Highest weight Harish-Chandra supermodules and their geometric realizations

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1. Introduction

During[2] 1955-56 Harish-Chandra published three papers in the American Journal of Mathematics devoted to understanding the theory of representations of a real semisimple Lie group, which are also (infinitesimally) highest weight modules [27]. These modules were constructed both infinitesimally and globally, the global modules realized as spaces of sections of certain holomorphic vector bundles on the associated symmetric space, which is hermitian symmetric. He constructed the matrix element defined by the highest weight vector, and verified its square integrability under suitable conditions. Under these conditions, the representations were in the discrete series and he obtained formulae for their formal degree and character, which showed a strong resemblance to the Weyl formulae in the finite dimensional case. These calculations convinced him of the structure of the discrete series in the general case, although he was still years away from resolving this puzzle completely.

The purpose of this paper is to generalize some aspects of this theory to the supersymmetric situation.

In the literature, several authors have discussed unitarizable infinitesimal Harish-Chandra modules, see for example [29, 16, 22] and Refs. within. In [29] Jakobsen classifies the unitarizable highest weight modules for Lie superalgebras, while Furutsu and Nishiyama in [22] focus on the case $\mathfrak{su}(p, q|n)$. A global realization of such modules for the conformal supergroup appears in [15] and in full generality later on in [1], where the author establishes an equivalence of categories between certain Harish-Chandra modules and the category of smooth Frechét representations of the supergroup whose module of K-finite vectors is Harish-Chandra. Our Theorem 1.2 can be read as an explicit realization of such equivalence. These representations appear classically in the space of holomorphic sections on hermitian spaces. An infinitesimal realization of such spaces in the supersetting is due first to Serganova in [43] and later on to Borthwick et al. in [6], though they do not construct the super Harish-Chandra modules associated with them.

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In the present work, we start by examining the infinitesimal setting, that is the Harish-Chandra highest weight supermodules over the complex field for $\mathfrak{g}$, a complex basic Lie superalgebra, $\mathfrak{g} \neq A(n,n)$, $\mathfrak{g}_1 \neq 0$. Let $\mathfrak{g}_r$ be a real form of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, be the complex super Cartan decomposition ($\mathfrak{k} = \mathfrak{k}_0$ the complexification of the maximal compact subalgebra $\mathfrak{k}_r$ of $\mathfrak{g}_r$). Assume $\mathfrak{h}_r \subset \mathfrak{k}_r$, that is $\mathrm{rk}(\mathfrak{k}) = \mathrm{rk}(\mathfrak{g})$, where $\mathfrak{h}$, the complexification of $\mathfrak{h}_r$, is a Cartan subalgebra (CSA) of both $\mathfrak{k}$ and $\mathfrak{g}$.

The main results for the infinitesimal theory are Theorems 2.9 and 2.12, that we summarize here.

Theorem 1.1. *Let $\lambda \in \mathfrak{h}^*$, $\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for all compact roots α. Let $U^\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F$ (F the finite dimensional representation of $\mathfrak{k}$ associated with λ, $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+$). Then*

1. *U^λ is the universal Harish-Chandra supermodule of highest weight λ.*
2. *U^λ has a unique irreducible quotient, which is the unique (up to isomorphism) irreducible highest weight Harish-Chandra supermodule with highest weight λ.*
3. *If $(\lambda + \rho)(H_\gamma) \leq 0$ for all $\gamma \in P_n$ and < 0 for γ isotropic, then U^λ is irreducible.*

The construction of U^λ is based on the existence of *admissible systems* for $\mathfrak{g}$. These are positive systems such that the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}$ stabilizes $\mathfrak{p}^\pm$, the sum of the positive (negative) non compact root spaces. In [13] is proven the existence of such systems using diagrammatic methods; here we exhibit them and take a more conceptual approach (Theorem 2.6). The existence of admissible systems is instrumental to provide an invariant complex structure on $\mathfrak{g}_r/\mathfrak{k}_r$, which is the infinitesimal counterpart of the complex structure on the homogeneous superspace G_r/K_r ($\mathfrak{g}_r = \mathrm{Lie}(G_r)$, $\mathfrak{k}_r = \mathrm{Lie}(\mathfrak{k}_r)$), that we will use to build the global Harish-Chandra representations. In the end of this part (Sec. 2.5), we obtain a character formula for the universal Harish-Chandra supermodule.

In the second part of the paper we proceed to study the geometric realization of the Harish-Chandra representation on the superspace of holomorphic sections of a vector bundle defined on the symmetric superspace G_r/K_r. We construct the infinite dimensional Harish-Chandra super representations of a real form G_r of a simple complex Lie supergroup G, whose infinitesimal version are the supermodules constructed in the first part of the paper. We first consider the Fréchet superspace of global sections of a (complex) line bundle L^χ on the quotient G/B^+, associated with the infinitesimal character λ (see Theorem 1.1), where B^+ is a Borel subsupergroup of G associated with an admissible system P. In order to prove that $L^\chi(G_rB^+) \neq \{0\}$, we use the existence of a global section of the projection $N^-B^+ \to N^-B^+/B^+$ which in turn ensures the existence of an isomorphism $\mathcal{O}(N^-) \to L^\chi(N^-B^+) : f \mapsto f^\sim$. Exactly as in the Harish-Chandra's theory, under generic conditions, we are able to go, from line bundles on G/B^+, to vector bundles over G_r/K_r. We also obtain the Harish-Chandra decomposition $P^-KP^+ \subset G$, $\mathfrak{p}^\pm = \mathrm{Lie}(P^\pm)$; however, while in the ordinary setting $P^\pm$ are abelian subgroups, in the supersetting this is no longer true. This is due to the fact that the superalgebras $\mathfrak{p}^\pm$ are not in general abelian. Nevertheless, we are able to give a thorough description of such representations. The main results for this part are in Theorems 4.27, 4.32 and Corollary 4.28 that we summarize here:

Theorem 1.2. *Let χ be a character of $B^+ \subset G$ corresponding to the infinitesimal character $\lambda \in \mathfrak{h}^*$. Let $L^\chi(G_rB^+) := \{f : G_rB^+ \longrightarrow \mathbb{C}^{1|1} \mid f_T(gb) = \chi_T(b)^{-1} f_T(g)\}$ and assume its reduced part $\widetilde{L^\chi(G_rB^+)} := \{\tilde{f} \mid f \in L^\chi(G_rB^+)\} \neq 0$. Let ℓ denote the natural left action of $\mathcal{U}(\mathfrak{g})$ on the sections in $L^\chi(G_rB^+)$. Then:*

(1) $L^\chi(G_rB^+)$ *contains an element* ψ *which is an analytic continuation of* $1^\sim$ *(the polynomial section corresponding to* 1*);*
(2) $F^{11} := \overline{\ell(\mathcal{U}(\mathfrak{g}))\psi} \subset L^\chi(G_rB^+)$ *is a Fréchet* G_r*-supermodule,* K_r*-finite and with* K_r*-finite part* $\ell(\mathcal{U}(\mathfrak{g}))\psi) = \mathcal{P}_\lambda^\sim$ *(the polynomial sections).*
(3) *the* K_r*-finite part* $\mathcal{P}_\lambda^\sim$ *is isomorphic to* $\pi_{-\lambda}$ *the irreducible representation with lowest weight* $-\lambda$*. In particular* $\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$ *for all compact positive roots* α*.*
(4) $\ell(\mathcal{U}(\mathfrak{g}))\psi$ *is the irreducible Harish-Chandra supermodule with highest weight* $-\lambda$ *with respect to the positive system* $-P$*.*

Vice-versa, if the center of $\mathfrak{k}$ *has positive dimension, and* $\lambda \in \mathfrak{h}^*$ *is integral,* K*-integrable and* $\lambda(H_\alpha) \geq 0$ *for* α *compact positive root, then* $L^\chi(\widetilde{G_rB^+}) \neq 0$*.*

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2. The infinitesimal theory

In this section we start with the discussion of the infinitesimal theory, namely the representation of the pair $(\mathfrak{g}_r, \mathfrak{k}_r)$, where $\mathfrak{g}_r$ is a real basic Lie superalgebra and $\mathfrak{k}_r$ the maximal compact subalgebra of its even part.

2.1. **Summary of results for basic Lie superalgebras.** Assume $\mathfrak{g}$ to be basic classical, $\mathfrak{g} \neq A(n,n)$, $\mathfrak{g}_1 \neq 0$, i.e. (see [31] Prop. 1.1):

$$A(m,n) \text{ with } m \neq n\,,\ B(m,n)\,,\ C(n)\,,\ D(m,n)\,,\ D(2,1;\alpha)\,,\ F(4)\,,\ G(3) \tag{1}$$

Let $\mathfrak{h} = \mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$ and let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}$.

Δ_0 and Δ_1 denote the set of *even* and *odd* roots respectively, where we say that a root α is even if $\mathfrak{g}_\alpha \cap \mathfrak{g}_0 \neq 0$ and similarly for the odd case.

The following proposition shows that many properties of the Cartan-Killing theory extend to the basic classical Lie superalgebras.

Proposition 2.1 ([31] Prop. 1.3, [30] Prop. 2.5.5)**.** *A basic classical Lie superalgebra in list (1) satisfies the following properties:*

(1) $\dim \mathfrak{g}_\alpha = 1$*, for all* $\alpha \in \Delta$*.*
(2) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ *if and only if* $\alpha,\ \beta \in \Delta$*,* $\alpha + \beta \notin \Delta$*.*
$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ *if* $\alpha,\ \beta,\ \alpha+\beta \in \Delta$*.*
(3) *If* $\alpha \in \Delta$*, then* $-\alpha \in \Delta$*.*
(4) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *for* $\alpha \neq -\beta$*. The form* $(\ ,\ )$ *determines a non degenerate pairing of* $\mathfrak{g}_\alpha$ *with* $\mathfrak{g}_{-\alpha}$ *and* $(\ ,\ )|_{\mathfrak{h}\times\mathfrak{h}}$ *is non degenerate.*
(5) $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$*, where* h_α *is defined by* $(h_\alpha, h) = \alpha(h)$ *and* $e_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$*.*
(6) *The bilinear form of* $\mathfrak{h}^*$ *defined by* $(\lambda,\mu) = (h_\lambda, h_\mu)$ *is non degenerate and* W*-invariant, where* W *is the Weyl group of* $\mathfrak{g}_0$*.*
(7) $k\alpha \in \Delta$ *for* $\alpha \neq 0$ *and* $k \neq \pm 1$ *if and only if* $\alpha \in \Delta_1$ *and* $(\alpha,\alpha) \neq 0$*. In this case* $k = \pm 2$*.*

The last point characterizes a root for which $(\alpha, \alpha) = 0$: it is an odd root α such that 2α is not a root. Such roots are called *isotropic roots* Notice that, for example, in $\mathfrak{sl}(m|n)$ all odd roots are isotropic.

Definition 2.2. Let the notation and the hypotheses be as above. We define a *Borel subsuperalgebra*, as a subsuperalgebra $\mathfrak{b} \subset \mathfrak{g}$, such that:

- $\mathfrak{b}_0$ is a Borel subalgebra of $\mathfrak{g}_0$;
- $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^+$ is a nilpotent ideal of $\mathfrak{b}$,

and $\mathfrak{b}$ is maximal with respect to these properties (see [40] pg 26 for more details).

Let us fix a Borel subsuperalgebra $\mathfrak{b}$. We say that a root α is *positive* if $\mathfrak{g}_\alpha \cap \mathfrak{n}^+ \neq (0)$, we say that it is *negative* if $\mathfrak{g}_\alpha \cap \mathfrak{n}^+ = (0)$. $\mathfrak{n}^+$ is the span of the positive root spaces and we define $\mathfrak{n}^-$ as the span of the negative ones. We define the *positive system* P as the set of positive roots. We say that a positive root is *simple* if it is indecomposable, that is, if we cannot write it as a sum of two positive roots. Let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be the set of simple roots, we also call Π a *fundamental system*.

We have the following result (see [31] Prop. 1.5).

Proposition 2.3. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra as above. Fix a Borel subsuperalgebra and let the notation be as above. Then*

(1) *The simple roots $\alpha_1, \dots, \alpha_r$ are linearly independent.*
(2) *We may choose elements $e_i \in \mathfrak{g}_{\alpha_i}$ and $f_i \in \mathfrak{g}_{-\alpha_i}$ and $h_i \in \mathfrak{h}$ such that $\{e_i, f_i, h_i\}_{i \in I}$ is the system of generators of $\mathfrak{g}$ satisfying the relations:*
$$[e_i, f_j] = \delta_{ij} h_i, \quad [h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j.$$
for a suitable non singular integral matrix $A = (a_{ij})_{i,j \in I}$ (the Cartan matrix of $\mathfrak{g}$).
(3) *$\mathfrak{n}^\pm$ are generated by the e_i's and f_i's respectively.*
(4) *$\Delta = P \coprod -P$ and P consists of the roots which are integral positive linear combinations of simple roots.*

As in the case with the ordinary Lie algebras, in the theory of finite dimensional representations of $\mathfrak{g}$, a fundamental role is played by the highest weight modules. These are defined with respect to the choice of a CSA $\mathfrak{h}$ of $\mathfrak{g}$ and a positive system P of roots of $(\mathfrak{g}, \mathfrak{h})$. They are parametrized by their highest weights. The universal highest weight modules are known as *super Verma modules* and are infinite dimensional. The irreducible highest weight modules are uniquely determined by their highest weights and are the unique irreducible quotients of the super Verma modules. The irreducible modules are finite dimensional if and only if the highest weight is dominant integral and it satisfies a condition which can be expressed in terms of all of the Borel subsuperalgebras containing $\mathfrak{h}$ (in the ordinary theory they are all conjugate, while in the super theory they are not). Furthermore, one obtains all irreducible finite dimensional representations of $\mathfrak{g}$ in this manner (see Refs. [30, 31, 40]).

Our goal is to describe the infinite dimensional highest weight supermodules, which are also $\mathfrak{k}$-finite, in the case $\mathrm{rk}(\mathfrak{k}) = \mathrm{rk}(\mathfrak{g}_0)$, so that the CSA $\mathfrak{h}$ of $\mathfrak{k}$ is also a CSA of $\mathfrak{g}_0^{ss}$, the semisimple part of $\mathfrak{g}_0$ (see Sec. 2.2 for the definition of the compact subalgebra $\mathfrak{k}_r$ and its complexification $\mathfrak{k}$). It is important to remark that in the ordinary setting, not every choice of positive system leads to infinite dimensional highest weight $\mathfrak{k}$-finite $\mathfrak{g}$ representations. The positive systems with this property are called *admissible* and they are characterized by the presence of *totally positive roots*, that is positive non compact roots, which stay positive under the adjoint action of $\mathfrak{k}$, (see Ref. [27]).

2.2. **Admissible Systems.** Let $\mathfrak{g}$ be a complex basic classical Lie superalgebra, $\mathfrak{g} \neq A(n,n)$ and $\mathfrak{g}_r$ a real form of $\mathfrak{g}$. We have the ordinary Cartan decomposition:

$$\mathfrak{g}_{r,0}^{ss} = \mathfrak{k}_{r,0}^{ss} \oplus \mathfrak{p}_{r,0}, \qquad \mathfrak{g}_0^{ss} = \mathfrak{k}_0^{ss} \oplus \mathfrak{p}_0, \qquad \mathfrak{p}_{r,0} = (\mathfrak{k}_{r,0}^{ss})^{\perp} \tag{2.1}$$

where we drop the index r to mean the complexification.

The ordinary real Lie algebra $\mathfrak{g}_{r,0}$ may not be semisimple; this happens for the type I Lie superalgebras, in which case $\mathfrak{g}_{r,0}$ has a one-dimensional center. We define $\mathfrak{k}_r$ as $\mathfrak{k}_{r,0}^{ss}$, when $\mathfrak{g}_{r,0}^{ss} = \mathfrak{g}_{r,0}$, that is when $\mathfrak{g}_{r,0}$ is semisimple, and we define $\mathfrak{k}_r = \mathfrak{k}_{r,0}^{ss} \oplus \mathfrak{c}(\mathfrak{g}_{r,0})$ if $\mathfrak{g}_{r,0}$ has center $\mathfrak{c}(\mathfrak{g}_{r,0})$. We assume $\mathfrak{k}_r$ to be of compact type and reductive. As usual we drop the index r to mean complexification.

Hence, we decompose $\mathfrak{g}$ as:

$$\mathfrak{g} = \mathfrak{k}_0 \oplus \mathfrak{p}, \qquad \mathfrak{p} := \mathfrak{p}_0 \oplus \mathfrak{g}_1. \tag{2.2}$$

Definition 2.4. We call the pair $(\mathfrak{g}_r, \mathfrak{k}_r)$, described above, a $(\mathfrak{g}_r, \mathfrak{k}_r)$ *pair of Lie superalgebras*. Note that $\mathfrak{k}_r = \mathfrak{k}_{r,0}$. We give the same definition for the complex pair of Lie superalgebras $(\mathfrak{g}, \mathfrak{k})$.

Let us now further assume that

$$\mathrm{rk}(\mathfrak{g}_{r,0}) = \mathrm{rk}(\mathfrak{k}_r)$$

Then we can choose a CSA $\mathfrak{h}_r = \mathfrak{h}_{r,0}$ of $\mathfrak{g}_{r,0}$ so that

$$\mathfrak{h}_r \subset \mathfrak{k}_r \subset \mathfrak{g}_r, \qquad \mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g} \tag{2}$$

$\mathfrak{h}$ is then a CSA of $\mathfrak{k}$, $\mathfrak{g}_0$ and $\mathfrak{g}$.

We fix a positive system $P = P_0 \cup P_1$ of roots for $(\mathfrak{g}, \mathfrak{h})$ and write $\alpha > 0$ interchangeably with $\alpha \in P$; P_0 denotes the even roots, while P_1 the odd roots in P.

We say that a root α is *compact* (resp. *non-compact*) if $\mathfrak{g}_\alpha \subseteq \mathfrak{k}$ (resp. $\mathfrak{g}_\alpha \subseteq \mathfrak{p}$). By (2), we have

$$\Delta = \Delta_k \cup \Delta_n$$

where Δ_k (resp.Δ_n) denotes the set of compact (resp. non-compact) roots.

We call P_k the set of positive compact roots and P_n the set of positive non-compact roots, $P = P_k \coprod P_n$. We define:

$$\mathfrak{p}^+ = \sum_{\alpha \in P_n} \mathfrak{g}_\alpha \subset \mathfrak{p}, \quad \mathfrak{p}^- = \sum_{\alpha \in -P_n} \mathfrak{g}_\alpha \subset \mathfrak{p}$$

that is $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) is the direct sum of the positive (resp. negative) non-compact root spaces.

Definition 2.5. We say that the positive system $P = P_0 \cup P_1$ is *admissible* if:

(1) $\mathfrak{p}^+$ is $\mathfrak{k}$-stable, that is, $[\mathfrak{k}, \mathfrak{p}^+] \subset \mathfrak{p}^+$;

(2) $\mathfrak{p}^+$ is a Lie subsuperalgebra, that is $[\mathfrak{p}^+, \mathfrak{p}^+] \subset \mathfrak{p}^+$.

The two conditions are equivalent to saying that $\mathfrak{k} \oplus \mathfrak{p}^+$ is a subsuperalgebra of $\mathfrak{g}$ and $\mathfrak{p}^+$ an ideal in $\mathfrak{k} + \mathfrak{p}^+$. Notice that these conditions imply that P_0 is an admissible system for $\mathfrak{g}_0^{ss}$ the semisimple part of $\mathfrak{g}_0$ and consequently that $\mathfrak{p}_0^+$ is abelian. If P is a positive system and P_0 is admissible, to verify P is admissible, we only need to check that:

$$[\mathfrak{p}_1^+, \mathfrak{p}_1^+] \subset \mathfrak{p}_0^+, \qquad [\mathfrak{k}, \mathfrak{p}_1^+] \subset \mathfrak{p}_1^+$$

We now want to prove the existence of admissible systems.

Theorem 2.6. *Let P_0 be an admissible system for $\mathfrak{g}_0^{ss}$, the semisimple part of $\mathfrak{g}_0$. Then $\mathfrak{g}$ has an admissible system P containing P_0.*

Proof. It is a known fact that if $\mathfrak{g}$ is a simple Lie algebra and $P = P_k \cup P_n$ is an admissible system, the only other admissible system containing P_k is $P_k \cup (-P_n)$. Due to the irreducibility of the $\mathfrak{k}$-modules $\mathfrak{p}_0^\pm$ and the fact each weight is multiplicity free, any other admissible system has the form:

$$P_k' \cup (\pm P_n)$$

for some compact positive system P_k'. In other words, the noncompact part of an admissible system is fixed modulo a sign.

Hence, if $\mathfrak{g}$ is a semisimple Lie algebra the admissible systems containing a given P_k are conjugate under the action of $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, where we have one copy of $\mathbb{Z}_2$ for each of the simple components of $\mathfrak{g}$. Hence, different admissible system are of the form:

$$\bigcup_i (P_{k,i}' \cup \epsilon_i P_{n,i})$$

where $P_{k,i}' \cup P_{n,i}$ is a fixed admissible system for the i-th factor and $\epsilon_i = \pm 1$.

