Invariant multicones for families of matrices

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Abstract

In this paper, we investigate sufficient conditions on the structure of the eigenspaces of a given finite family of matrices to assure the existence of an embedded pair of invariant *multicones*, which are the smallest and the biggest in a suitable and natural sense. Multicones, very similar structures to those known in the literature as 1-*multicones*, are quite natural generalizations of the classical cones. The conditions we find also suggest us a practical computational procedure for the actual construction of such invariant embedded pair.

Keywords Cone · Multicone · Duality · Matrix · Family of matrices · Invariant set · Leading eigenvalue · Leading eigenvector · Computational algorithm

Mathematics Subject Classification 15A18 · 15A48 · 52A30 · 52B55

1 Introduction

In the framework of the so-called *Perron–Frobenius theory* (see Perron [15,16] and Frobenius [9]), the study of endomorphisms admitting an *invariant cone* finds various generalizations by several authors in the direction of *families of matrices* $\mathcal{F}$ sharing a common invariant cone. Even starting from a finite family, infinite sets of matrices immediately occur since one has to deal with the (possibly infinite) semi-group $\Sigma(\mathcal{F})$ generated by the initial family.

It turned out soon that the existence of a common invariant cone for a family $\mathcal{F}$ simplifies the study of $\Sigma(\mathcal{F})$ and, in particular, of its *spectral characteristics*, such as the *joint spectral radius*, defined by Rota and Strang [20] about 60 years ago, and the *lower spectral radius*, also called the *joint spectral subradius*, defined by Gurvitz [12] about 20 years ago.

Such spectral characteristics of a family of matrices play an important role for the solution of many applicative problems (see, e.g. the survey by Jungers [14]). Therefore, in the last decades their study has been deepened, both from the theoretical and from the computational point of view. In particular, recently some papers have been devoted to analyse the properties

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of the lower spectral radius. We mention, for example, Guglielmi and Protasov [10] and Guglielmi and Zennaro [11]. Moreover, Bochi and Morris [3] have analysed the continuity properties of the lower spectral radius assuming the existence of invariant sets for $\mathcal{F}$ which are suitable generalizations of a cone, there called *k-multicones* (see also Avila et al. [1] and Bochi and Gourmelon [2]). Substantially, they are homogeneous $\mathcal{F}$-invariant subsets containing a k-dimensional subspace of $\mathbb{R}^d$ and no one of bigger dimension.

Since classical cones turn out to be connected components of 1-multicones, in [6] we have considered and studied similar structures using simply the term *multicone*, regardless of any requirement of invariance with respect to a matrix or a family of matrices. Roughly speaking, a multicone is a homogeneous set, symmetric with respect to the origin, consisting in the union of a finite number of cones (its *components*). Moreover, by using extensively a suitable notion of duality, we have extended the spectral analysis of a matrix having an invariant cone to one having an invariant multicone.

In the present paper we enlarge the study contained in [6] to families of matrices sharing an invariant multicone. Furthermore, we provide a constructive procedure for the actual detection of an *embedded pair* of invariant multicones under suitable assumptions on the family $\mathcal{F}$.

Our main motivation for detecting an embedded pair of invariant multicones is given by the fact that its existence should allow us to generalize the results given in [10,11] about the computation of the lower spectral radius to a significantly larger class of matrix families. Indeed, the theoretical and practical implementation of this idea will be the subject of a future paper.

In Sect. 2, we resume the main results of [6] which are used in the present paper, making it self-contained. In particular, we stress the strictly invariant case, which selects the class of matrices with only one simple *leading eigenvalue* (here called *asymptotically rank-one*).

In Sect. 3, we consider families $\mathcal{F}$ of matrices sharing an invariant multicone and, mostly, the asymptotically rank-one case, focussing on the properties of the *leading set* $\mathcal{L}(\mathcal{F})$, which gathers all the *leading eigenvectors*. Such families satisfy the so-called Leading set assumptions: substantially, each element of $\Sigma(\mathcal{F})$ is an asymptotically rank-one matrix and $\mathcal{L}(\mathcal{F})$ is disjoint from $\mathcal{H}(\mathcal{F})$, the set of the *secondary hyperplanes* of $\mathcal{F}$. This hypothesis, always assumed in the sequel, allows us to show, in Sect. 4, that the *fragmentation* of the *leading set* $\mathcal{L}(\mathcal{F})$, in the sense of being split into various subsets by the secondary hyperplanes, is finite.

In Sect. 5, we show that $\mathcal{L}(\mathcal{F})$ canonically generates the smallest multicone (in a suitable sense) which is invariant for a family $\mathcal{F}$, called the *leading multicone* and denoted by $K^{\mathcal{F}}_{mul}$. It also turns out to be the smallest invariant multicone having the minimum possible number of components.

In Sect. 6, we show that the leading multicone $K^{\mathcal{F}}_{mul}$ is embedded in another invariant multicone, called the *secondary multicone* and denoted by $\bar{K}^{\mathcal{F}}_{mul}$, which is the biggest invariant multicone for $\mathcal{F}$ with the same number of components.

In Sect. 7, we propose and theoretically justify a computational procedure directed to compute the smallest and the biggest invariant multicones $K^{\mathcal{F}}_{mul}$ and $\bar{K}^{\mathcal{F}}_{mul}$ for a family $\mathcal{F}$ of matrices satisfying the “Leading set assumptions” and, at the same time, the corresponding smallest and biggest invariant multicones for the transpose family $\mathcal{F}^{\mathrm{T}}$.

Eventually, in Sect. 8 we mention some open problems.

2 Preliminaries

We begin this section by recalling some known notions and results, using the same approach and terminology which have been adopted by Brundu and Zennaro [5,6].

We refer to $\mathbb{R}^d$ as a real vector space endowed with the Euclidean product, denoted by $x^\mathrm{T} y$ for any $x, y \in \mathbb{R}^d$, and with the induced norm $\|x\|^2 = x^\mathrm{T} x$.

The metric $d(x, y) = \|x - y\|$ and the topological structure of this Euclidean space are induced in this way.

In this framework, if U is a nonempty subset of $\mathbb{R}^d$, we denote by $\mathrm{cl}(U)$ its *closure*, by $\mathrm{conv}(U)$ the *convex hull* of U, by $\mathrm{int}(U)$ its *interior* and by ∂U its *boundary* as a subset of $\mathbb{R}^d$. We also set $\mathbb{R}_+ U := \left\{\alpha x \;\middle|\; \alpha \geq 0 \text{ and } x \in U\right\}$.

Moreover, any hyperplane H of $\mathbb{R}^d$ will be also denoted by $H = \{h\}^\perp$, where $U^\perp$ stands for the orthogonal set to $U \subseteq \mathbb{R}^d$ and h is a suitable nonzero vector.

Clearly, H splits $\mathbb{R}^d$ into two parts, say the *positive* and the *negative* semi-spaces, accordingly to the versus of the chosen h, i.e. $S_+^h := \{x \in \mathbb{R}^d \mid h^\mathrm{T} x \geq 0\}$ and $S_-^h := \{x \in \mathbb{R}^d \mid h^\mathrm{T} x \leq 0\}$.

Definition 1 Let K be a nonempty closed and convex set of $\mathbb{R}^d$ and consider the following conditions:

(c1) $\mathbb{R}_+ K \subseteq K$ (i.e. K is *positively homogeneous*);
(c2) $K \cap -K = \{0\}$ (i.e. K is *pointed* or *salient*);
(c3) $\mathrm{span}(K) = \mathbb{R}^d$ (i.e. K is *full* or *solid*).

We say that K is a *quasi-cone* if it verifies (c1). If, in addition, it verifies (c2), we say that K is a *cone*. Finally, if it satisfies all the above properties, we say that K is a *proper cone*.

If a cone K is not proper, we also say that it is a *degenerate cone*.

Note that a quasi-cone K is solid if and only if $\mathrm{int}(K) \neq \emptyset$.

If U is a nonempty subset of $\mathbb{R}^d$, we denote by $\mathrm{qcone}(U)$ the *quasi-cone generated* by U (i.e. the minimum quasi-cone containing U), which turns out to be

$$\mathrm{qcone}(U) = \mathrm{cl}\left(\mathrm{conv}\left(\mathbb{R}_+ U\right)\right) = \mathrm{cl}\left(\mathbb{R}_+ \mathrm{conv}(U)\right).$$

Whereas $\mathrm{qcone}(U)$ is defined for any set U, the smallest cone containing U may well not exist. Anyway, if it does exist, then it coincides with $\mathrm{qcone}(U)$ and is denoted by $\mathrm{cone}(U)$ and called the *cone generated by* U. Moreover, in this case it also holds that

$$\mathrm{cone}(U) = \mathrm{conv}\left(\mathrm{cl}\left(\mathbb{R}_+ U\right)\right) = \mathrm{cl}\left(\mathrm{conv}\left(\mathbb{R}_+ U\right)\right) = \mathrm{cl}\left(\mathbb{R}_+ \mathrm{conv}(U)\right), \tag{1}$$

i.e. the operators $\mathrm{cl}(\cdot)$ and $\mathrm{conv}(\cdot)$ can be interchanged.

Now let us recall the notion of *duality* and some of its properties.

Definition 2 If U is a nonempty set of $\mathbb{R}^d$, then

$$U^* := \left\{y \in \mathbb{R}^d \;\middle|\; y^\mathrm{T} x \geq 0 \quad \forall x \in U\right\}$$

is called the *dual set* of U. By convention, we also define $\emptyset^* := \mathbb{R}^d$.

Remark 1 Note that $\{0\}^* = \mathbb{R}^d$, $(\mathbb{R}^d)^* = \{0\}$ and, if $x \in \mathbb{R}^d \backslash \{0\}$, then $\{x\}^* = S_+^x = \{y \in \mathbb{R}^d \mid y^\mathrm{T} x \geq 0\}$ is the positive semi-space determined by x. Consequently, if U is a nonempty subset of $\mathbb{R}^d$, then

$$U^* = \bigcap_{x \in U} S_+^x .$$

Hence, U^* is closed, convex and positively homogeneous, i.e. U^* is a quasi-cone.

Proposition 1 *A quasi-cone K is a cone if and only if K^* is solid and, dually, K^* is a cone if and only if K is solid. In particular, K is a proper cone if and only if K^* is a proper cone.*

Besides the basic properties of the "geometric duality", the dual of a proper cone also fulfils the following relations.

Proposition 2 *Let K be a proper cone of $\mathbb{R}^d$. Then*

$$\mathrm{int}(K^*) = \left\{ y \in \mathbb{R}^d \;\middle|\; y^{\mathrm{T}} x > 0 \quad \forall x \in K \backslash \{0\} \right\} \tag{2}$$

and

$$K^* \backslash \{0\} = \left\{ y \in \mathbb{R}^d \;\middle|\; y^{\mathrm{T}} x > 0 \quad \forall x \in \mathrm{int}(K) \right\}.$$

Definition 3 Two positively homogeneous subsets U and V of $\mathbb{R}^d$ are said to be *(strictly) separated* if there exists a hyperplane $H = \{h\}^{\perp}$ such that

$$U \backslash \{0\} \subseteq \mathrm{int}\left(S_+^h\right) \quad \text{and} \quad V \backslash \{0\} \subseteq \mathrm{int}\left(S_-^h\right).$$

We say that H is a *separating hyperplane* for U and V.

If we only require

$$U \subseteq S_+^h \quad \text{and} \quad V \subseteq S_-^h,$$

we say that U and V are *weakly separated*.

Finally, we report a known "separation-type" theorem (see, e.g. Holmes [13] and Rockafellar [18]).

Theorem 1 *Any two cones $K^{(1)}$ and $K^{(2)}$ of $\mathbb{R}^d$ are (strictly) separated if and only if $K^{(1)} \cap K^{(2)} = \{0\}$.*

2.1 Multicones and their properties

Following Brundu and Zennaro [6], now we present the key definition of *multicone* passing through that of *symmetric cone*. We also review its main properties and report the most important related results.

Definition 4 Any subset of $\mathbb{R}^d$ of the form $K_{sym} = K \cup -K$, where K is a cone, is called *symmetric cone* of $\mathbb{R}^d$. We also conventionally say that K and $-K$ are the *positive* and the *negative* part of K_{sym} and denote them by K_+ and K_-, respectively.

Moreover, if K is proper, then K_{sym} is said to be *proper*, too.

Clearly, there exists a hyperplane H such that $H \cap K_{sym} = \{0\}$.

Definition 5 Consider a finite collection of symmetric cones $K_{sym}^{(1)}, \ldots, K_{sym}^{(r)}$ such that

(m1)

$$K_{sym}^{(i)} \cap K_{sym}^{(j)} = \{0\} \qquad \text{whenever } i \neq j; \tag{3}$$

(m2) there exists a hyperplane H of $\mathbb{R}^d$ such that

$$H \cap K_{sym}^{(i)} = \{0\} \quad \text{for all } i = 1, \dots, r.$$

Then the set

$$K_{mul} := \bigcup_{i=1}^{r} K_{sym}^{(i)}$$

is called a *multicone* of $\mathbb{R}^d$, the $K_{sym}^{(i)}$'s are its *symmetric components* and the $K_{\pm}^{(i)}$'s are its *(conic) components* (also denoted simply by $K^{(i)}$). The number r of symmetric components is called the *fragmentation index* of K_{mul}. Finally, H in (m2) is called a *splitting hyperplane* for K_{mul}.

Note that, as a particular case, a symmetric cone is a multicone with fragmentation index $r = 1$.

Given a multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$, we can choose a particular splitting hyperplane $H = \{h\}^{\perp}$ and label as positive the conic components contained in the positive semi-space S_{+}^{h}, i.e. we set

$$K_{+}^{(1)} \cup \cdots \cup K_{+}^{(r)} \backslash \{0\} \subseteq \operatorname{int}\left(S_{+}^{h}\right) \quad \text{and} \quad K_{-}^{(1)} \cup \cdots \cup K_{-}^{(r)} \backslash \{0\} \subseteq \operatorname{int}\left(S_{-}^{h}\right). \tag{4}$$

The above splitting hyperplane H, chosen once for ever, is called the *labelling hyperplane* of K_{mul}.

From Theorem 1, it turns out that, for any pair of conic components $K_{+}^{(i)}$ and $K_{+}^{(j)}$, condition (m1) of Definition 5 implies that there exists a separating hyperplane, say H_{ij}. But, in general, H_{ij} does not necessarily split K_{mul}.

Definition 6 If for all pairs of conic components $K_{+}^{(i)}$ and $K_{+}^{(j)}$ there exists a separating hyperplane H_{ij} which is also splitting, then K_{mul} is said to be *reduced*. Otherwise, it is said to be *nonreduced*.

The notion of reduced multicone is independent of the choice of the labelling hyperplane. In [6] we have proposed a procedure to canonically embed any nonreduced multicone K_{mul} into a reduced one, say $\tilde{K}_{mul}$, called the *reduction* of K_{mul}.

The following notion extends the analogous one given for cones and symmetric cones.

Definition 7 A multicone K_{mul} is said to be *proper* if each of its conic components is a proper cone. If we only require that $\operatorname{span}(K_{mul}) = \mathbb{R}^d$, we say that K_{mul} is *weakly proper*.

2.2 Duality of multicones

The concept of duality cannot be directly extended to symmetric cones. For instance, if K_{sym} is proper, its dual set (accordingly to Definition 2) is the trivial subspace $\{0\}$. For this reason, in [6] we introduced a suitably modified notion and symbol.

Definition 8 Let $K_{sym} = K_{+} \cup K_{-}$ be a symmetric cone of $\mathbb{R}^d$. Then the set

$$\begin{aligned} K_{sym}^{\dagger} &:= K_{+}^{*} \cup K_{-}^{*} = K_{+}^{*} \cup -K_{+}^{*} \\ &= \left\{ y \in \mathbb{R}^d \;\middle|\; y^{\mathrm{T}} x \geq 0 \;\forall x \in K_{+} \quad \text{or} \quad y^{\mathrm{T}} x \geq 0 \;\forall x \in K_{-} \right\} \end{aligned}$$

is called the *dual set* of K_{sym}.

Observe that, in general, $K_{sym}^{\dagger}$ is not a symmetric cone itself, but the union of two quasi-cones only (even if still symmetric with respect to the origin).

Example 1 Let $x \in \mathbb{R}^d \backslash \{0\}$ and let $K_{sym} := \text{cone}(\{x\}) \cup -\text{cone}(\{x\})$ (which is nothing but span($\{x\}$)). Then we have that (see Remark 1)

$$K_{sym}^{\dagger} = \text{cone}(\{x\})^* \cup -\text{cone}(\{x\})^* = S_+^x \cup -S_+^x = \mathbb{R}^d. \quad \diamond$$

Indeed, by Proposition 1, $K_{sym}^{\dagger}$ is a symmetric cone if and only if K_{sym} is proper and, in this case, $K_{sym}^{\dagger}$ is proper, too.

The most natural definition of the dual of a multicone is the intersection of the dual sets of its symmetric components.

Definition 9 Let $K_{mul} = \bigcup_{i=1}^r K_{sym}^{(i)}$ be a multicone. Then

$$K_{mul}^{\dagger} := \bigcap_{i=1}^{r} \left(K_{sym}^{(i)}\right)^{\dagger}$$

is called the *dual set* of K_{mul}.

In order to conveniently investigate the structure of the above dual set, we need the auxiliary notion of *associated quasi-cone*.

Consider a multicone $K_{mul} = \bigcup_{i=1}^r K_{sym}^{(i)}$ and the quasi-cone generated by the generic union of r conic components

$$\hat{K}_{\sigma_1 \ldots \sigma_r} := \text{qcone}\left(K_{\sigma_1}^{(1)} \cup \cdots \cup K_{\sigma_r}^{(r)}\right),$$

where $\sigma_i \in \{+, -\}$. We set

$$\Sigma := \left\{(\sigma_1, \ldots, \sigma_r) \in \{+, -\}^r \mid \hat{K}_{\sigma_1 \ldots \sigma_r} \neq \mathbb{R}^d\right\},$$

which clearly is the union of

$$\Sigma_+ := \{(+, \sigma_2, \ldots, \sigma_r) \in \Sigma\} \quad \text{and} \quad \Sigma_- := \{(-, \sigma_2, \ldots, \sigma_r) \in \Sigma\}.$$

Definition 10 Let $r^{\dagger}$ be the cardinality of Σ_+ and, for $k = 1, \ldots, r^{\dagger}$, set

$$\hat{K}_+^{(k)} := \hat{K}_{+\sigma_2(k)\cdots\sigma_r(k)}, \quad (+, \sigma_2(k), \ldots, \sigma_r(k)) \in \Sigma_+$$

and $\hat{K}_-^{(k)} := -\hat{K}_+^{(k)}$. We say that $\hat{K}_{\pm}^{(1)}, \ldots, \hat{K}_{\pm}^{(r^{\dagger})}$ are the *quasi-cones associated* with K_{mul} and $r^{\dagger}$ is called the *dual fragmentation index* of K_{mul}.

Since $K_{mul} \subseteq \hat{K}_+^{(k)} \cup \hat{K}_-^{(k)}$ for all $k = 1, \ldots, r^{\dagger}$, it obviously holds that

$$K_{mul} \subseteq \bigcap_{k=1}^{r^{\dagger}} \left(\hat{K}_+^{(k)} \cup \hat{K}_-^{(k)}\right). \tag{5}$$

Theorem 2 *Let $K_{mul} = \bigcup_{i=1}^r K_{sym}^{(i)}$ be a multicone and let $\hat{K}_{\pm}^{(1)}, \ldots, \hat{K}_{\pm}^{(r^{\dagger})}$ be its associated quasi-cones. Then*

$$K_{mul}^{\dagger} = \bigcup_{k=1}^{r^{\dagger}} \left(\hat{K}_+^{(k)}\right)^* \cup \left(\hat{K}_-^{(k)}\right)^*.$$

Furthermore, if K_{mul} is weakly proper, then all the $\left(\hat{K}_{\pm}^{(k)}\right)^*$'s are cones. In this case, by defining the symmetric cones

$$K_{sym}^{\dagger\,(k)} := \left(\hat{K}_{+}^{(k)}\right)^* \cup \left(\hat{K}_{-}^{(k)}\right)^*,$$

it holds that

$$K_{mul}^{\dagger} = \bigcup_{k=1}^{r^{\dagger}} K_{sym}^{\dagger\,(k)} \tag{6}$$

and at least one of the $K_{sym}^{\dagger\,(k)}$'s is proper.

Finally, if K_{mul} is proper, then $K_{mul}^{\dagger}$ is a multicone (clearly weakly proper).

In any case, even if K_{mul} is proper, its *dual multicone* $K_{mul}^{\dagger}$ may well have got some degenerate conic components, but not all. Thus, if we remove them, we obtain a proper multicone.

Definition 11 Let K_{mul} be a proper multicone. Then the set consisting of all the proper conic components of $K_{mul}^{\dagger}$ is called the *proper dual* multicone of K_{mul} and is denoted by $K_{mul}^{\times}$.

It is clear by definition that

$$\mathrm{int}(K_{mul}^{\times}) = \mathrm{int}(K_{mul}^{\dagger})$$

and, by equality (6), that $K_{mul}^{\times}$ is the union of all and only the *proper* symmetric cones $K_{sym}^{\dagger\,(k)}$. Therefore, by setting

$$\Sigma^{\times} := \left\{(\sigma_1, \ldots, \sigma_r) \in \{+, -\}^r \mid \hat{K}_{\sigma_1\ldots\sigma_r} \text{ is a proper cone}\right\},$$

whose cardinality is denoted by $2r^{\times}$, we obtain immediately the following fact.

