# Outer Approximations of Coherent Lower Probabilities Using Belief Functions

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**Abstract.** We investigate the problem of outer approximating a coherent lower probability with a more tractable model. In particular, in this work we focus on the outer approximations made by belief functions. We show that they can be obtained by solving a linear programming problem. In addition, we consider the subfamily of necessity measures, and show that in that case we can determine all the undominated outer approximations in a simple manner.

## 1 Introduction

*Coherent lower probabilities* are one of the most prominent models within imprecise probability theory [1]. They can be given a behavioural interpretation in terms of acceptable betting rates, thus extending Bruno de Finetti's work on subjective probability theory; at the same time, they are also equivalent to convex sets of probability measures (*credal sets*), meaning that they can be regarded as an epistemic model of imprecise information.

In spite of this, coherent lower probabilities also have a number of drawbacks that hinder their use in the practice. For instance, their associated credal sets do not possess a straightforward representation in terms of extreme points; and their extension to lower *previsions* of gambles is not unique in general. For these reasons, it becomes interesting to approximate a coherent lower probability by a more tractable model. In a previous contribution [2], we did so by means of 2-monotone lower probabilities, that overcome some of the issues mentioned above: there is a simple procedure to determine the number of extreme points of their associated credal sets [3], and they can be uniquely extended to gambles by means of the Choquet integral [4].

Although our previous results are promising, the use of 2-monotone capacities is not without issues; the most important one, in our view, is the lack of a compelling interpretation of 2-monotonicity. This has led us to study the approximation of coherent lower probabilities by means of *completely* monotone lower probabilities, or belief functions. They have a number of advantages: first, they have a clear interpretation from Shafer's Evidence Theory [5]; they can be equivalently represented by means of multi-valued mappings [6]; and still they are sufficiently general to include as particular cases many interesting models from imprecise probability theory, such as probability boxes [7] or possibility measures [8].

The rest of the contribution is organized as follows: after giving some preliminary concepts in Sect. 2, in Sect. 3 we deal with the problem of outer approximating a coherent lower probability. We recall our results for 2-monotone lower probabilities in Sect. 3.1, investigate the problem for belief functions in Sect. 3.2 and consider the particular case of possibility measures in Sect. 3.3. Some additional comments are given in Sect. 4. Due to space limitations, several results, comments as well as proofs have been omitted.

## 2 Preliminaries

Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ denote a finite universe with cardinality $n$. A *lower probability* on $\mathcal{P}(\mathcal{X})$ is a function $\underline{P} : \mathcal{P}(\mathcal{X}) \to [0,1]$. Under an epistemic interpretation, $\underline{P}(A)$ may be understood as a lower bound for the unknown probability $P_0(A)$ of the event $A$. In that case, the available information about the probability measure $P_0$ is given by the *credal set* associated with $\underline{P}$:

$$\mathcal{M}(\underline{P}) = \{P \text{ probability measure} \mid P(A) \geq \underline{P}(A)\ \forall A \subseteq \mathcal{X}\}.$$

The minimum requirement on $\underline{P}$ we shall consider in this paper is that the bounds it provides for every event can be attained by some probability in $\mathcal{M}(\underline{P})$.

**Definition 1.** [1] A lower probability $\underline{P}$ on $\mathcal{P}(\mathcal{X})$ is called *coherent* when its credal set $\mathcal{M}(\underline{P})$ is non-empty and $\underline{P}(A) = \min_{P \in \mathcal{M}(\underline{P})} P(A)$ for every $A \subseteq \mathcal{X}$.

The conjugate of a lower probability $\underline{P}$, denoted by $\overline{P}$, is called *upper probability* and it is given by $\overline{P}(A) = 1 - \underline{P}(A^c)$ for every $A \subseteq \mathcal{X}$. $\overline{P}(A)$ can be interpreted as an upper bound for the unknown probability of $A$. When $\underline{P}$ is coherent, $\overline{P}$ can also be computed by $\overline{P}(A) = \max\{P(A) \mid P \in \mathcal{M}(\underline{P})\}$ for every $A \subseteq \mathcal{X}$.

