Dam Rain and Cumulative Gain

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We consider a financial contract that delivers a single cash flow given by the terminal value of a cumulative gains process. The problem of modelling and pricing such an asset and associated derivatives is important, for example, in the determination of optimal insurance claims reserve policies, and in the pricing of reinsurance contracts. In the insurance setting, the aggregate claims play the role of the cumulative gains, and the terminal cash flow represents the totality of the claims payable for the given accounting period. A similar example arises when we consider the accumulation of losses in a credit portfolio, and value a contract that pays an amount equal to the totality of the losses over a given time interval. An expression for the value process of such an asset is derived as follows. We fix a probability space together with a pricing measure, and model the terminal cash flow by a random variable; next, we model the cumulative gains process by the product of the terminal cash flow and an independent gamma bridge process; finally, we take the filtration to be that generated by the cumulative gains process. An explicit expression for the value process is obtained by taking the discounted expectation of the future cash flow, conditional on the relevant market information. The price of an Arrow-Debreu security on the cumulative gains process is determined, and is used to obtain a closed-form expression for the price of a European-style option on the value of the asset at the given intermediate time. The results obtained make use of various remarkable properties of the gamma bridge process, and are applicable to a wide variety of financial products based on cumulative gains processes such as aggregate claims, credit portfolio losses, defined-benefit pension schemes, emissions, and rainfall.

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I. INTRODUCTION

There are a number of problems in finance and insurance that involve the analysis of accumulation processes—that is to say, processes representing cumulative gains or losses. The typical setup is as follows. We fix an accounting period $[0, T]$, where time 0 denotes the present. At time T a contract pays a random cash flow X_T, which is assumed to be positive and given by the terminal value of a process of accumulation. In the case of an insurance contract, for example, we consider the situation where a number of claims are made over the accounting period, and are then paid at T. The random variable X_T represents the totality of the payments made at T in settlement of claims arising over the accounting period. The problem facing the insurance firm is the valuation of the random cash flow. Let us write $\{S_t\}$ for the value process of the contract that pays X_T at T, and $\{\mathcal{F}_t\}$ for the filtration representing the flow of information available to market participants, and $\mathbb{Q}$ for the pricing measure, which we assume to have been established by the market. Then the value at t of the contract that pays X_T at T is

$$S_t = P_{tT}\mathbb{E}[X_T|\mathcal{F}_t], \tag{1}$$

where $\mathbb{E}[-]$ denotes expectation with respect to $\mathbb{Q}$, and P_{tT} denotes the discount factor, which for simplicity we take to be deterministic. One can interpret S_t as the reserve that the insurance firm requires at t to ensure that X_T will be payable at T. Alternatively, one can view S_t as the amount that would have to be paid at t in order for the insurance firm to relieve itself of the obligation to pay X_T, that is to say, to commute the relevant claims. Similarly, the cost C_{tT} at t of a simple stop-loss reinsurance contract that pays out $(X_T - K)^+$ at T for some fixed threshold K is given by

$$C_{tT} = P_{tT}\mathbb{E}[(X_T - K)^+|\mathcal{F}_t]. \tag{2}$$

We shall assume that $\{\mathcal{F}_t\}$ is generated by an aggregate claims process $\{\xi_t\}$, where for each t the random variable ξ_t represents the totality of claims known at t to be payable at T. The problem can then be stated as follows: given the history of claims over the accounting period up to time t, what is the appropriate reserve to allocate for settlement of these and any future claims arising in the accounting period? To obtain a solution to the problem we need to specify the aggregate claims process, then work out the reserve process $\{S_t\}$. Once we have the reserve process, we can value various types of reinsurance contracts.

Another example of an accumulation process comes from credit risk management. We consider a large credit portfolio, and let X_T denote the value of the accumulated losses at T. For instance, at time 0 a credit-card firm has a large number of customers, each with an outstanding balance payable in the accounting period. If a customer does not pay the balance by the required date, they will be deemed to be in default, and a loss will be registered. The random variable X_T will denote the totality of such losses. We assume that once a customer is in default, no further payments are made by that customer (this assumption can be relaxed in a more sophisticated model). The problem facing the credit-card firm is to determine what reserve policy to maintain, and what premium to charge over the base interest rate, to ensure that funds will be in hand to cover the default losses.

The purpose of this paper is to present a modelling framework for accumulation processes, and to establish explicit formulae for the associated valuation processes. In particular, we shall assume that $\{\xi_t\}$ takes the form

$$\xi_t = X_T \gamma_{tT}, \tag{3}$$

where $\{\gamma_{tT}\}$ is a gamma bridge over the interval $[0, T]$, independent of X_T. The motivation for the form of the accumulation process indicated above arises in two distinct lines of enquiry. The first relates to the idea that the gamma process might be used as a basis for describing the aggregate losses associated with insurance claims. This idea dates to the work of Hammersley (1955), Moran (1956), Gani (1957), Kendall (1957), and others, in connection with the theory of storage and dams. Moran (1956), in particular, observed that the amount of rainfall accumulating in a dam can be modelled by a gamma process, and Gani (1957) pointed out the relevance to insurance, the argument being that providing that the portfolio of events insured is sufficiently large, one can think of the arrival of claims as being analogous to the accumulation of dam rain. The gamma process has since then been investigated by Dufresne *et al.* (1991), Dufresne (1998), Dicksen & Waters (1993), and others, as a model for aggregate claims.

Let us therefore consider what results if we model the aggregate claims process as a $\mathbb{Q}$-gamma process. In other words, suppose we set $\xi_t = \kappa\gamma_t$, where κ is a constant and $\{\gamma_t\}$ is a standard gamma process under $\mathbb{Q}$, with mean and variance mt (see Section II for definitions). It follows that $\xi_t = X_T\gamma_{tT}$ where $X_T = \kappa\gamma_T$ and the process $\{\gamma_{tT}\}$ defined by $\gamma_{tT} = \gamma_t/\gamma_T$ is a standard gamma bridge over $[0, T]$. Moreover, by virtue of the special properties of the gamma process, we find that X_T is independent of $\{\gamma_{tT}\}$. We see that in the $\mathbb{Q}$-gamma model the aggregate claims process is the product of a gamma-distributed terminal cash flow and an independent gamma bridge. One can think of the gamma bridge as representing that aspect of the aggregate claims process that has no bearing on the terminal result. We are thus led to a multiplicative decomposition of the accumulation process into the product of a "signal" X_T and an independent "noise" $\{\gamma_{tT}\}$ carrying no information about X_T.

For such processes we are able to apply the techniques of information-based asset pricing developed in Brody *et al.* 2007a, 2007b, Hughston & Macrina 2007, Macrina 2006; and Rutkowski & Yu 2007. Indeed, through this second line of enquiry one is led to consider the more general situation where the terminal cash flow, instead of being gamma distributed, has a generic *a priori* distribution, and the claims process takes the form (3). The additive decomposition of the market information process in the case of the Brownian bridge noise considered in the references cited above is natural from the viewpoint of nonlinear filtering theory. The product representation of the gamma information process is equally natural, since many properties of the Brownian bridge that hold additively have striking multiplicative analogues for gamma bridges (Emery & Yor 2004, Yor 2007). The resulting model for the aggregate claims process is remarkably tractable, and we are able to derive explicit formulae both for the claims reserve process, and for the valuation of reinsurance contracts.

The paper is organised as follows. In Sections II, III, and IV, we outline a number of the properties of gamma processes and gamma bridges. The material covered in these sections is for the most part well known. However, since it is not easy to locate a systematic but elementary treatment of the gamma process and the associated bridge process, it will be useful to present some of the details here for the benefit of general readers. At the same time, we establish our notation and some results that will be applied in later sections. In Section V we derive an explicit expression for the value process of a contract that delivers the cash flow X_T at time T, when the market filtration is generated by the accumulation process (3). We show in Proposition 4 that $\{\xi_t\}$ has the Markov property, and then use the Bayes theorem to determine the conditional density of X_T, and finally the value process, which is given in Proposition 5. By use of the conditional density we are also able to obtain an expression for the value process of a simple stop-loss reinsurance contract. In Section VI we consider the valuation of general reinsurance contracts. In particular, we derive a formula for the value at time 0 of a contract that at some fixed time t gives the contract holder the option to commute the claim X_T by paying a fixed amount K at t. Such a contract takes the form of a European call option on the value of the reserve at t. An Arrow-Debrue method is introduced to simplify the calculations. The resulting formula for the option value is expressed in terms of the cumulative beta distribution. We examine in Section VII the case where X_T takes discrete values. When X_T is a binary random variable, the problem of option pricing can be solved completely. In Section VIII the material of Section VI is extended to determine an expression for the price process of an option on the value of an aggregate claim. In Section IX we conclude by returning to the case where X_T has a $\mathbb{Q}$-gamma distribution.

II. GAMMA PROCESSES AND ASSOCIATED MARTINGALES

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$. In our applications $\mathbb{Q}$ will always denote the pricing (risk-neutral) measure, but the material in this section, and the following two, does not depend on this interpretation. Equalities and inequalities among random variables are to be understood as holding except possibly on sets of measure zero. By a *standard gamma process* $\{\gamma_t\}_{0\le t<\infty}$ on $(\Omega, \mathcal{F}, \mathbb{Q})$ with growth rate m we mean a process with independent increments such that $\gamma_0 = 0$ and such that the random variable γ_t has a gamma distribution with mean and variance mt. More precisely, writing $G(x) = \mathbb{Q}[\gamma_t \le x]$ for the distribution of γ_t, and writing $g(x) = \mathrm{d}G(x)/\mathrm{d}x$, we have

$$g(x) = \mathbb{1}_{\{x>0\}} \frac{x^{mt-1}\mathrm{e}^{-x}}{\Gamma[mt]} \tag{4}$$

for the density of γ_t. Here $\Gamma[a]$ is the standard gamma function, which for $a > 0$ has the Eulerian representation:

$$\Gamma[a] = \int_0^\infty x^{a-1}\mathrm{e}^{-x}\mathrm{d}x. \tag{5}$$

It follows from the identity $\Gamma[a+1] = a\Gamma[a]$ satisfied by the gamma function that

$$\mathbb{E}[\gamma_t] = mt, \tag{6}$$

which justifies the interpretation of the parameter m as the *mean growth rate* of the process.

A straightforward calculation shows that the characteristic function for the gamma process is given by

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda\gamma_t}\right] = \frac{1}{(1-\mathrm{i}\lambda)^{mt}}, \tag{7}$$

valid for $t \ge 0$ and for $\lambda \in \mathbb{C}$ such that $\mathrm{Im}(\lambda) > -1$, from which the higher moments of γ_t can be deduced. We note that $\mathbb{E}[\gamma_t^2] = mt + m^2t^2$, and hence that $\mathrm{Var}[\gamma_t] = mt$. It follows as a consequence of the independent increments property that $\mathrm{Cov}[\gamma_t, \gamma_u] = mt$ for $u \ge t$.

