OKA PRINCIPLE FOR LEVI FLAT MANIFOLDS

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In memory of Paolo, colleague and friend

1. Introduction

The name of *Oka principle*, or *Oka-Grauert principle*, is traditionally used to refer to the holomorphic incarnation of the homotopy principle: on a Stein space, every problem that can be solved in the continuous category, can be solved in the holomorphic category as well; this line of thought originated from a paper by Oka [13], where he shows that a topologically trivial line bundle on a domain of holomorphy is holomorphically trivial. The underlying idea was further explored in Grauert's work on the classification of holomorphic fiber bundles [6, 7, 8]; subsequently, inspired by Gromov's seminal paper [9], Forstneric and others developed the Oka principle into a well formed and exhaustive theory (see [3]).

In this note, we begin the study of the same kind of questions on a Levi-flat manifold; more precisely, we try to obtain a classification of CR-bundles on a semiholomorphic foliation of type $(n, 1)$. Our investigation should only be considered a preliminary exploration, as it deals only with some particular cases, either in terms of regularity or bidegree of the bundle, and partial results.

In order to make our intent clearer, we anticipate some of the results and notions presented in the paper. We refer to Sections 2 and 3 for the precise definitions of the objects involved.

Given a (smooth) semiholomorphic foliation X of type (n, d) let

$$\mathsf{Vect}^{(m,l)}_{\mathsf{top}}(X) := \Big\{\text{topological vector bundles of birank } (m, l)\Big\}\,,$$

$$\mathsf{Vect}^{(m,l)}_{\mathsf{cr}}(X) := \Big\{\text{smooth CR vector bundles of birank } (m, l)\Big\}\,.$$

Elements of $\mathsf{Vect}^{(m,l)}_{\mathsf{cr}}(X)$ are called CR *vectors bundles*.

If X is real analytic let

$$\mathsf{Vect}^{(m,l)}_{\mathsf{cr,an}}(X) := \Big\{\text{real analytic CR vector bundles of birank } (m, l)\Big\}\,.$$

The CR analogous of the Oka-Grauert principle can be formulated in the following way:

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for strongly 1-complete semiholomorphic foliations X of type $(n,1)$ the natural map

$$\epsilon_X : \mathsf{Vect}_{\mathsf{cr}}^{(m,l)}(X) \longrightarrow \mathsf{Vect}_{\mathsf{top}}^{(m,l)}(X)$$

is a bijection.

Precise definitions can be found in section 2. Let us immediately give a counterexample to the real analytic case of the result.

Example 1.1. Let $X = \mathbb{C}\times\mathbb{R}$, $\mathcal{S}^\omega$ be the sheaf of germs of real analytic CR functions in X. Then, since $H^1(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) = 0$ we have

$$H^1(X,\mathcal{S}^\omega) \simeq H^1(X,\mathcal{S}^{\omega\,*})$$

i.e. the real analytic CR bundles of birank $(1,0)$ are (topologically trivial and) parametrized by the cohomology group $H^1(X,\mathcal{S}^\omega)$ which is $\neq 0$ (see [1]). Thus the Oka-Grauert principle fails in $\mathsf{Vect}_{\mathsf{cr,an}}^{(m,l)}(X)$.

As in the holomorphic category, the validity of the Oka-Grauert principle can be rephrased in terms or homotopy of CR maps.

Indeed, every CR vector bundle E of birank (m,l) embeds, as a topological bundle in the trivial bundle $X\times(\mathbb{C}^M\times\mathbb{R}^L)$ for some $M,L\in\mathbb{Z}$. Then, $\mathsf{E}\simeq f^*\mathsf{U}_{m,l}^{M,L}$, where $f: X\to G_{m,l}^{M,L}$ sends $x\in X$ to E_x, $G_{m,l}^{M,L}$ and $\mathsf{U}_{m,l}^{M,L}$ being appropriate CR versions of the Grassmannian and the universal bundle - we refer to section 3 for the definitions. Therefore surjectivity of ϵ_X will be be a consequence of the following assertion:

(i) every continuous map $f: X\to G_{m,l}^{M,L}$ is homotopic to a CR map.

As for injectivity let $\mathsf{E},\mathsf{E}'\in\mathsf{Vect}_{\mathsf{cr}}^{(m,l)}(X)$, $\{h_{ij}\}$, $\{h'_{ij}\}$ their cocycles (associated to a trivializing open covering $\{U_i\}$). Let $Z_i := U_i\times\mathsf{G}_{m,l}$ ($\mathsf{G}_{m,l}$ is an appropriate group of matrices, see section 2) and $Z := \amalg_i Z_i/\!\sim$ where $\sim$ identifies (x,v) to (x,v'), $x\in U_i\cap U_j$, where $v' = h'_{ij}(x)vh_{ji}(x)$.

Then, $Z\to X$ is a CR bundle with fiber $\mathsf{G}_{m,l}$ and as it easily seen, topological (respectively CR) isomorphisms $\mathsf{E}\simeq\mathsf{E}'$ correspond to continuous (respectively CR) sections of $Z\to X$. Therefore the sentence ϵ_X is injective will be a consequence of the following assertion:

(ii) every continuous section $X\to Z$ is homotopic to a CR section.

However, a number of technical difficulties arise, when trying to adapt this line of proof to the CR case, so our approach steers, in part, toward cohomological methods.

We present the main definitions in section 2, recalling the concepts of semiholomorphic foliation and CR-bundles from our previous work [12]; in section 3, we define the CR Grassmannian and we show that it has a natural structure of semiholomorphic foliation. Section 4 recalls the concept of 1-completeness from [12] and presents some vanishing results for cohomology.

Sections 5 and 6 contain the results we obtained on the Oka-Grauert principle. In section 5, we examine the case of real analytic vector bundles: as we pointed out, we cannot hope for an equivalence between topological and CR real analytic classifications, nonetheless the real analytic structure proves to be of help in proving that the topological and the CR smooth classifications are equivalent. We have the following result (see Theorems 5.1, 5.2 and 5.3):

Theorem. *Let X be a strongly 1-complete real analytic semiholomorphic foliation of type $(n, 1)$. Let $\mathsf{E} \to X$, $\mathsf{F} \to X$ be topologically equivalent real analytic CR bundles of birank (m, l). Then:*

(1) *if $\mathsf{E} \to X$ is topologically trivial, then it is (smoothly) CR trivial;*
(2) *if $l = 0$, $\mathsf{E} \to X$ and $\mathsf{F} \to X$ are (smoothly) CR equivalent;*
(3) *if $l = 1$ and $H^1(X, \mathcal{T}) = H^1(X, \mathbb{Z}) = 0$, then $\mathsf{E} \to X$ and $\mathsf{F} \to X$ are (smoothly) CR equivalent.*

The sheaf $\mathcal{T}$ is the sheaf of germs of smooth functions which are constant on the leaves of X.

In section 6, we tackle the problem for smooth CR bundles: we manage to give a complete answer for $(1, 0)$ bundles (Theorem 6.1), reducing the problem for $(1, 1)$ to a cohomological property (resembling of the approach to the classical problem presented in [2]). We also indicate a possible route to deal with the general case of (m, l) bundles, by reducing it to the corresponding problem for $(m, 0)$ and $(0, l)$ bundles.

2. Preliminaries on semiholomorphic foliations

We will briefly introduce the objects that we are going to use and study in this note; we refer also to [11, 12] for a broader discussion of semiholomorphic foliations.

2.1. **Main definitions.** We recall that a *semiholomorphic foliation of type* (n, d) is a (connected) smooth foliation X whose local models are subdomains $U_j = V_j \times B_j$ of $\mathbb{C}^n \times \mathbb{R}^d$ and whose local changes of coordinates $(z_k, t_k) \mapsto (z_j, t_j)$ are of the form

$$\begin{cases} z_j = f_{jk}(z_k, t_k) \\ t_j = g_{jk}(t_k), \end{cases} \tag{1}$$

where f_{jk}, g_{jk} are smooth and f_{jk} is holomorphic with respect to z_k. If we replace $\mathbb{R}^d$ by $\mathbb{C}^d$ and we suppose that f and h are holomorphic we get the notion of *holomorphic foliation of codimension d*.

Local coordinates $z_j^1, \ldots, z_j^n, t_j^1, \ldots, t_j^d$ satisfying (1) are called *local distinguished coordinates.*

A closed subset $Y \subset X$ is said to a *subsemiholomorphic foliation* of X if for every point $p \in Y$ there exist local distinguished coordinates $z^1, \dots, z^n, t^1, \dots, t^d$ on a neighborhood U of y in X such that

$$U \cap Y = \Big\{ x \in U : z_{m+1} = \ldots = z_n = t_{s+1} = \ldots = t_d = 0 \Big\}.$$

It follows that Y is a semiholomorphic foliation of type (m, s).