We can apply these considerations to the semisimple part $\mathfrak{g}_0^{ss}$ of the Lie superalgebra $\mathfrak{g}$, and notice that, if we show that there exists a super admissible system containing a particular admissible system P_0, then we are done. Indeed if we have

$$P = P_0 \cup P_1 = \bigcup_i (P_0)_{k,i} \cup (P_0)_{n,i} \cup (P_1)_{n,i}$$

then any other admissible system for $\mathfrak{g}_0^{ss}$ is of the form

$$\bigcup_i (P_0')_{k,i} \cup \epsilon_i (P_0)_{n,i}$$

and hence an easy check shows that

$$\bigcup_i (P_0')_{k,i} \cup \epsilon_i (P_0)_{n,i} \cup \epsilon_i (P_1)_{n,i}$$

is an admissible system for $\mathfrak{g}$.

The fact that, for a particular admissible system P_0 for $\mathfrak{g}_0^{ss}$, there exists an admissible system of $\mathfrak{g}$ containing P_0, will be proven in the remaining part of this section by a case by case analysis.

The Lie superalgebras of classical type and their real forms have been classified in [30] (see also [43, 42]), so we proceed to a case by case analysis. We briefly recall the ordinary setting.

The only simple basic classical real Lie algebras $\mathfrak{g}_{r,0}$ giving rise to an hermitian symmetric space, condition equivalent to have an admissible system, are:

- AIII. $\mathfrak{g}_{r,0} = \mathfrak{su}(p,q)$, $\mathfrak{g}_0 = \mathfrak{sl}_{p+q}(\mathbb{C})$, $\mathfrak{k}_{r,0} = \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus i\mathbb{R}$, $\mathfrak{k} = \mathfrak{sl}_p(\mathbb{C}) \oplus \mathfrak{sl}_q(\mathbb{C}) \oplus \mathbb{C}$.
- BDI ($q = 2$). $\mathfrak{g}_{r,0} = \mathfrak{so}_{\mathbb{R}}(p,2)$, $\mathfrak{g}_0 = \mathfrak{so}_{\mathbb{C}}(p+2)$, $\mathfrak{k}_{r,0} = \mathfrak{so}_{\mathbb{R}}(p) \oplus \mathfrak{so}_{\mathbb{R}}(2)$, $\mathfrak{k} = \mathfrak{so}_{\mathbb{C}}(p) \oplus \mathfrak{so}_{\mathbb{C}}(2)$.
- DIII. $\mathfrak{g}_{r,0} = \mathfrak{so}^*(2n)$, $\mathfrak{g}_0 = \mathfrak{so}_{\mathbb{C}}(2n)$, $\mathfrak{k}_{r,0} = \mathfrak{u}(n)$, $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$.
- CI. $\mathfrak{g}_{r,0} = \mathfrak{sp}_n(\mathbb{R})$, $\mathfrak{g}_0 = \mathfrak{sp}_n(\mathbb{C})$, $\mathfrak{k}_{r,0} = \mathfrak{u}(n)$, $\mathfrak{k} = \mathfrak{gl}_n(\mathbb{C})$.

We remark that there are two other Lie algebras corresponding to hermitian symmetric spaces, namely EIII and EVII, however neither of them will appear in the even part of a basic classical Lie superalgebra $\mathfrak{g}$, $\mathfrak{g}_1 \neq 0$.

Let us now proceed and examine the various cases in the super setting.

Type A. **Case:** $A(m-1,n-1) = \mathfrak{sl}_{m|n}(\mathbb{C})$, $m \neq n$. The even part is

$$\mathfrak{g}_0 = \mathfrak{sl}_m(\mathbb{C}) \oplus \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$$

We have 4 possible real forms of $A(m-1,n-1)_0$ that correspond to hermitian symmetric spaces. They are given by the following table (relative to the semisimple part of $\mathfrak{g}$):

type	$\mathfrak{sl}_m(\mathbb{C}), m = p+q$	$\mathfrak{sl}_n(\mathbb{C}), n = r+s$
non-compact	$\mathfrak{su}(p,q)$	$\mathfrak{su}(r,s)$
compact	$\mathfrak{su}(p+q)$	$\mathfrak{su}(r+s)$

Note that p, q, r, s can take also the value 0.

The only real Lie algebra which is the even part of a real form of $\mathfrak{g}$ is:

$$\mathfrak{g}_{r,0} = \mathfrak{su}(p,q) \oplus \mathfrak{su}(r,s) \oplus \mathfrak{u}(1)$$

(see [42]). We now describe the root system of $\mathfrak{g} = \mathfrak{sl}_{m|n}(\mathbb{C})$. Take as CSA $\mathfrak{h}$ the diagonal matrices:

$$\mathfrak{h} = \{\, d \;=\; \mathrm{diag}(a_1, \dots, a_m, b_1, \dots, b_n)\}$$

and define $\epsilon_i(d) = a_i$, $\delta_j(d) = b_j$, for $i = 1, \dots, m$ and $j = 1, \dots, n$. Choose the simple system:

$$\Pi = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n\}$$

We have 3 non-compact simple roots, 2 even and 1 odd:

$$\epsilon_p - \epsilon_{p+1}, \quad \delta_r - \delta_{r+1}, \quad \epsilon_m - \delta_1$$

According to the simple system, we have:

$$\begin{aligned} P_k =& \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq p\} \cup \{\epsilon_i - \epsilon_j \mid p+1 \leq i < j \leq m\} \\ & \cup \{\delta_i - \delta_j \mid 1 \leq i < j \leq r\} \cup \{\delta_i - \delta_j \mid r+1 \leq i < j \leq n\} \\ P_{n,0} =& \{\epsilon_i - \epsilon_j \mid 1 \leq i \leq p < j \leq m\} \cup \{\delta_i - \delta_j \mid 1 \leq i \leq r < j \leq n\} \\ P_{n,1} =& \{\epsilon_i - \delta_j \mid 1 \leq i \leq m \,,\, 1 \leq j \leq n\} \end{aligned}$$

The following checks are immediate:

(1) $[\mathfrak{k}_0, \mathfrak{p}_0^+] \subset \mathfrak{p}_0^+$, $[\mathfrak{k}_0, \mathfrak{p}_1^+] \subset \mathfrak{p}_1^+$.
(2) $[\mathfrak{p}_0^+, \mathfrak{p}_0^+] = 0$, $[\mathfrak{p}_0^+, \mathfrak{p}_1^+] \subset \mathfrak{p}_1^+$, $[\mathfrak{p}_1^+, \mathfrak{p}_1^+] = 0$.

Hence we have produced an admissible system.

Type B. **Case:** $B(m,n) = \mathrm{osp}_{\mathbb{C}}(2m+1|2n)$, $m \neq 0$. The even part is

$$B(m,n)_0 = \mathrm{osp}_{\mathbb{C}}(2m+1|2n)_0 = \mathfrak{so}_{\mathbb{C}}(2m+1) \oplus \mathfrak{sp}_n(\mathbb{C})$$

Reasoning as in the previous cases, that is reasoning on the ordinary setting and knowing the classification of the real forms there, we have only four possibilities for the choice of the real form of $B(m|n)_0$:

type	$\mathfrak{so}_{\mathbb{C}}(2m+1)$	$\mathfrak{sp}_n(\mathbb{C})$
non-compact	$\mathfrak{so}_{\mathbb{R}}(2,2m-1)$	$\mathfrak{sp}_n(\mathbb{R})$
compact	$\mathfrak{so}_{\mathbb{R}}(2m+1)$	$\mathfrak{sp}(n)$

The only case for which the real form $\mathfrak{g}_{r,0}$ extends to a real form of the Lie superalgebra $\mathfrak{g}$ is

$$\mathfrak{g}_{r,0} = \mathfrak{so}_{\mathbb{R}}(2,2m-1) \oplus \mathfrak{sp}_n(\mathbb{R})$$

(see [42]). The corresponding maximal compact subalgebra is

$$\mathfrak{k}_0 = \mathfrak{so}_{\mathbb{R}}(2) \oplus \mathfrak{so}_{\mathbb{R}}(2m-1) \oplus \mathfrak{u}(n)$$

Using the same notations as in [33], we choose the simple system:

$$\Pi = \{\epsilon_1 - \epsilon_2\,, \ldots, \,\epsilon_{m-1} - \epsilon_m\,, \,\epsilon_m\,, \,\delta_1 - \delta_2\,, \ldots, \,\delta_{n-1} - \delta_n\,, \,\delta_n - \epsilon_1\}$$

We have one simple non-compact even root $\epsilon_1 - \epsilon_2$ and one non-compact simple odd root: $\delta_n - \epsilon_1$. So we have:

$$\begin{aligned} P_{n,0} =& \{\epsilon_1 \pm \epsilon_j \mid 1 < j \leq m\} \cup \{\epsilon_1\} \cup \{\delta_i + \delta_j \mid 1 \leq i, j \leq n\} \\ P_{n,1} =& \{\delta_i \pm \epsilon_j \mid 1 \leq i \leq n\,,\, 1 \leq j \leq m\} \cup \{\delta_i \mid 1 \leq i \leq n\} \\ P_k =& \{\epsilon_i \pm \epsilon_j \mid 1 < i < j \leq m\} \cup \{\epsilon_i \mid 1 < i \leq m\} \cup \{\delta_i - \delta_j \mid 1 \leq i < j \leq n\} \end{aligned}$$

In order to check the $\mathrm{ad}(\mathfrak{k}_0)$ invariance of $\mathfrak{p}_0^+$ and $\mathfrak{p}_1^+$, we need to verify that summing one of the roots in $P_k \cup -P_k$ to one in $P_{n,0}$ or $P_{n,1}$ we remain in $P_{n,0}$ or $P_{n,1}$. The check is straightforward. Similarly one verifies that $\mathfrak{p}^+ = \mathfrak{p}_0^+ + \mathfrak{p}_1^+$ is a Lie subsuperalgebra, hence we have produced an admissible system.

Type B. **Case:** $B(0,n) = \mathrm{osp}_{\mathbb{C}}(1|2n)$. In this case, the even part is

$$B(0,n)_0 = \mathrm{osp}_{\mathbb{C}}(1|2n)_0 \cong \mathfrak{sp}_n(\mathbb{C})$$

The only real form of interest to us is:

$$\mathfrak{g}_{r,0} = \mathfrak{sp}_n(\mathbb{R})$$

corresponding to the maximal compact subalgebra $\mathfrak{k}_0 = \mathfrak{u}(n)$.

Choose the simple system:

$$\Pi = \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n\}$$

We have one simple non-compact odd root δ_n. So we have:

$$\begin{aligned} P_{n,0} =& \{\delta_i + \delta_j \mid 1 \le i, j \le n\} \\ P_{n,1} =& \{\delta_i \mid 1 \le i \le n\} \\ P_k =& \{\delta_i - \delta_j \quad 1 \le i < j \le n\} \end{aligned}$$

Both the checks for properties (1) and (2) in Def. 2.5 are immediate.

Type D. **Case:** $D(m,n) = \mathrm{osp}_{\mathbb{C}}(2m|2n)$. The even part is given by:

$$D(m,n)_0 = \mathrm{osp}_{\mathbb{C}}(2m|2n)_0 = \mathfrak{so}_{\mathbb{C}}(2m) \oplus \mathfrak{sp}_n(\mathbb{C})$$

The possibilities for the choice of the real form of $D(m,n)_0$ are:

type	$\mathfrak{so}_{\mathbb{C}}(2m+1)$	$\mathfrak{sp}_n(\mathbb{C})$
non-compact	$\mathfrak{so}_{\mathbb{R}}(2, 2m-2)$	$\mathfrak{sp}_n(\mathbb{R})$
compact	$\mathfrak{so}_{\mathbb{R}}(2m)$	$\mathfrak{sp}(n)$
non-compact	$\mathfrak{so}^*(2m)$	$\mathfrak{sp}_n(\mathbb{R})$
compact	$\mathfrak{so}_{\mathbb{R}}(2m)$	$\mathfrak{sp}_n(\mathbb{R})$

Again by [42] we have that the only possibilities for $\mathfrak{g}_{r,0}$ are:

$$\mathfrak{g}_{r,0} = \mathfrak{so}_{\mathbb{R}}(2, 2m-2) \oplus \mathfrak{sp}_n(\mathbb{R})$$

or

$$\mathfrak{g}_{r,0} = \mathfrak{so}^*(2m) \oplus \mathfrak{sp}_n(\mathbb{R})$$

The corresponding maximal compact subalgebra in the first case is

$$\mathfrak{k}_{r,0} = \mathfrak{so}_{\mathbb{R}}(2) \oplus \mathfrak{so}_{\mathbb{R}}(2m-2) \oplus \mathfrak{u}(n)$$

Choose the simple system:

$$\Pi = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_1\}$$

We have one simple non-compact even root $\epsilon_1 - \epsilon_2$ and one non-compact simple odd root: $\delta_n - \epsilon_1$.

$$\begin{aligned} P_{n,0} =& \{\epsilon_1 \pm \epsilon_j \mid 1 < j \le m\} \cup \{\delta_i + \delta_j \mid 1 \le i, j \le n\} \\ P_{n,1} =& \{\delta_i \pm \epsilon_j \mid 1 \le j \le m\,,\ 1 \le i \le n\} \\ P_k =& \{\epsilon_i \pm \epsilon_j \mid 1 < i < j \le m\} \cup \{\delta_i - \delta_j \mid 1 \le i < j \le n\} \end{aligned}$$

The calculation of $\mathrm{ad}(\mathfrak{k}_0)$-stability is exactly as before and similarly for the verification of the property (2) in Def. 2.5.

We now go to the second case.

The maximal compact subalgebra is

$$\mathfrak{k}_{r,0} = \mathfrak{u}(m) \oplus \mathfrak{u}(n)$$

Choose the same simple system as before:

$$\Pi = \{\epsilon_1 - \epsilon_2, \dots, \epsilon_{m-1} - \epsilon_m, \epsilon_{m-1} + \epsilon_m, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n, \delta_n - \epsilon_1\}$$

Now the simple non-compact even root $\epsilon_{m-1}+\epsilon_m$ while the non-compact simple odd root is as before $\delta_n-\epsilon_1$.

$$\begin{aligned}
P_{n,0} =& \{\epsilon_i+\epsilon_j \mid 1\le i<j\le m\}\cup\{\delta_i+\delta_j \mid 1\le i,j\le n\}\\
P_{n,1} =& \{\delta_i\pm\epsilon_j \mid 1\le i\le n\,,\, 1\le j\le m\}\\
P_k =& \{\epsilon_i-\epsilon_j \mid 1< i<j\le m\}\cup\{\delta_i-\delta_j \mid 1\le i<j\le n\}
\end{aligned}$$

The calculation of $\mathrm{ad}(\mathfrak{k}_0)$-stability is exactly as before and similarly for the verification of the property (2) in Def. 2.5.

Type C. **Case:** $C(n)=\mathrm{osp}_{\mathbb{C}}(2|2n-2)$. We have only one possible real form extending to a real form of the whole $\mathfrak{g}$ namely:

$$\mathfrak{g}_{r,0}=\mathfrak{so}_{\mathbb{R}}(2)\oplus\mathfrak{sp}_{n-1}(\mathbb{R})$$

Choose the simple system:

$$\Pi=\{\epsilon-\delta_1,\delta_1-\delta_2,\dots,\delta_{n-2}-\delta_{n-1},2\delta_{n-1}\}$$

We have one simple non-compact odd root $\epsilon_1-\delta_1$ and one non-compact simple even root: $2\delta_{n-1}$.

$$\begin{aligned}
P_{n,0} =& \{\delta_i+\delta_j \mid 1\le i,j\le n-1\}\\
P_{n,1} =& \{\epsilon\pm\delta_j \mid 1\le j\le n-1\}\\
P_k =& \{\delta_i-\delta_j \mid 1\le i<j\le n-1\}
\end{aligned}$$

The verification of the two properties (1) and (2) listed above is the same as in the B case.

We now examine the exceptional Lie superalgebras of classical type.

Case: $D(2,1;\alpha)$. We have two possible real forms of $D(2,1;\alpha)_0=\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})\oplus\mathfrak{sl}_2(\mathbb{C})$ admitting an extension to the whole $D(2,1;\alpha)$ (see [42]) namely:

$$\mathfrak{g}_{r,0}=\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_2(\mathbb{R}),\qquad\text{and}\qquad\mathfrak{g}_{r,0}=\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2).$$

We first examine the case $\mathfrak{g}_{r,0}=\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{sl}_2(\mathbb{R})$. The root system is:

$$\Delta_0=\{\pm2\epsilon_i \mid i=1,2,3\},\qquad\Delta_1=\{\pm\epsilon_1\pm\epsilon_2\pm\epsilon_3\}$$

All roots are non-compact and the positive roots corresponding to the simple system:

$$\Pi=\{\epsilon_1+\epsilon_2+\epsilon_3,-2\epsilon_2,-2\epsilon_3\}$$

form an admissible system. In fact the only thing to check is that $[\mathfrak{p}_1^+,\mathfrak{p}_1^+]\subset\mathfrak{p}_0^+$, where $P_{n,1}=\{\epsilon_1\pm\epsilon_2\pm\epsilon_3\}$ and $P_{n,0}$ consists of the positive even roots. This is immediate.

Consider now the case $\mathfrak{g}_{r,0}=\mathfrak{sl}_2(\mathbb{R})\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ and fix the positive system:

$$\Pi=\{\epsilon_1+\epsilon_2+\epsilon_3,-2\epsilon_2,-2\epsilon_3\}$$

where $\epsilon_1+\epsilon_2+\epsilon_3$ is non-compact, while $-2\epsilon_2$, $-2\epsilon_3$ are compact. We have that $\mathfrak{p}^+$ is spanned by the root spaces corresponding to the roots:

$$2\epsilon_1,\quad\epsilon_1+\epsilon_2+\epsilon_3,\quad\epsilon_1-\epsilon_2-\epsilon_3,\quad\epsilon_1-\epsilon_2+\epsilon_3,\quad\epsilon_1+\epsilon_2-\epsilon_3.$$

With such a choice we have that $\mathfrak{p}^+$ is $\mathrm{ad}(\mathfrak{k}_0)$-stable and $[\mathfrak{p}_1^+,\mathfrak{p}_1^+]\subset\mathfrak{p}_0^+$.

Case: $F(4)$. The root system is:

$$\Delta_0 = \{\pm\epsilon_i \pm \epsilon_j,\, \pm\epsilon_i, \pm\delta, \quad i = 1, 2, 3\}, \qquad \Delta_1 = \{1/2(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \delta)\}$$

ϵ_1, ϵ_2, ϵ_3 are the roots corresponding to $\mathfrak{so}_7(\mathbb{C})$, while δ corresponds to $\mathfrak{sl}_2(\mathbb{C})$. Choose the simple system (refer to [30] pg 53):

$$\Pi = \{1/2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta), -\epsilon_1, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$$

We have two real forms of the even part $F(4)_0 = \mathrm{sl}_2(\mathbb{C}) \oplus \mathfrak{so}_7(\mathbb{C})$ which extend to the whole $F(4)$:

$$\mathfrak{g}_{r,0} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_{\mathbb{R}}(7), \qquad \mathfrak{g}_{r,0} = \mathfrak{su}(2) \oplus \mathfrak{so}_{\mathbb{R}}(2, 5)$$

Let us first examine

$$\mathfrak{g}_{r,0} = \mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_{\mathbb{R}}(7), \qquad \mathfrak{k}_r = \mathfrak{so}_{\mathbb{R}}(2) \oplus \mathfrak{so}_{\mathbb{R}}(7)$$

We have that $1/2(\epsilon_1 + \epsilon_2 + \epsilon_3 + \delta)$ is the only non-compact simple root. With such a choice:

$$\begin{aligned} P_k &= \{\pm\epsilon_i - \epsilon_j,\, 1 \leq i < j \leq 3\} \cup \{-\epsilon_i,\, 1 \leq i \leq 3\} \\ P_{n,0} &= \{\delta\}, \\ P_{n,1} &= \{1/2(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 + \delta)\} \end{aligned}$$

One can easily check that such $\mathfrak{p}^+$ is ad($\mathfrak{k}$)-invariant and $[\mathfrak{p}_1^+, \mathfrak{p}_1^+] \subset \mathfrak{p}_0^+$.