An alternative expression for the characteristic function is given by the Lévy-Khinchine representation $\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda\gamma_t}\right] = \mathrm{e}^{-t\psi(\lambda)}$ for $\mathrm{Im}(\lambda) > -1$, where

$$\psi(\lambda) = m\ln(1-\mathrm{i}\lambda) = \int_0^\infty mx^{-1}\mathrm{e}^{-x}\left(1-\mathrm{e}^{\mathrm{i}\lambda x}\right)\mathrm{d}x, \tag{8}$$

which shows that the Lévy density associated with the gamma process is given by $mx^{-1}\mathrm{e}^{-x}$ for $x > 0$ (see, e.g., Protter 2005).

By use of the independent increments property we deduce that for $u \ge t \ge 0$ and for $a, b \in \mathbb{C}$ with $\mathrm{Im}(a+b) > -1$ and $\mathrm{Im}(b) > -1$ we have:

$$\begin{aligned}
\mathbb{E}\left[\mathrm{e}^{\mathrm{i}a\gamma_t+\mathrm{i}b\gamma_u}\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}(a+b)\gamma_t+\mathrm{i}b(\gamma_u-\gamma_t)}\right] \\
&= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}(a+b)\gamma_t}\right]\mathbb{E}\left[\mathrm{e}^{\mathrm{i}b(\gamma_u-\gamma_t)}\right] \\
&= \frac{1}{[1-\mathrm{i}(a+b)]^{mt}}\,\frac{1}{(1-\mathrm{i}\,b)^{m(u-t)}}.
\end{aligned} \tag{9}$$

In particular if we set $-a = b = \lambda$, we see that $\gamma_u - \gamma_t$ is gamma-distributed with parameter $m(u-t)$. It follows that the increments of $\{\gamma_t\}$ have a time-homogeneous probability law in the sense that $\gamma_{u+h} - \gamma_{t+h}$ has the same distribution as $\gamma_u - \gamma_t$.

Using the independent increments property it is a straightforward exercise to deduce that the processes $\{\gamma_t - mt\}$ and $\{\gamma_t^2 - 2mt\gamma_t + mt(mt-1)\}$ are martingales. More generally, for $\alpha > -1$ the process $\{L_t\}$ defined by

$$L_t = (1+\alpha)^{mt}\mathrm{e}^{-\alpha\gamma_t} \tag{10}$$

is a martingale, which can be verified by use of (9). We refer to this process as the exponential gamma martingale. It follows, by consideration of the corresponding power series in α, that for each term in the series we are able to obtain a martingale involving a polynomial expression in the gamma process. Suppose for $n \in \mathbb{N}$ and $k \in \mathbb{R}$ we define the so-called associated Laguerre polynomials $\{L_n^{(k)}(z)\}$ by setting

$$L_n^{(k)}(z) = z^{-k}\mathrm{e}^{z}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(z^{n+k}\mathrm{e}^{-z}\right). \tag{11}$$

Thus, we have $L_1^{(k)}(z) = -z + k + 1$, $L_2^{(k)}(z) = \frac{1}{2}[z^2 - 2(k+2)z + (k+1)(k+2)]$, and so on. The standard Laguerre polynomials, given by $L_n(z) = L_n^{(0)}(z)$, have the property that if Z is a standard exponentially distributed random variable, then $\mathbb{E}\left[L_n(Z)L_{n'}(Z)\right] = 0$ for $n \neq n'$ (cf. Wiener 1949). More generally, if Z has a gamma distribution with parameter $k+1$, i.e. such that $\mathbb{Q}[Z < z] = \int_0^z x^k \mathrm{e}^{-x}\mathrm{d}x/\Gamma[k+1]$, for $k > -1$, then $\mathbb{E}[L_n^{(k)}(Z)L_{n'}^{(k)}(Z)] = 0$ for $n \neq n'$. The significance of the associated Laguerre polynomials in the present context arises from the identity

$$(1+\alpha)^h \mathrm{e}^{-z\alpha} = \sum_{n=0}^{\infty} L_n^{(h-n)}(z)\,\alpha^n, \tag{12}$$

valid for $|\alpha| < 1$ and $h \geq 0$ (Erdélyi 1953), which gives us the required series expansion of the exponential gamma martingale in powers of α. In particular, by setting $h = mt$ and $z = \gamma_t$ in equation (12), we are able to deduce that for each value of n the process $\{L_n^{(mt-n)}(\gamma_t)\}$ is a martingale (cf. Schoutens 2000). For example, we have

$$L_1^{(mt-1)}(\gamma_t) = -(\gamma_t - mt), \qquad L_2^{(mt-2)}(\gamma_t) = \tfrac{1}{2}[\gamma_t^2 - 2mt\gamma_t + mt(mt-1)]. \tag{13}$$

So far we have confined the discussion to the case of the "standard" gamma process, for which $\mathbb{E}[\gamma_t] = mt$ and $\mathrm{Var}[\gamma_t] = mt$, for some value of m. We note that the ratio $(\mathbb{E}[\gamma_t])^2/\mathrm{Var}[\gamma_t]$ is dimensionless, and hence that m has the units of inverse time. For any fixed m we can choose the units of time so that $m = 1$ in those units (this is done implicitly, for example, in Yor 2007). We shall, however, take the units of time as fixed, and m as a model parameter.

For many applications it is useful also to consider a broader family of gamma processes, labelled by two parameters, which we shall call "scaled" gamma processes. By a scaled gamma process with growth rate μ and spread σ we mean a process $\{\Gamma_t\}_{0\leq t<\infty}$ with independent increments such that $\Gamma_0 = 0$ and such that Γ_t has a gamma distribution with mean μt and variance $\sigma^2 t$, where μ and σ are parameters. Defining $m = \mu^2/\sigma^2$ and $\kappa = \sigma^2/\mu$, we have $\mu = \kappa m$ and $\sigma^2 = \kappa^2 m$. One can think of m as a "standardised" growth rate, and κ as a "scale". The density of Γ_t is then given by

$$g_{\Gamma_t}(x) = \mathbb{1}_{\{x>0\}} \frac{\kappa^{-mt} x^{mt-1} \mathrm{e}^{-x/\kappa}}{\Gamma[mt]}. \tag{14}$$

It is straightforward to check that if $\{\Gamma_t\}$ is a scaled gamma process with standardised growth rate m and scale κ, then $\{\kappa^{-1}\Gamma_t\}$ is a standard gamma process, with growth rate m.

Now suppose that $\{\gamma_t\}$ is a standard gamma process on $(\Omega, \mathcal{F}, \mathbb{Q})$, let $\{\mathcal{G}_t\}$ denote the filtration generated by $\{\gamma_t\}$, and let $\mathbb{Q}^*$ denote the measure on $(\Omega, \mathcal{G}_T)$, for some fixed T, defined by the likelihood ratio

$$\left.\frac{\mathrm{d}\mathbb{Q}^*}{\mathrm{d}\mathbb{Q}}\right|_T = \kappa^{-mT} \exp\left(\frac{1-\kappa}{\kappa}\,\gamma_T\right) \tag{15}$$

for some $\kappa > 0$. Then $\{\gamma_t\}_{0\le t\le T}$ is a *scaled* gamma process on $(\Omega, \mathcal{G}_T, \mathbb{Q}^*)$, with scale parameter κ. Thus, $\mathbb{E}[\gamma_t] = mt$, $\mathrm{Var}[\gamma_t] = mt$, $\mathbb{E}^*[\gamma_t] = \kappa mt$, and $\mathrm{Var}^*[\gamma_t] = \kappa^2 mt$. This can be established by working out the joint characteristic function under $\mathbb{Q}^*$ of the increments $\gamma_t - \gamma_s,\ \gamma_s - \gamma_{s_1},\ \gamma_{s_1} - \gamma_{s_2},\ \cdots,\ \gamma_{s_{n-1}} - \gamma_{s_n}$ for $T \ge t \ge s \ge s_1 \ge s_2 \ge \cdots \ge s_n$ for each $n \in \mathbb{N}$, and showing that it factorises. We note that the change-of-measure density martingale arising in this example is obtained by taking the standard gamma exponential martingale (10) defined above, and setting $\alpha = (1-\kappa)/\kappa$.

The gamma process has been used as the basis of a number of different asset pricing models; see, for example, Madan & Seneta (1990), Madan and Milne (1991), Heston (1995), Madan *et al.* (1998), Carr *et al.* (2002), and Baxter (2007).

III. GAMMA BRIDGE PROCESSES

Let $\{\gamma_t\}_{0\le t<\infty}$ be a standard gamma process with growth rate m, and for fixed T define the process $\{\gamma_{tT}\}_{0\le t\le T}$ by setting

$$\gamma_{tT} = \frac{\gamma_t}{\gamma_T}. \tag{16}$$

Then clearly $\gamma_{0T} = 0$ and $\gamma_{TT} = 1$. We refer to $\{\gamma_{tT}\}$, thus defined, as the *standard gamma bridge* over $[0, T]$ associated with the gamma process $\{\gamma_t\}$. More generally, we refer to any process having the law of $\{\gamma_{tT}\}$ as a standard gamma bridge over $[0, T]$. It can be shown that the random variable γ_{tT} has a beta distribution. In particular, we have the following:

Proposition 1 *The density function of the random variable γ_{tT} is given by*

$$f(y) = \mathbb{1}_{\{0<y<1\}} \frac{y^{mt-1}(1-y)^{m(T-t)-1}}{B[mt, m(T-t)]}, \tag{17}$$

where

$$B[a, b] = \frac{\Gamma[a]\Gamma[b]}{\Gamma[a+b]}. \tag{18}$$

Proof. First we note that

$$\mathbb{Q}\left[\frac{\gamma_t}{\gamma_T} \leq y\right] = \mathbb{Q}\left[\frac{\gamma_t}{\gamma_T - \gamma_t} \leq \frac{y}{1-y}\right]. \tag{19}$$

Since γ_t and $\gamma_T - \gamma_t$ are independent, and γ_t has a gamma distribution with parameter mt, we have

$$\begin{aligned}
\mathbb{Q}\left[\frac{\gamma_t}{\gamma_T - \gamma_t} \leq \frac{y}{1-y}\right] &= \mathbb{Q}\left[\gamma_t \leq \frac{y}{1-y}(\gamma_T - \gamma_t)\right] \\
&= \mathbb{E}\left[\mathbb{Q}\left[\gamma_t \leq \frac{y}{1-y}(\gamma_T - \gamma_t)\middle|\gamma_T - \gamma_t\right]\right] \\
&= \frac{1}{\Gamma[mt]}\,\mathbb{E}\left[\int_0^{\frac{y}{1-y}(\gamma_T-\gamma_t)} x^{mt-1}\mathrm{e}^{-x}\mathrm{d}x\right]. \qquad (20)
\end{aligned}$$

