The massless Dirac equation from the continuum mechanics and microlocal analysis perspectives

A thesis submitted for the degree of
Doctor of Philosophy

by

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September 2012

Declaration

I, Olga Chervova, confirm that the work presented in this thesis is my own. Where information has been derived from other sources, I confirm that this has been indicated in the thesis.

"As far as the laws of mathematics refer to reality, they are not certain; as far as they are certain, they do not refer to reality."

Albert Einstein

"In mathematics you don't understand things. You just get used to them."

Johann von Neumann

Abstract

The thesis is concerned with the study of the massless Dirac equation.

In the first part we study the massless Dirac equation in dimension 1+3 in the stationary setting, i.e. when the spinor field oscillates harmonically in time. We suggest a new geometric interpretation for this equation. We think of our 3-dimensional space as an elastic continuum and assume that material points can experience no displacements, only rotations. This framework is a special case of the Cosserat theory of elasticity. Rotations of material points are described mathematically by attaching to each geometric point an orthonormal basis which gives a field of orthonormal bases called the coframe. As the dynamical variables we choose the coframe and a density. We choose a particular potential energy which is conformally invariant and then incorporate time into our action by subtracting kinetic energy. We prove that in the stationary setting our model is equivalent to a pair of massless Dirac equations.

In the second part we consider an elliptic self-adjoint first order pseudodifferential operator acting on columns of m complex-valued half-densities over a compact n-dimensional manifold. The eigenvalues of the principal symbol are assumed to be simple but no assumptions are made on their sign, so the operator is not necessarily semi-bounded. We study the spectral function and derive a two-term asymptotic formula. We then restrict our study to the case when $m = 2$, $n = 3$, the operator is differential and has trace-free principal symbol, and address the question: is our operator a massless Dirac operator? We prove that it is a massless Dirac operator if and only if, at every point, a) the subprincipal symbol is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function is zero.

Acknowledgements

I would like to express my sincere thanks to my amazing supervisor Professor Dima Vassiliev for the guidance and help he generously provided me with all the time.

I am truly grateful to my colleagues and friends James Burnett and Robert J. Downes for all the time we spent together during my UCL years.

I would also like to thank my beloved family for their support and encouragement.

Contents

Chapter 1

Introduction

1.1 The massless Dirac equation

The thesis is concerned with the study of the massless Dirac equation. It is the accepted mathematical model for a massless neutrino field.

The massless Dirac equation is a system of two complex first order partial differential equations for two complex-valued unknowns on a Lorentzian 4-manifold. Throughout this thesis we assume, for simplicity, that the Lorentzian manifold in question has the structure $\mathbb{R} \times M$ where M is a Riemannian 3-manifold.

The dynamical variable (unknown quantity) in the massless Dirac equation is a two-component complex-valued spinor field ξ which is a function of time $x^0 \in \mathbb{R}$ and local coordinates x^α on M. The explicit form of the massless Dirac equation is

$$i(\pm\sigma^0{}_{\dot{a}b}\partial_0 + \sigma^\alpha{}_{\dot{a}b}\nabla_\alpha)\xi^b = 0. \tag{1.1.1}$$

Here the σ are Pauli matrices, ∂_0 is the time derivative and ∇_α is the covariant spatial derivative, see Appendix 2.C for details. Summation in (1.1.1) is carried out over the tensor index $\alpha = 1, 2, 3$ as well as over the spinor index $b = 1, 2$. The use of the partial derivative $\partial_0 = \partial/\partial x^0$ in equation (1.1.1) is justified by the

fact that the time coordinate x^0 is fixed and we allow only changes of coordinates (x^1, x^2, x^3) which do not depend on x^0.

We see that the massless Dirac equation (1.1.1) is, indeed, a system of two ($\dot{a} = \dot{1}, \dot{2}$) complex linear partial differential equations on the 4-manifold $\mathbb{R} \times M$ for two complex unknowns ξ^b, $b = 1, 2$. The two choices of sign in (1.1.1) give two versions of the massless Dirac equation which differ by time reversal. Thus, we have a pair of massless Dirac equations.

The aim of the thesis is to have a fresh look at the massless Dirac equation (1.1.1) and to identify mathematical problems in other subject areas (i.e. other than theoretical physics and differential geometry) which generate the massless Dirac equation.

We found two new perspectives on the massless Dirac equation: a continuum mechanics interpretation and a microlocal analysis interpretation. Hence, the thesis consists of two parts. We describe below the main results from these two parts.

Note that the notation in the two parts of the thesis is somewhat different: in Chapter 2 it is in line with theoretical physics notation whereas in Chapter 3 it is in line with the notation of spectral theory. This makes a certain degree of repetition inevitable: say, in Appendix 3.A within Chapter 3 we redefine the massless Dirac equation (1.1.1) using spectral theoretic notation, without the explicit use of spinors.

1.2 Continuum mechanics interpretation of the massless Dirac equation

We will be interested in spinor fields of the form

$$\xi(x^0, x^1, x^2, x^3) = e^{-ip_0 x^0} \eta(x^1, x^2, x^3) \tag{1.2.1}$$

where

$$p_0 \neq 0 \tag{1.2.2}$$

is a real number. Substituting (1.2.1) into (1.1.1) we get the equation

$$\pm\, p_0 \sigma^0{}_{\dot{a}b} \eta^b + i \sigma^\alpha{}_{\dot{a}b} \nabla_\alpha \eta^b = 0 \tag{1.2.3}$$

which we shall call *the stationary massless Dirac equation*. Note that in equation (1.1.1) the spinor field ξ "lives" on the Lorentzian 4-manifold $\mathbb{R} \times M$ whereas in equation (1.2.3) the spinor field η "lives" on the Riemannian 3-manifold M. Thus, the stationary massless Dirac equation is the massless Dirac equation with time separated out.

We separated out time to simplify the problem while retaining most of its essential features. Note also that this separation of variables has a clear physical meaning: the real number p_0 appearing in (1.2.1) and (1.2.3) is quantum mechanical energy.

Our aim is to show that the stationary massless Dirac equation (1.2.3) can be reformulated in an alternative (but mathematically equivalent) way using instead of a spinor field a different set of dynamical variables.

We view our 3-manifold M as an elastic continuum. But it is not an ordinary elastic continuum, its material points possess a very special property. They cannot experience any displacements, they can only experience rotations. Moreover, different material points rotate independently.

To describe these rotations mathematically we attach an orthonormal basis to each geometric point of our manifold. It gives us a field of orthonormal bases or coframe. We denote the coframe as ϑ^j, $j = 1, 2, 3$, see Appendix 2.B for details.

As dynamical variables in our model we choose the coframe ϑ and a positive density ρ. They are functions of time x^0 and local coordinates (x^1, x^2, x^3) on M.

At a physical level, making the density ρ a dynamical variable means that we view our continuum more like a fluid rather than a solid. In other words, we allow the material to redistribute itself so that it finds its equilibrium distribution.

Note that the total number of real dynamical degrees of freedom contained in the coframe ϑ and positive density ρ is four, exactly as in a two-component complex-valued spinor field ξ. Moreover, it is known (see Appendix 2.D) that a coframe ϑ and a (positive) density ρ are geometrically equivalent to a nonvanishing spinor field ξ modulo the sign of ξ.

As a measure of rotational deformations we choose torsion, which is an approach going back to Einstein and Cartan. The torsion tensor is expressed via the coframe and its first partial derivatives, see [6] for details.

The crucial element in our construction is the choice of potential energy. It is known [6] that in the purely rotational setting the potential energy of a physically linear elastic continuum contains three quadratic terms, with three real parameters (elastic moduli) as factors. The three quadratic terms in potential energy correspond to the three irreducible pieces of torsion. It is not *a priori* clear what the elastic moduli of "world aether" are.

We choose a potential energy which feels only one piece of torsion, axial. This leaves us with a unique, up to rescaling by a positive constant, formula (2.1.6) for potential energy. This particular potential energy also has the remarkable property of conformal invariance, i.e. it is invariant under the rescaling of the 3-dimensional metric g by an arbitrary positive scalar function.

After the potential energy is chosen the remainder of our construction is straightforward. We incorporate time into our action in the standard Newtonian way, by subtracting kinetic energy. This gives us the Lagrangian density (2.1.13). As we are interested in comparing our mathematical model with the massless Dirac equation, we perform a change of dynamical variables and switch from coframe ϑ and density ρ to a spinor field ξ. Our Lagrangian density now takes the form (2.2.1). We write down the field equation (Euler–Lagrange equation) for our

Lagrangian density and observe that time separates out if we seek stationary solutions (1.2.1); this separation of variables is highly nontrivial because our field equation is nonlinear. After separation of variables our Lagrangian density takes the stationary form (2.3.7).

The main result presented in Chapter 2 is the following

Theorem 1.2.1. *A nonvanishing time-independent spinor field η is a solution of the field equation for our stationary Lagrangian density (2.3.7) if and only if it is a solution of one of the two stationary massless Dirac equations (1.2.3).*

Theorem 1.2.1 provides an elementary, in terms of Newtonian mechanics and elasticity theory, interpretation of the stationary massless Dirac equation. This interpretation is geometrically much simpler than the traditional one as the mathematical description of our model does not require the use of spinors, Pauli matrices or covariant differentiation.

The only technical assumption contained in the statement of Theorem 1.2.1 and its proof is that the density does not vanish which is equivalent to the spinor field not vanishing. At the moment we do not know how to drop this technical assumption. We can only remark that generically one would not expect a spinor field η "living" on a 3-manifold to vanish as this would mean satisfying four real equations $\operatorname{Re}\eta^1 = \operatorname{Im}\eta^1 = \operatorname{Re}\eta^2 = \operatorname{Im}\eta^2 = 0$ having at our disposal only three real variables x^α, $\alpha = 1, 2, 3$.

The crucial element of the proof of Theorem 1.2.1 is the observation that our Lagrangian density admits factorisation, see formula (2.4.3). Thus, our argument is similar to the original argument of Dirac, the difference being that we factorise the Lagrangian whereas Dirac factorised the field equation (Klein–Gordon equation). In our model factorising the field equation is impossible because the equation is nonlinear.

The results outlined above were published in [14], [13], [15]. The paper [8] contains results closely related to those outlined above.

1.3 Microlocal interpretation of the massless Dirac equation

In this part of the thesis we adopt an abstract spectral theoretic approach and view the stationary massless Dirac equation (1.2.3) as a special case of a spectral problem for a first order elliptic system.

We start with a general spectral problem:

$$Av = \lambda v. \tag{1.3.1}$$

Here A is a first order $m \times m$ elliptic formally self-adjoint classical pseudodifferential operator acting on a column of complex-valued half-densities $v = (v_1 \dots v_m)^T$ over a compact n-dimensional manifold M, λ is a spectral parameter.

We assume the coefficients of the operator A to be smooth. We also assume that the operator A is formally self-adjoint (symmetric): $\int_M w^* A v \, dx = \int_M (Aw)^* v \, dx$, for all smooth $v, w \in M \to \mathbb{C}^m$. Here and further on the star indicates Hermitian conjugation in $\mathbb{C}^m$ and $dx := dx^1 \dots dx^n$, where $x = (x^1, \dots, x^n)$ are local coordinates on M.

Let $A_1(x, \xi)$ be the principal symbol of the operator A. Here $\xi = (\xi_1, \dots, \xi_n)$ is the dual variable to the position variable x; in physics literature the ξ would be referred to as *momentum*. Our principal symbol A_1 is an $m \times m$ Hermitian matrix-function on $T'M := T^*M \setminus \{\xi = 0\}$ (i.e. on the cotangent bundle with the zero section removed).

Let $h^{(j)}(x, \xi)$, $j = 1, \dots, m$, be the eigenvalues of the principal symbol enumerated in increasing order. We assume these eigenvalues to be nonzero and simple. The use of the letter "h" for an eigenvalue of the principal symbol is motivated by the fact that later on it will take on the role of a Hamiltonian.

Let λ_k and $v_k = (v_{k1}(x) \ \dots \ v_{km}(x))^T$ be the eigenvalues and eigenfunctions of the operator A.

We study the spectral function, which is the real density defined as

$$e(\lambda, x, x) := \sum_{0<\lambda_k<\lambda} \|v_k(x)\|^2 \tag{1.3.2}$$

where $\|v_k(x)\|^2 := [v_k(x)]^* v_k(x)$ is the square of the Euclidean norm of the eigenfunction v_k evaluated at the point $x \in M$ and λ is a positive parameter (spectral parameter).

Our first result in this part of the thesis is the two-term asymptotic formula

$$e(\lambda, x, x) = a(x)\,\lambda^n + b(x)\,\lambda^{n-1} + o(\lambda^{n-1}), \tag{1.3.3}$$

where $a(x)$ and $b(x)$ are real densities which we write down explicitly, see formulae (3.1.21) and (3.1.22). We prove the asymptotic formula (1.3.3) under appropriate assumptions on Hamiltonian trajectories generated by the eigenvalues of the principal symbol $h^{(j)}(x, \xi)$, see Theorem 3.8.3.

The massless Dirac operator is the operator appearing in the LHS of formula (1.1.1) but without the dynamic term $\pm i\sigma^0{}_{\dot{a}b}\partial_0$. The operator A we have been studying is far more general than the massless Dirac operator. In order to provide a spectral-theoretic characterisation of the massless Dirac operator we need to make several additional assumptions. We assume that

$$m = 2 \quad \text{and} \quad \operatorname{tr} A_1 = 0, \tag{1.3.4}$$

$$\text{the operator } A \text{ is differential,} \tag{1.3.5}$$

$$n = 3. \tag{1.3.6}$$

We are finally in a position to examine the massless Dirac operator. Now, there is still the technical issue that the massless Dirac operator does not fit into our scheme because this is an operator acting on a 2-component complex-valued spinor (Weyl spinor) rather than a pair of complex-valued half-densities. However, under assumptions (1.3.4)–(1.3.6) our manifold is parallelizable and the

components of a spinor can be identified with half-densities. We call the resulting operator *the massless Dirac operator on half-densities*, see formula (3.A.30).

The massless Dirac operator on half-densities is an operator of the type described in this section (elliptic self-adjoint first order operator acting on a column of complex-valued half-densities) which, moreover, satisfies the additional assumptions (1.3.4), (1.3.5) and (1.3.6). We address the question: is a given operator A a massless Dirac operator?

The main result presented in Chapter 3 is the following

Theorem 1.3.1. *Let A be an elliptic self-adjoint first order pseudodifferential operator acting on columns of m complex-valued half-densities over a compact n-dimensional manifold. Suppose also that this operator satisfies the additional assumptions (1.3.4), (1.3.5) and (1.3.6). Then A is a massless Dirac operator on half-densities if and only if the following two conditions are satisfied at every point of the manifold M: a) the subprincipal symbol of the operator, $A_{\mathrm{sub}}(x)$, is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero.*

This part of the thesis has been published as a preprint [12].

Chapter 2

The stationary massless Dirac equation and Cosserat elasticity

2.1 Our model

In this section we describe in detail our mathematical model. At a basic level it was already sketched out in Section 1.2.

We need to write down the potential energy of a deformed Cosserat continuum. The natural measure of deformations caused by rotations of material points is the torsion tensor defined by the explicit formula

$$T := \delta_{jk} \vartheta^j \otimes d\vartheta^k, \tag{2.1.1}$$

where d denotes the exterior derivative. Here "torsion" means "torsion of the teleparallel connection" with "teleparallel connection" defined by the condition that the covariant derivative of each coframe element ϑ^j is zero; see Appendix A of [9] for a concise exposition.

Our construction of potential energy follows the logic of classical linear elasticity [32], the only difference being that instead of a rank 2 tensor (strain) we deal

with a rank 3 tensor (torsion). The logic of classical linear elasticity dictates that we must first decompose our measure of deformation (torsion) into irreducible pieces, with irreducibility understood in terms of invariance under changes of local coordinates preserving the metric $g_{\alpha\beta}$ at a given point $P \in M$. It is known [25] that torsion has three irreducible pieces labeled by the adjectives *axial*, *vector* and *tensor*. (Vector torsion is sometimes called trace torsion.) The general formula for the potential energy of a homogeneous isotropic linear elastic material contains squares of all irreducible pieces with some constant coefficients in front. Thus, the general formula for potential energy should contain three free parameters (elastic moduli).

We, however, choose to construct our potential energy using only one piece of torsion, namely, the axial piece given by the explicit formula

$$T^{\mathrm{ax}} := \frac{1}{3}\delta_{jk}\vartheta^j \wedge d\vartheta^k. \tag{2.1.2}$$

Comparing (2.1.2) with (3.1.35) we see that axial torsion has a very simple meaning: it is the totally antisymmetric part of the torsion tensor (T is antisymmetric only in the last pair of indices whereas T^{ax} is antisymmetric in all three). In other words, T^{ax} is a 3-form.

We chose the axial piece of torsion because it has two remarkable properties.

- The definition of axial torsion (2.1.2) is very simple in that it does not involve the metric. In a sense, axial torsion (3-form) is an analogue of the electromagnetic field tensor (2-form) from Maxwell's theory.
- Axial torsion possesses the property of conformal covariance, i.e. scales nicely under conformal rescalings of the metric. Indeed, it is easy to see that if we rescale our coframe as

$$\vartheta^j \mapsto e^h \vartheta^j, \tag{2.1.3}$$

where $h : M \to \mathbb{R}$ is an arbitrary scalar function, then our metric scales as

$$g_{\alpha\beta} \mapsto e^{2h} g_{\alpha\beta} \tag{2.1.4}$$

and axial torsion scales as

$$T^{\mathrm{ax}} \mapsto e^{2h} T^{\mathrm{ax}} \tag{2.1.5}$$

without the derivatives of h appearing. The fact that axial torsion is conformally covariant was previously observed by Yu. N. Obukhov [36] and J. M. Nester [34].

We take the potential energy of our continuum to be

$$P(x^0) := \int_M \|T^{\mathrm{ax}}\|^2 \rho \, dx^1 dx^2 dx^3. \tag{2.1.6}$$

It is easy to see that the potential energy (2.1.6) is conformally invariant: it does not change if we rescale our coframe as (2.1.3) and our density as

$$\rho \mapsto e^{2h} \rho. \tag{2.1.7}$$

This follows from formulae (2.1.5), (2.1.4) and

$$\|T^{\mathrm{ax}}\|^2 = \frac{1}{3!} T^{\mathrm{ax}}_{\alpha\beta\gamma} T^{\mathrm{ax}}_{\kappa\lambda\mu} g^{\alpha\kappa} g^{\beta\lambda} g^{\gamma\mu}.$$

We take the kinetic energy of our continuum to be

$$K(x^0) := \int_M \|\dot{\vartheta}\|^2 \rho \, dx^1 dx^2 dx^3, \tag{2.1.8}$$

where $\dot{\vartheta}$ is the 2-form

$$\dot{\vartheta} := \frac{1}{3} \delta_{jk} \vartheta^j \wedge \partial_0 \vartheta^k \tag{2.1.9}$$

(compare with (2.1.2)). The 2-form (2.1.9) can, of course, be written as

$$\dot{\vartheta} = \frac{2}{3} * \omega, \tag{2.1.10}$$

where

$$\omega := \frac{1}{2} * (\delta_{jk}\vartheta^j \wedge \partial_0 \vartheta^k) \tag{2.1.11}$$

is the (pseudo)vector of angular velocity. Hence, (2.1.8) is the standard expression for the kinetic energy of a homogeneous isotropic Cosserat continuum. In writing formula (2.1.8) we assumed homogeneity (properties of the material are the same at all points of the manifold M) and isotropy (properties of the material are invariant under rotations of the local coordinate system). We think of each material point as a uniform ball possessing a moment of inertia and without a preferred axis of rotation.

We now combine the potential energy (2.1.6) and kinetic energy (2.1.8) to form the action (variational functional) of our dynamic problem:

$$S(\vartheta, \rho) := \int_{\mathbb{R}} (P(x^0) - K(x^0))\, dx^0 = \int_{\mathbb{R}\times M} L(\vartheta, \rho)\, dx^0 dx^1 dx^2 dx^3, \tag{2.1.12}$$

where

$$L(\vartheta, \rho) := (\|T^{\mathrm{ax}}\|^2 - \|\dot{\vartheta}\|^2)\rho \tag{2.1.13}$$

is our Lagrangian density. Note that our construction of the action (2.1.12) out of potential and kinetic energies is Newtonian (compare with classical elasticity or even the harmonic oscillator in classical mechanics).

Our field equations (Euler–Lagrange equations) are obtained by varying the action (2.1.12) with respect to the coframe ϑ and density ρ. Varying with respect to the density ρ is easy: this gives the field equation $\|T^{\mathrm{ax}}\|^2 = \|\dot{\vartheta}\|^2$ which is equivalent to $L(\vartheta, \rho) = 0$. Varying with respect to the coframe ϑ is more difficult because we have to maintain the kinematic constraint (2.B.1); recall that the metric is assumed to be prescribed (fixed).

A technique for varying the coframe with kinematic constraint (2.B.1) was described in Appendix B of [9]. We, however, do not write down the field equations for the Lagrangian density (2.1.13) explicitly. We note only that they are highly nonlinear and do not appear to bear any resemblance to the linear massless Dirac equation (1.1.1).

Remark 2.1.1. *The 3-form T^{ax} and 2-form $\dot{\vartheta}$ are invariant under rigid rotations of the coframe, i.e. under special orthogonal transformations (2.B.3) with constant $O^j{}_k$. Hence, our Lagrangian density (2.1.13) is invariant under rigid rotations of the coframe and, accordingly, solutions of our field equations whose coframes differ by a rigid rotation can be collected into equivalence classes. Further on we view coframes differing by a rigid rotation as equivalent.*

2.2 Switching to the language of spinors

As pointed out in the previous section, varying the coframe subject to the kinematic constraint (2.B.1) is not an easy task. This technical difficulty can be overcome by switching to a different dynamical variable. Namely, it is known, see Appendix 2.D, that in dimension 3 a coframe ϑ and a (positive) density ρ are equivalent to a nonvanishing spinor field ξ modulo the sign of ξ. The great advantage of switching to a spinor field ξ is that there are no kinematic constraints on its components, so the derivation of field equations becomes absolutely straightforward.