Example 2.1. $\mathbb{C} \times \mathbb{R}$ is not a subsemiholomorphic foliation of $\mathbb{C}^2$ considered as a foliation of type $(2, 0)$ but it is if $\mathbb{C}^2$ is considered as a foliation of type $(1, 2)$.

We denote $\mathcal{S} = \mathcal{S}_X$ the sheaf of germs of smooth CR functions in X. If X is real analytic we define $\mathcal{S}^\omega$, the sheaf of germs of real analytic CR functions in X. For every open set U in X we consider on $\mathcal{S}_X(U)$ the Fréchet topology induced by $C^\infty(U)$. Clearly, $\mathcal{S}(U)$ is closed subset. If X, Y are semiholomorphic foliations of types (m, s), (m', s') a map $f : X \to Y$ is said to be CR if $f_* \mathcal{S}_Y \subset \mathcal{S}_X$. We denote $\mathcal{CR}(X, Y)$ the set of all CR maps $X \to Y$.

The sheaf of germs of smooth functions which are locally constant on the leaves is denoted by $\mathcal{T} = \mathcal{T}_X$. If X is real analytic, $\mathcal{T}^\omega \subset \mathcal{T}$ is the subset of those germs which are real analytic.

2.2. **Complexification.** A real analytic semiholomorphic foliation of type (n, d) can be *complexified*, essentially in a unique way: there exists a holomorphic foliation $\widetilde{X}$ of type (n, d) with a closed real analytic CR embedding $X \hookrightarrow \widetilde{X}$ (cfr. [14, Theorem 5.1]). In particular, X is a Levi flat submanifold of $\widetilde{X}$

In order to construct $\widetilde{X}$, we consider a covering by distinguished domains $\{U_j = V_j \times B_j\}$ and we complexify each B_j in such a way to obtain domains $\widetilde{U}_j$ in $\mathbb{C}^n \times \mathbb{C}^d$. The domains $\widetilde{U}_j$ are patched together by the local change of coordinates

$$\begin{cases} z_j = \widetilde{f}_{jk}(z_k, \tau_k) \\ \tau_j = \widetilde{g}_{jk}(\tau_k) \end{cases}$$

obtained complexifying the (vector) variable t_k by $\tau_k = t_k + i\theta_k$ in the real analytic functions f_{jk} and g_{jk} (cfr. (1)).

In the sequel, we will call complexification of X every neighborhood of X in $\widetilde{X}$.

2.3. **CR-bundles.** Let $\mathsf{G}_{m,l}$ be the group of matrices

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where $A \in \mathrm{GL}(\mathrm{m}, \mathbb{C})$, $B \in \mathrm{GL}(\mathrm{m}, \mathrm{l}, \mathbb{C})$ (the set of $m \times l$ matrices with complex entries) and $C \in \mathrm{GL}(\mathrm{l}, \mathbb{R})$. We set $\mathsf{G}_{\mathsf{m},\mathsf{0}} = \mathrm{GL}(\mathrm{m}, \mathbb{C})$, $\mathsf{G}_{\mathsf{0},\mathsf{l}} = \mathrm{GL}(\mathrm{l}, \mathbb{R})$.

To each matrix $M \in \mathsf{G}_{\mathsf{m},\mathsf{l}}$ we associate the linear transformation $\mathbb{C}^m \times \mathbb{R}^l \to \mathbb{C}^m \times \mathbb{R}^l$ given by

$$(\xi, u) \mapsto (A\xi + Bu, Cu),$$

$(\xi, u) \in \mathbb{C}^m \times \mathbb{R}^l$.

Let X be a semiholomorphic foliation of type (n, d). A CR *bundle of birank (m,l)* is a vector bundle $\pi : \mathsf{E} \to X$ whose cocycle with respect to a countable, trivializing distinguished covering $\{U_j\}$ is a set of smooth CR map $\gamma_{jk} : U_j \cap U_k \to \mathsf{G}_{\mathsf{m},\mathsf{l}}$

$$\gamma_{jk} = \begin{pmatrix} A_{jk} & B_{jk} \\ 0 & C_{jk} \end{pmatrix} \tag{2}$$

where $C_{jk} = C_{jk}(t)$ is a matrix with smooth entries and A_{jk}, B_{jk} are matrices with smooth CR entries. Thus E is foliated by complex leaves of dimension $n + m$ and real codimension $d + l$.

We denote $\mathcal{E} = \mathcal{E}_X$ the sheaf of germs of smooth CR sections of E and we set $\mathcal{E}^\omega = \mathcal{E} \cap C^\omega$ whenever X and E are real analytic. If U is an open subset of X the topology on $\mathcal{E}(U)$, the space of CR sections $s : U \to \mathsf{E}$ is defined as follows. We fix a trivializing distinguished covering $\{U_j\}_{\in\mathbb{N}}$ of X and let $\{\gamma_{jk}\}$ be the corresponding cocycle. Then $\mathcal{E}(U_j \cap U) \simeq \mathcal{CR}(U_j \cap U, \mathbb{C}^m \times \mathbb{R}^l)$ therefore $\mathcal{E}(U)$ identifies to

$$\Big\{(f_j) \in \prod_{j\in\mathbb{N}} \mathcal{CR}(U_j \cap U, \mathbb{C}^m \times \mathbb{R}^l) : f_{j|U_j\cap U_k\cap U} = \gamma_{jk} f_{k|U_j\cap U_k\cap U}\Big\}$$

which is (a closed subspace of) a Fréchet space. Let X be a semiholomorphic foliation of type (n, d). Then

- the tangent bundle $\mathsf{T}X$ of X is a CR-bundle of birank (n, d).
- the bundle $\mathsf{T}_{\mathcal{L}} = \mathsf{T}^{1,0}_{\mathcal{L}}X$ of the holomorphic tangent vectors to the leaves of X is a CR-bundle of type $(n, 0)$.
- the transverse bundle N_{tr} (to the leaves of X) is a CR-bundle of type $(0, d)$.

Observe that If X is embedded in a complex manifold Z, its transverse $\mathsf{T}Z/\mathsf{T}X$ is not a CR bundle in general.

If E is a real analytic CR bundle of type (m, l), given by the cocycle (2), then complexifying γ_{jk} by

$$\widetilde{\gamma}_{jk} = \begin{pmatrix} \widetilde{A}_{jk} & \widetilde{B}_{jk} \\ 0 & \widetilde{C}_{jk} \end{pmatrix}$$

and the fibre $\mathbb{C}^m \times \mathbb{R}^l$ by $\mathbb{C}^m \times \mathbb{C}^l$ we obtain a holomorphic vector bundle $\widetilde{\mathsf{E}} \to \widetilde{X}$ of rank (m, l). We will call $\widetilde{\mathsf{E}}$ the *complexification* of E. A section s of $\widetilde{\mathsf{E}}$ is locally given by a couple (f_i, g_i) where $f_i : U_i \to \mathbb{C}^m$, $g_i : U_i \to \mathbb{C}^l$ such that

$$\begin{cases} f_i = \widetilde{A}_{ij} f_j + \widetilde{B}_{ij} g_j \\ g_i = \widetilde{C}_{ij} g_j \end{cases}.$$

If $s = \{(f_i, g_i)\}_i$ is a section of $\widetilde{\mathsf{E}}$ then $s_{|X} := \{(f_{i|X}, \mathsf{Re}\, g_{i|X})\}_i$ is a section of E.

3. The CR Grassmannian

Let $\mathrm{Gr}_k(\mathbb{C}^d)$ (resp. $\mathrm{Gr}_k(\mathbb{R}^d)$) denote the grassmannian of the k-dimensional complex (real) subspaces of $\mathbb{C}^d$ (resp. $\mathbb{R}^d$).

Given two pairs (m, l),(M, L) of integers with $0 \leq m \leq M, 0 \leq l \leq L$, let
$\mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$ be the set of the real (linear) subspaces $V \subseteq \mathbb{C}^M \times \mathbb{R}^L$ such that

$$\dim_{\mathbb{R}} V = 2m + l, \quad \dim_{\mathbb{C}}(V \cap (\mathbb{C}^M \times \{0\})) = m.$$

By definition, $\mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$, also denoted $\mathrm{Gr}^{M,L}_{m,l}$, is the *Grassmannian variety* of the CR subspaces of $\mathbb{C}^M \times \mathbb{R}^L$ of type (m, l). We want to prove that $\mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$ is a semiholomorphic foliation.

We consider every $V \in \mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$ as a subspace of $\mathbb{C}^M \times \mathbb{C}^L = \mathbb{C}^{M+L}$ and we denote by $V^{\mathbb{C}} = V \oplus iV$; $V^{\mathbb{C}}$ is an $(m + l)$-dimensional complex subspace of so we have a map

$$\mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L) \to \mathrm{Gr}_{m+l}(\mathbb{C}^{M+L})$$

sending V in $V^{\mathbb{C}}$.