Therefore, the corresponding density is given by

$$\begin{aligned}
f(y) &= \frac{\mathrm{d}}{\mathrm{d}y}\mathbb{Q}\left[\frac{\gamma_t}{\gamma_T} \leq y\right] \\
&= \mathbb{1}_{\{0<y<1\}}\frac{1}{\Gamma[mt]}\,\mathbb{E}\left[\frac{\mathrm{d}}{\mathrm{d}y}\int_0^{\frac{y}{1-y}(\gamma_T-\gamma_t)} x^{mt-1}\mathrm{e}^{-x}\,\mathrm{d}x\right] \\
&= \mathbb{1}_{\{0<y<1\}}\frac{1}{\Gamma[mt]}\,\mathbb{E}\left[\frac{\gamma_T-\gamma_t}{(1-y)^2}\left(\frac{y}{1-y}(\gamma_T-\gamma_t)\right)^{mt-1}\mathrm{e}^{-\frac{y}{1-y}(\gamma_T-\gamma_t)}\right] \\
&= \mathbb{1}_{\{0<y<1\}}\frac{y^{mt-1}(1-y)^{-mt-1}}{\Gamma[mt]}\,\mathbb{E}\left[(\gamma_T-\gamma_t)^{mt}\mathrm{e}^{-\frac{y}{1-y}(\gamma_T-\gamma_t)}\right]. \qquad (21)
\end{aligned}$$

Now, since $\gamma_T - \gamma_t$ has a gamma distribution with parameter $m(T-t)$, for the expectation appearing in the line just above we obtain

$$\begin{aligned}
\mathbb{E}\left[(\gamma_T-\gamma_t)^{mt}\mathrm{e}^{-\frac{y}{1-y}(\gamma_T-\gamma_t)}\right] &= \frac{1}{\Gamma[m(T-t)]}\int_0^\infty x^{mt}\mathrm{e}^{-\frac{y}{1-y}x}x^{m(T-t)-1}\mathrm{e}^{-x}\mathrm{d}x \\
&= \frac{1}{\Gamma[m(T-t)]}\int_0^\infty x^{mT-1}\mathrm{e}^{-\frac{y}{1-y}x}\mathrm{d}x \\
&= \frac{(1-y)^{mT}}{\Gamma[m(T-t)]}\int_0^\infty u^{mT-1}\mathrm{e}^{-u}\mathrm{d}u \\
&= \frac{\Gamma[mT]}{\Gamma[m(T-t)]}\,(1-y)^{mT}, \qquad (22)
\end{aligned}$$

where in the last two steps we make the substitution $x = u(1-y)$ and use formula (5). Putting this result back into (21), we obtain (17), as desired. □

Let us calculate the moments of γ_{tT}. Bearing in mind the integral representation

$$\mathrm{B}[a,b] = \int_0^1 y^{a-1}(1-y)^{b-1}\mathrm{d}y \tag{23}$$

for the beta function, we deduce that

$$\mathbb{E}\left[\gamma_{tT}^n\right] = \frac{\mathrm{B}[mt+n, m(T-t)]}{\mathrm{B}[mt, m(T-t)]} \tag{24}$$

for $n > 0$. By use of (18) along with the identity $\Gamma[a+1] = a\Gamma[a]$ we find that $\mathbb{E}[\gamma_{tT}] = t/T$ and that $\mathbb{E}[\gamma_{tT}^2] = t(mt+1)/T(mT+1)$. It follows in particular that

$$\mathrm{Var}[\gamma_{tT}] = \frac{t(T-t)}{T^2(1+mT)}. \tag{25}$$

It is interesting to observe that the expectation of γ_{tT} does not depend on the growth rate m, and that the variance of γ_{tT} decreases in increasing m.

More generally, let us define the Pochhammer symbol by writing $(a)_0 = 1$ and $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$. Then we find that the moments of γ_{tT} are given by the expression $\mathbb{E}[\gamma_{tT}^n] = (mt)_n/(mT)_n$, and for the corresponding central moments we obtain

$$\mathbb{E}\left[(\gamma_{tT} - \mathbb{E}[\gamma_{tT}])^n\right] = \left(-\frac{t}{T}\right)^n F\left(-n, mt; mT; T/t\right), \tag{26}$$

where $F(a,b;c;z) = \sum_{k=0}^{\infty}(a)_k(b)_k z^k/[k!(c)_k]$ is the hypergeometric function (Erdélyi 1953).

IV. FURTHER PROPERTIES OF GAMMA BRIDGES

It is a remarkable property of the gamma process and the associated gamma bridge that the processes $\{\gamma_u\}_{u\geq T}$ and $\{\gamma_{tT}\}_{0\leq t\leq T}$ are independent. In particular, the random variables γ_T and $\gamma_{tT} = \gamma_t/\gamma_T$ are independent for $0 \leq t \leq T$. This property allows us to verify straightforwardly that $\{\gamma_t\}$ has the Markov property. To show that $\{\gamma_t\}$ has the Markov property we need to verify for $a > 0$ that

$$\mathbb{Q}\left[\gamma_t < a | \gamma_s, \gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n}\right] = \mathbb{Q}[\gamma_t < a|\gamma_s] \tag{27}$$

for all $t \geq s \geq s_1 \geq s_2 \geq \cdots \geq s_n \geq 0$, and for all $n \geq 1$. But clearly,

$$\begin{aligned} \mathbb{Q}\left[\gamma_t < a | \gamma_s, \gamma_{s_1}, \gamma_{s_2}, \ldots, \gamma_{s_n}\right] &= \mathbb{Q}\left[\gamma_t < a \,\middle|\, \gamma_s, \frac{\gamma_{s_1}}{\gamma_s}, \frac{\gamma_{s_2}}{\gamma_{s_1}}, \ldots\right] \\ &= \mathbb{Q}\left[\gamma_t < a | \gamma_s\right], \end{aligned} \tag{28}$$

since, according to the result to be established below, γ_{s_1}/γ_s, $\gamma_{s_2}/\gamma_{s_1}, \ldots$ are independent of γ_s and γ_t. It follows that the gamma process is Markovian. A similar argument shows that the gamma bridge has the Markov property. In particular, we have

$$\begin{aligned} \mathbb{Q}\left[\frac{\gamma_t}{\gamma_T} < a \,\middle|\, \frac{\gamma_s}{\gamma_T}, \frac{\gamma_{s_1}}{\gamma_T}, \frac{\gamma_{s_2}}{\gamma_T}, \ldots\right] &= \mathbb{Q}\left[\frac{\gamma_t}{\gamma_T} < a \,\middle|\, \frac{\gamma_s}{\gamma_T}, \frac{\gamma_{s_1}}{\gamma_s}, \frac{\gamma_{s_2}}{\gamma_{s_1}}, \ldots\right] \\ &= \mathbb{Q}\left[\frac{\gamma_t}{\gamma_T} < a \,\middle|\, \frac{\gamma_s}{\gamma_T}\right], \end{aligned} \tag{29}$$

since the random variables γ_{s_1}/γ_s, $\gamma_{s_2}/\gamma_{s_1}, \ldots$ are independent of γ_t/γ_T and γ_s/γ_T.

Proposition 2 *Let* $\{\gamma_t\}_{0\leq t<\infty}$ *be a standard gamma process. Then for* $T \geq t \geq 0$ *the random variables* γ_t/γ_T *and* γ_T *are independent.*

Proof. For the joint distribution of these random variables let us write

$$F(y,z) = \mathbb{Q}\left[\frac{\gamma_t}{\gamma_T} \leq y \ \cap \ \gamma_T \leq z\right]. \tag{30}$$

We note that this can be rearranged in the form

$$F(y,z) = \mathbb{Q}\left[\gamma_t \leq \frac{y}{1-y}(\gamma_T - \gamma_t) \ \cap \ \gamma_t \leq z - (\gamma_T - \gamma_t)\right]. \tag{31}$$

Conditioning with respect to $\gamma_T - \gamma_t$, we use the fact that γ_t and $\gamma_T - \gamma_t$ are independent, and that γ_t has a gamma distribution with parameter mt, to deduce that

$$F(y,z) = \frac{1}{\Gamma[mt]}\,\mathbb{E}\left[\int_0^\infty \mathbb{1}_{\left\{x\leq\frac{y}{1-y}(\gamma_T-\gamma_t)\right\}}\mathbb{1}_{\{x\leq z-(\gamma_T-\gamma_t)\}}x^{mt-1}\mathrm{e}^{-x}\mathrm{d}x\right]. \tag{32}$$

Differentiating each side of this relation with respect to y and z, we obtain the following expression for the joint density function:

$$\begin{aligned} f(y,z) \;=\; & \frac{1}{\Gamma[mt]}\,\mathbb{E}\left[\int_0^\infty \frac{\gamma_T-\gamma_t}{(1-y)^2}\,\delta\!\left(x - \frac{y}{1-y}(\gamma_T-\gamma_t)\right)\right. \\ & \qquad\qquad \left.\times\, \delta\big(x - [z-(\gamma_T-\gamma_t)]\big)x^{mt-1}\mathrm{e}^{-x}\mathrm{d}x\right]. \end{aligned} \tag{33}$$

Here we have used the relation $\partial_x \mathbb{1}_{\{x\leq a\}} = -\delta(x-a)$, where $\{\delta(z)\}_{z\in\mathbb{R}}$ denotes the Dirac distribution. Integrating out the first delta function we thus have

$$\begin{aligned} f(y,z) \;=\; & \frac{1}{\Gamma[mt]}\,\mathbb{E}\left[\frac{\gamma_T-\gamma_t}{(1-y)^2}\,\delta\!\left(\frac{y}{1-y}(\gamma_T-\gamma_t) - [z-(\gamma_T-\gamma_t)]\right)\right. \\ & \qquad\qquad \left.\times \left(\frac{y}{1-y}(\gamma_T-\gamma_t)\right)^{mt-1} \mathrm{e}^{-\frac{y}{1-y}(\gamma_T-\gamma_t)}\right], \end{aligned} \tag{34}$$

for $0 < y < 1$ and $z > 0$; and hence after some rearrangement we obtain

$$f(y,z) = \frac{y^{mt-1}(1-y)^{-mt-1}}{\Gamma[mt]}\,\mathbb{E}\left[(\gamma_T-\gamma_t)^{mt}\mathrm{e}^{-\frac{y}{1-y}(\gamma_T-\gamma_t)}\;\delta\!\left(\frac{\gamma_T-\gamma_t}{1-y} - z\right)\right]. \tag{35}$$

Now we introduce the Fourier representation

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\lambda x}\mathrm{d}\lambda \tag{36}$$

for the delta function, interpreted in a distributional sense, from which we deduce that

$$f(y,z) = \frac{y^{mt-1}(1-y)^{-mt-1}}{\Gamma[mt]}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\lambda z}\mathbb{E}\left[(\gamma_T-\gamma_t)^{mt}\mathrm{e}^{-\frac{y}{1-y}(\gamma_T-\gamma_t)}\mathrm{e}^{\mathrm{i}\lambda\frac{1}{1-y}(\gamma_T-\gamma_t)}\right]\mathrm{d}\lambda. \tag{37}$$