Proposition 3 *Assume, without loss of generality, that the first $r^{\times}$ elements in $\{K_{sym}^{\dagger\,(1)}, \ldots, K_{sym}^{\dagger\,(r^{\dagger})}\}$ are proper symmetric cones. Then we have*

$$K_{mul}^{\times} = \bigcup_{(\sigma_1,\ldots,\sigma_r)\in\Sigma^{\times}} \left(K_{\sigma_1\ldots\sigma_r}\right)^* = \bigcup_{k=1}^{r^{\times}} K_{sym}^{\dagger\,(k)}.$$

The number $r^{\times}$ is called *proper dual fragmentation index* of K_{mul} and coincides with the number of all its possible labellings.

Many properties of the duality of proper cones are inherited by the duality of proper multicones, possibly weakened somehow.

Proposition 4 *If K_{mul}, $K_{mul}^{(1)}$, $K_{mul}^{(2)}$ are proper multicones, then*

$$K_{mul}^{\dagger\dagger} \supseteq K_{mul} \quad \text{and} \quad K_{mul}^{\times\times} \supseteq K_{mul}, \tag{7}$$

$$\mathrm{int}(K_{mul}^{\dagger}) = \mathrm{int}(K_{mul}^{\times}) = \left\{y \in \mathbb{R}^d \mid y^{\mathrm{T}}x \neq 0 \quad \forall x \in K_{mul}\backslash\{0\}\right\} \tag{8}$$

and

$$K_{mul}^{\dagger}\backslash\{0\} = \left\{y \in \mathbb{R}^d \mid y^{\mathrm{T}}x \neq 0 \quad \forall x \in \mathrm{int}(K_{mul})\right\}.$$

Furthermore,

$$K_{mul}^{(1)} \subseteq K_{mul}^{(2)} \implies \left(K_{mul}^{(1)}\right)^{\dagger} \supseteq \left(K_{mul}^{(2)}\right)^{\dagger} \quad \text{and} \quad \left(K_{mul}^{(1)}\right)^{\times} \supseteq \left(K_{mul}^{(2)}\right)^{\times}$$

and

$$K_{mul}^{(1)} \backslash \{0\} \subseteq \operatorname{int}(K_{mul}^{(2)}) \implies \operatorname{int}\left((K_{mul}^{(1)})^{\dagger}\right) \supseteq \left(K_{mul}^{(2)}\right)^{\dagger} \backslash \{0\}. \tag{9}$$

The notion of proper dual multicone induces a low relevance of nonreduced multicones (see Definition 6).

Theorem 3 *Let $\tilde{K}_{mul}$ be the reduction of a proper multicone K_{mul}. Then*

$$K_{mul}^{\times} = \left(\tilde{K}_{mul}\right)^{\times} \tag{10}$$

and, consequently, their proper dual fragmentation indices coincide.

In dimension 2, any proper multicone K_{mul} verifies the equality $K_{mul}^{\times} = K_{mul}^{\dagger}$ and moreover the inclusions in (7) are always equalities. This is no longer true in dimension $d \geq 3$, and thus, we need the following definitions.

Definition 12 We say that a proper multicone K_{mul} is *reflexive* if

$$K_{mul}^{\dagger\dagger} = K_{mul}$$

and that it is *properly reflexive* if

$$K_{mul}^{\times\times} = K_{mul}.$$

The next two results regarding properly reflexive multicones are noteworthy.

Proposition 5 *Any properly reflexive proper multicone K_{mul} is also reduced.*

Proposition 6 *For any proper multicone K_{mul}, the proper dual $K_{mul}^{\times}$ is properly reflexive (i.e. $K_{mul}^{\times\times\times} = K_{mul}^{\times}$) and hence is reduced.*

2.3 Matrix invariance of multicones

Let $\mathbb{F}$ denote either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Throughout this paper, we denote by $\mathbb{F}^{d\times d}$ the space of the $d \times d$ matrices on $\mathbb{F}$.

If $A \in \mathbb{F}^{d\times d}$, we identify it with the corresponding endomorphism of $\mathbb{F}^d$ whose kernel and image will be denoted by $\ker(A)$ and $\operatorname{range}(A)$.

If $\lambda \in \mathbb{F}$ is an *eigenvalue* of A of *algebraic multiplicity* k, we denote by $V_\lambda := \ker(A - \lambda I)$ the associated *eigenspace* and by $W_\lambda := \ker((A - \lambda I)^k)$ the *generalized eigenspace* corresponding to λ. Clearly, $V_\lambda \subseteq W_\lambda$ are both invariant for A.

If A is a real matrix, we can take $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.

If $\lambda \in \mathbb{R}$, then W_λ is a linear subspace of $\mathbb{R}^d$ of real dimension k.

Otherwise, if $\lambda \in \mathbb{C}\backslash\mathbb{R}$, take $\mathbb{F} = \mathbb{C}$ and consider $W_\lambda \subseteq \mathbb{C}^d$. Since the conjugate of λ is an eigenvalue as well, set $U_{\mathbb{C}}(\lambda, \bar{\lambda}) := W_\lambda \oplus W_{\bar{\lambda}} \subseteq \mathbb{C}^d$. It turns out that $U_{\mathbb{R}}(\lambda, \bar{\lambda}) := U_{\mathbb{C}}(\lambda, \bar{\lambda}) \cap \mathbb{R}^d$ is a linear space of real dimension $2k$, invariant under the action of A.

Therefore, if $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$ and $\mu_1, \bar{\mu}_1 \ldots, \mu_s, \bar{\mu}_s \in \mathbb{C}\backslash\mathbb{R}$ are the distinct roots of the characteristic polynomial, then

$$\bigoplus_{i=1}^{r} W_{\lambda_i} \oplus \bigoplus_{i=1}^{s} U_{\mathbb{R}}(\mu_i, \bar{\mu}_i) = \mathbb{R}^d. \tag{11}$$

Finally, recall that the set $\sigma(A)$ of its (real or complex) eigenvalues is called the *spectrum* of A and the nonnegative real number

$$\rho(A) := \max_{\lambda \in \sigma(A)} |\lambda|$$

is called the *spectral radius* of A.

The eigenvalues whose modulus is $\rho(A)$ are called *leading eigenvalues*, and the corresponding eigenvectors are called *leading eigenvectors*. The remaining eigenvalues and eigenvectors are called *secondary eigenvalues* and *secondary eigenvectors*, respectively.

Let us draw our attention to matrices of the following type, already considered in [5,6].

Definition 13 A matrix $A \in \mathbb{R}^{d\times d}$ is said to be *asymptotically rank-one* if the following conditions hold:

(i) $\rho(A) > 0$;
(ii) either $\rho(A)$ or $-\rho(A)$ is a simple eigenvalue of A (denoted in the sequel by λ_A);
(iii) $|\lambda| < \rho(A)$ for any other eigenvalue λ of A.

Remark 2 A matrix A is asymptotically rank-one if and only if A^{T} is so.

The term "asymptotically rank-one" is inspired by the following known fact.

Proposition 7 *If A is an asymptotically rank-one matrix, then there exists*

$$\hat{A}^{\infty} := \lim_{k\to\infty} A^k/\lambda_A^k$$

and such limit is the rank-one matrix $\hat{A}^{\infty} = (v_A^{\mathrm{T}} h_A)^{-1} v_A h_A^{\mathrm{T}}$, where v_A and h_A are the (unique up to a scalar factor) leading eigenvectors of A and A^{T}, respectively.

Definition 14 A subset U of $\mathbb{R}^d$ is said to be *invariant under the action of the matrix A* on $\mathbb{R}^d$ (in short, *invariant for* A) if $A(U) \subseteq U$.

The following properties easily follow from (11).

Proposition 8 *Let $A \in \mathbb{R}^{d\times d}$ be an asymptotically rank-one matrix and let v_A be the eigenvector corresponding to the leading eigenvalue $\lambda_A = \lambda_1$. Then*

$$\mathbb{R}^d = V_A \oplus H_A, \qquad \text{with} \quad H_A := \bigoplus_{i=2}^{r} W_{\lambda_i} \ \oplus \bigoplus_{i=1}^{s} U_{\mathbb{R}}(\mu_i, \bar{\mu}_i), \tag{12}$$

where

(i) $V_A := V_{\lambda_A} = \operatorname{span}(v_A)$*;*
(ii) *the linear space H_A is a hyperplane of $\mathbb{R}^d$, invariant for A;*
(iii) $Ax \in H_A \implies x \in H_A$.

The next notions and results about matrix invariance of multicones have been introduced and proved in [6].

Definition 15 If A is an asymptotically rank-one matrix, then the eigenspace V_A will be called the *leading invariant line* of A, whereas the hyperplane H_A will be called *secondary invariant hyperplane* of A and we shall write $H_A = \{h_A\}^{\perp}$ for a suitable vector h_A.

Proposition 9 *If A is an asymptotically rank-one matrix, then h_A is the leading eigenvector of the transpose matrix A^{T}, i.e. $h_A = v_{A^{\mathrm{T}}}$ or, equivalently, $H_A = (V_{A^{\mathrm{T}}})^{\perp}$ or, equivalently $A^{\mathrm{T}} h_A = \lambda_A h_A$.*

Note that, by applying the foregoing result to the transpose matrix A^{T}, its secondary hyperplane $H_{A^{\mathrm{T}}}$ can be expressed by

$$H_{A^{\mathrm{T}}} = \{v_A\}^{\perp} = V_A^{\perp}. \tag{13}$$

Definition 16 We say that a weakly proper multicone K_{mul} is *strictly invariant under the action of the matrix A* (in short, *strictly invariant for A*) if

$$A(K_{mul} \backslash \{0\}) \subseteq \mathrm{int}(K_{mul}).$$

Not all the properties of invariant cones transfer directly to invariant symmetric cones and multicones. Indeed, in this more general context, the situation modifies as follows.

Lemma 1 *If K_{mul} is a strictly invariant multicone for A, then it holds that*

$$K_{mul} \cap \ker(A) = \{0\}. \tag{14}$$

Proposition 10 (localization) *Let a multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be invariant for a matrix A. Then for each symmetric component $K_{sym}^{(i)}$ there exists a symmetric component $K_{sym}^{(j)}$, possibly different from $K_{sym}^{(i)}$, such that*

$$A(K_{sym}^{(i)}) \subseteq K_{sym}^{(j)}. \tag{15}$$

Moreover, if

$$\mathrm{int}(K_{mul}) \cap \ker(A) = \emptyset, \tag{16}$$

then, for each proper conic component $K^{(i)}$, if any, there exists a conic component $K^{(j)}$ such that

$$A(K^{(i)}) \subseteq K^{(j)}.$$

The finiteness of the number of symmetric components gives immediately the following result.

Corollary 1 *Let a multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be invariant for a matrix A. Then there exist a positive integer $s \leq r$ and a symmetric component $K_{sym}^{(k)}$ such that*

$$A^s(K_{sym}^{(k)}) \subseteq K_{sym}^{(k)}.$$

Moreover, if K_{mul} is proper and (16) holds, then there exist a positive integer $p \leq 2r$ and a conic component $K^{(k)}$ such that

$$A^p(K^{(k)}) \subseteq K^{(k)}.$$

Then the classical spectral results concerning matrices admitting an invariant cone can be extended to the multicone case as follows.

Theorem 4 *Let a proper multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ be invariant for a matrix A. Then there exists an integer $s \leq r$ such that:*

(i) $\pm\rho(A)^s$ *is an eigenvalue of* A^s*;*
(ii) *the multicone* K_{mul} *contains a leading eigenvector* v *of* A^s *corresponding to such eigenvalue* $\pm\rho(A)^s$.

Theorem 5 *Let a proper multicone* K_{mul} *be invariant for a matrix* A *and let* (16) *hold. Then the dual multicone* $K^{\dagger}_{mul}$ *is invariant for the transpose matrix* A^{T} *and*

$$\mathrm{int}(K^{\dagger}_{mul}) \cap \ker(A^{\mathrm{T}}) = \emptyset. \tag{17}$$

Moreover, if the stronger property (14) *holds, then the proper dual multicone* $K^{\times}_{mul}$ *is invariant for* A^{T}*, too.*

In general, the above result cannot be reversed because not all proper multicones are reflexive and/or properly reflexive, as the inclusions in (7) only are assured and not vice versa (see [6] for more details).

In the case of asymptotically rank-one matrices (see Definition 13), some of the previous results take a more specific form.

Theorem 6 *Let a proper multicone* $K_{mul} = \bigcup_{i=1}^{r} K^{(i)}_{sym}$ *be invariant for an asymptotically rank-one matrix* A*. Then we have:*

(i) *there exists a symmetric component* $K^{(k)}_{sym}$ *such that* $A(K^{(k)}_{sym}) \subseteq K^{(k)}_{sym}$ *and* $v_A \in K^{(k)}_{sym}$*;*
(ii) $\mathrm{int}(K^{\dagger}_{mul}) \cap H_{A^{\mathrm{T}}} = \emptyset$*;*
(iii) *if also* (16) *holds, then the conic component* $K^{(k)}_{+}$ *is such that*

$$A(K^{(k)}_{+}) \subseteq \begin{cases} K^{(k)}_{+} & \text{if } \lambda_A > 0, \\ -K^{(k)}_{+} & \text{if } \lambda_A < 0, \end{cases}$$

so that $K^{(k)}_{+}$ *is invariant for* A^2*;*
(iv) *if* $K^{\times}_{mul}$ *is invariant for* A^{T}*, then*

$$\mathrm{int}(K_{mul}) \cap H_A = \emptyset.$$

The next result shows that asymptotically rank-one matrices play an important role.

Theorem 7 *If a matrix* A *admits a strictly invariant (weakly proper) multicone, then it is asymptotically rank-one.*

Finally we report the most significant refinements of the previous results in the strictly invariant case.

Proposition 11 *Let a weakly proper multicone* $K_{mul} = \bigcup_{i=1}^{r} K^{(i)}_{sym}$ *be strictly invariant for a matrix* A*. Then:*

(i) *for each proper conic component* $K^{(i)}$*, there exists a proper conic component* $K^{(j)}$*, possibly different, such that*

$$A(K^{(i)}\backslash\{0\}) \subseteq \mathrm{int}(K^{(j)}); \tag{18}$$

(ii) *there exists a proper conic component* $K^{(k)}$ *which is strictly* A^2*-invariant and, more precisely, such that*

$$A(K^{(k)}\backslash\{0\}) \subseteq \begin{cases} \mathrm{int}(K^{(k)}) & \text{if } \lambda_A > 0, \\ \mathrm{int}(-K^{(k)}) & \text{if } \lambda_A < 0 \end{cases}$$

and $v_A \in \mathrm{int}(K^{(k)})$.

Theorem 8 *Let a proper multicone K_{mul} be strictly invariant for a matrix A. Then:*

(i) *both $K_{mul}^{\dagger}$ and $K_{mul}^{\times}$ are strictly invariant for the transpose matrix A^{T};*
(ii) *both $K_{mul}^{\times\dagger}$ and $K_{mul}^{\times\times}$ are strictly invariant for A;*
(iii) $K_{mul}^{\dagger} \cap H_{A^{\mathrm{T}}} = K_{mul}^{\times} \cap H_{A^{\mathrm{T}}} = \{0\}$;
(iv) $K_{mul} \cap H_A = K_{mul}^{\dagger\dagger} \cap H_A = K_{mul}^{\times\times} \cap H_A = \{0\}$.

Corollary 2 *Let K_{mul} be a properly reflexive (proper) multicone. Then K_{mul} is strictly invariant for A if and only if the proper dual $K_{mul}^{\times}$ is strictly invariant for the transpose matrix A^{T}.*

2.4 Embedded pairs of multicones

We conclude this section by considering an *embedded pair* of multicones

$$K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)} \subseteq K'_{mul} = \bigcup_{k=1}^{s} K'^{(k)}_{sym}.$$

We can assume without any restriction that $K'^{(k)}_{sym}$, $k = 1, \ldots, t$, are those symmetric components of K'_{mul} for which

$$K'^{(k)}_{sym} \supseteq K_{sym}^{(i)} \qquad \text{for some } i \tag{19}$$

(possibly more than one) and that

$$K_{mul} \cap K'^{(k)}_{sym} = \{0\} \qquad \text{for all } k = t+1, \ldots, s. \tag{20}$$

Definition 17 If K_{mul} and K'_{mul} are an embedded pair of multicones, then $\bar{K}_{mul} := \bigcup_{k=1}^{t} K'^{(k)}_{sym}$ is called the *submulticone* of K'_{mul} *covering* K_{mul}.

Proposition 12 *Let $K_{mul} \subseteq K'_{mul}$ be an embedded pair of invariant multicones for a matrix A satisfying (19) and (20). Then the submulticone $\bar{K}_{mul}$ of K'_{mul} covering K_{mul} is invariant for A, too.*

Proof By Proposition 10, condition (15) holds for both multicones K_{mul} and K'_{mul}. Therefore, since each symmetric component of $\bar{K}_{mul}$ is also a symmetric component of K'_{mul}, (19) and (20) imply that (15) holds for $\bar{K}_{mul}$ as well. In turn, this fact clearly yields the invariance of $\bar{K}_{mul}$. □

3 Families of matrices with common invariant multicones

The results reported in the last part of the previous section are useful in order to appropriately treat *families of matrices*.

In the literature, some results on the existence of an invariant cone K for a given family of matrices are available (see, for example, Edwards et al. [8], Rodman et al. [19] and Protasov [17]). In particular, we mention that a theoretical characterization of finite *irreducible* families having an invariant cone K is given in [17].

It is also known that a family of matrices may have an invariant or a strictly invariant multicone with $r \geq 2$ symmetric components even if it does not admit any symmetric cone with such properties (see, for instance, Example 3 in Sect. 7).

In this paper, we investigate the existence of invariant and strictly invariant multicones for finite families of matrices in a geometric way, namely in terms of suitable conditions on the distribution of the leading and secondary eigenvectors of all the products of the elements of the family. Our study also includes the case of families containing singular matrices, which needs to take particular consideration of the location of the kernels.

Throughout this paper, $\mathcal{F} = \{A_1, \ldots, A_m\}$ denotes a finite family of real $d \times d$ matrices. For each $k \geq 1$, put $\Sigma_k(\mathcal{F}) := \{A_{i_k} \cdots A_{i_1} \mid i_1, \ldots, i_k \in \{1, \ldots, m\}\}$ the set of all products of *length* (or *degree*) k and

$$\Sigma(\mathcal{F}) := \bigcup_{k \geq 1} \Sigma_k(\mathcal{F})$$

the *product semi-group*.

Definition 18 We say that a cone (respectively, multicone) is *invariant* or *strictly invariant* for the family of matrices $\mathcal{F} = \{A_1, \ldots, A_m\}$ if it is so for each matrix A_i, $i = 1, \ldots, m$.

Remark 3 If a cone, or a multicone, is invariant for the family of matrices $\mathcal{F}$, then it is invariant for any $P \in \Sigma(\mathcal{F})$ as well. The same holds for strict invariance.

Dealing with families $\mathcal{F}$ of *nonsingular* matrices, very similar structures to multicones have been considered by Bochi and Morris [3] (see also Bochi and Gourmelon [2]). They introduced the definition of *k-multicone*, where k may take integer values between 1 and the dimension d of the matrices, which is more general than our definition of multicone. More precisely, a proper multicone in our sense is always a 1-multicone, whereas the opposite is not necessarily true. In fact, we require that the connected conic components of a multicone be convex, which is not assumed a priori in [3].

In [3] it has been proved that the existence of a strictly invariant 1-multicone is equivalent to the family $\mathcal{F}$ being 1-*dominated*, i.e. to the existence of constants $C > 0$ and $\tau \in (0, 1)$ such that

$$\frac{\sigma_2(P_k)}{\sigma_1(P_k)} \leq C\tau^k \quad \forall P_k \in \Sigma_k(\mathcal{F}) \quad \text{and} \quad \forall k \geq 1. \tag{21}$$

Here the $\sigma_i(P)$'s, $i = 1, \ldots, d$, denote the *singular values* of the matrix P, i.e. the square roots of the eigenvalues of the positive semidefinite matrix $P^{\mathrm{T}}P$ listed in nonincreasing order according to multiplicity.

It is worth remarking that, for families of matrices also including singular elements, the mentioned characterization may well fail (see Example 7.1 in [3]).

The following results are obvious consequences of the foregoing definitions and classical well-known spectral results, reported, e.g. in [5].

Proposition 13 *If a family $\mathcal{F}$ of matrices has an invariant proper cone K, then each product $P \in \Sigma(\mathcal{F})$ is such that:*

(i) *the spectral radius $\rho(P)$ is an eigenvalue of P;*
(ii) *K contains a leading eigenvector v corresponding to $\rho(P)$;*
(iii) *the secondary eigenvectors and generalized eigenvectors of P do not belong to* $\mathrm{int}(K)$.

Note that, usually, statement (iii) is not included in the standard formulations of the Perron–Frobenius theorem (see, e.g. Vandergraft [21]).

Proposition 14 *If a family $\mathcal{F}$ of matrices has a strictly invariant proper cone K, then each product $P \in \Sigma(\mathcal{F})$ satisfies the following conditions:*

(i) *P is an asymptotically rank-one matrix with $\lambda_P = \rho(P) > 0$;*
(ii) $\mathrm{int}(K)$ *contains the leading eigenvector v_P;*
(iii) *the secondary eigenvectors and generalized eigenvectors of P do not belong to K.*

As is observed in Remark 3, the existence of a (strictly) invariant cone or multicone for $\mathcal{F}$ affects all the matrices in the semi-group $\Sigma(\mathcal{F})$. Therefore, many of the results proved for matrices may be easily transferred to families.

