** Our  $W$ -stream algorithms can handle the case for  $s \leq h$  because it outputs extreme points on the fly. This output stream can be used as an input stream for another streaming algorithm, such as for diameter [37] and minimum enclosing rectangle [38], both of which rely on Shamos' rotating caliper method [37]. Theorems 2 and 3 imply Corollary 5.

► **Corollary 5.** *Given a set  $P$  of  $n$  points in  $\mathbb{R}^2$  where  $|ext(P)| = h$ , there exists a deterministic  $W$ -stream algorithm to compute the diameter and minimum enclosing rectangles of  $P$  in  $O(\lceil h/s \rceil \log n)$  passes using  $O(s)$  space and  $O(n \log^2 n)$  time for every  $s = \Omega(\log n)$ . Given randomness, the pass complexity can be reduced to  $O(h/s + \log n)$  w.h.p.*

**Approximate Convex Hulls.** Given the hardness result shown in Theorem 4, we know that one cannot have a constant-pass streaming algorithm that uses  $o(h)$  space to compute the convex hull. In view of this, to have constant-pass  $o(h)$ -space streaming algorithms, one may consider computing an approximate convex hulls. There are several results studying on how to efficiently find an approximate convex hull in the streaming model, based on a given error measurement. The error criterion varies from the Euclidean distance [24], and Hausdorff metric distance [29, 30], to the relative area error [35]. These algorithms use a single pass,  $O(s)$  space, and can bound the given error measurement by a function of  $s$ .

**Paper Organization.** In Section 2, we present a new  $O(n \log h)$ -time RAM algorithm to compute the convex hull. Then, in Section 3, we present a constant-pass streaming algorithm. In Section 4, we present two  $W$ -stream algorithms, both of which use  $O(s)$  space where  $s$  can be as small as  $O(\log n)$ . Finally, in Section 5, we generalize the previous lower bound result.

■ **Table 1** Categorization of four  $O(n \log h)$ -time algorithms for convex hull.

	Find $r$ hull edges, and recurse.	Find $r$ extreme points, and recurse.
$r = 1$	Kirkpatrick and Seidel 1986 [28]	Chan 1995 [11]
any $r \geq 1$	Chan and Chen 2007 [13]	This paper

## 2 Yet another $O(n \log h)$ -time algorithm in the RAM model

Our streaming algorithm is based on a RAM algorithm, which we present in this section. This RAM algorithm is a modification of Kirkpatrick and Seidel’s ultimate convex hull algorithm in the RAM model [28]. Chan and Chen’s streaming algorithm [13] is also based on Kirkpatrick and Seidel’s algorithm, and thus the structure of these two streaming algorithms have some similarities. The changes are made so that our streaming algorithm does not have to rely on solving linear programs, thus reducing the computation cost compared to Chan and Chen’s algorithm.

In what follows, we only discuss how to compute the upper hull because the lower hull can be computed analogously. Formally, computing the **upper hull**  $U(P)$  of a point set  $P$  means outputting that part of the extreme points  $v_1, v_2, \dots, v_t \in \text{ext}(P)$  in clockwise order so that  $v_1$  is the leftmost point in  $P$  and  $v_t$  is the rightmost point in  $P$ , tie-breaking by picking the point with the largest  $y$ -coordinate, so that all points in  $P$  lie below or on the line passing through  $v_i, v_{i+1}$  for each  $1 \leq i < t$ . Note that each of  $v_1, v_2, \dots, v_t$  has a unique  $x$ -coordinate, and each line that passes through  $v_i$  and  $v_{i+1}$  for  $1 \leq i < t$  has a finite slope.

Roughly speaking, Kirkpatrick and Seidel’s ultimate convex hull algorithm [28] evenly divides the point set into two subsets by a vertical line  $\ell : x = \mu$ , finds the hull edge in the upper hull that crosses  $\ell$ , and recurses on the two separated subsets. By appealing to the point-line duality, finding the crossing hull edge is equivalent to solving a linear program. Chan and Chen’s streaming algorithm is adapted from this implementation of the ultimate convex hull algorithm. Their algorithm evenly divides the point set into  $r + 1$  subsets for  $r \geq 1$  by  $r$  vertical lines, finds the hull edges in the upper hull that cross these vertical lines, and recurses on the  $r + 1$  separated subsets. Finding these  $r$  crossing hull edges is equivalent to solving  $r$  linear programs, where the constraint sets for each are the same but the objective functions are different.

In [11, Section 2], Chan gives another version of Kirkpatrick and Seidel’s ultimate convex hull algorithm, that finds a suitable (possibly random) extreme point, divides the point set into two by  $x$ -coordinate, and recurses. The extreme point can be found by elementary techniques. Our streaming algorithm is adapted from the latter algorithm. It finds  $r$  suitable extreme points for  $r \geq 1$ , divides the point set into  $r + 1$  subsets by  $x$ -coordinate, and recurses on each subset. Though this generalization sounds straightforward, finding the  $r$  suitable extreme points needs a different approach from that for finding a single suitable extreme point. We reduce finding these  $r$  suitable extreme points to computing the upper hulls of  $n/(r + 1)$  small point sets. This reduction is the key observation of our RAM algorithm and is described in detail in the subsequent paragraphs. These four algorithms are categorized in Table 1.