We now consider the case:

$$\mathfrak{g}_{r,0} = \mathfrak{su}(2) \oplus \mathfrak{so}_{\mathbb{R}}(2, 5), \qquad \mathfrak{k}_r = \mathfrak{su}(2) \oplus \mathfrak{so}_{\mathbb{R}}(2) \oplus \mathfrak{so}(5)$$

The simple root system described above is not suitable, since the corresponding even positive system is not admissible for the given real form (notice that both the roots $\epsilon_1 - \epsilon_2$ and $-\epsilon_3$ are positive and compare with case $B(m, n)$ discussed above). In order to obtain the correct simple system we need to transform it using the (unique) element w of the Weyl group such that $w(\Pi_0) = \Pi_0'$, where Π_0' leads to an admissible system for the real form $\mathfrak{sl}_2(\mathbb{R}) \oplus \mathfrak{so}_{\mathbb{R}}(2, 5)$ of $F(4)$:

$$\Pi_0 = \{-\epsilon_1, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\} \longmapsto \Pi_0' = \{\epsilon_3, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3\}$$

Now, taking the image of the simple system Π under w we obtain:

$$\Pi' := w(\Pi) = \{\epsilon_3, \epsilon_1 - \epsilon_2, \epsilon_2 - \epsilon_3, (1/2)(w(\epsilon_1 + \epsilon_2 + \epsilon_3) + \delta)\}$$

The odd positive roots are then characterized by having $+\delta$ and not $-\delta$ in their expression. We take as the simple non-compact roots:

$$1/2(w(\epsilon_1 + \epsilon_2 + \epsilon_3) + \delta), \qquad \epsilon_1 - \epsilon_2.$$

With such a choice we obtain:

$$\begin{aligned} P_{n,0} &= \{\delta,\, \epsilon_1,\, \epsilon_1 \pm \epsilon_j,\, 1 \leq i < j \leq 3\} \\ P_{n,1} &= \{1/2(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 + \delta)\} \\ P_k &= \{\epsilon_i \pm \epsilon_j,\, 1 < i < j \leq 3\} \cup \{\epsilon_i,\, i = 2, 3\} \end{aligned}$$

$\mathfrak{p}_0^+$ is ad($\mathfrak{k}$)-invariant (this is an immediate check, but it also comes from the ordinary theory), while $\mathfrak{p}_1^+$ is ad($\mathfrak{k}$)-invariant since no roots in P_k contain δ. Finally $[\mathfrak{p}_1^+, \mathfrak{p}_1^+] \subset \mathfrak{p}_0^+$.

Case: $G(3)$. The only real form of $G(3)_0$ with an admissible system is:

$$\mathfrak{sl}_2(\mathbb{R}) \oplus \mathcal{G}_2$$

where $\mathcal{G}_2$ is the compact form of G_2. Choose the simple system (refer to [30] pg 53):

$$\Pi = \{\delta + \epsilon_1, \epsilon_2, \epsilon_3 - \epsilon_2\}$$

where the linear functions ϵ_i correspond to G_2, $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and δ to A_1. The only simple non-compact root is $\delta + \epsilon_1$. We have:

$$\begin{aligned} P_k &= \{-\epsilon_1, \epsilon_i,\ 1 < i \leq 3\} \cup \{\epsilon_3 - \epsilon_2,\ \epsilon_2 - \epsilon_1,\ \epsilon_3 - \epsilon_1\} \\ P_{n,0} &= \{2\delta\} \\ P_{n,1} &= \{\delta,\ \delta \pm \epsilon_i,\ 1 \leq i \leq 3\} \end{aligned}$$

The properties (1) and (2) of Def. 2.5 are verified by an easy calculation.

No other Lie superalgebra of classical type satisfying our hypothesis admits a real form whose even part corresponds to an hermitian symmetric space, hence our case by case analysis ends here. ■

2.3. **Harish-Chandra modules.** Consider a $(\mathfrak{g}, \mathfrak{k})$ pair of Lie superalgebras as above, with rk $\mathfrak{g}$ = rk $\mathfrak{k}$ and $\mathfrak{k} = \mathfrak{k}_0$. We can choose a CSA $\mathfrak{h} = \mathfrak{h}_0$ so that

$$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$$

and $\mathfrak{h}$ will be a CSA of both $\mathfrak{k}$ and $\mathfrak{g}$. We fix a positive system $P = P_0 \cup P_1$ of roots for $(\mathfrak{g}, \mathfrak{h})$.

Definition 2.7. Let the complex super vector space V be a $\mathfrak{g}$-module. We say that V is a $(\mathfrak{g}, \mathfrak{k})$-*module* if

$$V = \sum_{\theta \in \Theta} V(\theta)$$

where the sum is algebraic and direct, Θ denotes the set of equivalence classes of the finite dimensional irreducible representations of $\mathfrak{k}$ and $V(\theta)$ is the sum of all representations occurring in V and belonging to the class $\theta \in \Theta$. We say that the $(\mathfrak{g}, \mathfrak{k})$-module V is an *Harish-Chandra module* (or *HC-module* for short) if each $V(\theta)$ is finite dimensional and V is finitely generated as $\mathcal{U}(\mathfrak{g})$ module.

We are interested in highest weight modules (with respect to P) which are also HC-modules (see Ref. [40] for an exhaustive introduction to highest weight modules).

Proposition 2.8. *Let U be a highest weight $\mathfrak{g}$-module with highest weight λ and highest weight vector v. The following are equivalent:*

(1) $\dim(\mathcal{U}(\mathfrak{k})v) < \infty$;
(2) *U is a $(\mathfrak{g}, \mathfrak{k})$-module;*
(3) *U is an HC-module.*

If these conditions are true, then $\mathcal{U}(\mathfrak{k})v$ is an irreducible $\mathfrak{k}$-module.

Proof. The proof is very similar to the classical case; we shall nevertheless rewrite it for clarity of exposition. By the ordinary theory we know that the action of $\mathfrak{k}$ (which is an ordinary reductive Lie algebra) is decomposable if and only if the center $\mathfrak{c}$ of $\mathfrak{k}$ acts semisimply. Given $\mathfrak{c} \subset \mathfrak{h}$, if U is an highest weight module (on which $\mathfrak{h}$ acts diagonally) we have that $\mathfrak{c}$ acts semisimply on U. We now show (1) $\implies$ (2). Let $U^{\mathfrak{k}}$ denote the $\mathfrak{k}$-finite vectors in U (i.e., those vectors lying in a finite dimensional $\mathfrak{k}$-stable subspace). It is easy to check $U^{\mathfrak{k}}$ is a submodule and since the highest weight vector $v \in U^{\mathfrak{k}}$, we have $U = U^{\mathfrak{k}}$ and this proves (2). We now show (2) $\implies$ (3). According to the previous definition, we need to show that $U(\theta)$

is finite dimensional. By contradiction, assume this is not the case and let μ a weight of $U(\theta)$. Such a weight occurs with infinite multiplicity, and we have $\dim(U(\theta)_\mu) = \infty$. This is in contradiction with the well known fact that in a highest weight module the weights occur with finite multiplicities. (3) $\implies$ (1) is straightforward. We now go about the proof of the irreducibility of $\mathcal{U}(\mathfrak{k})v$. As we remarked at the beginning $\mathfrak{c}$ acts as scalar: $cw = \lambda(c)w$, for $c \in \mathfrak{c}$ and $w \in \mathcal{U}(\mathfrak{k})v$. So we have $\mathcal{U}(\mathfrak{k})v = \mathcal{U}(\mathfrak{k}')v$ with $\mathfrak{k}' = [\mathfrak{k}, \mathfrak{k}]$. $\mathcal{U}(\mathfrak{k}')v$ is a finite dimensional highest weight module for the semisimple Lie algebra $\mathfrak{k}'$ and consequently it is irreducible. ∎

We now want to define the universal super HC-module of highest weight λ. Choose $P = P_k \cup P_n$, a positive admissible system.

Let $\lambda \in \mathfrak{h}^*$ be such that $\lambda(H_\alpha)$ is an integer ≥ 0 for all $\alpha \in P_k$. Let $F = F_\lambda$ be the irreducible finite dimensional module for $\mathfrak{k}$ of highest weight λ. Note that $\lambda(H_\beta)$ can be arbitrary for positive non-compact roots β. Write

$$\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{p}^+.$$

Recall that $[\mathfrak{k}, \mathfrak{p}^+] \subset \mathfrak{p}^+$ and so we can turn F into a left $\mathfrak{q}$-module by letting $\mathfrak{p}^+$ act trivially. Define

$$U^\lambda = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F \tag{2.3}$$

and view U^λ as a $\mathcal{U}(\mathfrak{g})$-module by left action $a(b \otimes f) = ab \otimes f$.

Let

$$\rho = \frac{1}{2} \sum_{\alpha \in P_0} \alpha - \frac{1}{2} \sum_{\alpha \in P_1} \alpha. \tag{2.4}$$

Theorem 2.9. *Let the notation be as above.*

1. *U^λ is the universal HC-module of highest weight λ.*
2. *U^λ has a unique irreducible quotient, which is the unique (up to isomorphism) irreducible highest weight Harish-Chandra module with highest weight λ.*

Proof. (1). Let f^λ be the highest weight vector for F. Then $1 \otimes f^\lambda$ is such that:

$$X_\alpha(1 \otimes f^\lambda) = 0, \quad \alpha > 0, \qquad \mathcal{U}(\mathfrak{g})(1 \otimes f^\lambda) = U^\lambda$$

hence $1 \otimes f^\lambda$ is the highest weight vector of U^λ. To prove universality, let V be a highest weight HC-module with highest weight vector v.The map

$$F \longrightarrow V, \quad uf^\lambda \longmapsto uv, \qquad (a \in \mathcal{U}(\mathfrak{g}),\, v \in F)$$

extends to a linear map $L : \mathcal{U}(\mathfrak{g}) \otimes_{\mathbb{C}} F \longrightarrow V$ which is onto. Since $L(au \otimes f) - L(a \otimes uf) = 0$ for all $a \in \mathcal{U}(\mathfrak{g})$, $u \in \mathcal{U}(\mathfrak{q})$, $f \in F$, it follows that L descends to a map $U^\lambda \longrightarrow V$ which is obviously a $\mathcal{U}(\mathfrak{g})$-module map.

(2) It follows from the standard theory of highest weight modules that U^λ has a unique irreducible quotient, which is a highest weight HC-module of highest weight λ. It is the unique irreducible highest weight HC-module of highest weight λ by universality (point 1). ∎

We shall now study the structure of U^λ as a $\mathfrak{q}$-module for arbitrary λ with $\lambda(H_\alpha)$ an integer ≥ 0 for all $\alpha \in P_k$. For this we need a standard lemma.

Lemma 2.10. *Let g be a field and A, B algebras over g. Suppose $B \subset A$, A is a free right B-module, F a left B-module, and $V = A \otimes_B F$. If (a_i) is a free basis for A as a right B-module, and $L = \sum_i g.a_i$, then the map taking $\ell \otimes_g f$ to $\ell \otimes_B f$ is a linear isomorphism of $L \otimes_g F$ with V.*

We regard $\mathcal{U}(\mathfrak{p}^-) \otimes F$ as a $\mathcal{U}(\mathfrak{p}^-)$-module by $a, b \otimes f \mapsto ab \otimes f$. Since $\mathfrak{p}^-$ is stable under $\operatorname{ad} \mathfrak{k}$ we may view $\mathcal{U}(\mathfrak{p}^-) \otimes F$ as a $\mathfrak{k}$-module also.

Corollary 2.11. *The map $\varphi : a \otimes f \mapsto a \otimes_{\mathcal{U}(\mathfrak{q})} f$ is a linear isomorphism of $\mathcal{U}(\mathfrak{p}^-) \otimes F$ with U^λ that intertwines both the actions of $\mathcal{U}(\mathfrak{p}^-)$ and $\mathcal{U}(\mathfrak{k})$. In particular, U^λ is a free $\mathcal{U}(\mathfrak{p}^-)$-module with basis $1 \otimes_{\mathcal{U}(\mathfrak{q})} f_j$ where (f_j) is a basis for F.*

Proof. Since $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{q}$ it follows that $a \otimes b \mapsto ab$ is a linear isomorphism of $\mathcal{U}(\mathfrak{p}^-) \otimes \mathcal{U}(\mathfrak{q})$ with $\mathcal{U}(\mathfrak{g})$. It is clear from this that $\mathcal{U}(\mathfrak{g})$ is a free right $\mathcal{U}(\mathfrak{q})$-module, and that any basis of $\mathcal{U}(\mathfrak{p}^-)$ is a free right $\mathcal{U}(\mathfrak{q})$-basis for $\mathcal{U}(\mathfrak{g})$. Lemma 2.10 now applies and shows that φ is an isomorphism. It obviously commutes with the action of $\mathcal{U}(\mathfrak{p}^-)$. The verification of the commutativity with respect to $\mathfrak{k}$ is also straightforward. ∎

2.4. **Irreducibility of the universal HC-module.** In this section we want to prove the following theorem which gives a sufficient condition for the irreducibility of the universal HC-module. For similar modules in the classical setting, see Theorem 3 in [27], IV (1955), pg 770.

Let our hypotheses and notation be as in Sec. 2.3.

Theorem 2.12. *Let $\mathfrak{g}$ be one of the complex basic Lie superalgebras in the list 1, and let U^λ be the universal Harish-Chandra module, with highest weight λ associated with the (finite dimensional) representation F of $\mathfrak{k}$, as defined by* (2.3)*. Let ρ be as in eq.* (2.4)*. If*

$$(\mathbf{2.5}) \qquad (\lambda+\rho)(H_\gamma) \leq 0 \quad \text{for all} \quad \gamma \in P_n \quad \text{and} \quad < 0 \quad \text{for} \quad \gamma \quad \text{isotropic},$$

then U^λ is irreducible.

The proof of this theorem relies on a result stated by V. Kac in [32] and proved by M. Gorelik in [24], that we give here, as Theorem 2.13, without proof (see also [40] Sec. 13.2). Here W denotes the Weyl group of $\mathfrak{g}_0$ and $S(\mathfrak{h})^W$ denotes the set of W-invariant symmetric tensors on $\mathfrak{h}$. In the following we will use the identification of $S(\mathfrak{h})$ with the superalgebra of polynomials $\operatorname{Pol}(\mathfrak{h}^*)$ without mention.

Theorem 2.13. *The Harish-Chandra isomorphism*

$$\beta \colon Z(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W$$

for basic Lie superalgebras identifies the center $Z(\mathfrak{g})$ of the universal enveloping algebra with the subalgebra $I(\mathfrak{h})$ of $S(\mathfrak{h})^W$:

$$I(\mathfrak{h}) = \{\phi \in S(\mathfrak{h})^W \mid \phi(\lambda + t\alpha) = \phi(\lambda),\ \forall\, \lambda \in \langle\alpha\rangle^\perp,\ \alpha \textit{ isotropic},\ \forall\, t \in \mathbb{C}\}$$

(a root α is isotropic if $\langle \alpha, \alpha \rangle = 0$).

This theorem allows us to show that when a weight λ is *typical* that is

$$\langle \lambda+\rho,\alpha\rangle \neq 0 \quad \forall\,\alpha\,\text{isotropic}$$

all of the maximal weights (with respect to a highest weight vector in a submodule) in the highest weight representation of highest weight λ are conjugate under the affine action of the Weyl group, that we denote as $s.\lambda$. Notice that the condition (2.5) implies that the weight λ in Theorem 2.12 is typical.

We now recall few results (see [40] Ch. 13).

Lemma 2.14. *Let $g = \prod_{\alpha\,isotropic} h_\alpha \in S(\mathfrak{h})$, where $h_\alpha \in \mathfrak{h}$ is defined by the property $h_\alpha(\mu) = \langle \mu, \alpha\rangle$. Then:*

(1) $\mathbb{C} + gS(\mathfrak{h})^W \subset I(\mathfrak{h}) \subset S(\mathfrak{h})^W$.
(2) $I(\mathfrak{h})_g = S(\mathfrak{h})^W_g \supset S(\mathfrak{h})^W$.

where $I(\mathfrak{h})_g$ and $S(\mathfrak{h})^W_g$ denote respectively the localizations of $I(\mathfrak{h})$ and $S(\mathfrak{h})$ at the set $G := \{g^k \mid k \geq 0\}$.

Proof. (1). Clearly $\mathbb{C} \subset I(\mathfrak{h})$ and any easy check shows that $gS(\mathfrak{h})^W$ is a subalgebra contained in $I(\mathfrak{h})$. Since $I(\mathfrak{h})$ is a subalgebra we have (1). As for (2), it follows from the inclusions in (1) by noticing that the localization at G is the same for $\mathbb{C} + gS(\mathfrak{h})^W$ and $S(\mathfrak{h})^W$. ■

We now introduce the *infinitesimal character*

$$\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C}$$

defined as $\chi_\lambda(z) = (\beta(z))(\lambda+\rho)$, where β is the Harish-Chandra isomorphism.

Proposition 2.15. *Let λ be typical. Then $\chi_\lambda = \chi_\mu$ implies $\mu = s.\lambda$ for some $s \in W$.*

Proof. The infinitesimal character χ_λ may be thought (via the HC isomorphism) as defined on $I(\mathfrak{h})$ since $Z(\mathfrak{g}) \cong I(\mathfrak{h})$, by Theorem 2.13. Since λ is typical, we have $\chi_\lambda(g) \neq 0$, hence we may extend (uniquely) χ_λ to $I(\mathfrak{h})_g = S(\mathfrak{h})^W_g \supset S(\mathfrak{h})^W$. Hence $\chi_\lambda = \chi_\mu$ on $S(\mathfrak{h})^W$. This implies $\mu = s.\lambda$ (from classical considerations, see, for example, [33], Ch. 5). ■

We now approach the proof of 2.12 with some lemmas.

Lemma 2.16. *Let λ be the highest weight as in 2.12 and let μ be a maximal weight of U^λ with respect to the admissible system $P = P_k \coprod P_n$. Then*

$$P^- = P_k \coprod -P_n \tag{2.6}$$

is also a positive system and $\lambda > \mu$ with respect to P_0^-.

Proof. The fact P^- is a positive system comes from the fact that $P^- = -s_0P$, for s_0 the longest element in W_k. In fact $-s_0(P) = -s_0(P_k) \coprod -s_0(P_n)$ and $-s_0(P_k) = P_k$, while $-s_0(P_n) = -P_n$. This is because P is chosen admissible, hence the roots in P_n represent the weights of the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}^+$, hence they are permuted by the action of W_k the Weyl group of $\mathfrak{k}$.

Now we turn to the second statement. Let v_λ and v_μ maximal vectors with weight λ and μ respectively. Hence $z \in Z(\mathfrak{g})$ acts as multiplication by the scalars $\chi_\lambda(z)$ and $\chi_\mu(z)$

respectively. Since v_λ is the highest weight vector, z acts as $\chi_\lambda(z)$ on the whole U^λ. Hence the two scalars $\chi_\lambda(z)$ and $\chi_\mu(z)$ have to be the same (since $\mathcal{U}(\mathfrak{g})v_\mu$ is a submodule of U^λ).

Then, by the previous lemma $s.\lambda = \mu$, that is $\mu + \rho = s(\lambda + \rho)$, for $s \in W$. Since the hypothesis of 2.12 guarantee that $\lambda + \rho$ is dominant with respect to P^-, by usual facts on groups of reflections (see [46] Appendix to Ch. 4) we have that $\lambda - \mu = \lambda + \rho - s(\lambda + \rho)$ is sum of simple roots, but since $\lambda - \mu$ is even, it will be the sum of simple roots of P_0^-. ■

We now make some remarks on the simple systems of P and P^-. Let

$$S_0 = \{\alpha_1, \dots \alpha_A, \beta_1, \dots \beta_B\}$$

be the simple system for $P_0 \subset P$ (P_0 the positive admissible even system contained in P). We denote by α_i the compact roots and by β_j the non-compact (even) roots.

Let us now consider S a simple system for P. Our simple system S is then written as:

$$S = \{\alpha_1, \dots \alpha_a, \beta_1, \dots \beta_b, \gamma_1, \dots \gamma_c\}$$

where the γ_i's are simple odd roots, while the number of compact and non-compact roots may change since we have introduced the odd roots, that is $a \leq A$, $b \leq B$ in general.

As a word of caution let us notice that S_0 may not be the even part of S, since in general $S_0 \not\subseteq S$.

Lemma 2.17. *The simple system S contains the same compact roots as S_0. In other words $a = A$.*

Proof. Let us assume by contradiction that, say, α_r is not in S. Then α_r is decomposable, so we can write is as $\alpha_r = \delta_1 + \delta_2$, where δ_1, δ_2 are odd (necessarily or otherwise α_r would be decomposable in the even positive system). But the positive system P is admissible, and this is not possible by the discussion after Definition 2.5. ■

Lemma 2.18. *Let S_0 be the simple system for P_0 as above. Then*

$$S_0^- = \{\alpha_1, \dots \alpha_A, -\beta_1', \dots - \beta_B'\}$$

is the simple system for the positive system P_0^- defined by (2.6) *and β_i' are the non-compact roots in $P_{n,0}$ which are the highest weights for the adjoint representation of $\mathfrak{k}$ on $\mathfrak{p}_0^+$ with respect to the positive system P. In particular there exist positive integers m_i such that $\beta_i' = \beta_i + \sum_{\alpha_i > 0} m_i \alpha_i$.*

Proof. This fact is entirely classical. ■

We now go to the proof of the main result.