We now need to substitute formulae (2.D.1), (2.D.3) and (2.D.4) into (2.1.2) and (2.1.9) to get explicit expressions for T^{ax} and $\dot{\vartheta}$ in terms of the spinor field ξ. The results are presented in Appendix 2.E. Namely, formula (2.E.1) gives the spinor representation of the 3-form T^{ax} whereas formulae (2.E.2) and (2.1.10) give the spinor representation of the 2-form $\dot{\vartheta}$. We also know the spinor representation for our density ρ, see formulae (2.D.1) and (2.D.2). Substituting all these into formula (2.1.13) we arrive at the following self-contained explicit spinor

representation of our Lagrangian density

$$L(\xi) = \frac{4}{9\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d}\Big([i(\bar{\xi}^{\dot{a}}\sigma^{\alpha}{}_{\dot{a}b}\nabla_{\alpha}\xi^b - \xi^b\sigma^{\alpha}{}_{\dot{a}b}\nabla_{\alpha}\bar{\xi}^{\dot{a}})]^2 \\ - \|i(\bar{\xi}^{\dot{a}}\sigma_{\alpha\dot{a}b}\partial_0\xi^b - \xi^b\sigma_{\alpha\dot{a}b}\partial_0\bar{\xi}^{\dot{a}})\|^2\Big)\sqrt{\det g}\,. \quad (2.2.1)$$

Here and further on we write our Lagrangian density and our action as $L(\xi)$ and $S(\xi)$ rather than $L(\vartheta,\rho)$ and $S(\vartheta,\rho)$, thus indicating that we have switched to spinors. The nonvanishing spinor field ξ is the new dynamical variable and it will be varied without any constraints.

Straightforward calculations show that the field equation for our Lagrangian density (2.2.1) is

$$-\frac{4i}{3}\big((*T^{\mathrm{ax}})\sigma^{\alpha}{}_{\dot{a}b}\nabla_{\alpha}\xi^b + \sigma^{\alpha}{}_{\dot{a}b}\nabla_{\alpha}((*T^{\mathrm{ax}})\xi^b)\big) \\ -\frac{8i}{9}\big(\omega_{\alpha}\sigma^{\alpha}{}_{\dot{a}b}\partial_0\xi^b + \sigma^{\alpha}{}_{\dot{a}b}\partial_0(\omega_{\alpha}\xi^b)\big) - \rho^{-1}L\sigma_{0\dot{a}b}\xi^b = 0, \quad (2.2.2)$$

where the geometric quantities $*T^{\mathrm{ax}}$, ω, ρ and L are expressed via the spinor field ξ in accordance with formulae (2.E.1), (2.E.2), (2.D.1), (2.D.2) and (2.2.1). The LHS of equation (2.2.2) is the spinor field $F_{\dot{a}}$ appearing in the formula for the variation of the action (2.1.12):

$$\delta S = \int_{\mathbb{R}\times M}(F_{\dot{a}}\delta\bar{\xi}^{\dot{a}} + \bar{F}_a\delta\xi^a)\sqrt{\det g}\; dx^0dx^1dx^2dx^3.$$

We shall refer to equation (2.2.2) as the *dynamic* field equation, with "dynamic" indicating that it contains the time derivative ∂_0.

2.3 Separating out time

Our dynamic field equation (2.2.2) is highly nonlinear and one does expect it to admit separation of variables. Nevertheless, we seek solutions of the form (1.2.1). Substituting formula (1.2.1) into formulae (2.E.1), (2.E.2), (2.D.1), (2.D.2) and

(2.2.1) and using the identity (2.C.5) we get

$$* T^{\mathrm{ax}} = -\frac{2i(\bar\eta^{\dot a}\sigma^{\alpha}{}_{\dot a b}\nabla_\alpha\eta^b - \eta^b\sigma^{\alpha}{}_{\dot a b}\nabla_\alpha\bar\eta^{\dot a})}{3\bar\eta^{\dot c}\sigma_{0\dot c d}\eta^d}\,, \tag{2.3.1}$$

$$\omega_\alpha = \frac{2p_0\bar\eta^{\dot a}\sigma_{\alpha\dot a b}\eta^b}{\bar\eta^{\dot c}\sigma_{0\dot c d}\eta^d}\,, \tag{2.3.2}$$

$$\rho = \bar\eta^{\dot a}\sigma_{0\dot a b}\eta^b\,\sqrt{\det g}\,, \tag{2.3.3}$$

$$L(\eta) = \frac{16}{9\bar\eta^{\dot c}\sigma_{0\dot c d}\eta^d}\Big(\Big[\frac{i}{2}(\bar\eta^{\dot a}\sigma^{\alpha}{}_{\dot a b}\nabla_\alpha\eta^b - \eta^b\sigma^{\alpha}{}_{\dot a b}\nabla_\alpha\bar\eta^{\dot a})\Big]^2 - (p_0\bar\eta^{\dot a}\sigma_{0\dot a b}\eta^b)^2\Big)\sqrt{\det g}\,. \tag{2.3.4}$$

Note that the geometric quantities (2.3.1)–(2.3.4) do not depend on time x^0, which simplifies the next step: substituting (1.2.1) into our dynamic field equation (2.2.2), using the identity (2.C.5) and dividing through by the common factor $e^{-ip_0x^0}$ we get

$$-\frac{4i}{3}\big((*T^{\mathrm{ax}})\sigma^{\alpha}{}_{\dot a b}\nabla_\alpha\eta^b + \sigma^{\alpha}{}_{\dot a b}\nabla_\alpha((*T^{\mathrm{ax}})\eta^b)\big) - \frac{32p_0^2}{9}\sigma_{0\dot a b}\eta^b - \rho^{-1}L\sigma_{0\dot a b}\eta^b = 0\,. \tag{2.3.5}$$

The remarkable feature of formulae (2.3.1)–(2.3.5) is that they do not contain dependence on time x^0. Thus, we have shown that our dynamic field equation (2.2.2) admits separation of variables, i.e. one can seek solutions in the form (1.2.1).

We shall refer to equation (2.3.5) as the *stationary* field equation, with "stationary" indicating that time x^0 has been separated out.

Consider now the action

$$S(\eta) := \int_M L(\eta)\,dx^1dx^2dx^3, \tag{2.3.6}$$

where $L(\eta)$ is our "stationary" Lagrangian density (2.3.4). It is easy to see that our stationary field equation (2.3.5) is the Euler–Lagrange equation for our "stationary" action (2.3.6).

In the remainder of the Chapter 2 we do not use the explicit form of the stationary field equation (2.3.5), dealing only with the stationary Lagrangian density (2.3.4)

and the stationary action (2.3.6). We needed the explicit form of field equations, dynamic and stationary, only to justify separation of variables.

It appears that the underlying group-theoretic reason for our nonlinear dynamic field equation (2.2.2) admitting separation of variables is the fact that our model is U(1)-invariant, i.e. it is invariant under the multiplication of the spinor field ξ by a complex constant of modulus 1. Hence, it is feasible that one could have performed the separation of variables argument without even writing down the explicit form of field equations.

We give for reference a more compact representation of our stationary Lagrangian density (2.3.4) in terms of axial torsion T^{ax} (see formula (2.3.1)) and density ρ (see formula (2.3.3)):

$$L(\eta) = \Big(\|T^{\mathrm{ax}}\|^2 - \frac{16}{9}p_0^2\Big)\rho\,. \tag{2.3.7}$$

Of course, formula (2.3.7) is our original formula (2.1.13) with time separated out. The choice of dynamical variables in the stationary Lagrangian density (2.3.7) is up to the user: one can either use the time-independent spinor field η or, equivalently, the corresponding time-independent coframe and time-independent density (the latter are related to η by formulae (2.D.1)–(2.D.4) with ξ replaced by η). The important thing is that now our dynamical variables are time-independent because we have separated out time.

The fact that we use the same notation L both for the dynamic and stationary Lagrangian densities should not cause problems as in all subsequent sections, apart form Section 2.8, we deal with the stationary case only.

2.4 Factorisation of our Lagrangian

Put

$$L_{\pm}(\eta) := \Big[\frac{i}{2}(\bar\eta^{\dot a}\sigma^{\alpha}{}_{\dot a b}\nabla_\alpha\eta^b - \eta^b\sigma^{\alpha}{}_{\dot a b}\nabla_\alpha\bar\eta^{\dot a}) \pm p_0\bar\eta^{\dot a}\sigma^{0}{}_{\dot a b}\eta^b\Big]\sqrt{\det g}\,. \tag{2.4.1}$$

This is the Lagrangian density for the stationary massless Dirac equation (1.2.3). Formula (2.4.1) can be written in more compact form as

$$L_{\pm}(\eta) = \Big(-\frac{3}{4} {*T^{\mathrm{ax}}} \mp p_0\Big)\rho, \tag{2.4.2}$$

where ${*T^{\mathrm{ax}}}$ is the Hodge dual of axial torsion, see formula (2.3.1), and ρ is the density, see formula (2.3.3). Comparing formulae (2.3.7) and (2.4.2) we get

$$L(\eta) = -\frac{32p_0}{9} \frac{L_+(\eta)\, L_-(\eta)}{L_+(\eta) - L_-(\eta)}\,. \tag{2.4.3}$$

Let us emphasise once again that throughout this chapter we assume that the density ρ does not vanish, which is, of course, equivalent to the spinor field not vanishing. In view of formulae (2.4.2) and (1.2.2) in the stationary case the assumption $\rho \neq 0$ can be equivalently rewritten as

$$L_+(\eta) \neq L_-(\eta) \tag{2.4.4}$$

so the denominator in (2.4.3) is nonzero.

Formula (2.4.3) is the centerpiece of this Chapter: it establishes the connection between Cosserat elasticity and the massless Dirac equation. Moreover, the fact that the RHS of formula (2.4.3) contains a product of two massless Dirac Lagrangian densities shows that we are essentially following Dirac's factorisation construction, the difference being that in the nonlinear setting we cannot factorise equations and have to settle for the next best thing — factorising the Lagrangian.

2.5 Proof of Theorem 1.2.1

Observe that the Lagrangian densities $L_\pm$ defined by formula (2.4.1) possess the property of scaling covariance:

$$L_\pm(e^h\eta) = e^{2h}L_\pm(\eta), \tag{2.5.1}$$

where $h : M \to \mathbb{R}$ is an arbitrary scalar function. In fact, the Lagrangian density of *any* formally selfadjoint (symmetric) linear first order partial differential operator has the scaling covariance property (2.5.1).

We claim that the statement of Theorem 1.2.1 follows from formulae (2.4.3) and (2.5.1). The proof presented below is an abstract one and does not depend on the physical nature of the dynamical variable η, the only requirement being that it is an element of a vector space so that scaling makes sense.

Note that formulae (2.4.3) and (2.5.1) imply that the Lagrangian density L possesses the property of scaling covariance, so all three of our Lagrangian densities, L, L_+ and L_-, have this property. Note also that if η is a solution of the field equation for some Lagrangian density $\mathcal{L}$ possessing the property of scaling covariance then $\mathcal{L}(\eta) = 0$. Indeed, let us perform a scaling variation of our dynamical variable

$$\eta \mapsto \eta + h\eta, \tag{2.5.2}$$

where $h : M \to \mathbb{R}$ is an arbitrary "small" scalar function with compact support. Then $0 = \delta \int \mathcal{L}(\eta) = 2 \int h\mathcal{L}(\eta)$ which holds for arbitrary h only if $\mathcal{L}(\eta) = 0$.

In the remainder of the proof the variations of η are arbitrary and not necessarily of the scaling type (2.5.2).

Suppose that η is a solution of the field equation for the Lagrangian density L_+. [The case when η is a solution of the field equation for the Lagrangian density L_- is handled similarly.] Then $L_+(\eta) = 0$ and, in view of formula (2.4.4), $L_-(\eta) \neq 0$.

Varying η we get

$$\delta \int L(\eta) = -\frac{32p_0}{9}\left(\int \frac{L_-(\eta)}{L_+(\eta)-L_-(\eta)}\,\delta L_+(\eta) + \int L_+(\eta)\,\delta\frac{L_-(\eta)}{L_+(\eta)-L_-(\eta)}\right)$$
$$= \frac{32p_0}{9}\int \delta L_+(\eta) = \frac{32p_0}{9}\,\delta\int L_+(\eta),$$

so,

$$\delta \int L(\eta) = \frac{32p_0}{9}\,\delta\int L_+(\eta)\,. \tag{2.5.3}$$

We assumed that η is a solution of the field equation for the Lagrangian density L_+ so $\delta\int L_+(\eta)=0$ and formula (2.5.3) implies that $\delta\int L(\eta)=0$. As the latter is true for an arbitrary variation of η this means that η is a solution of the field equation for the Lagrangian density L.

Suppose that η is a solution of the field equation for the Lagrangian density L. Then $L(\eta)=0$ and formula (2.4.3) implies that either $L_+(\eta)=0$ or $L_-(\eta)=0$; note that in view of (2.4.4) we cannot have simultaneously $L_+(\eta)=0$ and $L_-(\eta)=0$. Assume for definiteness that $L_+(\eta)=0$. [The case when $L_-(\eta)=0$ is handled similarly.] Varying η and repeating the argument from the previous paragraph we arrive at (2.5.3). We assumed that η is a solution of the field equation for the Lagrangian density L so $\delta\int L(\eta)=0$ and formula (2.5.3) implies that $\delta\int L_+(\eta)=0$. As the latter is true for an arbitrary variation of η this means that η is a solution of the field equation for the Lagrangian density L_+. □

2.6 Nonlinear second order PDEs which reduce to pairs of linear first order PDEs

In this section we give an abstract version of the construction presented in Section 2.5. This abstract version does not involve Cosserat elasticity or spinors and may be of interest to researchers in integrable systems. The material of this section is taken from [10].

Let $A_{\pm}$ be a pair of formally self-adjoint first order linear partial differential operator with smooth coefficients acting on smooth vector functions $u:\ \Omega \to \mathbb{C}^m$, $\Omega \subset \mathbb{R}^n$ is an open subset.

Put

$$L_{\pm}(u) = \mathrm{Re}(u^* A_{\pm} u). \tag{2.6.1}$$

It is easy to see that $L_{\pm}(u)$ is the Lagrangian density for the partial differential equation $A_{\pm}u = 0$. Indeed, this equation is the corresponding field equation (Euler-Lagrange equation) to the action (variational functional) $S_{\pm}(u) = \int\limits_{\Omega} L_{\pm}(u)dx^1...dx^n$.

Consider a new Lagrangian density

$$L(u) = \frac{L_+(u)L_-(u)}{L_+(u) - L_-(u)}. \tag{2.6.2}$$

The corresponding action for (2.6.2) is $S(u) = \int\limits_{\Omega} L(u)dx^1...dx^n$. Clearly, the field equation for the Lagrangian density (2.6.2) is second order and nonlinear.

Lemma 2.6.1. *Let $u:\ \Omega \to \mathbb{C}^m$ be a vector function satisfying the condition*

$$L_+(u) \neq L_-(u). \tag{2.6.3}$$

Then u is a solution of the field equation for the Lagrangian density L if and only if it is a solution of the equation $A_+(u) = 0$ or the equation $A_-(u) = 0$.

The proof of this lemma can be found in [10].

Let us deal with a simple example.

Consider the pair of first order linear ordinary differential equations

$$i\nabla u \pm u = 0. \tag{2.6.4}$$

Here we work on the real line $\mathbb{R}$ parametrised by the coordinate x. The dynamical variable (unknown quantity) is the scalar function $u : \mathbb{R} \to \mathbb{C}\backslash\{0\}$. Differentiation in x is denoted by ∇.

The corresponding Lagrangians are

$$L_{\pm}(u) := \frac{i}{2}(\bar{u}\nabla u - u\nabla\bar{u}) \pm |u|^2. \tag{2.6.5}$$

Equations (2.6.4) are simplified versions of the stationary massless Dirac equations (1.2.3) and Lagrangians (2.6.5) are simplified versions of the stationary massless Dirac Lagrangians (2.4.1). Note that the Lagrangians (2.6.5) possess the property of scaling covariance (2.5.1) where $h : \mathbb{R} \to \mathbb{R}$ is an arbitrary scalar function.

Put

$$L(u) := \frac{2L_+(u)L_-(u)}{L_+(u) - L_-(u)} = \left[\frac{i(\bar{u}\nabla u - u\nabla\bar{u})}{2|u|}\right]^2 - |u|^2. \tag{2.6.6}$$

The corresponding field equation (Euler–Lagrange equation) is

$$i\left\{\frac{(\nabla u)}{|u|} - \frac{u(\bar{u}\nabla u - u\nabla\bar{u})}{2|u|^3} + \nabla\frac{u}{|u|}\right\}\left[\frac{i(\bar{u}\nabla u - u\nabla\bar{u})}{2|u|}\right] - u = 0, \tag{2.6.7}$$

where the last ∇ in the curly brackets acts on all the terms to the right, including those in the square brackets. Equation (2.6.7) is a second order nonlinear ordinary differential equation which does not appear to bear any resemblance to the first order linear ordinary differential equations (2.6.4).

Let us switch to the polar representation of the complex function u:

$$u = re^{-i\varphi}, \tag{2.6.8}$$

where $r : \mathbb{R} \to (0, +\infty)$ and $\varphi : \mathbb{R} \to \mathbb{R}$ are the new dynamical variables (unknown quantities). Substituting formula (2.6.8) into equation (2.6.7) and multiplying by $e^{i\varphi}$ we arrive at the polar representation of our field equation:

$$2i(\nabla r)(\nabla\varphi) + r(\nabla\varphi)^2 + ir\nabla\nabla\varphi - r = 0.$$

Separating the real and imaginary parts we rewrite the latter as a system or real equations

$$r(\nabla\varphi)^2 - r = 0, \qquad 2(\nabla r)(\nabla\varphi) + r\nabla\nabla\varphi = 0,$$

which, in turn, is equivalent to

$$\nabla\varphi = \mp 1, \qquad \nabla r = 0. \tag{2.6.9}$$

This shows that a complex function u is a solution of equation (2.6.7) if and only if it is a solution of one of the two equations (2.6.4).

Of course, the explicit calculations carried out above were unnecessary because the toy model considered in this section is covered by Lemma 2.6.1. The point of these explicit calculations was to illustrate the degeneracy of field equations for Lagrangians of the form (2.6.2): looking at (2.6.9) one sees the absence of second derivatives.

2.7 Plane wave solutions

Suppose that $M = \mathbb{R}^3$ is Euclidean 3-space equipped with Cartesian coordinates $x = (x^1, x^2, x^3)$ and standard Euclidean metric (2.C.9). In this section we construct a special class of explicit solutions of the field equations for our Lagrangian density (2.1.13). This construction is presented in the language of spinors.

Let us choose Pauli matrices (2.C.10) and seek solutions of the form

$$\xi(x^0, x^1, x^2, x^3) = e^{-i(p_0 x^0 + p\cdot x)}\zeta, \tag{2.7.1}$$

where p_0 is a real number as in formulae (1.2.1) and (1.2.2), $p = (p_1, p_2, p_3)$ is a real constant covector and $\zeta \neq 0$ is a constant spinor. We shall call solutions of the type (2.7.1) *plane wave*. In seeking plane wave solutions what we are doing is separating out all the variables, namely, the time variable x^0 and the spatial variables $x = (x^1, x^2, x^3)$.

Our dynamic field equation (2.2.2) is highly nonlinear so it is not *a priori* clear that one can seek solutions in the form of plane waves. However, plane wave solutions (2.7.1) are a special case of stationary solutions (1.2.1) and these have already been analyzed in preceding sections. Namely, Theorem 1.2.1 gives us an algorithm for the calculation of all plane wave solutions (2.7.1) by reducing the problem to a pair of stationary massless Dirac equations (1.2.3) for the time-independent spinor field

$$\eta(x^1, x^2, x^3) = e^{-ip\cdot x}\zeta. \tag{2.7.2}$$

Substituting formulae (2.C.2), (2.C.10) and (2.7.2) into equation (1.2.3) we get

$$\begin{pmatrix} \mp p_0 + p_3 & p_1 - ip_2 \\ p_1 + ip_2 & \mp p_0 - p_3 \end{pmatrix} \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix} = 0. \tag{2.7.3}$$

The determinant of the matrix in the LHS of equation (2.7.3) is $p_0^2 - p_1^2 - p_2^2 - p_3^2$ so this system has a nontrivial solution ζ if and only if $p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0$. Our model is invariant under rotations of the Cartesian coordinate system (orthogonal transformations of the coordinate system preserving orientation) so without loss of generality we can assume that

$$p_1 = p_2 = 0, \qquad p_3 = \pm p_0, \tag{2.7.4}$$

where the $\pm$ sign is chosen to agree with that in equation (2.7.3), i.e. upper sign in (2.7.4) corresponds to upper sign in (2.7.3) and same for lower signs. Substituting formula (2.7.4) into equation (2.7.3) and recalling our assumption (1.2.2) we conclude that, up to scaling by a nonzero complex factor, we have

$$\zeta^d = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.7.5}$$

Combining formulae (2.7.1), (2.7.4) and (2.7.5) we conclude that for each real $p_0 \neq 0$ our model admits, up to a rotation of the coordinate system and complex scaling, two plane wave solutions and that these plane wave solutions are given

by the explicit formula

$$\xi^d = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ip_0(x^0 \pm x^3)}. \tag{2.7.6}$$

Let us now rewrite the plane wave solutions (2.7.6) in terms of our original dynamical variables, coframe ϑ and density ρ. Substituting formulae (2.C.2), (2.C.10) and (2.7.6) into formulae (2.D.1)–(2.D.4) we get $\rho = 1$ and

$$\vartheta^1{}_\alpha = \begin{pmatrix} \cos 2p_0(x^0 \pm x^3) \\ \sin 2p_0(x^0 \pm x^3) \\ 0 \end{pmatrix}, \vartheta^2{}_\alpha = \begin{pmatrix} -\sin 2p_0(x^0 \pm x^3) \\ \cos 2p_0(x^0 \pm x^3) \\ 0 \end{pmatrix}, \vartheta^3{}_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.7.7}$$

Note that scaling the spinor ζ by a nonzero complex factor is equivalent to scaling the density ρ by a positive real factor and time shift $x^0 \mapsto x^0 + \text{const}$.

We will now establish how many different (ones that cannot be continuously transformed into one another) plane wave solutions we have. To this end, we rewrite formula (2.7.7) in the form

$$\vartheta^1{}_\alpha = \begin{pmatrix} \cos 2|p_0|(x^0 + bx^3) \\ a \sin 2|p_0|(x^0 + bx^3) \\ 0 \end{pmatrix}, \vartheta^2{}_\alpha = \begin{pmatrix} -a \sin 2|p_0|(x^0 + bx^3) \\ \cos 2|p_0|(x^0 + bx^3) \\ 0 \end{pmatrix}, \vartheta^3{}_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{2.7.8}$$

where a and b can, independently, take values ± 1. It may seem that we have a total of 4 different plane wave solutions. Recall, however, that we can perform rigid rotations of the coframe and that we have agreed (see Remark 2.1.1 at the end of Section 2.1) to view coframes that differ by a rigid rotation as equivalent. Let us perform a rotation of the coordinate system

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \mapsto \begin{pmatrix} x^2 \\ x^1 \\ -x^3 \end{pmatrix}$$

simultaneously with a rigid rotation of the coframe

$$\begin{pmatrix} \vartheta^1 \\ \vartheta^2 \\ \vartheta^3 \end{pmatrix} \mapsto \begin{pmatrix} \vartheta^2 \\ \vartheta^1 \\ -\vartheta^3 \end{pmatrix}.$$

It is easy to see that the above transformations turn a solution of the form (2.7.8) into a solution of this form again only with

$$a \mapsto -a, \qquad b \mapsto -b.$$

Thus, the numbers a and b on their own do not characterize different plane wave solutions. Different plane wave solutions are characterized by the number $c := ab$ which can take two values, $+1$ and -1.