Given  $r$ , our algorithm partitions  $P$  arbitrarily into  $G_1, G_2, \dots, G_{n/(r+1)}$  so that each  $G_j$  has size in  $[1, r + 1]$ , and then computes the upper hull of each  $G_j$ . Let  $Q$  be the union of the slopes of the hull edges in the upper hull of  $G_1, G_2, \dots, G_{n/(r+1)}$ , which is a multiset. Let  $\sigma_k$  be the slope of rank  $k|Q|/(r + 1)$  in  $Q$ , for  $k \in [1, r]$ , in other words,  $\sigma_k$  is the  $k$ th  $(r + 1)$ -quantile in  $Q$ . To simplify the presentation, let  $\sigma_0 = -\infty$  and  $\sigma_{r+1} = \infty$ . Let  $s_k$  be

the extreme point in  $P$  that **supports** slope  $\sigma_k$ , for each  $k \in [0, r + 1]$ . That is, for every point  $p \in P$  draw a line passing through  $p$  with slope  $\sigma_k$ , and pick  $s_k$  as the point whose line has the highest  $y$ -intercept. We define  $s_0 = p_L$ , the point with the smallest  $x$ -coordinate, and  $s_{r+1} = p_R$ , the point with the largest  $x$ -coordinate. If any  $s_k$  has more than one candidates, pick the point that has the largest  $y$ -coordinate. Let  $x(p)$  denotes the  $x$ -coordinate of point  $p$ , and let  $\sigma(p, q)$  denote the slope of the line that passes through points  $p$  and  $q$ .

We use these  $s_1, s_2, \dots, s_r$  as the  $r$  **suitable** extreme points with which to refine  $P$  into  $P_1, P_2, \dots, P_{r+1}$  where we say the  $s_i$  are suitable in that each  $P_k$  has size bounded by  $O(|P|/(r + 1))$ . Initially, set  $P_k = \emptyset$  for all  $k \in [1, r + 1]$ . The refinement applies the **cascade-pruning** described in Lemma 7 on  $G_j$  for each  $j \in [1, n/(r + 1)]$ , which uses the known pruning technique stated in Lemma 6 as a building block, and works as follows:

- Step 1. Compute the extreme points  $v_1, v_2, \dots, v_t \in U(G_j)$  in clockwise order.
- Step 2. Set  $P_k \leftarrow P_k \cup \{v_i : i \in [\alpha, \beta], x(s_{k-1}) < x(v_i) < x(s_k)\}$  for each  $k \in [1, r + 1]$ , where  $v_\alpha$  (resp.  $v_\beta$ ) is the extreme point in  $G_j$  that supports  $\sigma_{k-1}$  (resp.  $\sigma_k$ ).

The pruning in Step 2 is two-fold. For any  $i < \alpha$ , if  $x(v_i) \leq x(s_{k-1})$ , then  $v_i$  cannot be placed in  $P_k$ . Otherwise  $x(v_i) > x(s_{k-1})$ , Case 2 of Lemma 7 applies. Again,  $v_i$  cannot be placed in  $P_k$ . Similarly,  $v_i$  for any  $i > \beta$  cannot be placed in  $P_k$  either. Finally, remove the points that lie below or on the line passing through  $s_{k-1}, s_k$  from  $P_k$  for each  $k \in [1, r + 1]$ .

► **Lemma 6** (Chan, [11]). *Given a point set  $P \subset \mathbb{R}^2$  and a slope  $\sigma$ , let  $s$  be the extreme point in  $P$  that supports  $\sigma$ . Then, for any pair of points  $p, q \in P$  where  $x(p) < x(q)$ ,*

- *Case 1. If  $\sigma(p, q) \leq \sigma$  and  $x(q) \leq x(s)$ , then  $q \notin U(P)$ .*
- *Case 2. If  $\sigma(p, q) \geq \sigma$  and  $x(p) \geq x(s)$ , then  $p \notin U(P)$ .*

► **Lemma 7** (Cascade-pruning). *Given a point set  $P \subset \mathbb{R}^2$  and a slope  $\sigma$ , let  $s$  be the extreme point in  $P$  that supports  $\sigma$ . Then, for any  $G \subseteq P$  whose  $U(G) = \{v_1, v_2, \dots, v_t\}$ ,  $x(v_1) < x(v_2) < \dots < x(v_t)$ , and where  $\delta \in [1, t]$  is such that  $v_\delta$  is the extreme point in  $G$  that supports  $\sigma$ , we have:*