One very interesting property that a coherent lower probability may satisfy is that of k-monotonicity.

**Definition 2.** [4] A lower probability $\underline{P} : \mathcal{P}(\mathcal{X}) \to [0,1]$ is *$k$-monotone* if for every $p \leq k$, and for every $A_1, \ldots, A_p \subseteq \mathcal{X}$ it holds that:

$$\underline{P}\big(\cup_{i=1}^{p} A_i\big) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,p\}} (-1)^{|I|+1} \underline{P}\big(\cap_{i \in I} A_i\big).$$

In particular, 2-monotone lower probabilities possess a number of interesting properties: for instance, the extreme points of their associated credal set can be easily determined using the permutations of the possibility space [3]; moreover, they have a unique extension as an expectation operator that preserves 2-monotonicity: their Choquet integral [9].

If $\underline{P}$ is $k$-monotone for every $k$, it is called *completely monotone*. It corresponds to a *belief function* within evidence theory, and we shall denote it *Bel* in this paper. The conjugate upper probability of a belief function is called *plausibility function* and we shall denote it $Pl$. A belief function can be equivalently expressed in terms of its *Möbius inverse*, which is given by [5]:

$$m(A) = \sum_{B \subseteq A} (-1)^{|A \setminus B|} Bel(B) \quad \forall A \subseteq \mathcal{X}.$$

This function $m$ satisfies $\sum_{A \subseteq \mathcal{X}} m(A) = 1$ and $m(A) \in [0,1]$ for every $A \subseteq \mathcal{X}$. Conversely, $m$ determines the belief function by:

$$Bel(A) = \sum_{B \subseteq A} m(B).$$

Given the Möbius inverse $m$, those events $A$ with strictly positive mass, $m(A) > 0$, are called *focal events*.

A particular case of plausibility functions are the possibility measures. They are connected to the theory of fuzzy sets.

**Definition 3.** [10] A *possibility measure* $\Pi : \mathcal{P}(\mathcal{X}) \to [0,1]$ is a function satisfying $\Pi(\emptyset) = 0$, $\Pi(\mathcal{X}) = 1$ and $\Pi(A \cup B) = \max\{\Pi(A), \Pi(B)\}$ for every $A, B \subseteq \mathcal{X}$.

A possibility measure is an instance of plausibility function, while its conjugate *necessity* measure is a belief function. They correspond to the particular case when the focal events are nested by set inclusion, meaning that for every two focal events $E_1, E_2$, either $E_1 \subseteq E_2$ or $E_2 \subseteq E_1$.

**Notation:** We shall denote by $\mathcal{C}_2, \mathcal{C}_\infty$ and $\mathcal{C}_\Pi$ the classes of 2-monotone lower probabilities, belief functions and possibility measures on $\mathcal{P}(\mathcal{X})$, respectively.

## 3 Outer Approximations of Coherent Lower Probabilities

In a recent paper [2] we investigated how to approximate a coherent lower probability $\underline{P}$ by a 2-monotone lower probability $\underline{Q}$ that at the same time (a) does not introduce new information; (b) is as close as possible to the original model. In this way, if $\mathcal{C}$ denotes a class of coherent lower probabilities, we said that $\underline{Q} \in \mathcal{C}$ is an *outer approximation* of $\underline{P}$ in $\mathcal{C}$ if $\underline{Q} \leq \underline{P}$, and it is called *undominated* if there is no $\underline{Q}' \in \mathcal{C}$ such that $\underline{Q} \lneq \underline{Q}' \leq \underline{P}$.

### 3.1 Outer Approximations in $\mathcal{C}_2$

One important issue is that of determining how close the outer approximation is to the original model. In [2], in addition to discussing other possibilities, we proposed to use the distance put forward by Baroni and Vicig in [11], given by

$$d(\underline{P}, \underline{Q}) := \sum_{E \subseteq \mathcal{X}} (\underline{P}(E) - \underline{Q}(E)). \tag{1}$$

If we interpret $\underline{P}(E) - \underline{Q}(E)$ as the additional imprecision introduced on $E$ when replacing $\underline{P}(E)$ with $\underline{Q}(E)$, then $d(\underline{P}, \underline{Q})$ can be understood as the total imprecision added by the outer approximation $\underline{Q}$.