In particular, Theorems 5 and 8 give rise to the following result, where the *transpose family* $\mathcal{F}^{\mathrm{T}} := \{A_1^{\mathrm{T}}, \ldots, A_m^{\mathrm{T}}\}$ is involved.

Proposition 15 *Let a proper multicone K_{mul} be invariant for a family of matrices $\mathcal{F} = \{A_1, \ldots, A_m\}$ and let condition (16) hold for any A_i. Then:*

(i) *$K_{mul}^{\dagger}$ is invariant for $\mathcal{F}^{\mathrm{T}}$ and condition (17) holds for any A_i^{T};*
(ii) *if, in particular, K_{mul} is strictly invariant for $\mathcal{F}$, then $K_{mul}^{\dagger}$ and $K_{mul}^{\times}$ are strictly invariant for $\mathcal{F}^{\mathrm{T}}$.*

Corollary 2 focuses on the case of reflexive multicones and here is the obvious extension to families of matrices.

Theorem 9 *Let K_{mul} be a properly reflexive (proper) multicone and let $\mathcal{F}$ be a family of matrices. Then K_{mul} is strictly invariant for $\mathcal{F}$ if and only if $K_{mul}^{\times}$ is strictly invariant for $\mathcal{F}^{\mathrm{T}}$.*

Now consider an embedded pair of multicones

$$K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)} \subseteq K'_{mul} = \bigcup_{k=1}^{s} K'^{(k)}_{sym}$$

satisfying conditions (19) and (20). Then Proposition 12 immediately extends to families.

Proposition 16 *Let $K_{mul} \subseteq K'_{mul}$ be an embedded pair of invariant multicones for a family of matrices $\mathcal{F}$. Then the submulticone $\bar{K}_{mul}$ of K'_{mul} covering K_{mul} is invariant for $\mathcal{F}$, too.*

As in the last part of Sect. 2, from now on throughout this section we shall work with asymptotically rank-one matrices only. Anyway, the next definition is necessary since, even if all the matrices A_i, $i = 1, \ldots, m$, are asymptotically rank-one, it is not guaranteed that all products $P \in \Sigma(\mathcal{F})$ are so.

Definition 19 We say that $\mathcal{F}$ is an *asymptotically rank-one* family of matrices if each product $P \in \Sigma(\mathcal{F})$ is so.

Remark 4 As a consequence of Remark 2, it turns out that a family $\mathcal{F}$ of matrices is asymptotically rank-one if and only if the transpose family $\mathcal{F}^{\mathrm{T}}$ is so.

If $\mathcal{F}$ is an asymptotically rank-one family of matrices, then, with reference to the notation introduced in Sect. 2 (see Proposition 8 and Definition 15), it is rather natural to define the *leading set* of the family $\mathcal{F}$ as

$$\mathcal{L}(\mathcal{F}) = \bigcup_{P \in \Sigma(\mathcal{F})} V_P$$

and the *secondary set* of the family $\mathcal{F}$ as

$$\mathcal{H}(\mathcal{F}) = \bigcup_{P \in \Sigma(\mathcal{F})} H_P.$$

Note that both $\mathcal{L}(\mathcal{F})$ and $\mathcal{H}(\mathcal{F})$ are homogeneous and symmetric sets.

The next results characterize the geometry of the leading and secondary sets with respect to a (strictly) invariant multicone.

Theorem 10 *If a proper multicone K_{mul} is invariant for an asymptotically rank-one family $\mathcal{F}$ of matrices, then it holds that*

$$\mathrm{cl}(\mathcal{L}(\mathcal{F})) \subseteq K_{mul} \tag{22}$$

and

$$\mathrm{int}(K^{\dagger}_{mul}) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F}^{\mathrm{T}})) = \emptyset. \tag{23}$$

Proof By Theorem 6-(i) we have that $v_P \in K_{mul}$ for any $P \in \Sigma(\mathcal{F})$. Therefore, $\mathcal{L}(\mathcal{F}) \subseteq K_{mul}$, and hence, since K_{mul} is closed, (22) is proved.

On the other hand, by Theorem 6-(ii) we have $\mathrm{int}(K^{\dagger}_{mul}) \cap H_{P^{\mathrm{T}}} = \emptyset$ for any $P \in \Sigma(\mathcal{F})$. Consequently, $\mathrm{int}(K^{\dagger}_{mul}) \cap \mathcal{H}(\mathcal{F}^{\mathrm{T}}) = \emptyset$, and hence, by standard topological arguments, we obtain (23). $\square$

The foregoing theorem establishes the mutual position of $K^{\dagger}_{mul}$ and $\mathcal{H}(\mathcal{F}^{\mathrm{T}})$ in (23). In order to obtain an analogous relationship between K_{mul} and $\mathcal{H}(\mathcal{F})$, we have to assume the further condition that K_{mul} does not intersect $\ker(P)$ for any $P \in \Sigma(\mathcal{F})$ [see Theorems 5 and 6-(iv)]. In principle, it seems to be hard to actually verify such a condition on infinitely many matrices, but the next lemma will show that it is sufficient to perform a finite number of computations only.

Lemma 2 *Let a multicone K_{mul} be invariant for a family $\mathcal{F} = \{A_1, \ldots, A_m\}$ and let*

$$K_{mul} \cap \ker(A_i) = \{0\}, \qquad i = 1, \ldots, m. \tag{24}$$

Then $K_{mul} \cap \ker(P) = \{0\}$ holds for any $P \in \Sigma(\mathcal{F})$.

Proof The proof is carried out by induction on the degree of the product P. The hypothesis (24) assures the validity of the thesis for the products of degree 1. Thus we assume it to hold for all $P \in \Sigma_k(\mathcal{F})$ and consider a product $Q \in \Sigma_{k+1}(\mathcal{F})$, which can be written in the form $Q = PA$ for some $P \in \Sigma_k(\mathcal{F})$ and $A \in \mathcal{F}$.

Now, if we suppose that $Qx = PAx = 0$ for some $x \in K_{mul}$, then we obtain $Ax \in K_{mul} \cap \ker(P)$ by the invariance of K_{mul}. Therefore, the inductive hypothesis implies that $Ax = 0$ and, consequently, that $x = 0$ because of (24). $\square$

Corollary 3 *If an asymptotically rank-one family $\mathcal{F}$ of matrices has an invariant proper multicone K_{mul} and satisfies condition (24), then $K^{\times}_{mul}$ is invariant for $\mathcal{F}^{\mathrm{T}}$ and*

$$\mathrm{int}(K_{mul}) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \emptyset. \tag{25}$$

Proof By Lemma 2 and Theorem 5 we have that $K^{\times}_{mul}$ is invariant for P^{T} for any $P \in \Sigma(\mathcal{F})$. Hence, by Theorem 6-(iv), we obtain

$$\mathrm{int}(K_{mul}) \cap H_P = \emptyset$$

for any $P \in \Sigma(\mathcal{F})$. Therefore, $\mathrm{int}(K_{mul}) \cap \mathcal{H}(\mathcal{F}) = \emptyset$ and, so, standard topological arguments yield (25). $\square$

Since the existence of a strictly invariant multicone implies that the family $\mathcal{F}$ is asymptotically rank-one (see Theorem 7), now we are in a position to prove a stronger version of Theorem 10. To this aim, we need first three technical preliminary results.

Lemma 3 *If K is a cone and A is a matrix such that $K \cap \ker(A) = \{0\}$, then*

(i) *$A(K)$ is a cone;*
(ii) *$A(K)\backslash\{0\} = A(K\backslash\{0\})$.*

In particular, if K is strictly invariant for A, then (i) and (ii) hold.

Proof (i) Since K is positively homogeneous and closed, its image $A(K)$ is closed (see Proposition 3 in Borwein and Moors [4]). It is immediate to see that $A(K)$ is convex and positively homogeneous.

We are left to show that it is pointed. For, let $x \in A(K) \cap (-A(K)) = A(K) \cap A(-K)$. Hence $x = Ay = A(-z)$ for suitable $y, z \in K$. Therefore, $y + z \in K \cap \ker(A)$ and so, by assumption, $y + z = 0$. This implies that $y = -z \in K \cap (-K) = \{0\}$, since K is pointed. Hence $x = 0$.

(ii) Obvious.

The last claim follows from the well-known fact that K strictly invariant for A implies $K \cap \ker(A) = \{0\}$. □

An immediate generalization of the above fact to multicones is the following.

Lemma 4 *If the multicone K_{mul} is strictly invariant for $\mathcal{F}$, then for all $i = 1, \ldots, m$,*

$$A_i(K_{mul})\backslash\{0\} = A_i(K_{mul}\backslash\{0\}) \subseteq \mathrm{int}(K_{mul}).$$

Lemma 5 *If the multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ is strictly invariant for $\mathcal{F}$, then for all $i = 1, \ldots, 2r$ the set*

$$W_i := K^{(i)} \cap \bigcup_{j=1}^{m} A_j(K_{mul})$$

is a (possibly empty) finite union of cones. In particular, its convex hull

$$\bar{K}^{(i)} := \mathrm{cvx}\,(W_i) \tag{26}$$

is a cone. Finally,

$$\bar{K}^{(i)}\backslash\{0\} \subseteq \mathrm{int}(K^{(i)}).$$

Proof The set W_i is a union of cones by Proposition 11-(i) and Lemma 3-(i). Moreover, if H is the labelling hyperplane of K_{mul}, then $W_i\backslash\{0\}$ is entirely contained in the interior of a semi-space defined by H. In this situation, the convex hull of a finite union of cones equals their sum which, in turn, is a cone (see Proposition 3.38 in [5]). Therefore, $\bar{K}^{(i)}$ is a cone.

Finally, since K_{mul} is strictly invariant for $\mathcal{F}$, then by Lemma 4

$$W_i\backslash\{0\} = K^{(i)} \cap \bigcup_{j=1}^{m} A_j(K_{mul}\backslash\{0\}) \subseteq K^{(i)} \cap \mathrm{int}(K_{mul}) = \mathrm{int}(K^{(i)}).$$

Therefore, since $\mathrm{cvx}(W_i)\backslash\{0\} = \mathrm{cvx}(W_i\backslash\{0\})$, and since $\mathrm{int}(K^{(i)})$ is convex, it turns out that

$$\bar{K}^{(i)}\backslash\{0\} = \mathrm{cvx}(W_i)\backslash\{0\} \subseteq \mathrm{int}(K^{(i)}),$$

So the proof is complete. □

Theorem 11 *If a proper multicone K_{mul} is strictly invariant for a family $\mathcal{F}$ of matrices, then the following facts hold:*

$$\mathrm{cl}(\mathcal{L}(\mathcal{F}))\backslash\{0\} \subseteq \mathrm{int}(K_{mul}), \tag{27}$$
$$K_{mul}^{\dagger} \cap \mathrm{cl}(\mathcal{H}(\mathcal{F}^{\mathrm{T}})) = \{0\}, \tag{28}$$
$$K_{mul} \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\}. \tag{29}$$

Proof Let us define the auxiliary set

$$\bar{K}_{mul} := \bigcup_{i=1}^{r} \bar{K}_{sym}^{(i)},$$

where $\bar{K}_{\pm}^{(i)}$, $i = 1, \ldots, r$ are defined in (26).

It is easy to see that $\bar{K}_{mul}$ is a submulticone of K_{mul} (not necessarily proper).

A straightforward computation shows that $A_j(K_{mul}) \subseteq \bar{K}_{mul}$ for any $j = 1, \ldots, m$.

The first consequence is that $A_j(\bar{K}_{mul}) \subseteq \bar{K}_{mul}$ for any $j = 1, \ldots, m$, i.e. $\bar{K}_{mul}$ is invariant for $\mathcal{F}$.

The second consequence is that

$$P(K_{mul}) \subseteq \bar{K}_{mul} \tag{30}$$

for any $P \in \Sigma(\mathcal{F})$. Since each leading eigenvector v_P belongs to K_{mul} (see Theorem 6), we also have that $v_P = \lambda_P^{-1} P v_P \in P(K_{mul})$ and, hence,

$$\mathcal{L}(\mathcal{F}) \subseteq \bigcup_{P \in \Sigma(\mathcal{F})} P(K_{mul}) \subseteq \bar{K}_{mul}.$$

Therefore, since $\bar{K}_{mul}$ is closed,

$$\mathrm{cl}(\mathcal{L}(\mathcal{F})) \subseteq \bar{K}_{mul}.$$

But K_{mul} is strictly invariant for $\mathcal{F}$, so that Lemma 5 yields

$$\bar{K}_{mul}\backslash\{0\} \subseteq \mathrm{int}(K_{mul}). \tag{31}$$

The two inclusions give (27).

Now observe that there surely exists a proper multicone $\tilde{K}_{mul}$ between $\bar{K}_{mul}$ and K_{mul} such that

$$\bar{K}_{mul}\backslash\{0\} \subseteq \mathrm{int}(\tilde{K}_{mul}) \subseteq \tilde{K}_{mul}\backslash\{0\} \subseteq \mathrm{int}(K_{mul}). \tag{32}$$

From Lemmas 1 and 2 and (30), for any $P \in \Sigma(\mathcal{F})$ we obtain

$$P(K_{mul}\backslash\{0\}) \subseteq \bar{K}_{mul}\backslash\{0\}$$

which, in turn, yields

$$P(\tilde{K}_{mul}\backslash\{0\}) \subseteq \bar{K}_{mul}\backslash\{0\}.$$

Therefore, $\tilde{K}_{mul}$ is strictly invariant for $\mathcal{F}$. On the other hand, the last inclusion in (32) and (9) yields

$$K_{mul}^{\dagger}\backslash\{0\} \subseteq \mathrm{int}(\tilde{K}_{mul}^{\dagger}).$$

Therefore, Theorem 10 applied to the pair $(\tilde{K}_{mul}, \mathcal{F})$ proves (28).

Finally, Proposition 15-(ii) tells us that $K_{mul}^{\times}$ is strictly invariant for $\mathcal{F}^{\mathrm{T}}$ and hence, by (28) applied to the pair $(K_{mul}^{\times}, \mathcal{F}^{\mathrm{T}})$, we obtain

$$K_{mul}^{\times\dagger} \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\}.$$

Since $K_{mul} \subseteq K_{mul}^{\times\times} \subseteq K_{mul}^{\times\dagger}$, we can conclude with (29). □

Remark 5 Note that (27) and (29) are stronger forms of (22) and (25), respectively.

Moreover, if K_{mul} satisfies (29), then all the hyperplanes belonging to $\mathcal{H}(\mathcal{F})$ are splitting for it (see Definition 5).

The following result is an immediate consequence of (27) and (29).

Corollary 4 *If a family $\mathcal{F}$ of matrices has a strictly invariant proper multicone, then*

$$\mathrm{cl}(\mathcal{L}(\mathcal{F})) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\}. \tag{33}$$

4 Geometric fragmentation and invariance of the leading set

This section is devoted to understand the geometry of the leading set $\mathcal{L}(\mathcal{F})$. Nevertheless, we start by proving a useful property concerning an arbitrary family $\mathcal{H}$ of hyperplanes, which will be later applied to the secondary set $\mathcal{H}(\mathcal{F})$.

Proposition 17 *Let $X \subset \mathbb{R}^d$ be a closed connected set and $\mathcal{H}$ be a family of hyperplanes such that*

$$X \cap \mathrm{cl}(\mathcal{H}) = \emptyset. \tag{34}$$

Then

$$\mathrm{conv}(X) \cap \mathrm{cl}(\mathcal{H}) = \emptyset.$$

Proof Clearly, $X \cap H = \emptyset$ for all $H = \{h\}^{\perp} \in \mathcal{H}$ and so, X being connected, $X \subseteq \mathrm{int}(S_+^h)$ (up to the sign of h).

By the convexity of $\mathrm{int}(S_+^h)$, we have that $\mathrm{conv}(X) \subseteq \mathrm{int}(S_+^h)$. Hence,

$$\mathrm{conv}(X) \cap \mathcal{H} = \emptyset. \tag{35}$$

By (34) it is enough to show that any point $x \in \mathrm{conv}(X) \backslash X$ does not belong to $\mathrm{cl}(\mathcal{H})$.

Observe that there exists a minimal set of distinct points $\{v_1, \ldots, v_n\} \subset X$ and corresponding scalars $t_1, \ldots, t_n \in [0, 1]$ such that

$$x = \sum_{i=1}^{n} t_i v_i \quad \text{and} \quad \sum_{i=1}^{n} t_i = 1. \tag{36}$$

Since $x \notin X$ and the set $\{v_1, \ldots, v_n\}$ is minimal, we have that $t_i \neq 0$ and $t_i \neq 1$ for all $i = 1, \ldots, n$. Moreover, the polytope

$$T := \mathrm{conv}(\{v_1, \ldots, v_n\})$$

clearly contains x and is contained in $\mathrm{conv}(X)$. In particular, by (35)

$$T \cap \mathcal{H} = \emptyset. \tag{37}$$

On the other hand, assumption (34) assures that there exist n open balls centred in $v_1, \ldots, v_n$, respectively, which do not intersect $\mathrm{cl}(\mathcal{H})$. Consequently, there exist n closed balls $B_i := B(v_i, \delta)$, $i = 1, \ldots, n$, of the same radius δ, such that

$$B_i \cap \mathcal{H} = \emptyset. \tag{38}$$

It is clear that the set $Y := B_1 \cup \ldots \cup B_n \cup T$ is closed and connected and that $Y \cap \mathcal{H} = \emptyset$ by (37) and (38). So, applying again the initial argument to Y, we obtain that (35) holds, i.e.

$$\mathrm{conv}(Y) \cap \mathcal{H} = \emptyset.$$

Finally, observe that the closed ball $B_x := B(x, \delta)$ is contained in $\mathrm{conv}(Y)$. In fact, if $y \in B_x$, then $y = x + w$, where w is a vector such that $\|w\| \le \delta$. Moreover, (36) implies that $y = \sum_{i=1}^n t_i y_i$, where $y_i := v_i + w$ and, clearly, $y_i \in B_i$ for all $i = 1, \ldots, n$.

Therefore, the open ball of radius $\delta/2$ and centred in x is contained in $\mathrm{conv}(Y)$ and does not meet $\mathcal{H}$. In other words, $x \in \mathrm{int}(\mathbb{R}^d \backslash \mathcal{H})$, i.e. $x \notin \mathrm{cl}(\mathcal{H})$ as required. □

Now we recall some known topological facts, whose proof may also be easily derived from Lemma 3.26 in [5].

Lemma 6 *If $X \subset \mathbb{R}^d$ is a compact set, then $\mathrm{conv}(X)$ is compact as well. In addition, if $0 \notin \mathrm{conv}(X)$, then $\mathbb{R}_+\mathrm{conv}(X)$ is closed and there exists $\mathrm{cone}(X)$ which satisfies the equalities*

$$\mathrm{cone}(X) = \mathbb{R}_+\mathrm{conv}(X) = \mathrm{conv}(\mathbb{R}_+X).$$

Corollary 5 *Let $X \subset \mathbb{R}^d$ be a compact connected set and $\mathcal{H}$ be a family of hyperplanes such that*

$$X \cap \mathrm{cl}(\mathcal{H}) = \emptyset.$$

Then

$$\mathrm{cone}(X) \cap \mathrm{cl}(\mathcal{H}) = \{0\}.$$

Proof By Proposition 17 we have $\mathrm{conv}(X) \cap \mathrm{cl}(\mathcal{H}) = \emptyset$. Hence, we immediately obtain $\mathbb{R}_+\mathrm{conv}(X) \cap \mathrm{cl}(\mathcal{H}) = \{0\}$ and, thus, since $0 \notin \mathrm{conv}(X)$, Lemma 6 concludes the proof. □

In order to try to reverse Theorem 11, we shall always assume the conditions we have seen to hold for a family of matrices sharing a strictly invariant multicone: first, to be asymptotically rank-one (by Theorem 7) and, second, to satisfy (33).

Both the above properties concern the leading eigenvalues and the geometry of the leading set $\mathcal{L}(\mathcal{F})$. So we shall refer to them as the "Leading set assumptions", together with the additional demand for *irreducibility* of $\mathcal{F}$.

For convenience of the reader, we recall this notion.

Definition 20 We say that a family of matrices $\mathcal{F} = \{A_1, \ldots, A_m\}$ is *reducible* if there exists a nonzero proper linear subspace $L \subset \mathbb{R}^d$ invariant for each A_i, $i = 1, \ldots, m$. Otherwise, we say that $\mathcal{F}$ is *irreducible*.

Note that irreducibility is not restrictive since reducibility only relocates many investigations into spaces of lower dimensions.

Remark 6 The above notion of reducibility given for a family $\mathcal{F}$ has nothing to do with the well-known notion of reducibility of a matrix A to upper triangular form via similarity transformations with permutation matrices.

Indeed, most of singleton families $\mathcal{F} = \{A\}$ are reducible, although the underlying matrix A may well be irreducible in the classical sense.

Assumption 1 (*Leading set assumptions*) The family $\mathcal{F}$ satisfies the following properties:

(i) $\mathcal{F}$ is irreducible;
(ii) $\mathcal{F}$ is asymptotically rank-one;
(iii) $\mathrm{cl}(\mathcal{L}(\mathcal{F})) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\}$.

First of all, we show that condition (iii) means that $\mathcal{L}(\mathcal{F})$ is finitely "fragmented" as illustrated by the following result.

Theorem 12 *Let the family $\mathcal{F}$ satisfy Assumption 1. Then there exists a proper multicone K_{mul} satisfying (27) and (29), i.e.*

$$\mathrm{cl}(\mathcal{L}(\mathcal{F}))\backslash\{0\} \subseteq \mathrm{int}(K_{mul})$$

and

$$K_{mul} \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\},$$

respectively.