Writing E for the result of the expectation appearing inside the integral above, and making use of the fact that $\gamma_T - \gamma_t$ is gamma distributed with parameter $m(T-t)$, we have

$$\begin{aligned} E &= \frac{1}{\Gamma[m(T-t)]} \int_0^\infty x^{mt} \mathrm{e}^{-\frac{y}{1-y}x} \mathrm{e}^{\mathrm{i}\lambda \frac{1}{1-y}x} x^{m(T-t)-1} \mathrm{e}^{-x} \mathrm{d}x \\ &= \frac{1}{\Gamma[m(T-t)]} \int_0^\infty x^{mT-1} \mathrm{e}^{-\frac{x}{1-y}} \mathrm{e}^{\mathrm{i}\lambda \frac{x}{1-y}} \mathrm{d}x \\ &= \frac{(1-y)^{mT}}{\Gamma[m(T-t)]} \int_0^\infty u^{mT-1} \mathrm{e}^{-u} \mathrm{e}^{\mathrm{i}\lambda u} \mathrm{d}u \\ &= \frac{(1-y)^{mT}}{\Gamma[m(T-t)]} \frac{\Gamma[mT]}{(1-\mathrm{i}\lambda)^{mT}}, \end{aligned} \tag{38}$$

where we have made use of (7) to deduce that the characteristic function of γ_T is $1/(1-\mathrm{i}\lambda)^{mT}$. Substituting (38) into (37) we obtain

$$f(y,z) = \frac{\Gamma[mT]}{\Gamma[mt]\Gamma[m(T-t)]} \, y^{mt-1}(1-y)^{m(T-t)-1} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{(1-\mathrm{i}\lambda)^{mT}} \, \mathrm{e}^{-\mathrm{i}\lambda z} \mathrm{d}\lambda, \tag{39}$$

and hence

$$f(y,z) = \frac{y^{mt-1}(1-y)^{m(T-t)-1}}{\mathrm{B}[mt, m(T-t)]} \, \frac{z^{mT-1}\mathrm{e}^{-z}}{\Gamma[mT]} \tag{40}$$

for $0 < y < 1$ and $z > 0$. Thus we see that the joint density for γ_t/γ_T and γ_T factorises into the product of a beta density for γ_t/γ_T and a gamma density for γ_T, as desired. □

The result of Proposition 2 is a special case of the following more general result:

Proposition 3 *Let $\{\gamma_t\}_{0\le t<\infty}$ be a standard gamma process. Then for $T \ge u \ge t \ge 0$ the random variables $(\gamma_u - \gamma_t)/(\gamma_T - \gamma_t)$ and $\gamma_T - \gamma_t$ are independent.*

Clearly, Proposition 2 follows as a special case of Proposition 3. The following lemma is a classical result (Lukacs 1955, Yeo & Milne 1991) which can be used as the basis of a proof of Proposition 3.

Lemma 1 *Let A and B be independent gamma-distributed random variables with parameters p and q, respectively. Then $A/(A+B)$ and $A+B$ are independent, $A/(A+B)$ has a* beta(p,q) *distribution, and $A+B$ has a* gamma$(p+q)$ *distribution.*

Proof. For independence it suffices to show that the joint Laplace transform of $A/(A+B)$ and $A+B$ factorises. In particular, for positive α, β we have

$$\mathbb{E}\left[\mathrm{e}^{-\alpha A/(A+B)-\beta(A+B)}\right] = \int_0^\infty \int_0^\infty \frac{a^{p-1}\mathrm{e}^{-a}}{\Gamma[p]} \frac{b^{q-1}\mathrm{e}^{-b}}{\Gamma[q]} \, \mathrm{e}^{-\alpha a/(a+b)-\beta(a+b)} \mathrm{d}a \, \mathrm{d}b. \tag{41}$$

Setting $x = a/(a+b)$ and $y = a+b$, we have $a = xy$ and $b = (1-x)y$, and hence $\mathrm{d}a\,\mathrm{d}b = y\,\mathrm{d}x\,\mathrm{d}y$. We see that

$$\begin{aligned} \mathbb{E}\left[\mathrm{e}^{-\alpha A/(A+B)-\beta(A+B)}\right] &= \int_0^1 \frac{x^{p-1}(1-x)^{q-1}}{\mathrm{B}[p,q]} \, \mathrm{e}^{-\alpha x} \mathrm{d}x \int_0^\infty \frac{y^{p+q-1}\mathrm{e}^{-y}}{\Gamma[p+q]} \, \mathrm{e}^{-\beta y} \mathrm{d}y \\ &= \mathbb{E}\left[\mathrm{e}^{-\alpha A/(A+B)}\right] \mathbb{E}\left[\mathrm{e}^{-\beta(A+B)}\right]. \end{aligned} \tag{42}$$

It follows that $A/(A+B)$ and $A+B$ are independent and have the distributions stated. □

The proof of Proposition 3 follows if we set $A=\gamma_u-\gamma_t$ and $B=\gamma_T-\gamma_u$. A proof of Proposition 2 is obtained if we set $A=\gamma_t$ and $B=\gamma_T-\gamma_t$.

V. VALUATION OF AGGREGATE CLAIMS

Our objective is to calculate the value at t of a contract that pays X_T at T. We assume that X_T is strictly positive and integrable. For simplicity of exposition, in this section we take X_T to be a continuous random variable; the adjustments required for the more general situation are straightforward. We assume that the default-free interest rate system is deterministic, that $\mathbb{Q}$ is the risk-neutral measure, and that the market filtration is generated by an aggregate claims process $\{\xi_t\}_{0\le t\le T}$ of the form $\xi_t = X_T\gamma_{tT}$, where $\{\gamma_{tT}\}$ is a standard gamma bridge under $\mathbb{Q}$, with parameter m, which we take to be independent of X_T. The value S_t of the contract at $t\le T$ is given by $S_t = P_{tT}\mathbb{E}\left[X_T \,|\, \mathcal{F}_t\right]$, where $\mathcal{F}_t=\sigma\left(\{\xi_s\}_{0\le s\le t}\right)$.

Proposition 4 *The aggregate claims process $\{\xi_t\}_{0\le t\le T}$ has the Markov property.*

Proof. For the Markov property we must verify that

$$\mathbb{Q}\left[\xi_t < a \,|\, \mathcal{F}_s\right] = \mathbb{Q}\left[\xi_t < a \,|\, \xi_s\right] \tag{43}$$

for all s, t such that $0\le s\le t\le T$. It suffices to establish that

$$\mathbb{Q}\left[\xi_t < a \,|\, \xi_s, \xi_{s_1}, \xi_{s_2}, \dots, \xi_{s_n}\right] = \mathbb{Q}\left[\xi_t < a \,|\, \xi_s\right] \tag{44}$$

for all $t\ge s\ge s_1\ge s_2\ge\cdots\ge s_n$, and for all $n\ge 1$. We use the representation $\{\gamma_{tT}\}=\{\gamma_t/\gamma_T\}$, where $\{\gamma_t\}$ is a standard gamma process with rate m. Then we have

$$\begin{aligned}\mathbb{Q}\left[\xi_t < a \,|\, \xi_s, \xi_{s_1}, \xi_{s_2}, \dots\right] &= \mathbb{Q}\left[\xi_t < a \,\middle|\, X_T\frac{\gamma_s}{\gamma_T}, X_T\frac{\gamma_{s_1}}{\gamma_T}, X_T\frac{\gamma_{s_2}}{\gamma_T}, \dots\right] \\ &= \mathbb{Q}\left[\xi_t < a \,\middle|\, X_T\frac{\gamma_s}{\gamma_T}, \frac{\gamma_{s_1}}{\gamma_s}, \frac{\gamma_{s_2}}{\gamma_{s_1}}, \dots\right].\end{aligned} \tag{45}$$

But $\gamma_{s_1}/\gamma_s, \gamma_{s_2}/\gamma_{s_1}, \dots$ are independent of ξ_t and ξ_s, which gives us the desired result. □

By virtue of the fact that $\{\xi_t\}$ has the Markov property and that X_T is $\mathcal{F}_T$-measurable we are able to simplify the expression for S_t so that it takes the form

$$S_t = P_{tT}\mathbb{E}\left[X_T \,|\, \xi_t\right]. \tag{46}$$

The conditional expectation appearing here can be carried out in closed form, leading to the following pricing formula:

Proposition 5 *The value S_t at time $t<T$ of the aggregate claim that pays the continuous random variable $X_T>0$ at time T is given by*

$$S_t = P_{tT}\frac{\int_{\xi_t}^{\infty} p(x)x^{2-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}{\int_{\xi_t}^{\infty} p(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}, \tag{47}$$

where $\{p(x)\}_{0<x<\infty}$ is the probability density of X_T.

Proof. The conditional expectation (46) can be written in the form

$$\mathbb{E}\left[X_T \,|\, \xi_t\right] = \int_0^\infty x\pi_t(x)\mathrm{d}x, \tag{48}$$

where $\{\pi_t(x)\}$ is the conditional density process for X_T, which by virtue of the Markov property of $\{\xi_t\}$ is given by

$$\pi_t(x) = \frac{\mathrm{d}}{\mathrm{d}x}\,\mathbb{Q}\left[X_T \le x \,|\, \xi_t\right]. \tag{49}$$

We can compute $\pi_t(x)$ by use of the following form of the Bayes formula:

$$\pi_t(x) = \frac{p(x)\rho\left(\xi_t \,|\, X_T = x\right)}{\int_0^\infty p(x)\rho(\xi_t \,|\, X_T = x)\mathrm{d}x}, \tag{50}$$

where $\rho\left(\xi_t \,|\, X_T = x\right)$ is the conditional density for ξ_t, valued at ξ_t. Specifically, we have

$$\begin{aligned} \rho\left(\xi \,|\, X_T = x\right) &= \frac{\mathrm{d}}{\mathrm{d}\xi}\,\mathbb{Q}\left[\xi_t \le \xi \,|\, X_T = x\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}\xi}\,\mathbb{Q}\left[X_T\gamma_{tT} \le \xi \,|\, X_T = x\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}\xi}\,\mathbb{Q}\left[\gamma_{tT} \le \frac{\xi}{x}\right]. \end{aligned} \tag{51}$$

Therefore, writing $\{f(y)\}_{0<y<1}$ for the density function of the random variable γ_{tT} we find

$$\begin{aligned} \rho\left(\xi \,|\, X_T = x\right) &= \frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^{\xi/x} f(y)\mathrm{d}y \\ &= \frac{1}{x}\, f\left(\frac{\xi}{x}\right). \end{aligned} \tag{52}$$