We have established that for a given positive frequency $|p_0|$ we have two essentially different types of plane wave solutions. These can be written, for example, as

$$\vartheta^1{}_\alpha = \begin{pmatrix} \cos 2|p_0|(x^0+x^3) \\ \pm \sin 2|p_0|(x^0+x^3) \\ 0 \end{pmatrix}, \vartheta^2{}_\alpha = \begin{pmatrix} \mp \sin 2|p_0|(x^0+x^3) \\ \cos 2|p_0|(x^0+x^3) \\ 0 \end{pmatrix}, \vartheta^3{}_\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (2.7.9)$$

The plane wave solutions (2.7.9) describe traveling waves of rotations. Both waves travel with the same velocity (speed of light) in the negative x^3-direction. The difference between the two solutions is in the direction of rotation of the coframe: if we fix the spatial coordinate x^3 and look at the evolution of (2.7.9) as a function of time x^0 or if we fix time x^0 and look at the evolution of (2.7.9) as a function of the spatial coordinate x^3 then one solution describes a clockwise rotation whereas the other solution describes an anticlockwise rotation. We identify one of the solutions (2.7.9) with a left-handed massless neutrino and the other with a right-handed massless antineutrino.

The bottom line is that our model gives the correct number, two, of distinct plane wave solutions.

2.8 Relativistic representation of our Lagrangian

In this section we work on the 4-manifold $\mathbb{R}\times M$ equipped with Lorentzian metric (2.A.1). This manifold is an extension of the original 3-manifold M. We use **bold** type for extended quantities.

We extend our coframe as

$$\boldsymbol{\vartheta}^{0}{}_{\boldsymbol{\alpha}} = \begin{pmatrix} 1 \\ 0_\alpha \end{pmatrix}, \tag{2.8.1}$$

$$\boldsymbol{\vartheta}^{j}{}_{\boldsymbol{\alpha}} = \begin{pmatrix} 0 \\ \vartheta^{j}{}_{\alpha} \end{pmatrix}, \qquad j = 1, 2, 3, \tag{2.8.2}$$

where the bold tensor index $\boldsymbol{\alpha}$ runs through the values 0, 1, 2, 3, whereas its non-bold counterpart α runs through the values 1, 2, 3. In particular, the 0_α in formula (2.8.1) stands for a column of three zeros.

Throughout this section our original 3-dimensional coframe ϑ is allowed to depend on time x^0 in an arbitrary (not necessarily harmonic) manner, as long as the kinematic constraint (2.B.1) is maintained. Thus, our only restriction on the choice of extended 4-dimensional coframe $\boldsymbol{\vartheta}$ is formula (2.8.1) which says that the zeroth element of the coframe is prescribed as the conormal to the original Riemannian 3-manifold M.

The extended metric (2.A.1) is expressed via the extended coframe (2.8.1) and (2.8.2) as

$$\mathbf{g} = \mathbf{o}_{\mathbf{jk}}\boldsymbol{\vartheta}^{\mathbf{j}} \otimes \boldsymbol{\vartheta}^{\mathbf{k}}, \tag{2.8.3}$$

where $\mathbf{o}_{\mathbf{jk}} = \mathbf{o}^{\mathbf{jk}} := \mathrm{diag}(-1, +1, +1, +1)$ (compare with formula (2.B.1)). The extended axial torsion is

$$\mathbf{T}^{\mathrm{ax}} := \frac{1}{3}\mathbf{o}_{\mathbf{jk}}\boldsymbol{\vartheta}^{\mathbf{j}} \wedge d\boldsymbol{\vartheta}^{\mathbf{k}} = \frac{1}{3}(-\underbrace{\boldsymbol{\vartheta}^{0} \wedge d\boldsymbol{\vartheta}^{0}}_{=0} + \boldsymbol{\vartheta}^{1} \wedge d\boldsymbol{\vartheta}^{1} + \boldsymbol{\vartheta}^{2} \wedge d\boldsymbol{\vartheta}^{2} + \boldsymbol{\vartheta}^{3} \wedge d\boldsymbol{\vartheta}^{3}), \tag{2.8.4}$$

where d denotes the exterior derivative on $\mathbb{R}\times M$ (compare with formula (2.1.2)). Formula (2.8.4) can be rewritten as

$$\mathbf{T}^{\mathrm{ax}} = T^{\mathrm{ax}} - \boldsymbol{\vartheta}^0 \wedge \dot{\vartheta} \tag{2.8.5}$$

with T^{ax} and $\dot{\vartheta}$ defined by formulae (2.1.2) and (2.1.9) respectively. Squaring (2.8.5) we get $\|\mathbf{T}^{\mathrm{ax}}\|^2 = \|T^{\mathrm{ax}}\|^2 - \|\dot{\vartheta}\|^2$ which implies that our Lagrangian density (2.1.13) can be rewritten as

$$L(\vartheta, \rho) = \|\mathbf{T}^{\mathrm{ax}}\|^2 \rho\,. \tag{2.8.6}$$

The point of the arguments presented in this section was to show that if one adopts the relativistic point of view then our Lagrangian density (2.1.13) takes the especially simple form (2.8.6). Formula (2.8.6) is also useful in that it allows us to see that our Lagrangian density is invariant under conformal rescalings of the 4-dimensional Lorentzian metric $\mathbf{g}$: the arguments from Section 2.1 (see formulae (2.1.3)–(2.1.5) and (2.1.7)) carry over to the 4-dimensional setting without change.

A consistent pursuit of the relativistic approach would require the variation of all four elements of the extended coframe, giving three extra dynamical degrees of freedom (Lorentz boosts in three directions). We do not do this in the thesis, assuming instead that the zeroth element of the extended coframe is specified by formula (2.8.1).

2.9 Discussion

The mathematical model presented in Section 2.1 is, effectively, a special case of the theory of teleparallelism [11, 51, 46]. Modern reviews of teleparallelism can be found in [25, 24, 23, 33, 5, 37]. The differences between our mathematical model and those commonly used in teleparallelism are as follows.

- We assume the metric to be prescribed (fixed) whereas in teleparallelism it is traditional to view the metric as a dynamical variable. In other words, in teleparallelism it is customary to view (2.B.1) not as a kinematic constraint but as a definition of the metric and, consequently, to vary the coframe without any constraints. This is not surprising as most, if not all, authors who contributed to teleparallelism came to the subject from General Relativity.

- We take the density of our continuum ρ to be a dynamical variable whereas in teleparallelism the tradition is to prescribe it as $\rho = \sqrt{\det g}$. Taking ρ to be a dynamical variable is, of course, equivalent to introducing an extra real positive scalar field $\rho/\sqrt{\det g}$ into our model

- We choose a very particular Lagrangian density (2.8.6) containing only one irreducible piece of torsion (axial) whereas in teleparallelism it is traditional to choose a more general Lagrangian containing all three pieces (axial, vector and tensor): see formula (26) in [25]. In choosing our particular Lagrangian density (2.8.6) we were guided by the principles of conformal invariance, simplicity and analogy with Maxwell's theory.

The main result of the Chapter 2 is Theorem 1.2.1 which establishes that in the stationary setting (prescribed harmonic oscillation in time) our mathematical model is equivalent to a pair of massless massless Dirac equations (1.1.1). The advantage of our approach is that it makes the massless Dirac equation look natural to someone with a continuum mechanics background. The downside is that our mathematical model is nonlinear which makes it look unnatural to someone with a quantum mechanical background.

The situation here has a certain similarity with integrable systems. Say, the Korteweg-de Vries equation (mathematical model of waves on shallow water surfaces) is nonlinear but the inverse scattering transform reduces it to the analysis of a spectral problem for a linear Sturm–Liouville operator. In the thesis we go

the other way round, reformulating the spectral problem for the linear massless Dirac operator as a nonlinear equation from continuum mechanics.

From a purely mathematical viewpoint Theorem 1.2.1 is unusual in that it states that a (particular) second order partial differential equation is equivalent to a pair of first order partial differential equations, which is actually hard to believe. Indeed, let us choose a 2-dimensional hypersurface S on the 3-manifold M and set a Cauchy problem on this surface. When dealing with a second order partial differential equation one expects to be able to prescribe the value of the spinor field η on the surface S as well as its normal derivative, whereas when dealing with a first order partial differential equation one expects to be able to prescribe the value of the spinor field η only (the value of the normal derivative of η on the surface S will be determined by the equation). This argument appears to show that there is no way a second order partial differential equation can be reduced to a pair of first order equations. However, our second order partial differential equation happens to be degenerate and does not admit the setting of a standard Cauchy problem. This degeneracy manifests itself in the property of scaling covariance of our stationary Lagrangian density (2.3.7), see Section 2.5 for details. Scaling covariance implies that our stationary Lagrangian density (2.3.7) vanishes on solutions of the (second order) field equation which means that the value of the spinor field η on the surface S and its normal derivative cannot be chosen independently. In order to allay fears that there is something inherently wrong with our construction we provide in Section 2.6 an elementary example showing by means of an explicit calculation that a second order differential equation with Lagrangian of the form (2.4.3) and (2.5.1) does indeed reduce to a pair of first order equations.

Our construction exhibits a certain similarity with the Riccati equation. Recall that the Riccati equation is a nonlinear first order differential equation which reduces to a linear second order differential equation. We go the other way round, reducing a nonlinear second order equation to a pair of linear first order

equations. However, unlike the Riccati equation, our construction works not only for ordinary differential equations but also for partial differential equations.

Theorem 1.2.1 leaves us with two issues unresolved.

A What can be said about the general case, when the spinor field ξ is an arbitrary function of all spacetime coordinates (x^0, x^1, x^2, x^3) and is not necessarily of the form (1.2.1)?

B What can be said about the relativistic version of our model described in Section 2.8?

The two issues are, of course, related: both arise because in formulating our basic model in Section 2.1 we adopted the Newtonian approach which specifies the time coordinate x^0 ("absolute time").

We plan to tackle issue A by means of perturbation theory. Namely, assuming the metric to be flat (as in Section 2.7), we start with a plane wave (2.7.1) and then seek the unknown spinor field ξ in the form

$$\xi(x^0, x^1, x^2, x^3) = e^{-i(p_0 x^0 + p\cdot x)}\zeta(x^0, x^1, x^2, x^3), \tag{2.9.1}$$

where ζ is a slowly varying spinor field. Here "slowly varying" means that second derivatives of ζ can be neglected compared to the first. Our conjecture is that the application of a formal perturbation argument will yield the massless Dirac equation (1.1.1) for the spinor field ξ.

We plan to tackle issue B by means of perturbation theory as well. The relativistic version of our model has three extra field equations corresponding to the three extra dynamical degrees of freedom (Lorentz boosts in three directions). Our conjecture is that if we take a solution of the nonrelativistic problem which is a perturbation of a plane wave (as in the previous paragraph) then, at a perturbative level, this solution will automatically satisfy the three extra field equations.

In other words, we conjecture that our nonrelativistic model possesses relativistic invariance at the perturbative level.

The detailed analysis of the two issues flagged up above could be the subject of additional research.

Appendices

2.A General notation

Our general notation mostly follows [9, 53], the only major difference being that we changed the signature of Lorentzian metric $\mathbf{g}_{\boldsymbol{\alpha\beta}}$ from $+---$ to $-+++$. The latter is more natural when promoting the Newtonian continuum mechanics approach.

Throughout the Appendix for Chapter 2 we work on a 3-manifold M equipped with local coordinates x^α, $\alpha = 1, 2, 3$, and prescribed positive metric $g_{\alpha\beta}$ which does not depend on time. We extend the Riemannian 3-manifold M to a Lorentzian 4-manifold $\mathbb{R} \times M$ by adding the time coordinate $x^0 \in \mathbb{R}$. The metric on $\mathbb{R} \times M$ is defined as

$$\mathbf{g}_{\boldsymbol{\alpha\beta}} = \begin{pmatrix} -1 & 0 \\ 0 & g_{\alpha\beta} \end{pmatrix}. \tag{2.A.1}$$

Here and further on we use **bold** type for extended quantities. Say, the use of bold type in tensor indices $\boldsymbol{\alpha}, \boldsymbol{\beta}$ appearing in the LHS of formula (2.A.1) indicates that these run through the values $0, 1, 2, 3$, whereas the use of normal type in tensor indices α, β appearing in the RHS of formula (2.A.1) indicates that these run through the values $1, 2, 3$.

All constructions presented in the Appendix for Chapter 2 are local so we do not make *a priori* assumptions on the geometric structure of $\{M, g\}$.

We use Greek letters for tensor (holonomic) indices and Latin letters for frame (anholonomic) indices.

We identify differential forms with covariant antisymmetric tensors. Given a pair of real covariant antisymmetric tensors P and Q of rank r we define their dot product as $P \cdot Q := \frac{1}{r!} P_{\alpha_1 \dots \alpha_r} Q_{\beta_1 \dots \beta_r} g^{\alpha_1 \beta_1} \dots g^{\alpha_r \beta_r}$. We also define $\|P\|^2 := P \cdot P$.

All our constructions are local and occur in a neighborhood of a given point P of the 3-manifold M. We allow only changes of local coordinates x^α, $\alpha = 1, 2, 3$, which preserve orientation.

Working in local coordinates with specified orientation allows us to define the Hodge star: we define the action of $*$ on a rank r antisymmetric tensor R as

$$(*R)_{\alpha_{r+1} \dots \alpha_3} := (r!)^{-1} \sqrt{\det g}\, R^{\alpha_1 \dots \alpha_r} \varepsilon_{\alpha_1 \dots \alpha_3}, \tag{2.A.2}$$

where ε is the totally antisymmetric quantity, $\varepsilon_{123} := +1$.

2.B Coframe notation

We view our 3-manifold M as an elastic continuum whose material points can experience no displacements, only rotations, with rotations of different material points being totally independent. The idea of rotating material points may seem exotic, however it has long been accepted in continuum mechanics within the Cosserat theory of elasticity [16]. This idea also lies at the heart of the theory of *teleparallelism* (= absolute parallelism = fernparallelismus), a subject promoted by A. Einstein and É. Cartan [11, 51, 46]. See Section 2.9 for more details.

Rotations of material points of the 3-dimensional elastic continuum are described mathematically by attaching to each geometric point of the manifold M an orthonormal basis, which gives a field of orthonormal bases called the *frame* or *coframe*, depending on whether one prefers dealing with vectors or covectors.

Our mathematical model will be built on the basis of exterior calculus (no need for covariant derivatives) so for us it will be more natural to use the coframe.

The coframe ϑ is a triple of orthonormal covector fields ϑ^j, $j = 1, 2, 3$, on the 3-manifold M. Each covector field ϑ^j can be written more explicitly as ${\vartheta^j}_\alpha$ where the tensor index $\alpha = 1, 2, 3$ enumerates the components. The orthonormality condition for the coframe can be represented as a single tensor identity

$$g = \delta_{jk} \vartheta^j \otimes \vartheta^k, \tag{2.B.1}$$

where δ_{jk} is the Kronecker delta. For the sake of clarity we repeat formula (2.B.1) giving tensor indices explicitly and performing summation over frame indices explicitly:

$$g_{\alpha\beta} = \delta_{jk} {\vartheta^j}_\alpha {\vartheta^k}_\beta = {\vartheta^1}_\alpha {\vartheta^1}_\beta + {\vartheta^2}_\alpha {\vartheta^2}_\beta + {\vartheta^3}_\alpha {\vartheta^3}_\beta,$$

where α and β run through the values $1, 2, 3$. We view the identity (2.B.1) as a kinematic constraint: the metric g is given (prescribed) and the coframe elements ϑ^j are chosen so that they satisfy (2.B.1), which leaves us with three real degrees of freedom at every point of M.

Coframes ϑ fall into two separate categories, depending on the sign of $\det {\vartheta^j}_\alpha$. We choose to work with coframes satisfying the condition

$$\det {\vartheta^j}_\alpha > 0. \tag{2.B.2}$$

Condition (2.B.2) means that orientation encoded in our coframe agrees with that encoded in our coordinate system.

An orthogonal transformation of a coframe is a linear map

$$\vartheta^j \mapsto \tilde{\vartheta}^j = {O^j}_k \vartheta^k, \tag{2.B.3}$$

where the ${O^j}_k$ are real scalar functions satisfying the condition $\delta_{ji} {O^j}_k {O^i}_r = \delta_{kr}$. Of course, orthogonal transformations map coframes into coframes, i.e. they

preserve the kinematic constraint (2.B.1). We call an orthogonal transformation special (or a rotation) if the ${O^j}_k$ satisfy the additional condition $\det {O^j}_k = +1$. Any two coframes satisfying condition (2.B.2) are related by a special orthogonal transformation (rotation).

2.C Spinor notation

Our spinor notation mostly follows [38], the difference being that we changed the signature of Lorentzian metric.

We use two-component complex-valued spinors (Weyl spinors) whose indices run through the values $1, 2$ or $\dot{1}, \dot{2}$. Complex conjugation makes the undotted indices dotted and vice versa.

Define the "metric spinor"

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \epsilon^{ab} = \epsilon^{\dot{a}\dot{b}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.C.1}$$

with the first index enumerating rows and the second enumerating columns. We will be using the spinor (3.A.6) for lowering and raising spinor indices.

We define

$$\sigma_{0\dot{a}b} = {\sigma_0}^{\dot{a}b} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ {\sigma^0}_{\dot{a}b} = \sigma^{0\dot{a}b} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.C.2}$$

The spinor (2.C.2) can also be used for raising and lowering spinor indices. This is a feature of the nonrelativistic setting, when we have a specified time coordinate $t = x^0$ and transformations of spatial local coordinates x^α, $\alpha = 1, 2, 3$, do not involve time.

Let $\mathfrak{v}$ be the real vector space of trace-free Hermitian 2×2 matrices $\sigma_{\dot{a}b}$. Pauli matrices $\sigma_{\alpha\dot{a}b}$, $\alpha = 1, 2, 3$, are a basis in $\mathfrak{v}$ satisfying

$$\sigma_{\alpha\dot{a}b}\sigma_{\beta}{}^{\dot{a}c} + \sigma_{\beta\dot{a}b}\sigma_{\alpha}{}^{\dot{a}c} = -2g_{\alpha\beta}\delta_{b}{}^{c}, \tag{2.C.3}$$

where $\sigma_{\beta}{}^{\dot{a}c} := \epsilon^{\dot{a}\dot{e}}\sigma_{\beta\dot{e}d}\epsilon^{cd}$. Note that formula (2.C.3) automatically implies an analogous formula for the extended metric (2.A.1):

$$\sigma_{\boldsymbol{\alpha}\dot{a}b}\sigma_{\boldsymbol{\beta}}{}^{\dot{a}c} + \sigma_{\boldsymbol{\beta}\dot{a}b}\sigma_{\boldsymbol{\alpha}}{}^{\dot{a}c} = -2\mathbf{g}_{\boldsymbol{\alpha}\boldsymbol{\beta}}\delta_{b}{}^{c}, \tag{2.C.4}$$

where the bold tensor indices $\boldsymbol{\alpha}, \boldsymbol{\beta}$ run through the values $0, 1, 2, 3$.

Of course, our Pauli matrices σ_{α}, $\alpha = 1, 2, 3$, are not uniquely defined: if $\sigma_{\alpha} = \sigma_{\alpha\dot{a}b}$ are Pauli matrices then so are the matrices $U^{*}\sigma_{\alpha}U$ where U is an arbitrary special ($\det U = 1$) unitary matrix-function. Note also that under coordinate transformations our Pauli matrices $\sigma_{\alpha a\dot{b}}$ transform as components of a covector: this is indicated by the Greek subscript α.

Let us mention a useful identity for Pauli matrices, very similar to (2.C.4) but with contraction over tensor indices instead of spinor ones:

$$\sigma_{\boldsymbol{\alpha}\dot{a}b}\sigma^{\boldsymbol{\alpha}}{}_{\dot{c}d} = -2\epsilon_{\dot{a}\dot{c}}\epsilon_{bd}\,. \tag{2.C.5}$$

We define the covariant derivatives of spinor fields as

$$\nabla_{\mu}\xi^{a} = \partial_{\mu}\xi^{a} + \Gamma^{a}{}_{\mu b}\xi^{b}, \qquad \nabla_{\mu}\xi_{a} = \partial_{\mu}\xi_{a} - \Gamma^{b}{}_{\mu a}\xi_{b},$$

$$\nabla_{\mu}\eta^{\dot{a}} = \partial_{\mu}\eta^{\dot{a}} + \bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}}\eta^{\dot{b}}, \qquad \nabla_{\mu}\eta_{\dot{a}} = \partial_{\mu}\eta_{\dot{a}} - \bar{\Gamma}^{\dot{b}}{}_{\mu\dot{a}}\eta_{\dot{b}},$$

where $\bar{\Gamma}^{\dot{a}}{}_{\mu\dot{b}} = \overline{\Gamma^{a}{}_{\mu b}}$ and μ runs through the values $1, 2, 3$. The explicit formula for the spinor connection coefficients $\Gamma^{a}{}_{\mu b}$ can be derived from the following two conditions:

$$\nabla_{\mu}\epsilon_{ab} = 0, \tag{2.C.6}$$

$$\nabla_{\mu}\sigma^{\alpha}{}_{\dot{a}b} = 0, \tag{2.C.7}$$

where

$$\nabla_\mu \sigma^\alpha{}_{\dot{a}b} = \partial_\mu \sigma^\alpha{}_{\dot{a}b} + \Gamma^\alpha{}_{\mu\beta}\sigma^\beta{}_{\dot{a}b} - \bar{\Gamma}^{\dot{c}}{}_{\mu\dot{a}}\sigma^\alpha{}_{\dot{c}b} - \Gamma^d{}_{\mu b}\sigma^\alpha{}_{\dot{a}d}$$

and $\Gamma^\beta{}_{\alpha\gamma} = \left\{ {\beta \atop \alpha\gamma} \right\} := \frac{1}{2} g^{\beta\delta}(\partial_\alpha g_{\gamma\delta} + \partial_\gamma g_{\alpha\delta} - \partial_\delta g_{\alpha\gamma})$ are the Christoffel symbols. Conditions (2.C.6), (2.C.7) give an overdetermined system of linear algebraic equations for $\operatorname{Re}\Gamma^a{}_{\mu b}$, $\operatorname{Im}\Gamma^a{}_{\mu b}$ the unique solution of which is

$$\Gamma^a{}_{\mu b} = -\frac{1}{4}\sigma_\alpha{}^{\dot{c}a}\left(\partial_\mu \sigma^\alpha{}_{\dot{c}b} + \Gamma^\alpha{}_{\mu\beta}\sigma^\beta{}_{\dot{c}b}\right). \tag{2.C.8}$$

Observe that the sign in the RHS of formula (2.C.8) is different from that of formula (A.9) in [38]. This is because we changed the signature of Lorentzian metric.