Proof Let $H = \{h\}^{\perp} \in \mathcal{H}(\mathcal{F})$ and let us denote by S^d the closed hypersphere of $\mathbb{R}^d$ of radius 1 centred in the origin. Moreover, let $\hat{\mathcal{S}} := S^d \cap S^h_+$ be the positive semisphere and define the sets $\hat{\mathcal{L}} := \mathrm{cl}(\mathcal{L}(\mathcal{F})) \cap \hat{\mathcal{S}}$ and $\hat{\mathcal{H}} := \mathrm{cl}(\mathcal{H}(\mathcal{F})) \cap \hat{\mathcal{S}}$.

Observe that $\hat{\mathcal{L}}$ and $\hat{\mathcal{H}}$ are closed subsets of S^d and hence are compact as well. Moreover, Assumption 1-(iii) implies that $\hat{\mathcal{L}} \cap \hat{\mathcal{H}} = \emptyset$, and thus, their Euclidean distance $\delta := \mathrm{dist}(\hat{\mathcal{L}}, \hat{\mathcal{H}})$ is strictly positive.

Consequently, the compactness of $\hat{\mathcal{L}}$ implies the existence of a finite family of open balls of $\mathbb{R}^d$, say $U_1, \ldots, U_s$, of radius $\delta/2$, centred in suitable points of $\hat{\mathcal{L}}$ such that

$$\left(\bigcup_{i=1}^{s} \mathrm{cl}(U_i)\right) \cap \hat{\mathcal{H}} = \emptyset \quad \text{and} \quad \left(\bigcup_{i=1}^{s} U_i\right) \supset \hat{\mathcal{L}}. \tag{39}$$

Note that the set

$$\bigcup_{i=1}^{s} \mathrm{cl}(U_i) \cap \hat{\mathcal{S}}$$

has got a finite number of connected components and denote them by $\hat{\mathcal{U}}_1, \ldots, \hat{\mathcal{U}}_t$.

Now set $K_+^{(i)} := \mathrm{cone}(\hat{\mathcal{U}}_i)$ and, with obvious notation, $K_{mul} := \bigcup_{i=1}^{\mathrm{T}} K_{sym}^{(i)}$. Observe that K_{mul} is a proper multicone and that H is one of its labelling hyperplanes. It is also clear that K_{mul} satisfies condition (27) by construction.

Finally, note that $\hat{\mathcal{U}}_i \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \emptyset$ by (39) and so, by Corollary 5,

$$K_+^{(i)} \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\}, \qquad i = 1, \ldots, t,$$

proving (29). □

The foregoing theorem shows that the set of all the multicones K_{mul} which satisfy conditions (27) and (29) is nonempty. A fortiori, there exist multicones verifying the weaker conditions (22) and (25), which will reveal to constitute the right environment where to look for invariant multicones.

Definition 21 We say that the minimum fragmentation index r of the multicones K_{mul} satisfying conditions (22) and (25), i.e.

$$\mathrm{cl}(\mathcal{L}(\mathcal{F})) \subseteq K_{mul} \tag{40}$$

and

$$\mathrm{int}(K_{mul}) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \emptyset, \tag{41}$$

is the *spectral fragmentation index* of the family $\mathcal{F}$ and that any of such multicones having r symmetric components is a *spectral multicone* for $\mathcal{F}$.

Lemma 7 *If K_{mul} is a spectral multicone for $\mathcal{F}$, then each of its conic components $K^{(i)}$ intersects the leading set $\mathcal{L}(\mathcal{F})\backslash\{0\}$.*

Proof If not, we could consider the multicone $\tilde{K}_{mul}$ obtained by suppressing all the conic components which do not intersect $\mathcal{L}(\mathcal{F})\backslash\{0\}$. Clearly, $\tilde{K}_{mul}$ would still fulfil (40) and (41) but with a smaller number of components, against the fact that K_{mul} is spectral. □

Proposition 18 *If K_{mul} is a spectral multicone for $\mathcal{F}$, then each two distinct conic components are weakly separated by a secondary hyperplane.*

Proof First observe that assumption (41) implies $H \cap \mathrm{int}(K_{mul}) = \emptyset$ for all $H \in \mathcal{H}(\mathcal{F})$.

Now assume by contradiction that $K^{(i)} \cup K^{(j)} \subseteq S_+^h$ for all $H \in \mathcal{H}(\mathcal{F})$.

If $K^{(j)} = -K^{(i)}$, then $K_{sym}^{(i)} \subseteq S_+^h$. So necessarily $K^{(i)}$ is a degenerate cone and $K_{sym}^{(i)} \subseteq H$. But $K_{sym}^{(i)}$ meets $\mathcal{L}(\mathcal{F})\backslash\{0\}$ by Lemma 7 and this contradicts Assumption 1-(iii).

Now let $K^{(j)} \neq -K^{(i)}$. Then, by a result of Holmes [13] (see also [6]), we obtain that $\mathrm{cl}(\mathrm{conv}(K^{(i)} \cup K^{(j)}))$ is a cone, say T_+. Thus we can consider the multicone $\tilde{K}_{mul}$ obtained by replacing $K^{(i)} \cup K^{(j)}$ by T_+ and the union of their opposites by $T_- := -T_+$.

Clearly, the fragmentation index of $\tilde{K}_{mul}$ is $r-1$. Therefore, if we show that $\tilde{K}_{mul}$ verifies (40) and (41), we obtain a contradiction.

Since $K_{mul} \subseteq \tilde{K}_{mul}$, relation (40) comes immediately.

In order to show also (41), it is enough to prove that

$$\mathrm{int}(T_+) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \emptyset. \tag{42}$$

From the assumption on $K^{(i)}$ and $K^{(j)}$, for each $H \in \mathcal{H}(\mathcal{F})$ we have

$$T_+ = \mathrm{cl}(\mathrm{conv}(K^{(i)} \cup K^{(j)})) \subseteq S_+^h,$$

since the half-space S_+^h is convex and closed. Therefore, $\mathrm{int}(T_+) \cap H = \emptyset$ for all H and hence, by standard topological arguments, $\mathrm{int}(T_+)$ being an open set, we obtain the requested equality (42). □

The next result makes the notion of fragmentation of the leading set $\mathcal{L}(\mathcal{F})$ more precise.

Proposition 19 *Let $K_{mul} = \bigcup_{i=1}^r K_{sym}^{(i)}$ be a spectral multicone and, for any $i = 1, \ldots, r$, let*

$$\mathcal{L}(\mathcal{F})_\pm^{(i)} := K_\pm^{(i)} \cap \mathcal{L}(\mathcal{F}).$$

Then, by denoting either of the sets $\mathcal{L}(\mathcal{F})_\pm^{(i)}$ simply by $\mathcal{L}(\mathcal{F})^{(i)}$, we have:

(i) *any two distinct sets $\mathcal{L}(\mathcal{F})^{(p)}$ and $\mathcal{L}(\mathcal{F})^{(q)}$ are separated by some hyperplane $H \in \mathcal{H}(\mathcal{F})$;*

(ii) *if* $J_{mul} = \bigcup_{i=1}^{r} J_{sym}^{(i)}$ *is another spectral multicone, then for any* $p = 1, \ldots, 2r$ *there exists* $q \in \{1, \ldots, 2r\}$ *such that* $K^{(p)} \cap \mathcal{L}(\mathcal{F}) = J^{(q)} \cap \mathcal{L}(\mathcal{F})$.

Proof (i) It comes from Proposition 18 and condition (iii) of Assumption 1.

(ii) Note that any conic component $J^{(q)}$ cannot intersect more than one of the sets $\mathcal{L}(\mathcal{F})^{(p)}$ because J_{mul} satisfies (41). Therefore, since also J_{mul} contains the entire $\mathcal{L}(\mathcal{F})$ and has exactly $2r$ conic components, each of them must contain just one entire set $\mathcal{L}(\mathcal{F})^{(q)}$, and so the thesis is proved.

□

Setting with obvious notation

$$\mathcal{L}(\mathcal{F})_{sym}^{(i)} := \mathcal{L}(\mathcal{F})_{+}^{(i)} \cup \mathcal{L}(\mathcal{F})_{-}^{(i)}$$

for any $i = 1, \ldots, r$, the previous result shows that the "fragmentation" of the leading set

$$\mathcal{L}(\mathcal{F}) := \bigcup_{i=1}^{r} \mathcal{L}(\mathcal{F})_{sym}^{(i)},$$

is independent of the particular spectral multicone containing it. It is also clear that the above decomposition is only induced by the secondary set $\mathcal{H}(\mathcal{F})$.

Definition 22 The sets $\mathcal{L}(\mathcal{F})_{sym}^{(i)}$ are called *essential symmetric components* of $\mathcal{L}(\mathcal{F})$ and the sets $\mathcal{L}(\mathcal{F})^{(i)}$ *essential conic components*.

Remark 7 Theorem 10 and Corollary 3 tell us that, if $\mathcal{F}$ is asymptotically rank-one, then any proper invariant multicone K_{mul} satisfying (24) necessarily fulfils also (40) and (41). Therefore, the number of its symmetric components is $\geq r$, where r is the spectral fragmentation index of $\mathcal{F}$.

In particular, if equality holds, K_{mul} is a spectral multicone and each of its symmetric (respectively, conic) components contains exactly one essential symmetric (respectively, conic) component $\mathcal{L}(\mathcal{F})_{sym}^{(i)}$ (respectively, $\mathcal{L}(\mathcal{F})^{(i)}$).

Theorem 13 *Let the family* $\mathcal{F}$ *satisfy Assumption* 1. *Then there exists a proper spectral multicone* $\tilde{K}_{mul}$ *satisfying* (27) *and* (29)*, i.e.*

$$\mathrm{cl}(\mathcal{L}(\mathcal{F}))\backslash\{0\} \subseteq \mathrm{int}(\tilde{K}_{mul})$$

and

$$\tilde{K}_{mul} \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\},$$

respectively.

Proof Let K_{mul} be the proper multicone given by Theorem 12.

For each $i = 1, \ldots, 2r$, consider the conic components of K_{mul} meeting $\mathcal{L}(\mathcal{F})^{(i)}$ and replace them by their conic hull. The obtained multicone $\tilde{K}_{mul}$ clearly verifies (27) and (29) (with an argument analogous to the one used in the proof of Proposition 18) and its fragmentation index is r. □

Now we are going to show the central result of this section, that is, $\mathrm{cl}(\mathcal{L}(\mathcal{F}))$ is invariant.

The next results are strictly connected to the well-known "power method" for the numerical approximation of the leading eigenvalue and eigenvector of an asymptotically rank-one matrix (see, e.g. Dahlquist and Björk [7]) and give a possible hint for an iterative construction of the set $\mathrm{cl}\,(\mathcal{L}(\mathcal{F}))$.

Lemma 8 *Let* $(R_n)_n$ *be a sequence of asymptotically rank-one matrices,* $(\lambda_n)_n$ *the corresponding leading eigenvalues and* $(x_n)_n$ *the corresponding leading eigenvectors with* $\|x_n\| = 1$. *Moreover, assume that*

$$\lim_{n\to\infty} R_n = R,$$

where R *is a rank-one matrix whose leading eigenvalue is* $\lambda \neq 0$ *and leading eigenvector is* x *with* $\|x\| = 1$. *Then it holds that*

$$\lim_{n\to\infty} \lambda_n = \lambda \quad \text{and} \quad \lim_{n\to\infty} x_n = x. \tag{43}$$

Proof Since R is a rank-one matrix, we have that $\dim(\ker(R)) = d - 1$ and, thus, its characteristic polynomial is $p(z) = (z - \lambda)z^{d-1}$. Analogously, since R_n is an asymptotically rank-one matrix, its characteristic polynomial is $p_n(z) = (z - \lambda_n)q_n(z)$, where $q_n(z)$ is a polynomial of degree $d - 1$ whose roots all have a modulus $< |\lambda_n|$. On the other hand, the coefficients of the characteristic polynomial are continuous functions of the matrix. Therefore, $\lim_{n\to\infty} p_n(z) = p(z)$ for each $z \in \mathbb{C}$ and, hence, $\lim_{n\to\infty} q_n(z) = z^{d-1}$ for each $z \in \mathbb{C}$ and the left-hand limit in (43) holds.

In order to prove the right-hand limit, consider the sequence $(x_n)_n$ and assume, without loss of generality, that $x^{\mathrm{T}} x_n \geq 0$. Since $\|x_n\| = 1$ for all n, possibly by extracting a suitable subsequence, we can assume that it converges to a limit vector y such that $\|y\| = 1$ and $x^{\mathrm{T}} y \geq 0$.

Now, each vector x_n may be written in the form

$$x_n = \alpha_n x + u_n,$$

where $\alpha_n \in \mathbb{R}$ and $u_n \in \ker(R)$ are uniquely determined. Therefore, $Rx_n = \lambda\alpha_n x$ and so

$$\lambda x_n = \alpha_n \lambda x + (\lambda - \lambda_n)x_n + (R_n - R)x_n.$$

On the other hand, letting $n \to \infty$ in both the above equalities, we clearly get $y = \alpha x + u$ for some $\alpha \in \mathbb{R}$ and $u \in \ker(R)$ and also $\lambda y = \alpha\lambda x$. Since $\lambda \neq 0$, $\|x\| = \|y\| = 1$ and $x^{\mathrm{T}} y \geq 0$, we conclude that $\alpha = 1$ and $y = x$.

Finally, if we assume that not all the sequences $(x_n)_n$ converge to x, then we can find another subsequence which is uniformly bounded away from x and which, by using the same arguments as before, is proved to necessarily converge to x, making the absurde. So the proof is complete. □

Lemma 9 *Let the family* $\mathcal{F}$ *of matrices satisfy Assumption 1 and let* $v \in \mathbb{R}^d$ *be such that* $v \notin \mathcal{H}(\mathcal{F})$. *Then*

$$\mathrm{cl}\,(\mathcal{L}(\mathcal{F})) \subseteq \mathrm{cl}\,(\Sigma(\mathcal{F})\mathrm{span}(v)). \tag{44}$$

In particular, if $v \in \mathcal{L}(\mathcal{F})$ *(i.e.* $v = v_Q$ *for some* $Q \in \Sigma(\mathcal{F})$*), the equality holds.*

Proof Let $v_P \in \mathcal{L}(\mathcal{F})$, where $P \in \Sigma(\mathcal{F})$ is an asymptotically rank-one matrix. Then, defining $\tilde{P} := P/\lambda_P$, we can apply Proposition 7 to P and get

$$\lim_{k\to\infty} \tilde{P}^k v = \left((v_P^{\mathrm{T}} h_P)^{-1} v_P h_P^{\mathrm{T}}\right) v = \alpha v_P, \tag{45}$$

where $\alpha := (v_P^{\mathrm{T}} h_P)^{-1} h_P^{\mathrm{T}} v$. Since $v \notin H_P$ by assumption, we have $h_P^{\mathrm{T}} v \neq 0$ and, so, $\alpha \neq 0$. Consequently,

$$v_P \in \mathrm{cl}\left(\bigcup_{k\geq 1} \tilde{P}^k \mathrm{span}(v)\right) = \mathrm{cl}\left(\bigcup_{k\geq 1} P^k \mathrm{span}(v)\right) \subseteq \mathrm{cl}\,(\Sigma(\mathcal{F})\mathrm{span}(v)),$$

where the equality follows from the fact that span(v) is homogeneous. Therefore, $\mathcal{L}(\mathcal{F}) \subseteq \mathrm{cl}\,(\Sigma(\mathcal{F})\mathrm{span}(v))$, and hence, we get (44).

Now consider the general element Pv_Q of $\Sigma(\mathcal{F})\mathrm{span}(v_Q)$ and show that $Pv_Q \in \mathrm{cl}\,(\mathcal{L}(\mathcal{F}))$. Note that $v_Q \notin H_P$ by Assumption 1-(iii). Therefore, (45) becomes

$$\lim_{k\to\infty} \tilde{P}^k v_Q = \beta v_P$$

for some $\beta \neq 0$. Still by Assumption 1-(iii), we have $\beta v_P \notin H_Q$. Consequently, for $\hat{k}$ sufficiently large it holds that $\tilde{P}^{\hat{k}} v_Q \notin H_Q$ and, so, $P^{\hat{k}} v_Q \notin H_Q$ either. Therefore, the same arguments used before, applied to the asymptotically rank-one matrix Q, lead to

$$\lim_{n\to\infty} \tilde{Q}^n P^{\hat{k}} v_Q = \gamma v_Q,$$

where $\tilde{Q} := Q/\lambda_Q$ and $\gamma = (v_Q^{\mathrm{T}} h_Q)^{-1} h_Q^{\mathrm{T}} P^{\hat{k}} v_Q \neq 0$, and in turn to

$$\lim_{n\to\infty} P\tilde{Q}^n P^{\hat{k}-1} P v_Q = \gamma P v_Q.$$

Now, all the products $R_n := P\tilde{Q}^n P^{\hat{k}-1}$ are asymptotically rank-one by Assumption 1-(ii). Furthermore, Q being asymptotically rank-one, Proposition 7 clearly implies that $R := \lim_{n\to\infty} P\tilde{Q}^n P^{\hat{k}-1}$ is a rank-one matrix and, consequently, that Pv_Q is its (unique) leading eigenvector. Therefore, since the (normalized) leading eigenvectors of $P\tilde{Q}^n P^{\hat{k}-1}$ and $PQ^n P^{\hat{k}-1}$ clearly coincide, by Lemma 8 we can conclude that $Pv_Q \in \mathrm{cl}\,(\mathcal{L}(\mathcal{F}))$ and, so, $\Sigma(\mathcal{F})\mathrm{span}(v_Q) \subseteq \mathrm{cl}\,(\mathcal{L}(\mathcal{F}))$. Passing to the closure completes the proof. □

The following result and its consequences come out directly.

Theorem 14 *Let the family $\mathcal{F}$ of matrices satisfy Assumption 1. Then the set* $\mathrm{cl}\,(\mathcal{L}(\mathcal{F}))$ *is invariant for $\mathcal{F}$.*

Corollary 6 *Let the family $\mathcal{F}$ satisfy Assumption 1. Then* $\mathrm{span}\,(\mathcal{L}(\mathcal{F})) = \mathbb{R}^d$.

Proof By Theorem 14, the linear space $\mathrm{span}\,(\mathcal{L}(\mathcal{F})) = \mathrm{span}\,(\mathrm{cl}\,(\mathcal{L}(\mathcal{F})))$ is invariant for $\mathcal{F}$ as well. The assumed irreducibility of $\mathcal{F}$ yields the result. □

Remark 8 Theorem 14 also implies that, if the family $\mathcal{F}$ of matrices satisfies Assumption 1, then

$$Pv_Q \notin \mathrm{cl}\,(\mathcal{H}(\mathcal{F})) \qquad \text{for all } P, Q \in \Sigma(\mathcal{F}).$$

This fact tells us that, in the second part of the proof of Lemma 9, we can choose $\hat{k} = 1$. In other words, Pv_Q is the leading eigenvector of the rank-one limit matrix $P\tilde{Q}^\infty := \lim_{n\to\infty} P\tilde{Q}^n$.

The next result is an immediate consequence of Theorem 14 and Remark 8.

Corollary 7 *Let the family $\mathcal{F}$ of matrices satisfy Assumption 1. Then the set* $\mathrm{cl}\,(\mathcal{L}(\mathcal{F}))\setminus\{0\}$ *is invariant for $\mathcal{F}$.*

Even if the whole leading set spans the ambient space $\mathbb{R}^d$, it is not difficult to see that this is not necessarily true for each single essential conic component.

Example 2 Consider the 2×2 family $\mathcal{F} = \{A, B\}$, where

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

are rank-one matrices. It holds that

$$\lambda_A = 1,\ v_A = [1, 0]^{\mathrm{T}},\ H_A = \ker(A) = \operatorname{span}\left([1, 1]^{\mathrm{T}}\right)$$

and

$$\lambda_B = 1,\ v_B = [0, 1]^{\mathrm{T}},\ H_B = \ker(B) = \operatorname{span}\left([1, -1]^{\mathrm{T}}\right).$$

It is immediate to see that all $P \in \Sigma(\mathcal{F})$ are rank-one matrices and that

$$\mathcal{L}(\mathcal{F}) = \operatorname{span}(\{v_A\}) \cup \operatorname{span}(\{v_B\}) \quad \text{and} \quad \mathcal{H}(\mathcal{F}) = H_A \cup H_B.$$

Moreover, $\operatorname{span}(\{v_A\})$ and $\operatorname{span}(\{v_B\})$ are separated by H_A and H_B. Therefore, either of them is an essential symmetric component of $\mathcal{L}(\mathcal{F})$ which does not span the whole ambient space $\mathbb{R}^2$. □

The following property will be crucial in the sequel.

Definition 23 We say that the family $\mathcal{F}$ is *$\mathcal{L}$-full* if

$$\operatorname{span}\left(\mathcal{L}(\mathcal{F})^{(i)}\right) = \mathbb{R}^d, \qquad i = 1, \ldots, 2r.$$

Remark 9 If $\mathcal{F}$ is $\mathcal{L}$-full, then each cone containing $\mathcal{L}(\mathcal{F})^{(i)}$ is proper.

We conclude this section with a result that generalizes Proposition 9 to an asymptotically rank-one family of matrices.