- *Case 1. If  $x(v_i) \leq x(s)$  for some  $i \in [\delta + 1, t]$ , then  $v_{\delta+1}, \dots, v_i \notin U(P)$ .*
- *Case 2. If  $x(v_i) \geq x(s)$  for some  $i \in [1, \delta - 1]$ , then  $v_i, \dots, v_{\delta-1} \notin U(P)$ .*

**Proof.** Observe that  $\sigma(v_j, v_{j+1}) \geq \sigma$  for all  $j \in [1, \delta - 1]$  and  $\sigma(v_{j-1}, v_j) \leq \sigma$  for all  $j \in [\delta + 1, t]$  because  $v_1, v_2, \dots, v_t$  are extreme points in  $U(G)$  in clockwise order and  $v_\delta$  is the extreme point in  $G$  that supports  $\sigma$ . Since there is an  $i \in [\delta + 1, t]$  such that  $x(v_i) \leq x(s)$ , we have  $x(v_j) \leq x(s)$  for each  $j \in [\delta + 1, i]$ . The above are exactly the conditions of Case 1 in Lemma 6 for all point pairs  $(v_{j-1}, v_j)$  whose  $j \in [\delta + 1, i]$ . Thus,  $v_j \notin U(P)$  for all  $j \in [\delta + 1, i]$ . The other case can be proved analogously. ◀

We get the exact bound for each  $P_k$  in Lemma 8, noting that  $|P_k| \leq \frac{3}{4}|P|$  for  $r = 1$ .

► **Lemma 8.**  $|P_k| \leq (\frac{2}{r+1} - \frac{1}{(r+1)^2})|P| \leq 2|P|/(r + 1)$  for each  $k \in [1, r + 1]$ .

**Proof.** To ensure that, for every  $k \in [1, r + 1]$ ,  $P_k$  is a small fraction of  $P$ , we use the cascade-pruning procedure described in Lemma 7. Let  $\{v_1, v_2, \dots, v_t\}$  be  $U(G_j)$  for some  $j \in [n/(r + 1)]$  where  $x(v_1) < x(v_2) < \dots < x(v_t)$ . Let  $v_{\alpha_j}$  (resp.  $v_{\beta_j}$ ) be the extreme point in  $G_j$  that supports  $\sigma_{k-1}$  (resp.  $\sigma_k$ ).

Let  $n_j$  be the number of points in  $P_k \cap G_j$ . Recall that  $P_k$  does not contain any  $v_i$  for any  $i \notin [\alpha_j, \beta_j]$ , and hence  $n_j \leq \beta_j - \alpha_j + 1$ . Observe that point pair  $(v_i, v_{i+1})$  has slope in the open interval  $(\sigma_{k-1}, \sigma_k)$  for each  $i \in [\alpha_j, \beta_j - 1]$ . Since  $\sigma_{k-1}$  (resp.  $\sigma_k$ ) is the

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**RAM Algorithm:** Compute the upper hull  $U(P)$  of  $P$ .

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1 Let  $G_1, G_2, \dots, G_{n/(r+1)}$  be any partition of  $P$  such that each  $G_j$  has size in  $[1, r+1]$ ;
2  $Q \leftarrow \emptyset$ ;
3 foreach  $G_j$  in the partition do
4   Compute the upper hull  $v_1, v_2, \dots, v_t$  of  $G_j$ ;
5   for  $i = 1$  to  $t - 1$  do
6      $\sigma \leftarrow$  the slope of the line passing through  $v_i, v_{i+1}$ ;
7      $Q \leftarrow Q \cup \{\sigma\}$ ;
8   end
9 end
10 for  $k = 1$  to  $r$  do
11    $\sigma_k \leftarrow$  the  $k|Q|/(r+1)$ -th smallest slope in  $Q$ ;
12    $s_k \leftarrow$  the extreme point in  $P$  that supports  $\sigma_k$ ;
13 end
14  $(s_0, \sigma_0, s_{r+1}, \sigma_{r+1}) \leftarrow (p_L, -\infty, p_R, \infty)$ ;
15 for  $k = 1$  to  $r + 1$  do
16    $P_k \leftarrow \emptyset$ ;
17   foreach  $G_j$  in the partition do
18     Compute the upper hull  $v_1, v_2, \dots, v_t$  of  $G_j$ ;
19     Find the extreme point  $v_\alpha$  (resp.  $v_\beta$ ) in  $G_j$  that supports  $\sigma_{k-1}$  (resp.  $\sigma_k$ );
20      $P_k \leftarrow P_k \cup \{v_\alpha, v_{\alpha+1}, \dots, v_\beta\}$ ;
21   end
22   Remove the points that lie below or on the line passing through  $s_{k-1}, s_k$  from  $P_k$ ;
23   if  $P_k \neq \emptyset$  then
24     Recurse on  $P_k \cup \{s_{k-1}, s_k\}$ ;
25   end
26 end