3.1. **Chart description.** The usual charts for the Grassmannian are given as follows: fix a basis $\{e_1, \ldots, e_{M+L}\}$ for $\mathbb{C}^{M+L}$ (so that the first M vectors span $\mathbb{C}^M$ and the second L span $\mathbb{C}^L$), for every $I \subseteq \{1, \ldots, M + L\}$ with $|I| = m + l$, let

$$V^0_I = \mathrm{span}_{\mathbb{C}}\{e_j,\ j \notin I\}$$

and consider

$$U_I = \{\Lambda \in \mathrm{Gr}_{m+l}(\mathbb{C}^{M+L})\ :\ \Lambda \cap V^0_I = \{0\}\}\ .$$

$\Lambda \in U_I$ is then represented by the unique $(M + L) \times (m + l)$ matrix A_Λ, with the $(m+l) \times (m+l)$ minor corresponding to the multi-index I is the identity and such that the image of $A_\Lambda : \mathbb{C}^{m+l} \to \mathbb{C}^{M+L}$ is Λ.

The other $(m + l)(M + L - m - l)$ entries of A_Λ are the local coordinates for U_I.

We cut out a real analytic subvariety $\mathcal{Z}$ of U_I with the following equations:

$$a_{jk} = 0 \qquad j = M + 1, \ldots, M + L, \quad k = 1, \ldots, m$$

$$a_{jk} = \overline{a}_{jk} \qquad j = M + 1, \ldots, M + L, \quad k = m + 1, \ldots, m + l\ .$$

We note that, if I contains less than m indexes in $[1, M]$, then no element of U_I can satisfy the first set of equations, because they would contradict the definition of U_I.

To calculate the dimension of $\mathcal{Z}$, let us note that, as the $(m+l) \times (m+l)$ minor defined by I is the identity, then some of the equations written above are redundant.

We stratify $\mathcal{Z}$ as follows; for $A \in \mathcal{Z}$, let A_s be the submatrix formed by the last l columns and L rows, then

$$\mathcal{Z}^r = \{A \in X \ : \ \mathrm{rk}(A_s) = r\} \ .$$

Obviously, the rank of A_s can never be more than l, as the rank of the submatrix given by the first m columns and M rows is at least m.

We note that $\mathcal{Z}^r$ does not intersect U_I if I contains less than r indices in $[M+1, M+L]$.

If we are at a point $x \in \mathcal{Z}^r$, we will have mr equations of the first set already satisfied and lr equations already satisfied in the second set.

Therefore,

$$\dim_{\mathbb{R}} \mathcal{Z}^r = 2m(M-m) + 2l(M-m) + l(L-l) \ .$$

The factors 2 take into account the fact that some of the equations describe complex manifolds.

So, $\mathcal{Z}^l$ is the top dimensional stratum of $\mathcal{Z}$, i.e. the regular points of the irreducible components of top dimension of $\mathcal{Z}$. It corresponds to $\mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$. By the set of equations describing $\mathcal{Z}$, it is obvious that its regular part has the structure of a semiholomorphic foliation.

3.2. **Matrix description.** As it is done with the usual Grassmannian, we describe the CR-Grassmannian as images of linear maps. Consider $A \in \mathsf{M}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$ with maximal rank, then

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$$

with A_1, A_2 with complex coefficients, A_3 with real coefficients; if A is of maximal rank, then the image of A sits in $\mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$.

$A, A' \in \mathsf{M}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$ have the same image if there exists an invertible $h \in \mathsf{G}_{m,l}$ such that $A = A'h$.

Incidentally,

$$\dim_{\mathbb{R}} \mathsf{M}_{m,l}(\mathbb{C}^M \times \mathbb{R}^S) = 2Mm + 2ML + Ll$$

and

$$\dim_{\mathbb{R}} \mathsf{G}_{m,l} = 2m^2 + 2ml - l^2$$

so

$$\dim_{\mathbb{R}} \mathsf{M}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)/\mathsf{G}_{m,l} = 2Mm + 2Ml + Ll - 2m^2 - 2ml - l^2 \ .$$

3.3. **Fibration description.** Let $V \in \mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$, $V_0 = V \cap (\mathbb{C}^M \times \{0\})$, $\dim_{\mathbb{C}} V_0 = m$. Fix a linear subspace $S \subset \mathbb{R}^L$ with $\dim_{\mathbb{R}} S = l$ and a $\mathbb{R}$-linear injective map $\Phi : S \to \mathbb{C}^M$. Then, let $\Gamma_\Phi = \{(\Phi v, v) : v \in S\}$.

We have $V = V_0 \oplus \Gamma_\Phi$. Obviously, given Φ, Φ',

$$V_0 \oplus \Gamma_\Phi = V_0 \oplus \Gamma_{\Phi'}$$

if and only if $\Gamma_{\Phi'} \subseteq V_0 \oplus \Gamma_\Phi$, i.e. if $\Phi'(S) \subseteq V_0 \oplus \Phi(S)$ (viewing V_0 as a subspace of $\mathbb{C}^M$), i.e. if the image of $\Phi - \Phi'$ is contained in V_0.

Let

$$\mathcal{H}(E, V_0) = \{\Phi : S \to \mathbb{C}^M\}/\{\Phi : S \to V_0\}$$

and fix a projection $p : \mathbb{C}^M \times \mathbb{R}^L \to \mathbb{R}^L$, then we have a map

$$\pi : \mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L) \longrightarrow \mathrm{Gr}_m(\mathbb{C}^M) \times \mathrm{Gr}_l(\mathbb{R}^L)$$

given by $V \mapsto (V \cap \mathbb{C}^M, p(V))$, such that $\pi^{-1}(V_0, S) = \mathcal{H}(V_0, S)$.

Again,

$$\dim_{\mathbb{R}} \mathrm{Gr}_m(\mathbb{C}^M) = 2m(M-m), \quad \dim_{\mathbb{R}} \mathrm{Gr}_l(\mathbb{R}^L) = l(L-l)$$

and

$$\dim_{\mathbb{R}} \mathcal{H}(V_0, S) = 2lM - 2lm = 2l(M-m)\ .$$

3.4. **The universal CR bundle $U_{m,l}(M, L)$.** The universal CR bundle $\mathsf{U}_{m,l}(M, L)$, also denoted $\mathsf{U}^{M,L}_{m,l}$ is defined by

$$\mathsf{U}^{M,L}_{m,l} = \Big\{(V, v) \in \mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L) \times \mathbb{C}^M \times \mathbb{R}^L : v \in V\Big\}.$$

Let $\mathsf{E} \to X$ be a CR bundle of type $(m.l)$ over a CR manifold of type (n, d) and let $f : X \to \mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$ denote the continuous map sending the point $x \in X$ to the subspace $\mathsf{E}_x \subset \mathbb{C}^M \times \mathbb{R}^L$ considered as an element of $\mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$. Then E is isomorphic (as a topological bundle) to $f^*\mathsf{U}^{M,L}_{m,l}$. Moreover, if $\{f_t\}_{t\in[0,1]}$ is a homotopy of continuous maps, the bundles $f_t^*\mathsf{U}^{M,L}_{m,l}$ are isomorphic to each other.

If $g : X \to \mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$ is a CR map then $g^*U^{M,L}_{m,l}$ is a CR bundle of birank (m, l); again, equivalence classes of CR bundles correspond to appropriate homotopy classes of CR maps into $\mathrm{Gr}_{m,l}(\mathbb{C}^M \times \mathbb{R}^L)$.

As it is well known, the Oka-Grauert results can be reformulated in terms of homotopies; however, this strategy of proof employs critically the compactness of the classical Grassmannian, whereas the CR Grassmannian is the set of regular points of a (compact) variety, hence non compact. Due to this difficulty, we will move in another direction.

4. Completeness and vanishing theorems

4.1. **1-complete foliations.** Let X be a semiholomorphic foliation X of type (n,d) and $\widetilde{X}$ its complexification. X is said to be *tangentially complete* if X carries a smooth exhaustion function $\phi : X \to \mathbb{R}^+$ which is plurisubharmonic along the leaves and such that $\sup\limits_X \phi = +\infty$.

Let N_{tr} be the transverse bundle to the leaves of X. A metric on the fibres of N_{tr} is an assignement of a distinguished covering $\{U_j\}$ of X and for every j a smooth map λ_j^0 from U_j to the space of symmetric positive $d \times d$ matrices such that

$$\lambda_k^0 = \frac{{}^t\partial g_{jk}}{\partial t_k} \lambda_j^0 \frac{\partial g_{jk}}{\partial t_k}.$$

Denoting ∂ and $\overline{\partial}$ the complex differentiation along the leaves of X, the local tangential forms

$$2\overline{\partial}\partial \log \lambda_j^0 - \overline{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0 = \frac{\lambda_j^0 \overline{\partial}\partial\lambda_j^0 - 2\overline{\partial}\lambda_j^0 \wedge \partial\lambda_j^0}{{\lambda_j^0}^3}$$

$$\overline{\partial}\partial \log \lambda_j^0 - \overline{\partial} \log \lambda_j^0 \wedge \partial \log \lambda_j^0 = \frac{2\lambda_j^0 \overline{\partial}\partial\lambda_j^0 - 3\overline{\partial}\lambda_j^0 \wedge \partial\lambda_j^0}{{\lambda_j^0}^3}$$

actually give global tangential forms ω, Ω.