Note that for the standard Euclidean metric

$$g_{\alpha\beta} = \operatorname{diag}(1,1,1) \tag{2.C.9}$$

the traditional choice of Pauli matrices is

$$\sigma_{1\dot{a}b} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_{2\dot{a}b} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_{3\dot{a}b} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.C.10}$$

2.D Correspondence between coframes and spinors

In dimension 3 a coframe ϑ and a (positive) density ρ are equivalent to a nonvanishing spinor field ξ modulo the sign of ξ in accordance with the formulae

$$s = \bar{\xi}^{\dot{a}}\sigma_{0\dot{a}b}\xi^b, \tag{2.D.1}$$

$$\rho = s\sqrt{\det g}\,, \tag{2.D.2}$$

$$(\vartheta^1 + i\vartheta^2)_\alpha = s^{-1}\epsilon^{\dot{c}\dot{b}}\sigma_{0\dot{b}a}\xi^a\sigma_{\alpha\dot{c}d}\xi^d, \tag{2.D.3}$$

$$\vartheta^3{}_\alpha = s^{-1}\bar{\xi}^{\dot{a}}\sigma_{\alpha\dot{a}b}\xi^b. \tag{2.D.4}$$

The above formulae are a special case of those from [22].

We assume that our Pauli matrices are chosen in such a way that the coframe ϑ defined by formulae (2.D.1), (2.D.3) and (2.D.4) satisfies condition (2.B.2) for all $\xi \neq 0$. Of course, the sign of $\det \vartheta^j{}_\alpha$ can always be changed by switching from original Pauli matrices to their complex conjugates.

Note that if we have the standard Euclidean metric (2.C.9), use traditional Pauli matrices (2.C.10) and take

$$\xi^a = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{2.D.5}$$

then formulae (2.D.1), (2.D.3) and (2.D.4) give us

$$\vartheta^j{}_\alpha = \delta^j{}_\alpha \,. \tag{2.D.6}$$

2.E Spinor representation of axial torsion and angular velocity

We show in this section that the Hodge dual of axial torsion (2.1.2) is expressed via the spinor field ξ as

$$*\, T^{\mathrm{ax}} = -\frac{2i(\bar{\xi}^{\dot{a}}\sigma^{\alpha}{}_{\dot{a}b}\nabla_\alpha \xi^b - \xi^b \sigma^{\alpha}{}_{\dot{a}b}\nabla_\alpha \bar{\xi}^{\dot{a}})}{3\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d} \tag{2.E.1}$$

and that the vector of angular velocity ω defined by formula (2.1.11) is expressed via the spinor field ξ as

$$\omega_\alpha = \frac{i(\bar{\xi}^{\dot{a}}\sigma_{\alpha\dot{a}b}\partial_0 \xi^b - \xi^b \sigma_{\alpha\dot{a}b}\partial_0 \bar{\xi}^{\dot{a}})}{\bar{\xi}^{\dot{c}}\sigma_{0\dot{c}d}\xi^d} \,. \tag{2.E.2}$$

Note that formulae (2.E.1) and (2.E.2) are invariant under the rescaling of our spinor field by an arbitrary nonvanishing real scalar function.

Formulae (2.E.1) and (2.E.2) are proved by direct substitution of formulae (2.D.1), (2.D.3) and (2.D.4) into (2.1.2) and (2.1.11) respectively. In order to simplify calculations we observe that the expressions in the left- and right-hand sides of formulae (2.E.1) and (2.E.2) have an invariant nature, hence it is sufficient to prove these formulae for standard Euclidean metric (2.C.9), traditional Pauli matrices (2.C.10) and at a point at which the spinor field takes the value (2.D.5).

We have

$$\xi^a = \begin{pmatrix} 1+\delta\xi^1 \\ \delta\xi^2 \end{pmatrix}, \quad (\vartheta^1 + i\vartheta^2)_\alpha = \begin{pmatrix} 1+\delta\xi^1 - \delta\bar{\xi}^{\dot{1}} \\ i + i\delta\xi^1 - i\delta\bar{\xi}^{\dot{1}} \\ -2\delta\xi^2 \end{pmatrix}, \quad \vartheta^3{}_\alpha = \begin{pmatrix} \delta\xi^2 + \delta\bar{\xi}^{\dot{2}} \\ -i\delta\xi^2 + i\delta\bar{\xi}^{\dot{2}} \\ 1 \end{pmatrix},$$

$$[\mathrm{curl}(\vartheta^1 + i\vartheta^2)]_\alpha = \begin{pmatrix} -2\nabla_2\xi^2 - \nabla_3(i\xi^1 - i\bar{\xi}^{\dot{1}}) \\ 2\nabla_1\xi^2 + \nabla_3(\xi^1 - \bar{\xi}^{\dot{1}}) \\ \nabla_1(i\xi^1 - i\bar{\xi}^{\dot{1}}) - \nabla_2(\xi^1 - \bar{\xi}^{\dot{1}}) \end{pmatrix}, \tag{2.E.3}$$

$$[\mathrm{curl}\,\vartheta^3]_\alpha = \begin{pmatrix} -\nabla_3(-i\xi^2 + i\bar{\xi}^{\dot{2}}) \\ \nabla_3(\xi^2 + \bar{\xi}^{\dot{2}}) \\ \nabla_1(-i\xi^2 + i\bar{\xi}^{\dot{2}}) - \nabla_2(\xi^2 + \bar{\xi}^{\dot{2}}) \end{pmatrix}, \tag{2.E.4}$$

$$[\partial_0(\vartheta^1 + i\vartheta^2)]_\alpha = \begin{pmatrix} \partial_0\xi^1 - \partial_0\bar{\xi}^{\dot{1}} \\ i\partial_0\xi^1 - i\partial_0\bar{\xi}^{\dot{1}} \\ -2\partial_0\xi^2 \end{pmatrix}, \tag{2.E.5}$$

$$[\partial_0\vartheta^3]_\alpha = \begin{pmatrix} \partial_0\xi^2 + \partial_0\bar{\xi}^{\dot{2}} \\ -i\partial_0\xi^2 + i\partial_0\bar{\xi}^{\dot{2}} \\ 0 \end{pmatrix}, \tag{2.E.6}$$

where $\mathrm{curl}\, u := *du$.

We rewrite the formulae for $*T^{\mathrm{ax}}$ and ω in the form

$$*T^{\mathrm{ax}} = \frac{1}{6}(\vartheta^1 - i\vartheta^2)\cdot\mathrm{curl}(\vartheta^1 + i\vartheta^2) + \frac{1}{6}(\vartheta^1 + i\vartheta^2)\cdot\mathrm{curl}(\vartheta^1 - i\vartheta^2) + \frac{1}{3}\vartheta^3\cdot\mathrm{curl}\,\vartheta^3, \tag{2.E.7}$$

$$\omega = \frac{1}{4}(\vartheta^1 - i\vartheta^2) \times \partial_0(\vartheta^1 + i\vartheta^2) + \frac{1}{4}(\vartheta^1 + i\vartheta^2) \times \partial_0(\vartheta^1 - i\vartheta^2) + \frac{1}{2}\vartheta^3 \times \partial_0\vartheta^3, \quad (2.E.8)$$

where $u \cdot v := u_\alpha v^\alpha$ (note the absence of complex conjugation) and $u \times v := *(u \wedge v)$. Substituting formulae (2.D.6), (2.E.3) and (2.E.4) into formula (2.E.7) we get

$$*T^{\mathrm{ax}} = -\frac{2i}{3}\Big[\nabla_3\xi^1 + (\nabla_1 - i\nabla_2)\xi^2 - \nabla_3\bar{\xi}^{\dot{1}} - (\nabla_1 + i\nabla_2)\bar{\xi}^{\dot{2}}\Big]$$

which coincides with the RHS of formula (2.E.1). Substituting formulae (2.D.6), (2.E.5) and (2.E.6) into formula (2.E.8) we get

$$\omega_\alpha = i \begin{pmatrix} \partial_0\xi^2 - \partial_0\bar{\xi}^{\dot{2}} \\ -i\partial_0\xi^2 - i\partial_0\bar{\xi}^{\dot{2}} \\ \partial_0\xi^1 - \partial_0\bar{\xi}^{\dot{1}} \end{pmatrix}$$

which coincides with the RHS of formula (2.E.2).

An alternative way of proving formulae of the type (2.E.1) and (2.E.2) is to choose Pauli matrices $\sigma_{\boldsymbol{\alpha}}$, $\boldsymbol{\alpha} = 0, 1, 2, 3$, in such a way that a given nonvanishing spinor field ξ takes the value (2.D.5) in some neighborhood of a given point (as opposed to only the point itself). This approach was adopted, for example, in [9, 17, 18, 19].

Chapter 3

Microlocal analysis of the massless Dirac operator

3.1 Main results

The aim of this chapter is to extend the classical results of [20] to systems. We are motivated by the following two observations.

- To our knowledge, all previous publications on systems give formulae for the second asymptotic coefficient that are either incorrect or incomplete (i.e. an algorithm for the calculation of the second asymptotic coefficient rather than an actual formula). The appropriate bibliographic review is presented in Section 3.13.
- Systems are fundamentally different from scalar operators in that spectral analysis of systems reveals a very rich geometric structure. An important example of an elliptic system is the massless Dirac operator which is examined in detail in this chapter.

Consider a first order classical pseudodifferential operator A acting on columns $v = \begin{pmatrix} v_1 & \dots & v_m \end{pmatrix}^T$ of complex-valued half-densities over a connected compact

n-dimensional manifold M. Throughout this chapter we assume that $m \geq 2$ and $n \geq 2$.

We assume the symbol of the operator A to be infinitely smooth. We also assume that the operator A is formally self-adjoint (symmetric): $\int_M w^* A v \, dx = \int_M (Aw)^* v \, dx$ for all infinitely smooth $v, w : M \to \mathbb{C}^m$. Here and further on the superscript * in matrices, rows and columns indicates Hermitian conjugation in $\mathbb{C}^m$ and $dx := dx^1 \dots dx^n$, where $x = (x^1, \dots, x^n)$ are local coordinates on M.

Let $A_1(x, \xi)$ be the principal symbol of the operator A. Here $\xi = (\xi_1, \dots, \xi_n)$ is the variable dual to the position variable x; in physics literature the ξ would be referred to as *momentum*. Our principal symbol A_1 is an $m \times m$ Hermitian matrix-function on $T'M := T^*M \setminus \{\xi = 0\}$, i.e. on the cotangent bundle with the zero section removed.

Let $h^{(j)}(x, \xi)$ be the eigenvalues of the principal symbol. We assume these eigenvalues to be nonzero (this is a version of the ellipticity condition) but do not make any assumptions on their sign. We also assume that the eigenvalues $h^{(j)}(x, \xi)$ are simple for all $(x, \xi) \in T'M$. The techniques developed in this part of the thesis do not work in the case when eigenvalues of the principal symbol have variable multiplicity, though they could probably be adapted to the case of constant multiplicity different from multiplicity 1. The use of the letter "h" for an eigenvalue of the principal symbol is motivated by the fact that later it will take on the role of a Hamiltonian, see formula (3.1.11).

We enumerate the eigenvalues of the principal symbol $h^{(j)}(x, \xi)$ in increasing order, using a positive index $j = 1, \dots, m^+$ for positive $h^{(j)}(x, \xi)$ and a negative index $j = -1, \dots, -m^-$ for negative $h^{(j)}(x, \xi)$. Here m^+ is the number of positive eigenvalues of the principal symbol and m^- is the number of negative ones. Of course, $m^+ + m^- = m$.

Under the above assumptions A is a self-adjoint operator, in the full functional analytic sense, in the Hilbert space $L^2(M; \mathbb{C}^m)$ (Hilbert space of square integrable complex-valued column "functions") with domain $H^1(M; \mathbb{C}^m)$ (Sobolev space of

complex-valued column "functions" which are square integrable together with their first partial derivatives) and the spectrum of A is discrete. These facts are easily established by constructing the parametrix (approximate inverse) of the operator $A + iI$. Note that for the special case of the massless Dirac operator a detailed examination of relevant functional analytic properties was performed in Chapter 4 of [21].

Let λ_k and $v_k = (v_{k1}(x) \ \dots \ v_{km}(x))^T$ be the eigenvalues and eigenfunctions of the operator A. The eigenvalues λ_k are enumerated in increasing order with account of multiplicity, using a positive index $k = 1, 2, \dots$ for positive λ_k and a nonpositive index $k = 0, -1, -2, \dots$ for nonpositive λ_k. If the operator A is bounded from below (i.e. if $m^- = 0$) then the index k runs from some integer value to $+\infty$; if the operator A is bounded from above (i.e. if $m^+ = 0$) then the index k runs from $-\infty$ to some integer value; and if the operator A is unbounded from above and from below (i.e. if $m^+ \neq 0$ and $m^- \neq 0$) then the index k runs from $-\infty$ to $+\infty$.

We will be studying the following three objects.

Object 1. Our first object of study is the *propagator*, which is the one-parameter family of operators defined as

$$U(t) := e^{-itA} = \sum_k e^{-it\lambda_k} v_k(x) \int_M [v_k(y)]^*(\,\cdot\,)\, dy\,, \tag{3.1.1}$$

$t \in \mathbb{R}$. The propagator provides a solution to the Cauchy problem

$$w|_{t=0} = v \tag{3.1.2}$$

for the dynamic equation

$$D_t w + Aw = 0\,, \tag{3.1.3}$$

where $D_t := -i\partial/\partial t$. Namely, it is easy to see that if the column of half-densities $v = v(x)$ is infinitely smooth, then, setting $w := U(t)\, v$, we get a time-dependent column of half-densities $w(t, x)$ which is also infinitely smooth and which satisfies

the equation (3.1.3) and the initial condition (3.1.2). The use of the letter "U" for the propagator is motivated by the fact that for each t the operator $U(t)$ is unitary.

Object 2. Our second object of study is the *spectral function* (1.3.2) defined in Section 1.3.

Object 3. Our third and final object of study is the *counting function*

$$N(\lambda) := \sum_{0<\lambda_k<\lambda} 1 \; = \int_M e(\lambda, x, x)\, dx\,. \tag{3.1.4}$$

In other words, $N(\lambda)$ is the number of eigenvalues λ_k between zero and λ.

It is natural to ask the question: why, in defining the spectral function (1.3.2) and the counting function (3.1.4), did we choose to perform summation over all *positive* eigenvalues up to a given positive λ rather than over all *negative* eigenvalues up to a given negative λ? There is no particular reason. One case reduces to the other by the change of operator $A \mapsto -A$. This issue will be revisited in Section 3.12.

Further on we assume that $m^+ > 0$, i.e. that the operator A is unbounded from above.

Our objectives are as follows.

Objective 1. We aim to construct the propagator (3.1.1) explicitly in terms of oscillatory integrals, modulo an integral operator with an infinitely smooth, in the variables t, x and y, integral kernel.

Objectives 2 and 3. We aim to derive, under appropriate assumptions on Hamiltonian trajectories, two-term asymptotics for the spectral function (1.3.2) and the counting function (3.1.4), i.e. formulae of the type (1.3.3) and

$$N(\lambda) = a\lambda^n + b\lambda^{n-1} + o(\lambda^{n-1}), \tag{3.1.5}$$

as $\lambda \to +\infty$. Obviously, here we expect the real constants a, b and real densities $a(x)$, $b(x)$ to be related in accordance with

$$a = \int_M a(x)\, dx, \tag{3.1.6}$$

$$b = \int_M b(x)\, dx. \tag{3.1.7}$$

It is well known that the above three objectives are closely related: if one achieves Objective 1, then Objectives 2 and 3 follow via Fourier Tauberian theorems [20, 45, 29, 44].

We are now in a position to state our main results.

Result 1. We construct the propagator as a sum of m oscillatory integrals

$$U(t) \overset{\operatorname{mod} C^\infty}{=} \sum_j U^{(j)}(t)\,, \tag{3.1.8}$$

where the phase function of each oscillatory integral $U^{(j)}(t)$ is associated with the corresponding Hamiltonian $h^{(j)}(x,\xi)$. The symbol of the oscillatory integral $U^{(j)}(t)$ is a complex-valued $m \times m$ matrix-function $u^{(j)}(t; y, \eta)$, where $y = (y^1, \dots, y^n)$ is the position of the source of the wave (i.e. this is the same y that appears in formula (3.1.1)) and $\eta = (\eta_1, \dots, \eta_n)$ is the corresponding dual variable (covector at the point y). When $|\eta| \to +\infty$, the symbol admits an asymptotic expansion

$$u^{(j)}(t; y, \eta) = u_0^{(j)}(t; y, \eta) + u_{-1}^{(j)}(t; y, \eta) + \dots \tag{3.1.9}$$

into components positively homogeneous in η, with the subscript indicating degree of homogeneity.

The formula for the principal symbol of the oscillatory integral $U^{(j)}(t)$ is known [43, 35] and reads as follows:

$$u_0^{(j)}(t;y,\eta) = [v^{(j)}(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))]\,[v^{(j)}(y,\eta)]^* \times \exp\left(-i\int_0^t q^{(j)}(x^{(j)}(\tau;y,\eta),\xi^{(j)}(\tau;y,\eta))\,d\tau\right), \quad (3.1.10)$$

where $v^{(j)}(z,\zeta)$ is the normalised eigenvector of the principal symbol $A_1(z,\zeta)$ corresponding to the eigenvalue (Hamiltonian) $h^{(j)}(z,\zeta)$, $(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))$ is the Hamiltonian trajectory originating from the point (y,η), i.e. solution of the system of ordinary differential equations (the dot denotes differentiation in t)

$$\dot{x}^{(j)} = h_\xi^{(j)}(x^{(j)},\xi^{(j)}), \qquad \dot{\xi}^{(j)} = -h_x^{(j)}(x^{(j)},\xi^{(j)}) \quad (3.1.11)$$

subject to the initial condition $(x^{(j)},\xi^{(j)})\big|_{t=0} = (y,\eta)$, $q^{(j)} : T'M \to \mathbb{R}$ is the function

$$q^{(j)} := [v^{(j)}]^* A_{\text{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\} - i[v^{(j)}]^*\{v^{(j)}, h^{(j)}\} \quad (3.1.12)$$

and

$$A_{\text{sub}}(z,\zeta) := A_0(z,\zeta) + \frac{i}{2}(A_1)_{z^\alpha\zeta_\alpha}(z,\zeta) \quad (3.1.13)$$

is the subprincipal symbol of the operator A, with the subscripts z^α and ζ_α indicating partial derivatives and the repeated index α indicating summation over $\alpha = 1,\dots,n$. Curly brackets in formula (3.1.12) denote the Poisson bracket on matrix-functions

$$\{P,R\} := P_{z^\alpha}R_{\zeta_\alpha} - P_{\zeta_\alpha}R_{z^\alpha} \quad (3.1.14)$$

and its further generalisation

$$\{P,Q,R\} := P_{z^\alpha}QR_{\zeta_\alpha} - P_{\zeta_\alpha}QR_{z^\alpha}\,. \quad (3.1.15)$$

As the derivation of formula (3.1.10) was previously performed only in theses [43, 35], we repeat it in Sections 3.2 and 3.3 of the thesis. Our derivation differs

slightly from that in [43] and [35].

Formula (3.1.10) is invariant under changes of local coordinates on the manifold M, i.e. elements of the $m \times m$ matrix-function $u_0^{(j)}(t; y, \eta)$ are scalars on $\mathbb{R} \times T'M$. Moreover, formula (3.1.10) is invariant under the transformation of the eigenvector of the principal symbol

$$v^{(j)} \mapsto e^{i\phi^{(j)}} v^{(j)}, \tag{3.1.16}$$

where

$$\phi^{(j)} : T'M \to \mathbb{R} \tag{3.1.17}$$

is an arbitrary smooth function. When some quantity is defined up to the action of a certain transformation, theoretical physicists refer to such a transformation as a *gauge transformation*. We follow this tradition. Note that our particular gauge transformation (3.1.16), (3.1.17) is quite common in quantum mechanics: when $\phi^{(j)}$ is a function of the position variable x only (i.e. when $\phi^{(j)} : M \to \mathbb{R}$) this gauge transformation is associated with electromagnetism.

Both Y. Safarov [43] and W.J. Nicoll [35] assumed that the operator A is semi-bounded from below but this assumption is not essential and their formula (3.1.10) remains true in the more general case that we are dealing with.

However, knowing the principal symbol (3.1.10) of the oscillatory integral $U^{(j)}(t)$ is not enough if one wants to derive two-term asymptotics (1.3.3) and (3.1.5). One needs information about $u_{-1}^{(j)}(t; y, \eta)$, the component of the symbol of the oscillatory integral $U^{(j)}(t)$ which is positively homogeneous in η of degree -1, see formula (3.1.9), but here the problem is that $u_{-1}^{(j)}(t; y, \eta)$ is not a true invariant in the sense that it depends on the choice of phase function in the oscillatory integral. We overcome this difficulty by observing that $U^{(j)}(0)$ is a pseudodifferential operator, hence, it has a well-defined subprincipal symbol $[U^{(j)}(0)]_{\text{sub}}$. We prove that

$$\operatorname{tr}[U^{(j)}(0)]_{\text{sub}} = -i\{[v^{(j)}]^*, v^{(j)}\} \tag{3.1.18}$$

and subsequently show that information contained in formulae (3.1.10) and (3.1.18) is sufficient for the derivation of two-term asymptotics (1.3.3) and (3.1.5).

Note that the RHS of formula (3.1.18) is invariant under the gauge transformation (3.1.16), (3.1.17).

Formula (3.1.18) plays a central role in Chapter 3. Sections 3.2 and 3.3 provide auxiliary material needed for the proof of formula (3.1.18), whereas the actual proof of formula (3.1.18) is given in Section 3.4.