In [2], we obtained undominated outer approximations in $\mathcal{C}_2$ by using a linear programming problem and minimizing the distance (1). Next proposition summarizes some of our results.

**Proposition 1.** [2] *Let $\underline{P}$ be a coherent lower probability, and let $\mathcal{C}_2'(\underline{P})$ denote the class of undominated outer approximations of $\underline{P}$ in $\mathcal{C}_2$.*

1. *$\mathcal{C}_2'(\underline{P})$ is non-empty, and may have infinite cardinality.*
2. *$\underline{Q}(\{x\}) = \underline{P}(\{x\})$ for every $x \in \mathcal{X}$ and every $\underline{Q} \in \mathcal{C}_2'(\underline{P})$.*
3. *$\underline{P}(A) = \max_{\underline{Q} \in \mathcal{C}_2'(\underline{P})} \underline{Q}(A)$ for every $A \subseteq \mathcal{X}$.*

### 3.2 Outer Approximations in $\mathcal{C}_\infty$

In this section, we outer approximate a coherent lower probability by means of a belief function. Similarly to our work in [2], we propose to obtain outer approximations that minimize the distance (1) between the initial lower probability $\underline{P}$ and the belief function: $d(\underline{P}, Bel) = \sum_{E \subseteq \mathcal{X}} (\underline{P}(E) - Bel(E))$. In terms of the Möbius inverse, this can be equivalently expressed as:

$$d(\underline{P}, Bel) = \sum_{E \subseteq \mathcal{X}} \left( \underline{P}(E) - \sum_{B \subseteq E} m(B) \right). \tag{2}$$

Let $\mathcal{C}_\infty'(\underline{P})$ denote the class of undominated outer approximations of $\underline{P}$ in $\mathcal{C}_\infty$.

**Proposition 2.** *Let $\underline{P} : \mathcal{P}(X) \to [0,1]$ be a coherent lower probability, and consider the problem of minimizing (2) where $m$ is subject to the following constraints:*

$$\sum_{B \subseteq \mathcal{X}} m(B) = 1, \qquad m(B) \geq 0 \quad \forall B \subseteq \mathcal{X}. \tag{LP-bel.1}$$

$$\sum_{B \subseteq E} m(B) \leq \underline{P}(E) \quad \forall E \subseteq \mathcal{X}. \tag{LP-bel.2}$$

1. *The feasible region of this linear programming problem is non-empty.*
2. *Any optimal solution of the linear programming problem belongs to $\mathcal{C}_\infty'(\underline{P})$.*
3. *If for a fixed event $A$ we add the constraint*

$$\sum_{B \subseteq A} m(B) = \underline{P}(A), \tag{LP-bel.3A}$$

*then the feasible region of the new linear programming problem is non-empty, any optimal solution $Bel$ belongs to $\mathcal{C}_\infty'(\underline{P})$ and satisfies $Bel(A) = \underline{P}(A)$.*

4. *If $\mathcal{C}''_\infty(\underline{P})$ denotes the union, for every $A \subseteq X$, of the sets of belief functions that minimize* (2) *subject to* (LP-bel.1)–(LP-bel.3A), *then for any event $E$ it holds that $\underline{P}(E) = \max_{\underline{Q} \in \mathcal{C}''_\infty(\underline{P})} \underline{Q}(E)$.*

This result parallels much of our work in [2]: it tells us that we can obtain undominated outer approximations by means of linear programming, and that we can guarantee the equality $Bel(A) = \underline{P}(A)$ for a fixed event $A$ just by adding the constraint (LP-bel.3A). Some detailed comments about the complexity associated with solving the linear programming problem (LP-bel.1)–(LP-bel.2) in Property 2 can be found in [12].