These forms play the role of the curvature forms of a holomorphic vector bundle on a complex manifold.

In [11, 12] the following definitions were given:

- X is said to be *tranversally 1-complete* (*strongly tranversally 1-complete*) if a metric on the fibres of $N_{\mathrm{tr}|U}$ can be chosen in such a way that the hermitian form associated to $i\omega_{|U}$ $(i\Omega_{|U})$ has at least $n-q+1$ positive eigenvalues;
- X is said to be 1–*complete* (*strongly* 1–*complete*) if it is tangentially and transversally 1-complete (tangentially and strongly transversally 1-complete).

From now on strongly 1-completeness is intended with respect to given exhaustion function ϕ and metric on the fibres of N_{tr}.

Assume now that X is real analytic, strongly $1-$*complete* and let $\widetilde{X}$ be the complexification of X. Then (see [11, Theorems 3.1, 3.3]) we have the following

- there exist an open neighbourhood U of X in $\widetilde{X}$ and a smooth function $u : U \to \mathbb{R}$ such that $u \geq 0$, u is plurisubharmonic in U, strongly plurisubharmonic on $U \setminus X$ and $X = \{u = 0\}$;
- for every $c > 0$, $\overline{X}_c = \{\phi \leq c\}$ is a Stein compact of $\widetilde{X}$ and every smooth CR function on a neighbourhood of $\overline{X}_c$ in X can be approximated in the C^∞ topology by smooth CR functions in X.

4.2. **Vanishing theorems.** Assume that $\widetilde{X}$ is divided by X into two connected components $X_\pm$. Let $\overline{\mathcal{O}}_\pm$ be the sheaf of germs of holomorphic functions in $X_\pm$ smooth up to $\overline{X}_\pm := X_\pm \cup X$ and extend it on $\widetilde{X}$ by 0. Then jump formula for CR functions gives rise to the following (Mayer-Vietoris) exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \overline{\mathcal{O}}_+ \oplus \overline{\mathcal{O}}_- \longrightarrow \mathcal{S} \longrightarrow 0 \tag{3}$$

where $\mathcal{O} := \mathcal{O}_{\widetilde{X}}$ and $\overline{\mathcal{O}}_+ \oplus \overline{\mathcal{O}}_- \to \mathcal{S}$ is defined by $(f \oplus g) = f_{|X} - g_{|X}$. Then, using the results recalled in 4.1 and Kohn's regularity theorem for $\overline{\partial}$ the following was proved in [11, Theorems 4.1, 4.2].

Theorem 4.1. *If X is real analytic strongly 1-complete semiholomorphic foliation of type $(n, 1)$ or a closed, orientable, smooth Levi flat hypersurface in a (connected) Stein manifold D then*

$$H^q(X, \mathcal{S}) = 0$$

for every $q \geq 1$.

Remark 4.1. *Vanishing for $\mathcal{S}^\omega$ (the sheaf of germs of real analytic CR functions in X) fails to be true in general as proved by Andreotti-Nacinovich (see* [1]*).*

In particular, we obtain the following

Theorem 4.2. *Let X be a real analytic strongly 1-complete foliation of type $(n, 1)$ and $\mathcal{T}$ be the sheaf of germs of smooth functions which are locally constant on the leaves. Then*

$$H^q(X, \mathcal{T}) = 0$$

for $q > n$.

Proof. Let Ω^p_{tg} be the sheaf of germs of smooth CR tangential p-forms and $\overline{\partial}_{\mathrm{tg}}$ the tangential operator $\Omega^p_{\mathrm{tg}} \to \Omega^{p+1}_{\mathrm{tg}}$. In view of the tangential Dolbeault-Grothendieck and [11, Theorem 5.2]

$$0 \to \mathcal{T} \to \Omega^0_{\mathrm{tg}} \overset{\overline{\partial}_{\mathrm{tg}}}{\to} \Omega^1_{\mathrm{tg}} \overset{\overline{\partial}_{\mathrm{tg}}}{\to} \ldots \overset{\overline{\partial}_{\mathrm{tg}}}{\to} \Omega^n_{\mathrm{tg}} \to 0$$

is an acyclic resolution of $\mathcal{T}$. Then, de Rham theorem and $\Omega^q_{\mathrm{tg}} = 0$ for $q > n$ imply $H^q(X, \mathcal{T}) = 0$ for $q > n$. $\square$

Remark 4.2. *Cohomology groups $H^q(X, \mathcal{T})$ are in general infinite dimensional. They vanish, for instance, if the foliation has a* parameters space *i.e. the space of the leaves of X is a smooth curve T (see e.g.* [10]*).*

4.2.1. *Vanishing for real analytic CR bundles.* The above method still applies to real analytic CR bundles.

Let $\pi : \mathsf{E} \to X$ be a real analytic CR bundle of birank (m, l) and $\mathcal{E}$ the sheaf of germs of (smooth) sections of a E

Let $\widetilde{\mathsf{E}} \to \widetilde{X}$ be the complexification of E and $\widetilde{\mathcal{E}}$ the sheaf of germs of holomorphic sections of $\widetilde{\mathsf{E}}$. Then it divides $\widetilde{X}$ into two domains $\widetilde{X}_\pm$ whose boundary is X. Denote $\widetilde{\mathcal{E}}_\pm := \widetilde{\mathcal{E}}_{|X_\pm}$ and $\overline{\mathcal{E}}_\pm$ the subsheaf of $\widetilde{\mathcal{E}}_\pm$ on $X_\pm$ of germs holomorphic sections which are smooth up to X extended on $\widetilde{X}$ by 0. We have again the following (Mayer-Vietoris) exact sequence

$$0 \longrightarrow \widetilde{\mathcal{E}} \xrightarrow{\alpha} \overline{\mathcal{E}}_+ \oplus \overline{\mathcal{E}}_- \xrightarrow{\beta} \widetilde{\mathcal{E}}_X \longrightarrow 0$$

where $\widetilde{\mathcal{E}}_X$ is the sheaf of germs of CR section of $\widetilde{\mathsf{E}}_{|X}$ and $\beta(s \oplus \sigma) = s_{|X} - \sigma_{|X}$.

Theorem 4.3. *Let X be an orientable analytic strongly 1-complete semiholomorphic foliation of type $(n, 1)$, $\pi : \mathsf{E} \to X$ a real analytic CR bundle of birank (m, l). Then, if $l = 0$*

i) *$H^q(X, \mathcal{E}) = 0$ for $q \geq 1$;*
ii) *$\Gamma(X, \mathcal{E})$ generates $\mathcal{E}_x$ for every $x \in X$.*

If $l \geq 1$, i) *and* ii) *are still true provided X satisfies the additional condition*

$$H^1(X, \mathcal{T}) = 0.$$

Proof. (Sketch) For $l = 0$ statement ii) is the content of [11, Theorems 5.2]. In the case $l > 0$ we argue as in [11, Theorems 4.1].

Let $X = \bigcup_{r \geq 1}^{+\infty} X_r$ where $\overline{X}_r = \{\phi \leq r\}$. Every $\overline{X}_r$ is a Stein compact so, by the methods developed in [5] and [11, Theorem 4.1] we show that $H^q(\overline{X}_r, \widetilde{\mathcal{E}}_X) = 0$ for $q \geq 1$, $r \geq 1$. It follows that $\widetilde{\mathcal{E}}_{\overline{X}_r}$ is generated by $\Gamma(\overline{X}_r, \widetilde{\mathcal{E}}_{X_r})$. Arguing as in [11, Theorem 4.1] and using Freeman's approximation theorem (cfr. [4, Theorem 1.3]) finally we obtain the vanishing $H^q(X, \widetilde{\mathcal{E}}_X) = 0$ for $q \geq 1$ and from this it easy to derive the vanishing $H^q(X, \mathcal{E}) = 0$ for $q \geq 1$. Likewise we prove that $H^q(X, \mathcal{M}_x\mathcal{E}) = 0$ for $q \geq 1$ where $\mathcal{M}_x$ is the sheaf of germs of CR functions vanishing at a point $x \in \overline{X}_r$. Then ii) follows as in the classical case. □

Remark 4.3. *The condition $H^1(X, \mathcal{T}) = 0$ is necessary. Indeed, let E be the trivial bundle $X \times \mathbb{R}$ considered as CR bundle of type $(0, 1)$. Then $\mathcal{E} = \mathcal{T}$.*

4.2.2. *Vanishing for CR bundles.* The general case of a smooth CR bundle $\pi : \mathsf{E} \to X$ of birank $(m, 0)$ is much more involved. In what follows we just outline the main points of the proof which heavily depends on the thesis work by Sebbar [5, 15, 16]. In order for Sebbar's results to apply in our situation, we examine two coherence conditions.

