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Spectral Properties for Polynomial and Matrix Operators Involving Demicompactness Classes

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Abstract: The first aim of this paper is to show that a polynomially demicompact operator satisfying certain conditions is demicompact. Furthermore, we give a refinement of the Schmoëger and the Rakocević essential spectra of a closed linear operator involving the class of demicompact ones. The second aim of this work is devoted to provide some sufficient conditions on the inputs of a closable block operator matrix to ensure the demicompactness of its closure. An example involving the Caputo derivative of fractional of order α is provided. Moreover, a study of the essential spectra and an investigation of some perturbation results.

Key words: Matrix operator, Demicompact linear operator, Fredholm and semi-Fredholm operators, Perturbation theory, Essential spectra.

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1. Introduction

Let X and Y be two Banach spaces. The set of all closed densely defined (resp. bounded) linear operators acting from X into Y is denoted by $\mathcal{C}(X,Y)$ (resp. $\mathcal{L}(X,Y)$). We denote by $\mathcal{K}(X,Y)$ the subset of compact operators of $\mathcal{L}(X,Y)$. For $T \in \mathcal{C}(X,Y)$, we use the following notations: $\alpha(T)$ is the dimension of the kernel $\mathcal{N}(T)$ and $\beta(T)$ is the codimension of the range $\mathcal{R}(T)$ in Y. The next sets of upper semi-Fredholm, lower semi-Fredholm, Fredholm and semi-Fredholm operators from X into Y are, respectively, defined by:

$$\Phi_+(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \alpha(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi_-(X,Y) = \{T \in \mathcal{C}(X,Y) \text{ such that } \beta(T) < \infty \text{ and } \mathcal{R}(T) \text{ closed in } Y\},$$
$$\Phi(X,Y) := \Phi_-(X,Y) \cap \Phi_+(X,Y)$$

and

$$\Phi_\pm(X,Y) := \Phi_-(X,Y) \cup \Phi_+(X,Y).$$

For $T \in \Phi_{\pm}(X,Y)$, the index is defined as $i(T) := \alpha(T) - \beta(T)$. A complex number λ is in $\Phi_{+T}, \Phi_{-T}, \Phi_{\pm T}$ or Φ_T if $\lambda - T$ is in $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_{\pm}(X,Y)$ or $\Phi(X,Y)$, respectively. If $X = Y$, then $\mathcal{L}(X,Y)$, $\mathcal{C}(X,Y)$, $\mathcal{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi_{\pm}(X,Y)$ are replaced by $\mathcal{L}(X)$, $\mathcal{C}(X)$, $\mathcal{K}(X)$, $\Phi(X)$, $\Phi_+(X)$, $\Phi_-(X)$ and $\Phi_{\pm}(X)$, respectively. If $T \in \mathcal{C}(X)$, we denote by $\rho(T)$ the resolvent set of T and by $\sigma(T)$ the spectrum of T. Let $T \in \mathcal{C}(X)$. For $x \in \mathcal{D}(T)$, the graph norm $\|.\|_T$ of x is defined by $\|x\|_T = \|x\| + \|Tx\|$. It follows from the closedness of T that $X_T := (\mathcal{D}(T), \|.\|_T)$ is a Banach space. Clearly, for every $x \in \mathcal{D}(T)$ we have $\|Tx\| \leq \|x\|_T$, so that $T \in \mathcal{L}(X_T, X)$. A linear operator B is said to be T-defined if $\mathcal{D}(T) \subseteq \mathcal{D}(B)$. If the restriction of B to $\mathcal{D}(T)$ is bounded from X_T into X, we say that B is T-bounded. An operator $T \in \mathcal{L}(X,Y)$ is said to be weakly compact if $T(B)$ is, relatively weakly compact in Y for every bounded set $B \subset X$. The family of weakly compact operators from X into Y is denoted by $\mathcal{W}(X,Y)$. If $X = Y$, the family of weakly compact operators on X which is denoted by $\mathcal{W}(X) := \mathcal{W}(X,X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.

Definition 1.1. Let X and Y be two Banach spaces and let $F \in \mathcal{L}(X,Y)$. The operator F is called:

(a) Fredholm perturbation if $T + F \in \Phi(X,Y)$ whenever $T \in \Phi(X,Y)$.

(b) Upper semi-Fredholm perturbation if $T + F \in \Phi_+(X,Y)$ whenever $T \in \Phi_+(X,Y)$.

(c) Lower semi-Fredholm perturbation if $T + F \in \Phi_-(X,Y)$ whenever $T \in \Phi_-(X,Y)$.

The set of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y), \mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively.

The concept of demicompactness appeared in the literature since 1966 in order to discuss fixed points. It was introduced by W. V. Petryshyn [16] as follows:

Definition 1.2. An operator $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ is said to be demicompact if for every bounded sequence $(x_n)_n$ in $\mathcal{D}(T)$ such that $x_n - Tx_n \to x \in X$, there exists a convergent subsequence of $(x_n)_n$. The family of demicompact operators on X is denoted by $\mathcal{DC}(X)$.

It is clear that the sum, the product of demicompact operators and the product of a complex number by a demicompact operator are not necessarily demicompact. W. V. Petryshyn [16] and W. Y. Akashi [1] used the class of demicompact operators to obtain some results on Fredholm perturbation. In fact, in 1966 W. V. Petryshyn [16] studied various conditions on a continuous 1-set-contractive map T of a real Banach space X, which ensure the surjectivity. In the same paper, the author generalized to k-set-contractions the results obtained in [10] for Lipschitzian pseudo-contractive maps. In 1983 W. Y. Akashi [1] generalized some known results in the classical theory of linear Fredholm operators in which the compact operators played a fundamental role. For this the author introduced a new class of operators containing the class of compact operators. Recently, W. Chaker, A. Jeribi and B. Krichen [4] continued this study to investigate the essential spectra of densely defined linear operators. In the same work, it was proved that for a closed operator T, if T is demicompact, then $I-T$ is an upper semi-Fredholm operator and if μT is demicompact for all $\mu \in [0,1]$, then $I-T$ is a Fredholm operator with index zero. In 2014, B. Krichen [11] gave a generalization of this notion by introducing the class of relative demicompact linear operators with respect to a given linear operator.

The theory of block operator matrices arise in various areas of mathematics and its applications: in systems theory as Hamiltonians (see [7]), in the discretization of partial differential equations as large partitioned matrices due to sparsity patterns, in saddle point problems in non-linear analysis (see [3]), in evolution problems as linearization of second order Cauchy problems and as linear operators describing coupled systems of partial differential equations. Such systems occur widely in mathematical physics, e.g. in fluid mechanics (see [6]), magnetohydrodynamics (see [15]), and quantum mechanics (see [21]). In all these applications, the spectral properties of the corresponding block operator matrices are of vital importance as they govern for instance the time evolution and hence the stability of the underlying physical systems. From the most important works on the spectral theory of block operator matrices, we mention [9], in which the author developed the essential spectra of 2×2 and 3×3 block operator matrices. We also mention [22], in which it was presented a wide panorama of methods to investigate the essential spectra of block operator matrices. In this paper, we will study the demicompactness properties of the following matrix operator L_0 acting on the Banach space product $X \times X$ which is defined by:

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

In general, the entries of L_0 are unbounded. The operator A acts on the Banach space X and has the domain $\mathcal{D}(A)$, D acts on the same Banach space