The main difference with Property 1 is that undominated outer approximations in $\mathcal{C}''_\infty(\underline{P})$ may not agree with $\underline{P}$ on singletons, and also they may not determine the same order on $\mathcal{X}$. Since belief functions are in particular 2-monotone, any outer approximation in $\mathcal{C}_\infty$ is also an outer approximation in $\mathcal{C}_2$. However, we do not have the inclusion $\mathcal{C}'_\infty(\underline{P}) \subseteq \mathcal{C}'_2(\underline{P})$: an undominated outer approximation in $\mathcal{C}_\infty$ may be dominated in $\mathcal{C}_2$, as we shall see in Example 1.

### 3.3 Outer Approximations in $\mathcal{C}_\Pi$

We focus now on the subfamily of belief functions given by necessity measures. Taking conjugacy into account, a necessity measure $N^*$ outer approximates a coherent lower probability $\underline{P}$ if and only if its conjugate possibility measure $\Pi^*$ outer approximates the conjugate upper probability $\overline{P}$ of $\underline{P}$, in the sense that $\overline{P}(A) \leq \Pi^*(A)$ for every $A \subseteq \mathcal{X}$. Since possibility measures appear more frequently in the literature than necessity measures, we shall formulate the problem in this equivalent manner.

Let $\mathcal{C}'_\Pi(\overline{P})$ denote the class of possibility measures $\Pi^*$ that outer approximate $\overline{P}$ and are *non-dominating* in $\mathcal{C}_\Pi(\overline{P})$, meaning that there is no other $\Pi'$ in $\mathcal{C}_\Pi(\overline{P})$ such that $\overline{P} \leq \Pi' \lneq \Pi^*$. Our next result characterizes this class.

**Proposition 3.** *Let $\overline{P} : \mathcal{P}(\mathcal{X}) \to [0,1]$ be a coherent upper probability satisfying $\overline{P}(\{x_i\}) > 0$ for any $x_i \in \mathcal{X}$. For any permutation $\sigma$ of $\{1, \ldots, n\}$, define $\Pi_\sigma : \mathcal{P}(\mathcal{X}) \to [0,1]$ by:*

$$\Pi_\sigma(\{x_{\sigma(1)}\}) = \overline{P}(\{x_{\sigma(1)}\}) \text{ and}$$

$$\Pi_\sigma(\{x_{\sigma(i)}\}) = \max_{A \in \mathcal{A}_{\sigma(i)}} \overline{P}\left(A \cup \{x_{\sigma(i)}\}\right), \text{ where for every } i > 1\text{:}$$

$$\mathcal{A}_{\sigma(i)} = \left\{ A \subseteq \{x_{\sigma(1)}, \ldots, x_{\sigma(i-1)}\} \mid \overline{P}\left(A \cup \{x_{\sigma(i)}\}\right) > \max_{x \in A} \Pi_\sigma(\{x\}) \right\},$$

*and let $\Pi_\sigma(A) = \max_{x \in A} \Pi_\sigma(\{x\})$ for every other $A \subseteq \mathcal{X}$. Then:*

1. *$\mathcal{C}'_\Pi(\overline{P}) = \{\Pi_\sigma : \sigma \in S_n\}$, where $S_n$ is the set of permutations of $\{1, \ldots, n\}$.*
2. *For every event $A \subseteq \mathcal{X}$, $\overline{P}(A) = \min_{\sigma \in S_n} \Pi_\sigma(A)$.*

This result provides us with a simple constructive method for obtaining the undominated outer approximations of $\overline{P}$ in $\mathcal{C}_\Pi$. We also deduce that there are at most $n!$ different undominated outer approximations. It is not difficult to show that this bound is tight.