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$(k-1)|Q|/(r+1)$ -th largest slope (resp. the  $k|Q|/(r+1)$ -th largest slope) in  $Q$ ,  $Q$  has at most  $|Q|/(r+1)$  slopes in the open interval  $(\sigma_{k-1}, \sigma_k)$ . This yields that

$$\sum_{j=1}^{n/(r+1)} n_j - 1 \leq \frac{|Q|}{r+1} \Rightarrow \sum_{j=1}^{n/(r+1)} n_j \leq \frac{|Q|}{r+1} + \frac{n}{r+1} \leq \frac{r|P|}{(r+1)^2} + \frac{|P|}{r+1}$$

The last inequality holds because  $|Q| \leq r|P|/(r+1)$ , and it establishes that the number of points from all  $G_j$ 's that comprise  $P_k$  for each  $k \in [1, r+1]$  is at most  $2|P|/(r+1)$ .  $\blacktriangleleft$

For each  $k \in [1, r+1]$ , if  $P_k \neq \emptyset$ , then our algorithm recurses on  $P_k \cup \{s_{k-1}, s_k\}$ . This ensures that every subproblem has an input that contains some intermediate extreme point(s), i.e. not the leftmost and rightmost extreme points, and any two subproblems where one is not an ancestor or a descendant of the other have an empty intersection in their intermediate extreme point set. As a result,

► **Lemma 9.** *Our algorithm has  $O(h)$  leaf subproblems.*

Here we analyze the running time of the RAM algorithm for the case of  $r = O(1)$  and defer the discussion for the case of  $r = \omega(1)$  until the section on streaming algorithms. Let  $T_C$  be the recursive computation tree of the RAM algorithm. The root of  $T_C$  represents the initial problem of the recursive computation. Every node in  $T_C$  has at most  $r+1$  child nodes, each of which represents a recursive subproblem.

For a computation node with the input point set  $P$  whose  $|P| < r$ , we use any  $O(|P| \log r)$ -time algorithm to compute the convex hull. Otherwise, we need to compute  $|P|/(r+1)$  convex hulls of point sets of size at most  $r+1$ , which runs in  $O(|P| \log r)$  time (Lines 1-9). In addition, the quantile selection in  $Q$  has the running time  $O(|Q| \log r) = O(|P| \log r)$  (Line 11). The  $r$  suitable extreme points can be found in  $O(|P| \log r)$  time by Lemma 15 (Line 12). The pruning procedure can be done in  $O(|P| \log r)$  time by a simple merge (Lines 15-26). Hence, each computation node needs  $O(|P| \log r)$  time.

Since each child subproblem has an input set  $P_k \cup \{s_{k-1}, s_k\}$  of size at most  $2|P|/(r+1)+2$  (Lemma 8), the running time of child subproblem is an  $(2/(r+1))$ -fraction of its parent subproblem. Hence,  $T_C$  is an  $(2/(r+1))$ -**fading** computation tree where Edelsbrunner and Shi [17] define a recursive computation tree to be  $\alpha$ -fading for some  $\alpha < 1$  if the running time of a child subproblem is an  $\alpha$ -fraction of its parent. In [11], Chan extends Edelsbrunner and Shi's results and obtains that, if an  $\alpha$ -fading recursive computation tree has  $L$  leaf nodes and the total running time of the nodes on each level is at most  $F$ , then the recursive computation tree has total running time  $O(F \log L)$ . Our algorithm has  $O(h)$  leaf nodes (Lemma 9) and  $O(|P| \log r)$  time for the computation nodes on each level because two subproblems on the same level have their inputs only intersected at one of their extreme points. We get:

► **Theorem 10.** *The RAM algorithm runs in  $O(n \log h \log r)$  time, and for  $r = O(1)$  it is an  $O(n \log h)$ -time algorithm.*

### 3 A Simpler and Faster Streaming Algorithm

In this section, we show how to adapt our RAM algorithm to the streaming model. Our streaming algorithm is the same as our RAM algorithm, but we execute the subproblems on  $T_C$  in BFS order. That is, starting from the root of  $T_C$ , all subproblems on  $T_C$  of the same level are solved together in a round, then their invoked subproblems are solved together in the next round, and so on. We will see in a moment that our algorithm needs to scan the input  $O(1)$  times for each round. Therefore, to have an  $O(1)$ -pass streaming algorithm, our approach requires  $r = n^\delta$  for some positive constant  $\delta < 1$ . By setting  $r = n^\delta$ , we have:

► **Lemma 11.** *By setting the parameter  $r$  to be  $n^\delta$  for any constant  $\delta \in (0, 1)$ , the recursive computation tree  $T_C$  has  $O(\delta^{-1}h)$  nodes.*

**Proof.** This lemma holds because  $T_C$  has depth  $O(\log_r n) = O(\delta^{-1})$  by Lemma 8 and  $T_C$  has  $O(h)$  leaf nodes by Lemma 9. ◀