Proof. (Theorem 2.12) Let us assume by contradiction that U^λ has a submodule M. Let μ be a maximal weight of such submodule, so we may as well replace M with the cyclic module generated by a weight vector v_μ. We start by showing that $\lambda - \mu$ is a sum of compact roots.

Since U^λ is a highest weight module with respect to P, we have that $\lambda - \mu$ is sum of simple roots of P. On the other hand, by Lemma 2.16 we also have that $\lambda - \mu$ is the sum of simple roots of P_0^-

$$S_0^- = \{\alpha_1 \dots \alpha_A, -\beta_1', \dots - \beta_B'\}$$

where $\beta_i' = \beta_i + \sum_{\alpha_i > 0} m_i \alpha_i$ (by Lemma 2.18).

Hence we can write:

$$\lambda - \mu = \sum_{i=1}^{A} a_i\alpha_i + \sum_{j=1}^{b} b_j\beta_j + \sum_{k=1}^{c} c_k\gamma_k, \qquad a_i, b_j, c_k \in \mathbb{Z}_{\geq 0}$$

and

$$\lambda - \mu = \sum_{i=1}^{A} a'_i\alpha_i - \sum_{j=1}^{B} b'_j\beta_j \qquad b'_j \in \mathbb{Z}_{\geq 0}$$

where in the second expression we use β'_j and then we substitute its expression in terms of β_j and compact roots.

By comparing the two expressions we have:

$$\sum_{i=1}^{A} a''_i\alpha_i + \sum_{j=1}^{b}(b_j + b'_j)\beta_j + \sum_{j=b+1}^{B} b'_j\beta_j + \sum c_k\gamma_k = 0$$

where the coefficients are all positive, with the exception of the a''_i's.

For $j = b+1, \dots B$ we have:

$$\beta_j = \sum_{i=1}^{A} m_{ji}\alpha_i + \sum_{k=1}^{b} n_{jk}\beta_k + \sum_{r=1}^{c} p_{jr}\gamma_r$$

If we substitute, we get:

$$\sum_{i=1}^{A} a'''_i\alpha_i + \sum_{l=1}^{b}(b_l + b'_l + \sum_{j=b+1}^{B} b'_j n_{jl})\beta_l + \sum_{k=1}^{c}(c_k + \sum_{j=b+1}^{B} b'_j p_{jk})\gamma_k = 0$$

where the coefficients are all positive, with the exception of the a'''_i's.

Hence we obtain $b_l = b'_l = c_k = 0$, that is $\lambda - \mu$ is the sum of compact roots.

Now we go back to the highest weight vector v_μ of the submodule $M \subseteq \mathcal{U}(\mathfrak{g})v_\lambda$. v_μ is a linear combination of $X_{-\theta_1} \dots X_{-\theta_m} v_\lambda$ where each θ_j is in P.

$$\lambda - \mu = \theta_1 + \dots + \theta_m$$

Writing each θ_j as a linear combination of $\alpha_i (1 \leq i \leq a)$, $\beta_j (1 \leq j \leq b)$ and $\gamma_k (1 \leq k \leq c)$ with integer coefficients ≥ 0, and noting that $\lambda - \mu$ does not involve the β_j and γ_k, we conclude that each θ_j does not involve any β_j or γ_k. In other words, $v_\mu \in \mathcal{U}(\mathfrak{k})v_\lambda$. But then v_μ must be a multiple of v_λ, showing that $M = U^\lambda$. ■

2.5. **Super Character.** In this section we compute the character for the universal Harish-Chandra module U^λ described in the previous sections.

Let the notation be as above. If M is an $\mathfrak{h}$-module with finite multiplicities, its character is given by:

$$\mathrm{ch}(M) = \sum_{\mu \in \mathfrak{h}^*} (\dim V_\mu) e^\mu$$

By Corollary 2.11 we have the $\mathfrak{h}$-module isomorphism

$$U^\lambda \cong \mathcal{U}(\mathfrak{p}^-) \otimes F$$

This implies

$$\mathrm{ch}(U^\lambda) = \mathrm{ch}(\mathcal{U}(\mathfrak{p}^-))\mathrm{ch}(F)$$

We further know that $\mathcal{U}(\mathfrak{p}^-) \simeq \mathcal{U}(\mathfrak{p}_0^-) \otimes \wedge(\mathfrak{p}_1^-)$, as $\mathfrak{h}$-modules, hence we can immediately write:

$$\mathrm{ch}(U^\lambda) = \mathrm{ch}(\mathcal{U}(\mathfrak{p}_0^-))\,\mathrm{ch}(\wedge(\mathfrak{p}_1^-))\,\mathrm{ch}(F)$$

Let us quickly recall the following well known expressions

$$\mathrm{ch}(\mathcal{U}(\mathfrak{p}_0^-)) = \prod_{\eta\in P_{n,0}} (1-e^{-\eta})^{-1} = \frac{e^{\rho_{n,0}}}{\Delta_{n,0}}$$

$$\mathrm{ch}(F) = \frac{\sum_{s\in W_k} \epsilon(s) e^{s(\lambda+\rho_{k,0})}}{e^{\rho_{k,0}}\prod_{\eta\in P_{k,0}}(1-e^{-\eta})} = \frac{\sum_{s\in W_k} \epsilon(s) e^{s(\lambda+\rho_{k,0})}}{\Delta_{k,0}}$$

where $\epsilon(s) = \det(s)$, and the Weyl denominators are defined as:

$$\Delta_{n,0} = e^{\rho_{n,0}} \prod_{\eta\in P_{n,0}} (1-e^{-\eta})$$

$$\Delta_{k,0} = e^{\rho_{k,0}} \prod_{\eta\in P_{k,0}} (1-e^{-\eta})$$

where

$$\rho_{n,0} = (1/2)\sum_{\alpha\in P_{n,0}} \alpha, \qquad \rho_{k,0} = (1/2)\sum_{\alpha\in P_{k,0}} \alpha.$$

and $P_{n,0}$, $P_{k,0}$ correspond respectively to the non-compact and compact roots in the (admissible) positive system P_0 of $\mathfrak{g}_0$. Hence:

$$\mathrm{ch}(F)\mathrm{ch}(\mathcal{U}(\mathfrak{p}_0^-)) = \left(\sum_{s\in W_k} \epsilon(s)\frac{e^{s(\lambda+\rho_{k,0})}}{\Delta_{k,0}}\right)\frac{e^{\rho_{n,0}}}{\Delta_{n,0}} = \sum_{s\in W_k}\epsilon(s)\frac{e^{s(\lambda+\rho_{k,0})+\rho_{n,0}}}{\Delta_{k,0}\Delta_{n,0}}$$

(for the classical expression of $\mathrm{ch}(F)$, see, for example, [46, Ch.4]). Notice that for each $s \in W_k$, $s(\lambda+\rho_{k,0})+\rho_{n,0} = s(\lambda+\rho_{k,0}+\rho_{n,0})$, hence we write:

$$\mathrm{ch}(F)\mathrm{ch}(\mathcal{U}(\mathfrak{p}_0^-)) \;=\; \sum_{s\in W_k}\epsilon(s)\frac{e^{s(\lambda+\rho_0)}}{\Delta_0}$$

where $\rho_0 = (1/2)\sum_{\alpha\in P_0}\alpha$ and $\Delta_0 = \Delta_{k,0}\Delta_{n,0}$.

We now compute $\mathrm{ch}(\wedge(\mathfrak{p}_1^-))$.

$$\mathrm{ch}(\wedge(\mathfrak{p}_1^-)) = \prod_{\eta\in P_{1,n}} (1+e^{-\eta})$$

where $P_{1,n}$ are all the positive non-compact roots. Notice that in a PBW basis the odd variables appear at most with degree one. Hence:

$$\mathrm{ch}(U^\lambda) \;=\; \sum_{s\in W_k}\epsilon(s)\frac{e^{s(\lambda+\rho_0)}}{\Delta_0}\prod_{\eta\in P_{1,n}}(1+e^{-\eta})$$

3. Preliminaries on supergeometry

In this section we discuss few facts of supergeometry, we need in the following. We refer the reader for a complete basic treatment of this subject to [8], [34], [36], [38], [47]. We are interested in the *analytic* category of supermanifolds, both real and complex, denoted by $(\text{smflds})_{\mathbb{R}}$ and $(\text{smflds})_{\mathbb{C}}$ respectively, or by (smflds) when the statement holds in both categories.

Definition 3.1. Let $M = (\widetilde{M}, \mathcal{O}_M)$ and $N = (\widetilde{N}, \mathcal{O}_N)$ be connected supermanifolds, i.e. their reduced spaces $\widetilde{M}$ and $\widetilde{N}$ are connected. Suppose $\pi : M \to N$ is a morphism such that

(1) $\widetilde{\pi} : \widetilde{M} \to \widetilde{N}$ is surjective and is a covering map;
(2) for each $x \in \widetilde{N}$ we may choose an open submanifold $\widetilde{U} \subseteq \widetilde{N}$, $x \in \widetilde{U}$, such that $\widetilde{\pi}^{-1}(\widetilde{U}) = \bigsqcup_i \widetilde{V}_i$, where each $\widetilde{V}_i \to \widetilde{U}$ is an analytic isomorphism; furthermore if V_i is the open subsupermanifold of N defined by $\widetilde{V}_i$, then, for each i, $\pi_{|_{V_i}} : V_i \to U$ is an analytic isomorphism;

then we say that (M, π) is a *covering space* of N.

Remark 3.2. If $\pi : M \to N$ is a covering, then $\widetilde{\pi} : \widetilde{M} \to \widetilde{N}$ is a covering. Moreover, if $\tau \colon M \to N$ is a morphism, it is a covering map if (and only if) $\widetilde{\tau} \colon \widetilde{M} \to \widetilde{N}$ is a covering map, $\dim M = \dim N$ and $\mathrm{d}\pi$ is surjective everywhere on M.

Suppose that G is a SLG. We denote with G_e the subsuperLiegroup (subSLG for short) of G whose reduced space is the identity component of G. (For the relevant notions about quotients see [7] and also [8]).

Lemma 3.3. *Suppose G is an analytic connected SLG and $A \subseteq G$ is a closed analytic subSLG of G, A_e its identity component. Then, in the commutative diagram*

$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & G/A \\ & G/A_e \arrow[ur] & \end{tikzcd}$$

the map $G/A_e \to G/A$ is a covering morphism. Moreover

(1) *if D is the discrete even group A/A_e (note that A_e is open and normal in A), it acts (from the right) on G/A_e and commutes with the projection $G/A_e \to G/A$, acting transitively on the fibers.*
(2) *if we work over $\mathbb{R}$ and either G/A or G/A_e has a complex structure compatible with the real analytic structure, the other can be equipped with a unique complex structure compatible with the real analytic structure, such that $G/A_e \to G/A$ is a complex analytic morphism and so a covering map of the complex supermanifolds.*

Remark 3.4. It follows from the above that if B is a subSLG of G such that $B_e = A_e$, then the existence of a complex structure on G/B, G-invariant and compatible with the real analytic one, implies the existence of such a complex structure on G/A_e and hence on all the G/C with $C_e = A_e$. Moreover, if $B \subset C$, $G/B \to G/C$ is a covering map for the complex structures.

Remark 3.5. If M and N are real analytic supermanifolds and $\pi\colon M \to N$ is a covering map, then a complex structure on N can be lifted to one on M in an obvious fashion. In general, to push down a complex structure from M to N is more complicated, however in the situation considered above, we can do it, since there is a discrete even group acting as a group of super isomorphisms on M, commuting with π, acting transitively on the fibers of π.

We end this section with two results on SLG that we shall need in the sequel.

We first notice that if M and N are supermanifolds and $\psi\colon M \to N$ is a submersion, then $\widetilde{\psi}\colon \widetilde{M} \to \widetilde{N}$ is an open mapping. Hence $\widetilde{\psi}(\widetilde{M})$ is open in $\widetilde{N}$ and defines the open subsupermanifold of N given by

$$\psi(M) := \left(\widetilde{\psi}(\widetilde{M}), \mathcal{O}_N|_{\widetilde{\psi}(\widetilde{M})}\right) \tag{3}$$

Lemma 3.6. *Let M be a SLG (real or complex), A_1 and A_2 closed subSLG with* $\mathrm{Lie}(A_1) + \mathrm{Lie}(A_2) = \mathrm{Lie}(M)$. *Consider the map*

$$\alpha:\, A_1 \times A_2 \overset{i_1\times i_2}{\hookrightarrow} M \times M \overset{\mu}{\to} M$$

where $i_j : A_i \longrightarrow M$ denotes the canonical embedding of A_i in M and μ is the multiplication of the supergroup M. Then

1. *We have $\widetilde{A}_1\widetilde{A}_2 = \widetilde{\alpha}(\widetilde{A}_1 \times \widetilde{A}_2)$ is open in $\widetilde{M}$ and defines an open subsupermanifold of M, which we write as $\alpha(A_1 \times A_2)$ or A_1A_2.*
2. *If $\widetilde{A}_1\cap\widetilde{A}_2 = \{e\}$ and Lie$(A_1)\cap$Lie$(A_2) = \{0\}$ then α is an analytic super isomorphism of $A_1 \times A_2$ with A_1A_2*
3. *A_1 acts transitively on A_1A_2/A_2 and its stabilizer at $\pi(e)$ ($\pi\colon A_1A_2 \to A_1A_2/A_2$ is the natural map) is $A_1 \cap A_2$.*

Proof. The map, at the functor of points level, is

$$\alpha_T\colon (a_1, a_2) \longmapsto a_1a_2 \in M(T),\ a_i \in A_i(T) \subset M(T)\,,\ A_1(T)A_2(T) \subset M(T),\ T \in (\text{smflds})$$

We first notice that at the topological point $(e, e) \in \widetilde{A_1 \times A_2}$ (e denoting the identity element),

$$(\mathrm{d}\alpha)_{(e,e)}(X_1, X_2) = (\mathrm{d}i_1)_e(X_1) + (\mathrm{d}i_2)_e(X_2)$$

Since i_1 and i_2 are injective immersions and $\mathrm{Lie}(A_1) + \mathrm{Lie}(A_2) = \mathrm{Lie}(M)$, we have that α is a submersion at (e, e). For proving that α is a submersion at any $(\overline{a}_1, \overline{a}_2) \in \widetilde{A}_1 \times \widetilde{A}_2$, it is enough to notice that the diagram

$$\begin{tikzcd} A_1 \times A_2 \arrow[r, "\alpha"] \arrow[d, "t"'] & G \arrow[d, "s"] \\ A_1 \times A_2 \arrow[r, "\alpha"] & G \end{tikzcd}$$

is commutative, where t and s are given by $t = \ell(\overline{a}_1) \times r(\overline{a}_2)$, $s = \ell(\overline{a}_1) \circ r(\overline{a}_2)$, $\ell\,$, r being left and right translations.

For proving 2., note that $\mathrm{Lie}(\widetilde{A}_1) \cap \mathrm{Lie}(\widetilde{A}_2) = \{0\}$, hence $\mathrm{Lie}(M)_0 = \mathrm{Lie}(\widetilde{A}_1) \oplus \mathrm{Lie}(\widetilde{A}_2)$. As $\widetilde{A}_1 \cap \widetilde{A}_2 = \{e\}$, $\widetilde{\alpha}$ is an analytic isomorphism. Since $\mathrm{Lie}(A_1) + \mathrm{Lie}(A_2) = \mathrm{Lie}(M)$, and

$\mathrm{Lie}(A_1) \cap \mathrm{Lie}(A_2) = \{0\}$, $\dim(A_1 \times A_2) = \dim M$. So α is a covering map. As $\widetilde{\alpha}$ is an isomorphism, so is α.

Let us come to 3. The statements are clear at the reduced level. Hence, for transitivity of A_1 on A_1A_2/A_2, we must show that $d\pi_e(\mathrm{Lie}(A_1)) = T_{\pi(e)}(A_1A_2/A_2)$. But $d\pi_e(\mathrm{Lie}(A_1)) = \{0\}$ from the theory of quotient spaces, which also gives $d\pi_e(\mathrm{Lie}(A_1A_2)) = T_{\pi(e)}(A_1A_2/A_2)$. Hence, as $\mathrm{Lie}(A_1) + \mathrm{Lie}(A_2) = \mathrm{Lie}(A_1A_2)$, we must have $d\pi_e(\mathrm{Lie}(A_1)) = T_{\pi(e)}(A_1A_2/A_2)$.

To find the stabilizer at $\pi(e)$ of the action of A_1 on A_1A_2/A_2, let N be the stabilizer. Clearly $\widetilde{N} = \widetilde{A}_1 \cap \widetilde{A}_2$. Since the kernel of $d\pi_e$ is $\mathrm{Lie}(A_2)$, the kernel of $d\pi'_e$ (where $\pi' = \pi_{|_N}$) is precisely the space of those vectors in $T_e(A_2)$ which are tangent to A_1 at e, namely $T_e(A_1) \cap T_e(A_2) \simeq \mathrm{Lie}(A_1) \cap \mathrm{Lie}(A_2)$. So $N = A_1 \cap A_2$.

■

If X is a complex supermanifold, let us denote with $X^{\mathbb{R}}$ its underlying real supermanifold (see [14, 10]).

Lemma 3.7. *Let G be a connected complex matrix Lie supergroup,* $\mathrm{Lie}(G) = \mathfrak{g}$. *Let G_r be a connected real form of G,* $\mathrm{Lie}(G_r) = \mathfrak{g}_r$*, so that $\mathfrak{g}$ is the complexification of $\mathfrak{g}_r$. Let R be a closed subsupergroup of G_r,* $\mathrm{Lie}(R) = \mathfrak{r}$*.*
Let $\mathfrak{q}$ be a complex Lie subsuperalgebra of $\mathfrak{g}$ such that

- $\mathfrak{g}_r + \mathfrak{q}^{\mathbb{R}} = \mathfrak{g}^{\mathbb{R}}$*;*
- $\mathfrak{g}_r \cap \mathfrak{q}^{\mathbb{R}} = \mathfrak{r}$*.*

where $\mathfrak{q}^{\mathbb{R}}$ and $\mathfrak{g}^{\mathbb{R}}$ are the complex Lie superalgebras $\mathfrak{q}$ and $\mathfrak{g}$ viewed as real Lie superalgebras. Assume that the analytic subSLG Q defined by $\mathfrak{q}$ in G is closed. Let R_1 be the SLG with reduced group $\widetilde{Q} \cap \widetilde{G_r}$ and Lie superalgebra $\mathfrak{r}$.

1. *$G_rQ^{\mathbb{R}}$ is an open subsupermanifold of $G^{\mathbb{R}}$. Hence, $G_rQ := (\widetilde{G}\widetilde{Q}, \mathcal{O}_G|_{\widetilde{G}\widetilde{Q}})$ is an open subsupermanifold of the complex supergroup G. If π is the natural map $G \to G/Q$, $M := \pi(G_rQ)$ is an open subsupermanifold of the complex supermanifold G/Q. We also write $M = G_rQ/Q$.*
2. *The natural action of G_r on $M^{\mathbb{R}}$ is transitive; if its stabilizer at $\pi(e)$ is R_1, then $G_r/R_1 \simeq M^{\mathbb{R}}$. Hence G_r/R_1 acquires naturally a complex supermanifold structure.*
3. *If R' is any closed subSLG of G_r with Lie$(R') = \mathfrak{r}$, then G_r/R' acquires a natural G_r-invariant complex (super) structure, compatible with the real analytic one.*

Proof. 1. By our hypothesis G_r and $Q^{\mathbb{R}}$ are subsupergroups of $G^{\mathbb{R}}$ such that $\mathfrak{g}_r + \mathfrak{q}^{\mathbb{R}} = \mathfrak{g}^{\mathbb{R}}$. Hence by Lemma 3.6 we are done. 2. Immediate from 3. of Lemma 3.6. 3. Follows from Remark 3.4.

■

3.1. **Associated Super Vector Bundles and Super Fréchet Representations.** For the relevant material about super vector bundles we refer to [14]. Here we recall that a *super vector bundle* $\mathcal{V}$ of rank $p|q$ on a supermanifold M is a locally free sheaf of rank $p|q$ over M, that is for each $x \in \widetilde{M}$, there exist U open such that $\mathcal{V}(U) \cong \mathcal{O}_M(U)^{p|q} := \mathcal{O}_M(U) \otimes k^{p|q}$. $\mathcal{V}$ is a sheaf of $\mathcal{O}_M$ modules and at each $x \in \widetilde{M}$, the stalk $\mathcal{V}_x$ is a $\mathcal{O}_{M,x}$-module. We define the *fiber* of $\mathcal{V}$ at the point x as the vector superspace $\mathcal{V}_x/m_x\mathcal{V}_x$, where m_x is the maximal ideal of $\mathcal{O}_{M,x}$.

We briefly recall the definition of associated super vector bundle in the language of SHCP. These are super vector bundles on G/H associated to finite dimensional H-representations, where H is a closed subSLG of G.