An $\mathcal{S}_X$-module $\mathcal{F}$ is said to be *coherent* if for every point x of X and for every integer $d \geq 0$ there exist an open neighborhood U of x and an exact sequence of $\mathcal{S}_U$-modules

$$\mathcal{S}_U^{p_d} \longrightarrow \cdots \longrightarrow \mathcal{S}_U^{p_0} \longrightarrow \mathcal{F}_U \longrightarrow 0,$$

where $\mathcal{S}_U = \mathcal{S}_X|_U$ and p_i are non-negative integers.

An $\overline{\mathcal{O}}_+ \oplus \overline{\mathcal{O}}_-$-module $\mathcal{F}$ on $\widetilde{X}$ is said to be *coherent* if for every point x of X and for every integer $d \geq 0$ there exist an open neighborhood U of x and an exact sequence of $(\overline{\mathcal{O}}_+ \oplus \overline{\mathcal{O}}_-)_U$-modules

$$(\overline{\mathcal{O}}_+^{\,p_d} \oplus \overline{\mathcal{O}}_-^{\,p_d})_U \longrightarrow \cdots \longrightarrow (\overline{\mathcal{O}}_+^{\,p_0} \oplus \overline{\mathcal{O}}_-^{\,p_0})_U \longrightarrow \mathcal{F}_U \longrightarrow 0,$$

where p_i are non-negative integers.

Let $\mathcal{F}$ be a coherent $\mathcal{S}$-module and extend it by on $\widetilde{X}$ by 0. Then, the jump formula (3) for the sheaves $\mathcal{S}^r$

$$0 \longrightarrow \mathcal{O}^r \longrightarrow \overline{\mathcal{O}}_+^{\,r} \oplus \overline{\mathcal{O}}_-^{\,r} \xrightarrow{\alpha} \mathcal{S}^r \longrightarrow 0$$

and the results proved by Sebbar in [15, III-3, §1] allow us to prove that

1) $\mathcal{F}$ is a coherent $\overline{\mathcal{O}}_+ \oplus \overline{\mathcal{O}}_-$-module;
2) $H^q(\overline{X}_c, \mathcal{F}) = 0$ for every $q \geq 1$, $c > 0$.

Arguing as in [11, Theorems 3.1, 3.3] (using the approximation for CR functions (see 4.1)) we then obtain the vanishing

$$H^q(\overline{X}, \mathcal{F}) = 0$$

for every $q \geq 1$. In particular

Theorem 4.4. *Let X be an orientable analytic strongly 1-complete semiholomorphic foliation of type $(n, 1)$ and $\mathsf{E} \to X$ a smooth CR bundle of birank $(m, 0)$. Then*

i) $H^q(X, \mathcal{E}) = 0$ *for* $q \geq 1$*;*
ii) $\Gamma(X, \mathcal{E})$ *generates* $\mathcal{E}_x$ *for every* $x \in X$*.*

5. The Oka-Grauert principle for real analytic CR bundles

Theorem 5.1. *Let X be a strongly 1-complete real analytic semiholomorphic foliation of type $(n, 1)$, $\mathsf{E} \to X$ a real analytic CR bundle of birank (m, l). If $\mathsf{E} \to X$ is topologically trivial then it is CR trivial.*

Proof. Let $\widetilde{X}$ be the complexification of X and $\widetilde{\mathsf{E}}$ the complexification of E on a neighborod of X. If E is topologically trivial then $\widetilde{\mathsf{E}}$ is topologically trivial on a neighborod of X (take n continuous sections

$s_1, \dots, s_N : X \to E$ linearly independent at every point $x \in X$ and extend them by continuous sections $\tilde{s}_1, \dots, \tilde{s}_N$ of $\widetilde{\mathsf{E}}$ on a neighborhood of X).

Since $\overline{X}_c = \{\phi \le c\}$, $c > 0$, is a Stein compact of $\widetilde{X}$ ([11, Theorem 3.3]), Grauert theorem implies that $\widetilde{E}$ is holomorphically trivial on a neighborhood of $\overline{X}_c$ in $\widetilde{E}$ and so E is CR trivial on a neighborhood of $\overline{X}_c$ in X.

Let $\mathcal{E}(U) := \Gamma(U, \mathcal{E})$. Consider $X = \bigcup_{j=1}^{+\infty} X_j$. Then for every j the image of the restriction map $r_j : \mathcal{E}(X) \to \mathcal{E}(X_j)$ is everywhere dense. For this is enough to show that the restrictions $r_j^{j+1} : \mathcal{E}(X_{j+1}) \to \mathcal{E}(X_j)$ are dense image. Let $s \in \mathcal{E}(X_j)$. Since E is CR trivial on a neighborhood of $\overline{X}_{j+1}$ in X, there exist CR sections $s_1, s_2, \dots, s_{m+l}$ on $\overline{X}_{j+1}$ such that $s = \sum_{k=1}^{m+l} a_k s_k$ where the $a_k^{'s}$ are CR functions on X_j. The density of the image of r_j is achieved applying an approximation result [11, Theorem 3.3].

To conclude the proof, let F_j be the space of the global sections $s = (s_1, \dots, s_{m+l}) \in \mathcal{E}^{\mathsf{cr}}(X)^{\oplus(m+l)}$ such that $s_1, \dots, s_{m+l}$ generate $\mathsf{E}_{|\overline{X}_j}$, $j = 1, 2, \dots$: clearly all the F_j are open in the Fréchet space $\mathcal{E}^{\mathsf{cr}}(X)^{\oplus(m+l)}$ and $F_1 \supset F_2 \supset \cdots$; let us prove that they are everywhere dense. Indeed, let $s = (s_1, \dots, s_{m+l}) \in \mathcal{E}(X)^{\oplus(m+l)}$ and fix a seminorm $\|.\|_{\alpha,K}$ on $\mathcal{E}(X)^{\oplus(m+l)}$ with $K = \overline{X}_{j_0}$, $j_o > j$. Let

$$U(\alpha, K, \varepsilon) = \Big\{ s' \in \mathcal{E}(X)^{\oplus m+l} : \|s' - s\|_{\alpha,K} < \varepsilon \Big\}.$$

Let $\sigma = (\sigma_1, \dots, \sigma_{m+l})$ where $\sigma_1, \dots, \sigma_{m+l}$ generate $E_{|\overline{X}_{j_0}}$. Then, $s_p = \sum_{q=1}^{m+l} a_{pq}\sigma_q$, $1 \le p \le m+l$, with $a_{pq} \in \mathcal{S}(\overline{X}_{j_0})$. Approximating $\sigma_1, \dots, \sigma_{m+l}$ and the a_{pq} we obtain global sections $\tilde{\sigma}_1, \dots, \tilde{\sigma}_{m+l}$ which generate $\mathsf{E}_{|\overline{X}_{j_0}}$ and such that $\|\tilde{\sigma} - s\|_{\alpha,\overline{X}_{j_0}} < \varepsilon$, $\tilde{\sigma} = (\tilde{\sigma}_1, \dots, \tilde{\sigma}_{m+l})$, and $\tilde{\sigma} \in \mathcal{E}^{\mathsf{cr}}(X)^{\oplus m+l}$ i.e. $\tilde{\sigma} \subset \Big(F_{j_0} \cap U(\alpha, K, \varepsilon)\Big) \subset \Big(F_j \cap U(\alpha, K, \varepsilon)\Big)$. This show that sets F_j, $j \ge 1$ are everywhere sense in $\mathcal{E}(X)^{\oplus m+l}$, so, in view of Baire Lemma, $\cap_{j\ge1} F_j \ne \varnothing$. It follows that there are global sections $\sigma_1, \dots, \sigma_{m+l}$ which generate $\mathcal{E}(X)^{\oplus m+l}$ and consequently that E is CR trivial. □

Theorem 5.2. *Let X be a strongly 1-complete real analytic semiholomorphic foliation of type (n, d). Then two topologically equivalent real analytic CR bundles of birank $(m, 0)$ are CR equivalent.*

Proof. Let $\mathsf{E} \to X$, $\mathsf{F} \to X$ be topologically equivalent CR bundles of birank $(m, 0)$. Let $\mathcal{U} = \{U_j\}_{j\in\mathbb{N}}$ be covering of X by balls of X such that $\mathsf{E}_{|U_j}$ and $\mathsf{F}_{|U_j}$ are trivial for every $j \in \mathbb{N}$ and $\{h_{ij} : U_i \cap U_j \to \mathbb{C}^*\}$, $\{g_{ij} : U_i \cap U_j \to \mathbb{C}^*\}$ the cocycles of E, F respectively, associated to $\mathcal{U}$.