We assign a unique identifier  $z \in [1, |T_C|]$  to each of  $|T_C| = O(\delta^{-1}h)$  subproblems. Let  $S_z$  be the subproblem on  $T_C$  whose identifier is  $z$ . For each  $z \in [1, |T_C|]$ ,  $S_z$  has input point set  $P_z$ .  $P_z$  is a subsequence of  $P$  and is given to  $S_z$  as an input stream of  $|P_z|$  points. Our algorithm will generate  $P_z$  more than once for  $S_z$  to access, for all  $z \in [1, |T_C|]$ . The data structures used in  $S_z$  also are suffixed with  $z$ . To compute  $S_z$ , naively we need  $O(|P_z|)$  space. We will see in a moment that given  $P_z$ , how to solve  $S_z$  using  $O(r \log r |P_z|)$  space in  $O(r \log |P_z| + |P_z| \log r)$  time. We will also see how to generate the input for all the subproblems on  $T_C$  of depth  $d > 0$  in  $O(1)$  passes. We now establish all these claims, after which we will be ready to prove Theorem 1. We decompose  $S_z$  into the following three subtasks and describe the algorithms for the subtasks in the subsequent subsections. (1) Given  $P_z$ , obtain the  $r$  quantile slopes  $\sigma_1, \sigma_2, \dots, \sigma_r$ . (2) Given  $P_z$  and  $\sigma_1, \sigma_2, \dots, \sigma_r$ , obtain the  $r$  suitable extreme points  $s_1, s_2, \dots, s_r$ . (3) After the ancestor subproblems of  $S_z$  (excluding  $S_z$ ) are all solved, generate  $P_z$ .

### 3.1 Obtaining the $r$ quantile slopes

To find the  $r$  quantile slopes for  $S_z$  (Lines 1-11 in the RAM algorithm) using small space, we use a Greenwald and Khanna [21] quantile summary structure, abbreviated as  $QS_z$ . This summary is a data structure that supports two operations: insert a slope ( $QS_z.\text{insert}(\sigma)$ ) and query for (an estimate of) the  $t$ -th smallest slope ( $QS_z.\text{query}(t)$ ) in  $Q_z$ . Given access to  $QS_z$ , we do not have to store the entire  $P_z$  to obtain the  $r$  quantile slopes. Instead, we invoke  $QS_z.\text{insert}(\sigma)$  for each slope  $\sigma \in Q_z$ . After updating all slopes in  $Q_z$ , we obtain an estimate of the  $(r+1)$ -quantile of  $Q_z$  by invoking  $QS_z.\text{query}(k|Q_z|/(r+1))$  for all  $k \in [1, r]$ .

$QS_z.\text{query}(k|Q_z|/(r+1))$  returns an estimate  $\hat{\sigma}_k$  that has an additive error  $c|Q_z|$  in the rank, where  $c$  is a parameter to be determined. We set  $c = \varepsilon/(r+1)$  for some constant  $\varepsilon > 0$  so that the additive error cannot increase the depth of  $T_C$  by more than a constant factor. Precisely, because the obtained  $\hat{\sigma}_k$  has the rank in the range  $[(k-\varepsilon)|Q_z|/(r+1), (k+\varepsilon)|Q_z|/(r+1)]$  for each  $k \in [1, r]$ , we need to replace Lemma 8 with Corollary 12. Such a replacement increases the depth of  $T_C$  from  $O(\log_r n) = O(\delta^{-1})$  to  $O(\log_{r/(1+\varepsilon)} n) = O(\delta^{-1}) + o(1)$ .

► **Corollary 12.**  $|P_k| \leq (\frac{2+2\varepsilon}{r+1} - \frac{1}{(r+1)^2})|P| \leq 2(1+\varepsilon)|P|/(r+1)$  for each  $k \in [1, r+1]$ .

The summary  $QS_z$  needs  $O(\frac{1}{c} \log(c|Q_z|))$  space, and therefore the space usage for each subproblem is  $O((r/\varepsilon) \log((\varepsilon/r)|Q_z|))$ . In [39], it shows that Greenwald and Khanna's quantile summary needs  $O(\log |Q_z|)$  time for an update and  $O(\log r + \log \log(|Q_z|/r))$  for a query. Because  $S_z$  conducts  $O(r)$  updates and  $O(r)$  queries, we get:

► **Lemma 13.** *Given  $P_z$ , some streaming algorithm can find the  $r$  approximate quantile slopes in  $Q_z$  to within any  $O(1)$  factor in  $O(r \log(|P_z|+r))$  time using  $O(r \log(|P_z|/r))$  space.*

### 3.2 Obtaining the $r$ suitable extreme points

To find the  $r$  suitable extreme points in  $P_z$  (Line 12 in the RAM algorithm), a naive implementation, which would update the supporting points of  $\hat{\sigma}_k$  for all  $k \in [1, r]$  once for each point  $p \in P_z$ , needs  $O(r|P_z|)$  running time. To reduce the running time to the claimed time complexity  $O(r \log |P_z| + |P_z| \log r)$ , we need the following observation.