Definition 3.8. Let G be a SLG, H a closed subSLG, σ a finite finite-dimensional complex representation of H on V, with $\sigma = (\widetilde{\sigma}, \rho^\sigma)$ in the language of SHCP's. Consider the sheaf over $\widetilde{G}/\widetilde{H}$

$$\mathcal{A}(U) := \mathcal{O}_G(\widetilde{p}^{-1}(U)) \otimes V, \qquad U \subset \widetilde{G}/\widetilde{H}$$

where $p : G \to G/H$ is the canonical projection. We define the super vector bundle associated with σ as

$$U \longmapsto \mathcal{A}_{SHCP}(U) \tag{4}$$

where:

$$\mathcal{A}_{SHCP}(U) := \left\{ f \in \mathcal{A}(U) \mid \left\{ \begin{array}{ll} (r_h^* \otimes 1)f = (1 \otimes \widetilde{\sigma}(h)^{-1})(f) & \forall h \in \widetilde{H} \\ (D_X^L \otimes 1)f = (1 \otimes \rho^\sigma(-X))f & \forall X \in \mathfrak{h}_1 \end{array} \right. \right\} \tag{5}$$

One can prove that the previous definition defines a super vector bundle over G/H with typical fiber V.

The following definition is a natural generalization of the one given in [12], and it can be found in [1]. We refer to [50] for the classical result.

Definition 3.9. Let G be a complex or real Lie supergroup. We say we have a *representation* of G in the complex Fréchet vector superspace F if:

- $\widetilde{G}$ acts on $F = F_0 \oplus F_1$ and:
$$\widetilde{\pi} \colon \widetilde{G} \longrightarrow \mathrm{Aut}(F_0) \times \mathrm{Aut}(F_1)$$
is an ordinary Frechét representation preserving parity.
- Denote with $C\underline{\mathrm{End}}(C^\infty(\widetilde{\pi}))$ the algebra of continuous linear endomorphisms of the space of smooth vectors of $C^\infty(\widetilde{\pi})$ endowed with the Fréchet relative topology inherited from $C^\infty(\widetilde{G}; F)$. There is an even linear map
$$\rho^\pi : \mathfrak{g} \longrightarrow C\underline{\mathrm{End}}(C^\infty(\widetilde{\pi}))$$
such that
  1. $\rho^\pi|_{\mathfrak{g}_0} = \mathrm{d}\widetilde{\pi}$
  2. $\rho^\pi([X, Y]) = \rho^\pi(X)\rho^\pi(Y) - (-1)^{|X|+|Y|}\rho^\pi(Y)\rho^\pi(X)$
  3. $\rho^\pi(\mathrm{Ad}(g)X) = \widetilde{\pi}(g)\rho^\pi(X)\widetilde{\pi}(g)^{-1}$

We have: $F^\infty = F_0^\infty \oplus F_1^\infty$.

It is not difficult to prove the following proposition.

Proposition 3.10. *Let G and H be as above. Let $\sigma = (\widetilde{\sigma}, \rho^\sigma)$ be an H-representation in the language of SHCP. If $U \subset \widetilde{G/H}$ is a G-invariant open subset. The assignment:*

1. $\widetilde{G} \times \mathcal{A}^\sigma(U) \longrightarrow \mathcal{A}^\sigma(U), \qquad g, f \mapsto l_{g^{-1}}^* f$
2. $\mathfrak{g} \longrightarrow \mathrm{End}(\mathcal{A}^\sigma(U)), \qquad X \mapsto D_{-X}^R$

gives a representation of G on the Fréchet superspace $\mathcal{A}^\sigma(U)$, where as usual $D_{-X}^R = (1 \otimes X)\mu^*$ and the element $X \in \mathfrak{g}$ is interpreted as a left invariant vector field.

We now turn to examine the decomposition of a super Fréchet representations of a supergroup with respect to the action of an ordinary compact Lie subgroup.

Let H be a real Lie supergroup, and U an ordinary compact Lie subgroup in H, $T \subset U$ a maximal torus. Notice that the following treatment applies also to the case $T = U$.

Assume H acts on a Fréchet superspace F via the representation R according to Definition 3.9. Notice that the restriction of the representation R to U automatically preserves the $\mathbb{Z}_2$–grading $F = F_0 \oplus F_1$. Let τ be a character of an irreducible representation of U, that we can assume unitary. We define the operator:

$$P(\tau) = d(\tau) \int_U \tau(k)^{-1} R(k) \mathrm{d}k, \qquad \text{with} \quad \int_U \mathrm{d}k = 1 \tag{6}$$

where $d(\tau)$ is the degree of τ (namely the dimension of the irreducible representation associated with τ). We define $F(\tau)$ as the closed subspace of F stable under U and on which U acts according to the irreducible representation with character τ. $F(\tau)$ is called the *isotypic subspace* corresponding to the character τ.

We stress that, in the whole section, $F^\infty = F_0^\infty \oplus F_1^\infty$ denotes the space of smooth vectors for the representation R of H. When we want to consider smooth vectors for the restriction of R to a subgroup U of G we will add a subscript F_U^∞. Clearly, one has $F^\infty \subseteq F_U^\infty$. The following is a standard result, see [26] or [50, Sections 4.4.2 and 4.4.3].

Proposition 3.11. *Let R be a representation of the compact Lie group U on the Frechét superspace $F = F_0 \oplus F_1$. Then:*

1. *the operator $P(\tau)$ defined by (6), is an even continuous projection onto the isotypic subspace $F(\tau) = F(\tau)_0 \oplus F(\tau)_1$.*
2. *$F(\tau)$ is a closed subsuperspace of F and it consists of the algebraic sum of the linear subsuperspaces on which U acts irreducibly according to the (irreducible) representation with character τ. Furthermore the $F(\tau)$ are linearly independent.*
3. *$P(\tau)P(\tau') = 0$, if $\tau \neq \tau'$.*
4. *$P(\tau)$ commutes with the U action and with any continuous endomorphism of F commuting with U.*
5. *On the space of smooth vectors we have $\sum_\tau P(\tau) = \mathrm{id}_{|F^\infty}$, that is any $f \in F^\infty$ is expressed as $\sum_\tau f_\tau$, which is called the* Fourier series *of f. Furthermore, such series converges uniformly.*
6. *Let $F^0 := \sum F(\tau)$ (algebraic sum). Then $F^0 \subset F^\infty$ and both are dense in F.*

When necessary we shall stress the fact that the decomposition of F is under the U-action by writing F_U^0 and $F_U(\tau)$, similarly we write $P_U(\tau)$ for the operator defined in (6).

Definition 3.12. We say that a representation R as above is *U-finite* if every $F(\tau)$ is finite dimensional.

The following is a standard lemma, that we leave to the reader as an exercise.

Lemma 3.13. *Let the notation and setting be as above.*

1. *Let $\widehat{F}^0 = \sum_\tau L_\tau$ be a dense subspace in F, where the sum is algebraic, the subspaces L_τ are all finite dimensional and $L_\tau \subset F_U(\tau)$. Then $L_\tau = F_U(\tau)$ for all τ and $\widehat{F}^0 = F_U^0 \subset F^\infty$. Hence, F is U-finite.*

(2) *Let U' be a compact subgroup of U and assume that F is U'-finite. Then F is also U-finite and $F^0_U = F^0_{U'}$.*

4. Representations of the Supergroup

The objective of this section is to construct representations of a real supergroup G_r which correspond infinitesimally to the highest weight Harish-Chandra modules.

Let $\mathfrak{g}$ be as in list (1) and let $\mathfrak{g}_r$ be a real form of $\mathfrak{g}$ (see [42, 9]). By the ordinary theory, we know that, since $\mathfrak{g}_0$ is either semisimple or with a one-dimensional center, the simply connected corresponding ordinary Lie group $\widetilde{G}$ is a matrix Lie and algebraic group. Then, the SHCP $G = (\widetilde{G}, \mathfrak{g})$ (see [8] Ch. 11 and [17]) can be viewed either as an algebraic or an analytic complex SHCP. Hence G is a complex analytic matrix supergroup and $G^{\mathbb{R}}$, the supergroup G viewed as a real supergroup (see [14, 10]), is also a real analytic matrix supergroup. Let G_r be the real analytic supergroup corresponding to the real subsuperalgebra $\mathfrak{g}_r$ of $\mathfrak{g}^{\mathbb{R}}$ (the superalgebra $\mathfrak{g}$ viewed as real superalgebra). Also G_r is a matrix real Lie supergroup and we will refer to G as the complexification of G_r and we will refer to G_r as a real form of G.

Fix $\mathfrak{h}$ and $\mathfrak{h}_r$ CSA of $\mathfrak{g}$ and $\mathfrak{g}_r$ respectively, $\mathfrak{h}$ the complexification of $\mathfrak{h}_r$. $K_r = \widetilde{K_r}$ is the maximal compact in $\widetilde{G_r}$, $A_r = \widetilde{A_r}$ the (ordinary) torus, $A_r \subset G_r$, while $\mathfrak{k}_r$, $\mathfrak{h}_r$ the respective Lie superalgebras. We drop the index r to mean the complexifications. We assume:

$$\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}, \qquad \mathfrak{h}_r \subset \mathfrak{k}_r \subset \mathfrak{g}_r.$$

Hence our CSA $\mathfrak{h} = \mathfrak{h}_0$. Let Δ be the root system corresponding to $(\mathfrak{g}, \mathfrak{h})$ and fix P a positive system. Let us define $\mathfrak{b}^{\pm}$ and $\mathfrak{n}^{\pm}$ the *Borel and nilpotent subsuperalgebras*:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha, \qquad \mathfrak{b}^{\pm} := \mathfrak{h} \oplus \sum_{\alpha\in\pm P} \mathfrak{g}_\alpha, \qquad \mathfrak{n}^{\pm} := \sum_{\alpha\in\pm P} \mathfrak{g}_\alpha.$$

We call $B^{\pm}$ *Borel subsupergroup* and $N^{\pm}$ *unipotent subsupergroup*, their corresponding analytic Lie supergroups in G. In particular, $B^{\pm}$and $N^{\pm}$ are connected and are algebraic subsupergroups of G.

4.1. **Maximal torus and big cell in Lie supergroups of classical type.**

In this section we want to study the connected ordinary Lie group $A \subset G$, called a *maximal torus* of G, with associated Lie algebra $\mathrm{Lie}(A) = \mathfrak{h}$ the CSA of $\mathfrak{g}$, and its relation with the supergroups $N^{\pm}$. In particular we introduce the supermanifold $\Gamma := N^- A N^+ \subset G$, called the *big cell*, which plays a key role in what follows. We observe first that the (ordinary) torus A normalizes $N^{\pm}$, as it happens for the ordinary setting.

Proposition 4.1. *Let $\mathfrak{m}_r$ be a real form of the Lie superalgebra $\mathfrak{g} = \mathrm{Lie}(G)$ containing $\mathfrak{h}_r$ the CSA of $\mathfrak{g}_r$. Then: $\mathfrak{m}_r + \mathfrak{b}^+ = \mathfrak{g}$ and in particular $M_r(B^+)^{\mathbb{R}}$ is open subsupermanifold of $G^{\mathbb{R}}$, where M_r is the connected subSLG of G_r determined by $\mathfrak{m}_r$.*

Proof. Since $\mathfrak{m}_r$ is a real form of $\mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{m}_r \oplus i\mathfrak{m}_r$. This is equivalent to say that there exists an antilinear involution $\widetilde{\ }: \mathfrak{g} \to \mathfrak{g}$ whose set of fixed points is $\mathfrak{m}_r$. Moreover, since $\mathfrak{h}_r$ is contained in $\mathfrak{k}_r$ we have that all the roots are imaginary when restricted to $\mathfrak{h}_r$. These facts imply that $\widetilde{\mathfrak{g}_\alpha} = \mathfrak{g}_{-\alpha}$.

In order to prove our statement it is enough to show that $X_{-\alpha}$ and $iX_{-\alpha}$ belong to $\mathfrak{m}_r + \mathfrak{n}^+$. We have that:

$$X_{-\alpha} = \widetilde{X_\alpha} = (X_\alpha + \widetilde{X_\alpha}) - X_\alpha \in \mathfrak{m}_r + \mathfrak{n}^+$$
$$iX_{-\alpha} = -i\widetilde{X_\alpha} = (-i\widetilde{X_\alpha} + iX_\alpha) - iX_\alpha \in \mathfrak{m}_r + \mathfrak{n}^+$$

Hence, by Lemma 3.6, we obtain our result. ■

Proposition 4.2. *Let the notation be as above. Then we have that:*

(1) $\widetilde{\Gamma} := \widetilde{N^- A N^+}$ *is open in* $\widetilde{G}$.

(2) $\widetilde{A}$, $\widetilde{N^\pm}$ *are closed and* $\widetilde{N^\pm} \cap \widetilde{A} = \{1\}$.

(3) *The morphism*

$$N^- \times A \times N^+ \longrightarrow G, \quad (n^-, h, n^+) \longmapsto n^- h n^+, \quad n \in N^\pm(T), \quad h \in A(T), T \in (\mathrm{smflds})_\mathbb{C}$$

is an analytic isomorphism onto its image $N^- A N^+$ *which is an open subsupermanifold of* G.

Proof. (1) and (2) are statements of ordinary geometry. (3) Consider the morphism $\phi : A \times N^+ \longrightarrow AN^+ \subset G$. AN^+ is a Lie supergroup, since N^+ is normalized by A and $\mathrm{Lie}(AN^+) = \mathfrak{h} + \mathfrak{n}^+$. Hence ϕ is a diffeomorphism onto its image (apply Lemma 3.6). We now apply again Lemma 3.6 to the map $\psi : N^- \times AN^+ \to G$. ψ is a diffeomorphism onto its image, which is an open subsupermanifold of G. ■

Remark 4.3. The image of the multiplication morphism $A \times N^+ \longrightarrow AN^+ \subset G$ is a Lie supergroup. Since its reduced space is $\widetilde{B^+} = \widetilde{AN^+}$ and its Lie superalgebra $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^-$, we have that $B^+ = AN^+$ (and similarly $B^- = AN^-$), where B^+ is the unique connected subsupergroup of G with Lie superalgebra $\mathfrak{b}^+$.

Definition 4.4. Let the notation be as above. We define the *big cell* in G as the open subsupermanifold of G:

$$\Gamma := N^- A N^+ \subset G$$

Its underlying topological space $\widetilde{\Gamma} = \widetilde{N^- A N^+}$ is open and dense in G.

Proposition 4.5. *Let the notation be as above. Then we have that:*

(1) $G_r(B^\pm)^\mathbb{R}$ *are open real subsupermanifolds in* $G^\mathbb{R}$*;* $G_r B^\pm = (\widetilde{G_r}\widetilde{B^\pm}, \mathcal{O}_G|_{\widetilde{G_r}\widetilde{B^\pm}})$ *are open complex subsupermanifolds in* G*;*

(2) $G_r/A_r \cong G_r B^\pm / B^\pm$ *acquires a* $\widetilde{G_r}$ *invariant complex structure.*

(3) N^- *is a section for* $\Gamma \to \Gamma/B^+$*, the left action of* A *reads:*

$$A \times \Gamma/B^+ \longrightarrow \Gamma/B^+, \ (h, nB^+(T)) \longmapsto hnh^{-1}B^+(T), \ n \in N^\pm(T), \ h \in A(T), \ T \in (\mathrm{smflds})_\mathbb{C}$$

Proof. (1) Due to Lemma 3.6, $\mathfrak{g}_r + \mathfrak{b}^\pm = \mathfrak{g}$, hence the map $\alpha \colon G_r \times B^\pm \to G$ is a subsupermersion and the subsupermanifolds $G_r(B^\pm)^\mathbb{R}$ are open in $G^\mathbb{R}$.

(2) If we prove that $\mathfrak{g}_r \cap \mathfrak{b}^\pm = \mathfrak{h}_r$ and $\widetilde{G_r} \cap \widetilde{B^+} = \widetilde{A_r}$ we can apply Lemma 3.7 and conclude. Let us hence proceed to prove these facts. We have

$$\mathfrak{g}_r \cap \mathfrak{b}^\pm = \mathfrak{h}_r \,.$$

Indeed, let $X \mapsto X\tilde{}$ be the conjugation of $\mathfrak{g}$ associated with $\mathfrak{g}_r$ as described in the proof of Prop 4.1. Consider the case of $\mathfrak{b}^+$ for definiteness. $X \in \mathfrak{b}^+$ can be written as $X = \sum c_i H_i + \sum_{\alpha\in\Delta^+} d_\alpha X_\alpha$ with $H_i \in \mathfrak{h}_r$, $X_\alpha \in \mathfrak{g}_\alpha$, and $c_i, d_\alpha \in \mathbb{C}$. Then $X\tilde{} = \sum_i \overline{c}_i H_i + \sum_{\alpha\in\Delta^+} \overline{d}_\alpha X_\alpha\tilde{}$. Since $\mathfrak{g}_\alpha\tilde{} = \mathfrak{g}_{-\alpha}$ (see the proof of Prop 4.1), we have that $X\tilde{} = X$ if and only if $X \in \mathfrak{h}_r$.
Since by assumption the SLG G is simply connected, there exists an antiautomorphism $\sigma\colon G \to G$ such that $(\mathrm{d}\sigma)_e(X) = X\tilde{}$. By the ordinary theory we have that $\widetilde{G_r} \cap \widetilde{B^+} = \widetilde{A_r}$. Indeed, let $a = hn^+ \in \widetilde{G_r} \cap \widetilde{B^+}$, with $h \in \widetilde{A_r}$ and $n^+ \in \widetilde{B^+}$. Hence $hn^+ = \widetilde{\sigma}(h)n^-$ (where $n^- := \widetilde{\sigma}(n^+)$), that is $\widetilde{\sigma}(h)^{-1}hn^+ = n^-$, so that $h = \widetilde{\sigma}(h)$, $n^+ = n^- = 1$, so $a \in \widetilde{A_r}$.

(3) Since the big cell $\Gamma \subset G$ is right B^+-invariant and open, and the canonical projection $p\colon G \to G/B^+$ is a submersion, we can define the open subsupermanifold of G/B^+:

$$\Gamma/B^+ := (\widetilde{\Gamma/B^+}, \mathcal{O}_{G/B^+}|_{\widetilde{\Gamma/B^+}})$$

We have a N^- equivariant diffeomorphism $N^- \longrightarrow \Gamma/B^+$, $n^- \mapsto n^- B^+(T)$, $n^- \in N^-(T)$, $T \in (\text{smflds})_{\mathbb{C}}$. In fact, by the ordinary theory we have a diffeomorphisms of the underlying differentiable manifolds and the differential at the identity is an isomorphism: $\mathfrak{n}^- \cong \mathfrak{g}/\mathfrak{b}^+$. ∎

4.2. **Line bundles on G/B^+.** Let us consider a character χ_r of the classical real maximal torus A_r inside the real supergroup G_r. This character uniquely extends to an holomorphic character of A and has the form

$$\begin{aligned} \chi : A &\longrightarrow \mathbb{C}^\times \\ exp(H) &\longmapsto e^{\lambda(H)} \end{aligned}$$

for an integral weight $\lambda \in \mathfrak{h}^*$ (i.e. a weight such that $\lambda(H_\gamma) \in \mathbf{Z}$ for all roots γ).

We can trivially extend the character χ of A to a character of the Borel subsupergroup B^+, since we know $B^+ = AN^+$. We denote with (χ_0, λ) the corresponding representation in the SHCP formalism.

The character $\chi = e^\lambda$ defines according to 3.1 an holomorphic line bundle on $\widetilde{G/B^+}$ that we denote with L^χ or L_λ depending on the convenience. If $p : G \longrightarrow G/B^+$ we have:

$$L^\chi(U) = L_\lambda(U) = \left\{ f \in \mathcal{O}_G(p^{-1}(U)) \mid \begin{cases} r_b^* f = \chi_0(b)^{-1}(f) & \forall b \in \widetilde{B^+} \\ D_X^L f = \lambda(-X) f & \forall X \in \mathfrak{b}^+ \end{cases} \right\} \tag{4.1}$$

We can equivalently write:

$$L^\chi(U) = L_\lambda(U) = \left\{ f : p^{-1}(U) \to \mathbb{C}^{1|1} \mid \ f_T(gb) = \chi_T(b)^{-1} f_T(g),\ b \in B^+(T)\,,\, g \in U(T) \right\}$$

We now turn our attention to the Frechét superspace $F := L^\chi(\widetilde{\Gamma/B^+})$, where $\Gamma = N^-AN^+$ is the big cell in the complex supergroup G. Γ is neither stable under G-action nor under the G_r-action, however, as any neighbourhood of the identity, it is stable under the action of $\mathcal{U}(\mathfrak{g})$ and we want to study such representation.

Proposition 4.6. *The restriction of the holomorphic line bundle L^χ to $\widetilde{\Gamma/B^+} = \widetilde{N^-AN^+}/B^+$ is trivial:*

$$L^\chi(\widetilde{\Gamma/B^+}) \simeq \mathcal{O}_{G/B^+}(\widetilde{\Gamma/B^+})$$

In particular, there is a canonical identification between:

$$F = L^\chi(\widetilde{\Gamma/B^+}) \simeq \mathcal{O}(N^-) \tag{7}$$

between the sections on the big cell of the line bundle L^χ and the holomorphic functions on N^-.