Hence by Proposition 1 we have

$$\begin{aligned} \rho\left(\xi \,|\, X_T = x\right) &= \frac{1}{x}\,\mathbb{1}_{\{x>\xi\}}\frac{(\xi/x)^{mt-1}(1-\xi/x)^{m(T-t)-1}}{\mathrm{B}\left[mt, m(T-t)\right]} \\ &= \mathbb{1}_{\{x>\xi\}}\xi^{mt-1}\frac{x^{1-mT}(x-\xi)^{m(T-t)-1}}{\mathrm{B}\left[mt, m(T-t)\right]}. \end{aligned} \tag{53}$$

The conditional probability density function $\{\pi_t(x)\}_{0\le t<T,\, x>0}$ for X_T is thus given by

$$\pi_t(x) = \mathbb{1}_{\{x>\xi_t\}}\frac{p(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}}{\int_{\xi_t}^\infty p(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\mathrm{d}x}, \tag{54}$$

from which the desired result (47) follows at once. □

With these results at hand we are also in a position to price a simple stop-loss reinsurance policy. For such a policy the value process is given by (2), and hence we have

$$\begin{aligned} C_{tT} &= P_{tT}\int_0^\infty (x-K)^+\pi_t(x)\mathrm{d}x \\ &= P_{tT}\frac{\int_{\xi_t}^\infty (x-K)^+ p(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}{\int_{\xi_t}^\infty p(x)x^{1-mT}(x-\xi_t)^{m(T-t)-1}\,\mathrm{d}x}. \end{aligned} \tag{55}$$

It should be evident that once a time t has been reached such that $\xi_t \geq K$, then $C_{uT} = P_{uT}(S_t - K)$ for all u such that $t \leq u \leq T$. In other words, once a sufficient number of claims have accumulated the option is sure to expire in-the-money.

VI. VALUATION OF GENERAL REINSURANCE CONTRACTS

In the previous section we showed how one works out the reserve process for an aggregate claim that pays X_T at T, and we were also able to determine the value process of a stop-loss contract that pays $(X_T - K)^+$ at T. In this section we consider the more general situation of a contract that at a fixed time $t < T$ allows the policy holder the option of commuting the claim X_T in exchange for a pre-fixed settlement K. Let us write C_{0t} for the value at time 0 of such an option; then clearly we have

$$C_{0t} = P_{0t}\mathbb{E}\left[(S_t - K)^+\right], \tag{56}$$

where S_t is the value at t of the claim that pays X_T at T. With reference to Proposition 5, it will be useful to introduce a function $S(t, y)$ for $0 \leq t < T$ and $y \geq 0$ by setting

$$S(t,y) = P_{tT}\frac{\int_y^\infty p(x)x^{2-mT}(x-y)^{m(T-t)-1}\mathrm{d}x}{\int_y^\infty p(x)x^{1-mT}(x-y)^{m(T-t)-1}\mathrm{d}x}. \tag{57}$$

Then the value of the claim is given by $S_t = S(t, \xi_t)$, and the value of the option can be written in the form

$$C_{0t} = P_{0t}\mathbb{E}\left[(S(t,\xi_t) - K)^+\right]. \tag{58}$$

Since the payout of the option is a function of ξ_t, one way of working out the expectation in (58) is to obtain an expression for the price $A_{0t}(y)$ of an Arrow-Debreu security that pays $\delta(\xi_t - y)$ at t, where $y \geq 0$ is a parameter. Thus we have

$$A_{0t}(y) = P_{0t}\mathbb{E}[\delta(\xi_t - y)], \tag{59}$$

and for the option we can write

$$C_{0t} = \int_0^\infty A_{0t}(y)\,[S(t,y) - K]^+\,\mathrm{d}y. \tag{60}$$

We shall calculate $A_{0t}(y)$ and use the result to determine the expectation (58). We state the result of this calculation first, the proof of which is given at the end of this section.

Proposition 6 *The price $A_{0t}(y)$ at time 0 of an Arrow-Debrue security that pays $\delta(\xi_t - y)$ at t is given by*

$$A_{0t}(y) = P_{0t}\frac{y^{mt-1}}{B[mt, m(T-t)]}\int_y^\infty p(x)\, x^{1-mT}(x-y)^{m(T-t)-1}\mathrm{d}x. \tag{61}$$

By comparing (57) and (61) we observe that the integral term in (61) cancels with the denominator in the expression for $S(t, y)$. After some rearrangement we thus obtain

$$C_{0t} \;=\; \int_0^\infty \frac{P_{0t}\, y^{mt-1}}{\mathrm{B}[mt, m(T-t)]}\left[\int_y^\infty p(x)\,(xP_{tT} - K)\, x^{1-mT}(x-y)^{m(T-t)-1}\mathrm{d}x\right]^+ \mathrm{d}y. \tag{62}$$

We are now left with the task of finding the critical values at which the argument of the max-function in the integrand of (62) vanishes. Suppose that $S(t, y)$ is monotonic in y; then there is at most a single critical value y^*, obtained by solving the following equation:

$$\int_{y^*}^\infty p(x)\,(xP_{tT} - K)\; x^{1-mT}(x-y^*)^{m(T-t)-1}\mathrm{d}x = 0. \tag{63}$$

The lower limit of the outer integration in the expression for C_{0t} above can then be changed, and we have

$$C_{0t} \;=\; \int_{y^*}^\infty \frac{P_{0t}\, y^{mt-1}}{\mathrm{B}[mt, m(T-t)]}\left[\int_y^\infty p(x)\,(xP_{tT} - K)\, x^{1-mT}(x-y)^{m(T-t)-1}\mathrm{d}x\right] \mathrm{d}y. \tag{64}$$

This expression simplifies further if we swap the order of integration as follows:

$$C_{0t} \;=\; \frac{P_{0t}}{\mathrm{B}[mt, m(T-t)]}\int_{y^*}^\infty \int_{y^*}^x p(x)\,(xP_{tT} - K)\, y^{mt-1}x^{1-mT}(x-y)^{m(T-t)-1}\mathrm{d}y\,\mathrm{d}x. \tag{65}$$

Making the substitution $y = xz$ we then obtain

$$C_{0t} = \frac{P_{0t}}{\mathrm{B}[mt, m(T-t)]}\int_{y^*}^\infty p(x)\,(xP_{tT} - K)\int_{y^*/x}^1 z^{mt-1}(1-z)^{m(T-t)-1}\mathrm{d}z\,\mathrm{d}x. \tag{66}$$

Let us now introduce the complementary beta distribution function $\mathcal{B}(u)$ with parameters mt and $m(T-t)$ by the expression:

$$\mathcal{B}(u) = \frac{\int_u^1 z^{mt-1}(1-z)^{m(T-t)-1}\mathrm{d}z}{\int_0^1 z^{mt-1}(1-z)^{m(T-t)-1}\mathrm{d}z}. \tag{67}$$

We call this the "complementary" distribution because the integration ranges from u to 1. Clearly, the denominator in (67) is $B[mt, m(T-t)]$. We thus find that the integration over the variable z in (66) combines with the factor $B[mt, m(T-t)]$ appearing of that expression to give a cumulative beta distribution function, and for the option price we have

$$C_{0t} = P_{0t}\int_{y^*}^\infty p(x)\,(x\,P_{tT} - K)\,\mathcal{B}(y^*/x)\,\mathrm{d}x. \tag{68}$$

We remark, incidentally, that a sufficient condition for $S(t,y)$ to be monotonic in y for fixed t is $m(T-t)>1$. To see this, we differentiate $S(t,y)$ with respect to y, assuming the stated condition, and after some rearrangement we obtain

$$\frac{\partial S(t,y)}{\partial y}=P_{tT}\left[m(T-t)-1\right]\left(\frac{\int_y^\infty p(x)\alpha^2(x)\mathrm{d}x\int_y^\infty p(x)\beta^2(x)\mathrm{d}x}{\left(\int_y^\infty p(x)\alpha(x)\beta(x)\mathrm{d}x\right)^2}-1\right), \tag{69}$$

where $\alpha^2(x)=x^{1-mT}(x-y)^{m(T-t)}$ and $\beta^2(x)=x^{1-mT}(x-y)^{m(T-t)-2}$. If $m(T-t)>1$, then the integrals exist, and it follows on account of the Schwartz inequality that $\partial S(t,y)/\partial y>0$.

Proof of Proposition 6. It suffices to determine the expectation $\mathbb{E}[\delta(\xi_t-y)]$. By use of the Fourier representation (36) we can write

$$\mathbb{E}[\delta(\xi_t-y)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\lambda y}\,\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda\xi_t}\right]\mathrm{d}\lambda. \tag{70}$$

Since X_T and γ_{tT} are independent, it follows from the tower property that

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda\xi_t}\right] &= \mathbb{E}\Big[\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda X_T\gamma_{tT}}|X_T\right]\Big]\\ &= \int_0^\infty p(x)\,\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda x\gamma_{tT}}\right]\mathrm{d}x\\ &= \int_0^\infty p(x)\,\phi(\lambda x)\,\mathrm{d}x,\end{aligned} \tag{71}$$

where $\phi(\nu)=\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\nu\gamma_{tT}}\right]$ is the characteristic function of γ_{tT}. We deduce that

$$\mathbb{E}[\delta(\xi_t-y)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\lambda y}\int_{x=0}^{\infty}p(x)\,\phi(\lambda x)\,\mathrm{d}x\,\mathrm{d}\lambda. \tag{72}$$

Thus, by interchanging the order of integration and using the fact that the inverse Fourier transform of the characteristic function is the density function we have

$$\begin{aligned}\mathbb{E}[\delta(\xi_t-y)] &= \int_{x=0}^{\infty}p(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\lambda y}\phi(\lambda x)\,\mathrm{d}\lambda\right]\mathrm{d}x\\ &= \int_{x=0}^{\infty}p(x)\frac{1}{x}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\,\nu\,y/x}\phi(\nu)\,\mathrm{d}\nu\right]\mathrm{d}x\\ &= \int_0^\infty p(x)\frac{1}{x}f\left(\frac{y}{x}\right)\mathrm{d}x,\end{aligned} \tag{73}$$

where f is the density function of γ_{tT}. Substituting the expression (17) for f into (73) we find that

$$\begin{aligned}\mathbb{E}\left[\delta(\xi_t-y)\right] &= \int_0^\infty p(x)\,\frac{1}{x}\,\mathbb{1}_{\{x>y\}}\frac{(y/x)^{mt-1}(1-y/x)^{m(T-t)-1}}{\mathrm{B}[mt,m(T-t)]}\,\mathrm{d}x\\ &= \frac{y^{mt-1}}{\mathrm{B}[mt,m(T-t)]}\int_y^\infty p(x)\,x^{1-mT}(x-y)^{m(T-t)-1}\mathrm{d}x,\end{aligned} \tag{74}$$

which verifies the claim. $\square$

We remark that the price of the Arrow-Debrue security can be put in the form

$$A_{0t}(y) = P_{0t}\frac{\int_0^1 p(y/u)\, u^{mt-2}(1-u)^{m(T-t)-1}\mathrm{d}u}{\int_0^1 u^{mt-1}(1-u)^{m(T-t)-1}\mathrm{d}u}, \tag{75}$$

by use of which the normalisation $\int_0^\infty A_{0t}(y)\mathrm{d}y = P_{0t}$ can be checked. It follows also from (75) that the characteristic function $\Phi_\xi(\lambda)$ of ξ_t is given by the beta average of the characteristic function ϕ_X of X_T:

$$\Phi_\xi(\lambda) = \frac{\int_0^1 \phi_X(\lambda u)u^{mt-1}(1-u)^{m(T-t)-1}\mathrm{d}u}{\int_0^1 u^{mt-1}(1-u)^{m(T-t)-1}\mathrm{d}u}. \tag{76}$$