X and is defined on $\mathcal{D}(D)$ and the intertwining operator B (resp. D) is defined on $\mathcal{D}(B)$ (resp. $\mathcal{D}(D)$) and acts from X into itself. Below, we shall assume that $\mathcal{D}(A) \subset \mathcal{D}(C)$ and $\mathcal{D}(B) \subset \mathcal{D}(D)$. Note in general that the operator L_0 is neither closed nor closable operator even if its entries are closed operators. In [2] it was proved that under some conditions, L_0 is closable and its closure is denoted by L. In the literature, many important results were obtained concerning the spectral theory of this type of operators. We mention from these works the paper [22] in which the authors investigated the essential spectra of the matrix operator by means of an abstract nonzero two-sided ideal. The central aim of this work is to use the concept of demicompactness to investigate the essential spectra of L, the closure of L_0. More precisely, we are concerned with the following essential spectra:

$$\begin{aligned}
\sigma_{e_1}(T) &= \{\alpha \in \mathbb{C} \text{ such that } \alpha - T \notin \Phi_+(X)\} := \mathbb{C}\backslash\Phi_{+T},\\
\sigma_{e_4}(T) &= \{\alpha \in \mathbb{C} \text{ such that } \alpha - T \notin \Phi(X)\} := \mathbb{C}\backslash\Phi_{T},\\
\sigma_{e_5}(T) &= \mathbb{C}\backslash\rho_5(T),\\
\sigma_{e_7}(T) &= \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap}(T+K),\\
\sigma_{e_8}(T) &= \bigcap_{K\in\mathcal{K}(X)} \sigma_{\delta}(T+K),
\end{aligned}$$

where

$$\begin{aligned}
\rho_5(T) &= \{\alpha \in \mathbb{C} \text{ such that } \alpha \in \Phi_T \text{ and } i(\alpha - T) = 0\},\\
\sigma_{ap}(T) &= \Big\{\lambda \in \mathbb{C} \text{ such that } \inf_{x\in D(T);\|x\|=1} \|(\lambda - T)x\| = 0\Big\},
\end{aligned}$$

and

$$\sigma_{\delta}(T) = \{\lambda \in \mathbb{C} \text{ such that } \lambda - T \text{ is not surjective}\}.$$

The subsets $\sigma_{e_1}(\cdot)$ and $\sigma_{e_4}(\cdot)$ are, respectively the Gustafson and the Wolf essential spectra [8]. $\sigma_{e_5}(\cdot)$ is the Schechter essential spectrum [18]. $\sigma_{e_7}(\cdot)$ is the essential approximate point spectrum or the Schmoëger essential spectrum and $\sigma_{e_8}(\cdot)$ is the essential defect spectrum or the Rakocević essential spectrum (see for instance [9, 17, 18, 19, 23]). Note that for $T \in \mathcal{C}(X)$, we have:

$$\sigma_{e_1}(T) \subset \sigma_{e_4}(T) \subset \sigma_{e_5}(T) = \sigma_{e_7}(T) \cup \sigma_{e_8}(T),$$

and

$$\sigma_{e_1}(T) \subset \sigma_{e_7}(T).$$

Let us recall the following lemma whose the proof can be found in [9].

LEMMA 1.1. *Let $T \in \mathcal{C}(X)$, then*

(i) $\lambda \notin \sigma_{e_7}(T)$ *if, and only if,* $(\lambda - T) \in \Phi_+(X)$ *and* $i(\lambda - T) \leq 0$.

(ii) $\lambda \notin \sigma_{e_8}(T)$ *if, and only if,* $(\lambda - T) \in \Phi_-(X)$ *and* $i(\lambda - T) \geq 0$.

PROPOSITION 1.1. ([19]) *Let $T \in \mathcal{C}(X)$, then*

$$\lambda \notin \sigma_{e_5}(T) \text{ if, and only if, } (\lambda - T) \in \Phi(X) \text{ and } i(\lambda - T) = 0.$$

This paper is organized in the following way. In Section 2, we recall some definitions and results needed in the rest of the paper. In Section 3, we show that under some conditions, a polynomially demicompact operator is demicompact and we give an example involving the Caputo derivative of fractional of order α. In Section 4, we give a fine description of the essential approximate point spectrum and the essential defect spectrum. In Section 5, we prove in Proposition 5.1 that under some conditions, μL is demicompact for each $\mu \in \rho(A)$ and we give, in Theorem 5.3, a necessary condition for which $I - L$ is an upper semi-Fredholm operator on a Banach space with the Dunford-Pettis property (see Definition 2.1). In Section 6, we investigate the essential spectra of the matrix operator L.

2. PRELIMINARY RESULTS

We start this section by recalling some Fredholm results related with demicompact operators.

THEOREM 2.1. ([4]) *Let $T \in \mathcal{C}(X)$. If T is demicompact, then $I - T$ is an upper semi-Fredholm operator.*

THEOREM 2.2. ([4]) *Let $T \in \mathcal{C}(X)$. If μT is demicompact for each $\mu \in [0, 1]$, then $I - T$ is a Fredholm operator of index zero.*

THEOREM 2.3. ([4]) *Let $T : D(T) \subseteq X \longrightarrow X$ be a closed linear operator. If T is a 1-set-contraction then μT is demicompact for each $\mu \in [0, 1)$.*

In the next, we give a lemma which shows that, in a special Banach Space X, the sum of demicompact and weakly compact operators is demicompact. To this end, we recall the following definition.

Definition 2.1. A Banach space X is said to have the Dunford-Pettis property (in short DP property) if every bounded weakly compact operator T from X into another Banach space Y transforms weakly compact sets on X into norm-compact sets on Y.

Remark 2.1. It was proved in [13] that if X is Banach space with DP property, then

$$\mathcal{W}(X) \subset \mathcal{F}(X).$$

Lemma 2.1. *Let X be a Banach space with DP property. If $A \in \mathcal{DC}(X)$ and $B \in \mathcal{W}(X)$, then $A + B \in \mathcal{DC}(X)$.*

Proof. Let $(x_n)_n$ be a bounded sequence in $\mathcal{D}(A)$ such that $((I - A - B)x_n)_n$ converges. Since $B \in \mathcal{W}(X)$, then there exists a subsequence of $(x_n)_n$, still denoted $(x_n)_n$, such that the operator $(Bx_n)_n$ is weakly convergent. We deduce from the fact that X has DP property, that $(Bx_n)_n$ has a convergent subsequence and therefore $((I - A)x_n)_n$ has also a convergent subsequence. Using demicompactness of A, we infer that $(x_n)_n$ has a convergent subsequence and we conclude that $A + B$ is demicompact. ∎

3. Polynomially demicompact operators

It was shown in [12] that a polynomially compact operator T, element of $\mathcal{P}(X) := \{T \in \mathcal{L}(X)$ such that there exists a nonzero complex polynomial $P(z) = \sum_{r=0}^{p} a_r z^r$ satisfying $P(1) \neq 0$, $P(1) - a_0 \neq 0$, and $P(T) \in \mathcal{K}(X)\}$, is demicompact. In this section, we show that this result remains valid for a broader class of polynomially demicompact operators on X. To this end we let $\mathcal{PDC}(X)$ be the set defined by $\mathcal{PDC}(X) := \{T \in \mathcal{L}(X)$ such that there exists a nonzero complex polynomial $P(z) = \sum_{r=0}^{p} a_r z^r$ satisfying $P(1) \neq 0$ and $\frac{1}{P(1)} P(T) \in \mathcal{DC}(X)\}$. We note that $\mathcal{PDC}(X)$ contains the set $\mathcal{P}(X)$.

Theorem 3.1. *If $T \in \mathcal{PDC}(X)$, then T is demicompact.*

Proof. We first give the following relation that we will use in the proof. Since $I - T$ commutes with I, Newton's binomial formula allows us to write

$$T^j = I + \sum_{i=1}^{j} (-1)^i C_j^i (I - T)^i.$$

By making some simple calculations, we may write

$$P(T) = P(1)I + \sum_{j=1}^{p} a_j \left(\sum_{i=1}^{j} (-1)^i C_j^i (I-T)^i \right). \tag{3.1}$$