Let us elaborate briefly on the geometric meaning of the RHS of (3.1.18) (a more detailed exposition is presented in Section 3.5). The eigenvector of the principal symbol is defined up to a gauge transformation (3.1.16), (3.1.17) so it is natural to introduce a U(1) connection on $T'M$ as follows: when parallel transporting an eigenvector of the principal symbol along a curve in $T'M$ we require that the derivative of the eigenvector along the curve be orthogonal to the eigenvector itself. This is equivalent to the introduction of an (intrinsic) electromagnetic field on $T'M$, with the $2n$-component real quantity

$$i\,(\,[v^{(j)}]^*v^{(j)}_{x^\alpha}\,,\,[v^{(j)}]^*v^{(j)}_{\xi_\gamma}\,) \tag{3.1.19}$$

playing the role of the electromagnetic covector potential. Our quantity (3.1.19) is a 1-form on $T'M$, rather than on M itself as is the case in "traditional" electromagnetism. The above U(1) connection generates curvature which is a 2-form on $T'M$, an analogue of the electromagnetic tensor. Out of this curvature 2-form one can construct, by contraction of indices, a real scalar. This scalar curvature is the expression appearing in the RHS of formula (3.1.18).

Observe now that $\sum_j U^{(j)}(0)$ is the identity operator on half-densities. The subprincipal symbol of the identity operator is zero, so formula (3.1.18) implies

$$\sum_j \{[v^{(j)}]^*, v^{(j)}\} = 0. \tag{3.1.20}$$

One can check the identity (3.1.20) directly, without constructing the oscillatory integrals $U^{(j)}(t)$: it follows from the fact that the $v^{(j)}(x,\xi)$ form an orthonormal basis, see end of Section 3.5 for details. We mentioned the identity (3.1.20) in order to highlight, once again, the fact that the curvature effects we have identified are specific to systems and do not have an analogue in the scalar case.

Results 2 and 3. We prove, under appropriate assumptions on Hamiltonian trajectories (see Theorems 3.8.3 and 3.8.4 for details), asymptotic formulae (1.3.3) and (3.1.5) with

$$a(x) = \sum_{j=1}^{m^+} \int\!\!\!\!\!\!- \limits_{h^{(j)}(x,\xi)<1} đ\xi\,, \tag{3.1.21}$$

$$\begin{aligned} b(x) = -n \sum_{j=1}^{m^+} \int\!\!\!\!\!\!- \limits_{h^{(j)}(x,\xi)<1} \Big(& [v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} \\ & - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\} + \frac{i}{n-1} h^{(j)} \{[v^{(j)}]^*, v^{(j)}\} \Big)(x,\xi)\, đ\xi\,, \end{aligned} \tag{3.1.22}$$

and a and b expressed via the above densities (3.1.21) and (3.1.22) as (3.1.6) and (3.1.7). In (3.1.21) and (3.1.22) $đ\xi$ is shorthand for $đ\xi := (2\pi)^{-n}\, d\xi = (2\pi)^{-n}\, d\xi_1 \dots d\xi_n$, and the Poisson bracket on matrix-functions $\{\cdot,\cdot\}$ and its further generalisation $\{\cdot,\cdot,\cdot\}$ are defined by formulae (3.1.14) and (3.1.15) respectively.

To our knowledge, formula (3.1.22) is a new result. Note that in [43] this formula (more precisely, its integrated over M version (3.1.7)) was written incorrectly, without the curvature terms $-\frac{ni}{n-1}\int h^{(j)}\{[v^{(j)}]^*, v^{(j)}\}$. See also Section 3.13 where we give a more detailed bibliographic review.

It is easy to see that the right-hand sides of (3.1.21) and (3.1.22) behave as densities under changes of local coordinates on the manifold M and that these expressions are invariant under gauge transformation (3.1.16), (3.1.17) of the eigenvectors of the principal symbol. Moreover, the right-hand sides of (3.1.21) and (3.1.22) are unitarily invariant, i.e. invariant under transformation of the

operator

$$A \mapsto RAR^*, \tag{3.1.23}$$

where

$$R : M \to \mathrm{U}(m) \tag{3.1.24}$$

is an arbitrary smooth unitary matrix-function. The fact that the RHS of (3.1.22) is unitarily invariant is non-trivial: the appropriate calculations are presented in Section 3.9. The observation that without the curvature terms $-\frac{ni}{n-1}\int h^{(j)}\{[v^{(j)}]^*, v^{(j)}\}$ (as in [43]) the RHS of (3.1.22) is not unitarily invariant was a major motivating factor in the writing of this chapter.

We will now start making additional assumptions which will, in the end, allow us to provide a simple spectral theoretic characterisation of the massless Dirac operator.

Additional assumption 1: We assume that statement (1.3.4) is true.

In this case we can simplify notation by denoting the positive eigenvalue of the principal symbol by h^+, the corresponding eigenvector by $v^+ = \begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix}$ and Hamiltonian trajectories by $(x^+(t;y,\eta), \xi^+(t;y,\eta))$. Obviously, the other eigenvalue of the principal symbol is $-h^+$, the corresponding eigenvector is $\begin{pmatrix} -\bar{v}_2^+ \\ \bar{v}_1^+ \end{pmatrix}$ and Hamiltonian trajectories are $(x^+(-t;y,\eta), \xi^+(-t;y,\eta))$ (time reversal). Note that in theoretical physics the antilinear transformation

$$\begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix} \overset{\mathrm{C}}{\mapsto} \begin{pmatrix} -\bar{v}_2^+ \\ \bar{v}_1^+ \end{pmatrix} \tag{3.1.25}$$

is referred to as *charge conjugation* [7].

Moreover, in this case the two scalar invariants, $\{[v^+]^*, A_1 - h^+, v^+\}$ and $h^+\{[v^+]^*, v^+\}$, appearing in formula (3.1.22) cease being independent and become related as $\{[v^+]^*, A_1 - h^+, v^+\} = -2h^+\{[v^+]^*, v^+\}$. Hence, formulae (3.1.21) and (3.1.22)

simplify and now read

$$a(x) = \int\!\!\!\!\!\!\diagup_{h^+(x,\xi)<1} đ\xi\,, \tag{3.1.26}$$

$$b(x) = -n \int\!\!\!\!\!\!\diagup_{h^+(x,\xi)<1} \Big([v^+]^* A_{\mathrm{sub}} v^+ + \frac{n}{n-1} i h^+ \{[v^+]^*, v^+\}\Big)(x,\xi)\, đ\xi\,. \tag{3.1.27}$$

Additional assumption 2: We assume that statement (1.3.5) is true.

In this case there are three further simplifications.

Firstly, the dimension of the manifold can only be $n = 2$ or $n = 3$. This follows from the ellipticity condition and the fact that the dimension of the real vector space of trace-free Hermitian 2×2 matrices is 3.

Secondly, the subprincipal symbol A_{sub} does not depend on the dual variable ξ (momentum) and is a function of x (position) only.

Thirdly, we acquire a geometric object, the *metric*. Indeed, the determinant of the principal symbol is a negative definite quadratic form

$$\det A_1(x,\xi) = -g^{\alpha\beta}\xi_\alpha\xi_\beta \tag{3.1.28}$$

and the coefficients $g^{\alpha\beta}(x)$, $\alpha,\beta = 1,\dots,n$, appearing in (3.1.28) can be interpreted as the components of a (contravariant) Riemannian metric. This implies, in particular, that our Hamiltonian (positive eigenvalue of the principal symbol) takes the form

$$h^+(x,\xi) = \sqrt{g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta} \tag{3.1.29}$$

and the x-components of our Hamiltonian trajectories become geodesics. Moreover, formulae (3.1.26) and (3.1.6) simplify and now read

$$a(x) = (2\pi)^{-n}\,\omega_n \sqrt{\det g_{\alpha\beta}(x)}\,, \tag{3.1.30}$$

$$a = (2\pi)^{-n}\,\omega_n \operatorname{Vol} M\,, \tag{3.1.31}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$ and $\operatorname{Vol} M$ is the n-dimensional volume of the Riemannian manifold M.

Additional assumption 3: We assume that statement (1.3.6) is true.

In this case there are three more simplifications.

Firstly, the manifold M is bound to be parallelizable (and, hence, orientable). The relevant argument is presented in the beginning of Section 3.10. From this point we work only in local coordinates with prescribed orientation.

Secondly, we acquire the identity

$$\det g^{\alpha\beta} = -\frac{1}{4}\big[\operatorname{tr}\big((A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}\big)\big]^2 \tag{3.1.32}$$

which allows us to define the topological invariant

$$\mathbf{c} := -\frac{i}{2}\sqrt{\det g_{\alpha\beta}}\ \operatorname{tr}\big((A_1)_{\xi_1}(A_1)_{\xi_2}(A_1)_{\xi_3}\big). \tag{3.1.33}$$

The number $\mathbf{c}$ defined by formula (3.1.33) can take only two values, $+1$ or -1, and describes the orientation of the principal symbol $A_1(x,\xi)$ relative to the chosen orientation of local coordinates, see formula (3.10.4) for a more natural geometric definition. In calling the number $\mathbf{c}$ a topological invariant we are referring to the topology of deformations of the elliptic trace-free principal symbol $A_1(x,\xi)$ rather than the deformations of the manifold M itself.

Thirdly, we acquire a new differential geometric object, namely, a *teleparallel connection*. This is an affine connection defined as follows. Suppose we have a covector η based at the point $y \in M$ and we want to construct a parallel covector ξ based at the point $x \in M$. This is done by solving the linear system of equations

$$A_1(x,\xi) = A_1(y,\eta). \tag{3.1.34}$$

Equation (3.1.34) is equivalent to a system of three real linear algebraic equations for the three real unknowns, components of the covector ξ, and it is easy to

see that this system has a unique solution. It is also easy to see that the affine connection defined by formula (3.1.34) preserves the Riemannian norm of covectors, i.e. $g^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta = g^{\alpha\beta}(y)\,\eta_\alpha\eta_\beta$, hence, it is metric compatible. The parallel transport defined by formula (3.1.34) does not depend on the curve along which we transport the (co)vector, so our connection has zero curvature. The word "teleparallel" (parallel at a distance) is used in theoretical physics to describe metric compatible affine connections with zero curvature. This terminology goes back to the works of A. Einstein and É. Cartan [51, 46, 11], though Cartan preferred to use the term "absolute parallelism" rather than "teleparallelism".

The teleparallel connection coefficients $\Gamma^\alpha{}_{\beta\gamma}(x)$ can be written down explicitly in terms of the principal symbol, see formula (3.10.7), and this allows us to define yet another geometric object — the torsion tensor

$$T^\alpha{}_{\beta\gamma} := \Gamma^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\gamma\beta}\,. \tag{3.1.35}$$

Further on we raise and lower indices of the torsion tensor using the metric. Torsion is a rank three tensor antisymmetric in the last two indices. Because we are working in dimension three, it is convenient, as in [6], to apply the Hodge star in the last two indices and deal with the rank two tensor

$$\overset{*}{T}{}^\alpha{}_\beta := \frac{1}{2}\,T^{\alpha\gamma\delta}\,\varepsilon_{\gamma\delta\beta}\,\sqrt{\det g_{\mu\nu}} \tag{3.1.36}$$

rather than with the rank three tensor T. Here ε is the totally antisymmetric quantity, $\varepsilon_{123} := +1$.

The teleparallel connection is a simpler geometric object than the U(1) connection because the coefficients of the teleparallel connection do not depend on the dual variable (momentum), i.e. they are "functions" on the base manifold M. The relationship between the two connections is established in Section 3.10 where we show that the scalar curvature of the U(1) connection is expressed via the torsion

of the teleparallel connection and the metric as

$$-i\{[v^+]^*, v^+\}(x,\xi) = \frac{\mathbf{c}}{2}\,\frac{\overset{*}{T}{}^{\alpha\beta}(x)\,\xi_\alpha\xi_\beta}{(g^{\mu\nu}(x)\,\xi_\mu\xi_\nu)^{3/2}}\,. \tag{3.1.37}$$

Integration of both terms appearing in formula (3.1.27) can now be carried out explicitly, giving

$$\int\limits_{h^+(x,\xi)<1} ([v^+]^* A_{\text{sub}} v^+)(x,\xi)\,đ\xi = \frac{1}{12\pi^2}\big(\operatorname{tr} A_{\text{sub}}\,\sqrt{\det g_{\alpha\beta}}\,\big)(x)\,, \tag{3.1.38}$$

$$-i\int\limits_{h^+(x,\xi)<1} h^+\{[v^+]^*, v^+\}(x,\xi)\,đ\xi = \frac{\mathbf{c}}{36\pi^2}\big(\operatorname{tr}\overset{*}{T}\,\sqrt{\det g_{\alpha\beta}}\,\big)(x)\,, \tag{3.1.39}$$

where $\operatorname{tr}\overset{*}{T} := \overset{*}{T}{}^{\alpha}{}_{\alpha}$. Note that $\operatorname{tr}\overset{*}{T}$ corresponds to one of the three irreducible pieces of torsion, namely, the piece which is labelled by theoretical physicists by the adjective "axial", see [6, 25] for details; it is interesting that this is exactly the irreducible piece of torsion which is used when one models the neutrino [14] or the electron [10] by means of Cosserat elasticity. Formula (3.1.39) follows immediately from (3.1.37), whereas formula (3.1.38) is somewhat less obvious. In order to see where formula (3.1.38) comes from one has to write the orthogonal projection $v^+(x,\xi)\,[v^+(x,\xi)]^*$ as $v^+(x,\xi)\,[v^+(x,\xi)]^* = \frac{1}{2h^+(x,\xi)}(A_1(x,\xi) + h^+(x,\xi)\,I)$ and use the fact that the principal symbol $A_1(x,\xi)$ is an odd function of ξ.

Substituting (1.3.6), (3.1.38) and (3.1.39) into (3.1.27) we get

$$b(x) = \frac{1}{8\pi^2}\big(\big(\mathbf{c}\operatorname{tr}\overset{*}{T} - 2\operatorname{tr} A_{\text{sub}}\big)\sqrt{\det g_{\alpha\beta}}\,\big)(x)\,. \tag{3.1.40}$$

An explicit self-contained expression for $\operatorname{tr}\overset{*}{T}$ is given in formula (3.10.28).

Note that the two traces appearing in formula (3.1.40) have a different meaning: $\operatorname{tr}\overset{*}{T}$ is the trace of a 3×3 tensor, whereas $\operatorname{tr} A_{\text{sub}}$ is the trace of a 2×2 matrix.

We now turn our attention to the massless Dirac operator. This operator is defined in Appendix 3.A, see formula (3.A.3), and it does not fit into our scheme

because this is an operator acting on a 2-component complex-valued spinor (Weyl spinor) rather than a pair of complex-valued half-densities. However, on a parallelizable manifold components of a spinor can be identified with half-densities. We call the resulting operator *the massless Dirac operator on half-densities*. The explicit formula for the massless Dirac operator on half-densities is (3.A.30).

The massless Dirac operator on half-densities is an operator of the type described in this section (elliptic self-adjoint first order operator acting on a column of complex-valued half-densities) which, moreover, satisfies the additional assumptions (1.3.4), (1.3.5) and (1.3.6). We address the question: is a given operator A a massless Dirac operator? The answer is given by the Theorem 1.3.1 which we prove in Section 3.11.

Theorem 1.3.1 warrants the following remarks.

- In stating Theorem 1.3.1 we did not make any assumptions on Hamiltonian trajectories (loops). The second asymptotic coefficient (3.1.40) is, in itself, well-defined irrespective of how many loops we have. If one wishes to reformulate the asymptotic formula (1.3.3) in such a way that it remains valid without assumptions on the number of loops, this can easily be achieved, say, by taking a convolution with a function from Schwartz space $\mathcal{S}(\mathbb{R})$. See Theorem 3.7.1 for details.

- Conditions a) and b) in Theorem 1.3.1 are invariant under special unitary transformation, i.e. transformation of the operator (3.1.23) where $R = R(x)$ is an arbitrary smooth special unitary matrix-function. This is not surprising as the massless Dirac operator is designed around the concept of SU(2) invariance, see Property 4 in Appendix 3.A.

- Condition b) in Theorem 1.3.1 is actually invariant under the action of a broader group: the unitary matrix-function appearing in formula (3.1.23) does not have to be special.

3.2 Algorithm for the construction of the propagator

We construct the propagator as a sum of m oscillatory integrals (3.1.8) where each integral is of the form

$$U^{(j)}(t) = \int e^{i\varphi^{(j)}(t,x;y,\eta)}\, u^{(j)}(t;y,\eta)\, \varsigma^{(j)}(t,x;y,\eta)\, d_{\varphi^{(j)}}(t,x;y,\eta)\,(\,\cdot\,)\, dy\, đ\eta\,. \quad (3.2.1)$$

Here we use notation from the book [45], only adapted to systems. Namely, the expressions appearing in formula (3.2.1) have the following meaning.

- The function $\varphi^{(j)}$ is a phase function, i.e. a function $\mathbb{R} \times M \times T'M \to \mathbb{C}$ positively homogeneous in η of degree 1 and satisfying the conditions

$$\varphi^{(j)}(t,x;y,\eta) = (x - x^{(j)}(t;y,\eta))^{\alpha}\, \xi^{(j)}_{\alpha}(t;y,\eta) + O(|x - x^{(j)}(t;y,\eta)|^2), \quad (3.2.2)$$

$$\operatorname{Im} \varphi^{(j)}(t,x;y,\eta) \geq 0, \quad (3.2.3)$$

$$\det \varphi^{(j)}_{x^{\alpha}\eta_{\beta}}(t, x^{(j)}(t;y,\eta);y,\eta) \neq 0. \quad (3.2.4)$$

  Recall that according to Corollary 2.4.5 from [45] we are guaranteed to have (3.2.4) if we choose a phase function

$$\begin{aligned}\varphi^{(j)}(t,x;y,\eta) = {} & (x - x^{(j)}(t;y,\eta))^{\alpha}\, \xi^{(j)}_{\alpha}(t;y,\eta) \\ & + \frac{1}{2} C^{(j)}_{\alpha\beta}(t;y,\eta)\, (x - x^{(j)}(t;y,\eta))^{\alpha}\, (x - x^{(j)}(t;y,\eta))^{\beta} \\ & + O(|x - x^{(j)}(t;y,\eta)|^3) \end{aligned} \quad (3.2.5)$$

  with complex-valued symmetric matrix-function $C^{(j)}_{\alpha\beta}$ satisfying the strict inequality $\operatorname{Im} C^{(j)} > 0$ (our original requirement (3.2.3) implies only the non-strict inequality $\operatorname{Im} C^{(j)} \geq 0$). Note that even though the matrix-function $C^{(j)}_{\alpha\beta}$ is not a tensor, the inequalities $\operatorname{Im} C^{(j)} \geq 0$ and $\operatorname{Im} C^{(j)} > 0$ are invariant under transformations of local coordinates x; see Remark 2.4.9 in [45] for details.

- The quantity $u^{(j)}$ is the symbol of our oscillatory integral, i.e. a complex-valued $m\times m$ matrix-function $\mathbb{R}\times T'M \to \mathbb{C}^{m^2}$ which admits the asymptotic expansion (3.1.9). The symbol is the unknown quantity in our construction.

- The quantity $d_{\varphi^{(j)}}$ is defined in accordance with formula (2.2.4) from [45] as

$$d_{\varphi^{(j)}}(t,x;y,\eta) := (\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta})^{1/4} = |\det\varphi^{(j)}_{x^\alpha\eta_\beta}|^{1/2}\, e^{\,i\arg(\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta})/4}. \quad (3.2.6)$$

  Note that in view of (3.2.4) our $d_{\varphi^{(j)}}$ is well-defined and smooth for x close to $x^{(j)}(t;y,\eta)$. It is known [45] that under coordinate transformations $d_{\varphi^{(j)}}$ behaves as a half-density in x and as a half-density to the power -1 in y.

  In formula (3.2.6) we wrote $(\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta})^{1/4}$ rather than $(\det\varphi^{(j)}_{x^\alpha\eta_\beta})^{1/2}$ in order to make this expression truly invariant under coordinate transformations. Recall that local coordinates x and y are chosen independently and that η is a covector based at the point y. Consequently, $\det\varphi^{(j)}_{x^\alpha\eta_\beta}$ changes sign under inversions of local coordinates x or y, whereas $\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta}$ retains sign under inversions.

  The choice of (smooth) branch of $\arg(\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta})$ is assumed to be fixed. Thus, for a given phase function $\varphi^{(j)}$ formula (3.2.6) defines the quantity $d_{\varphi^{(j)}}$ uniquely up to a factor $e^{ik\pi/2}$, $k = 0,1,2,3$. Observe now that if we set $t = 0$ and choose the same local coordinates for x and y, we get $\varphi^{(j)}_{x^\alpha\eta_\beta}(0,y;y,\eta) = I$. This implies that we can fully specify the choice of branch of $\arg(\det{}^2\varphi^{(j)}_{x^\alpha\eta_\beta})$ by requiring that $d_{\varphi^{(j)}}(0,y;y,\eta) = 1$.

  The purpose of the introduction of the factor $d_{\varphi^{(j)}}$ in (3.2.1) is twofold.

  (a) It ensures that the symbol $u^{(j)}$ is a function on $\mathbb{R}\times T'M$ in the full differential geometric sense of the word, i.e. that it is invariant under transformations of local coordinates x and y.

  (b) It ensures that the principal symbol $u^{(j)}_0$ does not depend on the choice of phase function $\varphi^{(j)}$. See Remark 2.2.8 in [45] for more details.

- The quantity $\varsigma^{(j)}$ is a smooth cut-off function $\mathbb{R}\times M\times T'M\to\mathbb{R}$ satisfying the following conditions.

  (a) $\varsigma^{(j)}(t,x;y,\eta)=0$ on the set $\{(t,x;y,\eta):\ |h^{(j)}(y,\eta)|\le 1/2\}$.

  (b) $\varsigma^{(j)}(t,x;y,\eta)=1$ on the intersection of a small conic neighbourhood of the set

  $$\{(t,x;y,\eta):\ x=x^{(j)}(t;y,\eta)\} \tag{3.2.7}$$

  with the set $\{(t,x;y,\eta):\ |h^{(j)}(y,\eta)|\ge 1\}$.

  (c) $\varsigma^{(j)}(t,x;y,\lambda\eta)=\varsigma^{(j)}(t,x;y,\eta)$ for $|h^{(j)}(y,\eta)|\ge 1,\ \lambda\ge 1$.

- It is known (see Section 2.3 in [45] for details) that Hamiltonian trajectories generated by a Hamiltonian $h^{(j)}(x,\xi)$ positively homogeneous in ξ of degree 1 satisfy the identity

  $$(x^{(j)}_\eta)^{\alpha\beta}\xi^{(j)}_\alpha=0, \tag{3.2.8}$$

  where $(x^{(j)}_\eta)^{\alpha\beta}:=\partial(x^{(j)})^\alpha/\partial\eta_\beta$. Formulae (3.2.2) and (3.2.8) imply

  $$\varphi^{(j)}_\eta(t,x^{(j)}(t;y,\eta);y,\eta)=0. \tag{3.2.9}$$

  This allows us to apply the stationary phase method in the neighbourhood of the set (3.2.7) and disregard what happens away from it.