VII. DISCRETE CASH FLOWS

Thus far we have considered the case for which the terminal cash flow is a continuous random variable. In this section we consider the example for which X_T takes values in a discrete set $\{x_i\}_{i=1,\ldots,n}$. The corresponding *a priori* probabilities will be denoted $\{p_i\}$. The calculation presented in Section V holds and we obtain, instead of (47), the following expression for the value process:

$$S_t = P_{tT}\frac{\sum_i p_i x_i^{2-mT}(x_i-\xi_t)^{m(T-t)-1}\mathbb{1}_{\{\xi_t<x_i\}}}{\sum_i p_i x_i^{1-mT}(x_i-\xi_t)^{m(T-t)-1}\mathbb{1}_{\{\xi_t<x_i\}}}. \tag{77}$$

It is straightforward to verify that expression (77) converges to the correct terminal value as t approaches T. To see this, suppose that for some $\omega\in\Omega$ the value of X_T is x_k. Then for that choice of ω we have

$$S_t = P_{tT}\frac{\sum_i p_i x_i^{2-mT}(x_i-x_k\gamma_{tT})^{m(T-t)-1}\mathbb{1}_{\{x_i>x_k\gamma_{tT}\}}}{\sum_i p_i x_i^{1-mT}(x_i-x_k\gamma_{tT})^{m(T-t)-1}\mathbb{1}_{\{x_i>x_k\gamma_{tT}\}}}, \tag{78}$$

and hence, after some rearrangement,

$$S_t \;=\; P_{tT}\frac{p_k x_k^{1-mt} + \sum_{i\neq k} p_i x_i^{2-mT}\left(\frac{1-\gamma_{tT}}{x_i-x_k\gamma_{tT}}\right)^{1-m(T-t)}\mathbb{1}_{\{x_i>x_k\gamma_{tT}\}}}{p_k x_k^{-mt} + \sum_{i\neq k} p_i x_i^{1-mT}\left(\frac{1-\gamma_{tT}}{x_i-x_k\gamma_{tT}}\right)^{1-m(T-t)}\mathbb{1}_{\{x_i>x_k\gamma_{tT}\}}}. \tag{79}$$

It follows at once that $S_T = x_k$.

We proceed now to value a reinsurance contract that pays $(S_t-K)^+$ at time t. For this purpose we need the price of an Arrow-Debreu security with payoff $\delta(\xi_t-y)$ at t. In the discrete case the Arrow-Debreu price is given by

$$A_{0t}(y) = P_{0t}\frac{y^{mt-1}}{\mathrm{B}[mt, m(T-t)]}\sum_{i=0}^{n} p_i x_i^{1-mT}(x_i-y)^{m(T-t)-1}\mathbb{1}_{\{x_i>y\}}. \tag{80}$$

Substituting (80) and the function

$$S(t,y) = P_{tT}\frac{\sum_i p_i x_i^{2-mT}(x_i-y)^{m(T-t)-1}\mathbb{1}_{\{x_i>y\}}}{\sum_i p_i x_i^{1-mT}(x_i-y)^{m(T-t)-1}\mathbb{1}_{\{x_i>y\}}} \tag{81}$$

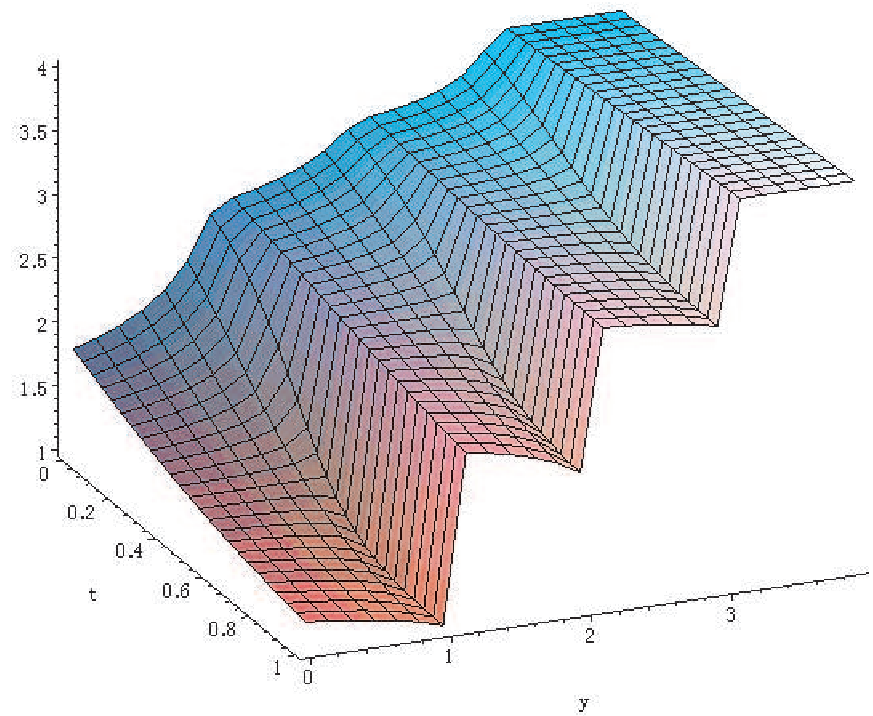

FIG. 1: The value function $S(t, y)$ associated with the reserve price in the case of a discrete cash flow at time T taking four possible values. The parameters are chosen such that $\{x_1, x_2, x_3, x_4\} = \{1, 2, 3, 4\}$, $\{p_1, p_2, p_3, p_4\} = \{0.5, 0.2, 0.2, 0.1\}$, $m = 2.0$, $r = 5\%$, and $T = 1$. For a given time t the value function represents the reserve required if the aggregate claims amount to y.

into (60) we obtain, after some rearrangement,

$$C_{0t} = \frac{P_{0t}}{B[mt, m(T-t)]} \int_0^\infty y^{mt-1} \left[\sum_{i=1}^n p_i x_i^{1-mT} (x_i - y)^{m(T-t)-1} \mathbb{1}_{\{x_i > y\}} (P_{tT} x_i - K) \right]^+ \mathrm{d}y. \quad (82)$$

A discrete version of formula (69) shows that $S(t, y)$ is increasing in y if $m(T-t) > 1$, and decreasing in y for $y \in (x_k, x_{k+1})$ for each $k = 1, \ldots, n-1$ if $m(T-t) < 1$. See Figure 1 for the typical behaviour of $S(t, y)$ when X_T takes four possible values. For fixed t there is at most a single critical value $y = y^*$ for which $S(t, y) = K$, when $y \neq x_k$ for all k. We thus have three scenarios to consider, namely: (I) $S(t, y)$ is increasing in y at $y = y^*$; (II) the critical value y^* is at $y = x_k$ for some k; and (III) $S(t, y)$ is decreasing in y at $y = y^*$.

We therefore analyse the price of the reinsurance contract in these different scenarios. In case (I) the integrand in (82) is nonzero when $y \in (y^*, \infty)$, and we have

$$C_{0t} = \frac{P_{0t}}{B[mt, m(T-t)]} \sum_{i=1}^{n} p_i x_i^{1-mT} (P_{tT} x_i - K) \int_{y^*}^{\infty} y^{mt-1} (x_i - y)^{m(T-t)-1} \mathbb{1}_{\{x_i > y\}} \mathrm{d}y. \quad (83)$$

The y integration in (83) can be carried out by observing that for $x_i > y^*$ we have

$$\begin{aligned} \int_{y^*}^{\infty} y^{mt-1}(x_i - y)^{m(T-t)-1} \mathbb{1}_{\{x_i > y\}} \mathrm{d}y &= x_i^{mT-2} \int_{y^*}^{x_i} \left(\frac{y}{x_i}\right)^{mt-1} \left(1 - \frac{y}{x_i}\right)^{m(T-t)-1} \mathrm{d}y \\ &= x_i^{mT-1} \int_{y^*/x_i}^{1} z^{mt-1}(1-z)^{m(T-t)-1} \mathrm{d}z, \end{aligned} \quad (84)$$

where we have made the substitution $y = x_i z$. Therefore, the price of the reinsurance contract can be expressed in terms of the complementary beta distribution function with parameters mt and $m(T-t)$:

$$C_{0t} = P_{0t} \sum_{i=1}^{n} \mathbb{1}_{\{x_i > y^*\}} p_i (P_{tT} x_i - K) \mathcal{B}(y^*/x_i). \quad (85)$$

If there is no critical value in the range (x_k, x_{k+1}), then $y^* = x_k$ for some k. Hence the pricing formula in case (II) is identical to the result obtained in (85), with $y^* = x_k$. In case (III) there are two distinct regions for which the integrand in (82) is nonzero. These are given by $y \in [x_k, y^*)$ and $y \geq x_{k+1}$ for some k. Hence the pricing formula is similar to that obtained in (85), except there are additional terms arising from the integration over the range $[x_k, y^*)$.