We start now proving our theorem. To this end we let $T \in \mathcal{PDC}(X)$, then there exists a nonzero complex polynomial P such that $P(1) \neq 0$. We shall prove that T is a demicompact operator. To do so, it suffices from Theorem 2.1 in [5] to establish that $I-T$ is an upper semi-Fredholm operator. First, we prove that $\alpha(I-T) < \infty$. We let $x \in \mathcal{N}(I-T)$, then $Tx = x$ and therefore

$$P(T)x = \sum_{j=0}^{p} a_j T^j x = P(1)x.$$

Hence, $x \in \mathcal{N}(I - \frac{1}{P(1)}P(T))$ which implies that $\mathcal{N}(I-T) \subset \mathcal{N}(I - \frac{1}{P(1)}P(T))$. Since $\frac{1}{P(1)}P(T)$ is demicompact, we deduce that $\alpha(I - \frac{1}{P(1)}P(T)) < \infty$ and as consequence, $\alpha(I-T) < \infty$. In order to complete the proof, we will check that $\mathcal{R}(I-T)$ is closed. Indeed, since $\mathcal{N}(I-T)$ is finite dimensional, then there exists from Lemma 5.1 in [19] a closed subspace X_0 of X such that

$$X = \mathcal{N}(I-T) \oplus X_0.$$

Next, we let T_0 be the restriction of $I-T$ to X_0. Then, T_0 is continuous and we shall see that $\mathcal{N}(T_0)= \{0\}$. Since X_0 is also closed and $(I-T)(X_0) = T_0(X_0) = (I-T)(X) = \mathcal{R}(I-T)$, we need only to prove that $T_0(X_0)$ is closed. To this end, we shall prove that $T_0^{-1} : T_0(X_0) \longrightarrow X_0$ is continuous. By linearity, it is equivalent to that T_0^{-1} is continuous at 0. Assume the contrary, for every $n \in \mathbb{N}$, there exists a sequence $(x_n)_n$ in X_0 which does not converge to 0 such that $(I-T)(x_n)$ converges to 0. Then, we can find $\varepsilon > 0$ such that $\|x_n\| \geq \varepsilon > 0$ for all $n \in \mathbb{N}$. Then,

$$\frac{1}{\|x_n\|} \leq \frac{1}{\varepsilon} \quad \text{for all } \ n \in \mathbb{N}.$$

It is clear that $y_n := x_n/\|x_n\|$ has a norm equal to 1 and $(I-T)(y_n) \to 0$. This together with the relation (3.1) leads to

$$\big(P(T) - P(1)\big)y_n \to 0.$$

Since $P(1) \neq 0$, then

$$\left(I - \frac{1}{P(1)}P(T) \right) y_n \to 0.$$

Using the demicompactness of $\frac{1}{P(1)}P(T)$, we deduce that $(y_n)_n$ admits a converging subsequence to an element y in X_0, verifying $\|y\| = 1$. Using the closedness of $I - T$, we get $(I - T)y = 0$, which implies that $y \in \mathcal{N}(I - T)$. This contradict the fact $X_0 \cap \mathcal{N}(I - T) = \{0\}$ and $\|y\| = 1$, which achieves the proof. ∎

Remark 3.1. The converse of Theorem 3.1 is not true, in fact if we take a demicompact operator T such that $-T$ is not demicompact, then T^2 is not demicompact.

Example. Before giving the example we recall the following definition and theorems.

Definition 3.1. The Caputo derivative of fractional order α of function $x \in \mathcal{C}^m$ is defined as

$$_C D_{0,t}^{(\alpha)} x(t) = D_{0,t}^{(-m+\alpha)} \frac{d^m}{dt^m} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) d\tau,$$

in which $m - 1 < \alpha < m \in \mathbb{N}$ and Γ is the well-known Euler Gamma function.

Theorem 3.2. *[14] If $x(t) \in C^1[0,T]$, for $T > 0$ then*

$$_C D_{0,t}^{(\alpha_2)}\ {}_C D_{0,t}^{(\alpha_1)} x(t) = {}_C D_{0,t}^{(\alpha_1)}\ {}_C D_{0,t}^{(\alpha_2)} x(t) = {}_C D_{0,t}^{(\alpha_1+\alpha_2)} x(t);\ t \in [0,T],$$

where α_1 and $\alpha_2 \in \mathbb{R}_+$ and $\alpha_1 + \alpha_2 \leq 1$.

Theorem 3.3. *[14] If $x(t) \in C^m[0,T]$, for $T > 0$ then*

$$_C D_{0,t}^{(\alpha)} x(t) = {}_C D_{0,t}^{(\alpha_n)} \cdots {}_C D_{0,t}^{(\alpha_2)}\ {}_C D_{0,t}^{(\alpha_1)} x(t);\ t \in [0,T],$$

where $\alpha = \sum_{i=1}^n \alpha_i$; $\alpha_i \in (0,1]$, $m - 1 \leq \alpha < m \in \mathbb{N}$ and there exists $i_k < n$, such that $\sum_{j=1}^{i_k} \alpha_j = k$, and $k = 1, 2, \ldots, m-1$.

Let C_ω be the space of continuous ω-periodic functions $x : \mathbb{R} \longrightarrow \mathbb{R}$ and C'_ω the space of continuously differentiable ω-periodic functions $x : \mathbb{R} \longrightarrow \mathbb{R}$. C_ω equipped with the maximum norm $\|\cdot\|_\infty$ and C'_ω with the norm given by $\|\cdot\|_\infty^1 = \max\{\|u\|_\infty, \|u\|'_\infty\}$ for $u \in \mathcal{C}'_\omega$ are Banach spaces. Let us consider the following differential equation:

$$x'(t) = a(t)x'(t - h_1) + b(t)x(t - h_2) + f(t).$$

Here, a and b are continuous ω-periodic functions such that $|a(t)| < k$, $(k < \infty)$, where $k < \frac{1}{\omega}$ if $\omega > 2$ or $k < \frac{1}{2}$ if $\omega \leq 2$; $f \in \mathcal{C}_\omega$ is a given function and $x \in \mathcal{C}'_\omega$ is an unknown function. This equation can be rewritten in the operator from

$$Gx - Ax = f,$$

where $G : \mathcal{C}'_\omega \longrightarrow \mathcal{C}_\omega$ is given by the formula

$$(Gx)(t) = x'(t),$$

and the operator $A : \mathcal{C}'_\omega \to \mathcal{C}_\omega$ by the formula

$$(Ax)(t) = a(t)x'(t - h_1) + b(t)x(t - h_2).$$

Let us consider the polynomial $P(X) = X^n$ and the operator $T = {}_C D^{(\frac{1}{n})}$; $n \in \mathbb{N}\backslash\{0\}$, where ${}_C D^{(\frac{1}{n})}$ is the Caputo derivative of fractional order $\frac{1}{n}$. Applying Theorem 3.3, we get

$$P(T) = T^n(x) = [{}_C D^{(\frac{1}{n})}]^n x(t) = x'(t).$$

Clearly, $P(T)$ is bounded linear operator with $\|P(T)\| = 1$ and therefore, $P(T)$ is 1-set-contractive. Hence, using Theorem 2.3, we get

$$\mu_C D^{(\frac{1}{n})} \in \mathcal{DC}(X) \ \forall \ \mu \in [0, 1[.$$

4. Characterization of Schmoëger and Rakocević essential spectra

The aim of this section is to give a refinement of the essential approximate point spectrum and the essential defect spectrum. For this, let X be a Banach space and $T \in \mathcal{C}(X)$. Let us consider the following sets $\Lambda_X, \Upsilon_T(X)$, and $\Psi_T(X)$, respectively, defined by:

$$\Lambda_X = \big\{J \in \mathcal{L}(X) \text{ such that } \mu J \text{ is demicompact for all } \mu \in [0,1]\big\},$$

$$\Upsilon_T(X) = \big\{K \in \mathcal{L}(X) \text{ such that } \forall \lambda \in \rho(T+K),\ -(\lambda - T - K)^{-1}K \in \Lambda_X\big\},$$

$$\Psi_T(X) = \big\{K \text{ is } T\text{-bounded such that } \forall \lambda \in \rho(T+K),\\ - K(\lambda - T - K)^{-1} \in \Lambda_X\big\}.$$

We also denote:

$$\sigma_r(T) := \bigcap_{K \in \Upsilon_T(X)} \sigma_{ap}(T+K) \quad \text{and} \quad \sigma_l(T) := \bigcap_{K \in \Psi_T(X)} \sigma_\delta(T+K).$$