Our task now is to construct the symbols $u^{(j)}_0(t;y,\eta)$, $j=1,\dots,m$, so that our oscillatory integrals $U^{(j)}(t)$, $j=1,\dots,m$, satisfy the dynamic equations

$$(D_t+A(x,D_x))\,U^{(j)}(t)\overset{\mathrm{mod}\,C^\infty}{=}0 \tag{3.2.10}$$

and initial condition

$$\sum_j U^{(j)}(0)\overset{\mathrm{mod}\,C^\infty}{=}I\,, \tag{3.2.11}$$

where I is the identity operator on half-densities; compare with formulae (3.1.3), (3.1.2) and (3.1.8). Note that the pseudodifferential operator A in formula

(3.2.10) acts on the oscillatory integral $U(t)$ in the variable x; say, if A is a differential operator this means that in order to evaluate $A\,U^{(j)}(t)$ one has to perform the appropriate differentiations of the oscillatory integral (3.2.1) in the variable x. Following the conventions of Section 3.3 of [45], we emphasise the fact that the pseudodifferential operator A in formula (3.2.10) acts on the oscillatory integral $U(t)$ in the variable x by writing this pseudodifferential operator as $A(x, D_x)$, where $D_{x^\alpha} := -i\partial/\partial x^\alpha$.

We examine first the dynamic equation (3.2.10). We have

$$(D_t + A(x, D_x))\, U^{(j)}(t) = F^{(j)}(t)\,,$$

where $F^{(j)}(t)$ is the oscillatory integral

$$F^{(j)}(t) = \int e^{i\varphi^{(j)}(t,x;y,\eta)}\, f^{(j)}(t, x; y, \eta)\, \varsigma^{(j)}(t, x; y, \eta)\, d_{\varphi^{(j)}}(t, x; y, \eta)\, (\ \cdot\ )\, dy\, đ\eta$$

whose matrix-valued amplitude $f^{(j)}$ is given by the formula

$$f^{(j)} = D_t u^{(j)} + \big(\varphi_t^{(j)} + (d_{\varphi^{(j)}})^{-1}(D_t d_{\varphi^{(j)}}) + s^{(j)}\big)\, u^{(j)}, \tag{3.2.12}$$

where the matrix-function $s^{(j)}(t, x; y, \eta)$ is defined as

$$s^{(j)} = e^{-i\varphi^{(j)}}(d_{\varphi^{(j)}})^{-1}\, A(x, D_x)\, (e^{i\varphi^{(j)}} d_{\varphi^{(j)}})\,. \tag{3.2.13}$$

Theorem 18.1 from [48] gives us the following explicit asymptotic (in inverse powers of η) formula for the matrix-function (3.2.13):

$$s^{(j)} = (d_{\varphi^{(j)}})^{-1} \sum_{\boldsymbol{\alpha}} \frac{1}{\boldsymbol{\alpha}!}\, A^{(\boldsymbol{\alpha})}(x, \varphi_x^{(j)})\, (D_z^{\boldsymbol{\alpha}} \chi^{(j)})\big|_{z=x}\,, \tag{3.2.14}$$

where

$$\chi^{(j)}(t, z, x; y, \eta) = e^{i\psi^{(j)}(t,z,x;y,\eta)} d_{\varphi^{(j)}}(t, z; y, \eta), \tag{3.2.15}$$

$$\psi^{(j)}(t, z, x; y, \eta) = \varphi^{(j)}(t, z; y, \eta) - \varphi^{(j)}(t, x; y, \eta) - \varphi_{x^\beta}^{(j)}(t, x; y, \eta)\, (z-x)^\beta. \tag{3.2.16}$$

In formula (3.2.14)

- $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n)$ is a multi-index (note the bold font which we use to distinguish multi-indices and individual indices), $\boldsymbol{\alpha}! := \alpha_1! \cdots \alpha_n!$, $D_z^{\boldsymbol{\alpha}} := D_{z^1}^{\alpha_1} \cdots D_{z^n}^{\alpha_n}$, $D_{z^\beta} := -i\partial/\partial z^\beta$,
- $A(x, \xi)$ is the full symbol of the pseudodifferential operator A written in local coordinates x,
- $A^{(\boldsymbol{\alpha})}(x, \xi) := \partial_\xi^{\boldsymbol{\alpha}} A(x, \xi)$, $\partial_\xi^{\boldsymbol{\alpha}} := \partial_{\xi_1}^{\alpha_1} \cdots \partial_{\xi_n}^{\alpha_n}$ and $\partial_{\xi_\beta} := \partial/\partial\xi_\beta$.

When $|\eta| \to +\infty$ the matrix-valued amplitude $f^{(j)}(t, x; y, \eta)$ defined by formula (3.2.12) admits an asymptotic expansion

$$f^{(j)}(t, x; y, \eta) = f_1^{(j)}(t, x; y, \eta) + f_0^{(j)}(t, x; y, \eta) + f_{-1}^{(j)}(t, x; y, \eta) + \dots \qquad (3.2.17)$$

into components positively homogeneous in η, with the subscript indicating degree of homogeneity. Note the following differences between formulae (3.1.9) and (3.2.17).

- The leading term in (3.2.17) has degree of homogeneity 1, rather than 0 as in (3.1.9). In fact, the leading term in (3.2.17) can be easily written out explicitly

$$f_1^{(j)}(t, x; y, \eta) = (\varphi_t^{(j)}(t, x; y, \eta) + A_1(x, \varphi_x^{(j)}(t, x; y, \eta))) \, u_0^{(j)}(t; y, \eta)\,, \quad (3.2.18)$$

  where $A_1(x, \xi)$ is the (matrix-valued) principal symbol of the pseudodifferential operator A.
- Unlike the symbol $u^{(j)}(t; y, \eta)$, the amplitude $f^{(j)}(t, x; y, \eta)$ depends on x.

We now need to exclude the dependence on x from the amplitude $f^{(j)}(t, x; y, \eta)$. This can be done by means of the algorithm described in subsection 2.7.3 of [45]. We outline this algorithm below.

Working in local coordinates, define the matrix-function $\varphi^{(j)}_{x\eta}$ in accordance with $(\varphi^{(j)}_{x\eta})_\alpha{}^\beta := \varphi^{(j)}_{x^\alpha\eta_\beta}$ and then define its inverse $(\varphi^{(j)}_{x\eta})^{-1}$ from the identity $(\varphi^{(j)})_\alpha{}^\beta[(\varphi^{(j)}_{x\eta})^{-1}]_\beta{}^\gamma := \delta_\alpha{}^\gamma$. Define the "scalar" first order linear differential operators

$$L^{(j)}_\alpha := [(\varphi^{(j)}_{x\eta})^{-1}]_\alpha{}^\beta\,(\partial/\partial x^\beta), \qquad \alpha = 1,\dots,n. \tag{3.2.19}$$

Note that the coefficients of these differential operators are functions of the position variable x and the dual variable ξ. It is known, see part 2 of Appendix E in [45], that the operators (3.2.19) commute: $L^{(j)}_\alpha L^{(j)}_\beta = L^{(j)}_\beta L^{(j)}_\alpha$, $\alpha,\beta = 1,\dots,n$.

Denote $L^{(j)}_{\boldsymbol{\alpha}} := (L^{(j)}_1)^{\alpha_1}\cdots(L^{(j)}_n)^{\alpha_n}$, $(-\varphi^{(j)}_\eta)^{\boldsymbol{\alpha}} := (-\varphi^{(j)}_{\eta_1})^{\alpha_1}\cdots(-\varphi^{(j)}_{\eta_n})^{\alpha_n}$, and, given an $r\in\mathbb{N}$, define the "scalar" linear differential operator

$$\mathfrak{P}^{(j)}_{-1,r} := i(d_{\varphi^{(j)}})^{-1}\,\frac{\partial}{\partial\eta_\beta}\,d_{\varphi^{(j)}}\left(1+\sum_{1\le|\boldsymbol{\alpha}|\le 2r-1}\frac{(-\varphi^{(j)}_\eta)^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!\,(|\boldsymbol{\alpha}|+1)}\,L^{(j)}_{\boldsymbol{\alpha}}\right)L^{(j)}_\beta\,, \tag{3.2.20}$$

where $|\boldsymbol{\alpha}| := \alpha_1+\ldots+\alpha_n$ and the repeated index β indicates summation over $\beta = 1,\dots,n$.

Recall Definition 2.7.8 from [45]: the linear operator L is said to be positively homogeneous in η of degree $p\in\mathbb{R}$ if for any $q\in\mathbb{R}$ and any function f positively homogeneous in η of degree q the function Lf is positively homogeneous in η of degree $p+q$. It is easy to see that the operator (3.2.20) is positively homogeneous in η of degree -1 and the first subscript in $\mathfrak{P}^{(j)}_{-1,r}$ emphasises this fact.

Let $\mathfrak{S}^{(j)}_0$ be the (linear) operator of restriction to $x = x^{(j)}(t;y,\eta)$,

$$\mathfrak{S}^{(j)}_0 := (\,\cdot\,)|_{x=x^{(j)}(t;y,\eta)}\,, \tag{3.2.21}$$

and let

$$\mathfrak{S}^{(j)}_{-r} := \mathfrak{S}^{(j)}_0(\mathfrak{P}^{(j)}_{-1,r})^r \tag{3.2.22}$$

for $r = 1,2,\ldots$. Observe that our linear operators $\mathfrak{S}^{(j)}_{-r}$, $r = 0,1,2,\ldots$, are positively homogeneous in η of degree $-r$. This observation allows us to define

the linear operator

$$\mathfrak{S}^{(j)} := \sum_{r=0}^{+\infty} \mathfrak{S}^{(j)}_{-r} \,, \tag{3.2.23}$$

where the series is understood as an asymptotic series in inverse powers of η.

According to subsection 2.7.3 of [45], the dynamic equation (3.2.10) can now be rewritten in the equivalent form

$$\mathfrak{S}^{(j)} f^{(j)} = 0 \,, \tag{3.2.24}$$

where the equality is understood in the asymptotic sense, as an asymptotic expansion in inverse powers of η. Recall that the matrix-valued amplitude $f^{(j)}(t, x; y, \eta)$ appearing in (3.2.24) is defined by formulae (3.2.12)–(3.2.16).

Substituting (3.2.23) and (3.2.17) into (3.2.24) we obtain a hierarchy of equations

$$\mathfrak{S}^{(j)}_0 f^{(j)}_1 = 0, \tag{3.2.25}$$

$$\mathfrak{S}^{(j)}_{-1} f^{(j)}_1 + \mathfrak{S}^{(j)}_0 f^{(j)}_0 = 0, \tag{3.2.26}$$

$$\mathfrak{S}^{(j)}_{-2} f^{(j)}_1 + \mathfrak{S}^{(j)}_{-1} f^{(j)}_0 + \mathfrak{S}^{(j)}_0 f^{(j)}_{-1} = 0,$$

$$\dots$$

positively homogeneous in η of degree 1, 0, -1, ... These are the *transport* equations for the determination of the unknown homogeneous components $u^{(j)}_0(t; y, \eta)$, $u^{(j)}_{-1}(t; y, \eta)$, $u^{(j)}_{-2}(t; y, \eta)$, ..., of the symbol of the oscillatory integral (3.2.1).

Let us now examine the initial condition (3.2.11). Each operator $U^{(j)}(0)$ is a pseudodifferential operator, only written in a slightly nonstandard form. The issues here are as follows.

- We use the invariantly defined phase function $\varphi^{(j)}(0, x; y, \eta) = (x-y)^\alpha \, \eta_\alpha + O(|x-y|^2)$ rather than the linear phase function $(x-y)^\alpha \, \eta_\alpha$ written in local coordinates.

- When defining the (full) symbol of the operator $U^{(j)}(t)$ we excluded the variable x from the amplitude rather than the variable y. Note that when dealing with pseudodifferential operators it is customary to exclude the variable y from the amplitude; exclusion of the variable x gives the dual symbol of a pseudodifferential operator, see subsection 2.1.3 in [45]. Thus, at $t = 0$, our symbol $u^{(j)}(0; y, \eta)$ resembles the dual symbol of a pseudodifferential operator rather than the "normal" symbol.

- We have the extra factor $d_{\varphi^{(j)}}(0, x; y, \eta)$ in our representation of the operator $U^{(j)}(0)$ as an oscillatory integral.

The (full) dual symbol of the pseudodifferential operator $U^{(j)}(0)$ can be calculated in local coordinates in accordance with the following formula which addresses the issues highlighted above:

$$\sum_{\boldsymbol{\alpha}} \frac{(-1)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} \left(D_x^{\boldsymbol{\alpha}} \partial_\eta^{\boldsymbol{\alpha}} u^{(j)}(0; y, \eta)\, e^{i\omega^{(j)}(x;y,\eta)}\, d_{\varphi^{(j)}}(0, x; y, \eta)\right)\Big|_{x=y} , \tag{3.2.27}$$

where $\omega^{(j)}(x; y, \eta) = \varphi^{(j)}(0, x; y, \eta) - (x - y)^\beta \eta_\beta$. Formula (3.2.27) is a version of the formula from subsection 2.1.3 of [45], only with the extra factor $(-1)^{|\boldsymbol{\alpha}|}$. The latter is needed because we are writing down the dual symbol of the pseudodifferential operator $U^{(j)}(0)$ (no dependence on x) rather than its "normal" symbol (no dependence on y).

The initial condition (3.2.11) can now be rewritten in explicit form as

$$\sum_j \sum_{\boldsymbol{\alpha}} \frac{(-1)^{|\boldsymbol{\alpha}|}}{\boldsymbol{\alpha}!} \left(D_x^{\boldsymbol{\alpha}} \partial_\eta^{\boldsymbol{\alpha}} u^{(j)}(0; y, \eta)\, e^{i\omega^{(j)}(x;y,\eta)}\, d_{\varphi^{(j)}}(0, x; y, \eta)\right)\Big|_{x=y} = I , \tag{3.2.28}$$

where I is the $m \times m$ identity matrix. Condition (3.2.28) can be decomposed into components positively homogeneous in η of degree $0, -1, -2, \ldots$, giving us a hierarchy of initial conditions. The leading (of degree of homogeneity 0) initial condition reads

$$\sum_j u_0^{(j)}(0; y, \eta) = I , \tag{3.2.29}$$

whereas lower order initial conditions are more complicated and depend on the choice of our phase functions $\varphi^{(j)}$.

3.3 Leading transport equations

Formulae (3.2.21), (3.2.18), (3.2.2), (3.1.11) and the identity $\xi_\alpha h^{(j)}_{\xi_\alpha}(x,\xi) = h^{(j)}(x,\xi)$ (consequence of the fact that $h^{(j)}(x,\xi)$ is positively homogeneous in ξ of degree 1) give us the following explicit representation for the leading transport equation (3.2.25):

$$\left[A_1\big(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta)\big) - h^{(j)}\big(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta)\big)\right] u_0^{(j)}(t;y,\eta) = 0. \tag{3.3.1}$$

Here, of course, $h^{(j)}\big(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta)\big) = h^{(j)}(y,\eta)$.

Equation (3.3.1) implies that

$$u_0^{(j)}(t;y,\eta) = v^{(j)}(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))\,[w^{(j)}(t;y,\eta)]^T, \tag{3.3.2}$$

where $v^{(j)}(z,\zeta)$ is the normalised eigenvector of the principal symbol $A_1(z,\zeta)$ corresponding to the eigenvalue $h^{(j)}(z,\zeta)$ and $w^{(j)} : \mathbb{R}\times T'M \to \mathbb{C}^m$ is a column-function, positively homogeneous in η of degree 0, that remains to be found. Formulae (3.2.29) and (3.3.2) imply the following initial condition for the unknown column-function $w^{(j)}$:

$$w^{(j)}(0;y,\eta) = \overline{v^{(j)}(y,\eta)}. \tag{3.3.3}$$

We now consider the next transport equation in our hierarchy, equation (3.2.26). We will write down the two terms appearing in (3.2.26) separately.

In view of formulae (3.2.18) and (3.2.20)–(3.2.22), the first term in (3.2.26) reads

$$\mathfrak{S}^{(j)}_{-1} f^{(j)}_1 = \\ i \left[(d_{\varphi^{(j)}})^{-1} \frac{\partial}{\partial \eta_\beta} d_{\varphi^{(j)}} \left(1 - \frac{1}{2} \varphi^{(j)}_{\eta_\alpha} L^{(j)}_\alpha \right) \left(L^{(j)}_\beta \big(\varphi^{(j)}_t + A_1(x, \varphi^{(j)}_x) \big) \right) u^{(j)}_0 \right] \Big|_{x = x^{(j)}}, \tag{3.3.4}$$

where we dropped, for the sake of brevity, the arguments $(t; y, \eta)$ in $u^{(j)}_0$ and $x^{(j)}$, and the arguments $(t, x; y, \eta)$ in $\varphi^{(j)}_t$, $\varphi^{(j)}_x$, $\varphi^{(j)}_\eta$ and $d_{\varphi^{(j)}}$. Recall that the differential operators $L^{(j)}_\alpha$ are defined in accordance with formula (3.2.19) and the coefficients of these operators depend on $(t, x; y, \eta)$.

In view of formulae (3.2.12)–(3.2.17) and (3.2.21), the second term in (3.2.26) reads

$$\mathfrak{S}^{(j)}_0 f^{(j)}_0 = D_t u^{(j)}_0 \\ + \left[(d_{\varphi^{(j)}})^{-1} \left(D_t + (A_1)_{\xi_\alpha} D_{x^\alpha} \right) d_{\varphi^{(j)}} + A_0 - \frac{i}{2} (A_1)_{\xi_\alpha \xi_\beta} C^{(j)}_{\alpha\beta} \right] \Big|_{x = x^{(j)}} u^{(j)}_0 \\ + \big[A_1 - h^{(j)} \big] u^{(j)}_{-1}, \tag{3.3.5}$$

where

$$C^{(j)}_{\alpha\beta} := \varphi^{(j)}_{x^\alpha x^\beta} \Big|_{x = x^{(j)}} \tag{3.3.6}$$

is the matrix-function from (3.2.5). In formulae (3.3.5) and (3.3.6) we dropped, for the sake of brevity, the arguments $(t; y, \eta)$ in $u^{(j)}_0$, $u^{(j)}_{-1}$, $C^{(j)}_{\alpha\beta}$ and $x^{(j)}$, the arguments $(x^{(j)}(t; y, \eta), \xi^{(j)}(t; y, \eta))$ in A_0, A_1, $(A_1)_{\xi_\alpha}$, $(A_1)_{\xi_\alpha \xi_\beta}$ and $h^{(j)}$, and the arguments $(t, x; y, \eta)$ in $d_{\varphi^{(j)}}$ and $\varphi^{(j)}_{x^\alpha x^\beta}$.

Looking at (3.3.4) and (3.3.5) we see that the transport equation (3.2.26) has a complicated structure. Hence, in this section we choose not to perform the analysis of the full equation (3.2.26) and analyse only one particular subequation of this equation. Namely, observe that equation (3.2.26) is equivalent to m subequations

$$\big[v^{(j)} \big]^* \big[\mathfrak{S}^{(j)}_{-1} f^{(j)}_1 + \mathfrak{S}^{(j)}_0 f^{(j)}_0 \big] = 0, \tag{3.3.7}$$

$$\left[v^{(l)}\right]^* \left[\mathfrak{S}_{-1}^{(j)} f_1^{(j)} + \mathfrak{S}_0^{(j)} f_0^{(j)}\right] = 0, \qquad l \neq j, \tag{3.3.8}$$

where we dropped, for the sake of brevity, the arguments $(x^{(j)}(t;y,\eta), \xi^{(j)}(t;y,\eta))$ in $\left[v^{(j)}\right]^*$ and $\left[v^{(l)}\right]^*$. In the remainder of this section we analyse (sub)equation (3.3.7) only.

Equation (3.3.7) is simpler than each of the $m-1$ equations (3.3.8) for the following two reasons.

- Firstly, the term $\left[A_1 - h^{(j)}\right] u_{-1}^{(j)}$ from (3.3.5) vanishes after multiplication by $\left[v^{(j)}\right]^*$ from the left. Hence, equation (3.3.7) does not contain $u_{-1}^{(j)}$.

- Secondly, if we substitute (3.3.2) into (3.3.7), then the term with

$$\partial [d_{\varphi^{(j)}} w^{(j)}(t;y,\eta)]^T / \partial \eta_\beta$$

vanishes. This follows from the fact that the scalar function

$$\left[v^{(j)}\right]^* \big(\varphi_t^{(j)} + A_1(x, \varphi_x^{(j)})\big) v^{(j)}$$

has a second order zero, in the variable x, at $x = x^{(j)}(t;y,\eta)$. Indeed, we have

$$\begin{aligned}
&\left[\frac{\partial}{\partial x^\alpha} \left[v^{(j)}\right]^* \big(\varphi_t^{(j)} + A_1(x, \varphi_x^{(j)})\big) v^{(j)}\right]\Big|_{x=x^{(j)}} \\
&\qquad = \left[v^{(j)}\right]^* \left[\big(\varphi_t^{(j)} + A_1(x, \varphi_x^{(j)})\big)_{x^\alpha}\right]\Big|_{x=x^{(j)}} v^{(j)} \\
&\qquad = \left[v^{(j)}\right]^* \big(-h_{x^\alpha}^{(j)} - C_{\alpha\beta}^{(j)} h_{\xi_\beta}^{(j)} + (A_1)_{x^\alpha} + C_{\alpha\beta}^{(j)} (A_1)_{\xi_\beta}\big) v^{(j)} \\
&\qquad = \left[v^{(j)}\right]^* (A_1)_{x^\alpha} v^{(j)} - h_{x^\alpha}^{(j)} + C_{\alpha\beta}^{(j)} \big(\left[v^{(j)}\right]^* (A_1)_{\xi_\beta} v^{(j)} - h_{\xi_\beta}^{(j)}\big) = 0\,,
\end{aligned}$$

where in the last two lines we dropped, for the sake of brevity, the arguments $(x^{(j)}(t;y,\eta), \xi^{(j)}(t;y,\eta))$ in $(A_1)_{x^\alpha}$, $(A_1)_{\xi_\beta}$, $h_{x^\alpha}^{(j)}$, $h_{\xi_\beta}^{(j)}$, and the argument $(t;y,\eta)$ in $C_{\alpha\beta}^{(j)}$ (the latter is the matrix-function from formulae (3.2.5) and (3.3.6)). Throughout the above argument we used the fact that our $\left[v^{(j)}\right]^*$ and $v^{(j)}$ do not depend on x: their argument is $(x^{(j)}(t;y,\eta), \xi^{(j)}(t;y,\eta))$.