Proof. In order to prove the triviality of the line bundle L^χ over $\widetilde{\Gamma/B^+}$, we have to construct a section $s\colon \Gamma/B^+ \to \Gamma$. This is the content of Prop. 4.5, (3). The isomorphism (7) is easily established using the correspondence between sections of the associated bundle L^χ and the B^+-equivariant mappings $N^-B^+ \to \mathbb{C}^{1|1}$, as in the classical setting (see, for example, [39]). More precisely, let

$$\kappa\colon N^- \times B^+ \xrightarrow{\simeq} N^-B^+$$

be the isomorphism established in Prop. 4.2. We have maps $\eta\colon L^\chi(\widetilde{\Gamma/B^+}) \to \mathcal{O}(N^-)$ and $\zeta\colon \mathcal{O}(N^-) \to L^\chi(\widetilde{\Gamma/B^+})$ given by

$$\begin{aligned} \eta\colon L^\chi(\widetilde{\Gamma/B^+}) &\longrightarrow \mathcal{O}(N^-) \\ f &\longrightarrow G_f := \eta(f) \end{aligned} \qquad\qquad \begin{aligned} \zeta\colon \mathcal{O}(N^-) &\longrightarrow L^\chi(\widetilde{\Gamma/B^+}) \\ G &\longrightarrow f_G := \zeta(G) \end{aligned}$$

where G_f and f_G are the morphisms defined as follows

$$G_f\colon N^- \xrightarrow{i} N^- \times B^+ \xrightarrow{\kappa} N^-B^+ \xrightarrow{f} \mathbb{C}^{1|1}$$

and

$$f_G\colon N^-B^+ \xrightarrow{\kappa^{-1}} N^- \times B^+ \xrightarrow{G\times\chi} \mathbb{C}^{1|1} \times \mathbb{C}^\times \longrightarrow \mathbb{C}^{1|1}$$

η and ζ gives the desired isomorphism, we leave to the reader the standard checks involved. ∎

Remark 4.7. Since the Frechét topology on L^χ is defined through local trivializations, by the previous Proposition we have that the identification $L^\chi(\widetilde{\Gamma/B^+}) \simeq \mathcal{O}(N^-)$ is also an isomorphism of Frechét superspaces, an in the ordinary case.

To ease the notation we shall also write $L^\chi(\Gamma)$ in place of $L^\chi(\widetilde{\Gamma/B^+})$.

Let t_α denote the global homogeneous exponential coordinates on N^- (see [35, 23] for the definition of exponential). By a classical result, if $\widetilde{N}$ is a connected nilpotent Lie group, $\mathcal{U}(\mathfrak{n}_0)$ preserves the ordinary polynomials $\mathcal{P}(\widetilde{N})$ on $\widetilde{N}$. Let $\mathcal{P} = \mathcal{P}(\widetilde{N}^-) \otimes \wedge(\mathfrak{n}_1^-)$. We thus have the natural identifications:

$$\mathcal{P} = \mathcal{P}(\widetilde{N}^-) \otimes \wedge(\mathfrak{n}_1^-) = \underline{\mathrm{Hom}}_{\mathfrak{n}_0}(\mathcal{U}(\mathfrak{n}^-), \mathcal{P}(\widetilde{N}^-)) \subset \underline{\mathrm{Hom}}_{\mathfrak{n}_0}(\mathcal{U}(\mathfrak{n}^-), C^\infty(\widetilde{N}^-))$$

Notice that $\mathcal{P}$ are the polynomials in the indeterminates t_α.

We now want to study in detail the action of A_r, the ordinary torus in G_r, on the polynomials $\mathcal{P}$ in $\mathcal{O}(N^-)$ and the corresponding superalgebra $\mathcal{P}^\sim$ in $F = L^\chi(\Gamma)$.

Proposition 4.6 allows us to obtain immediately the following corollary (we shall also see it later as a consequence of Lemma 4.12 in the next section).

Proposition 4.8. *$\mathcal{P}$ is dense in $\mathcal{O}(N^-)$ and $\mathcal{P}^\sim$ is dense in F.*

Proof. In view of the definition of the topology on F, it is enough to prove that $\mathcal{P}$ is dense in $\mathcal{O}(N^-)$. The proof goes as in the ordinary setting, since N^- is analytically isomorphic to $\mathbb{C}^{m|n}$ via the exponential morphism, for suitable $m|n$. ∎

4.3. **The action of the maximal torus on the polynomials on the big cell.** In this section we introduce two natural actions c and l of the ordinary Lie group A_r on the big cell $\Gamma = N^- A N^+$, together with the actions i and ℓ they induce on the Frechét superspace $L^\chi(\Gamma)$. We notice for future reference that both actions coincide on the quotient Γ/B^+.

In the definition of the two actions we use the isomorphism (see Prop. 4.2)

$$\kappa\colon N^- \times B^+ \xrightarrow{\simeq} \Gamma$$

Let us start with the action c related to the coniugation. Since A_r acts on N^- by conjugation (see Prop. 4.5), we have a global action of A_r on Γ defined as:

$$c\colon A_r \times \Gamma \xrightarrow{1_{A_r}\times\kappa^{-1}} A_r \times (N^- \times B^+) \xrightarrow{\mathrm{conj}\times 1_{B^+}} N^- \times B^+ \xrightarrow{\kappa} \Gamma \tag{8}$$

which in the functor of points notation reads

$$a \cdot (n^- b^+) = (a n^- a^{-1}) b^+, \quad a \in \widetilde{A_r},\, n^- \in N^-(T), b^+ \in B^+(T).$$

Since A_r also acts on B^+ by left translation l', we can define the left action of A_r on Γ as

$$l_a = \kappa \circ (\mathrm{conj}_a \times l'_a) \circ \kappa^{-1}$$

or, in the functor of points notation,

$$a \cdot (n^- b^+) = (a n^- a^{-1}) a \cdot b^+. \tag{9}$$

Both actions commute with right translations by B^+ and hence define representations of A_r on $L^\chi(\Gamma)$

$$i,\, \ell\colon A_r \times L^\chi(\Gamma) \longrightarrow L^\chi(\Gamma)$$

where $i_a(f) = c^*_{a^{-1}}(f)$ and $\ell_a(f) = l^*_{a^{-1}}(f)$ for all $a \in \widetilde{A_r}$, and all $f \in L^\chi(\Gamma)$.

These representations are most easily written in the functor of points notation as

$$i_a(f)(n^- b^+) = f((a^{-1} n^- a) b^+)$$
$$\ell_a(f)(n^- b^+) = f((a^{-1} n^- a) a^{-1} b^+)$$

The above formulas further simplify using the identification

$$L^\chi(\Gamma) \simeq \mathcal{O}(N^-)\,,$$

we leave the details to the reader.

Lemma 4.9. *Let the notation be as above. Then*

(1) $\ell_a f = \chi(a)(i_a f)$
(2) $i_a t_\alpha = \chi_\alpha(a) t_\alpha \qquad \forall a \in \widetilde{A_r}$

Proof. (1) follows immediately from (9). For (2) let $n = \exp(\sum_{\beta\in P} y_\beta X_{-\beta})$ in $N^-(T)$, then the result comes from the following formal calculation:

$$\begin{aligned} t_\alpha(a^{-1}na) &= t_\alpha\big(\exp(\sum_{\beta\in P} y_\beta Ad(a)X_{-\beta})\big) = t_\alpha(\exp(\sum_{\beta\in P} y_\beta \chi_\beta(a)X_{-\beta}) \\ &= \chi_\alpha(a)t_\alpha(n), \end{aligned}$$

where $a \in \widetilde{A_r}$, $y_\beta \in \mathbb{C}$ and the t_α are the polynomial coordinates on N^- (see Sec. 4.2). ∎

To ease the notation we shall also write $a \cdot f$ in place of $\ell_a(f)$.

Proposition 4.10. *Let $\mathcal{P}$ be the polynomial superalgebra generated by the t_α in $\mathcal{O}(N^-)$ and let $\mathcal{P}^\sim$ be the corresponding submodule in F. A_r acts on $\mathcal{P}^\sim$ and we have that:*

$$a \cdot (t_{\alpha_1}^{r_{\alpha_1}} \dots t_{\alpha_s}^{r_{\alpha_s}})^\sim = \chi_{\lambda+\sum r_{\alpha_i}\alpha_i}(a)\,(t_{\alpha_1}^{r_{\alpha_1}} \dots t_{\alpha_s}^{r_{\alpha_s}})^\sim$$

Hence $\mathcal{P}^\sim$ decomposes into the sum of eigenspaces $\mathcal{P}^\sim_d$ for the action of A_r, where d ranges in D^+ the semigroup in $\mathfrak{h}^*$ generated by the positive roots:

$$\mathcal{P}^\sim = \oplus_{d\in D^+}\mathcal{P}^\sim_d, \qquad \mathcal{P}^\sim_d = \oplus_{\sum r_{\alpha_i}\alpha_i = d}\,\mathbb{C}\cdot(t_{\alpha_1}^{r_{\alpha_1}} \dots t_{\alpha_s}^{r_{\alpha_s}})^\sim$$

A similar decomposition holds also for $\mathcal{P}$.

Proof. This is a simple calculation, similar to the one in Lemma 4.9. ∎

Corollary 4.11. *The maximal torus A_r acts on the Frechét superspace $L^\chi(\Gamma)$ and we have:*

(1) $F(\tau) \neq 0$ *if and only if* $\tau = \chi_{-\lambda+d}$ *for some* $d = \sum_{m_\alpha\in\mathbf{Z}_{\geq 0},\alpha\in P} m_\alpha\alpha$.
(2)

$$F(\chi_{\lambda+d}) = \mathcal{P}^\sim_{\lambda+d}$$

and

$$\dim(F(\chi_{\lambda+d})) = \#\left\{ r = (r_\alpha) \mid \sum_{r_\alpha\in\mathbf{Z}_{\geq 0},\alpha\in P} r_\alpha\alpha = d \right\}$$

Proof. (1) and (2) are consequences of Lemma 3.13. The computation of the dimension is straightforward. ∎

We now want to prove the fact that the spectrum of A_r remains unchanged when we change the open set we are considering in a suitable way. We shall first prove a general lemma.

Let T be an ordinary compact torus acting on a finite dimensional complex vector superspace V. By a classical result we have the action of T on V is via characters τ_i's and, with a suitable choice of a basis of V, reads as follows

$$\begin{aligned} T \times V &\longrightarrow V \\ t, (v_1, \dots, v_m, \nu_1, \dots, \nu_n) &\longmapsto (\tau_1(t)v_1, \dots \tau_m(t)v_m, \tau_{m+1}(t)\nu_1, \dots, \tau_{m+n}(t)\nu_n) \end{aligned}$$

We can easily transport this action to the space of polynomial functions $\mathrm{Pol}(V)$ on V and obtain the following action:

$$t \cdot \sum a_{IJ} z^I \xi^J = \sum a_{IJ} \tau^I(t)^{-1}\tau^J(t)^{-1} z^I \xi^J$$

using the multiindex notation $I = (i_1 \dots, i_n)$ with possibly repeated indices, $J = (j_1, \dots, j_n)$ with no repeated indices.

T has also a natural action on the holomorphic sections of the structural sheaf on V, $\mathcal{O}_V(U)$, where U is a T invariant open set in V. If $g \in \mathcal{O}_V(U)$, we know we can view such g as a morphism $g : U \longrightarrow \mathbb{C}^{1|1}$. If $\rho_t(u) := t \cdot u$ is the action of T on U, we define:

$$t \cdot g = g \circ \rho_{t^{-1}}$$

Notice that such action agrees with the previously defined action on the polynomials.

We define $\mathrm{Pol}(\tau)$ the space of polynomials transforming according to the character τ, that is:

$$\mathrm{Pol}(\tau) = \{p \in \mathrm{Pol}(V) \,\big|\, t \cdot p = \tau(t)p\}$$

We define also $\mathrm{Pol}(U) = \mathrm{Pol}(V)|_U$, for any open $U \subset V$.

Lemma 4.12. *Let T be an ordinary compact torus acting on a finite dimensional complex vector superspace V. For any character τ of T, we assume that $\dim(\mathrm{Pol}(\tau)) < \infty$. Then any open connected subset U of V which is T-invariant and contains the origin, is such that $\mathrm{Pol}(U)$ is dense in $\mathcal{O}_V(U)$.*

Proof. We may assume that $V = \mathbb{C}^{m|n}$ with T-action

$$t, (z_1, ..., \xi_{m+n}) \longmapsto (f_1(t)z_1, ..., f_{m+n}(t)\xi_{m+n})$$

where the f_j are characters of T. Let U be an open connected subset of $\mathbb{C}^{m|n}$ containing the origin and stable under T. The action of T induces an action on $\mathcal{O}_V(U)$. It is enough to prove that the closure of $\mathrm{Pol}(\mathbb{C}^{m|n})$ contains $\mathcal{O}_V(U)^\infty$ the smooth vectors in $\mathcal{O}_V(U)$ with respect to the T action, since we know such space is dense in $\mathcal{O}_V(U)$. Since the Fourier series of any g in $\mathcal{O}_V(U)$ converges to g (see Prop. 3.11 (5) and (6)), it is enough to show that any eigenfunction of T in $\mathcal{O}_V(U)$ is a polynomial. Suppose $g \neq 0$ is in $\mathcal{O}_V(U)$ such that $t^{-1} \cdot g = f(t)g$ for all $t \in T$ and $u \in U$, f being a character of T. Since $0 \in U$ we can expand g as a power series $g(u) = \sum c_r u^r$ in a polydisk, where u comprehends even and odd coordinates and we are using the multiindex notation. Notice that the action of T preserves the polidisks, and we have

$$(t^{-1}.g)(u) = g(tu) = \sum_r c_r(tu)^r = \sum_r c_r f^r(t) u^r = f(t)g = \sum_r c_r f(t) u^r$$

Then $c_r f^r = c_r f$ whenever $c_r \neq 0$, because $t^{-1} \cdot g = f(t)g$. So only the r with $f = f^r$ appear in the expansion of g. We claim that there are only finitely many such r; once this claim is proven we are done, because g is a linear combination of the monomials u^r with $f^r = f$, hence g is a polynomial. To prove the claim, note that all such u^r are eigenfunctions for T for the eigencharacter f, and by assumption, there are only finitely many of these. ∎

We want to apply the previous lemma in a case that is of interest to us.

Define now $\Gamma_1 = G_r B^+/B^+$ and $\Gamma_2 = (\Gamma \cap G_r B^+)^0/B^+$ (the suffix "0" denotes the connected component of the identity). These are open sets in G/B^+ which are invariant under the A_r action. Let us denote with $\mathcal{P}^\sim(\Gamma_2)$, the set $(\mathcal{P}|_{\Gamma_2'})^\sim$ where $\Gamma_2' \cong \Gamma_2$ in the isomorphism of analytic supermanifolds $N^- \cong \Gamma/B^+$.

Corollary 4.13. *Let the notation be as above.*

(1) $\overline{\mathcal{P}} = \mathcal{O}(N^-)$, *where the nilpotent supergroup* N^- *is interpreted as a vector superspace via the identification* $\mathfrak{n}^- \cong N^-$ *via the exponential morphism.*
(2) $\overline{\mathcal{P}^\sim} = L^\chi(\Gamma)$, $\Gamma = N^-AN^+$ *the big cell in* G.
(3) $\overline{\mathcal{P}^\sim}(\Gamma_2) = L^\chi(\Gamma_2)$.

Proof. (1) We apply Lemma 4.12. The torus A_r acts on N^- through the action c given by (8). The condition $\dim \mathrm{Pol}(\tau) < \infty$ is checked with a calculation completely similar to that of Corollary 4.11. (2) is a consequence of the isomorphism (7). (3) follows again from Lemma 4.12. ■

Define now $F = L^\chi(\Gamma)$, $F^1 = L^\chi(\Gamma_1)$, $F^2 = L^\chi(\Gamma_2)$. Notice that on F and F^2 we do not have any G or G_r action, only F^1 is a G_r module in a natural way.

Corollary 4.14. *Let the notation be as above.*

(1) *The restriction morphism* $F^1 \longrightarrow F^2$ *is a continuous injection.*
(2) *Under the restriction,* $F^1(\tau) \subset F^2(\tau)$ *for characters* τ *of* A_r.
(3) $F^2(\tau) = F(\tau)|_{\Gamma_2}$.

Proof. (1) and (2) are clear, (1) because Γ_2 is open in Γ_1 and of the analytic continuation principle, which holds also in the supersetting, while (2) is a simple check. Now we go to (3). The space of polynomials $\mathrm{Pol}(\Gamma)$ on Γ is dense in F and by Corollary 4.11 we have $F^0 = \mathrm{Pol}(\Gamma)$. $\mathrm{Pol}(\Gamma_2) = \mathrm{Pol}(\Gamma)|_{\Gamma_2}$ is dense in F^2 by the previous corollary. Since $\mathrm{Pol}(\Gamma_2)$ is dense in F^2, we have that the restriction of $F(\tau)$ to Γ_2 is dense in $F^2(\tau)$ and since $F(\tau)$ is finite dimensional we have $F(\tau)|_{\Gamma_2} = F^2(\tau)$. ■

4.4. **The action of $\mathcal{U}(\mathfrak{g})$ on $L^\chi(\Gamma)$.** We start by defining the natural action of $\mathcal{U}(\mathfrak{g})$ on the holomorphic functions on any neighbourhood W of the identity of the supergroup G.

Definition 4.15. Let $W \subset G$ be an open neighbourhood of the identity 1_G in G. There are two well defined actions of $\mathfrak{g}$, hence of $\mathcal{U}(\mathfrak{g})$, on $\mathcal{O}(W)$ that read as follows:

$$\ell(X)f = (-X \otimes 1)\mu^*(f), \qquad X \in \mathfrak{g}$$
$$\partial(X)f = (1 \otimes X)\mu^*(f)$$

Proposition 4.16. *Let* U *be open in* $\widetilde{G/B^+}$. *Then* ℓ *and* ∂ *are well defined actions on* $\mathcal{O}(U)$ *and they commute with each other.*

Proof. Immediate. ■

The natural action of $\mathcal{U}(\mathfrak{g})$ on $L^\chi(N^-B^+)$ is algebraic, hence it preserves $\mathcal{P}^\sim$. The proof is analogous to the classical one and we leave the details to the reader.

Proposition 4.17. *The action* ℓ *of* $\mathcal{U}(\mathfrak{g})$ *on* $L^\chi(U)$, $p^{-1}(U) \subset \Gamma$ *leaves* $\mathcal{P}^\sim$ *invariant.*

We now want to establish a fundamental pairing between a certain Verma module and the space of polynomials inside $L^\chi(\Gamma)$. We start by recalling the notion of pairing between $\mathfrak{g}$-modules.

Definition 4.18. Let M_1, M_2 be two modules for $\mathfrak{g}$. By a $\mathfrak{g}$-*pairing* between them we mean a bilinear form $\langle \cdot\, , \cdot \rangle$ on $M_1 \times M_2$ with the property that:

$$\langle X m_1, m_2 \rangle = \langle m_1, -(-1)^{|m_1||X|} X m_2 \rangle, \qquad m_i \in M_i,\ X \in \mathfrak{g}.$$

Since the M_i are modules for $\mathcal{U}(\mathfrak{g})$ this implies that

$$\langle X_1 \dots X_r m_1, m_2\rangle = \langle m_1, (-1)^{r+|m_1|(|X_1|+\dots+|X_r|)+l_{odd}(w)} X_r \dots X_1 m_2\rangle$$

$$m_i \in M_i,\ X_j \in \mathfrak{g}.$$

where $l_{odd}(w)$ is the (minimum) number of odd transpositions appearing in the permutation $w : (1, \dots, r) \mapsto (r, \dots, 1)$. The map $X \mapsto -X$ of $\mathfrak{g}$ is an involutive anti-automorphism of $\mathfrak{g}$. It extends uniquely to an involutive anti-automorphism $u \mapsto u^T$ of $\mathcal{U}(\mathfrak{g})$. The $\mathfrak{g}$-pairing requirement is equivalent to

$$\langle um_1, m_2\rangle = \langle m_1, (-1)^{|u||m_1|} u^T m_2\rangle, \qquad m_i \in M_i,\ u \in \mathcal{U}(\mathfrak{g}).$$

We refer to this as a $\mathcal{U}(\mathfrak{g})$-pairing also. The pairing is said to be *non- singular* if $\langle m_1, m_2\rangle = 0$ for all m_2 (resp. for all m_1) implies that $m_1 = 0$ (resp. $m_2 = 0$).

Proposition 4.19. *Let u and v in $\mathcal{U}(\mathfrak{g})$, $f \in \mathcal{P}^{\sim}$.*

(1) $(\partial(u^T)f)(1_G) \ = \ (\ell(u)f)(1_G)$
(2) $\partial(u)\ell(v)(f)(1_G) = (-1)^{|u||v|}\ell(v)\partial(u)(f)(1_G)$

where 1_G denotes the identity element in G.