Proposition 20 *Let the family $\mathcal{F}$ of matrices satisfy Assumption 1. Then*

$$\operatorname{cl}(\mathcal{H}(\mathcal{F})) = \bigcup_{h \in \mathcal{L}(\mathcal{F}^{\mathrm{T}}))\backslash\{0\}} \{h\}^{\perp}.$$

Proof If $x \in \operatorname{cl}(\mathcal{H}(\mathcal{F}))\backslash\{0\}$, there exists a sequence $(x_n)_n$ such that $\lim_{n\to\infty} x_n = x$, where $x_n \in H_{P_n}$ and $P_n \in \Sigma(\mathcal{F})$. Thus, for each n, we can consider a leading eigenvector $h_{P_n} = v_{P_n^{\mathrm{T}}} \in \mathcal{L}(\mathcal{F}^{\mathrm{T}})$ (see Proposition 9). Obviously

$$h_{P_n}^{\mathrm{T}} x_n = 0 \tag{46}$$

and we can assume

$$\|h_{P_n}\| = 1. \tag{47}$$

Since the sequence $(h_{P_n})_n$ is uniformly bounded, it is not restrictive to suppose that there exists a vector h with $\|h\| = 1$ such that $\lim_{n\to\infty} h_{P_n} = h$, so that $h \in \operatorname{cl}(\mathcal{L}(\mathcal{F}^{\mathrm{T}}))\backslash\{0\}$. On the other hand, by (46) we obtain

$$h^{\mathrm{T}} x = (h - h_{P_n})^{\mathrm{T}} x + h_{P_n}^{\mathrm{T}}(x - x_n)$$

and hence, by using the Cauchy–Schwartz inequality and (47), we get

$$|h^{\mathrm{T}} x| \le \|h - h_{P_n}\| \cdot \|x\| + \|x - x_n\|$$

which, for $n \to \infty$, yields $h^{\mathrm{T}} x = 0$.

Conversely, let $x \neq 0$ such that $h^{\mathrm{T}} x = 0$ for some $h \in \mathrm{cl}(\mathcal{L}(\mathcal{F}^{\mathrm{T}})) \backslash \{0\}$. Then there exists a sequence $(h_{P_n})_n$ converging to h and such that $h_{P_n} \in \mathcal{L}(\mathcal{F}^{\mathrm{T}}) \backslash \{0\}$.

Now, for each n we consider the orthogonal projection $\hat{x}_n$ of x onto the hyperplane H_{P_n}. Clearly, Pythagoras' theorem yields

$$\|\hat{x}_n\|^2 = \|x\|^2 - \|x - \hat{x}_n\|^2 < \|x\|^2. \tag{48}$$

Hence, the sequence $(\hat{x}_n)_n$ is uniformly bounded and, consequently, it is not restrictive to assume that it converges to a vector $\hat{x} \in \mathrm{cl}(\mathcal{H}(\mathcal{F}))$.

Assume by contradiction that

$$x - \hat{x} \neq 0. \tag{49}$$

Since $0 \neq x - \hat{x}_n \in (H_{P_n})^{\perp}$, we get $h_{P_n} = \alpha_n (x - \hat{x}_n)$ for some $\alpha_n \neq 0$. Thus, the convergence of h_{P_n} and $\hat{x}_n$ implies the existence of $\lim_{n \to \infty} \alpha_n$, say α, clearly nonzero, and that $h = \alpha(x - \hat{x})$. Since $h^{\mathrm{T}} x = 0$, then $\hat{x}^{\mathrm{T}} x = \|x\|^2$ and, therefore,

$$\|x\|^4 = (\hat{x}^{\mathrm{T}} x)^2 \leq \|\hat{x}\|^2 \cdot \|x\|^2 = \|x\|^4 - \|x\|^2 \cdot \|x - \hat{x}\|^2 < \|x\|^4,$$

where the second equality follows from (48) for $n \to \infty$ and the last strict inequality from $x - \hat{x} \neq 0$ and $x \neq 0$.

We conclude that (49) cannot hold and, hence, that $x = \hat{x} \in \mathrm{cl}(\mathcal{H}(\mathcal{F}))$. □

Corollary 8 *Given a family of matrices $\mathcal{F}$, the following conditions are equivalent:*

(i) $\mathrm{cl}\,(\mathcal{L}(\mathcal{F})) \cap \mathrm{cl}\,(\mathcal{H}(\mathcal{F})) = \{0\}$*;*
(ii) $h^{\mathrm{T}} v \neq 0$ for all $v \in \mathrm{cl}(\mathcal{L}(\mathcal{F})) \backslash \{0\}$ and $h \in \mathrm{cl}(\mathcal{L}(\mathcal{F}^{\mathrm{T}})) \backslash \{0\}$*;*
(iii) $\mathrm{cl}\left(\mathcal{L}(\mathcal{F}^{\mathrm{T}})\right) \cap \mathrm{cl}\left(\mathcal{H}(\mathcal{F}^{\mathrm{T}})\right) = \{0\}$.

In particular, the family $\mathcal{F}$ of matrices satisfies Assumption 1 if and only if the transpose family $\mathcal{F}^{\mathrm{T}}$ does so.

Proof Conditions (i) and (ii) are equivalent by Proposition 20. Moreover, since the latter condition is clearly symmetric in $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ and since $(\mathcal{F}^{\mathrm{T}})^{\mathrm{T}} = \mathcal{F}$, conditions (ii) and (iii) are also equivalent by Proposition 20 applied to $\mathcal{F}^{\mathrm{T}}$.

To conclude the proof, we observe that, as is easy to see, $\mathcal{F}$ is irreducible if and only if $\mathcal{F}^{\mathrm{T}}$ is so and that $\mathcal{F}$ is asymptotically rank-one if and only if $\mathcal{F}^{\mathrm{T}}$ is so (see Remark 4). □

As a consequence of the previous results, we have the following corollary [generalizing Proposition 8-(iii)].

Corollary 9 *Let the family $\mathcal{F}$ of matrices satisfy Assumption 1. Then*

$$\{x \in \mathbb{R}^d \mid Px \in \mathrm{cl}(\mathcal{H}(\mathcal{F})) \text{ for some } P \in \Sigma(\mathcal{F})\} \subseteq \mathrm{cl}(\mathcal{H}(\mathcal{F})).$$

Proof If $Px \in \mathrm{cl}(\mathcal{H}(\mathcal{F}))$, by Proposition 20 there exists

$$h \in \mathrm{cl}\left(\mathcal{L}(\mathcal{F}^{\mathrm{T}})\right) \backslash \{0\} \tag{50}$$

such that

$$(P^{\mathrm{T}} h)^{\mathrm{T}} x = h^{\mathrm{T}} P x = 0. \tag{51}$$

Consequently, $x \in \{P^{\mathrm{T}} h\}^{\perp}$ and thus, again by Proposition 20, we are left to show that $P^{\mathrm{T}} h \in \mathrm{cl}\left(\mathcal{L}(\mathcal{F}^{\mathrm{T}})\right) \backslash \{0\}$.

But this holds since, by Corollary 8, also the transpose family $\mathcal{F}^{\mathrm{T}}$ satisfies Assumption 1 and, therefore, by Corollary 7 applied to the transpose family, $\mathrm{cl}\left(\mathcal{L}(\mathcal{F}^{\mathrm{T}})\right) \backslash \{0\}$ is invariant for $\mathcal{F}^{\mathrm{T}}$. □

5 Existence of the "smallest" invariant multicone

In this section, we prove the existence of an invariant multicone for a finite family of matrices $\mathcal{F}$ under the "Leading set assumptions". Observe that Theorem 10 forces us to look for it among the multicones containing $\mathcal{L}(\mathcal{F})$, i.e. satisfying condition (40).

Let the family $\mathcal{F}$ of matrices be $\mathcal{L}$-full and satisfy Assumption 1 and let r be its spectral fragmentation index. We set

$$K_+^{\mathcal{F}\,(i)} := \text{cone}\left(\mathcal{L}(\mathcal{F})_+^{(i)}\right) \quad \text{and} \quad K_-^{\mathcal{F}\,(i)} := \text{cone}\left(\mathcal{L}(\mathcal{F})_-^{(i)}\right) \tag{52}$$

and

$$K_{mul}^{\mathcal{F}} := \bigcup_{i=1}^{r} K_{sym}^{\mathcal{F}\,(i)}. \tag{53}$$

Since $\mathcal{F}$ is $\mathcal{L}$-full, all the cones (52) are proper by Remark 9.

Proposition 21 *In the above assumptions, the following properties hold:*

(i) $K_{mul} \supseteq K_{mul}^{\mathcal{F}}$ *for any spectral multicone* K_{mul}*;*
(ii) $K_{mul}^{\mathcal{F}}$ *is a proper multicone, spectral itself.*

Thus $K_{mul}^{\mathcal{F}}$ *is the smallest spectral multicone for* $\mathcal{F}$.

Proof (i) By Remark 7, each conic component of a spectral multicone K_{mul} contains exactly one essential conic component. Then $K^{(i)} \supseteq K^{\mathcal{F}(i)}$ for all i.
(ii) Note first that Theorem 13 assures the existence of spectral multicones. Therefore, from the previous inclusion and from the fact that $K_{mul}^{\mathcal{F}}$ is a union of symmetric cones, it follows that $K_{mul}^{\mathcal{F}}$ verifies (m1) and (m2) of Definition 5 and thus is a multicone itself.

Moreover, by (52) and (53), $K_{mul}^{\mathcal{F}}$ has got as many conic components as the number of essential conic components of $\mathcal{L}(\mathcal{F})$, that is $2r$.

We are left to show that $K_{mul}^{\mathcal{F}}$ verifies (40) and (41). Observe first that, by (52), $\text{cl}\left(\mathcal{L}(\mathcal{F})^{(i)}\right) \subseteq K^{\mathcal{F}\,(i)}$ for all $i = 1, \ldots, 2r$, and this implies (40). Finally,

$$\text{int}(K_{mul}^{\mathcal{F}}) \cap \text{cl}(\mathcal{H}(\mathcal{F})) = \emptyset$$

comes immediately from (i) and from the fact that each spectral multicone verifies (41). □

Definition 24 We say that the multicone $K_{mul}^{\mathcal{F}}$ is the *leading multicone* of $\mathcal{F}$.

Now we show that $K_{mul}^{\mathcal{F}}$ is not only spectral, but also verifies a stronger property than (41).

Proposition 22 *The leading multicone* $K_{mul}^{\mathcal{F}}$ *satisfies condition* (29)*, i.e.*

$$K_{mul}^{\mathcal{F}} \cap \text{cl}(\mathcal{H}(\mathcal{F})) = \{0\}, \tag{54}$$

and is reduced.

Proof Property (54) follows from Theorem 13 and Proposition 21-(i).

In order to prove the last claim, recall that $K_{mul}^{\mathcal{F}}$ is spectral by Proposition 21-(ii). Thus, by Proposition 18, for any $i \neq j$ there exists a hyperplane $H_{ij} \in \mathcal{H}(\mathcal{F})$ which weakly separates $K^{\mathcal{F}\,(i)}$ from $K^{\mathcal{F}\,(j)}$.

On the other hand, by (54) we necessarily have that

$$K^{\mathcal{F}\,(i)} \cap H_{ij} = \{0\} = K^{\mathcal{F}\,(j)} \cap H_{ij}.$$

Therefore, H_{ij} strictly separates $K^{\mathcal{F}(i)}$ from $K^{\mathcal{F}(j)}$.

Finally, since $H_{ij} \in \mathcal{H}(\mathcal{F})$, it is a splitting hyperplane for $K^{\mathcal{F}}_{mul}$ by Remark 5. So $K^{\mathcal{F}}_{mul}$ is reduced (see Definition 6). ◻

In order to show the invariance of $K^{\mathcal{F}}_{mul}$, we need a preliminary result which better focuses the invariance of $\mathrm{cl}(\mathcal{L}(\mathcal{F}))$ (see Theorem 14).

Let us use the following notation: consider the decomposition of $\mathcal{L}(\mathcal{F})$ as the union of its $2r$ essential conic components $\mathcal{L}(\mathcal{F})^{(i)}$. So, setting $L_i := \mathrm{cl}(\mathcal{L}(\mathcal{F})^{(i)})$, we can write

$$\mathrm{cl}(\mathcal{L}(\mathcal{F})) = \bigcup_{i=1}^{2r} L_i.$$

Lemma 10 *Assume that $\mathcal{F}$ verifies Assumption 1 and let $P \in \Sigma(\mathcal{F})$. Then for any $i \in \{1, \ldots, 2r\}$ there exists $j \in \{1, \ldots, 2r\}$ such that*

$$P(L_i) \subseteq L_j.$$

Proof If it were not so, there would exist two nonzero vectors $v_1, v_2 \in L_i$ such that $Pv_1 \in L_j \backslash \{0\}$ and $Pv_2 \in L_k \backslash \{0\}$, with $j \neq k$. By Proposition 19, there exists $H_Q \in \mathcal{H}(\mathcal{F})$ which weakly separates L_i from L_j. Therefore, there exists a vector y such that

$$P(\beta v_1 + (1-\beta)v_2) = \beta P v_1 + (1-\beta) P v_2 = y \in H_Q.$$

Let $x := \beta v_1 + (1-\beta)v_2$. Note that $x \in \mathrm{conv}(L_i \backslash \{0\}) = \mathrm{conv}(L_i)\backslash\{0\}$, as is easy to see. Moreover, since $\mathcal{L}(\mathcal{F})^{(i)}$ is positively homogeneous, equality (1) yields

$$\mathrm{conv}(L_i) = \mathrm{conv}(\mathrm{cl}(\mathcal{L}(\mathcal{F})^{(i)})) = K^{\mathcal{F}(i)}$$

and, so, $x \in K^{\mathcal{F}(i)} \backslash \{0\}$.

On the other hand, $Px \in \mathrm{cl}(\mathcal{H}(\mathcal{F}))$. So, by Corollary 9 it follows that $x \in \mathrm{cl}(\mathcal{H}(\mathcal{F}))$. Hence

$$x \in K^{\mathcal{F}}_{mul} \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\},$$

where the above equality follows from Proposition 22, and this is impossible since $x \neq 0$.

◻

Theorem 15 *Let the family $\mathcal{F}$ be $\mathcal{L}$-full and satisfy Assumption 1. Then the leading multicone $K^{\mathcal{F}}_{mul}$ is invariant for $\mathcal{F}$.*

Proof Clearly, it is enough to show that, if $P \in \Sigma(\mathcal{F})$, then for any $i \in \{1, \ldots, 2r\}$ there exists $j \in \{1, \ldots, 2r\}$ such that

$$P(K^{\mathcal{F}(i)}) \subseteq K^{\mathcal{F}(j)}.$$

Since $K^{\mathcal{F}(i)} = \mathrm{cl}(\mathrm{conv}(\mathcal{L}(\mathcal{F})^{(i)}))$, we have

$$P(K^{\mathcal{F}(i)}) \subseteq \mathrm{cl}(P(\mathrm{conv}(\mathcal{L}(\mathcal{F})^{(i)}))) \subseteq \mathrm{cl}(\mathrm{conv}(P(\mathcal{L}(\mathcal{F})^{(i)}))) \subseteq K^{\mathcal{F}(j)}$$

where the first and the second inclusions come from the continuity and the linearity of the map P, respectively, while the last one comes from Lemma 10. ◻

Corollary 10 *Let the family $\mathcal{F}$ be $\mathcal{L}$-full and satisfy Assumption 1. Then both $(K^{\mathcal{F}}_{mul})^{\dagger}$ and $(K^{\mathcal{F}}_{mul})^{\times}$ are invariant for $\mathcal{F}^{\mathrm{T}}$.*

Proof Equation (54) means that (14) and, hence, also (16) hold for any matrix of $\mathcal{F}$. Therefore, we can apply Proposition 15-(i) to the multicone $K_{mul}^{\mathcal{F}}$, invariant for $\mathcal{F}$, obtaining that $(K_{mul}^{\mathcal{F}})^{\dagger}$ is invariant for $\mathcal{F}^{\mathrm{T}}$.

Moreover, Corollary 3 implies that $(K_{mul}^{\mathcal{F}})^{\times}$ is invariant for $\mathcal{F}^{\mathrm{T}}$, too. □

6 Existence of the "biggest" invariant multicone

As illustrated in the previous sections, the leading multicone $K_{mul}^{\mathcal{F}}$ is, in some sense, the "smallest" invariant multicone which can be naturally associated with the family $\mathcal{F}$ of matrices. Now, starting again from the essential fragmentation of $\mathcal{L}(\mathcal{F})$, we want to define an invariant multicone, in some sense the "biggest", containing $\mathcal{L}(\mathcal{F})$ and not intersecting $\mathrm{cl}(\mathcal{H}(\mathcal{F}))$ but on the boundary.

To this purpose, consider an $\mathcal{L}$-full family $\mathcal{F}$ satisfying Assumption 1 and, for each $i = 1, \dots, r$, the essential component $\mathcal{L}(\mathcal{F})_{+}^{(i)}$ of the leading set. Moreover, for any $P \in \Sigma(\mathcal{F})$, denote by $S_{P+}^{(i)}$ the closed semi-space determined by H_P containing $\mathcal{L}(\mathcal{F})_{+}^{(i)}$.

Consider now a spectral multicone $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ and observe that, for any $P \in \Sigma(\mathcal{F})$, its positive conic components satisfy

$$\mathcal{L}(\mathcal{F})_{+}^{(i)} \subseteq K_{+}^{(i)} \subseteq S_{P+}^{(i)},$$

since K_{mul} verifies (40) and (41). Hence,

$$K_{+}^{(i)} \subseteq \bigcap_{P \in \Sigma(\mathcal{F})} S_{P+}^{(i)}.$$

Therefore, it comes natural to define the set

$$\bar{K}_{+}^{\mathcal{F}\,(i)} := \bigcap_{P \in \Sigma(\mathcal{F})} S_{P+}^{(i)}. \tag{55}$$

Remark 10 From the above argument, it is clear that, if $K_{mul} = \bigcup_{i=1}^{r} K_{sym}^{(i)}$ is a spectral multicone, then

$$K_{+}^{(i)} \subseteq \bar{K}_{+}^{\mathcal{F}\,(i)}$$

for all $i = 1, \dots, r$. Obviously, the same holds for the opposite components.

Proposition 23 *For each $i = 1, \dots, r$, the set $\bar{K}_{+}^{\mathcal{F}\,(i)}$ is a proper cone and verifies*

$$K_{+}^{\mathcal{F}\,(i)} \backslash \{0\} \subseteq \mathrm{int}(\bar{K}_{+}^{\mathcal{F}\,(i)}), \tag{56}$$

$$\mathrm{int}(\bar{K}_{+}^{\mathcal{F}\,(i)}) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \emptyset \tag{57}$$

and

$$\partial \bar{K}_{+}^{\mathcal{F}\,(i)} \subset \mathrm{cl}(\mathcal{H}(\mathcal{F})). \tag{58}$$

Proof With reference to Definition 1, it is clear that the set $\bar{K}_{+}^{\mathcal{F}\,(i)}$ is positively homogeneous. Now assume by contradiction that it is not salient, i.e. there exists $x \neq 0$ such that $\mathrm{span}(x) \subset \bar{K}_{+}^{\mathcal{F}\,(i)}$, which means that $\mathrm{span}(x) \subset S_{P+}^{(i)}$ for all $P \in \Sigma(\mathcal{F})$ and, consequently, $\mathrm{span}(x) \subset H_P$ for all $P \in \Sigma(\mathcal{F})$.

On the other hand, by Proposition 9, we have that

$$v_{P^{\mathrm{T}}}^{\mathrm{T}} x = 0 \quad \text{for all } P^{\mathrm{T}} \in \Sigma(\mathcal{F}^{\mathrm{T}}),$$

but this is impossible because $\mathcal{L}(\mathcal{F}^{\mathrm{T}})$ spans the whole $\mathbb{R}^d$ by Corollaries 8 and 6 applied to the transpose family $\mathcal{F}^{\mathrm{T}}$. Finally, $\bar{K}_+^{\mathcal{F}(i)}$ is solid since it contains $\mathcal{L}(\mathcal{F})_+^{(i)}$ (see Remark 9), so it is a proper cone.

In order to show (56), just observe that $K_+^{\mathcal{F}(i)} \subseteq \bar{K}_+^{\mathcal{F}(i)}$. Now, using (54) and (58), we obtain $K_+^{\mathcal{F}(i)} \cap \partial \bar{K}_+^{\mathcal{F}(i)} = \{0\}$ as required.

In order to show (57), let us recall that

$$\mathrm{int}\left(\bar{K}_+^{\mathcal{F}(i)}\right) \subseteq \bigcap_{P \in \Sigma(\mathcal{F})} \mathrm{int}\left(S_{P+}^{(i)}\right).$$

In particular, for each $x \in \mathrm{int}\left(\bar{K}_+^{\mathcal{F}(i)}\right)$ it holds that $x \notin H_P$ for all $P \in \Sigma(\mathcal{F})$. Therefore, $x \notin \mathcal{H}(\mathcal{F})$ and, so, $\mathrm{int}(\bar{K}_+^{\mathcal{F}(i)}) \cap \mathcal{H}(\mathcal{F}) = \emptyset$. By standard topological arguments we get the required equality.

Now let us prove that, if $x \in \bar{K}_+^{\mathcal{F}(i)} \setminus \mathrm{cl}(\mathcal{H}(\mathcal{F}))$, then $x \in \mathrm{int}(\bar{K}_+^{\mathcal{F}(i)})$.