► **Observation 14.** *For any non-singleton set  $G$  whose extreme points in the upper hull  $U(G)$  from left to right are  $v_1, v_2, \dots, v_t$ , the point in  $G$  that supports a given slope  $\sigma$  is*

$$s = \begin{cases} v_1 & \text{if } \sigma > \sigma(v_1, v_2) \\ v_t & \text{if } \sigma < \sigma(v_{t-1}, v_t) \\ v_i & \text{if } \sigma(v_{i-1}, v_i) \geq \sigma \geq \sigma(v_i, v_{i+1}) \text{ for some } i \in [2, t-2] \end{cases}$$

To find the extreme points in  $P_z$  that supports  $\hat{\sigma}_k$  for all  $k \in [1, r]$ , we compute the extreme points  $v_1, v_2, \dots, v_t$  in  $P_z$  from left to right, generate a (sorted) list  $\ell_A$  of slopes  $\sigma(v_1, v_2), \sigma(v_2, v_3), \dots, \sigma(v_{t-1}, v_t)$ , and merge  $\ell_A$  with another (sorted) list  $\ell_B$  of the approximate  $(r+1)$ -quantile slopes  $\hat{\sigma}_1, \hat{\sigma}_2, \dots, \hat{\sigma}_r$ . By Observation 14, the point  $\hat{s}_k$  in  $P_z$  that supports  $\hat{\sigma}_k$  for each  $k \in [1, r]$  can be easily determined by the its predecessor and successor in  $\ell_A$ . Scanning the merged list suffices to get  $\hat{s}_1, \hat{s}_2, \dots, \hat{s}_k$ . Though the above reduces the time complexity to  $O(r + |P_z| \log |P_z|)$ , the space complexity  $O(|P_z|)$  is much higher than the claimed space complexity  $O(r \log r |P_z|)$  for  $r \ll |P_z|$ . To remedy, again, we reduce this problem to computing the upper hulls of  $|P_z|/(r+1)$  smaller point sets. First, we partition  $P_z$  arbitrarily into  $G_1, G_2, \dots, G_{|P_z|/(r+1)}$  so that each group  $G_i$  has size  $|G_i| \in [1, r+1]$  points. Then, for each  $G_i$  we apply the above accordingly. We get:

► **Lemma 15.** *Given  $P_z$  and sorted  $\sigma_1, \dots, \sigma_r$ , some streaming algorithm can find the extreme points in  $P_z$  that support  $\sigma_i$  for all  $i \in [1, r]$  in  $O(r + |P_z| \log r)$  time using  $O(r)$  space.*



### 3.3 Generating the input point set $P_z$ for each subproblem $S_z$

Recall that we execute the subproblems in  $T_C$  in BFS order. Upon executing the subproblems of depth  $d$  for any  $d > 0$ , all the subproblems of depth  $< d$  are done and the associated  $r$  quantile slopes and  $r$  suitable extreme points are memoized in memory. For  $d = 0$ , we need to generate the input for the initial problem  $S_o$ , i.e.  $P$ , so scanning over  $P$  suffices.

Given the associated  $r$  quantile slopes and  $r$  suitable extreme points for all the subproblems of depth less than  $d$ , to generate the input point sets for all the subproblems of depth  $d$ , we can directly execute Lines 15-26 in the RAM algorithm for all the subproblems of depth less than  $d$  and ignore Lines 1-14 because the intermediate values, the quantile slopes and suitable extreme points, are already computed and kept in memory. Initially, we allocate a buffer  $B_z$  of size  $r + 1$  for each subproblem  $S_z$  of depth less than  $d$  so as to temporarily store the incoming input points, i.e. points in  $P_z$ . Then, we scan  $P$  on the input tape once and for each input point  $p$  in  $P$ , we place  $p$  in the buffer  $B_o$  of  $S_o$ . Once any buffer  $B_z$  gets full or the input terminates, we let  $B_z$  be some  $G_i$ , a part in the partition of  $P_z$ , and apply the pruning procedure stated in Lines 15-26 in the RAM algorithm. Those points that survive the pruning are flushed, one by one, into the buffers of  $S_z$ 's child subproblems. We apply the above iteratively until we reach the end of the input tape. The space usage counted on each  $S_z$  is  $O(|B_z|) = O(r)$  and the overall running time to generate the input point set for all the subproblems of depth  $d > 0$  is  $O(dn \log r)$  because all the subproblems of each depth  $i \in [1, d - 1]$  computes the upper hull of points sets, disjoint subsets of  $P$ . Hence, we get:

► **Lemma 16.** *Some streaming algorithm can generate the input for all depth- $d$  subproblems on  $T_C$  for each  $d \in [0, \text{depth}(T_C)]$  using  $O(1)$  passes,  $O(hr)$  space, and  $O(dn \log r)$  time.*