Theorem 4.1. *Let $T \in \mathcal{C}(X)$, we have*

$$\sigma_{e_7}(T) = \sigma_r(T),$$

and

$$\sigma_{e_8}(T) = \sigma_l(T).$$

Proof. We first should remark that

$$\lambda - T = (\lambda - T - K)[I + (\lambda - T - K)^{-1}K], \tag{4.1}$$

and

$$\lambda - T = [I + K(\lambda - T - K)^{-1}](\lambda - T - K). \tag{4.2}$$

Let us notice that for $T \in \mathcal{C}(X)$, and K be a T-bounded operator such that $\lambda \in \rho(T + K)$, then, according to closed graph theorem (Lemma 2.1 in [18]), $K(\lambda - T - K)^{-1}$ is a closed linear operator defined on X and then bounded. We start by showing that $\sigma_{e_7}(T) \subset \sigma_r(T)$ (resp. $\sigma_{e_8}(T) \subset \sigma_l(T)$). For $\lambda \notin \sigma_r(T)$ (resp. $\lambda \notin \sigma_l(T)$), there exists $K \in \Upsilon_T(X)$(resp. $K \in \Psi_T(X)$) such that $\lambda - T - K$ is injective (resp. surjective). It follows that $\lambda - T - K \in \Phi_+(X)$, (resp. $\Phi_-(X)$) and $i(\lambda - T - K) \leq 0$, (resp. $i(\lambda - T - K) \geq 0$). Now, since $K \in \Upsilon_T(X)$, (resp. $K \in \Psi_T(X)$), $-(\lambda - T - K)^{-1}K \in \Lambda_X$, (resp. $-K(\lambda - T - K)^{-1} \in \Lambda_X$), whenever $\lambda \in \rho(T+K)$. Using Theorem 2.2 we show that $I + (\lambda - T - K)^{-1}K$, (resp. $I + K(\lambda - T - K)^{-1}$) is a Fredholm operator and $i(I + (\lambda - T - K)^{-1}K) = 0$, (resp. $i(I + K(\lambda - T - K)^{-1}) = 0$). Which implies that $(I + (\lambda - T - K)^{-1}K) \in \Phi_+(X)$, (resp. $(I + K(\lambda - T - K)^{-1} \in \Phi_-(X)$ and $i(I + (\lambda - T - K)^{-1}K) \leq 0$, (resp. $i(I + K(\lambda - T - K)^{-1}) \geq 0$). Hence, applying Theorem 5.26 (resp. 5.30) in [19] on (4.1) (resp. (4.2)), we obtain $\lambda - T \in \Phi_+(X)$ (resp. $\Phi_-(X)$) and $i(\lambda - T) \leq 0$ (resp. $i(\lambda - T) \geq 0$). Thanks to Lemma 1.1, we conclude that $\lambda \notin \sigma_{e_7}(T)$ (resp. $\lambda \notin \sigma_{e_8}(T)$). Conversely, remark that $\mathcal{K}(X) \subset \Upsilon(X)$ (resp. $\mathcal{K}(X) \subset \Psi(X)$). In fact, if $K \in \mathcal{K}(X)$ and $\lambda \in \rho(T + K)$, then $-\mu(\lambda - T - K)^{-1}K \in \mathcal{K}(X) \subset \mathcal{DC}(X)$ (resp. $-\mu K(\lambda - T - K)^{-1} \in \mathcal{K}(X) \subset \mathcal{DC}(X)$). Hence, $\sigma_r(T) \subset \sigma_{e_7}(T)$ (resp. $\sigma_l(T) \subset \sigma_{e_8}(T)$). ∎

Corollary 4.1. *Let $T \in \mathcal{C}(X)$ and let $\Gamma(X)$ be a subset of X containing $\mathcal{K}(X)$. Then,*

(i) *if $\Gamma(X) \subset \Upsilon_T(X)$, then* $\sigma_{e_7}(T) = \displaystyle\bigcap_{K \in \Gamma(X)} \sigma_{ap}(T + K)$

(ii) *if* $\Gamma(X) \subset \Psi(X)$, *then* $\sigma_{e_8}(T) = \bigcap_{K\in\Gamma(X)} \sigma_\delta(T+K)$.

Proof. Since $\mathcal{K}(X) \subset \Gamma(X) \subset \Upsilon_T(X)$ (resp. $\mathcal{K}(X) \subset \Gamma(X) \subset \Psi(X)$), we obtain

$$\bigcap_{K\in\Upsilon_T(X)} \sigma_{ap}(T+K) \subset \bigcap_{K\in\Gamma(X)} \sigma_{ap}(T+K) \subset \bigcap_{K\in\mathcal{K}(X)} \sigma_{ap}(T+K) := \sigma_{e_7}(T),$$

(resp.

$$\bigcap_{K\in\Psi_T(X)} \sigma_\delta(T+K) \subset \bigcap_{K\in\Gamma(X)} \sigma_\delta(T+K) \subset \bigcap_{K\in\mathcal{K}(X)} \sigma_\delta(T+K) := \sigma_{e_8}(T)\ ).$$

The use of Theorem 4.1 allows us to conclude that

$$\sigma_{e_7}(T) = \bigcap_{K\in\Gamma(X)} \sigma_{ap}(T+K),$$

and

$$\sigma_{e_8}(T) = \bigcap_{K\in\Gamma(X)} \sigma_\delta(T+K).$$

Hence, we get the desired result. ∎

5. Demicompactness results for operator matrices

In this section, we are concerned with some new results which can be used to determinate the essential spectra of the matrix operator L, the closure of L_0, on the space $X \times X$, where X is a Banach space. In the product space $X \times X$, we consider an operator which is formally defined by a matrix

$$L_0 := \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{5.1}$$

where the operator A acts on X and has domain $\mathcal{D}(A)$, D is defined on $\mathcal{D}(D)$ and acts on the Banach space X, and the intertwining operator B (resp. C) is defined on the domain $\mathcal{D}(B)$ (resp., $\mathcal{D}(D)$) and acts on X. In the following, it is always assumed that the entries of this matrix satisfy the following conditions, introduced in [20].

$(H1)$ A is closed, densely defined linear operator on X with nonempty resolvent set $\rho(A)$.

$(H2)$ The operator B is a densely defined linear operator on X and for (hence all) $\mu \in \rho(A)$, the operator $(A-\mu)^{-1}B$ is closable. (In particular, if B is closable, then $(A-\mu)^{-1}B$ is closable).

$(H3)$ The operator C satisfies $\mathcal{D}(A) \subset \mathcal{D}(C)$, and for some (hence all) $\mu \in \rho(A)$, the operator $C(A-\mu)^{-1}$ is bounded. (In particular, if C is closable, then $C(A-\mu)^{-1}$ is bounded).

$(H4)$ The lineal $\mathcal{D}(B)\cap\mathcal{D}(D)$ is dense in X and for some (hence all) $\mu \in \rho(A)$, the operator $D - C(A-\mu)^{-1}B$ is closable. We will denote by $S(\mu)$ its closure.

Remark 5.1. (i) Under the assumptions $(H1)$ and $(H2)$, we infer that for each $\mu \in \rho(A)$ the operator $G(\mu) := \overline{(A-\mu)^{-1}B}$ is bounded on X.

(ii) From the assumption $(H3)$, it follows that the operator: $F(\mu) := C(A-\mu)^{-1}$ is bounded on X.

We recall the following result which describes the operator L_0.

THEOREM 5.1. ([2]) *Let conditions $(H1)$-$(H3)$ be satisfied and the lineal $\mathcal{D}(B)\cap\mathcal{D}(D)$ be dense in X. Then, the operator L_0 is closable and the closure L of L_0 is given by:*

$$L = \mu - \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \mu - A & 0 \\ 0 & \mu - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}. \tag{5.2}$$

Or, spelled out,

$$\begin{aligned} L:\ \mathcal{D}(L) \subset (X\times X) &\longrightarrow X\times X \\ \begin{pmatrix} x \\ y \end{pmatrix} &\longrightarrow L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} A\big(x+G(\mu)y\big) - \mu G(\mu) \\ C\big(x+G(\mu)y\big) - S(\mu)y \end{pmatrix}, \end{aligned}$$

with

$$\mathcal{D}(L) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in X\times X \text{ such that } x + G(\mu)y \in \mathcal{D}(A) \text{ and } y \in \mathcal{D}\big(S(\mu)\big) \right\}.$$

Note that the description of the operator L does not depend on the choice of the point $\mu \in \rho(A)$.