As an example of a discrete cash flow we consider the binary case where X_T can take the values x_0, x_1. In this situation the critical value $y^* < x_0$ can be worked out by solving

$$p_0(P_{tT} x_0 - K) x_0^{1-mT} (x_0 - y^*)^{m(T-t)-1} = p_1(K - P_{tT} x_1) x_1^{1-mT} (x_1 - y^*)^{m(T-t)-1} \quad (86)$$

for y^*. A short calculation shows that

$$y^* = \frac{\theta x_1 - x_0}{\theta - 1}, \quad \text{where} \quad \theta = \left[\frac{p_1(K - P_{tT} x_1)}{p_0(P_{tT} x_0 - K)} \left(\frac{x_1}{x_0}\right)^{1-mT}\right]^{\frac{1}{m(T-t)-1}}. \quad (87)$$

It follows that the price of a reinsurance contract in the case of a binary payoff is given by

$$C_{0t} = p_0(P_{0T} x_0 - P_{0t} K) \mathcal{B}(y^*/x_0) + p_1(P_{0T} x_1 - P_{0t} K) \mathcal{B}(y^*/x_1). \quad (88)$$

VIII. OPTION PRICE PROCESS

We generalise now the analysis of Section VI to derive an expression for the price process of a call option on the value of the reserve S_t at time t associated with the claim X_T. As before, we let K be the strike. Then the value of the option at time $s \leq t$ is given by

$$C_{st} = P_{st} \mathbb{E}\left[(S(t, \xi_t) - K)^+ | \xi_s\right]. \quad (89)$$

Once again we find it convenient to obtain first the price process for the Arrow-Debreu security. This is on account of the relation

$$\begin{aligned} C_{st} &= P_{st}\mathbb{E}\left[\int_0^\infty \delta(\xi_t - y)\,(S(t,y)-K)^+\,\mathrm{d}y \,\middle|\, \xi_s \right] \\ &= P_{st}\int_0^\infty \mathbb{E}\left[\delta(\xi_t - y)|\xi_s\right](S(t,y)-K)^+\,\mathrm{d}y \\ &= \int_0^\infty A_{st}(y)\,(S(t,y)-K)^+\,\mathrm{d}y, \end{aligned} \tag{90}$$

where $S(t,y)$ is defined as in (57), and $\{A_{st}\}_{0\le s\le t\le T}$ is given by

$$A_{st}(y) = P_{st}\mathbb{E}\left[\delta(\xi_t - y)|\xi_s\right]. \tag{91}$$

By taking the conditional expectation we obtain the following result:

Proposition 7 *The price process $\{A_{st}(y)\}_{0\le s\le t\le T}$ of the Arrow-Debreu security that pays out $\delta(\xi_t - y)$ at t is given by*

$$A_{st}(y) = P_{st}\frac{\mathbb{1}_{\{y>\xi_s\}}(y-\xi_s)^{m(t-s)-1}}{B[m(t-s),m(T-t)]}\frac{\int_y^\infty p(x)\,x^{1-mT}(x-y)^{m(T-t)-1}\mathrm{d}x}{\int_{\xi_s}^\infty p(x)\,x^{1-mT}(x-\xi_s)^{m(T-s)-1}\mathrm{d}x}, \tag{92}$$

where $y \ge \xi_s$ and $\{p(x)\}$ is the probability density of X_T.

This result is established later in this section. By substitution of (92) in (90) we see that the price process of the option is given by

$$\begin{aligned} C_{st} &= \frac{P_{st}}{B[m(t-s),m(T-t)]\int_{\xi_s}^\infty p(x)\,x^{1-mT}(x-\xi_s)^{m(T-s)-1}\mathrm{d}x} \\ &\quad \times \int_{y=\xi_s}^\infty (y-\xi_s)^{m(t-s)-1}\left[\int_y^\infty p(x)x^{1-mT}(x-y)^{m(T-t)-1}(P_{tT}x-K)\mathrm{d}x\right]^+\mathrm{d}y. \end{aligned} \tag{93}$$

Assuming that there is only one critical value y^* that solves (63), we find that the integration over y in (93) vanishes for y smaller than y^*. In this case, we can lift the max-function in the integrand, and by interchanging the order of integration we obtain

$$\begin{aligned} C_{st} &= \frac{P_{st}}{B[m(t-s),m(T-t)]\int_{\xi_s}^\infty p(x)\,x^{1-mT}(x-\xi_s)^{m(T-s)-1}\mathrm{d}x} \\ &\quad \times P_{tT}\int_{x=y^*}^\infty p(x)x^{1-mT}(P_{tT}x-K)\int_{y=y^*}^x (y-\xi_s)^{m(t-s)-1}(x-y)^{m(T-t)-1}\,\mathrm{d}y\,\mathrm{d}x. \end{aligned} \tag{94}$$

Let us analyse the y integration. Making the substitution $y = z + \xi_s$ we find that

$$\int_{y=y^*}^x (y-\xi_s)^{m(t-s)-1}(x-y)^{m(T-t)-1}\,\mathrm{d}y = \int_{z=y-\xi_s}^{x-\xi_s} z^{m(t-s)-1}(x-\xi_s-z)^{m(T-t)-1}\mathrm{d}z. \tag{95}$$

A further change of variable obtained by setting $w = z/(x-\xi_s)$ gives

$$\int_{z=y-\xi_s}^{x-\xi_s} z^{m(t-s)-1}(x-\xi_s-z)^{m(T-t)-1}\mathrm{d}z$$
$$= (x-\xi_s)^{m(T-s)-1}\int_{w=\frac{y^*-\xi_s}{x-\xi_s}}^{1} w^{m(t-s)-1}(1-w)^{m(T-t)-1}\mathrm{d}w. \quad (96)$$

We see that together with the beta function in the denominator of (94) the integral term in the right side of (96) gives rise to a complementary beta distribution function. Therefore, the call price can be written in the form

$$C_{st} = P_{st}\int_{x=y^*}^{\infty} \frac{p(x)x^{1-mT}(x-\xi_s)^{m(T-s)-1}}{\int_{\xi_s}^{\infty} p(x)\, x^{1-mT}(x-\xi_s)^{m(T-s)-1}\mathrm{d}x}\,(P_{tT}x-K)\,\mathcal{B}\left(\frac{y^*-\xi_s}{x-\xi_s}\right)\mathrm{d}x. \quad (97)$$

Finally, we observe that the quotient in the integrand is the conditional density $\pi_s(x)$. The call price at time $s \leq t$ thus reduces to the following expression:

$$C_{st} = P_{st}\int_{x=y*}^{\infty} \pi_s(x)(xP_{tT}-K)\,\mathcal{B}\left(\frac{y^*-\xi_s}{x-\xi_s}\right)\mathrm{d}x. \quad (98)$$

As in the case of the initial price of the option, the range of integration in (98) must be modified appropriately if there is more than one critical value for which (63) is satisfied. We now proceed to derive the expression for the Arrow-Debreu price process.

Proof of Proposition 7. By use of the Fourier representation (36) we have

$$\mathbb{E}\left[\delta(\xi_t-y)|\xi_s\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda y}\,\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda\xi_t}\middle|\,\xi_s\right]\mathrm{d}\lambda. \quad (99)$$

To determine the conditional expectation $\mathbb{E}\mathrm{e}^{\mathrm{i}\lambda\xi_t}|\xi_s]$ the following result is handy:

Lemma 2 *Let $\{\xi_t\}_{0\leq t\leq T}$ be given by $\xi_t = X_T\gamma_{tT}$, where $\{\gamma_{tT}\}$ is a gamma bridge and X_T is an independent positive random variable. Then for fixed s such that $0 \leq s \leq t \leq T$ we have*

$$\xi_t = \xi_s + Z_T\delta_{tT}, \quad (100)$$

where $Z_T = (1-\gamma_{sT})X_T$, and where the process $\{\delta_{tT}\}_{s\leq t\leq T}$, defined by

$$\delta_{tT} = \frac{\gamma_{tT}-\gamma_{sT}}{1-\gamma_{sT}}, \quad (101)$$

is a gamma bridge over the interval $t \in [s,T]$ and is independent of ξ_s and Z_T.

By use of (100) and the tower property we find that

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda\xi_t}\middle|\,\xi_s\right] &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda(\xi_s+Z_T\delta_{tT})}\middle|\,\xi_s\right] \\ &= \mathrm{e}^{\mathrm{i}\lambda\xi_s}\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda Z_T\delta_{tT}}\middle|\,\xi_s\right] \\ &= \mathrm{e}^{\mathrm{i}\lambda\xi_s}\mathbb{E}\left[\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda Z_T\delta_{tT}}\middle|\xi_s,\delta_{tT}\right]\middle|\,\xi_s\right]. \end{aligned} \quad (102)$$

Since $Z_T = X_T - \xi_s$, and since $\{\delta_{tT}\}$ is independent of ξ_s and X_T, the inner expectation can be carried out explicitly by use of the conditional density for X_T, and we obtain

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}\lambda\xi_t}\middle|\,\xi_s\right] = \mathrm{e}^{\mathrm{i}\lambda\xi_s}\mathbb{E}\left[\int_{x=\xi_s}^{\infty} \mathrm{e}^{\mathrm{i}\lambda(x-\xi_s)\delta_{tT}}\pi_s(x)\mathrm{d}x\middle|\,\xi_s\right]. \tag{103}$$

By substituting (103) in (99) we deduce that

$$\begin{aligned}\mathbb{E}\left[\delta(\xi_t - y)|\xi_s\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda(y-\xi_s)} \int_{x=\xi_s}^{\infty} \Phi_\delta[\lambda(x-\xi_s)]\,\pi_s(x)\,\mathrm{d}x\,\mathrm{d}\lambda \\ &= \int_{x=\xi_s}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\lambda(y-\xi_s)}\,\Phi_\delta[\lambda(x-\xi_s)]\,\mathrm{d}\lambda\right)\pi_s(x)\,\mathrm{d}x, \end{aligned} \tag{104}$$

where Φ_δ is the characteristic function for δ_{tT}. By use of the substitution $z = \lambda(x - \xi_s)$ we then find that

$$\begin{aligned}\mathbb{E}\left[\delta(\xi_t - y)|\xi_s\right] &= \int_{x=\xi_s}^{\infty} \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\frac{y-\xi_s}{x-\xi_s}z}\,\Phi_\delta(z)\mathrm{d}z\right)\frac{1}{x-\xi_s}\,\pi_s(x)\,\mathrm{d}x \\ &= \int_{x=\xi_s}^{\infty} \frac{\pi_s(x)}{x-\xi_s}\,f_\delta\left(\frac{y-\xi_s}{x-\xi_s}\right)\mathrm{d}x, \end{aligned} \tag{105}$$

where f_δ is the probability density of δ_{tT}. Since δ_{tT} is beta distributed with parameters $m(t-s)$ and $m(T-t)$, we deduce, after some rearrangement, the expression obtained in (92) for the Arrow-Debreu price. □