Substituting (3.3.4), (3.3.5) and (3.3.2) into (3.3.7) we get

$$(D_t + p^{(j)}(t;y,\eta))\,[w^{(j)}(t;y,\eta)]^T = 0\,, \tag{3.3.9}$$

where

$$\begin{aligned} p^{(j)} = i\,[v^{(j)}]^* \left[\frac{\partial}{\partial\eta_\beta}\left(1-\frac{1}{2}\varphi^{(j)}_{\eta_\alpha}L^{(j)}_\alpha\right)\left(L^{(j)}_\beta\big(\varphi^{(j)}_t + A_1(x,\varphi^{(j)}_x)\big)\right)v^{(j)}\right]\Big|_{x=x^{(j)}} \\ - i[v^{(j)}]^*\{v^{(j)},h^{(j)}\} + \left[(d_{\varphi^{(j)}})^{-1}\left(D_t + h^{(j)}_{\xi_\alpha}D_{x^\alpha}\right)d_{\varphi^{(j)}}\right]\Big|_{x=x^{(j)}} \\ + [v^{(j)}]^*\left(A_0 - \frac{i}{2}(A_1)_{\xi_\alpha\xi_\beta}C^{(j)}_{\alpha\beta}\right)v^{(j)}. \end{aligned} \tag{3.3.10}$$

Note that the ordinary differential operator in the LHS of formula (3.3.9) is a scalar one, i.e. it does not mix up the different components of the column-function $w^{(j)}(t;y,\eta)$. The solution of the ordinary differential equation (3.3.9) subject to the initial condition (3.3.3) is

$$w^{(j)}(t;y,\eta) = \overline{v^{(j)}(y,\eta)}\exp\left(-i\int_0^t p^{(j)}(\tau;y,\eta)\,d\tau\right). \tag{3.3.11}$$

Comparing formulae (3.3.2), (3.3.11) with formula (3.1.10) we see that in order to prove the latter we need only to establish the scalar identity

$$p^{(j)}(t;y,\eta) = q^{(j)}(x^{(j)}(t;y,\eta),\xi^{(j)}(t;y,\eta))\,, \tag{3.3.12}$$

where $q^{(j)}$ is the function (3.1.12). In view of the definitions of the quantities $p^{(j)}$ and $q^{(j)}$, see formulae (3.3.10) and (3.1.12), and the definition of the subprincipal symbol (3.1.13), proving the identity (3.3.12) reduces to proving the identity

$$\begin{aligned} &\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\}(x^{(j)},\xi^{(j)}) = \\ &-2\,[v^{(j)}(x^{(j)},\xi^{(j)})]^*\left[\frac{\partial}{\partial\eta_\beta}\left(1-\frac{1}{2}\varphi^{(j)}_{\eta_\alpha}L^{(j)}_\alpha\right)\left(L^{(j)}_\beta\big(\varphi^{(j)}_t + A_1(x,\varphi^{(j)}_x)\big)\right)v^{(j)}(x^{(j)},\xi^{(j)})\right]\Big|_{x=x^{(j)}} \\ &\qquad + 2\left[(d_{\varphi^{(j)}})^{-1}\left(\partial_t + h^{(j)}_{\xi_\alpha}\partial_{x^\alpha}\right)d_{\varphi^{(j)}}\right]\Big|_{x=x^{(j)}} \\ &\qquad + [v^{(j)}(x^{(j)},\xi^{(j)})]^*\left((A_1)_{x^\alpha\xi_\alpha} + (A_1)_{\xi_\alpha\xi_\beta}C^{(j)}_{\alpha\beta}\right)v^{(j)}(x^{(j)},\xi^{(j)}). \end{aligned} \tag{3.3.13}$$

Note that the expressions in the LHS and RHS of (3.3.13) have different structure. The LHS of (3.3.13) is the generalised Poisson bracket $\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\}$, see (3.1.15), evaluated at $z = x^{(j)}(t; y, \eta)$, $\zeta = \xi^{(j)}(t; y, \eta)$, whereas the RHS of (3.3.13) involves partial derivatives (in η) of $v^{(j)}(x^{(j)}(t; y, \eta), \xi^{(j)}(t; y, \eta))$ (Chain Rule). In writing (3.3.13) we also dropped, for the sake of brevity, the arguments $(t, x; y, \eta)$ in $\varphi_t^{(j)}$, $\varphi_x^{(j)}$, $\varphi_\eta^{(j)}$, $d_{\varphi^{(j)}}$ and the coefficients of the differential operators $L_\alpha^{(j)}$ and $L_\beta^{(j)}$, the arguments $(x^{(j)}, \xi^{(j)})$ in $h_{\xi_\alpha}^{(j)}$, $(A_1)_{x^\alpha \xi_\alpha}$ and $(A_1)_{\xi_\alpha \xi_\beta}$, and the arguments $(t; y, \eta)$ in $x^{(j)}$, $\xi^{(j)}$ and $C_{\alpha\beta}^{(j)}$.

Before performing the calculations that will establish the identity (3.3.13) we make several observations that will allow us to simplify these calculations considerably.

Firstly, our function $p^{(j)}(t; y, \eta)$ does not depend on the choice of the phase function $\varphi^{(j)}(t, x; y, \eta)$. Indeed, if $p^{(j)}(t; y, \eta)$ did depend on the choice of phase function, then, in view of formulae (3.3.2) and (3.3.11) the principal symbol of our oscillatory integral $U^{(j)}(t)$ would depend on the choice of phase function, which would contradict Theorem 2.7.11 from [45]. Here we use the fact that operators $U^{(j)}(t)$ with different j cannot compensate each other to give an integral operator whose integral kernel is infinitely smooth in t, x and y because all our $U^{(j)}(t)$ oscillate in t in a different way: $\varphi_t^{(j)}(t, x^{(j)}(t; y, \eta); y, \eta) = -h^{(j)}(y, \eta)$ and we assumed the eigenvalues $h^{(j)}(y, \eta)$ of our principal symbol $A_1(y, \eta)$ to be simple.

Secondly, the arguments (free variables) in (3.3.13) are $(t; y, \eta)$. We fix an arbitrary point $(\tilde{t}; \tilde{y}, \tilde{\eta}) \in \mathbb{R} \times T'M$ and prove formula (3.3.13) at this point. Put $(\xi_\eta^{(j)})_\alpha{}^\beta := \partial(\xi^{(j)})_\alpha / \partial \eta_\beta$. According to Lemma 2.3.2 from [45] there exists a local coordinate system x such that $\det(\xi_\eta^{(j)})_\alpha{}^\beta \neq 0$. This opens the way to the use of the linear phase function

$$\varphi^{(j)}(t, x; y, \eta) = (x - x^{(j)}(t; y, \eta))^\alpha \, \xi_\alpha^{(j)}(t; y, \eta) \tag{3.3.14}$$

which will simplify calculations to a great extent. Moreover, we can choose a local coordinate system y such that

$$(\xi_\eta^{(j)})_\alpha{}^\beta(\tilde{t};\tilde{y},\tilde{\eta}) = \delta_\alpha{}^\beta \tag{3.3.15}$$

which will simplify calculations even further.

The calculations we are about to perform will make use of the symmetry

$$(x_\eta^{(j)})^{\gamma\alpha}(\xi_\eta^{(j)})_\gamma{}^\beta = (x_\eta^{(j)})^{\gamma\beta}(\xi_\eta^{(j)})_\gamma{}^\alpha \tag{3.3.16}$$

which is an immediate consequence of formula (3.2.8). Formula (3.3.16) appears as formula (2.3.3) in [45] and the accompanying text explains its geometric meaning. Note that at the point $(\tilde{t};\tilde{y},\tilde{\eta})$ formula (3.3.16) takes the especially simple form

$$(x_\eta^{(j)})^{\alpha\beta}(\tilde{t};\tilde{y},\tilde{\eta}) = (x_\eta^{(j)})^{\beta\alpha}(\tilde{t};\tilde{y},\tilde{\eta}). \tag{3.3.17}$$

Our calculations will also involve the quantity $\varphi^{(j)}_{\eta_\alpha\eta_\beta}(\tilde{t},\tilde{x};\tilde{y},\tilde{\eta})$ where $\tilde{x} := x^{(j)}(\tilde{t};\tilde{y},\tilde{\eta})$. Formulae (3.3.14), (3.2.8), (3.3.15) and (3.3.17) imply

$$\varphi^{(j)}_{\eta_\alpha\eta_\beta}(\tilde{t},\tilde{x};\tilde{y},\tilde{\eta}) = -(x_\eta^{(j)})^{\alpha\beta}(\tilde{t};\tilde{y},\tilde{\eta}). \tag{3.3.18}$$

Further on we denote $\tilde{\xi} := \xi^{(j)}(\tilde{t};\tilde{y},\tilde{\eta})$.

With account of all the simplifications listed above, we can rewrite formula (3.3.13), which is the identity that we are proving, as

$$\begin{aligned}
\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\}(\tilde{x},\tilde{\xi}) = \\
- 2[\tilde{v}^{(j)}]^*\Big[\frac{\partial^2}{\partial x^\alpha \partial \eta_\alpha}\big(A_1(x,\xi^{(j)}) - h^{(j)}(\tilde{y},\eta) \\
- (x-x^{(j)})^\gamma h^{(j)}_{x^\gamma}(x^{(j)},\xi^{(j)})\big)\, v^{(j)}(x^{(j)},\xi^{(j)})\Big]\Big|_{(x,\eta)=(\tilde{x},\tilde{\eta})} \\
- (\tilde{x}^{(j)}_\eta)^{\alpha\beta}\,[\tilde{v}^{(j)}]^*\Big[\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\big(A_1(x,\xi^{(j)}) - h^{(j)}(\tilde{y},\eta) \\
- (x-x^{(j)})^\gamma h^{(j)}_{x^\gamma}(x^{(j)},\xi^{(j)})\big)\, v^{(j)}(x^{(j)},\xi^{(j)})\Big]\Big|_{(x,\eta)=(\tilde{x},\tilde{\eta})} \\
+ [\tilde{v}^{(j)}]^*(\tilde{A}_1)_{x^\alpha\xi_\alpha}\tilde{v}^{(j)} - \tilde{h}^{(j)}_{x^\alpha\xi_\alpha} - \tilde{h}^{(j)}_{x^\alpha x^\beta}(\tilde{x}^{(j)}_\eta)^{\alpha\beta}\,, \qquad (3.3.19)
\end{aligned}$$

where $\tilde{v}^{(j)} = v^{(j)}(\tilde{x},\tilde{\xi})$, $\tilde{x}^{(j)}_\eta = x^{(j)}_\eta(\tilde{t};\tilde{y},\tilde{\eta})$, $(\tilde{A}_1)_{x^\alpha\xi_\alpha} = (A_1)_{x^\alpha\xi_\alpha}(\tilde{x},\tilde{\xi})$, $\tilde{h}^{(j)}_{x^\alpha\xi_\alpha} = h^{(j)}_{x^\alpha\xi_\alpha}(\tilde{x},\tilde{\xi})$, $\tilde{h}^{(j)}_{x^\alpha x^\beta} = h^{(j)}_{x^\alpha x^\beta}(\tilde{x},\tilde{\xi})$, $x^{(j)} = x^{(j)}(\tilde{t};\tilde{y},\eta)$ and $\xi^{(j)} = \xi^{(j)}(\tilde{t};\tilde{y},\eta)$.

Note that the last two terms in the RHS of (3.3.19) originate from the term with $d_{\varphi^{(j)}}$ in (3.3.13): we used the fact that $d_{\varphi^{(j)}}$ does not depend on x and that

$$\big[(d_{\varphi^{(j)}})^{-1}\partial_t d_{\varphi^{(j)}}\big]\big|_{(t,x;y,\eta)=(\tilde{t},\tilde{x};\tilde{y},\tilde{\eta})} = -\frac{1}{2}\big(\tilde{h}^{(j)}_{x^\alpha\xi_\alpha} + \tilde{h}^{(j)}_{x^\alpha x^\beta}(\tilde{x}^{(j)}_\eta)^{\alpha\beta}\big). \qquad (3.3.20)$$

Formula (3.3.20) is a special case of formula (3.3.21) from [45].

Note also that the term $-h^{(j)}(\tilde{y},\eta)$ appearing (twice) in the RHS of (3.3.19) will vanish after being acted upon with the differential operators $\frac{\partial^2}{\partial x^\alpha \partial \eta_\alpha}$ and $\frac{\partial^2}{\partial x^\alpha \partial x^\beta}$ because it does not depend on x.

We have

$$\begin{aligned}
[\tilde{v}^{(j)}]^*\,\Big[\frac{\partial^2}{\partial x^\alpha \partial \eta_\alpha}\big(A_1(x,\xi^{(j)}) - (x-x^{(j)})^\gamma h^{(j)}_{x^\gamma}(x^{(j)},\xi^{(j)})\big)\, v^{(j)}(x^{(j)},\xi^{(j)})\Big]\Big|_{(x,\eta)=(\tilde{x},\tilde{\eta})} \\
= [\tilde{v}^{(j)}]^*(\tilde{A}_1)_{x^\alpha\xi_\alpha}\tilde{v}^{(j)} - \tilde{h}^{(j)}_{x^\alpha\xi_\alpha} - \tilde{h}^{(j)}_{x^\alpha x^\beta}(\tilde{x}^{(j)}_\eta)^{\alpha\beta} \\
+ [\tilde{v}^{(j)}]^*\big((\tilde{A}_1)_{x^\alpha} - \tilde{h}^{(j)}_{x^\alpha}\big)\big(\tilde{v}^{(j)}_{\xi_\alpha} + \tilde{v}^{(j)}_{x^\beta}(\tilde{x}^{(j)}_\eta)^{\alpha\beta}\big), \qquad (3.3.21)
\end{aligned}$$

$$[\tilde{v}^{(j)}]^* \left[\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\big(A_1(x,\xi^{(j)}) - (x-x^{(j)})^\gamma h^{(j)}_{x^\gamma}(x^{(j)},\xi^{(j)})\big)\, v^{(j)}(x^{(j)},\xi^{(j)})\right]\Big|_{(x,\eta)=(\tilde{x},\tilde{\eta})} = [\tilde{v}^{(j)}]^*(\tilde{A}_1)_{x^\alpha x^\beta}\tilde{v}^{(j)}\,, \quad (3.3.22)$$

where $(\tilde{A}_1)_{x^\alpha} = (A_1)_{x^\alpha}(\tilde{x},\tilde{\xi})$, $\tilde{h}^{(j)}_{x^\alpha} = h^{(j)}_{x^\alpha}(\tilde{x},\tilde{\xi})$, $\tilde{v}^{(j)}_{\xi_\alpha} = v^{(j)}_{\xi_\alpha}(\tilde{x},\tilde{\xi})$ and $\tilde{v}^{(j)}_{x^\beta} = v^{(j)}_{x^\beta}(\tilde{x},\tilde{\xi})$. We also have

$$[\tilde{v}^{(j)}]^*\big((\tilde{A}_1)_{x^\alpha} - \tilde{h}^{(j)}_{x^\alpha}\big)\tilde{v}^{(j)}_{x^\beta} + [\tilde{v}^{(j)}]^*\big((\tilde{A}_1)_{x^\beta} - \tilde{h}^{(j)}_{x^\beta}\big)\tilde{v}^{(j)}_{x^\alpha} = \tilde{h}^{(j)}_{x^\alpha x^\beta} - [\tilde{v}^{(j)}]^*(\tilde{A}_1)_{x^\alpha x^\beta}\tilde{v}^{(j)}. \quad (3.3.23)$$

Using formulae (3.3.23) and (3.3.17) we can rewrite formula (3.3.21) as

$$\begin{aligned}[\tilde{v}^{(j)}]^* \left[\frac{\partial^2}{\partial x^\alpha \partial \eta_\alpha}\big(A_1(x,\xi^{(j)}) - (x-x^{(j)})^\gamma h^{(j)}_{x^\gamma}(x^{(j)},\xi^{(j)})\big)\, v^{(j)}(x^{(j)},\xi^{(j)})\right]\Big|_{(x,\eta)=(\tilde{x},\tilde{\eta})} \\ = [\tilde{v}^{(j)}]^*(\tilde{A}_1)_{x^\alpha\xi_\alpha}\tilde{v}^{(j)} - \tilde{h}^{(j)}_{x^\alpha\xi_\alpha} + [\tilde{v}^{(j)}]^*\big((\tilde{A}_1)_{x^\alpha} - \tilde{h}^{(j)}_{x^\alpha}\big)\tilde{v}^{(j)}_{\xi_\alpha} \\ - \frac{1}{2}\big([\tilde{v}^{(j)}]^*(\tilde{A}_1)_{x^\alpha x^\beta}\tilde{v}^{(j)} + \tilde{h}^{(j)}_{x^\alpha x^\beta}\big)(\tilde{x}^{(j)}_\eta)^{\alpha\beta}. \quad (3.3.24)\end{aligned}$$

Substituting (3.3.24) and (3.3.22) into (3.3.19) we see that all the terms with $(\tilde{x}^{(j)}_\eta)^{\alpha\beta}$ cancel out and we get

$$\begin{aligned}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\}(\tilde{x},\tilde{\xi}) = \\ - [\tilde{v}^{(j)}]^*\big((\tilde{A}_1)_{x^\alpha\xi_\alpha} - \tilde{h}^{(j)}_{x^\alpha\xi_\alpha}\big)\tilde{v}^{(j)} - 2[\tilde{v}^{(j)}]^*\big((\tilde{A}_1)_{x^\alpha} - \tilde{h}^{(j)}_{x^\alpha}\big)\tilde{v}^{(j)}_{\xi_\alpha}. \quad (3.3.25)\end{aligned}$$

Thus, the proof of the identity (3.3.13) has been reduced to the proof of the identity (3.3.25).

Observe now that formula (3.3.25) no longer has Hamiltonian trajectories present in it. This means that we can drop all the tildes and rewrite (3.3.25) as

$$\begin{aligned}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\} = \\ - [v^{(j)}]^*\big(A_1 - h^{(j)}\big)_{x^\alpha\xi_\alpha} v^{(j)} - 2[v^{(j)}]^*\big(A_1 - h^{(j)}\big)_{x^\alpha} v^{(j)}_{\xi_\alpha}\,, \quad (3.3.26)\end{aligned}$$

where the arguments are (x,ξ). We no longer need to restrict our consideration

to the particular point $(x,\xi)=(\tilde{x},\tilde{\xi})$: if we prove (3.3.26) for an arbitrary $(x,\xi)\in T'M$ we will prove it for a particular $(\tilde{x},\tilde{\xi})\in T'M$.

The proof of the identity (3.3.26) is straightforward. We note that

$$[v^{(j)}]^*(A_1-h^{(j)})_{x^\alpha\xi_\alpha}v^{(j)}= \\ -[v^{(j)}]^*(A_1-h^{(j)})_{x^\alpha}v^{(j)}_{\xi_\alpha}-[v^{(j)}]^*(A_1-h^{(j)})_{\xi_\alpha}v^{(j)}_{x^\alpha} \quad (3.3.27)$$

and substituting (3.3.27) into (3.3.26) reduce the latter to the form

$$\{[v^{(j)}]^*,A_1-h^{(j)},v^{(j)}\}= \\ [v^{(j)}]^*\big(A_1-h^{(j)}\big)_{\xi_\alpha}v^{(j)}_{x^\alpha}-[v^{(j)}]^*\big(A_1-h^{(j)}\big)_{x^\alpha}v^{(j)}_{\xi_\alpha}. \quad (3.3.28)$$

But

$$[v^{(j)}]^*\big(A_1-h^{(j)}\big)_{x^\alpha}=-[v^{(j)}_{x^\alpha}]^*\big(A_1-h^{(j)}\big), \quad (3.3.29)$$

$$[v^{(j)}]^*\big(A_1-h^{(j)}\big)_{\xi_\alpha}=-[v^{(j)}_{\xi_\alpha}]^*\big(A_1-h^{(j)}\big). \quad (3.3.30)$$

Substituting (3.3.29) and (3.3.30) into (3.3.28) we get

$$\{[v^{(j)}]^*,A_1-h^{(j)},v^{(j)}\}=[v^{(j)}_{x^\alpha}]^*\big(A_1-h^{(j)}\big)v^{(j)}_{\xi_\alpha}-[v^{(j)}_{\xi_\alpha}]^*\big(A_1-h^{(j)}\big)v^{(j)}_{x^\alpha}$$

which agrees with the definition of the generalised Poisson bracket (3.1.15).

3.4 Proof of formula (3.1.18)

In this section we prove formula (3.1.18). Our approach is as follows.

We write down explicitly the transport equations (3.3.8) at $t=0$, i.e.

$$\big[v^{(l)}\big]^*\ \big[\mathfrak{S}^{(j)}_{-1}f^{(j)}_1+\mathfrak{S}^{(j)}_0f^{(j)}_0\big]\Big|_{t=0}=0, \qquad l\neq j. \quad (3.4.1)$$

We use the same local coordinates for x and y and we assume all our phase functions to be linear, i.e. we assume that for each j we have (3.3.14). Using linear phase functions is justified for small t because we have $(\xi_\eta^{(j)})_\alpha{}^\beta(0;y,\eta) = \delta_\alpha{}^\beta$ and, hence, $\det \varphi^{(j)}_{x^\alpha \eta_\beta}(t,x;y,\eta) \neq 0$ for small t. Writing down equations (3.4.1) for linear phase functions is much easier than for general phase functions (3.2.2).

Using linear phase functions has the additional advantage that the initial condition (3.2.28) simplifies and reads now $\sum_j u^{(j)}(0;y,\eta) = I$. In view of (3.1.9), this implies, in particular, that

$$\sum_j u_{-1}^{(j)}(0) = 0. \tag{3.4.2}$$

Here and further on in this section we drop, for the sake of brevity, the arguments (y,η) in $u_{-1}^{(j)}$.

Of course, the formula we are proving, formula (3.1.18), does not depend on our choice of phase functions. It is just easier to carry out calculations for linear phase functions.

We will show that (3.4.1) is a system of complex linear algebraic equations for the unknowns $u_{-1}^{(j)}(0)$. The total number of equations (3.4.1) is $m^2 - m$. However, for each j and l the LHS of (3.4.1) is a row of m elements, so (3.4.1) is, effectively, a system of $m(m^2-m)$ scalar equations.

Equation (3.4.2) is a single matrix equation, so it is, effectively, a system of m^2 scalar equations.

Consequently, the system (3.4.1), (3.4.2) is, effectively, a system of m^3 scalar equations. This is exactly the number of unknown scalar elements in the m matrices $u_{-1}^{(j)}(0)$.

In the remainder of this section we write down explicitly the LHS of (3.4.1) and solve the linear algebraic system (3.4.1), (3.4.2) for the unknowns $u_{-1}^{(j)}(0)$. This will allow us to prove formula (3.1.18).