Clearly, there exists $\delta > 0$ such that $B(x, \delta) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \emptyset$. In particular, $B(x, \delta) \cap H_p = \emptyset$ for all $P \in \Sigma(\mathcal{F})$. Therefore, $B(x, \delta) \subseteq S_{P+}^{(i)}$ for all $P \in \Sigma(\mathcal{F})$ and so (58) is proved. $\square$

Consequently, with $\bar{K}_-^{\mathcal{F}(i)} := -\bar{K}_+^{\mathcal{F}(i)}$, $i = 1, \ldots, r$, we obtain the symmetric cones

$$\bar{K}_{sym}^{\mathcal{F}(i)} := \bar{K}_+^{\mathcal{F}(i)} \cup \bar{K}_-^{\mathcal{F}(i)}. \tag{59}$$

Instead of proving directly that $\bigcup_{i=1}^r \bar{K}_{sym}^{\mathcal{F}(i)}$ verifies conditions (m1) and (m2) of Definition 5, i.e. that it is a multicone, we shall prove that each $\bar{K}_{sym}^{\mathcal{F}(i)}$ is a symmetric component of another multicone. To the aim of doing this, we consider the transpose family $\mathcal{F}^{\mathrm{T}} = \{A_1^{\mathrm{T}}, \ldots, A_m^{\mathrm{T}}\}$ and the fragmentation of its leading set $\mathcal{L}(\mathcal{F}^{\mathrm{T}})$ into its essential conic components $\mathcal{L}(\mathcal{F}^{\mathrm{T}})^{(k)}$, $k = 1, \ldots, 2s$, i.e.

$$\mathcal{L}(\mathcal{F}^{\mathrm{T}}) = \bigcup_{k=1}^{2s} \mathcal{L}(\mathcal{F}^{\mathrm{T}})^{(k)}.$$

The number s of essential symmetric components of $\mathcal{L}(\mathcal{F}^{\mathrm{T}})$ (i.e. the spectral fragmentation index of $\mathcal{F}^{\mathrm{T}}$) does not necessarily equal r, the one of $\mathcal{L}(\mathcal{F})$.

Recall that Assumption 1 on $\mathcal{F}$ is transmitted to $\mathcal{F}^{\mathrm{T}}$ (see Corollary 8). On the contrary, it is not difficult to convince oneself that the property of being $\mathcal{L}$-full is not transmitted. Therefore, from now on it will be necessary to assume explicitly that also $\mathcal{F}^{\mathrm{T}}$ is $\mathcal{L}$-full, i.e. that

$$\mathrm{span}\left(\mathcal{L}(\mathcal{F}^{\mathrm{T}})^{(k)}\right) = \mathbb{R}^d, \qquad k = 1, \ldots, 2s.$$

The above hypothesis allows us to perform the construction of the leading multicone also for $\mathcal{F}^{\mathrm{T}}$, obtaining $K_{mul}^{\mathcal{F}^{\mathrm{T}}}$ and, in particular, by Proposition 22, that

$$K_{mul}^{\mathcal{F}^{\mathrm{T}}} \cap \mathrm{cl}(\mathcal{H}(\mathcal{F}^{\mathrm{T}})) = \{0\}.$$

In this framework, now we investigate the relations between the conic components of the dual multicone $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\dagger}$ and the cones $\bar{K}_{\pm}^{\mathcal{F}(i)}$.

Lemma 11 *Let $K^{\mathcal{F}(i)}$ and $K^{\mathcal{F}^{\mathrm{T}}(k)}$ be any two conic components of the leading multicones of $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$, respectively. Then for all $v \in K^{\mathcal{F}(i)}\backslash\{0\}$ and for all $h_1, h_2 \in K^{\mathcal{F}^{\mathrm{T}}(k)}\backslash\{0\}$ it holds that*

$$(v^{\mathrm{T}}h_1)(v^{\mathrm{T}}h_2) > 0. \tag{60}$$

Proof Consider $h_1, h_2 \in \mathrm{cl}(\mathcal{L}(\mathcal{F}^{\mathrm{T}})^{(k)})\backslash\{0\}$. By Proposition 22 applied to the family $\mathcal{F}^{\mathrm{T}}$, these vectors are not separated by $\mathrm{cl}(\mathcal{H}(\mathcal{F}^{\mathrm{T}}))$.

Therefore, using the expression of $\mathrm{cl}(\mathcal{H}(\mathcal{F}^{\mathrm{T}}))$ given in Proposition 20, we get (60) for all $v \in \mathrm{cl}(\mathcal{L}((\mathcal{F}^{\mathrm{T}})^{\mathrm{T}}))\backslash\{0\} = \mathrm{cl}(\mathcal{L}(\mathcal{F}))\backslash\{0\}$. In particular, (60) holds for all $v \in \mathrm{cl}(\mathcal{L}(\mathcal{F})^{(i)})\backslash\{0\}$ and $h_1, h_2 \in \mathrm{cl}(\mathcal{L}(\mathcal{F}^{\mathrm{T}})^{(k)})\backslash\{0\}$.

We complete the proof by using (1) and standard convexity arguments. □

Lemma 12 *For any $i = 1, \ldots, r$ and $k = 1, \ldots, 2r^{\dagger}$ one of the following two possibilities necessarily occurs:*

$$K^{\mathcal{F}^{\mathrm{T}}(k)}\backslash\{0\} \subseteq \mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*}) \quad \text{or} \quad K^{\mathcal{F}^{\mathrm{T}}(k)}\backslash\{0\} \subseteq \mathrm{int}((K_{-}^{\mathcal{F}(i)})^{*}). \tag{61}$$

Consequently,

$$\mathrm{cone}\left(K_{mul}^{\mathcal{F}^{\mathrm{T}}} \cap \mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*})\right) \cup \mathrm{cone}\left(K_{mul}^{\mathcal{F}^{\mathrm{T}}} \cap \mathrm{int}((K_{-}^{\mathcal{F}(i)})^{*})\right)$$

is a symmetric cone associated with $K_{mul}^{\mathcal{F}^{\mathrm{T}}}$.

Proof Observe that (61) follows from Lemma 11 and implies that $\mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*})$ induces a partition of $K_{mul}^{\mathcal{F}^{\mathrm{T}}}$ in two unions of conic components, symmetric to each other. By the convexity of the cone $\mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*})$, we obtain the last claim (see Definition 10). □

Theorem 16 *Let both the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ be $\mathcal{L}$-full and satisfy Assumption 1. Then each $\bar{K}_{sym}^{\mathcal{F}(i)}$ is a symmetric component of the proper dual multicone $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\times}$.*

Proof First observe that $\bar{K}_{+}^{\mathcal{F}(i)}$ is a proper cone (see Proposition 23). Therefore, by Theorem 2 (which describes $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\dagger}$ in terms of associated cones) and by Lemma 12, it is enough to show that, for all $i = 1, \ldots, r$,

$$\bar{K}_{+}^{\mathcal{F}(i)} = \left(\mathrm{cone}\left(K_{mul}^{\mathcal{F}^{\mathrm{T}}} \cap \mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*})\right)\right)^{*}. \tag{62}$$

Note first that by definition $S_{P+}^{(i)} = \{h_P^{(i)}\}^{*}$ where $(h_P^{(i)})^{\mathrm{T}}v > 0$ for all $v \in \mathrm{cl}(\mathcal{L}(\mathcal{F})_{+}^{(i)})\backslash\{0\}$. It is easy to see that this last condition is equivalent to $(h_P^{(i)})^{\mathrm{T}}v > 0$ for all $v \in K_{+}^{\mathcal{F}(i)}\backslash\{0\}$, i.e. to $h_P^{(i)} \in \mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*})$.

On the other hand, $h_P^{(i)} = v_{P^{\mathrm{T}}} \in \mathcal{L}(\mathcal{F}^{\mathrm{T}})$, so

$$\bar{K}_{+}^{\mathcal{F}(i)} = \bigcap_{h \in W_i} \{h\}^{*}, \quad \text{where} \quad W_i := \mathcal{L}(\mathcal{F}^{\mathrm{T}}) \cap \mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*}),$$

and hence, by Remark 1,

$$\bar{K}_{+}^{\mathcal{F}(i)} = \left(\mathcal{L}(\mathcal{F}^{\mathrm{T}}) \cap \mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*})\right)^{*}.$$

Finally recall that, as can be found in [6],

$$\left(\mathcal{L}(\mathcal{F}^{\mathrm{T}}) \cap \mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*})\right)^{*} = \left(\mathrm{cone}\left(\mathcal{L}(\mathcal{F}^{\mathrm{T}}) \cap \mathrm{int}((K_{+}^{\mathcal{F}(i)})^{*})\right)\right)^{*}.$$

Therefore, in order to prove (62), it is enough to show that

$$\operatorname{cone}\left(\mathcal{L}(\mathcal{F}^{\mathrm{T}}) \cap \operatorname{int}((K_+^{\mathcal{F}(i)})^*)\right) = \operatorname{cone}\left(K_{mul}^{\mathcal{F}^{\mathrm{T}}} \cap \operatorname{int}((K_+^{\mathcal{F}(i)})^*)\right)$$

and this holds by using (61) and (1). □

At this point we can introduce the multicone collecting the symmetric cones defined in (59).

Definition 25 The proper multicone

$$\bar{K}_{mul}^{\mathcal{F}} := \bigcup_{i=1}^{r} \bar{K}_{sym}^{\mathcal{F}(i)}$$

is called the *secondary multicone* of $\mathcal{F}$.

In analogy to what we saw for the leading multicone $K_{mul}^{\mathcal{F}}$, we have the following facts concerning the secondary multicone.

Proposition 24 *Let both the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ be $\mathcal{L}$-full and satisfy Assumption 1. Then the following properties hold:*

(i) *$K_{mul} \subseteq \bar{K}_{mul}^{\mathcal{F}}$ for any spectral multicone K_{mul};*
(ii) *the proper multicone $\bar{K}_{mul}^{\mathcal{F}}$ is spectral itself.*

Thus $\bar{K}_{mul}^{\mathcal{F}}$ is the biggest spectral multicone for $\mathcal{F}$.

Proof (i) It immediately follows from Remark 10.
(ii) The fragmentation index of $\bar{K}_{mul}^{\mathcal{F}}$ is the right one by construction. Moreover, (40) follows from (56) and (41) from (57). □

Now we show that $\bar{K}_{mul}^{\mathcal{F}}$ is not only spectral, but also verifies a stronger property than (40).

Proposition 25 *The secondary multicone $\bar{K}_{mul}^{\mathcal{F}}$ satisfies condition (27), i.e.*

$$\operatorname{cl}(\mathcal{L}(\mathcal{F}))\backslash\{0\} \subseteq \operatorname{int}(\bar{K}_{mul}^{\mathcal{F}}),$$

and is reduced.

Proof The inclusion follows immediately from (56) and from the definition of $K_{mul}^{\mathcal{F}}$. The reducibility follows from Theorem 16 and Proposition 6. □

Proposition 23 and Theorem 16 also give rise to the chain of inclusions

$$K_{mul}^{\mathcal{F}}\backslash\{0\} \subseteq \operatorname{int}(\bar{K}_{mul}^{\mathcal{F}}) \subseteq \operatorname{int}((K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\times}) \tag{63}$$

and, with reference to Definition 17, clearly assure the following result.

Proposition 26 *Let both the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ be $\mathcal{L}$-full and satisfy Assumptions 1. Then $\bar{K}_{mul}^{\mathcal{F}}$ is the submulticone of $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\times}$ covering $K_{mul}^{\mathcal{F}}$.*

Indeed, the second inclusion in (63) may be actually strict. If this is the case, then $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\times}$ have got some more symmetric components than $\bar{K}_{mul}^{\mathcal{F}}$ and precisely those which do not intersect the leading set $\mathcal{L}(\mathcal{F})$.

Remark 11 Assuming as usual that both $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ are $\mathcal{L}$-full and satisfy Assumption 1, we can apply Theorem 15 to the family $\mathcal{F}^{\mathrm{T}}$ obtaining that $K_{mul}^{\mathcal{F}^{\mathrm{T}}}$ is invariant for $\mathcal{F}^{\mathrm{T}}$. Consequently, (23) yields

$$\mathrm{int}((K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\times}) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \emptyset$$

since $(\mathcal{F}^{\mathrm{T}})^{\mathrm{T}} = \mathcal{F}$, and this is consistent to and confirms (57).

We conclude this section with the last of the main results of the paper.

Theorem 17 *Let both the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ be $\mathcal{L}$-full and satisfy Assumption 1. Then the multicones $\bar{K}_{mul}^{\mathcal{F}}$, $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\dagger}$ and $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\times}$ are invariant for $\mathcal{F}$.*

Moreover, also the open set $\mathrm{int}(\bar{K}_{mul}^{\mathcal{F}})$ is invariant for $\mathcal{F}$.

Proof We have that $K_{mul}^{\mathcal{F}^{\mathrm{T}}}$ is invariant for $\mathcal{F}^{\mathrm{T}}$ (see Remark 11). Thus, Corollary 10 implies that $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\dagger}$ and $(K_{mul}^{\mathcal{F}^{\mathrm{T}}})^{\times}$ are invariant for $(\mathcal{F}^{\mathrm{T}})^{\mathrm{T}} = \mathcal{F}$. Consequently, the invariance of $\bar{K}_{mul}^{\mathcal{F}}$ follows from the invariance of $K_{mul}^{\mathcal{F}}$, inclusions (63) and Proposition 16.

Finally, let $x \in \mathrm{int}(\bar{K}_{mul}^{\mathcal{F}})$. If $Px \notin \mathrm{int}(\bar{K}_{mul}^{\mathcal{F}})$, then $Px \in \partial(\bar{K}_{mul}^{\mathcal{F}})$ and hence, by (58), $Px \in \mathrm{cl}(\mathcal{H}(\mathcal{F}))$. Therefore, by Corollary 9, $x \in \mathrm{cl}(\mathcal{H}(\mathcal{F}))$ contradicting (57). $\square$

7 Basic computational procedure

We conclude the paper with the outline of a procedure able to compute the "smallest" and the "biggest" invariant multicones $K_{mul}^{\mathcal{F}}$ and $\bar{K}_{mul}^{\mathcal{F}}$ (and also $K_{mul}^{\mathcal{F}^{\mathrm{T}}}$ and $\bar{K}_{mul}^{\mathcal{F}^{\mathrm{T}}}$) of a family $\mathcal{F}$ of matrices under Assumption 1 in the case that both $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ are $\mathcal{L}$-full.

The algorithm we are going to propose is of iterative type. Although it is assured to converge in infinitely many iterations, it would be desirable that it ends successfully in a finite number of them. Therefore, we shall give a criterion to recognize this nice occurrence, which is actually feasible.

The idea is to provide finite systems of generators (i.e. sets of *edges*) of certain *polyhedral* multicones, say K_{mul} and $\bar{K}_{mul}$, which approximate or, possibly, coincide with $K_{mul}^{\mathcal{F}}$ and $\bar{K}_{mul}^{\mathcal{F}}$.

Unfortunately, apart from the irreducibility of $\mathcal{F}$ (very easy to check), in most of practical cases we do not know a priori whether the hypotheses made above are satisfied or not and, therefore, the iterations might not converge. For this reason, the algorithm is designed in such a way that the assumptions on the matrices can be monitored in real time while running. Should we find that one of them fails to hold, the procedure would be stopped without any output. In particular, whenever we process a matrix $P \in \Sigma(\mathcal{F})$, we verify whether it is asymptotically rank-one or not.

Even when the algorithm ends successfully in a finite number of iterations, it furnishes an embedded pair of invariant multicones without assuring that the family $\mathcal{F}$ is asymptotically rank-one and, consequently, without assuring the existence of $\mathcal{L}(\mathcal{F})$ and $\mathcal{H}(\mathcal{F})$ (and hence the existence of $K_{mul}^{\mathcal{F}}$ and $\bar{K}_{mul}^{\mathcal{F}}$). Therefore, a fortiori, we are not sure that the computed multicones K_{mul} and $\bar{K}_{mul}$ actually coincide with $K_{mul}^{\mathcal{F}}$ and $\bar{K}_{mul}^{\mathcal{F}}$. This is due to the fact that we are not able to establish whether a family $\mathcal{F}$ is asymptotically rank-one or not from the sole knowledge of a finite subset of $\Sigma(\mathcal{F})$.

In the light of the foregoing discussion, now we present the theoretical background of our algorithm, assuming that the family $\mathcal{F}$ is irreducible but not necessarily asymptotically rank-one. This requirement is imposed on the generators A_i of $\mathcal{F}$ and on the various matrices $P \in \Sigma(\mathcal{F})$ involved by the procedure only. Also the separation between the leading and the secondary set is checked limited to the same matrices.

Let us first recall that, for any asymptotically rank-one matrix P, we have $\mathbb{R}^d = V_P \oplus H_P$, where V_P is the leading eigenspace, H_P is the secondary hyperplane and it holds that $V_{P^{\mathrm{T}}} = H_P^{\perp}$ [see (13)].

Consider finite subsets $\tilde{\Sigma}$ and $\tilde{\Omega}$ of $\Sigma(\mathcal{F})$ and a pair of (finite) sets $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}^{\mathrm{T}}$ satisfying the following properties:

(p1) any $P \in \tilde{\Sigma} \cup \tilde{\Omega}$ is asymptotically rank-one;
(p2) $\tilde{\mathcal{L}} = \tilde{\mathcal{V}} \cup \widetilde{\mathcal{W}}$, where

$$\tilde{\mathcal{V}} := \{V_P \mid P \in \tilde{\Sigma}\}$$

and $\widetilde{\mathcal{W}}$ is a finite subset (possibly empty) of

$$\left\{Q(V_R) \nsubseteq \mathcal{L}(\mathcal{F}) \mid Q \in \tilde{\Omega} \text{ and } R \in \tilde{\Sigma}\right\};$$

(p3) $\tilde{\mathcal{L}}^{\mathrm{T}} = \tilde{\mathcal{V}}^{\mathrm{T}} \cup \widetilde{\mathcal{W}}^{\mathrm{T}}$, where

$$\tilde{\mathcal{V}}^{\mathrm{T}} := \{V_{P^{\mathrm{T}}} \mid P \in \tilde{\Sigma}\}$$

and $\widetilde{\mathcal{W}}^{\mathrm{T}}$ is a finite subset (possibly empty) of

$$\{\bar{Q}^{\mathrm{T}}(V_{\bar{R}^{\mathrm{T}}}) \nsubseteq \mathcal{L}(\mathcal{F}^{\mathrm{T}}) \mid \bar{Q} \in \tilde{\Omega} \text{ and } \bar{R} \in \tilde{\Sigma}\}.$$

Remark 12 If Assumption 1 holds, property (p1) is necessarily true.

Note also that $\widetilde{\mathcal{W}}$ and $\widetilde{\mathcal{W}}^{\mathrm{T}}$ are not necessarily related, although $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{L}}^{\mathrm{T}}$ are strictly linked to each other.

Then we define

$$\tilde{\mathcal{H}} := \bigcup_{h \in \tilde{\mathcal{L}}^{\mathrm{T}} \setminus \{0\}} \{h\}^{\perp} \quad \text{and} \quad \tilde{\mathcal{H}}^{\mathrm{T}} := \bigcup_{v \in \tilde{\mathcal{L}} \setminus \{0\}} \{v\}^{\perp}$$

and assume that

(p4) $\tilde{\mathcal{L}} \cap \tilde{\mathcal{H}} = \{0\}$ and $\tilde{\mathcal{L}}^{\mathrm{T}} \cap \tilde{\mathcal{H}}^{\mathrm{T}} = \{0\}$.

Remark that, since $\tilde{\Sigma}$ and $\tilde{\Omega}$ are finite, property (p4) may be actually checked.

Reasoning like in the proof of Lemma 9 and in Remark 8, we easily find that, if $R \in \tilde{\Sigma}$ (assuming without restriction that $\lambda_R = 1$), a vector Qv_R belonging to $\widetilde{\mathcal{W}}$ is leading eigenvector of the rank-one limit matrix QR^{∞} of the sequence of products QR^n as $n \to \infty$, i.e. $Qv_R = v_{QR^{\infty}}$. In fact, property (p4) assures that $Qv_R \notin H_R$.

Moreover, still by using similar arguments it is easy to see that the vector $h_{QR^{\infty}}$ coincides with h_R, the leading eigenvector of R^{T}. Since $v_R \in \tilde{\mathcal{V}}$, it holds that $h_{QR^{\infty}} = h_R \in \tilde{\mathcal{V}}^{\mathrm{T}} \subseteq \tilde{\mathcal{L}}^{\mathrm{T}}$.

Summarizing, the vectors of $\widetilde{\mathcal{W}}$ are leading eigenvectors of rank-one limit matrices whose secondary hyperplanes are orthogonal to vectors of $\tilde{\mathcal{V}}^{\mathrm{T}} \subseteq \tilde{\mathcal{L}}^{\mathrm{T}}$. Hence, such hyperplanes are already included in $\tilde{\mathcal{H}}$, i.e.

$$Q(V_R) = V_{QR^{\infty}} \subseteq \tilde{\mathcal{L}} \quad \Longrightarrow \quad H_{QR^{\infty}} = \{h_R\}^{\perp} \subseteq \tilde{\mathcal{H}}.$$

Vice versa, the hyperplane $\{Qv_R\}^{\perp} = \{v_{QR^{\infty}}\}^{\perp}$, which is contained in $\tilde{\mathcal{H}}^{\mathrm{T}}$, is the secondary hyperplane of the rank-one limit matrix $(QR^{\infty})^{\mathrm{T}} = (R^{\mathrm{T}})^{\infty} Q^{\mathrm{T}}$ of the sequence of transpose products $(R^{\mathrm{T}})^n Q^{\mathrm{T}}$ as $n \to \infty$. However, similarly as before, we can see that the leading eigenvector $v_{(R^{\mathrm{T}})^\infty Q^{\mathrm{T}}}$ of $(R^{\mathrm{T}})^\infty Q^{\mathrm{T}}$ is equal to $v_{R^{\mathrm{T}}} \in \tilde{\mathcal{V}}^{\mathrm{T}} \subseteq \tilde{\mathcal{L}}^{\mathrm{T}}$.