**Proof of Theorem 1.** For  $r = n^\delta$ ,  $T_C$  has  $O(\delta^{-1}h)$  nodes and depth  $O(\delta^{-1})$  by Lemma 11, 8. Hence, the space complexity of our streaming algorithm is the sum of  $O(\delta^{-1}h)$  times the space complexity in Lemma 13, 15, and  $O(\delta^{-1})$  times the space complexity in Lemma 16. The overall space complexity is  $O(\delta^{-1}hn^\delta \log n)$ . One can obtain the space bound  $O(\min\{\delta^{-1}hn^\delta \log n, n\})$  by checking whether  $\delta^{-1}hn^\delta \log n > n$  before proceeding to the subproblems on the next depth, where  $\delta^{-1}h$  is the number of subproblems executed so far and thus  $\delta^{-1}h = O(\delta^{-1})$ . If so, we compute the convex hull by a RAM algorithm. Analogously, we have that the pass (resp. time) complexity of our streaming algorithm is  $O(\delta^{-1})$  (resp.  $O(\delta^{-2}n \log n)$ ). ◀

## 4 A W-Stream Algorithm Of Nearly-Optimal Pass-Space Tradeoff

Demetrescu et al. [15] establish a general scheme to convert PRAM algorithms to W-stream algorithms. Theorem 17 is an implication of their main result.

► **Theorem 17** (Demetrescu et al. [15]). *If there exists a PRAM algorithm that uses  $m$  processors to compute the convex hull of  $n$  given points in  $t$  rounds, then there exists an  $O(s)$ -space  $O(mt/s)$ -pass W-stream algorithm to compute the convex hull.*

There is a long line of research that studies how to compute the convex hull of  $n$  given points efficiently in parallel [1, 3, 4, 14, 19, 23]. In particular, Akl's PRAM algorithm [1] uses  $O(n^\varepsilon)$  processors and runs in  $O(n^{1-\varepsilon} \log h)$  time for any  $\varepsilon \in (0, 1)$ . Converting Akl's PRAM algorithm to a W-stream algorithm by Theorem 17, we have:

► **Corollary 18.** *There exists an  $O((n/s) \log h)$ -pass W-stream algorithm that can compute the convex hull of  $n$  given points using  $O(s)$  space.*

The optimal work, i.e. the total number of primitive operations that the processors perform, for any parallel algorithm in the algebraic decision tree model to compute the convex hull is  $O(n \log h)$  [23, 28]. Therefore the W-stream algorithm stated in Corollary 18 is already the best possible among those W-stream algorithms that are converted from a PRAM algorithm in the algebraic decision tree model by Theorem 17. However, in this Section, we will show that such a tradeoff between pass complexity and space usage is suboptimal by devising a W-stream algorithm that has a better pass-space tradeoff. Together with the results shown in Section 5, we have that the pass-space tradeoff of our W-stream algorithm is nearly optimal.

#### 4.1 Deterministic W-stream Algorithm

Our deterministic W-stream algorithm is the same as our streaming algorithm, except for the following differences:

- We set  $r = 1$  (rather than  $r = n^\delta$ ) for our deterministic W-stream algorithm. Thus, by Corollary 12  $\text{depth}(T_C)$  increases from  $O(\delta^{-1})$  to  $O(\log n)$ , but the space usage of subproblem  $S_z$  decreases from  $O(n^\delta \log n)$  to  $O(\log n)$  for each  $z \in [1, |T_C|]$ . Moreover, if the extreme point in the input  $P$  that supports the approximate median slope is the leftmost point  $p_L$  or the rightmost point  $p_R$ , i.e. the degenerate case, we replace it with the extreme point that supports  $\sigma(p_L, p_R)$ . In this way, each subproblem on  $T_C$  has a unique extreme point and therefore the number of subproblems on  $T_C$  is  $O(h)$ .
- Our streaming algorithm executes the subproblems on  $T_C$  in BFS order, that is, all subproblems of depth  $d$  are executed in a round for each  $d \in [0, \text{depth}(T_C)]$ . In contrast, our deterministic W-stream algorithm refines a single round into subrounds, in each of which it takes care of  $O(s/\log n)$  subproblems, so as to bound the working space by  $O(s)$ .
- Note that algorithms in the W-stream model are capable of modifying the input tape. Formally, while scanning the input tape in the  $i$ -th pass, algorithms can write something on a write-only output stream; in the  $(i+1)$ -th pass, the input tape read by algorithms is the output tape written in the  $i$ -th pass. Hence, our deterministic W-stream algorithm is able to assign an attribute to each point  $p \in P$  to indicate that  $p$  is an input of a certain subproblem. Moreover, our deterministic W-stream algorithm can write down the parameters for every subproblem on the output tape. In each subround, our deterministic W-stream algorithm needs to scan the input twice. The first pass is used to load the parameters of subproblems to be solved in the current subround. The second pass is used to scan the input tape and process the points that are the input points for the subproblems to be solved in the current subround.