Remark 5.2. Let $\lambda \in \mathbb{C}$. It follows from (5.2) that

$$\begin{aligned}\lambda - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} \lambda - A & 0 \\ 0 & \lambda - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (\lambda - \mu) M(\mu) \\ &:= UV(\lambda)W - (\lambda-\mu)M(\mu),\end{aligned} \tag{5.3}$$

where

$$M(\mu) = \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}.$$

PROPOSITION 5.1. *Let L_0 the matrix operator defined in* (5.1) *satisfies* $(H1)$-$(H4)$ *and let L be its closure. Suppose that there is $\mu \neq 0$ such that $\frac{1}{\mu} \in \rho(A)$. If the operator $\mu S(\frac{1}{\mu})$ is demicompact, then μL is a demicompact operator.*

Proof. Let $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n \in \mathcal{D}(L)$ be a bounded sequence such that

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} := (I - \mu L) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Recalling the factorization (5.2), one has

$$L = \frac{1}{\mu} I - \begin{pmatrix} I & 0 \\ F\big(\frac{1}{\mu}\big) & I \end{pmatrix} \begin{pmatrix} \frac{1}{\mu} - A & 0 \\ 0 & \frac{1}{\mu} - S\big(\frac{1}{\mu}\big) \end{pmatrix} \begin{pmatrix} I & G\big(\frac{1}{\mu}\big) \\ 0 & I \end{pmatrix}.$$

Then,

$$\begin{pmatrix} x_n' \\ y_n' \end{pmatrix} = \begin{pmatrix} I & 0 \\ F\big(\frac{1}{\mu}\big) & I \end{pmatrix} \begin{pmatrix} I - \mu A & 0 \\ 0 & I - \mu S\big(\frac{1}{\mu}\big) \end{pmatrix} \begin{pmatrix} I & G\big(\frac{1}{\mu}\big) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

It follows that

$$\begin{pmatrix} I & 0 \\ -F\big(\frac{1}{\mu}\big) & I \end{pmatrix} \begin{pmatrix} x_n' \\ y_n' \end{pmatrix} = \begin{pmatrix} I - \mu A & 0 \\ 0 & I - \mu S\big(\frac{1}{\mu}\big) \end{pmatrix} \begin{pmatrix} I & G\big(\frac{1}{\mu}\big) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Therefore, we get the following system:

$$\begin{cases} (I - \mu A)^{-1} x_n' = x_n + G\big(\frac{1}{\mu}\big) y_n. \\ -F\big(\frac{1}{\mu}\big) x_n' + y_n' = \big(I - \mu S\big(\frac{1}{\mu}\big)\big) y_n. \end{cases} \tag{5.4}$$

The use of the second equation of the system (5.4) allows us to conclude that $(I - \mu S(\frac{1}{\mu}))y_n$ is convergent. This together with the demicompactness of $\mu S(\frac{1}{\mu})$ show that $(y_n)_n$ has a convergent subsequence. Since $G(\frac{1}{\mu})$ and $(I-\mu A)^{-1}$ are bounded operators, we infer that $(x_n)_n$ has a convergent subsequence, which proves the demicompactness of μL and this shows our claim. ∎

For more generalization, we give the following result.

Theorem 5.2. *Let L_0 the operator defined in* (5.1) *satisfies* $(H1)$-$(H4)$ *and let L be its closure. Suppose that for a certain $\mu \in \rho(A)$, there is $\lambda \in \mathbb{C}\backslash\{0\}$ such that $\lambda A \in \mathcal{DC}(X)$. Then, if $F(\mu) \in \mathcal{K}(X)$ and $\lambda S(\mu) \in \mathcal{DC}(X)$, we have that $\lambda L \in \mathcal{DC}(X \times X)$.*

Proof. Take the following bounded sequence $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in \mathcal{D}(L)$ such that

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} := (I - \lambda L)\begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Let $\mu \in \rho(A)$ be such that there is a complex nonzero number λ verifying $\lambda A \in \mathcal{DC}(X)$. Thanks to Remark 5.2, one has

$$\frac{1}{\lambda} - L = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}\begin{pmatrix} \frac{1}{\lambda} - A & 0 \\ 0 & \dfrac{1}{\lambda} - S(\mu) \end{pmatrix}\begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - \left(\frac{1}{\lambda} - \mu\right) M(\mu),$$

where

$$M(\mu) = \begin{pmatrix} 0 & G(\mu) \\ F(\mu) & F(\mu)G(\mu) \end{pmatrix}.$$

Thus,

$$I - \lambda L = \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}\begin{pmatrix} I - \lambda A & 0 \\ 0 & I - \lambda S(\mu) \end{pmatrix}\begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (1 - \lambda\mu)M(\mu).$$

Therefore,

$$\begin{aligned} \begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = & \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix}\begin{pmatrix} I - \lambda A & 0 \\ 0 & I - \lambda S(\mu) \end{pmatrix}\begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \\ & - (1 - \lambda\mu)M(\mu)\begin{pmatrix} x_n \\ y_n \end{pmatrix}. \end{aligned} \tag{5.5}$$

Observe that (5.5) is equivalent to

$$\begin{pmatrix} I & 0 \\ -F(\mu) & I \end{pmatrix}\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} + (1-\lambda\mu)\begin{pmatrix} I & 0 \\ -F(\mu) & I \end{pmatrix} M(\mu)\begin{pmatrix} x_n \\ y_n \end{pmatrix}$$
$$= \begin{pmatrix} I-\lambda A & 0 \\ 0 & I-\lambda S(\mu) \end{pmatrix}\begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Moreover, by making some simple calculations, we may show that

$$\begin{pmatrix} x'_n \\ -F(\mu)x'_n + y'_n \end{pmatrix} + \begin{pmatrix} (I-\lambda\mu)G(\mu)y_n \\ (I-\lambda\mu)F(\mu)x_n \end{pmatrix} = \begin{pmatrix} (I-\lambda A)x_n + (I-\lambda A)G(\mu)y_n \\ \big(I-\lambda S(\mu)\big)y_n \end{pmatrix},$$

in equivalent way,

$$\begin{cases} x'_n - \lambda(\mu - A)G(\mu)y_n = (I-\lambda A)x_n. \\ -F(\mu)x'_n + y'_n + (I-\lambda\mu)F(\mu)x_n = (I-\lambda S(\mu))y_n. \end{cases} \tag{5.6}$$

We deduce from the fact that $F(\mu) \in \mathcal{K}(X)$ and $(x_n)_n$ is bounded, that $(1-\lambda\mu)F(\mu)x_n$ has a convergent subsequence. Hence, from the second equation of system (5.6), we infer that $(I-\lambda S(\mu))y_n$ has a convergent subsequence. Using the demicompactness of $\lambda S(\mu)$, we deduce that there exists a convergent subsequence of $(y_n)_n$. Now, since $G(\mu)$ and $\mu - A$ are bounded, we conclude from the first equation of system (5.6) that $(I-\lambda A)x_n$ has a convergent subsequence. This together with the fact that λA is demicompact allows us to conclude that $(x_n)_n$ has a convergent subsequence. Therefore, there exists a subsequence of $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n$ which converges on $\mathcal{D}(L)$. Thus, λL is demicompact. ∎

THEOREM 5.3. *Let X be a Banach space with DP property. Assume that the operator L_0 defined in* (5.1) *and acting on $X \times X$ satisfies* $(H1)$-$(H4)$ *and denote L its closure. Suppose that $\mu \in \rho(A)$, $G(\mu) \in \mathcal{W}(X)$ and $F(\mu) \in \mathcal{F}_+(X)$. If the operators A and $S(\mu)$ are demicompact, then $I - L$ is an upper semi-Fredholm operator.*

Proof. Let $\mu \in \rho(A)$ be such that $G(\mu) \in \mathcal{W}(X)$. Since $F(\mu)$ is bounded, then the product $F(\mu)G(\mu) \in \mathcal{W}(X)$. Therefore, we can deduce from Remark 2.1 that $F(\mu)G(\mu) \in \mathcal{F}_+(X)$. This together with the fact that $F(\mu) \in$