Summarizing again, the hyperplanes of $\tilde{\mathcal{H}}^{\mathrm{T}}$ of the type $\{Qv_R\}^\perp$ are secondary hyperplanes of rank-one limit matrices whose leading eigenvectors belong to $\tilde{\mathcal{V}}^{\mathrm{T}} \subseteq \tilde{\mathcal{L}}^{\mathrm{T}}$. Hence, such leading eigenvectors are already included in $\tilde{\mathcal{L}}^{\mathrm{T}}$, i.e.

$$\{Qv_R\}^\perp = H_{(R^{\mathrm{T}})^\infty Q^{\mathrm{T}}} \subseteq \tilde{\mathcal{H}}^{\mathrm{T}} \quad \Longrightarrow \quad V_{(R^{\mathrm{T}})^\infty Q^{\mathrm{T}}} = V_{R^{\mathrm{T}}} \subseteq \tilde{\mathcal{L}}^{\mathrm{T}}.$$

Of course, similar arguments apply to the vectors $\bar{Q}^{\mathrm{T}} h_{\bar{R}} \in \tilde{\mathcal{L}}^{\mathrm{T}}$, so that

$$\bar{Q}^{\mathrm{T}}(V_{\bar{R}^{\mathrm{T}}}) = V_{\bar{Q}^{\mathrm{T}}(\bar{R}^{\mathrm{T}})^\infty} \subseteq \tilde{\mathcal{L}}^{\mathrm{T}} \quad \Longrightarrow \quad H_{\bar{Q}^{\mathrm{T}}(\bar{R}^{\mathrm{T}})^\infty} = \{v_{\bar{R}}\}^\perp \subseteq \tilde{\mathcal{H}}^{\mathrm{T}}$$

and

$$\{\bar{Q}^{\mathrm{T}} v_{\bar{R}}\}^\perp = H_{\bar{R}^\infty \bar{Q}} \subseteq \tilde{\mathcal{H}} \quad \Longrightarrow \quad V_{\bar{R}^\infty \bar{Q}} = V_{\bar{R}} \subseteq \tilde{\mathcal{L}}.$$

Remark 13 If Assumption 1 holds, from the above discussion it appears that

$$\tilde{\mathcal{L}} \subseteq \mathrm{cl}(\mathcal{L}(\mathcal{F})) \quad \text{and} \quad \tilde{\mathcal{L}}^{\mathrm{T}} \subseteq \mathrm{cl}(\mathcal{L}(\mathcal{F}^{\mathrm{T}})) \tag{64}$$

and, dually, that

$$\tilde{\mathcal{H}} \subseteq \mathrm{cl}(\mathcal{H}(\mathcal{F})) \quad \text{and} \quad \tilde{\mathcal{H}}^{\mathrm{T}} \subseteq \mathrm{cl}(\mathcal{H}(\mathcal{F}^{\mathrm{T}})). \tag{65}$$

Therefore, in this case, property (p4) is necessarily true.

In Sect. 4 we saw that the decomposition of $\mathcal{L}(\mathcal{F})$ into its essential symmetric components is only induced by the secondary set $\mathcal{H}(\mathcal{F})$. In the same way, we consider the (uniquely determined) decomposition of $\tilde{\mathcal{L}}$ induced by $\tilde{\mathcal{H}}$. We denote it by

$$\tilde{\mathcal{L}} = \bigcup_{i=1}^{\tilde{r}} \left(\tilde{\mathcal{L}}_+^{(i)} \cup \tilde{\mathcal{L}}_-^{(i)} \right),$$

where the positive and the negative parts, called again the *essential components*, are determined by the choice of a particular hyperplane $\hat{H} \in \tilde{\mathcal{H}}$.

Remark 14 If Assumption 1 holds, by (64) and (65) we have that $\tilde{r} \leq r$, where r is the spectral fragmentation index of $\mathcal{F}$.

We also make the following assumption:

(p5) $\mathrm{span}(\tilde{\mathcal{L}}_\pm^{(i)}) = \mathbb{R}^d, \quad i = 1, \ldots, \tilde{r}$.

In our terminology, *polyhedral multicones* are those having polyhedral cones as conic components. Recall that a *polyhedral cone* can be defined either as the cone generated by a finite number of half-lines or, equivalently, as a finite intersection of semi-spaces.

In analogy to the notion of $K_{mul}^{\mathcal{F}}$, we introduce the corresponding multicone associated with $\tilde{\mathcal{L}}$.

Definition 26 The multicone

$$K_{mul} := \bigcup_{i=1}^{\tilde{r}} K_{sym}^{(i)},$$

where $K_{sym}^{(i)} := K_+^{(i)} \cup K_-^{(i)}$ with

$$K_+^{(i)} := \mathrm{cone}(\tilde{\mathcal{L}}_+^{(i)}) \quad \text{and} \quad K_-^{(i)} := \mathrm{cone}(\tilde{\mathcal{L}}_-^{(i)}), \tag{66}$$

is called the *partial leading multicone* corresponding to $\tilde{\mathcal{L}}$.

Note that, since $\tilde{\mathcal{L}}$ is the union of a finite number of straight lines, K_{mul} is of polyhedral type and so we can find, uniquely determined, a minimum subset of them, say $\tilde{\mathcal{E}}$, such that, with $\tilde{\mathcal{E}}_\pm^{(i)} := \tilde{\mathcal{E}} \cap \tilde{\mathcal{L}}_\pm^{(i)}$, $i = 1, \ldots, \tilde{r}$, relations (66) become

$$K_+^{(i)} := \mathrm{cone}(\tilde{\mathcal{E}}_+^{(i)}) \quad \text{and} \quad K_-^{(i)} := \mathrm{cone}(\tilde{\mathcal{E}}_-^{(i)}). \tag{67}$$

Remark that, in the geometric terminology, the half-lines constituting $\tilde{\mathcal{E}}_\pm^{(i)}$ are the *edges* of the cone $K_\pm^{(i)}$. Hence, for the sake of brevity, $\tilde{\mathcal{E}}$ will be called the set of the *edges* of K_{mul}.

As done before, we give the notion analogous to $\bar{K}_{mul}^{\mathcal{F}}$ in the polyhedral case.

Definition 27 Consider the set

$$\bar{K}_{mul} := \bigcup_{i=1}^{\tilde{r}} \bar{K}_{sym}^{(i)},$$

where

$$\bar{K}_{sym}^{(i)} := \bar{K}_+^{(i)} \cup \bar{K}_-^{(i)}, \qquad \bar{K}_+^{(i)} := \bigcap_{h \in \tilde{\mathcal{L}}^{\mathrm{T}} \setminus \{0\}} S_{h+}^{(i)}, \tag{68}$$

and, as usual, for any $h \in \tilde{\mathcal{L}}^{\mathrm{T}} \setminus \{0\}$, we denote by $S_{h+}^{(i)}$ the closed semi-space determined by $H = \{h\}^\perp$ containing $K_+^{(i)}$. Such a set will be called the *partial secondary multicone* corresponding to $\tilde{\mathcal{L}}$.

Note that similar arguments to those used in the proof of Proposition 23 allow us to conclude that the sets $\bar{K}_+^{(i)}$ are proper cones.

Similarly as in Sect. 6, the forthcoming Proposition 27 assures that $\bar{K}_{mul}$ is actually a multicone (in particular, that the cones $\bar{K}_+^{(i)}$ satisfy conditions (m1) and (m2) of Definition 5). Moreover, it is polyhedral since each of its conic components is a finite intersection of semi-spaces.

As done before, starting from the pair of sets $\tilde{\mathcal{L}}^{\mathrm{T}}$ and $\tilde{\mathcal{H}}^{\mathrm{T}}$, we arrive at defining the essential components, which we assume to be such that

(p6) $\mathrm{span}(\tilde{\mathcal{L}}_\pm^{T\,(i)}) = \mathbb{R}^d, \qquad i = 1, \ldots, \tilde{s}.$

Remark 15 Here, if Assumption 1 holds, still by (64) and (65) we have that $\tilde{s} \leq s$, where s is the spectral fragmentation index of $\mathcal{F}^{\mathrm{T}}$.

In analogy with the previous Definitions 26 and 27, we introduce the multicones K_{mul}^{T} and $\bar{K}_{mul}^{\mathrm{T}}$, referred to the transpose family $\mathcal{F}^{\mathrm{T}}$.

Remark 16 Similarly to what we saw in Proposition 23, the just defined partial leading and secondary multicones fulfil the following properties:

$$K_{mul} \setminus \{0\} \subseteq \mathrm{int}(\bar{K}_{mul}), \qquad \mathrm{int}(\bar{K}_{mul}) \cap \tilde{\mathcal{H}} = \emptyset \quad \text{and} \quad \partial \bar{K}_{mul} \subseteq \tilde{\mathcal{H}}$$

and

$$K_{mul}^{\mathrm{T}} \setminus \{0\} \subseteq \mathrm{int}(\bar{K}_{mul}^{\mathrm{T}}), \qquad \mathrm{int}(\bar{K}_{mul}^{\mathrm{T}}) \cap \tilde{\mathcal{H}}^{\mathrm{T}} = \emptyset \quad \text{and} \quad \partial \bar{K}_{mul}^{\mathrm{T}} \subseteq \tilde{\mathcal{H}}^{\mathrm{T}}.$$

Definition 28 Any pair $(\tilde{\mathcal{L}}, \tilde{\mathcal{L}}^{\mathrm{T}})$ enjoying the properties (p1–p6) is called a *consistent pair of sets of leading eigenvectors for the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$*.

Reasoning in the same way as in Sect. 6, we obtain the following analogues of Theorem 16 and Proposition 26.

Proposition 27 *Let $(\tilde{\mathcal{L}}, \tilde{\mathcal{L}}^{\mathrm{T}})$ be a consistent pair of sets of leading eigenvectors for the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$. Then each symmetric component of the partial secondary multicones $\bar{K}_{mul}$ (respectively, $\bar{K}^{\mathrm{T}}_{mul}$) is a symmetric component of the proper dual $(K^{\mathrm{T}}_{mul})^{\times}$ (respectively, $K^{\times}_{mul}$).*

Proposition 28 *Let $(\tilde{\mathcal{L}}, \tilde{\mathcal{L}}^{\mathrm{T}})$ be a consistent pair of sets of leading eigenvectors for the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$. Then $\bar{K}_{mul}$ is the submulticone of $(K^{\mathrm{T}}_{mul})^{\times}$ covering K_{mul} and, dually, that $\bar{K}^{\mathrm{T}}_{mul}$ is the submulticone of $K^{\times}_{mul}$ covering K^{T}_{mul}.*

Moreover, the following result is similar to Theorem 17 even if requires the invariance of the partial leading multicones.

Theorem 18 *Let $(\tilde{\mathcal{L}}, \tilde{\mathcal{L}}^{\mathrm{T}})$ be a consistent pair of sets of leading eigenvectors for the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$. If the partial leading multicones K_{mul} and K^{T}_{mul} are invariant for $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$, respectively, then also $\bar{K}_{mul}$, $(K^{\mathrm{T}}_{mul})^{\dagger}$ and $(K^{\mathrm{T}}_{mul})^{\times}$ are invariant for $\mathcal{F}$ and, dually, $\bar{K}^{\mathrm{T}}_{mul}$, $K^{\dagger}_{mul}$ and $K^{\times}_{mul}$ are invariant for $\mathcal{F}^{\mathrm{T}}$.*

Anyway, we cannot prove the invariance of $\mathrm{int}(\bar{K}_{mul})$ and $\mathrm{int}((K^{\mathrm{T}}_{mul})^{\dagger})$ because the analogue of Corollary 9 is not assured to hold for $\tilde{\mathcal{H}}$ if this set does not equal $\mathrm{cl}(\mathcal{H}(\mathcal{F}))$.

The foregoing theorem obviously suggests to stop the algorithm when it produces a pair $(K_{mul}, K^{\mathrm{T}}_{mul})$ of invariant multicones for $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$.

In addition, the next result shows that, in case of $\mathcal{F}$ being asymptotically rank-one, the output $(K_{mul}, K^{\mathrm{T}}_{mul})$ of this procedure coincides with the pair $(K^{\mathcal{F}}_{mul}, K^{\mathcal{F}^{\mathrm{T}}}_{mul})$ itself.

Theorem 19 *Let the family $\mathcal{F}$ be asymptotically rank-one and let $(\tilde{\mathcal{L}}, \tilde{\mathcal{L}}^{\mathrm{T}})$ be a consistent pair of sets of leading eigenvectors for $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$. If the partial leading multicones K_{mul} and K^{T}_{mul} are invariant for $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$, respectively, then:*

(i) $\mathrm{cl}(\mathcal{L}(\mathcal{F})) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \{0\}$ *(hence, Assumption 1 holds);*
(ii) *$\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ are $\mathcal{L}$-full;*
(iii) $K_{mul} = K^{\mathcal{F}}_{mul}$ *and* $K^{\mathrm{T}}_{mul} = K^{\mathcal{F}^{\mathrm{T}}}_{mul}$*;*
(iv) $\bar{K}_{mul} = \bar{K}^{\mathcal{F}}_{mul}$ *and* $\bar{K}^{\mathrm{T}}_{mul} = \bar{K}^{\mathcal{F}^{\mathrm{T}}}_{mul}$.

Proof By Theorem 10 we have that (40) holds for $\mathcal{F}$, i.e.

$$\mathrm{cl}(\mathcal{L}(\mathcal{F})) \subseteq K_{mul}$$

and, since $(\mathcal{F}^{\mathrm{T}})^{\mathrm{T}} = \mathcal{F}$, also that (23) holds for $\mathcal{F}^{\mathrm{T}}$, i.e.

$$\mathrm{int}((K^{\mathrm{T}}_{mul})^{\dagger}) \cap \mathrm{cl}(\mathcal{H}(\mathcal{F})) = \emptyset. \tag{69}$$

Finally, Remark 16 and Proposition 28 yield

$$K_{mul} \backslash \{0\} \subseteq \mathrm{int}((K^{\mathrm{T}}_{mul})^{\dagger}), \tag{70}$$

so that (i) is proved.

Concerning (ii), $\mathcal{F}$ is $\mathcal{L}$-full by property (p5) and $\mathcal{F}^{\mathrm{T}}$ is $\mathcal{L}$-full by property (p6).

In order to show (iii), observe that inclusions (69) and (70) imply the validity of condition (16) for K_{mul} for all $P \in \Sigma(\mathcal{F})$. Thus, K_{mul} being invariant for $\mathcal{F}$, Remark 7 tells us that $\tilde{r} \geq r$. Since the opposite inequality holds in any case (see Remark 14), we conclude that $\tilde{r} = r$. Finally, again Remark 7 assures that K_{mul} is a spectral multicone for $\mathcal{F}$. Hence, in particular, $K_{mul} \supseteq K_{mul}^{\mathcal{F}}$.

To show the opposite inclusion, just observe that not only $\tilde{\mathcal{L}} \subseteq \mathrm{cl}(\mathcal{L}(\mathcal{F}))$ holds, but also $\tilde{\mathcal{L}}^{(i)} \subseteq \mathrm{cl}(\mathcal{L}(\mathcal{F})^{(i)})$, for all i, still by Remark 7. Thus (iii) is proved.

Finally, by (iii) and Propositions 26 and 28, the proof of (iv) is completed. □

The localization result (see Proposition 10) implies that the invariance of K_{mul} for $\mathcal{F}$ is equivalent to the fact that, for each conic component $K^{(i)}$ and for each $A \in \mathcal{F}$, there exists a conic component $K^{(j)}$ such that

$$A(K^{(i)}) \subseteq K^{(j)}.$$

However, in this (polyhedral) case, thanks to (67), the above condition can be checked just by verifying that

$$A(\tilde{\mathcal{E}}^{(i)}) \subseteq K^{(j)}. \tag{71}$$

Observe that the "global" condition $A(\tilde{\mathcal{E}}) \subseteq K_{mul}$ is weaker than the validity of (71) for all $i = 1, \ldots, \tilde{r}$. In fact, it is not sufficient to assure the invariance of K_{mul}, and hence, it cannot be assumed as a stopping criterion for the algorithm.

Of course, the same arguments can be repeated for $\tilde{\mathcal{L}}^{\mathrm{T}}$ and the corresponding partial leading multicone K_{mul}^{T}.

In order to allow a more efficient computation, it is useful to also observe that Proposition 27 implies

$$\bar{K}_{mul} \subseteq (K_{mul}^{\mathrm{T}})^{\dagger} \quad \text{and} \quad \bar{K}_{mul}^{\mathrm{T}} \subseteq K_{mul}^{\dagger}$$

and hence, by (8),

$$\mathrm{int}(\bar{K}_{mul}) \cap \{h\}^{\perp} = \emptyset \quad \forall\, h \in K_{mul}^{\mathrm{T}} \backslash \{0\} \tag{72}$$

and, dually, that

$$\mathrm{int}(\bar{K}_{mul}^{\mathrm{T}}) \cap \{v\}^{\perp} = \emptyset \quad \forall\, v \in K_{mul} \backslash \{0\}.$$

In fact, for the actual construction of $\bar{K}_{mul}$ defined by (68), condition (72) suggests that it is sufficient to confine ourselves to consider the semi-spaces $S_{h+}^{(i)}$ for $h \in \tilde{\mathcal{E}}^{\mathrm{T}}$, the set of the edges of K_{mul}^{T}, i.e.

$$\bar{K}_{+}^{(i)} = \bigcap_{h \in \tilde{\mathcal{E}}^{\mathrm{T}} \backslash \{0\}} S_{h+}^{(i)}, \quad i = 1, \ldots, \tilde{r}. \tag{73}$$

Analogously, we have that

$$\bar{K}_{+}^{\mathrm{T}\,(\mathrm{i})} = \bigcap_{v \in \tilde{\mathcal{E}} \backslash \{0\}} S_{v+}^{(i)}, \quad i = 1, \ldots, \tilde{s}. \tag{74}$$

Now we are in a position to present our iterative algorithm (see Algorithm 1).

Note that the way Algorithm 1 selects the products P used to define the sets $\tilde{\mathcal{L}}_k$ and $\tilde{\mathcal{L}}_k^{\mathrm{T}}$ automatically assures that the corresponding sets $\tilde{\Sigma}_k$ and $\tilde{\Omega}_k$ verify the inclusion $\tilde{\Omega}_k \subseteq \tilde{\Sigma}_k$. Therefore, it is sufficient to monitor property (p1) on the elements of $\tilde{\Sigma}_k$, which is, in this case, nothing but $\Sigma_k(\mathcal{F})$.