In this result, we are assuming that $\overline{P}(\{x_i\}) > 0$ for every $x_i \in \mathcal{X}$. This assumption is not restrictive: if we consider the set $\mathcal{X}^* = \{x \in \mathcal{X} \mid \overline{P}(\{x\}) > 0\}$, that is bound to be non-empty due to the coherence of $\overline{P}$, there exists a one-to-one correspondence between the credal sets $\mathcal{M}_1 := \{P : P(A) \leq \overline{P}(A) \; \forall A \subseteq \mathcal{X}\}$ and $\mathcal{M}_2 := \{P : P(A) \leq \overline{P}(A) \; \forall A \subseteq \mathcal{X}^*\}$, because any $P \in \mathcal{M}_1$ satisfies $P(\mathcal{X} \setminus \mathcal{X}^*) = 0$. As a consequence, any non-dominating outer approximation $\Pi^*$ of the restriction of $\overline{P}$ to $\mathcal{P}(\mathcal{X}^*)$ can be extended to a non-dominating outer approximation $\Pi'$ of $\overline{P}$, simply by making $\Pi'(\{x\}) = \Pi^*(\{x\})$ if $x \in \mathcal{X}^*$, $\Pi'(\{x\}) = 0$ if $x \in \mathcal{X} \backslash \mathcal{X}^*$ and $\Pi'(A) = \max_{x \in A} \Pi'(\{x\}) \; \forall A \subseteq \mathcal{X}$.

*Remark 1.* A somewhat related procedure to that in Property 3 was considered by Dubois and Prade in [13] and [14, Sect. 3.3] with the name of *Optimal Mass Allocation Procedure*; they used it to deal with the problem of outer approximating belief functions by means of possibility measures. In their formulation, given a permutation $\sigma$, they consider the nested family of events $E_j^\sigma = \{x_{\sigma(1)}, \ldots, x_{\sigma(j)}\}$ for $j = 1, \ldots, n$. If $A_1, \ldots, A_k$ are the focal events of the initial belief function to be outer approximated, for every $i = 1, \ldots, k$ they define the value $f_\sigma(i) = \min\{j \mid A_i \subseteq E_j^\sigma\}$, and from it they define the mass of $E_j^\sigma$ by:

$$m^\sigma(E_j^\sigma) = \sum_{i: f_\sigma(i) = j} m(A_i), \quad \forall j = 1, \ldots, n.$$

It holds that $m^\sigma(E_1^\sigma) + \ldots + m^\sigma(E_n^\sigma) = 1$ and $E_1^\sigma \subseteq \ldots \subseteq E_n^\sigma$, so $m^\sigma$ defines a possibility measure by means of the formula $\Pi(A) = \sum_{E_j^\sigma \cap A \neq \emptyset} m^\sigma(E_j^\sigma)$. Although this possibility measure does not coincide with the one we have denoted $\Pi_\sigma$ in Property 3, in the end both procedures give rise to all elements in $\mathcal{C}_\Pi'(\overline{P})$. Note, nevertheless, that the procedure in [14] may, unlike ours, also produce dominating outer approximations. ♦

Although Property 3 provides a procedure for determining non-dominating outer approximations in $\mathcal{C}_\Pi$, we should be aware that the non-dominating outer approximations in $\mathcal{C}_\Pi$ may be conjugate to necessity measures that are dominated in $\mathcal{C}_\infty$, as our next example shows:

*Example 1.* Let us consider a four-element space $\mathcal{X}$ and the lower probability $\underline{P}$ given in Table 1. To see that it is coherent, note that it is the lower envelope of the probabilities $(0.1, 0, 0.4, 0.5)$, $(0.4, 0.1, 0.2, 0.3)$ and $(0.3, 0.3, 0, 0.4)$. If we minimize Eq. (2) with constraints (LP-bel.1)–(LP-bel.2), we obtain the optimal solutions $Bel_1$ and $Bel_2$ as well as their convex combinations. If we add the additional constraint (LP-bel.3A) with $A = \{x_3, x_4\}$, we obtain a linear programming problem with infinite solutions; one of them is $Bel_3$. Table 1 also gives an undominated 2-monotone lower probability $\underline{Q}$ that outer approximates $\underline{P}$. It holds that $Bel_2$ is dominated by $\underline{Q}$, whence we see that $Bel_2$ is an undominated outer approximation of $\underline{P}$ in $\mathcal{C}_\infty$, but not in $\mathcal{C}_2$.