$\mathcal{F}_+(X)$ and $G(\mu) \in \mathcal{W}(X) \subset \mathcal{F}_+(X)$ give us $M(\mu) \in \mathcal{F}_+(X \times X)$. Next, according to (5.3), we have for $\lambda = 1$:

$$\begin{aligned} I - L &= \begin{pmatrix} I & 0 \\ F(\mu) & I \end{pmatrix} \begin{pmatrix} I - A & 0 \\ 0 & I - S(\mu) \end{pmatrix} \begin{pmatrix} I & G(\mu) \\ 0 & I \end{pmatrix} - (1-\mu)M(\mu) \\ &:= UV(1)W - (1-\mu)M(\mu). \end{aligned}$$

Since A and $S(\mu)$ are demicompact and thanks to Theorem 2.1, the operators $I - A$ and $I - S(\mu)$ are upper semi-Fredholm, hence $V(1) \in \Phi_+(X \times X)$. The boundedness of the operators U and W and their inverses gives us that $UV(1)W$ is an upper semi-Fredholm operator. Owing to the fact that $M(\mu) \in \mathcal{F}_+(X \times X)$, it follows that $I - L$ is an upper semi-Fredholm operator. ∎

THEOREM 5.4. *Let X be a Banach space with DP property. Assume that the operator L_0 defined in* (5.1) *acting on the product space $X \times X$ satisfies* $(H1)$-$(H4)$ *and denote L its closure. Suppose that $[1, +\infty[\subset \rho(A)$. Then, if there exists a complex number λ such that $\lambda D \in \mathcal{DC}(X)$ and $C(I - \lambda A)^{-1}B \in \mathcal{W}(X)$, we have that $\lambda L \in \mathcal{DC}(X \times X)$.*

Proof. We assume that the assumption holds and we take $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n$ a bounded sequence in $\mathcal{D}(L)$ which verifies

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} := (I - \lambda L) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \to \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},$$

where $[1, +\infty[\subset \rho(A)$. According to the Frobenius-Schur factorization, one has

$$\lambda L = I - \begin{pmatrix} I & 0 \\ F_\lambda(1) & I \end{pmatrix} \begin{pmatrix} I - \lambda A & 0 \\ 0 & I - S_\lambda(1) \end{pmatrix} \begin{pmatrix} I & G_\lambda(1) \\ 0 & I \end{pmatrix},$$

where $F_\lambda(1) = \lambda C(\lambda A - I)^{-1}$, $S_\lambda(1) = \overline{\lambda D - \lambda^2 C(\lambda A - I)^{-1}B}$ and $G_\lambda(1) = \overline{\lambda(\lambda A - I)^{-1}B}$. It follows that

$$\begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = \begin{pmatrix} I & 0 \\ F_\lambda(1) & I \end{pmatrix} \begin{pmatrix} I - \lambda A & 0 \\ 0 & I - S_\lambda(1) \end{pmatrix} \begin{pmatrix} I & G_\lambda(1) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

thus,

$$\begin{pmatrix} I & 0 \\ -F_\lambda(1) & I \end{pmatrix} \begin{pmatrix} x'_n \\ y'_n \end{pmatrix} = \begin{pmatrix} I - \lambda A & 0 \\ 0 & I - S_\lambda(1) \end{pmatrix} \begin{pmatrix} I & G_\lambda(1) \\ 0 & I \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

which allows us to get the following system

$$\begin{cases} x_n' = (I - \lambda A)x_n + (I - \lambda A)G_\lambda(1)y_n. \\ -F_\lambda(1)x_n' + y_n' = \big(I - S_\lambda(1)\big)y_n. \end{cases} \tag{5.7}$$

Since, $\lambda D \in \mathcal{DC}(X)$ and $C(\lambda A - I)^{-1}B \in \mathcal{W}(X)$, we infer by the use of Lemma 2.1 that the operator $\lambda D - \lambda^2 C(\lambda A - I)^{-1}B$ is demicompact. Now, it is easy to show that if a closable operator is demicompact, then its closure is also demicompact. Consequently, $S_\lambda(1)$ is a demicompact operator. Moreover, it should be observed that the second equation of the system (5.7) implies the convergence of $\big((I - S_\lambda(1))y_n\big)_n$, hence $(y_n)_n$ has a convergent subsequence. Next, since $G_\lambda(1)$ is bounded and $(I - \lambda A)$ is invertible and has a bounded inverse, the first equation of the system (5.7) implies that $(x_n)_n$ has a convergent subsequence. Therefore, there exists a convergent subsequence of $\begin{pmatrix} x_n \\ y_n \end{pmatrix}_n$ which converges in $\mathcal{D}(L)$. Hence, the demicompactness of λL is proved. ∎

The following corollary gives a sufficient condition to guarantee the demicompactness of L, the closure of the closable matrix operator L_0.

COROLLARY 5.1. *Let X be a Banach space with DP property. Assume that the operator L_0 defined in* (5.1) *and acting on $X \times X$ satisfies* $(H1)$-$(H4)$ *and denote L its closure. Suppose that $[1, +\infty[\subset \rho(A)$. Then, if $D \in \mathcal{DC}(X)$ and $C(I - A)^{-1}B \in \mathcal{DC}(X)$, we have that $L \in \mathcal{DC}(X \times X)$.*

Proof. The proof is a direct application of Theorem 5.4 for $\lambda = 1$. ∎

6. ESSENTIAL SPECTRA OF MATRIX OPERATORS BY MEANS OF DEMICOMPACTNESS

We start this section by giving some notations that we will need in the proof. Let L_0 be the matrix operator defined in (5.1). Assume that L_0 satisfies $(H1)$-$(H4)$ and denote L its closure. Let $\alpha \in \mathbb{C}\backslash\{0\}$ and we suppose that $[1, +\infty[\subset \rho(A)$. Applying Remark 5.2 on the operator $\frac{1}{\alpha}L$ and for the case $\lambda = 1$, one has

$$\begin{aligned} I-\frac{1}{\alpha}L &= \begin{pmatrix} I & 0 \\ F_{\frac{1}{\alpha}}(\mu) & I \end{pmatrix} \begin{pmatrix} I-\frac{1}{\alpha}A & 0 \\ 0 & I-S_{\frac{1}{\alpha}}(\mu) \end{pmatrix} \begin{pmatrix} I & G_{\frac{1}{\alpha}}(\mu) \\ 0 & I \end{pmatrix} \\ &\quad -(1-\mu)M_{\frac{1}{\alpha}}(\mu) \\ &:= U_{\frac{1}{\alpha}}V_{\frac{1}{\alpha}}W_{\frac{1}{\alpha}}-(1-\mu)M_{\frac{1}{\alpha}}(\mu), \end{aligned} \tag{6.1}$$

where

$$M_{\frac{1}{\alpha}}(\mu) := \begin{pmatrix} 0 & G_{\frac{1}{\alpha}}(\mu) \\ F_{\frac{1}{\alpha}}(\mu) & F_{\frac{1}{\alpha}}(\mu)G_{\frac{1}{\alpha}}(\mu) \end{pmatrix}, \quad F_{\frac{1}{\alpha}}(\mu) := \frac{1}{\alpha}C\left(\frac{1}{\alpha}A-\mu\right)^{-1},$$

$$G_{\frac{1}{\alpha}}(\mu) := \frac{1}{\alpha}\left(\frac{1}{\alpha}A-\mu\right)^{-1}B \quad \text{and} \quad S_{\frac{1}{\alpha}}(\mu) := \overline{\frac{1}{\alpha}D-\frac{1}{\alpha^2}C\left(\frac{1}{\alpha}A-\mu\right)^{-1}B}.$$

THEOREM 6.1. *Let X be a Banach space with DP property. Assume that the matrix operator L_0 defined in* (5.1) *satisfies* $(H1)$-$(H4)$ *and denote L its closure. Suppose that $[1,+\infty[\subset \rho(A)$, then we have:*