Data: $\mathcal{F} = \{A_1, \ldots, A_m\}$
begin
  compute V_A and $V_{A^{\mathrm{T}}}$ for each $A \in \mathcal{F}$;
  define $\tilde{\mathcal{L}}_1 := \bigcup_{A \in \mathcal{F}} V_A$ and $\tilde{\mathcal{L}}_1^{\mathrm{T}} := \bigcup_{A \in \mathcal{F}} V_{A^{\mathrm{T}}}$;
  compute $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_1^{\mathrm{T}}$ and the essential components of $\tilde{\mathcal{L}}_1$ and $\tilde{\mathcal{L}}_1^{\mathrm{T}}$;
  set k:=1;
  if (p4) is not satisfied **stop**;
  while $\tilde{\mathcal{L}}_k$ *does not satisfy (p5)* or $\tilde{\mathcal{L}}_k^{\mathrm{T}}$ *does not satisfy (p6)* **do**
    compute V_P and $V_{P^{\mathrm{T}}}$ for each $P \in \Sigma_{k+1}(\mathcal{F})$;
    if not all $P \in \Sigma_{k+1}(\mathcal{F})$ are asymptotically rank-one **stop**;
    define $\tilde{\mathcal{L}}_{k+1} := \tilde{\mathcal{L}}_k \cup \bigcup_{P \in \Sigma_{k+1}(\mathcal{F})} V_P$ and $\tilde{\mathcal{L}}_{k+1}^{\mathrm{T}} := \tilde{\mathcal{L}}_k^{\mathrm{T}} \cup \bigcup_{P \in \Sigma_{k+1}(\mathcal{F})} V_{P^{\mathrm{T}}}$;
    set $k := k + 1$;
    compute $\tilde{\mathcal{H}}_k$ and $\tilde{\mathcal{H}}_k^{\mathrm{T}}$ and the essential components of $\tilde{\mathcal{L}}_k$ and $\tilde{\mathcal{L}}_k^{\mathrm{T}}$;
    if (p4) is not satisfied **stop**;
  end
  set $\kappa_1 := k$;
  compute the edges $\tilde{\mathcal{E}}_{\kappa_1}$ and $\tilde{\mathcal{E}}_{\kappa_1}^{\mathrm{T}}$ of the corresponding partial leading multicones K_{κ_1} and $K_{\kappa_1}^{\mathrm{T}}$;
  set $i := 1$;
  while K_{κ_i} *is not invariant for* $\mathcal{F}$ or $K_{\kappa_i}^{\mathrm{T}}$ *is not invariant for* $\mathcal{F}^{\mathrm{T}}$ **do**
    set $\tilde{\mathcal{L}}_{\kappa_i} := \tilde{\mathcal{L}}_{\kappa_i} \cup \mathcal{F}(\tilde{\mathcal{E}}_{\kappa_i})$ and $\tilde{\mathcal{L}}_{\kappa_i}^{\mathrm{T}} := \tilde{\mathcal{L}}_{\kappa_i}^{\mathrm{T}} \cup \mathcal{F}^{\mathrm{T}}(\tilde{\mathcal{E}}_{\kappa_i}^{\mathrm{T}})$;
    compute V_P and $V_{P^{\mathrm{T}}}$ for each product $P \in \Sigma_{\kappa_i+1}(\mathcal{F})$;
    if not all $P \in \Sigma_{\kappa_i+1}(\mathcal{F})$ are asymptotically rank-one **stop**;
    define $\tilde{\mathcal{L}}_{\kappa_i+1} := \tilde{\mathcal{L}}_{\kappa_i} \cup \bigcup_{P \in \Sigma_{\kappa_i+1}(\mathcal{F})} V_P$ and $\tilde{\mathcal{L}}_{\kappa_i+1}^{\mathrm{T}} := \tilde{\mathcal{L}}_{\kappa_i}^{\mathrm{T}} \cup \bigcup_{P \in \Sigma_{\kappa_i+1}(\mathcal{F})} V_{P^{\mathrm{T}}}$;
    set $k := \kappa_i + 1$;
    compute $\tilde{\mathcal{H}}_k$ and $\tilde{\mathcal{H}}_k^{\mathrm{T}}$ and the essential components of $\tilde{\mathcal{L}}_k$ and $\tilde{\mathcal{L}}_k^{\mathrm{T}}$;
    if (p4) is not satisfied **stop**;
    compute the edges $\tilde{\mathcal{E}}_k$ and $\tilde{\mathcal{E}}_k^{\mathrm{T}}$ of the corresponding partial leading multicones K_k and K_k^{T};
    while $\tilde{\mathcal{L}}_k$ *does not satisfy (p5)* or $\tilde{\mathcal{L}}_k^{\mathrm{T}}$ *does not satisfy (p6)* **do**
      compute V_P and $V_{P^{\mathrm{T}}}$ for each product $P \in \Sigma_{k+1}(\mathcal{F})$;
      if not all $P \in \Sigma_{k+1}(\mathcal{F})$ are asymptotically rank-one **stop**;
      define $\tilde{\mathcal{L}}_{k+1} := \tilde{\mathcal{L}}_k \cup \bigcup_{P \in \Sigma_{k+1}(\mathcal{F})} V_P$ and $\tilde{\mathcal{L}}_{k+1}^{\mathrm{T}} := \tilde{\mathcal{L}}_k^{\mathrm{T}} \cup \bigcup_{P \in \Sigma_{k+1}(\mathcal{F})} V_{P^{\mathrm{T}}}$;
      set $k := k + 1$;
      compute $\tilde{\mathcal{H}}_k$ and $\tilde{\mathcal{H}}_k^{\mathrm{T}}$ and the essential components of $\tilde{\mathcal{L}}_k$ and $\tilde{\mathcal{L}}_k^{\mathrm{T}}$;
      if (p4) is not satisfied **stop**;
      compute the edges $\tilde{\mathcal{E}}_k$ and $\tilde{\mathcal{E}}_k^{\mathrm{T}}$ of the corresponding partial leading multicones K_k and K_k^{T};
    end
    set $i := i + 1$ and $\kappa_i := k$;
  end
  compute $\bar{K}_{\kappa_i}$ and $\bar{K}_{\kappa_i}^{\mathrm{T}}$ using (73) and (74), respectively;
  Result: $K_{\kappa_i}, \bar{K}_{\kappa_i}, K_{\kappa_i}^{\mathrm{T}}, \bar{K}_{\kappa_i}^{\mathrm{T}}$
end

Algorithm 1: Basic iterative algorithm

Observe also that the way of selecting the sets $\tilde{\mathcal{L}}_k$ and $\tilde{\mathcal{L}}_k^{\mathrm{T}}$ precisely fulfils the requirements (p2) and (p3).

Moreover, something can be pointed out.

On the one hand, if the procedure ends successfully in a finite number of iterations, we obtain a pair $(K_{\kappa_i}, K_{\kappa_i}^{\mathrm{T}})$ of invariant polyhedral multicones for $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$. But (unless this fact is known a priori for some other reasons) even in this case we are not guaranteed that $\mathcal{F}$ is asymptotically rank-one.

Nevertheless, if it is so, the computed invariant pair equals the pair of leading multicones $(K_{mul}^{\mathcal{F}}, K_{mul}^{\mathcal{F}^{\mathrm{T}}})$ by virtue of Theorem 19.

On the other hand, if the procedure never ends, nothing can be said.

However, as before, if $\mathcal{F}$ is asymptotically rank-one, then the sequence of computed pairs of partial leading multicones (K_k, K_k^{T}) clearly converge to $(K_{mul}^{\mathcal{F}}, K_{mul}^{\mathcal{F}^{\mathrm{T}}})$ as $k \to \infty$.

At the end of the fair, the main target of our procedure, i.e. to find a pair of invariant polyhedral multicones, is reached as long as the algorithm stops. Our dedication to consider also theoretical results under the hypothesis that $\mathcal{F}$ is asymptotically rank-one is justified by the fact that it is a necessary condition for the existence of a strictly invariant multicone.

We just mention that many tricks may be used aiming at making Algorithm 1 more efficient, which we do not point out in our summary of the procedure, but leave to the interested reader.

For example, as the most obvious, in order to avoid underflow/overflow occurrences, it is always advisable to normalize the computed eigenvectors and all their images in the elements of the families $\mathcal{F}$ and $\mathcal{F}^{\mathrm{T}}$ (see line 17).

Moreover, still rather obvious, one has not to recompute the leading eigenvector $v_{P^n} = v_P$ and the secondary hyperplane $H_{P^n} = H_P$ of a power P^n whenever the matrix P has already been involved by the algorithm. Nor, analogously, the leading eigenvector and the secondary hyperplane of a cyclic permutation QP whenever the matrix PQ has already been processed, since $v_{QP} = Qv_{PQ}$ and $H_{QP} = QH_{PQ}$.

It is also clear that it is important to use an efficient method to select the essential components of $\tilde{\mathcal{L}}_k$ and $\tilde{\mathcal{L}}_k^{\mathrm{T}}$ (see lines 1, 11, 22, 30) and the edges $\tilde{\mathcal{E}}_k$ and $\tilde{\mathcal{E}}_k^{\mathrm{T}}$ of the corresponding partial leading multicones K_k and K_k^{T} (see lines 14, 24, 32). Regarding this task, we observe that, apart from the trivial case of dimension $d = 2$, it is difficult to establish a priori how many the edges are.

Keeping well in mind that, in general, the number κ_i of iterations needed to conclude successfully the procedure (if any) is not predictable a priori, in order to give a rough estimate of the computational cost of each iteration, we first observe that the most time-consuming *elementary* operation consists in the computation of the eigenspaces of a new product P (i.e. v_P and H_P). In turn, this operation is increasingly expensive with the dimension d of the matrices. Thus, the choice of a suitable numerical method, in accordance with the particular structure of the matrices, is crucial for the overall efficiency of the algorithm. Clearly, also the cardinality m of the family $\mathcal{F}$ makes the difference in that, to a first approximation, an upper bound to the cost of the $k+1$st iteration equals m times the cost of the kth one. In fact, one has to perform the exhaustive analysis of all the matrices of $\Sigma_{k+1}(\mathcal{F})$, whose cardinality is m times that of $\Sigma_k(\mathcal{F})$.

Finally, it is worth remarking that, in view of what above, the overall computational complexity may reveal to be rather high (exponential in κ_i). Therefore, it would be interesting to study some strategies which, without reducing the chances of final success, allow us to disregard large subsets of $\Sigma_{k+1}(\mathcal{F})$ on the basis of the structure of the just computed multicones K_k and K_k^{T}.

Example 3 Consider the 2×2 matrix family $\mathcal{F} = \{A, B\}$, where

$$A = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -\frac{2}{9} & \frac{5}{9} \\ -\frac{5}{2} & \frac{11}{2} \end{bmatrix}$$

are asymptotically rank-one with $\lambda_A = 3$ and $\lambda_B = 5.2460\ldots$.

Running Algorithm 1 results in the following steps.

(s1)

$$\tilde{\mathcal{L}}_1 = \tilde{\mathcal{L}}_1^{(1)} \cup \tilde{\mathcal{L}}_1^{(2)} \quad \text{and} \quad \tilde{\mathcal{L}}_1^{\mathrm{T}} = \tilde{\mathcal{L}}_1^{\mathrm{T}\,(1)} \cup \tilde{\mathcal{L}}_1^{\mathrm{T}\,(2)},$$

where $\tilde{\mathcal{L}}_1^{(1)} = V_A$, $\tilde{\mathcal{L}}_1^{(2)} = V_B$, $\tilde{\mathcal{L}}_1^{\mathrm{T}\,(1)} = V_{A^{\mathrm{T}}}$, $\tilde{\mathcal{L}}_1^{\mathrm{T}\,(2)} = V_{B^{\mathrm{T}}}$, so that (p5) and (p6) are not satisfied (see Fig. 1).

(s2)

$$\tilde{\mathcal{L}}_2 = \tilde{\mathcal{L}}_2^{(1)} \cup \tilde{\mathcal{L}}_2^{(2)} \cup \tilde{\mathcal{L}}_2^{(3)} \quad \text{and} \quad \tilde{\mathcal{L}}_2^{\mathrm{T}} = \tilde{\mathcal{L}}_2^{\mathrm{T}\,(1)} \cup \tilde{\mathcal{L}}_2^{\mathrm{T}\,(2)} \cup \tilde{\mathcal{L}}_2^{\mathrm{T}\,(3)},$$

where $\tilde{\mathcal{L}}_2^{(1)} = V_A$, $\tilde{\mathcal{L}}_2^{(2)} = V_{AB}$, $\tilde{\mathcal{L}}_2^{(3)} = V_B \cup V_{BA}$, $\tilde{\mathcal{L}}_2^{\mathrm{T}\,(1)} = V_{A^{\mathrm{T}}}$, $\tilde{\mathcal{L}}_2^{\mathrm{T}\,(2)} = V_{B^{\mathrm{T}}} \cup V_{(AB)^{\mathrm{T}}}$, $\tilde{\mathcal{L}}_2^{\mathrm{T}\,(3)} = V_{(BA)^{\mathrm{T}}}$, so that (p5) and (p6) are not satisfied again (see Fig. 2).

(s3)

$$\tilde{\mathcal{L}}_3 = \tilde{\mathcal{L}}_3^{(1)} \cup \tilde{\mathcal{L}}_3^{(2)} \cup \tilde{\mathcal{L}}_3^{(3)} \quad \text{and} \quad \tilde{\mathcal{L}}_3^{\mathrm{T}} = \tilde{\mathcal{L}}_3^{\mathrm{T}\,(1)} \cup \tilde{\mathcal{L}}_3^{\mathrm{T}\,(2)} \cup \tilde{\mathcal{L}}_3^{\mathrm{T}\,(3)},$$

where $\tilde{\mathcal{L}}_3^{(1)} = V_A \cup V_{A^2B}$, $\tilde{\mathcal{L}}_3^{(2)} = V_{AB} \cup V_{ABA} \cup V_{AB^2}$, $\tilde{\mathcal{L}}_3^{(3)} = V_B \cup V_{BA} \cup V_{BA^2} \cup V_{BAB} \cup V_{B^2A}$, $\tilde{\mathcal{L}}_3^{\mathrm{T}\,(1)} = V_{A^{\mathrm{T}}} \cup V_{(BA^2)^{\mathrm{T}}}$, $\tilde{\mathcal{L}}_3^{\mathrm{T}\,(2)} = V_{B^{\mathrm{T}}} \cup V_{(AB)^{\mathrm{T}}} \cup V_{(A^2B)^{\mathrm{T}}} \cup V_{(BAB)^{\mathrm{T}}} \cup V_{(AB^2)^{\mathrm{T}}}$, $\tilde{\mathcal{L}}_3^{\mathrm{T}\,(3)} = V_{(BA)^{\mathrm{T}}} \cup V_{(ABA)^{\mathrm{T}}} \cup V_{(B^2A)^{\mathrm{T}}}$, so that (p5) and (p6) are eventually satisfied (see Fig. 3).

(s4) The edges of K_3 are $\tilde{\mathcal{E}}_3^{(1)} = V_A \cup V_{A^2B}$, $\tilde{\mathcal{E}}_3^{(2)} = V_{AB} \cup V_{ABA}$, $\tilde{\mathcal{E}}_3^{(3)} = V_{BA} \cup V_{BA^2}$ and those of K_3^{T} are $\tilde{\mathcal{E}}_3^{\mathrm{T}\,(1)} = V_{A^{\mathrm{T}}} \cup V_{(BA^2)^{\mathrm{T}}}$, $\tilde{\mathcal{E}}_3^{\mathrm{T}\,(2)} = V_{(AB)^{\mathrm{T}}} \cup V_{(A^2B)^{\mathrm{T}}}$, $\tilde{\mathcal{E}}_3^{\mathrm{T}\,(3)} = V_{(BA)^{\mathrm{T}}} \cup V_{(ABA)^{\mathrm{T}}}$.

(s5) By analysing condition (71), it turns out that

$$\begin{aligned} &A(K_{3+}^{(1)}) \subseteq K_{3+}^{(1)}, \quad A(K_{3+}^{(2)}) \subseteq K_{3+}^{(1)}, \quad A(K_{3+}^{(3)}) \subseteq K_{3+}^{(2)}, \\ &B(K_{3+}^{(1)}) \subseteq K_{3-}^{(3)}, \quad B(K_{3+}^{(2)}) \subseteq K_{3+}^{(3)}, \quad B(K_{3+}^{(3)}) \subseteq K_{3+}^{(3)}, \end{aligned}$$

and, analogously, that

$$\begin{aligned} &A^{\mathrm{T}}(K_{3+}^{T\,(1)}) \subseteq K_{3+}^{T\,(1)}, \quad A^{\mathrm{T}}(K_{3+}^{T\,(2)}) \subseteq K_{3+}^{T\,(3)}, \quad A^{\mathrm{T}}(K_{3+}^{T\,(3)}) \subseteq K_{3+}^{T\,(1)}, \\ &B^{\mathrm{T}}(K_{3+}^{T\,(1)}) \subseteq K_{3+}^{T\,(2)}, \quad B^{\mathrm{T}}(K_{3+}^{T\,(2)}) \subseteq K_{3+}^{T\,(2)}, \quad B^{\mathrm{T}}(K_{3+}^{T\,(3)}) \subseteq K_{3+}^{T\,(2)}. \end{aligned}$$

Therefore, K_3 and K_3^{T} are invariant for $\mathcal{F}$, and hence, the algorithm ends successfully.

(s6) With $S_P^{(i)} := S_{h_P+}^{(i)}$, by using (73) it turns out that

$$\bar{K}_{3+}^{(1)} = S_A^{(1)} \cap S_{A^2B}^{(1)}, \quad \bar{K}_{3+}^{(2)} = S_{AB}^{(1)} \cap S_{ABA}^{(1)}, \quad \bar{K}_{3+}^{(3)} = S_{BA}^{(1)} \cap S_{BA^2}^{(1)},$$

and, with $S_{P^{\mathrm{T}}}^{(i)} := S_{v_P+}^{(i)}$, by using (74) we have that

$$\begin{aligned} &\bar{K}_{3+}^{\mathrm{T}\,(1)} = S_{A^{\mathrm{T}}}^{(1)} \cap S_{(BA^2)^{\mathrm{T}}}^{(1)}, \quad \bar{K}_{3+}^{\mathrm{T}\,(2)} = S_{(AB)^{\mathrm{T}}}^{(1)} \cap S_{(A^2B)^{\mathrm{T}}}^{(1)}, \\ &\bar{K}_{3+}^{\mathrm{T}\,(3)} = S_{(BA)^{\mathrm{T}}}^{(1)} \cap S_{(ABA)^{\mathrm{T}}}^{(1)}. \end{aligned}$$

□

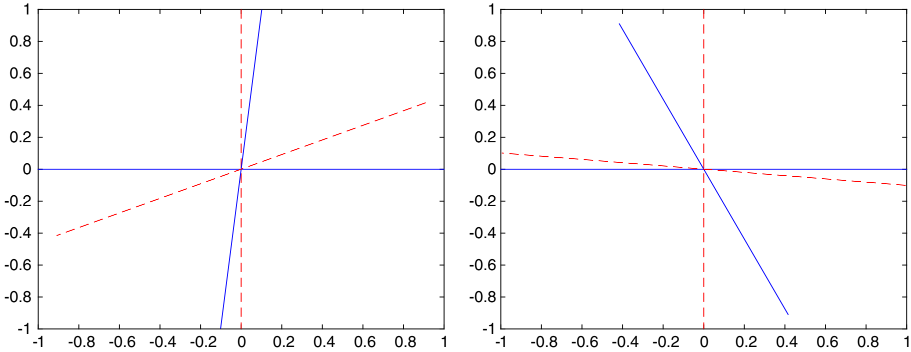

Fig. 1 Left figure: $\tilde{\mathcal{L}}_1$ (solid) and $\tilde{\mathcal{H}}_1$ (dashed). Right figure: $\tilde{\mathcal{L}}_1^{\mathrm{T}}$ (solid) and $\tilde{\mathcal{H}}_1^{\mathrm{T}}$ (dashed)

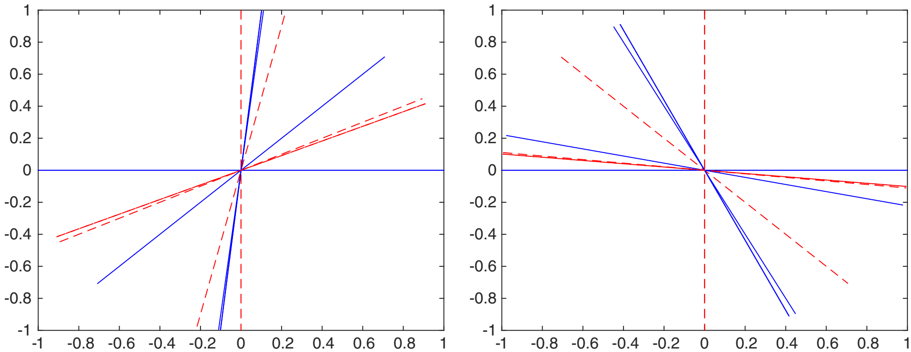

Fig. 2 Left figure: $\tilde{\mathcal{L}}_2$ (solid) and $\tilde{\mathcal{H}}_2$ (dashed). Right figure: $\tilde{\mathcal{L}}_2^{\mathrm{T}}$ (solid) and $\tilde{\mathcal{H}}_2^{\mathrm{T}}$ (dashed)

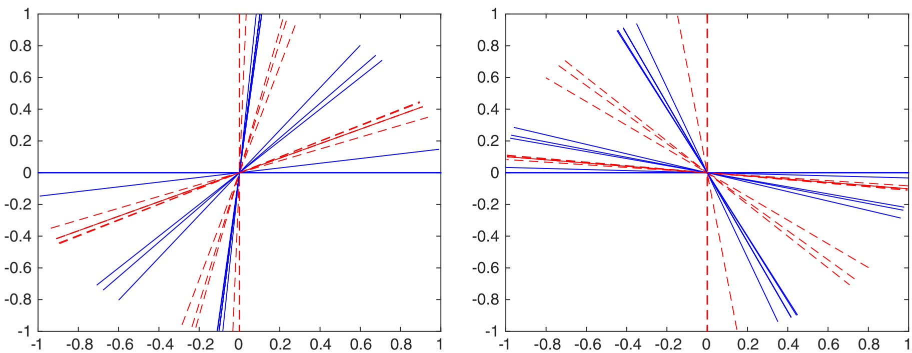

Fig. 3 Left figure: $\tilde{\mathcal{L}}_3$ (solid) and $\tilde{\mathcal{H}}_3$ (dashed). Right figure: $\tilde{\mathcal{L}}_3^{\mathrm{T}}$ (solid) and $\tilde{\mathcal{H}}_3^{\mathrm{T}}$ (dashed)

8 Conclusions and open problems

One of the main goals of this paper has been the detection of sufficient conditions on the structure of the eigenspaces of a given finite family $\mathcal{F}$ of matrices to assure the existence of (the embedded pair of) the "smallest" and the "biggest" invariant multicones $K_{mul}^{\mathcal{F}}$ and

$\bar{K}^{\mathcal{F}}_{mul}$. The conditions found suggest an effective computational procedure for such invariant embedded pair.

Our study can lead to a generalization to multicones of the so-called *Barabanov antinorm* for a family of matrices (see Guglielmi and Protasov [10] and Guglielmi and Zennaro [11]), which will be the subject of a future paper.

Anyway, some interesting questions still remain open.

The cited conditions (assumed in Sect. 4) guarantee the existence of an embedded pair of invariant multicones for $\mathcal{F}$. The natural question arises: are they sufficient to assure also the existence of a *strictly* invariant multicone? In other words, to assure 1-dominance? [see (21)]

In general, are the "smallest" and the "biggest" invariant multicones $K^{\mathcal{F}}_{mul}$ and $\bar{K}^{\mathcal{F}}_{mul}$ reflexive?

An important question from the computational point of view: are the conditions assumed in Sect. 4 sufficient to always assure that $K^{\mathcal{F}}_{mul}$ and $\bar{K}^{\mathcal{F}}_{mul}$ are of polyhedral type?

Finally, what of the theory developed in the present paper could be saved and adapted/extended to more general cases of matrix families which fail to be asymptotically rank-one?

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