Let $\widetilde{\mathsf{E}} \to \widetilde{X}$, $\widetilde{\mathsf{F}} \to \widetilde{X}$ be the respective complexifications; then, they are topologically equivalent on a neighbourhood of X in $\widetilde{X}$.

Let $X = \bigcup_{r\geq 1}^{+\infty} X_r$ where $\overline{X}_r = \{\phi \leq r\}$. Since $\overline{X}_r$ is a Stein compact (see subsection 4.2), by Grauert Theorem for every r there exists a Stein neighborhood U_r of $\overline{X}_r$ in $\widetilde{X}$ such that $\widetilde{\mathsf{E}}_{|U_r} \overset{\text{hol}}{\simeq} \widetilde{\mathsf{F}}_{|U_r}$. Moreover, for every compact subset $K \subset X$ there exist sections $u_1, u_2, \ldots, u_N \in \mathcal{E}_{|K}$, $N = N(K)$, which generate $\mathcal{E}$ on K.

Consider the sheaf $\mathsf{Hom}(\mathcal{E}, \mathcal{F})$; it is a Fréchet sheaf. Every local isomorphism $\mathsf{E} \overset{\text{cr}}{\simeq} \mathsf{F}$ defines a local section of $\mathsf{Hom}(\mathcal{E}, \mathcal{F})$.

Let s be a section in $\mathsf{Hom}(\mathcal{E}, \mathcal{F})(U)$, $U \subset X$ which defines an isomorphism $\mathcal{E}_{|U} \simeq \mathcal{F}_{|U}$. If s' near s (in the Fréchet topology on the space of sections $\mathsf{Hom}(\mathcal{E}, \mathcal{F})(U)$) then s' is a local isomorphism.

Let $s_1 \in \mathsf{Hom}(\mathcal{E}, \mathcal{F})(\overline{X}_1)$ be the section determined by the isomorphism $\widetilde{\mathsf{E}}_{|U_1} \overset{\text{hol}}{\simeq} \widetilde{\mathsf{F}}_{|U_1}$. Then there exists δ_1 such that if a section $s' \in \mathsf{Hom}(\mathcal{E}, \mathcal{F})(\overline{X}_1)$ satisfies $\|s' - s_1\|_{\overline{X}_1} < \delta_1$, where $\|s' - s_1\|$ is the distance in the Fréchet topology; then s' is an isomorphism on $\overline{X}_1$.

Fix sections $u_1, u_2, \ldots, u_N \in \mathcal{E}_{|\overline{X}_2}$, which generate $\mathcal{E}$ on $\overline{X}_2$ and let $s' \in \mathsf{Hom}(\mathcal{E}, \mathcal{F})(\overline{X}_2)$ be an isomorphism $\mathcal{E}_{|\overline{X}_2} \simeq \mathcal{F}_{|\overline{X}_2}$ (determined by the isomorphism $\widetilde{\mathsf{E}}_{|U_2} \overset{\text{hol}}{\simeq} \widetilde{\mathsf{F}}_{|U_2}$). Then $s'(u_1), s'(u_2), \ldots, s'(u_m)$ generate $\mathcal{F}_{|\overline{X}_2}$ hence on one has

$$s(u_i) = \sum_{k=1}^{m} A'_{ki} s'(u_k)$$

where the A'_{ki} are (smooth) CR functions on $\overline{X}_1$ and the matrix $A' := (A'_{ki})$ is of maximal rank m on $\overline{X}_1$.

Since $\overline{X}_2$ is $\mathcal{O}(U_2)$-convex we can approximate A' by matrices of holomorphic functions of rank m on $\overline{X}_1$. In view of[3, Theorem 5.4.4], A' approximates by matrices of holomorphic functions of rank m on $\overline{X}_2$ so that we can choose a matrix A of CR functions of maximal rank m on $\overline{X}_2$ in such a way to have $\|As' - s_1\| < 2^{-1}\delta_1$. Set $s_2 := As'$, then s_2 is an isomorphism on X_2. Let $\delta_2 < \delta_1$ and s_3 be an isomorphism on X_3 such that $\|s_3 - s_2\| < 2^{-1}\delta_2$ and so on. By this procedure for every $k \in \mathbb{N}$ we construct a section $s_k \in \mathsf{Hom}(\mathcal{E}, \mathcal{F})(\overline{X}_k)$ which is a isomorphism on $\overline{X}_k$ and

$$\|s_{k+1} - s_k\| < 2^{-k}\delta_k$$

and $\delta_1 > \delta_2 > \cdots$. By standard techniques, $s_k \to s \in \mathsf{Hom}(\mathcal{E}, \mathcal{F})(X)$ and for every k the restriction of s to $\overline{X}_k$ is an isomorphism. This shows that s is an isomorphism $\mathcal{E} \overset{\text{CR}}{\simeq} \mathcal{F}$. □

With the techniques developed in the next section, we can proof an analogue result for $(m, 1)$-bundles, with an extra assumption:

Theorem 5.3. *Let X be a strongly 1-complete real analytic semiholomorphic foliation of type (n, d). If*

$$H^1(X, \mathcal{T}) = H^1(X, \mathbb{Z}) = 0 \ ,$$

then two topologically equivalent real analytic CR bundles of birank $(m, 1)$ are CR equivalent.

6. The Oka-Grauert principle for smooth CR bundles

6.1. The case of birank $(1, 0)$-bundles.

We have the following

Theorem 6.1. *Let X be a strongly 1-complete real analytic semiholomorphic foliation of type $(n, 1)$. Then*

$$\mathsf{Vect}^{(1,0)}_{\mathsf{cr}}(X) \simeq \mathsf{Vect}^{(1,0)}_{\mathsf{top}}(X).$$

Proof. The proof is the same as that of Oka for Stein manifolds. We know that

$$\mathsf{Vect}^{(1,0)}_{\mathsf{top}}(X) \simeq H^1(X, \mathcal{C}^*), \ \ \mathsf{Vect}^{(1,0)}_{\mathsf{cr}}(X) \simeq H^1(X, \mathcal{S}^*)$$

where $\mathcal{C} = \mathcal{C}_X$ is the sheaf of germs of continuous functions in X).

Consider the exact diagram of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{S} & \xrightarrow{\exp 2\pi i} & \mathcal{S}^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C} & \xrightarrow{\exp 2\pi i} & \mathcal{C}^* & \longrightarrow & 0 \end{array}$$

where $\mathcal{C} = \mathcal{C}_X$ is the sheaf of germs of continuous functions in X. Since $H^q(X, \mathcal{C}) = 0$, $q \geq 1$, and $H^q(X, \mathcal{S}) = 0$, $q \geq 1$, (see [11, Theorems 4.1]) passing to cohomology we get the following exact diagram of groups

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(X, \mathcal{S}^*) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & 0 \ . \\ \downarrow & & \downarrow & & \simeq\downarrow & & \downarrow \\ 0 & \longrightarrow & H^1(X, \mathcal{C}^*) & \longrightarrow & H^2(X, \mathbb{Z}) & \longrightarrow & 0 \end{array}$$

whence

$$\mathsf{Vect}^{(1,0)}_{\mathsf{cr}}(X) \simeq H^1(X, \mathcal{S}^*) \simeq H^1(X, \mathcal{C}^*) \simeq \mathsf{Vect}^{(1,0)}_{\mathsf{top}}(X).$$

In particular

a) two topologically equivalent CR vector bundles of birank $(1, 0)$ over X are CR equivalent;
b) every topological vector bundle of type $(1, 0)$ over X admits an equivalent CR vector bundle structure.

□

6.2. **The case of $(1,1)$-bundles.** The result we aim at is the following.

Theorem 6.2. *Let X be a strongly 1-complete real analytic semiholomorphic foliation of type $(n,1)$. Assume that*

$$H^1(X,\mathcal{T}) = H^1(X,\mathbb{Z}) = 0.$$

Then

$$\epsilon_X : \mathsf{Vect}^{(1,1)}_{\mathsf{cr}}(X) \longrightarrow \mathsf{Vect}^{(1,1)}_{\mathsf{top}}(X)$$

is a bijection.

However, we will not present a complete proof, but only a reduction to a cohomological problem, with the same general strategy presented by Cartan in [2] for the classical Oka-Grauert results. To be more precise,

(i) in Proposition 6.4 we show that a topologically trivial CR bundle of birank $(m,1)$ is CR trivial,
(ii) in Proposition 6.6 we show that the map ϵ_X is injective,
(iii) in the last part of subsection 6.2, we reduce the surjectivity to a cohomological problem.