Let us now apply the procedure in Property 3 to obtain the possibility measure associated with the permutation $\sigma_1 = (1,2,3,4)$. First of all, we define $\Pi_{\sigma_1}(\{x_1\}) = \overline{P}(\{x_1\}) = 0.4$. Then:

$$\mathcal{A}_2 = \{A \subseteq \{x_1\} \mid \overline{P}(A \cup \{x_2\}) > \max_{x \in A} \Pi_{\sigma_1}(\{x\})\} = \{\emptyset, \{x_1\}\}, \text{ and}$$
$$\Pi_{\sigma_1}(\{x_2\}) = \max\{\overline{P}(\emptyset \cup \{x_2\}), \overline{P}(\{x_1\} \cup \{x_2\})\} = \overline{P}(\{x_1, x_2\}) = 0.6.$$

Iterating the procedure,

$$\mathcal{A}_3 = \{A \subseteq \{x_1, x_2\} \mid \overline{P}(A \cup \{x_3\}) > \max_{x \in A} \Pi_{\sigma_1}(\{x\})\} = \{\emptyset, \{x_1\}, \{x_1, x_2\}\},$$

whence $\Pi_{\sigma_1}(\{x_3\}) = \overline{P}(\{x_1, x_2, x_3\}) = 0.7$, and finally, $\Pi_{\sigma_1}(\{x_4\}) = 1$. The associated possibility measure is depicted in Table 1. Its conjugate necessity measure $N_{\sigma_1}$ is dominated by $Bel_3$. ♦

**Table 1.** Coherent lower probability from Example 1 and its outer approximations.

| $A$ | $\underline{P}(A)$ | $\overline{P}(A)$ | $\underline{Q}$ | $Bel_1$ | $Bel_2$ | $Bel_3$ | $\Pi_{\sigma_1}$ | $N_{\sigma_1}$ |
|---|---|---|---|---|---|---|---|---|
| $\{x_1\}$ | 0.1 | 0.4 | 0.1 | 0.1 | 0.1 | 0.1 | 0.4 | 0 |
| $\{x_2\}$ | 0 | 0.3 | 0 | 0 | 0 | 0 | 0.6 | 0 |
| $\{x_3\}$ | 0 | 0.4 | 0 | 0 | 0 | 0 | 0.7 | 0 |
| $\{x_4\}$ | 0.3 | 0.5 | 0.3 | 0.3 | 0.3 | 0.3 | 1 | 0.3 |
| $\{x_1, x_2\}$ | 0.1 | 0.6 | 0.1 | 0.1 | 0.1 | 0.1 | 0.6 | 0 |
| $\{x_1, x_3\}$ | 0.3 | 0.6 | 0.3 | 0.2 | 0.3 | 0.1 | 0.7 | 0 |
| $\{x_1, x_4\}$ | 0.6 | 0.7 | 0.5 | 0.6 | 0.5 | 0.6 | 1 | 0.3 |
| $\{x_2, x_3\}$ | 0.3 | 0.4 | 0.2 | 0.3 | 0.2 | 0.2 | 0.7 | 0 |
| $\{x_2, x_4\}$ | 0.4 | 0.7 | 0.4 | 0.3 | 0.4 | 0.3 | 1 | 0.3 |
| $\{x_3, x_4\}$ | 0.4 | 0.9 | 0.4 | 0.3 | 0.3 | 0.4 | 1 | 0.4 |
| $\{x_1, x_2, x_3\}$ | 0.5 | 0.7 | 0.5 | 0.5 | 0.5 | 0.4 | 0.7 | 0 |
| $\{x_1, x_2, x_4\}$ | 0.6 | 1 | 0.6 | 0.6 | 0.6 | 0.6 | 1 | 0.3 |
| $\{x_1, x_3, x_4\}$ | 0.7 | 1 | 0.7 | 0.7 | 0.7 | 0.7 | 1 | 0.4 |
| $\{x_2, x_3, x_4\}$ | 0.6 | 0.9 | 0.6 | 0.6 | 0.6 | 0.6 | 1 | 0.6 |
| $\mathcal{X}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