Proof. (1). It is enough to prove for $u = X$ and $v = Y$ both in $\mathfrak{g}$. We can rewrite our equality as:

$$(\epsilon \otimes 1)(1 \otimes -X)\mu^*(f) \ = \ (1 \otimes \epsilon)(-X \otimes 1)\mu^*(f)$$

where ϵ is the counit morphism: $\epsilon(f) = f(1_G) \quad \forall f \in \mathcal{P}^{\sim}$. We have:

$$(\epsilon \otimes 1)(1 \otimes X)\mu^*(f) \ = \ (1 \otimes X)(\epsilon \otimes 1)\mu^*(f) \ = \ X(f)$$

since $(\epsilon \otimes 1)\mu^*(f) = f$. On the other hand:

$$(1 \otimes \epsilon)(X \otimes 1)\mu^*(f) = (X \otimes 1)(1 \otimes \epsilon)\mu^*(f) = X(f).$$

(2). Again it is enough to prove for $u = X$ and $v = Y$ both in $\mathfrak{g}$.

$$\begin{aligned}\partial(X)\ell(Y)(f)(1_G) &= (\epsilon \otimes 1)(1 \otimes X)\mu^*(-Y \otimes 1)\mu^*(f) = \\ &= (-1)^{|X||Y|}(-Y \otimes X)\mu^*(f)(1_G).\end{aligned}$$

because $(\epsilon \otimes 1)\mu^*(f) = f$. Similarly

$$\begin{aligned}\ell(Y)\partial(X)(f)(1_G) &= (1 \otimes \epsilon)(-Y \otimes 1)\mu^*(1 \otimes X)\mu^*(f) = \\ &= (-Y \otimes X))\mu^*(f)(1_G).\end{aligned}$$

∎

Lemma 4.20. *Let $\lambda \in \mathfrak{h}^*$, and let*

$$\mathcal{M}_\lambda := \sum_{\alpha>0} \mathcal{U}(\mathfrak{g})\mathfrak{g}_\alpha + \sum_{H\in\mathfrak{h}} \mathcal{U}(\mathfrak{g})(H + \lambda(H))$$

then $\mathcal{M}_\lambda$ is a left ideal and

$$\mathcal{U}(\mathfrak{g}) = \mathcal{M}_\lambda \oplus \mathcal{U}(\mathfrak{n}^-)$$

Proof. For the ordinary setting this is Lemma 4.6.6 in [46]. As for the supersetting it is the same. ∎

Theorem 4.21. *There is a non-singular $\mathcal{U}(\mathfrak{g})$-pairing between $\mathcal{P}^\sim$ and the Verma module V_λ. Moreover every non-zero submodule of $\mathcal{P}^\sim$ contains the element $1^\sim$ corresponding to the constant function $1 \in \mathcal{P}$. In particular, the submodule $\mathcal{I}^\sim$ of $\mathcal{P}^\sim$ generated by $1^\sim$ is irreducible and is the unique irreducible submodule of $\mathcal{P}^\sim$. Finally, $\mathcal{I}^\sim$ is the unique irreducible module of lowest weight $-\lambda$.*

Proof. The proof is the same as for the ordinary setting, let us sketch it. We first define:

$$<,>: \mathcal{P}^\sim \times \mathcal{U}(\mathfrak{g}) \longrightarrow \mathbb{C}, \qquad < f, u >:= (-1)^{|u||f|}(\partial(u)f)(1_G)$$

In order for $<,>$ to be a $\mathfrak{g}$-pairing, we need to verify:

$$< \ell(c)f, u > = < f, (-1)^{|f||c|}c^T u >, \qquad c, u \in \mathcal{U}(\mathfrak{g}), f \in \mathcal{P}^\sim$$

where $(\cdot)^T$ denotes the antiautomorphism of $\mathcal{U}(\mathfrak{g})$ induced by $X \mapsto -X$ with $X \in \mathfrak{g}$. We have by the previous proposition:

$$\begin{aligned} < \ell(c)f, u > &= (-1)^{|u|(|c|+|f|)}(\partial(u)\ell(c)f)(1_G) = (-1)^{|u||f|}(\ell(c)\partial(u)f)(1_G) = \\ &= (-1)^{|u||f|}(\partial(c^T)\partial(u)f)(1_G) = \\ &= (-1)^{|u||f|}(\partial(c^T u)f)(1_G) =< f, (-1)^{|f||c|}c^T u > \end{aligned}$$

By Lemma 4.20, in order to prove that the bilinear map $<,>$ descends to a $\mathfrak{g}$-pairing between $\mathcal{P}^\sim$ and V_λ we need to prove that

$$u \in \mathcal{M}_\lambda \quad \Longleftrightarrow \quad (\partial(u)f)(1_G) = 0$$

For sufficiency, we notice that $< f, X_\alpha >= \partial(X_\alpha)(f)(1_G) = D^L_{X_\alpha}(f) = \lambda(-X_\alpha) = 0$ by (4.1). Again by (4.1), we have that $< f, H >= D^L_H(f) = -\lambda(H)f(1_G)$. For necessity, suppose that $(\partial(u)f)(1_G) = 0$ for each $f \in \mathcal{P}^\sim$. By Lemma 4.20 it is enough to notice that for each $X \in \mathcal{U}(\mathfrak{n}^-)$ there exists a polynomial $p \in \mathcal{P}$ such that $D^L_X p^\sim(1_G) \neq 0$. So we have obtained a nonsingular pairing

$$\mathcal{P}^\sim_\lambda \subset V^*_{-\lambda}$$

They are both weight spaces, for each weight the corresponding weight spaces having the same dimension (See Corollary 4.11), hence they are isomorphic.

More explicitly, the functions $(t^r)^\sim = (t^{r_1}_{\alpha_1} \dots t^{r_m}_{\alpha_m})^\sim$, corresponding to the coordinate polynomials t^r on N^-, are weight vectors for the action of $\mathfrak{h}$ for the weight $r - \lambda$. Hence $\mathcal{P}^\sim$ is a weight module with the multiplicities defined in Sec. 4.3. We shall prove that every non-zero ℓ-invariant subspace W of $\mathcal{P}^\sim$ contains the vector $1^\sim$ defined by the constant function 1 on N^-. Now W is a sum of weight spaces and if it does not contain $1^\sim$, then W is contained in the sum of all weight spaces corresponding to the weights $-\lambda + r$ where $r = (r_i)$ with some $r_i > 0$. Now $< \ell(H)m_1, m_2 >= - < m_1, \partial(H)m_2 >$, for all $H \in \mathfrak{h}$, $m_1 \in \mathcal{P}^\sim$, $m_2 \in V_\lambda$. This shows that the weight space of $\mathcal{P}^\sim$ for the weight θ is orthogonal to the weight space of V_λ for the weight ϕ unless $\theta = -\phi$. Let v be a non-zero vector of highest weight λ in V_λ. Since W is contained in the span of weights other than $-\lambda$, we have $< W, v >= 0$. Hence, for all $g \in \mathcal{U}(\mathfrak{g})$, $w \in W$ we have $< \ell(g)w, v >= 0$. So $< w, g^T v >= 0$ for all $g \in \mathcal{U}(\mathfrak{g})$. But v is cyclic for V_λ and so we have $< w, V_\lambda >= 0$ for all $w \in W$. This means that $W = 0$, contradicting the hypothesis that $W \neq 0$. Thus every non-zero submodule of $\mathcal{P}^\sim$ contains the submodule $\mathcal{I}^\sim$ generated by $1^\sim$. This submodule is then the unique irreducible submodule of $\mathcal{P}^\sim$. The weights of $\mathcal{I}^\sim$ are of the form $-\lambda + d$ where d is a positive integral linear combination of the simple roots, and $1^\sim$ has weight $-\lambda$. It is then clear that $1^\sim$ is the lowest weight of $\mathcal{I}^\sim$. This fact, together with its irreducibility, characterizes it uniquely.

■

4.5. **Harish-Chandra decomposition.** From now on we fix a positive admissible system P for $\mathfrak{g}$ (see Sec. 2.2).

Let P^+ be the analytic supergroup corresponding to the subsuperalgebra $\mathfrak{p}^+ = \sum_{\beta \in P} \mathfrak{g}_\beta$. Similarly define P^-. Let K be the (ordinary) analytic subgroup of G corresponding to the Lie superalgebra $\mathfrak{k} = \mathfrak{k}_0$.

Proposition 4.22. *The morphism $\phi : P^- \times K \times P^+ \longrightarrow G$, defined as $(p^-, k, p^+) \mapsto p^- k p^+$ in the functor of points notation, is a complex analytic isomorphism of $P^- \times K \times P^+$ onto an open set $\Omega \subset G$.*

Proposition 4.23. (1) *$G_r K^{\mathbb{R}} (P^+)^{\mathbb{R}}$ is an open subsupermanifold in $(P^- K P^+)^{\mathbb{R}}$.*

(2) *$G_r K P^+ = (\widetilde{G_r}\widetilde{K}\widetilde{P^+}, \mathcal{O}_{P^-KP^+}|_{\widetilde{G_r}\widetilde{K}\widetilde{P^+}})$ is a complex open subsupermanifold in $P^- K P^+$.*

(3) *$\widetilde{G_r} \cap \widetilde{K}\widetilde{P^+} = \widetilde{K_r}$.*

Proof. For the first statement, observe that $\mathfrak{g}_r \oplus \mathfrak{k} \oplus \mathfrak{p}^+ = \mathfrak{g}$, by Prop. 4.1, because $\mathfrak{k} \oplus \mathfrak{p}^+ \supset \mathfrak{b}$. Hence by Lemma 3.6 we have that $G_r K P^+$ is open in G and since $\widetilde{G_r}\widetilde{K}\widetilde{P^+} \subset \widetilde{P^-}\widetilde{K}\widetilde{P^+}$ (see [28] pg. 389), we have $G_r K P^+$ is open in $P^- K P^+$. The second statement is topological, so it is true because of the ordinary theory. ■

We now turn to the construction of the complex structure of G_r/K_r.

Proposition 4.24. *We have $G_r/K_r \cong (G_r K P^+ / K P^+)^{\mathbb{R}}$. Hence and G_r/K_r acquires a natural $\widetilde{G_r}$ invariant complex structure.*

Proof. This is an immediate consequence of Lemma 3.6 (2) and (3), together with Prop. 4.23. ■

4.6. **Harish-Chandra representations and their geometric realization.** In this section we give a global realization of the Harish-Chandra infinitesimal representations studied in Sec. 2.3.

Definition 4.25. Let the complex vector superspace V be a $\mathfrak{g}$-module via the representation π. We say that V is a $(\mathfrak{g}_r, K_r)$-*module* if there exists a representation π_{K_r} of K_r such that

(1) $\pi(\mathrm{Ad}(k)X) = \pi_{K_r}(k)\pi(X)\pi_{K_r}(k)^{-1}$
(2) $V = \sum_\tau V(\tau)$ where the sum is algebraic and direct and $V(\tau)$ is the span of all the linear finite dimensional subspaces corresponding to the irreducible representation associated with the K_r-character τ.

We say that V is a $(\mathfrak{g}_r, \mathfrak{k}_r)$-*module* if

$$V = \sum_{\theta \in \Theta} V(\theta)$$

where the sum is algebraic and direct, Θ denotes the set of equivalence classes of the finite dimensional irreducible representations of $\mathfrak{k}$ and $V(\theta)$ is the sum of all representation occurring in V lying in one of such classes $\theta \in \Theta$.

We say that the $(\mathfrak{g}_r, K_r)$-module V is a *Harish-Chandra module* (or HC-module for short) if each $V(\tau)$ is finite dimensional and V is finitely generated as $\mathcal{U}(\mathfrak{g})$ module. Similarly we can define also the notion of Harish-Chandra modules for $(\mathfrak{g}_r, \mathfrak{k}_r)$-modules.

We say that a vector is K_r-*finite* if it lies in a finite dimensional K_r stable subspace.

We now want to study the action of G_r on a superspace of sections of the line bundle L^χ over $\widetilde{G/B^+}$. Since $\widetilde{G_rB^+}$ is open in $\widetilde{G}$ (see Lemma 3.7), we can consider $L^\chi(G_rB^+)$ and since G_r acts on the left on G_rB^+ we have a well defined action of G_r on the Frechét superspace $L^\chi(G_rB^+/B^+)$:

$$\begin{cases}(g\cdot f) = l^*_{g^{-1}}f & g\in\widetilde{G_r}\\ X.f = D^R_{-X}f & X\in\mathfrak{g}\end{cases}$$

Next lemma is a simple generalization of Theorem 11, pg 312 in [48] and holds in a general setting.

Lemma 4.26. *Let the notation be as above. Let F be a Frechét representation of G_r on which G_r acts via $\pi = (\pi_0, \rho)$. If v is a weakly analytic vector for $\widetilde{G_r}$, then*

$$\overline{\mathcal{U}(\mathfrak{g})v} \subseteq F$$

is the smallest closed G_r-invariant subspace of F containing v.

Proof. For each $X \in \mathcal{U}(\mathfrak{g})$ and $\lambda \in F^*$ (the topological dual of F), define the function $f_{X,\lambda}\colon \widetilde{G_r} \to \mathbb{C}$:

$$f_{X,\lambda}(g) = \lambda(\pi_0(g)\rho(X)v)$$

Let λ be such that $\lambda = 0$ on $\overline{\mathcal{U}(\mathfrak{g})v}$. It is easily checked that the infinitesimal action of $\mathcal{U}(\mathfrak{g}_{0,r})$ preserves the analytic vectors hence we obtain

$$Zf_{X,\lambda}(1_G) = 0 \quad \text{ for each } Z \in \mathcal{U}(\mathfrak{g}_0)$$

Since $\widetilde{G_r}$ is connected, we conclude that $f_{X,\lambda} = 0$ on $\widetilde{G_r}$. Hence by the Hahn-Banach theorem we conclude that $\text{span}\{\pi_0(G)\rho(X)v\}$ is contained in $\overline{\mathcal{U}(\mathfrak{g})v}$. Since this is true for all $X \in \mathcal{U}(\mathfrak{g})$ we conclude that $\overline{\mathcal{U}(\mathfrak{g})v}$ is $\widetilde{G_r}$ invariant. Since

$$\rho(Y)\pi_0(g)\rho(X)v = \pi_0(g)\rho((g^{-1}Y)X)v$$

it is also clear that it is the smallest G_r-invariant subspace of F containing v. ∎

Let us introduce the notation:

$$\widetilde{L^\chi(U)} := \{\widetilde{f}\,|\, f \in L^\chi(U)\} \tag{4.2}$$

which is meaningful because L^χ is a subsheaf of the structural sheaf.

Theorem 4.27. *Let $S = G_rB^+/B^+$ and assume $\widetilde{F^1} := \widetilde{L^\chi(S)} \neq 0$. Then:*

1. *F^1 contains an element ψ which is an analytic continuation of $1^\sim$;*
2. *$F^{11} := \overline{\ell(\mathcal{U}(\mathfrak{g}))\psi} \subset L^\chi(S)$ is a Fréchet G_r-module, K_r-finite and with K_r-finite part $\ell(\mathcal{U}(\mathfrak{g}))\psi) = \mathcal{P}^\sim_\lambda$.*
3. *the K_r-finite part $\mathcal{P}^\sim_\lambda$ is isomorphic to $\pi_{-\lambda}$ the irreducible representation with lowest weight $-\lambda$. In particular $\lambda(H_\alpha) \in \mathbb{Z}_{\geq 0}$ for all compact positive roots α.*

Proof. We first establish the K_r-finitess of F^1. From (2) of Corollary 4.14 it follows that the subspace $F^1(\tau)$ injects (through the restriction morphism) in $F^2(\tau)$, where F^2 denotes $L^\chi(\Gamma_2)$ (we recall that $\Gamma_2 = (G_r B^+ \cap \Gamma)^0/B^+)$. From (3) of Corollary 4.14, we know that $\dim F^2(\tau) = \dim F(\tau)$. By (3) of Corollary 4.13 and (2) of Corollary 4.11, we finally obtain $\dim F(\tau) < +\infty$. Hence F^1 is A_r-finite. By Corollary 3.13 the A_r finiteness implies in our case the K_r-finiteness. Hence F^1 is K_r-finite

We now go to the proof of (1). Assume that the K_r-finite part $(F^1)^0 = \sum F^1(\tau)$ does not include the weight $-\lambda$, in other words we assume there is no analytic continuation of $1^\sim$ to S. By Corollary 4.13 and Corollary 4.14, $(F^1)^0$ is isomorphic to a subset of the set of polynomials in the t_α (see Sec. 4.2 for the notation). Since $1^\sim \in F^2(-\lambda) \supseteq F^1(-\lambda)$ we have that all the elements f in $(F^1)^0$ are zero when evaluated at 1_G. Hence, by the density of $(F^1)^0$ in F^1, all the elements in F^1 vanish at 1_G. Using the G_r action it follows that $\tilde{f} = 0$ for all $f \in F^1$.

As for (2), $F^{11} := \overline{\ell(\mathcal{U}(\mathfrak{g}))\psi}$ is a Frechét superspace, since it is a closed subspace of a Frechét superspace. The fact that F^{11} is a G_r-module follows from Lemma 4.26. Hence F^{11} is a G_r submodule of F^1, and it is K_r-finite since it is a submodule of the K_r finite module F^1. $\mathcal{J} = \ell(\mathcal{U}(\mathfrak{g}))\psi$ is clearly a highest weight $\mathcal{U}(\mathfrak{g})$ module. Since F^{11} is the closure of the K_r-finite subspace $\mathcal{J}$ subspace, its K_r-finite part is precisely $\mathcal{J}$.

(3). We know that $\mathcal{J} \subset (F^1)^0 \hookrightarrow (F^2)^0 \simeq F^0$. Clearly $\mathcal{J} \hookrightarrow \mathcal{I}^\sim := \mathcal{U}(\mathfrak{g})1^\sim \subset F^0$, but since by 4.21 $\mathcal{I}^\sim$ is irreducible, we have $\mathcal{J} = \mathcal{I}^\sim$. $\mathcal{J}$ is the irreducible lowest weight module of lowest weight $-\lambda$ or equivalently $\mathcal{J}$ is the irreducible highest weight module of highest weight $-\lambda$ with respect to the positive system $-P$. The K_r-finiteness of $\mathcal{J}$ implies that $-\lambda(H_{-\alpha}) \geq 0$, hence our result.

■

Corollary 4.28. *Let the notation be as above. Then $\mathcal{J} = \mathcal{U}(\mathfrak{g})\psi \subset (F^1)^0$ is the irreducible Harish-Chandra module with highest weight $-\lambda$ with respect to the positive system $-P$.*

Proof. This is an immediate consequence of Proposition 2.8. ■

Definition 4.29. We say that a dominant integral weight λ is K*-integrable* if the $\mathfrak{k}$ irreducible representation associated with λ can be lifted to K.

Definition 4.30. An holomorphic character χ_λ of A is K*-integrable* if λ is dominant integral for the positive compact roots and if the associated $\mathfrak{k}$ representation can be lifted to K. In this case, we also say that λ is K-integrable.

As in the classical case we have the following Lemma.

Lemma 4.31. *If λ is K-integrable then the associated representation of K is finite-dimensional and holomorphic.*

Proof. This fact is entirely classical and it is proved in [27]. ■

Theorem 4.32. *Let the notation be as above. Assume the following:*

- $\dim(\mathfrak{c}) \geq 1$.
- $\lambda \in \mathfrak{h}^*$ *is integral and* $\lambda(H_\alpha) \geq 0$ *for all* α *compact positive root.*
- λ *is* K*-integrable.*

Then $L^\chi(\widetilde{G_r B^+}) \neq 0$ and F^{11} is a G_r representation whose K_r-finite part is the lowest weight representation $\pi_{-\lambda}$.

Proof. It is enough to show that $L^\chi(\widetilde{G_r B^+}) \neq 0$, since (2) is an immediate consequence of Theorem 4.27. Let σ_λ be the finite dimensional irreducible representation of K with highest weight λ on the vector space V. Let v_λ be the corresponding highest weight vector. We can define the coefficient of the representation σ_λ corresponding to v_λ that is the nonzero section $a_{11} : K \longrightarrow \mathbb{C}$, $a_{11}(k) = (\sigma_\lambda(k)v_\lambda)_{v_\lambda}$, that is the v_λ component of $\sigma_\lambda(k)v_\lambda$ corresponding to the weight decomposition of V. Using Prop. 4.22, we can extend a_{11} to a nonzero section in $\mathcal{O}(P^-KP^+)$. Since G_r is embedded into $(P^-KP^+)^{\mathbb{R}}$ and $a_{11}(1_G) = 1$ we obtain a non zero section of G_r, that is $a_{11} \in \mathcal{O}(G_r)$. It is immediate to verify:

$$r_b^* a_{11} = \chi_0^\lambda(b)^{-1} a_{11}, \quad b \in \widetilde{B^+}, \qquad D_X^L a_{11} = -\lambda(X) a_{11}, \quad X \in \mathfrak{b}^+$$

so that $a_{11} \in L^\chi(S)$ as requested. ■

4.7. **The Siegel superspace.** To illustrate the theory developed so far, we want to give an example, interesting by itself, where the various geometrical tools developed so far (admissible systems, symmetric superspaces,...) come into play.