**Proof of Theorem 2.** Suppose there are  $h_d$  subproblems of depth  $d$  on  $T_C$  for each  $d \in [0, \text{depth}(T_C)]$ , then our deterministic W-stream algorithm has to execute  $\sum_d \left\lceil \frac{h_d}{\lfloor s/\Theta(\log n) \rfloor} \right\rceil = O(\lceil h/s \rceil \log n)$  subrounds for any  $s = \Omega(\log n)$ . Because our deterministic W-stream algorithm scans the input tape twice for each subround, the pass complexity is  $O(\lceil h/s \rceil \log n)$ .

As shown in Section 3, subproblem  $S_z$  needs  $O(|P_z| \log |P_z|)$  running time. Since the input of subproblems of depth  $d$  on  $T_C$  are disjoint subsets of  $P$ , for each  $d \in [0, |T_C|]$ . We get that the time complexity is  $O(n \log^2 n)$ . ◀

#### 4.2 Randomized W-stream Algorithm

Observe that for  $r = 1$ , finding the  $r$  approximate quantile slopes in  $Q_z$  is exactly finding the approximate median slope in  $Q_z$ . Our algorithms mentioned previously all use Greenwald and Khanna quantile summary structure, which needs  $O(\log n)$  space. In our randomized

W-stream algorithm, we replace the Greenwald and Khanna quantile summary with a random slope in  $Q_z$ , thereby reducing the space usage to  $O(1)$ . As noted by Bhattacharya and Sen [7], such a replacement cannot increase the depth of  $T_C$  by more than a constant factor w.h.p.

**Proof of Theorem 3.** Similar to the arguments used in the proof of Theorem 2, the pass complexity of our randomized W-stream algorithm is  $\sum_{d \in [0, \text{depth}(T_C)]} \left\lceil \frac{h_d}{\lfloor s/\Theta(1) \rfloor} \right\rceil = O(h/s + \log n)$  for any  $s = \Omega(1)$  w.h.p. and the time complexity is  $O(n \log^2 n)$  w.h.p. ◀

## 5 Unconditional Lower Bound

In this section, we will show that any streaming (or W-stream) algorithm that can compute the convex hull with success rate  $> 2/3$  using  $O(s)$  space requires  $\Omega(\lceil h/s \rceil)$  passes (i.e. Theorem 4). This establishes the near-optimality of our proposed algorithms. We note here that the lower bound holds even if the output is  $|ext(P)|$ , rather than the set  $ext(P)$ .

We construct a point set  $U$  so that it is hard to compute the convex hull of point set  $P = Q \cup \{(1, 0), (-1, 0)\}$  for all  $Q \subseteq U$ . Let  $C_1, C_2$  be concentric half circles. The radius of  $C_1$  equals 1 and that of  $C_2$  is any value in  $(k, 1)$  for some  $k$  to be determined later. Let  $a_0, a_1, \dots, a_{n+1}$  be points distributed evenly on  $C_1$  so that  $a_0 = (1, 0)$  and  $a_{n+1} = (-1, 0)$ . Define  $b_0, b_1, \dots, b_{n+1}$  on  $C_2$  similarly. Let  $k$  be the distance between the origin  $O$  and the line  $\overleftrightarrow{a_i a_{i+2}}$  for any  $i \in [0, n-1]$ . Let  $U$  be the set  $\{a_i : i \in [1, n]\} \cup \{b_i : i \in [1, n]\}$ .

We need the following geometric property of points in  $U$  for the hardness proof.

► **Lemma 19.** *For every  $Q \subseteq U$ , let  $R = ext(Q \cup \{(1, 0), (-1, 0)\})$ . We have that (1)  $a_i \in Q \Rightarrow a_i \in R$ , and (2)  $(b_i \in Q \Rightarrow b_i \in R)$  iff  $a_i \notin Q$ .*

**Proof.** Due to space constraints, we defer the proof to the full version of this paper [18]. ◀

Lemma 19 implies the fact that, for every  $Q \subseteq U$ ,  $|ext(Q \cup \{(1, 0), (-1, 0)\})| = |Q| + 2$  if and only if  $a_i$  and  $b_i$  are not both contained in  $Q$  for each  $i$ . Given this fact, we are ready to perform a reduction from the *set disjointness* problem (a two-party communication game) to computing the convex hull in the streaming (and W-stream) model. Set disjointness is defined as follows. Alice has a private  $(\alpha n)$ -size subset  $A$  of  $[n]$ , and Bob has another private  $(\alpha n)$ -size subset  $B$  of  $[n]$  for some constant  $\alpha < 1/2$ . The goal is to answer whether  $A$  and  $B$  have a non-empty intersection. Based on the hardness result of set-disjointness, due to Kalyanasundaram and Schintger [27], we are ready to prove Theorem 4.

**Proof of Theorem 4.** Due to space constraints, we defer the proof to the full version of this paper [18]. ◀

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