This example also shows that the non-dominating outer approximations in $\mathcal{C}_\Pi$ do not preserve the order between the events, in the sense that $\overline{P}(A) = \overline{P}(B) \not\Rightarrow \Pi(A) = \Pi(B)$ and $\overline{P}(A) < \overline{P}(B) \not\Rightarrow \Pi(A) \leq \Pi(B)$. To see this, it suffices to compare $\overline{P}$ and $\Pi_{\sigma_1}$ on singletons. A procedure for defining non-dominating outer approximations in $\mathcal{C}_\Pi$ that preserve the ordered preferences between the events can be found in [11, Sect. 6.3].

## 4 Conclusions

In this paper, we have investigated the problem of outer approximating a coherent lower probability by means of belief functions. We have focused on those belief functions that are at the same time as close as possible to the initial model, while not adding new information, and we have shown that we can obtain these by means of a linear programming problem, and that they allow us to retrieve the initial coherent lower probability.

In the particular case of possibility measures we have provided a constructive procedure for obtaining the non-dominating outer approximations, proving thus that their number is upper bounded by $n!$. Our procedure is related to the optimal mass allocation procedure of Dubois and Prade.

As future lines of research, we would like to consider other particular families of belief functions, such as probability boxes, and to look at the representation in terms of multi-valued mappings. In addition, we would like to investigate how to elicit an outer approximation among all of the possible ones.

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## References

1. Walley, P.: Statistical Reasoning with Imprecise Probabilities. Chapman and Hall, London (1991)
2. Montes, I., Miranda, E., Vicig, P.: 2-monotone outer approximations of coherent lower probabilities. Int. J. Approximate Reasoning. **101**, 181–205 (2018)
3. Shapley, L.S.: Cores of convex games. Int. J. Game Theor. **1**, 11–26 (1971)
4. Choquet, G.: Theory of capacities. Annales de l'Institut Fourier **5**, 131–295 (1953–1954)
5. Shafer, G.: A Mathematical Theory of Evidence. Princeton University Press, Princeton (1976)
6. Nguyen, H.T.: On random sets and belief functions. J. Math. Anal. Appl. **65**(3), 531–542 (1978)
7. Troffaes, M.C.M., Destercke, S.: Probability boxes on totally preordered spaces for multivariate modelling. Int. J. Approximate Reasoning **52**(6), 767–791 (2011)
8. Dubois, D., Prade, H.: Possibility theory: qualitative and quantitative aspects. In: Smets, P. (ed.) Handbook on Defeasible Reasoning and Uncertainty Management Systems. Volume 1: Quantified Representation of Uncertainty and Imprecision, pp. 169–226. Kluwer Academic Publishers, Dordrecht (1998)
9. de Cooman, G., Troffaes, M.C.M., Miranda, E.: $n$-Monotone exact functionals. J. Math. Anal. Appl. **347**, 143–156 (2008)
10. Dubois, D., Prade, H.: Possibility Theory. Plenum Press, New York (1988)
11. Baroni, P., Vicig, P.: An uncertainty interchange format with imprecise probabilities. Int. J. Approximate Reasoning **40**, 147–180 (2005)
12. Quaeghebeur, E.: Completely monotone outer approximations of lower probabilities on finite possibility spaces. In: Li, S., Wang, X., Okazaki, Y., Kawabe, J., Murofushi, T., Guan, L. (eds.) Nonlinear Mathematics for Uncertainty and its Applications. Advances in Intelligent and Soft Computing, vol. 100. Springer, Heidelberg (2011). https://doi.org/10.1007/978-3-642-22833-9_20

13. Dubois, D., Prade, H.: Fuzzy sets and statistical data. Eur. J. Oper. Res. **25**(3), 345–356 (1986)
14. Dubois, D., Prade, H.: Consonant approximations of belief functions. Int. J. Approximate Reasoning **4**(5–6), 419–449 (1990)