(i) *If for all $\alpha \in \mathbb{C}\backslash\{0\}$, the operators $\frac{1}{\alpha}D \in \mathcal{DC}(X)$, $\frac{1}{\alpha^2}C\big(I-\frac{1}{\alpha}A\big)^{-1}B \in \mathcal{W}(X)$ and $M_{\frac{1}{\alpha}}(\mu) \in \mathcal{F}_+(X\times X)$, then*

$$\sigma_{e_1}(L)\backslash\{0\} = \sigma_{e_1}(A)\backslash\{0\} \cup \sigma_{e_1}(\alpha S_{\frac{1}{\alpha}}(\mu))\backslash\{0\}.$$

(ii) *If for all $\lambda \in [0,1]$ and $\alpha \in \mathbb{C}\backslash\{0\}$ the operators $\frac{\lambda}{\alpha}D \in \mathcal{DC}(X)$, $C(I-\frac{\lambda}{\alpha}A)^{-1}B \in \mathcal{W}(X)$ and $M(\mu) \in \mathcal{F}(X\times X)$, then*

$$\sigma_{e_i}(L)\backslash\{0\} = \sigma_{e_i}(A)\backslash\{0\} \cup \sigma_{e_i}\big(\alpha S_{\frac{1}{\alpha}}(\mu)\big)\backslash\{0\}, \text{ where } i \in \{4,5\},$$

and

$$\sigma_{e_i}(L)\backslash\{0\} \subseteq \sigma_{e_i}(A)\backslash\{0\} \cup \sigma_{e_i}\big(\alpha S_{\frac{1}{\alpha}}(\mu)\big)\backslash\{0\}, \text{ where } i \in \{7,8\}.$$

Proof. (i) Let $\alpha \in \mathbb{C}\backslash\{0\}$ be such that $\alpha \notin \sigma_{e_1}(L)$. Then,

$$\alpha - L = \alpha\left(I-\frac{1}{\alpha}L\right) \in \Phi_+(X\times X). \tag{6.2}$$

Clearly, $\alpha I \in \Phi_+(X \times X)$. We get then the following equivalence

$$\alpha - L \in \Phi_+(X \times X) \Longleftrightarrow \left(I - \frac{1}{\alpha}L\right) \in \Phi_+(X \times X).$$

Since $\frac{1}{\alpha}D \in \mathcal{DC}(X)$ and $\frac{1}{\alpha^2}C\big(I - \frac{1}{\alpha}A\big)^{-1}B \in \mathcal{W}(X)$, it follows from Corollary 5.1 that the operator $\frac{1}{\alpha}L$ is demicompact. Hence, thanks to Theorem 2.1, the operator $I - \frac{1}{\alpha}L \in \Phi_+(X \times X)$. Using the fact that $M_{\frac{1}{\alpha}}(\mu) \in \mathcal{F}_+(X \times X)$, we infer that $I - \frac{1}{\alpha}L \in \Phi_+(X \times X)$ if, and only if, the operator $U_{\frac{1}{\alpha}}V_{\frac{1}{\alpha}}(\mu)W_{\frac{1}{\alpha}}$ is such too. Now, observe that $U_{\frac{1}{\alpha}}$ and $W_{\frac{1}{\alpha}}$ are invertible and have bounded inverses, hence $I - \frac{1}{\alpha}L \in \Phi_+(X \times X)$ if, and only if, $V_{\frac{1}{\alpha}}(\mu)$ has this property, if and only if, $I - \frac{1}{\alpha}A \in \Phi_+(X)$ and $I - S_{\frac{1}{\alpha}}(\mu) \in \Phi_+(X)$. Which is equivalent to that $\alpha - A \in \Phi_+(X)$ and $\alpha - \alpha S_{\frac{1}{\alpha}} \in \Phi_+(X)$. Thus,

$$\sigma_{e_1}(L)\backslash\{0\} = \sigma_{e_1}(A)\backslash\{0\} \cup \sigma_{e_1}\big(\alpha S_{\frac{1}{\alpha}}(\mu)\big)\backslash\{0\}.$$

(ii) We claim that

$$\sigma_{e_4}(L)\backslash\{0\} = \sigma_{e_4}(A)\backslash\{0\} \cup \sigma_{e_4}\big(\alpha S_{\frac{1}{\alpha}}(\mu)\big)\backslash\{0\}.$$

For this purpose, take $\alpha \in \mathbb{C}\backslash\{0\}$. Since $\alpha I \in \Phi(X)$, then $\alpha - L \in \Phi(X \times X)$ if, and only if, the operator $(I - \frac{1}{\alpha}L) \in \Phi(X \times X)$. Next, since $\frac{\lambda}{\alpha}D \in \mathcal{DC}(X)$ and $C(I - \frac{\lambda}{\alpha}A)^{-1}B \in \mathcal{W}(X)$ for all $\lambda \in [0, 1]$, we deduce from Theorem 5.4 that the operator $\frac{\lambda}{\alpha}L$ is demicompact. Hence, according to Theorem 2.2, we have $I - \frac{1}{\alpha}L \in \Phi(X \times X)$. Using (6.1) and the fact that $M_{\frac{1}{\alpha}}(\mu) \in \mathcal{F}(X \times X)$, we infer that $I - \frac{1}{\alpha}L$ is a Fredholm operator if, and only if, the operator $U_{\frac{1}{\alpha}}V_{\frac{1}{\alpha}}(\mu)W_{\frac{1}{\alpha}}$ is such too. Now, observe that $U_{\frac{1}{\alpha}}$ and $W_{\frac{1}{\alpha}}$ are invertible and have bounded inverses, hence $I - \frac{1}{\alpha}L \in \Phi(X \times X)$ if, and only if, $V_{\frac{1}{\alpha}}(\mu)$ has this property if, and only if, $I - \frac{1}{\alpha}A \in \Phi(X)$ and $I - S_{\frac{1}{\alpha}}(\mu) \in \Phi(X)$. Thus the desired result follows.

Now, we prove the same equality for the Schechter's essential spectrum. To this end, we take $\alpha \in \mathbb{C}\backslash\{0\}$. It is easy to see that $\alpha - L \in \Phi(X \times X)$ and $i(\alpha - L) = 0$ if, and only if, the operator $(I - \frac{1}{\alpha}L) \in \Phi(X \times X)$ and $i(I - \frac{1}{\alpha}L) = 0$. Since $\frac{\lambda}{\alpha}D \in \mathcal{DC}(X)$ and $C(I - \frac{\lambda}{\alpha}A)^{-1}B \in \mathcal{W}(X)$ for all $\lambda \in [0, 1]$, it follows from Theorem 5.4 that the operator $\frac{\lambda}{\alpha}L$ is demicompact. Hence, according to Theorem 2.2, the operator $I - \frac{1}{\alpha}L \in \Phi(X \times X)$ and $i(I - \frac{1}{\alpha}L) = 0$. Using (6.1) and the fact that $M_{\frac{1}{\alpha}}(\mu) \in \mathcal{F}(X \times X)$, we infer that $I - \frac{1}{\alpha}L$ is a Fredholm operator with index zero if, and only if, the operator $U_{\frac{1}{\alpha}}V_{\frac{1}{\alpha}}(\mu)W_{\frac{1}{\alpha}}$ is such too. Note that $U_{\frac{1}{\alpha}}$ and $W_{\frac{1}{\alpha}}$ are invertible and have bounded inverses, then $I-\frac{1}{\alpha}L$ is Fredholm with index zero if, and only if, $V_{\frac{1}{\alpha}}(\mu)$ has this property, if and only if, $I-\frac{1}{\alpha}A$ and $I-\frac{1}{\alpha}S(\mu)$ are Fredholm operator with index zero. Therefore, $\alpha - A \in \Phi(X)$ and $i(\alpha - A) = 0$ and $\alpha - \alpha S_{\frac{1}{\alpha}}(\mu) \in \Phi(X)$ and $i(\alpha - \alpha S_{\frac{1}{\alpha}}(\mu)) = 0$. Hence $\alpha \notin \sigma_{e_5}(A)\backslash\{0\} \cap \sigma_{e_5}(\alpha S_{\frac{1}{\alpha}}(\mu))\backslash\{0\}$. Thus,

$$\sigma_{e_5}(A)\backslash\{0\} \cup \sigma_{e_5}(\alpha S_{\frac{1}{\alpha}}(\mu))\backslash\{0\} \subseteq \sigma_{e_5}(L)\backslash\{0\}. \tag{6.3}$$