Consider the closed complex analytic subsupergroup P of the complex orthosymplectic supergroup $\mathrm{Osp}(m|2n)$ defined, via its functor of points, as:

$$P(T) = \left\{ \begin{pmatrix} a & 0 & \alpha_2 \\ b_{11}\alpha_2^t a & b_{11} & b_{12} \\ 0 & 0 & (b_{11}^t)^{-1} \end{pmatrix} \mid \begin{cases} a^t a = 1 \\ (b_{11}^{-1} b_{12}) - (b_{11}^{-1} b_{12})^t = \alpha_2^t \alpha_2 \end{cases} \right\}$$

where $a \in \mathrm{GL}(m)(T)$, $b_{11} \in \mathrm{GL}(n)(T)$, $b_{12} \in \mathrm{M}(n)(T)$, $\alpha_2 \in \mathrm{M}(m|0,0|n)(T)$, $T \in (\mathrm{smflds})_{\mathbb{C}}$, (GL, M denoting respectively the general linear (super)group and the (super)matrices).

As for any closed analytic subsupergroup of an analytic Lie supergroup it is possible to construct the quotient $\mathrm{Osp}(m|2n)/P$. This is a complex analytic supermanifold, that we call the *super Lagrangian* and denote it by $\mathcal{L}$. Notice that, by the very definition, the reduced manifold of $\mathcal{L}$ is $\widetilde{\mathcal{L}}$, the ordinary Lagrangian manifold in $\mathrm{Sp}(2n, \mathbb{C})$, and we have a natural transitive action of $\mathrm{Osp}(m|2n)$ on the supermanifold $\mathcal{L}$. We also define $\mathcal{L}_{\mathrm{f}}$ as the open subsupermanifold of $\mathcal{L}$ corresponding to the open subset $\widetilde{\mathcal{L}_{\mathrm{f}}}$ of $\widetilde{\mathcal{L}}$:

$$\begin{aligned} \widetilde{\mathcal{L}_{\mathrm{f}}} &= \left\{ \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \mid Z_1^t Z_2 \text{ symmetric, } \det(Z_2) \neq 0 \right\} \Big/ \mathrm{GL}(n)(\mathbb{C}) \\ &\cong \left\{ \begin{pmatrix} Z \\ 1 \end{pmatrix} \mid Z \,\text{symmetric} \right\} = \{X + iY \mid X, Y \in \mathrm{M}(2n, n)(\mathbb{R}), \text{ symmetric}\}. \end{aligned}$$

We now want to characterize the functor of points of $\mathcal{L}_{\mathrm{f}}$. We start observing that we can always choose *uniquely* a representative of the class

$$\textbf{(4.3)} \qquad \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} P(T) \in \mathcal{L}_{\mathrm{f}}, \quad \text{in the form} \quad \begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

This is equivalent, to to find z, ζ, u, v, w, ξ depending on α_i, β_i and a, b_{ij} in the following equation:

$$\begin{pmatrix} 1 & 0 & -\zeta \\ 0 & 0 & 1 \\ \zeta^t & -1 & z-\zeta^t\zeta \end{pmatrix} \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} u & 0 & \xi \\ v\xi^t u & v & w \\ 0 & 0 & (v^t)^{-1} \end{pmatrix}$$

Notice that there is no loss of generality in assuming $b_{21} = 1$. The check the solutions are unique and compatible with the conditions defining $\mathrm{Osp}(m|2n)$ is a direct calculation. The values obtained are:

$$u = a - \alpha_1\beta_2, \quad \xi = \alpha_2 - \alpha_1 b_{22}, \quad v = 1, \quad w = b_{22}, \quad z = b_{11}, \quad \zeta = \alpha_1$$

Proposition 4.33. *The T-points of the supermanifold $\mathcal{L}_{\mathrm{f}}$ are identified with the matrices in $Osp(m|n)(T)$ of the form:*

$$\mathcal{L}_{\mathrm{f}}(T) \simeq \left\{ \begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} \mid \zeta^t\zeta + z^t - z = 0 \right\}. \quad \textit{Hence} \quad \mathcal{L}_{\mathrm{f}} \cong \mathbb{C}^{\frac{n^2+n}{2}|mn}.$$

Proof. Let us choose a suitable open cover $\{T_i\}_{i\in I}$ of T, so that

$$\mathcal{L}(T_i) = (\mathrm{Osp}(m|2n)/P)(T_i) = \mathrm{Osp}(m|2n)(T_i)/P(T_i).$$

We can then write

$$\mathcal{L}_{\mathrm{f}}(T_i) = \left\{ \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} P(T_i) \,|\, b_{21} \,\text{invertible} \right\}$$

By (4.3), we can write:

$$\textbf{(4.4)} \qquad \begin{pmatrix} a & \alpha_1 & \alpha_2 \\ \beta_1 & b_{11} & b_{12} \\ \beta_2 & b_{21} & b_{22} \end{pmatrix} P(T_i) = \begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} P(T_i), \qquad \zeta^t\zeta + z^t - z = 0$$

$\mathcal{L}_{\mathrm{f}}$ is then defined by $n(n-1)/2$ equations in $\mathbb{C}^{n^2|mn}$:

$$\sum_k \zeta_{ki}\zeta_{kj} + z_{ji} - z_{ij} = 0, \qquad 1 \le i < j \le n$$

∎

Definition 4.34. We define *Siegel superspace* the open supermanifold of $\mathcal{L}_{\mathrm{f}}$ corresponding to the complex open subset:

$$\widetilde{\mathcal{S}} = \{Z \,|\, Z = X + iY,\, X, Y \in \mathrm{M}(2n,n)(\mathbb{R})\,\text{symmetric},\, Y > 0\} \subset \widetilde{\mathcal{L}_{\mathrm{f}}}$$

By the Chart Theorem (see Ch. 4 in [8]) we have that a T-point of the Siegel superspace $\widetilde{\mathcal{S}}$ corresponds to a choice of two matrices ζ and z with entries in $\mathcal{O}(T)$ such that their values at all topological points of $\widetilde{T}$ land in $\widetilde{\mathcal{S}}$. In other words, $\mathcal{S}(T)$ consists of the following elements in $\mathcal{L}_{\mathrm{f}}(T)$:

$$\mathcal{S}(T) = \left\{ \begin{pmatrix} 1 & \zeta & 0 \\ \zeta^t & z & -1 \\ 0 & 1 & 0 \end{pmatrix} \mid \begin{cases} \zeta^t\zeta + z^t - z = 0 \\ z = x + iy,\, \widetilde{y}(t) > 0,\, \forall t \in \widetilde{T} \end{cases} \right\} \subset \mathcal{L}_{\mathrm{f}}(T)$$

We now want to realize the Siegel superspace as a real homogeneous supermanifold (for our notation see [10]).

Consider the natural action of the real orthosymplectic supergroup $\mathrm{Osp}(m|2n,\mathbb{R})$ on the quotient $\mathrm{Osp}(m|2n)/P$ and restrict it to $\mathcal{S}$:

$$\mathrm{Osp}(m|2n,\mathbb{R})(T)\times\mathcal{S}(T)\longrightarrow\mathcal{S}(T)$$

$$\begin{pmatrix} a & \alpha_1 & \alpha_2\\ \beta_1 & b_{11} & b_{12}\\ \beta_2 & b_{21} & b_{22}\end{pmatrix},\ \begin{pmatrix} 1 & \zeta & 0\\ -\zeta^t & z & -1\\ 0 & 1 & 0\end{pmatrix}\longmapsto\begin{pmatrix} 1 & \frac{a\zeta+\alpha_1 z+\alpha_2}{\beta_2\zeta+b_{21}z+b_{22}} & 0\\ \left(\frac{a\zeta+\alpha_1 z+\alpha_2}{\beta_2\zeta+b_{21}z+b_{22}}\right)^t & \frac{\beta_1\zeta+b_{11}z+b_{12}}{\beta_2\zeta+b_{21}z+b_{22}} & -1\\ 0 & 1 & 0\end{pmatrix}$$

Theorem 4.35. *$Osp(m|2n,\mathbb{R})$ acts transitively on the Siegel superspace and the stabilizer of the topological point $(iI,0)\in\widetilde{\mathcal{S}}$ is the subgroup:*

$$K_{\mathbf{r}}(T)=Stab(iI,0)(T)=\left\{\begin{pmatrix} a & 0 & 0\\ 0 & b_{11} & b_{12}\\ 0 & -b_{12} & b_{11}\end{pmatrix}\in Osp(m|2n,\mathbb{R})(T)\right\},\qquad T\in(\mathrm{smflds})_{\mathbb{R}}$$

which is compact and coincides with its reduced group: $(K_{\mathbf{r}})_{red}=K_{\mathbf{r}}$ and it is equal to $\mathrm{O}(m)\times\mathrm{U}(n)$. *So we have the isomorphism as real supermanifolds:*

$$\mathcal{S}\cong Osp(m|2n,\mathbb{R})/K_{\mathbf{r}}.$$

Proof. The action of $\widetilde{\mathrm{Osp}(m|2n,\mathbb{R})}$ on $\widetilde{\mathcal{S}}$ is transitive. Consider the supermanifold morphism $a_p:\mathrm{Osp}(m|2n,\mathbb{R})\longrightarrow\mathcal{S}$, $a_p(g)=g\cdot(0,iI)$. The differential $(da_p)_I$ at the identity is surjective, hence the result follows (see also Prop. 9.1.4 in [8]). ∎

The form we have found for $K_{\mathbf{r}}$ is not suitable for Lie superalgebra calculations, so we need to transform $\mathcal{S}$, so that also $K_{\mathbf{r}}$ transforms accordingly. We shall do this via the *super Cayley transform*.

Consider the following linear transformation:

$$L=\begin{pmatrix} 1 & 0 & 0\\ 0 & i/\sqrt{2i} & i/\sqrt{2i}\\ 0 & -1/\sqrt{2i} & 1/\sqrt{2i}\end{pmatrix}\in\widetilde{\mathrm{Osp}(m|2n)}$$

where we write 1 in place of the identity matrix.

Define the open subsupermanifold $\mathcal{D}=(\widetilde{\mathcal{D}},\mathcal{O}_{\mathcal{L}_{\mathrm{f}}|_{\widetilde{\mathcal{D}}}})$ of $\mathcal{L}_{\mathrm{f}}$ with topological space:

$$\widetilde{\mathcal{D}}=\left\{\begin{pmatrix}0\\ z\\ 1\end{pmatrix}\mid z\in\mathrm{M}(n,n)(\mathbb{C})\text{ symmetric, } 1-z\overline{z}>0\right\}$$

Proposition 4.36. *The linear transformation L induces a supermanifold diffeomorphism:*

$$\phi_T:\mathcal{D}(T)\longrightarrow\mathcal{S}(T)$$

$$\begin{pmatrix}\eta\\ z\\ 1\end{pmatrix}\longmapsto\begin{pmatrix}\sqrt{2i}\eta(1-z)^{-1}\\ i(z+1)(1-z)^{-1}\\ 1\end{pmatrix}$$

Proof. Let us take a generic element in $\mathcal{L}_\mathrm{f}(T)$ and multiply it by L:

$$\begin{pmatrix}1 & 0 & 0\\ 0 & i/\sqrt{2i} & i/\sqrt{2i}\\ 0 & -1/\sqrt{2i} & 1/\sqrt{2i}\end{pmatrix}\begin{pmatrix}\eta\\ z\\ 1\end{pmatrix} = \begin{pmatrix}\eta\\ \frac{i}{\sqrt{2i}}(z+1)\\ \frac{1}{\sqrt{2i}}(1-z)\end{pmatrix} \sim \begin{pmatrix}\sqrt{2i}\eta(1-z)^{-1}\\ i(z+1)(1-z)^{-1}\\ 1\end{pmatrix}$$

The map ϕ can be extended on the lagrangian $\mathcal{L}_\mathrm{f}$ (except at the locus $z=1$) and it is differentiable on $\widetilde{\mathcal{L}_\mathrm{f}}\setminus\{z=1\}$. Since $\widetilde{\phi}$ is an homeomorphism when restricted to $\widetilde{\mathcal{D}}$ and the differential $\mathrm{d}\phi$ is surjective, the result follows (see [8]). ■

We call the diffeomorphism ϕ the *super Cayley transform*. Define:

$$\mathrm{Osp}_\mathcal{D}(m|2n)(T) := L^{-1}\mathrm{Osp}(m|2n,\mathbb{R})(T)L, \qquad K_\mathcal{D}(T) := L^{-1}K_\mathbf{r}(T)L$$

Proposition 4.37. *$Osp_\mathcal{D}(m|2n)$ is a real form of the orthosymplectic supergroup and its functor of points is explicitly given by:*

$$Osp_\mathcal{D}(m|2n)(T) = \left\{\begin{pmatrix}a_0 & \alpha_1 & -i\overline{\alpha}_1\\ \beta_1 & b_{11} & b_{12}\\ i\overline{\beta}_1 & \overline{b_{12}} & \overline{b_{11}}\end{pmatrix}\right\} \subset Osp(m|2n)^\mathbb{R}(T)$$

where $a_0 \in O(m)$, T is a real supermanifold and, as usual, $Osp(m|2n)^\mathbb{R}$ is the complex orthosymplectic supergroup viewed as a real supergroup.

$K_\mathcal{D}$ is a real form of the compact group $K_\mathbf{r}$ and it is given by

$$K_\mathcal{D} = \left\{\begin{pmatrix}a_0 & 0 & 0\\ 0 & b_{11} & 0\\ 0 & 0 & \overline{b}_{11}\end{pmatrix} \,\middle|\, b_{11}\overline{b}_{11}^t = 1,\, a_0 \in O(m)\right\}$$

Proof. The conjugation defining $\mathrm{Osp}_\mathcal{D}(m|2n)$ inside $\mathrm{Osp}(m|2n)$ is:

$$\begin{pmatrix}a & \alpha_1 & \alpha_2\\ \beta_1 & b_{11} & b_{12}\\ \beta_2 & b_{21} & b_{22}\end{pmatrix} \longmapsto \begin{pmatrix}\bar{a} & -i\overline{\alpha}_2 & -i\overline{\alpha}_1\\ i\overline{\beta}_2 & \overline{b_{22}} & \overline{b_{21}}\\ i\overline{\beta}_1 & \overline{b_{12}} & \overline{b_{11}}\end{pmatrix}$$

The statement about $K_\mathcal{D}$ is entirely classical and known. ■

Proposition 4.38. *$Osp_\mathcal{D}(m|2n)$ acts transitively on $\mathcal{D}$ and $K_\mathcal{D}$ is the stabilizer of the topological point $(1,0)$. Hence*

$$\mathcal{D} \cong Osp_\mathcal{D}(m|2n)/K_\mathcal{D}.$$

Proof. It the same as for 4.35. ■

We now compute the real Lie superalgebras of $\mathrm{Osp}_\mathcal{D}(m|2n)$ and $K_\mathcal{D}$.

Proposition 4.39. *We have that*

$$\mathrm{osp}_\mathcal{D}(m|2n) = \left\{\begin{pmatrix}x & \xi & -i\overline{\xi}\\ -i\overline{\xi^t} & y_{11} & y_{12}\\ -\xi^t & \overline{y_{12}} & \overline{y_{11}}\end{pmatrix} \mid \begin{cases} y_{12} \text{ symmetric,}\\ x = -\overline{x}^t,\ y_{11} = -\overline{y_{11}^t}\end{cases}\right\}$$

$$\mathfrak{k}_\mathcal{D} = \mathrm{Lie}(K_\mathcal{D}) = \left\{\begin{pmatrix}x & 0 & 0\\ 0 & y_{11} & 0\\ 0 & 0 & \overline{y_{11}}\end{pmatrix}\right\}$$

Proof. The conjugation defining $\mathrm{osp}_{\mathcal{D}}(m|2n)$ is obtained as follows:

$X \in \mathrm{osp}_{\mathcal{D}}(m|2n)$ if and only if $X \in \mathrm{osp}(m|2n)$ and $F\overline{X} = XF$ where $F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}$. An easy calculation shows the result. ■

For the complex Lie superalgebra (see Sec. 2.2) $\mathrm{osp}(m|2n)$ we have the admissible system $P = P_k \cup P_{n,0} \cup P_{n,1}$, where:

$$P_{n,0} = \{\epsilon_1 \pm \epsilon_j \mid 1 < j \leq m\} \cup \{\epsilon_1\} \cup \{\delta_i + \delta_j \mid 1 \leq i, j \leq n\}$$

$$P_{n,1} = \{\delta_i \pm \epsilon_j \mid 1 \leq i \leq n \, , \, 1 \leq j \leq m\} \cup \{\delta_i \mid 1 \leq i \leq n\}$$

$$P_k = \{\epsilon_i \pm \epsilon_j \mid 1 < i < j \leq m\} \cup \{\epsilon_i \mid 1 < i \leq m\} \cup \{\delta_i - \delta_j \mid 1 \leq i < j \leq n\}$$

with

$$\Pi = \{\epsilon_1 - \epsilon_2 \, , \dots , \, \epsilon_{m-1} - \epsilon_m \, , \, \epsilon_m \, , \, \delta_1 - \delta_2 \, , \dots , \, \delta_{n-1} - \delta_n \, , \, \delta_n - \epsilon_1\}$$

the simple system with one simple non-compact even root $\epsilon_1 - \epsilon_2$ and one non-compact simple odd root: $\delta_n - \epsilon_1$.

Proposition 4.40. *The complex Lie superalgebra* $\mathrm{osp}(m|2n)$ *is the vector space direct sum of three Lie subsuperalgebras:*

$$\mathrm{osp}(m|2n) = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-,$$

$$\mathfrak{k} = \sum_{\alpha \in P_k \cup -P_k} \mathfrak{g}_\alpha, \quad \mathfrak{p}^+ = \sum_{\alpha \in P_n} \mathfrak{g}_\alpha, \quad \mathfrak{p}^- = \sum_{\alpha \in -P_n} \mathfrak{g}_\alpha.$$

where $\mathfrak{k} = \mathbb{C} \otimes \mathrm{Lie}(K_{\mathcal{D}})$ *and*

$$\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & 0 & \xi \\ \xi^t & 0 & u \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & -\eta & 0 \\ 0 & 0 & 0 \\ \eta^t & v & 0 \end{pmatrix} \right\}.$$

We can now express explicitly the Harish-Chandra decomposition for $\mathrm{Osp}(m|2n)$, proven in Prop. 4.22.

Let P^- and P^+ be the complex subsupergroups of the complex orthosymplectic supergroup $\mathrm{Osp}(m|2n)$ defined via their functor of points as:

$$P^+ = \left\{ \begin{pmatrix} 1 & 0 & \xi \\ \xi^t & 1 & u \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad P^- = \left\{ \begin{pmatrix} 1 & -\eta & 0 \\ 0 & 1 & 0 \\ \eta^t & v & 1 \end{pmatrix} \right\}$$

Most immediately $\mathfrak{p}^\pm = \mathrm{Lie}(P^\pm)$. Notice that while in the ordinary setting we have that the groups $\widetilde{P}^\pm$ are abelian, in the supersetting, this is no longer true.

By Prop. 4.22 we have that the supermanifold $P^- K P^+$ is open in $\mathrm{Osp}(m|2n)$.

By its very construction $\mathcal{D}$ is a complex supermanifold and it has a natural action of $\mathrm{Osp}_{\mathcal{D}}(m|2n)$. Notice that

$$J \circ \mathrm{ad}(X)|_{\mathfrak{p}_{\mathcal{D}}} = \mathrm{ad}(X)|_{\mathfrak{p}_{\mathcal{D}}} \circ J$$

with J the almost complex structure at the identity coset, $J : \mathfrak{p}_{\mathcal{D}} \longrightarrow \mathfrak{p}_{\mathcal{D}}$, where we identify $\mathfrak{p}_{\mathcal{D}} = T_{K_{\mathcal{D}}}(\mathrm{Osp}_{\mathcal{D}}(m|2n)/K_{\mathcal{D}})$.

Proposition 4.41. *Let* $J = \mathrm{ad}(c)|_{\mathrm{osp}_{\mathcal{D}}(m|n)_0} + \mathrm{ad}(2c)|_{\mathrm{osp}_{\mathcal{D}}(m|n)_1}$, *where* c *is the element in the center of* $\mathfrak{k}_{\mathcal{D}}$*:*

$$c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix}$$

Then $J|_{\mathfrak{p}_{\mathcal{D}}}$ *defines a complex structure on* $\mathfrak{p}_{\mathcal{D}}$*, which corresponds to the* $\widetilde{Osp_{\mathcal{D}}(m|2n)}$ *invariant complex structure on* $Osp_{\mathcal{D}}(m|2n)/K_{\mathcal{D}}$*, as in Prop. 4.24.*

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