We do not present the proof of the cohomological property needed to conclude, but we believe it can be achieved by techniques similar to the ones employed in the classical case in [2].

We first prove the following

Lemma 6.3. *Let $\mathsf{E} \to X$, $\mathsf{F} \to X$ topologically equivalent CR bundles of birank $(m,1)$. Then they can be defined by cocycles $\{h_{ij}\}$, $\{g_{ij}\}$ of the form*

$$h_{ij} = \begin{pmatrix} F_{ij} & B_{ij} \\ 0 & 1 \end{pmatrix}, \; g_{ij} = \begin{pmatrix} F_{ij} & G_{ij} \\ 0 & H_{ij} \end{pmatrix}. \tag{4}$$

Proof. Suppose that $\mathsf{E} \to X$, $\mathsf{F} \to X$ are topologically equivalent CR bundles of birank $(m,1)$ and let

$$h_{ij} = \begin{pmatrix} A_{ij} & B_{ij} \\ 0 & C_{ij} \end{pmatrix}, \; g_{ij} = \begin{pmatrix} F_{ij} & G_{ij} \\ 0 & H_{ij} \end{pmatrix} \tag{5}$$

the cocycles of E and F respectively.

By hypothesis there are continuous maps $a_i; U_i \to \mathsf{G}_{1,1}$

$$a_i = \begin{pmatrix} M_i & S_i \\ 0 & P_i \end{pmatrix}$$

such that

$$h_{ij} = a_i^{-1} g_{ij} a_j$$

on $U_{ij}; = U_i \cap U_j$ where P_i is constant on the leaves.

In particular, (5) gives the following identities

$$\begin{cases} A_{ij} = M_i^{-1} F_{ij} M_j \\ C_{ij} = P_i^{-1} H_{ij} P_j. \end{cases} \tag{6}$$

Then, again by [11, Theorems 4.1, 4.2], we can suppose that M_i and M_j are CR functions. So, $\{h_{ij}\}$ is CR equivalent to the cocycle

$$\begin{pmatrix} M_i & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} A_{ij} & B_{ij} \\ 0 & C_{ij} \end{pmatrix}\begin{pmatrix} M_j^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} F_{ij} & \star \\ 0 & C_{ij} \end{pmatrix}$$

thus we may assume E and F are defined by the CR cocycles

$$h_{ij} = \begin{pmatrix} F_{ij} & B_{ij} \\ 0 & C_{ij} \end{pmatrix}, \; g_{ij} = \begin{pmatrix} F_{ij} & G_{ij} \\ 0 & H_{ij} \end{pmatrix}.$$

respectively. From here it is easy to conclude. □

By Lemma 6.3 we may suppose that E, F are defined by the cocycles (4).

If $\{h_{ij}\} \overset{\text{top}}{\simeq} \{g_{ij}\}$,$\{H_{ij}\}$ is coboundary with values in $\mathbb{R}^*$ so $\{H_{ij}\} = \exp(H_i - H_j)$ with $H_i : U_i \to \mathbb{C}$, $H_j : U_j \to \mathbb{C}$ continuous and constant on the leaves. Since F_{ij}/Hij is real valued function

$$\mathsf{Im}(H_i - H_j) = \pi N_{ij}$$

with $N_{ij} \in \mathbb{Z}$. Clearly, $\{N_{ij}\}$ is a 1-cocycle with values in the constant sheaf $\mathbb{Z}$ hence $N_{ij} = N_i - N_j$, $N_i, N_j \in \mathbb{Z}$, by the hypothesis $H^1(X, \mathbb{Z}) = 0$. Then $\mathsf{Im}\, H_i - \pi N_i = \mathsf{Im}\, H_j - \pi N_j$ for evey i, j, so $\{\mathsf{Im}\, H_i - \pi N_i\}$ is a continuous function u in X and

$$\begin{aligned} H_{ij} &= \exp(\mathsf{Re}\, H_i + i\pi N_i + iu)\exp(-(\mathsf{Re}\, c_j + i\pi N_j + iu) = \\ &\quad \exp(\mathsf{Re}\, H_i + i\pi N_i))\exp(-(\mathsf{Re}\, H_j + i\pi N_j). \end{aligned}$$

It follows that

$$\exp(i(N_j - N_i))H_{ij} = \exp H_i \exp(-H_j)$$

whence that the function $H_i - H_j$ is smooth and constant on the leaves. Clearly. $\{H_i - H_j\}$ is a 1-cocycle with values in the sheaf $\mathcal{T}$ so, by hypothesis $H^1(X, \mathcal{T}) = 0$, there exist smooth functions $\psi_i : U_i \to \mathbb{R}$ which are constant on the leaves and such that $H_i - H_j = \psi_i - \psi_j$. Then the smooth functions $P_i := \exp(\psi_i + iN_i\pi)$ are real valued, constant on the leaves and

$$H_{ij} = \frac{\exp(\psi_i + iN_i\pi)}{\exp(\psi_j + iN_j\pi)} = \frac{P_i}{P_j}.$$

It follows that

$$g_{ij} = \begin{pmatrix} F_{ij} & G_{ij} \\ 0 & P_i^{-1}/P_j \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & P_i^{-1} \end{pmatrix}\begin{pmatrix} F_{ij} & G_{ij} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & P_j \end{pmatrix}.$$

Therefore we may assume that

$$h_{ij} = \begin{pmatrix} F_{ij} & B_{ij} \\ 0 & 1 \end{pmatrix}, \; g_{ij} = \begin{pmatrix} F_{ij} & G_{ij} \\ 0 & 1 \end{pmatrix}. \tag{7}$$

Proposition 6.4. *Let* $\mathsf{E} \to X$ *be a topologically trivial CR bundle of birank* $(m, 1)$*. Assume that*

$$H^1(X, \mathcal{T}) = H^1(X, \mathbb{Z}) = 0.$$

Then $\mathsf{E} \to X$ *is CR trivial.*

Proof. With the notation introduced above, if $\{g_{ij}\}$ is topologically trivial then F_{ij} is also topologically trivial, and consequently, since $H^{-1}(X, \mathcal{S}) = 0$, we have $F_{ij} = F_i^{-1}F_j$ for some CR functions F_i. It follows that

$$\begin{pmatrix} F_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_{ij} & B_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_j & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & B'_{ij} \\ 0 & 1 \end{pmatrix} := h'_{ij}$$

i.e. $\{h'_{ij}\} \overset{\text{cr}}{\simeq} \{h_{ij}\}$. Cocycle condition for h'_{ij} implies that B'_{ij} is a 1-cocycle with values in $\mathcal{S}$ wence a coboundary: $B'_{ij} = B_i - B_j$. It follows that

$$\begin{pmatrix} 1 & B_i \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & B'_{ij} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & B_j \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

that E is CR trivial. □

Let us prove now that $\{h_{ij}\} \overset{\text{top}}{\simeq} \{g_{ij}\}$ implies $\{h_{ij}\} \overset{\text{cr}}{\simeq} \{g_{ij}\}$. We can assume that the cocycles are given by (7).

By hypothesis there are continuous maps $c_i : U_i \to \mathsf{G}_{1,1}$

$$c_i = \begin{pmatrix} F_i & G_i \\ 0 & H_i \end{pmatrix}$$

such that

$$h_{ij} = c_i^{-1} g_{ij} c_j$$

on $U_i \cap U_j$ where H_i is constant on the leaves. Moreover, in view of the particular form of h_{ij} and g_{ij}, we have $F_i = F_j = F$, $H_i = H_j = H$, where F and H are continuous global functions in X and H is an invertible real valued function constant on the leaves. Moreover, multiplying by H^{-1} we may assume

$$c_i = \begin{pmatrix} F & G_i \\ 0 & 1 \end{pmatrix}; \tag{8}$$

thus

$$c_i^{-1} c_j = \begin{pmatrix} 1 & -F^{-1}G_i + F^{-1}G_j \\ 0 & 1 \end{pmatrix} \tag{9}$$

From (4) and (8), we deduce that E and F have structure group $\mathsf{G} := \mathbb{C}^* \times \mathbb{C}$, identified with the group of the matrices

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

$a \in \mathbb{C}^*$, $b \in \mathbb{C}$ with product $(a, b) \cdot (c, d) = (ac, ad + b)$

The Lie algebra $\mathfrak{g}$ of G is $\mathbb{C}\times\mathbb{C}$ with bracket

$$[(\zeta,\eta),(\zeta',\eta')] = (0,\zeta\eta'-\eta\zeta').$$

If $\xi=(\zeta,\eta)\in\mathfrak{g}$, $\lambda\in\mathbb{C}$, then $\exp\lambda\xi=(\exp\lambda\zeta,\lambda\eta)$. Then $\{c_i\}$ gives a section of the CR bundle $Z\to X$ and we have to prove that

> (ii) every continuous section $X\to Z$ is homotopic to a CR section.