Proof of Lemma 2. The decomposition (100) can be verified by direct calculation if one sets $\{\gamma_{tT}\} = \{\gamma_t/\gamma_T\}$, where $\{\gamma_t\}$ is a standard gamma process. To see that $\{\delta_{tT}\}_{s\le t\le T}$ is, for fixed s, a gamma bridge over $[s,T]$ it suffices to note that $\delta_{tT} = (\gamma_t - \gamma_s)/(\gamma_T - \gamma_s)$ and that $\{\gamma_t - \gamma_s\}_{s\le t<\infty}$ is a gamma process. In particular, we observe that the independent increments property holds, and that $\gamma_t - \gamma_s$ is gamma distributed with mean $m(t-s)$. Finally, to see that $\{\delta_{tT}\}$ is independent of ξ_s and Z_T it suffices to show that δ_{tT}, γ_s and $\gamma_T - \gamma_s$ are independent. We have:

$$\begin{aligned}\mathbb{Q}\big(\{\delta_{tT} < a\}\cap\{\gamma_T-\gamma_s < b\}\cap\{\gamma_s < c\}\big) &= \mathbb{E}\big[\mathbb{1}_{\{\delta_{tT}<a\}}\mathbb{1}_{\{\gamma_T-\gamma_s<b\}}\mathbb{1}_{\{\gamma_s<c\}}\big] \\ &= \mathbb{E}\big[\mathbb{E}\big[\mathbb{1}_{\{\delta_{tT}<a\}}\mathbb{1}_{\{\gamma_T-\gamma_s<b\}}\mathbb{1}_{\{\gamma_s<c\}}\,|\,\gamma_s\big]\big] \\ &= \mathbb{E}\big[\mathbb{E}\big[\mathbb{1}_{\{\delta_{tT}<a\}}\mathbb{1}_{\{\gamma_T-\gamma_s<b\}}\,|\,\gamma_s\big]\,\mathbb{1}_{\{\gamma_s<c\}}\big] \\ &= \mathbb{E}\big[\mathbb{E}\big[\mathbb{1}_{\{\delta_{tT}<a\}}\mathbb{1}_{\{\gamma_T-\gamma_s<b\}}\big]\,\mathbb{1}_{\{\gamma_s<c\}}\big] \\ &= \mathbb{E}\big[\mathbb{1}_{\{\delta_{tT}<a\}}\mathbb{1}_{\{\gamma_T-\gamma_s<b\}}\big]\,\mathbb{E}\big[\mathbb{1}_{\{\gamma_s<c\}}\big] \\ &= \mathbb{E}\big[\mathbb{1}_{\{\delta_{tT}<a\}}\big]\,\mathbb{E}\big[\mathbb{1}_{\{\gamma_T-\gamma_s<b\}}\big]\,\mathbb{E}\big[\mathbb{1}_{\{\gamma_s<c\}}\big] \end{aligned} \tag{106}$$

In going from the fourth to the fifth line we have used the fact that γ_s is independent of δ_{tT} and $\gamma_T - \gamma_s$, which can be checked by use of the independent increments property of $\{\gamma_t\}$. In going from the sixth to seventh line we have used Lemma 1 together with the fact that we can write $\delta_{tT} = B/(A+B)$ and $\gamma_T - \gamma_s = A + B$, with $A = \gamma_T - \gamma_t$ and $B = \gamma_t - \gamma_s$, from which it follows that δ_{tT} and $\gamma_T - \gamma_s$ are independent. □

The result of Lemma 2 leads to the following observation concerning the model calibration. Suppose that the aggregate claims process is given, and that we reinitialise the model at some specified intermediate time. We would like the dynamics of the model moving forward from that intermediate time to be consistently represented by an aggregate claims process of the same type. Indeed, it follows from Lemma 2 that the process $\{\eta_t\}_{s\leq t\leq T}$ defined by

$$\eta_t = Z_T \delta_{tT} \tag{107}$$

is an aggregate claims process spanning the time interval $[s, T]$. The random variable Z_T can be thought of as representing the information about X_T that is "not yet revealed" at time s. The idea is that at time s the value of ξ_s is known, and the "new" gains process $\{\eta_t\}_{s\leq t\leq T}$ begins to reveal the value of Z_T in such a way that $\eta_s = 0$ and $\eta_T = Z_T$.

Alternatively, at time s we can use the knowledge of ξ_s to compute the "new" *a priori* density for X_T. Thus, at time s the *a priori* density $p(x)$ for X_T is replaced by the appropriate *a posteriori* density $\pi_s(x)$. On account of the relation $Z_T = X_T - \xi_s$ we have

$$\mathbb{Q}[Z_T < z|\xi_s] = \mathbb{Q}[X_T < z + \xi_s|\xi_s], \tag{108}$$

from which it follows that the conditional density of Z_T is given at time s by $\pi_s(z + \xi_s)$. We can think of $\pi_s(\xi_s + z)$ as a "new" *a priori* density, now for the random variable Z_T. Given this density we calculate the conditional probability $\mathbb{Q}[Z_T < z|\eta_t]$ for $t \in [s, T]$. By the method used to establish Proposition 4 and the probability law for the gamma bridge $\{\delta_{tT}\}$ we deduce that the associated density function is given by

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathbb{Q}[Z_T < z|\eta_t] = \mathbb{1}_{\{z>\eta_t\}} \frac{\pi_s(\xi_s + z) z^{1-m(T-s)} (z - \eta_t)^{m(T-t)-1}}{\int_{\eta_t}^{\infty} \pi_s(\xi_s + z) z^{1-m(T-s)} (z - \eta_t)^{m(T-t)-1} \mathrm{d}z}, \tag{109}$$

from which we see that the value process can be represented in the following form:

$$S_t = P_{tT}\left[\xi_s + \frac{\int_{\eta_t}^{\infty} \pi_s(\xi_s + z) z^{2-m(T-s)} (z - y)^{m(T-t)-1} \mathrm{d}z}{\int_{\eta_t}^{\infty} \pi_s(\xi_s + z) z^{1-m(T-s)} (z - y)^{m(T-t)-1} \mathrm{d}z}\right]. \tag{110}$$

Making the substitution $z = x - \xi_s$ and also substituting $\eta_t = \xi_t - \xi_s$, this expression reduces to the value process obtained in (47).

IX. EXAMPLE: GAMMA-DISTRIBUTED CASH FLOW

When the terminal payout X_T of the cumulative gains process (3) is gamma distributed with mean κmT and variance $\kappa^2 mT$ for some choice of κ, the resulting value process $\{S_t\}$ has an especially simple structure. In particular, we are lead back to the "$\mathbb{Q}$-gamma" model discussed in the introduction. This can be seen as follows. Let $\{\gamma_t\}$ be a standard gamma process with rate m, and let $\{\gamma_{tT}\}$ be the associated gamma bridge. Then X_T and $\kappa\gamma_T$ have the same distribution; but since γ_T and $\{\gamma_{tT}\}$ are independent, it follows that $\{X_T\gamma_{tT}\}$ and $\{\kappa\gamma_T\gamma_{tT}\}$ have the same probability law; therefore, $\{\xi_t\}$ has the same law as $\{\kappa\gamma_t\}$, and hence is a $\mathbb{Q}$-gamma process, with scale κ and standard growth rate m. The fact that ξ_t is gamma distributed can be verified directly as follows. The characteristic function of X_T is $\phi_X(\lambda) = (1 - \mathrm{i}\kappa\lambda)^{-mT}$. Substituting this into (76) and setting $z = (1 - u)/(1 - \mathrm{i}\kappa\lambda u)$, we deduce that $\Phi_\xi(\lambda) = (1 - \mathrm{i}\kappa\lambda)^{-mt}$, which is the characteristic function of a gamma distributed random variable with mean κmt and variance $\kappa^2 mt$.

It is interesting to note that although the $\mathbb{Q}$-gamma process has independent increments, the cumulative gains process (3) has dependent increments. In particular, for the covariance of ξ_s and $\xi_t - \xi_s$ in the general case we have

$$\mathrm{Cov}[\xi_s, \xi_t - \xi_s] = \frac{ms(t-s)}{T(mT+1)}\,\mathbb{E}[X_T^2] - \frac{s(t-s)}{T^2}\,(\mathbb{E}[X_T])^2\,. \tag{111}$$

Hence a necessary condition for independent increments is given by $(\mathbb{E}[X_T])^2 = mT\,\mathrm{Var}[X_T]$.

We conclude the paper by working out in some detail the value processes for various claims in the $\mathbb{Q}$-gamma model. For the density of X_T we have $g_{\Gamma_T}(x)$, where $g_{\Gamma_t}(x)$ is defined in (14). Substituting the expression for the density function into (47) and carrying out the relevant integration, we are led to the following expression for the reserve process:

$$S_t = P_{tT}\big(\xi_t + \kappa m(T-t)\big). \tag{112}$$

Therefore, $\{S_t\}$ in this case is a linear function of $\{\xi_t\}$. We observe that $S_0 = P_{0T}\kappa m T$ and that $S_T = X_T$, as required. An alternative derivation of (112) is as follows. Since $\{\xi_t\}$ is a gamma process with scale parameter κ and standardised growth rate m, by the Markov property we have $S_t = P_{tT}\mathbb{E}[\xi_T|\xi_t]$, and (112) follows immediately as a consequence of the independent increments property of the gamma process.

These relations lead to simplifications in the valuation of contingent claims. Let us work out, for example, the value C_{tT} at time t of a simple stop-loss reinsurance contract that pays out $\max(X_T - K, 0)$ at T for some fixed threshold K. In the $\mathbb{Q}$-gamma model we have

$$C_{tT} = P_{tT}\mathbb{E}[(\xi_T - K)^+|\xi_t], \tag{113}$$

and hence by use of the independent increments property we deduce that

$$\begin{aligned} C_{tT} &= P_{tT}\int_{(K-\xi_t)/\kappa}^{\infty} (\kappa z + \xi_t - K)\frac{z^{m(T-t)-1}\mathrm{e}^{-z}}{\Gamma[m(T-t)]}\,\mathrm{d}z \\ &= P_{tT}\left[\kappa\frac{\Gamma[m(T-t)+1,(K-\xi_t)/\kappa]}{\Gamma[m(T-t)]} - (K-\xi_t)\frac{\Gamma[m(T-t),(K-\xi_t)/\kappa]}{\Gamma[m(T-t)]}\right], \end{aligned} \tag{114}$$

where $\Gamma[a,z] = \int_z^\infty x^{a-1}\mathrm{e}^{-x}\mathrm{d}x$ denotes the incomplete gamma integral.

We proceed to calculate the associated Arrow-Debreu price A_{st} in this model. By substituting (112) in (92) we deduce that

$$A_{st}(y) = P_{st}\frac{\kappa^{-m(t-s)}}{\Gamma[m(t-s)]}\,(y-\xi_s)^{m(t-s)-1}\exp\left(-\frac{1}{\kappa}(y-\xi_s)\right). \tag{115}$$

It follows by use of (112) that the price at time s of a reinsurance contract with payout $(S_t - K)^+$ at t is

$$\begin{aligned} C_{st} &= P_{st}\mathbb{E}_s[(S_t-K)^+] \\ &= \int_0^\infty A_{st}(y)\,[P_{tT}(y+\kappa m(T-t)) - K]^+\,\mathrm{d}y \\ &= P_{sT}\left[\frac{\Gamma[m(t-s)+1,\kappa^{-1}R_s]}{\Gamma[m(t-s)]} - \kappa^{-1}R_s\frac{\Gamma[m(t-s),\kappa^{-1}R_s]}{\Gamma[m(t-s)]}\right], \end{aligned} \tag{116}$$

where $R_s = P_{tT}^{-1}K - (S_s + \kappa m(t-s))$.

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