Conversely, let $0 \neq \alpha \notin \sigma_{e_5}(A) \cap \sigma_{e_5}(\alpha S_{\frac{1}{\alpha}}(\mu))$, then $\alpha - A \in \Phi(X)$ and $i(\alpha - A) = 0$ and $\alpha - \alpha S_{\frac{1}{\alpha}}(\mu) \in \Phi(X)$ and $i(\alpha - \alpha S_{\frac{1}{\alpha}}(\mu)) = 0$. Which is equivalent to write $I - \frac{1}{\alpha}A \in \Phi(X)$ and $i(I-\frac{1}{\alpha}A) = 0$ and $I - S_{\frac{1}{\alpha}}(\mu) \in \Phi(X)$ and $i(I - S_{\frac{1}{\alpha}}(\mu)) = 0$. The boundedness of the operators $U_{\frac{1}{\alpha}}$ and $W_{\frac{1}{\alpha}}$ and their inverses and the fact that $M_{\frac{1}{\alpha}}(\mu) \in \mathcal{F}(X \times X)$ give us that $I - \frac{1}{\alpha}L \in \Phi(X\times X)$ and $i(I-\frac{1}{\alpha}L) = 0$. Therefore, $\alpha - L \in \Phi(X \times X)$ and $i(\alpha - L) = 0$, hence $\alpha \notin \sigma_{e_5}(L)\backslash\{0\}$. This immediately shows that

$$\sigma_{e_5}(L)\backslash\{0\} \subseteq \sigma_{e_5}(A)\backslash\{0\} \cup \sigma_{e_5}\big(\alpha S_{\frac{1}{\alpha}}(\mu)\big)\backslash\{0\}. \tag{6.4}$$

Now, the use of (6.3) and (6.4) makes us to conclude that

$$\sigma_{e_5}(L)\backslash\{0\} = \sigma_{e_5}(A)\backslash\{0\} \cup \sigma_{e_5}\big(\alpha S_{\frac{1}{\alpha}}(\mu)\big)\backslash\{0\}.$$

We give now the proof for $i = 7$. Note that the case $i = 8$ can be checked in the same manner. Let $\alpha \in \mathbb{C}\backslash\{0\}$, we have proved for $i = 5$ that $I - \frac{1}{\alpha}L \in \Phi(X \times X)$ and $i(I - \frac{1}{\alpha}L) = 0$. This implies that $I - \frac{1}{\alpha}L \in \Phi_+(X \times X)$ and $i(I - \frac{1}{\alpha}L) \leq 0$. If $\alpha \notin \sigma_{e_7}(A) \cap \sigma_{e_7}(\alpha S_{\frac{1}{\alpha}}(\mu))$, then $\alpha - A \in \Phi_+(X)$ and $i(\alpha - A) \leq 0$ and $\alpha - S_{\frac{1}{\alpha}}(\mu) \in \Phi_+(X)$ and $i(\alpha - \alpha S_{\frac{1}{\alpha}}(\mu)) \leq 0$. It remains to get $I - \frac{1}{\alpha}A \in \Phi_+(X)$ and $i(I - \frac{1}{\alpha}A) \leq 0$ and $I - S_{\frac{1}{\alpha}}(\mu) \in \Phi_+(X)$ and $i(I - S_{\frac{1}{\alpha}}(\mu)) \leq 0$. Since $U_{\frac{1}{\alpha}}$ and $W_{\frac{1}{\alpha}}$ are invertible and have bounded inverses and using the fact that $M_{\frac{1}{\alpha}}(\mu) \in \mathcal{F}_+(X \times X)$, we infer that $I - \frac{1}{\alpha}L \in \Phi_+(X \times X)$ and $i(I - \frac{1}{\alpha}L) \leq 0$. Therefore, $\alpha - L \in \Phi_+(X \times X)$ and $i(\alpha - L) \leq 0$. Now, by applying Lemma 1.1, we conclude that $\alpha \notin \sigma_{e_7}(L)\backslash\{0\}$ and then,

$$\sigma_{e_7}(L)\backslash\{0\} \subseteq \sigma_{e_7}(A)\backslash\{0\} \cup \sigma_{e_7}\big(\alpha S_{\frac{1}{\alpha}}(\mu)\big)\backslash\{0\}.$$

Hence, the theorem is proved. ∎

References

[1] W.Y. Akashi, On the perturbation theory for Fredholm operators, *Osaka J. Math.* **21** (3) (1984), 603 – 612.

[2] F.V. Atkinson, H. Langer, R. Mennicken, A.A. Shkalikov, The essential spectrum of some matrix operators, *Math. Nachr.* **167** (1994), 5 – 20.

[3] M. Benzi, G.H. Golub, J. Liesen, Numerical solution of saddle point problems, *Acta Numer.* **14** (2005), 1 – 137.

[4] W. Chaker, A. Jeribi, B. Krichen, Demicompact linear operators, essential spectrum and some perturbation results, *Math. Nachr.* **288** (13) (2015), 1476 – 1486.

[5] W. Chaker, A. Jeribi, B. Krichen, Some Fredholm theory results around relative demicompactness concept, Preprint, 2017.

[6] S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability", The International Series of Monographs on Physics, Clarendon Press, Oxford, 1961.

[7] R.F. Curtain, H. Zwart, An introduction to infinite-dimensional linear systems theory, Texts in Applied Mathematics 21, Springer-Verlag, New York, 1995.

[8] K. Gustafson, J. Weidmenn, On the essential spectrum, *J. Math. Anal. Appl.* **25** (1969), 121 – 127.

[9] A. Jeribi, "Spectral Theory snd Applications of Linear Operators and Block Operators Matrices", Springer, Cham, 2015.

[10] W.A. Kirk, Remarks on pseudo-contractive mappings, *Proc. Amer. Math. Soc.* **25** (1970), 820 – 823.

[11] B. Krichen, Relative essential spectra involving relative demicompact unbounded linear operators, *Acta Math. Sci. Ser. B Engl. Ed.* **34** (2) (2014), 546 – 556.

[12] K. Latrach, A. Jeribi, Some results on Fredholm operators, essential spectra, and application, *J. Math. Anal. Appl.* **225** (2) (1998), 461 – 485.

[13] K. Latrach, Essential spectra on spaces with the Dunford-Pettis property, *J. Math. Anal. Appl.* **223** (2) (1999), 607 – 622.

[14] C. Li, W. Deng, Remarks on fractional derivatives, *Appl. Math. Comput.* **187** (2) (2007), 777 – 784.

[15] A.E. Lifschitz, "Magnetohydrodynamics and Spectral Theory", Developments in Electromagnetic Theory and Applications, 4, Kluwer Academic Publishers Group, Dordrecht, 1989.

[16] W.V. Petryshyn, Remarks on condensing and k-set contractive mappings, *J. Math. Anal. Appl.* **39** (1972), 717 – 741.

[17] V. Rakočević, On one subset of M. Schechter's essential spectrum, *Mat. Vesnik* **5(18)(33)** (4) (1981), 389 – 391.

[18] M. Schechter, On the essential spectrum of an arbitrary operator, I, *J. Math. Anal. Appl.* **13** (1966), 205 – 215.

[19] M. Schechter, "Principles of Functional Analysis", Academic Press, New York-London, 1971.

[20] A.A. Shkalikov, On the essential spectrum of matrix operators, *Math. Notes* **58** (6) (1995), 1359 – 1362.

[21] B. Thaller, "The Dirac Equation", Texts and Monographs in Physics, Springer-Verlag, Berlin, 1992.

[22] C. Tretter, "Spectral Theory of Block Operator Matrices and Applications", Imperial College Press, London, 2008

[23] F. Wolf, On the invariance of the essential spectrum under a change of boundary conditions of partial differential boundary operators, *Nederl. Akad. Wetensch. Proc. Ser. A = Indag. Math.* **62** (1959), 142 – 147.