This reduces to a cohomological problem.

We follow [2, **4**] taking into account (9). Let $\mathsf{G}_0=\{1\}\times\mathbb{C}\times\{1\}\simeq\mathbb{C}$ ($(1,a,1)\cdot(1,b,1)=(1,a+b,1)$, $(1,a,1)^{-1}=(1,-a,1)$ in G_0). For every open subset U of X, let $\mathcal{F}(U)$ be the topological group of the continuous maps $f:I\to C^0(U,\mathsf{G}_0)$ satisfying: $f(0)=\mathbf{1}$ and $f(1)\in C^{\mathsf{cr}}(U,\mathsf{G}_0)$. Both are topological groups (with a Fréchet topology).

Then, arguing as in [2, **4**], (ii) will be a consequence of the following

Theorem 6.5. *If X is a real analytic strongly 1-complete semiholomorphic foliation of type $(n,1)$. Then*

$$H^1(X,\mathcal{F})=0.$$

Proof. We have to prove the following: given an open covering $\{U_i\}_i$ of X and continuous functions $a_{ij}:U_{ij}\times I:\to\mathbb{C}$, $a_{ij}=a_{ij}(x,\lambda)$, such that

$$a_{ij}+a_{jk}+a_{ki}=0,\ \ ,a_{ij}(\cdot,0)=0,\ \ a_{ij}(\cdot,1)\in\mathcal{S}(U_{ij})$$

then there exist continuous functions $a_i:U_i\times I\to\mathbb{C}$ such that

$$a_{ij}=a_i-a_j,\ \ ,a_i(\cdot,0)=0,\ \ a_i(\cdot,1)\in\mathcal{S}(U_i).$$

The proof of this theorem is achieved adapting to our case the method followed in [2, **4**], using the results recalled in Section 4.1. □

Thus, we obtained the following

Proposition 6.6. *In our hypotheses, the map*

$$\epsilon_X:\mathsf{Vect}^{(1,1)}_{\mathsf{cr}}(X)\longrightarrow\mathsf{Vect}^{(1,1)}_{\mathsf{top}}(X)$$

is injective.

It remains to prove that ϵ_X is onto i.e. if $\{h_{ij}\}$ is a continuous cocycle with values in $\mathsf{G}_{1,1}$, $h_{ij}:Uij\to\mathsf{G}_{1,1}$ then there exist continuous maps $c_i:U_i\to\mathsf{G}_{1,1}$ such that the cocycle $c_i^{-1}h_{ij}c_j$ is CR.

We may assume that

$$h_{ij}=\begin{pmatrix} F_{ij} & B_{ij}\\ 0 & 1\end{pmatrix}.$$

Let $\mathsf{G}=\mathbb{C}^*\times\mathbb{C}\times\{1\}\simeq\mathbb{C}^*\times\mathbb{C}$ (with product $(a,b)\cdot(a',b')=(aa',ab'+b)$ and $(a,b)^{-1}=(a^{-1},-a^{-1}b)$). Denote $\mathcal{G}^{\mathrm{cont}}$ (respectively $\mathcal{G}^{\mathrm{cr}}$) the sheaf of germs of continuos (respectively CR) maps with values in G. Then showing that the cocycle $c_i^{-1}h_{ij}c_j$ is CR is equivalent to prove that the natural map

$$H^1(X, \mathcal{G}^{\mathrm{cr}}) \longrightarrow H^1(X, \mathcal{G}^{\mathrm{cont}}) \tag{10}$$

is onto.

So, we have reduced the proof of Theorem 6.2 to proving that the map (10) is surjective.

6.3. **The general case.** The case (m, l) essentially reduces to the cases $(m, 0)$, $(0, l)$. Indeed, let $\mathsf{E}, \mathsf{F} \in \mathsf{Vect}_{\mathsf{cr}}^{(\mathsf{m},\mathsf{l})}(\mathsf{X})$ be topologically equivalent with respective cocycles

$$h_{ij} = \begin{pmatrix} A_{ij} & B_{ij} \\ 0 & C_{ij} \end{pmatrix}, \; g_{ij} = \begin{pmatrix} F_{ij} & G_{ij} \\ 0 & H_{ij} \end{pmatrix}.$$

Then $\{A_{ij}\}$, $\{F_{ij}\}$ are topologically equivalent CR cocycles of type $(m, 0)$ and $\{C_{ij}\}$, $\{H_{ij}\}$ of type $(0, l)$ (see (6)) Moreover, $\{A_{ij}\}$, $\{F_{ij}\}$ are are 1-cocycle with values in $\mathrm{GL(m,\mathbb{C})}$ with, by hypothesis, are topologically equivalent. If the case $(m, 0)$ is already solved there exist CR maps M_i, P_i such that

$$F_{ij} = M_i^{-1}A_{ij}M_j, \;\; H_{ij} = P_i^{-1}C_{ij}P_j$$

It follows that $\{g_{ij}\}$ is CR equivalent to Taking

$$c_i^{-1} = \begin{pmatrix} I & 0 \\ 0 & P_i^{-1} \end{pmatrix} \begin{pmatrix} M_i^{-1} & 0 \\ 0 & I \end{pmatrix}, \; c_j = \begin{pmatrix} M_j & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P_j \end{pmatrix}$$

we have

$$c^{-1}g_{ij}c_j = \begin{pmatrix} A_{ij} & \star \\ 0 & C_{ij} \end{pmatrix}$$

so we may assume

$$h_{ij} = \begin{pmatrix} A_{ij} & B_{ij} \\ 0 & C_{ij} \end{pmatrix}, \; g_{ij} = \begin{pmatrix} A_{ij} & G_{ij} \\ 0 & C_{ij} \end{pmatrix}.$$

Now, topological equivalence implies that there are continuous maps $c_i : X \to \mathsf{G}_{m,l}$

$$c_i = \begin{pmatrix} F & G_i \\ 0 & H \end{pmatrix}$$

such that

$$h_{ij} = c_i^{-1}g_{ij}c_j.$$

Moreover

$$c_i^{-1}c_j = \begin{pmatrix} I & -F^{-1}G_i + F^{-1}G_j \\ 0 & I \end{pmatrix}. \tag{11}$$

$c = \{c_i\}$ gives a section of the CR bundle $Z \to X$ and we have to prove that

(ii) c is homotopic to a CR section.

This reduces again to a cohomological problem.

We follow [2, **4**] taking into account (11). Let $\mathsf{G}_0 = \{\mathbf{1}\} \times \mathbb{C}^{m+l} \times \{\mathbf{1}\} \simeq \mathbb{C}^{m+l}$ $((\mathbf{1}, a, \mathbf{1}) \cdot (\mathbf{1}, b, \mathbf{1}) = (\mathbf{1}, a+b, \mathbf{1})$, $(\mathbf{1}, a, \mathbf{1})^{-1} = (\mathbf{1}, -a, \mathbf{1})$ in G_0). Let $\mathrm{I} = [0, 1]$. For every open subset U of X, let $\mathcal{F}(U)$ be the topological group of the continuous maps $f : \mathrm{I} \to C^0(U, \mathsf{G}_0)$ satisfying: $f(0) = \mathsf{I}$ and $f(1) \in C^{\mathsf{cr}}(U, \mathsf{G}_0)$. Both are topological groups (with a Fréchet topology).

Injectivity of the map

$$\epsilon_X : \mathsf{Vect}^{(m,l)}_{\mathsf{cr}}(X) \longrightarrow \mathsf{Vect}^{(m,l)}_{\mathsf{top}}(X)$$

will be a consequence of the following

Theorem 6.7. *If X is a real analytic strongly 1-complete semiholomorphic foliation of type $(n, 1)$. Then*

$$H^1(X, \mathcal{F}) = 0.$$

As for surjectivity denote $\mathcal{G}^{\mathrm{cont}}$ respectively $\mathcal{G}^{\mathrm{cr}}$ the sheaf on X of germs of continuous maps with values in $\mathsf{G}_{m,l}$ and that one of germs of CR maps with values in $\mathsf{G}_{m,l}$.

Then it must be prove that

$$H^1(X, \mathcal{G}^{\mathrm{cr}}) \longrightarrow H^1(X, \mathcal{G}^{\mathrm{cont}})$$

is onto.

Observe that if the cases $(m, 0)$, $(0, l)$ are solved we may assume that every cohomology class in $H^1(X, \mathcal{G}^{\mathrm{cont}})$ is represented by a cocycle

$$h_{ij} = \begin{pmatrix} A_{ij} & B_{ij} \\ 0 & C_{ij} \end{pmatrix}$$

where A_{ij} and C_{ij} are CR, B_{ij} is continuous and satisfies the cocycle condition

$$A_{ij}B_{ji} + B_{ij}C_{ji} = 0.$$

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