Before starting explicit calculations we observe that equations (3.4.1) can be equivalently rewritten as

$$P^{(l)}\ \left[\mathfrak{S}^{(j)}_{-1}f^{(j)}_1+\mathfrak{S}^{(j)}_0 f^{(j)}_0\right]\Big|_{t=0}=0,\qquad l\neq j, \tag{3.4.3}$$

where $P^{(l)}:=[v^{(l)}(y,\eta)]\,[v^{(l)}(y,\eta)]^*$ is the orthogonal projection onto the eigenspace corresponding to the (normalised) eigenvector $v^{(l)}(y,\eta)$ of the principal symbol. We will deal with (3.4.3) rather than with (3.4.1). This is simply a matter of convenience.

3.4.1 Part 1 of the proof of formula (3.1.18)

Our task in this subsection is to calculate the LHS of (3.4.3). In our calculations we use the explicit formula (3.1.10) for the principal symbol $u^{(j)}_0(t;y,\eta)$ which was proved in Section 3.3.

At $t=0$ formula (3.3.4) reads

$$\left[\mathfrak{S}^{(j)}_{-1}f^{(j)}_1\right]\Big|_{t=0}=i\left[\frac{\partial^2}{\partial x^\alpha\eta_\alpha}\big(A_1(x,\eta)-h^{(j)}(y,\eta)-(x-y)^\gamma h^{(j)}_{y^\gamma}(y,\eta)\big)P^{(j)}(y,\eta)\right]\Big|_{x=y}$$

which gives us

$$\left[\mathfrak{S}^{(j)}_{-1}f^{(j)}_1\right]\Big|_{t=0}=i\left[(A_1-h^{(j)})_{y^\alpha\eta_\alpha}P^{(j)}+(A_1-h^{(j)})_{y^\alpha}P^{(j)}_{\eta_\alpha}\right]. \tag{3.4.4}$$

In the latter formula we dropped, for the sake of brevity, the arguments (y,η).

At $t=0$ formula (3.3.5) reads

$$\left[\mathfrak{S}^{(j)}_0 f^{(j)}_0\right]\Big|_{t=0}=-i\{v^{(j)},h^{(j)}\}[v^{(j)}]^*+\left(A_0-q^{(j)}+\frac{i}{2}h^{(j)}_{y^\alpha\eta_\alpha}\right)P^{(j)}$$
$$+\,[A_1-h^{(j)}]u^{(j)}_{-1}(0)\,, \tag{3.4.5}$$

where $q^{(j)}$ is the function (3.1.12) and we dropped, for the sake of brevity, the arguments (y,η). Note that in writing down (3.4.5) we used the fact that

$$\left[(d_{\varphi^{(j)}})^{-1}\partial_t d_{\varphi^{(j)}}\right]\Big|_{(t,x;y,\eta)=(0,y;y,\eta)} = -\frac{1}{2}h^{(j)}_{y^\alpha\eta_\alpha}(y,\eta)\,,$$

compare with formula (3.3.20).

Substituting formulae (3.4.4) and (3.4.5) into (3.4.3) we get

$$(h^{(l)}-h^{(j)})P^{(l)}u^{(j)}_{-1}(0)+P^{(l)}B^{(j)}_0=0,\qquad l\neq j, \tag{3.4.6}$$

where

$$B^{(j)}_0=\left(A_0-q^{(j)}-\frac{i}{2}h^{(j)}_{y^\alpha\eta_\alpha}+i(A_1)_{y^\alpha\eta_\alpha}\right)P^{(j)}-ih^{(j)}_{\eta_\alpha}P^{(j)}_{y^\alpha}+i(A_1)_{y^\alpha}P^{(j)}_{\eta_\alpha}. \tag{3.4.7}$$

The subscript in $B^{(j)}_0$ indicates the degree of homogeneity in η.

3.4.2 Part 2 of the proof of formula (3.1.18)

Our task in this subsection is to solve the linear algebraic system (3.4.6), (3.4.2) for the unknowns $u^{(j)}_{-1}(0)$.

It is easy to see that the unique solution to the system (3.4.6), (3.4.2) is

$$u^{(j)}_{-1}(0)=\sum_{l\neq j}\frac{P^{(l)}B^{(j)}_0+P^{(j)}B^{(l)}_0}{h^{(j)}-h^{(l)}}\,. \tag{3.4.8}$$

Summation in (3.4.8) is carried out over all l different from j.

3.4.3 Part 3 of the proof of formula (3.1.18)

Our task in this subsection is to calculate $[U^{(j)}(0)]_{\rm sub}$.

We have

$$[U^{(j)}(0)]_{\text{sub}} = u_{-1}^{(j)}(0) - \frac{i}{2}P^{(j)}_{y^\alpha\eta_\alpha}. \tag{3.4.9}$$

Here the sign in front of $\frac{i}{2}$ is opposite to that in (3.1.13) because the way we write $U^{(j)}(0)$ is using the dual symbol.

Substituting (3.4.8) and (3.4.7) into (3.4.9) we get

$$\begin{aligned}
[U^{(j)}(0)]_{\text{sub}} &= -\frac{i}{2}P^{(j)}_{y^\alpha\eta_\alpha} + \sum_{l\neq j}\frac{1}{h^{(j)}-h^{(l)}} \\
&\qquad \times \big(P^{(l)}[(A_0 + i(A_1)_{y^\alpha\eta_\alpha})P^{(j)} - ih^{(j)}_{\eta_\alpha}P^{(j)}_{y^\alpha} + i(A_1)_{y^\alpha}P^{(j)}_{\eta_\alpha}] \\
&\qquad\quad + P^{(j)}[(A_0 + i(A_1)_{y^\alpha\eta_\alpha})P^{(l)} - ih^{(l)}_{\eta_\alpha}P^{(l)}_{y^\alpha} + i(A_1)_{y^\alpha}P^{(l)}_{\eta_\alpha}]\big) \\
&= \sum_{l\neq j}\frac{P^{(l)}A_{\text{sub}}P^{(j)} + P^{(j)}A_{\text{sub}}P^{(l)}}{h^{(j)}-h^{(l)}} + \frac{i}{2}\Big(-P^{(j)}_{y^\alpha\eta_\alpha} + \sum_{l\neq j}\frac{G_{jl}}{h^{(j)}-h^{(l)}}\Big),
\end{aligned} \tag{3.4.10}$$

where

$$\begin{aligned}
G_{jl} := P^{(l)}&[(A_1)_{y^\alpha\eta_\alpha}P^{(j)} - 2h^{(j)}_{\eta_\alpha}P^{(j)}_{y^\alpha} + 2(A_1)_{y^\alpha}P^{(j)}_{\eta_\alpha}] \\
&+ P^{(j)}[(A_1)_{y^\alpha\eta_\alpha}P^{(l)} - 2h^{(l)}_{\eta_\alpha}P^{(l)}_{y^\alpha} + 2(A_1)_{y^\alpha}P^{(l)}_{\eta_\alpha}].
\end{aligned}$$

We have

$$\begin{aligned}
G_{jl} &= 2P^{(l)}\{A_1, P^{(j)}\} + 2P^{(j)}\{A_1, P^{(l)}\} \\
&\qquad + P^{(l)}[(A_1 - h^{(j)})_{y^\alpha\eta_\alpha}P^{(j)} + 2(A_1 - h^{(j)})_{\eta_\alpha}P^{(j)}_{y^\alpha}] \\
&\qquad + P^{(j)}[(A_1 - h^{(l)})_{y^\alpha\eta_\alpha}P^{(l)} + 2(A_1 - h^{(l)})_{\eta_\alpha}P^{(l)}_{y^\alpha}] \\
&= 2P^{(l)}\{A_1, P^{(j)}\} + 2P^{(j)}\{A_1, P^{(l)}\} - P^{(l)}\{A_1 - h^{(j)}, P^{(j)}\} - P^{(j)}\{A_1 - h^{(l)}, P^{(l)}\} \\
&\qquad + P^{(l)}[(A_1 - h^{(j)})_{y^\alpha\eta_\alpha}P^{(j)} + (A_1 - h^{(j)})_{\eta_\alpha}P^{(j)}_{y^\alpha} + (A_1 - h^{(j)})_{y^\alpha}P^{(j)}_{\eta_\alpha}] \\
&\qquad + P^{(j)}[(A_1 - h^{(l)})_{y^\alpha\eta_\alpha}P^{(l)} + (A_1 - h^{(l)})_{\eta_\alpha}P^{(l)}_{y^\alpha} + (A_1 - h^{(l)})_{y^\alpha}P^{(l)}_{\eta_\alpha}] \\
&\qquad\qquad = P^{(l)}\{A_1 + h^{(j)}, P^{(j)}\} + P^{(j)}\{A_1 + h^{(l)}, P^{(l)}\} \\
&\qquad\qquad - P^{(l)}(A_1 - h^{(j)})P^{(j)}_{y^\alpha\eta_\alpha} - P^{(j)}(A_1 - h^{(l)})P^{(l)}_{y^\alpha\eta_\alpha} \\
&\qquad\qquad = P^{(l)}\{A_1 + h^{(j)}, P^{(j)}\} + P^{(j)}\{A_1 + h^{(l)}, P^{(l)}\} \\
&\qquad\qquad\qquad - P^{(l)}(h^{(l)} - h^{(j)})P^{(j)}_{y^\alpha\eta_\alpha} - P^{(j)}(h^{(j)} - h^{(l)})P^{(l)}_{y^\alpha\eta_\alpha}
\end{aligned}$$

$$= P^{(l)}\{A_1+h^{(j)}, P^{(j)}\}+P^{(j)}\{A_1+h^{(l)}, P^{(l)}\}+(h^{(j)}-h^{(l)})(P^{(l)}P^{(j)}_{y^\alpha\eta_\alpha}-P^{(j)}P^{(l)}_{y^\alpha\eta_\alpha})\,,$$

so formula (3.4.10) can be rewritten as

$$[U^{(j)}(0)]_{\rm sub} = \frac{i}{2}\Big(-P^{(j)}_{y^\alpha\eta_\alpha} + \sum_{l\neq j}(P^{(l)}P^{(j)}_{y^\alpha\eta_\alpha} - P^{(j)}P^{(l)}_{y^\alpha\eta_\alpha})\Big)$$
$$+\frac{1}{2}\sum_{l\neq j}\frac{P^{(l)}(2A_{\rm sub}P^{(j)} + i\{A_1+h^{(j)}, P^{(j)}\}) + P^{(j)}(2A_{\rm sub}P^{(l)} + i\{A_1+h^{(l)}, P^{(l)}\})}{h^{(j)}-h^{(l)}}\,. \tag{3.4.11}$$

But

$$\sum_{l\neq j}(P^{(l)}P^{(j)}_{y^\alpha\eta_\alpha} - P^{(j)}P^{(l)}_{y^\alpha\eta_\alpha}) = \Big(\sum_{l\neq j}P^{(l)}\Big)P^{(j)}_{y^\alpha\eta_\alpha} - P^{(j)}\Big(\sum_{l\neq j}P^{(l)}\Big)_{y^\alpha\eta_\alpha}$$
$$= (I-P^{(j)})P^{(j)}_{y^\alpha\eta_\alpha} - P^{(j)}(I-P^{(j)})_{y^\alpha\eta_\alpha} = P^{(j)}_{y^\alpha\eta_\alpha},$$

so formula (3.4.11) can be simplified to read

$$[U^{(j)}(0)]_{\rm sub}$$
$$= \frac{1}{2}\sum_{l\neq j}\frac{P^{(l)}(2A_{\rm sub}P^{(j)} + i\{A_1+h^{(j)}, P^{(j)}\}) + P^{(j)}(2A_{\rm sub}P^{(l)} + i\{A_1+h^{(l)}, P^{(l)}\})}{h^{(j)}-h^{(l)}}\,. \tag{3.4.12}$$

3.4.4 Part 4 of the proof of formula (3.1.18)

Our task in this subsection is to calculate $\operatorname{tr}[U^{(j)}(0)]_{\rm sub}$.

Formula (3.4.12) implies

$$\operatorname{tr}[U^{(j)}(0)]_{\rm sub} = \frac{i}{2}\operatorname{tr}\sum_{l\neq j}\frac{P^{(l)}\{A_1, P^{(j)}\} + P^{(j)}\{A_1, P^{(l)}\}}{h^{(j)}-h^{(l)}}\,. \tag{3.4.13}$$

Put $A_1 = \sum_k h^{(k)}P^{(k)}$ and observe that

- terms with the derivatives of h vanish and

- the only k which may give nonzero contributions are $k = j$ and $k = l$.

Thus, formula (3.4.13) becomes

$$\operatorname{tr}[U^{(j)}(0)]_{\mathrm{sub}} = \frac{i}{2}\operatorname{tr}\sum_{l\neq j}\frac{1}{h^{(j)}-h^{(l)}}$$
$$\times\big(h^{(j)}[P^{(l)}\{P^{(j)},P^{(j)}\}+P^{(j)}\{P^{(j)},P^{(l)}\}]+h^{(l)}[P^{(l)}\{P^{(l)},P^{(j)}\}+P^{(j)}\{P^{(l)},P^{(l)}\}]\big). \tag{3.4.14}$$

We claim that

$$\begin{aligned}\operatorname{tr}(P^{(l)}\{P^{(j)},P^{(j)}\}) &= \operatorname{tr}(P^{(j)}\{P^{(j)},P^{(l)}\})\\ &= -\operatorname{tr}(P^{(l)}\{P^{(l)},P^{(j)}\}) = -\operatorname{tr}(P^{(j)}\{P^{(l)},P^{(l)}\})\\ &= [v^{(l)}]^*\{v^{(j)},[v^{(j)}]^*\}v^{(l)}\\ &= ([v^{(l)}]^*v^{(j)}_{y^\alpha})([v^{(j)}_{\eta_\alpha}]^*v^{(l)}) - ([v^{(l)}]^*v^{(j)}_{\eta_\alpha})([v^{(j)}_{y^\alpha}]^*v^{(l)}).\end{aligned} \tag{3.4.15}$$

These facts are established by writing the orthogonal projections in terms of the eigenvectors and using, if required, the identities

$$[v^{(l)}_{y^\alpha}]^*v^{(j)} + [v^{(l)}]^*v^{(j)}_{y^\alpha} = 0, \qquad [v^{(l)}_{\eta_\alpha}]^*v^{(j)} + [v^{(l)}]^*v^{(j)}_{\eta_\alpha} = 0,$$

$$[v^{(j)}_{y^\alpha}]^*v^{(l)} + [v^{(j)}]^*v^{(l)}_{y^\alpha} = 0, \qquad [v^{(j)}_{\eta_\alpha}]^*v^{(l)} + [v^{(j)}]^*v^{(l)}_{\eta_\alpha} = 0.$$

In view of the identities (3.4.15) formula (3.4.14) can be rewritten as

$$\begin{aligned}\operatorname{tr}[U^{(j)}(0)]_{\mathrm{sub}} &= i\operatorname{tr}\sum_{l\neq j}P^{(l)}\{P^{(j)},P^{(j)}\}\\ &= i\operatorname{tr}(\{P^{(j)},P^{(j)}\} - P^{(j)}\{P^{(j)},P^{(j)}\}) = -i\operatorname{tr}(P^{(j)}\{P^{(j)},P^{(j)}\}).\end{aligned} \tag{3.4.16}$$

It remains only to simplify the expression in the RHS of (3.4.16). We have

$$\begin{aligned}
\mathrm{tr}(P^{(j)}\{P^{(j)},P^{(j)}\}) &= \{[v^{(j)}]^*, v^{(j)}\} \\
&\quad + [([v^{(j)}]^* v^{(j)}_{y^\alpha})([v^{(j)}]^* v^{(j)}_{\eta_\alpha}) - ([v^{(j)}]^* v^{(j)}_{\eta_\alpha})([v^{(j)}]^* v^{(j)}_{y^\alpha})] \\
&\quad + [([v^{(j)}_{y^\alpha}]^* v^{(j)})([v^{(j)}_{\eta_\alpha}]^* v^{(j)}) - ([v^{(j)}_{\eta_\alpha}]^* v^{(j)})([v^{(j)}_{y^\alpha}]^* v^{(j)})] \\
&\quad + [([v^{(j)}]^* v^{(j)}_{y^\alpha})([v^{(j)}_{\eta_\alpha}]^* v^{(j)}) - ([v^{(j)}]^* v^{(j)}_{\eta_\alpha})([v^{(j)}_{y^\alpha}]^* v^{(j)})] \\
&= \{[v^{(j)}]^*, v^{(j)}\} + [([v^{(j)}]^* v^{(j)}_{y^\alpha})([v^{(j)}_{\eta_\alpha}]^* v^{(j)}) - ([v^{(j)}]^* v^{(j)}_{\eta_\alpha})([v^{(j)}_{y^\alpha}]^* v^{(j)})] \\
&= \{[v^{(j)}]^*, v^{(j)}\} - [([v^{(j)}]^* v^{(j)}_{y^\alpha})([v^{(j)}]^* v^{(j)}_{\eta_\alpha}) - ([v^{(j)}]^* v^{(j)}_{\eta_\alpha})([v^{(j)}]^* v^{(j)}_{y^\alpha})] \\
&= \{[v^{(j)}]^*, v^{(j)}\}. \quad (3.4.17)
\end{aligned}$$

Formulae (3.4.16) and (3.4.17) imply formula (3.1.18).

3.5 U(1) connection

In the preceding Sections 3.2–3.4 we presented technical details of the construction of the propagator. We saw that the eigenvectors of the principal symbol, $v^{(j)}(x,\xi)$, play a major role in this construction. As pointed out in Section 3.1, each of these eigenvectors is defined up to a U(1) gauge transformation (3.1.16), (3.1.17). In the end, the full symbols (3.1.9) of our oscillatory integrals $U^{(j)}(t)$ do not depend on the choice of gauge for the eigenvectors $v^{(j)}(x,\xi)$. However, the effect of the gauge transformation (3.1.16), (3.1.17) is not as trivial as it may appear at first sight. We will show in this section that the gauge transformation (3.1.16), (3.1.17) show up, in the form of invariantly defined curvature, in the lower order terms $u^{(j)}_{-1}(t;y,\eta)$ of the symbols of our oscillatory integrals $U^{(j)}(t)$. More precisely, we will show that the RHS of formula (3.1.18) is the scalar curvature of a connection associated with the gauge transformation (3.1.16), (3.1.17). Further on in this section, until the very last paragraph, the index j enumerating eigenvalues and eigenvectors of the principal symbol is assumed to be fixed.

Consider a smooth curve $\Gamma \subset T'M$ connecting points (y,η) and (x,ξ). We write this curve in parametric form as $(z(t),\zeta(t))$, $t \in [0,1]$, so that $(z(0),\zeta(0)) = (y,\eta)$ and $(z(1),\zeta(1)) = (x,\xi)$. Put

$$w(t) := e^{i\phi(t)} v^{(j)}(z(t),\zeta(t))\,, \tag{3.5.1}$$

where $\phi : [0,1] \to \mathbb{R}$ is an unknown function which is to be determined from the condition

$$iw^*\dot{w} = 0 \tag{3.5.2}$$

with the dot indicating the derivative with respect to the parameter t. Substituting (3.5.1) into (3.5.2) we get an ordinary differential equation for ϕ which is easily solved, giving

$$\begin{aligned}\phi(1) = \phi(0) + \int_0^1 (\dot{z}^\alpha(t)\, P_\alpha(z(t),\zeta(t)) + \dot{\zeta}_\gamma(t)\, Q^\gamma(z(t),\zeta(t)))\, dt \\ = \phi(0) + \int_\Gamma (P_\alpha dz^\alpha + Q^\gamma d\zeta_\gamma)\,,\end{aligned} \tag{3.5.3}$$

where

$$P_\alpha := i[v^{(j)}]^* v^{(j)}_{z^\alpha}, \qquad Q^\gamma := i[v^{(j)}]^* v^{(j)}_{\zeta_\gamma}. \tag{3.5.4}$$

Note that the $2n$-component real quantity (P_α, Q^γ) is a covector field (1-form) on $T'M$. This quantity already appeared in Section 3.1 as formula (3.1.19).

Put $f(y,\eta) := e^{i\phi(0)}$, $f(x,\xi) := e^{i\phi(1)}$ and rewrite formula (3.5.3) as

$$f(x,\xi) = f(y,\eta)\, e^{i\int_\Gamma (P_\alpha dz^\alpha + Q^\gamma d\zeta_\gamma)}. \tag{3.5.5}$$

Let us identify the group U(1) with the unit circle in the complex plane, i.e. with $f \in \mathbb{C}$, $|f| = 1$. We see that formulae (3.5.5) and (3.5.4) give us a rule for the parallel transport of elements of the group U(1) along curves in $T'M$. This is the natural U(1) connection generated by the normalised field of columns of

complex-valued scalars

$$v^{(j)}(z,\zeta) = \begin{pmatrix} v_1^{(j)}(z,\zeta) & \dots & v_m^{(j)}(z,\zeta) \end{pmatrix}^T. \tag{3.5.6}$$

Recall that the Γ appearing in formula (3.5.5) is a curve connecting points (y,η) and (x,ξ), whereas the $v^{(j)}(z,\zeta)$ appearing in formulae (3.5.4) and (3.5.6) enters our construction as an eigenvector of the principal symbol of our $m \times m$ matrix pseudodifferential operator A.

In practice, dealing with a connection is not as convenient as dealing with the covariant derivative ∇. The covariant derivative corresponding to the connection (3.5.5) is determined as follows. Let us view the (x,ξ) appearing in formula (3.5.5) as a variable which takes values close to (y,η), and suppose that the curve Γ is a short straight (in local coordinates) line segment connecting the point (y,η) with the point (x,ξ). We want the covariant derivative of our function $f(x,\xi)$, evaluated at (y,η), to be zero. Examination of formula (3.5.5) shows that the unique covariant derivative satisfying this condition is

$$\nabla_\alpha := \partial/\partial x^\alpha - iP_\alpha(x,\xi), \qquad \nabla^\gamma := \partial/\partial \xi_\gamma - iQ^\gamma(x,\xi). \tag{3.5.7}$$

We define the curvature of our U(1) connection as

$$R := -i \begin{pmatrix} \nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha & \nabla_\alpha\nabla^\delta - \nabla^\delta\nabla_\alpha \\ \nabla^\gamma\nabla_\beta - \nabla_\beta\nabla^\gamma & \nabla^\gamma\nabla^\delta - \nabla^\delta\nabla^\gamma \end{pmatrix}. \tag{3.5.8}$$

It may seem that the entries of the $(2n) \times (2n)$ matrix (3.5.8) are differential operators. They are, in fact, operators of multiplication by "scalar functions". Namely, the more explicit form of (3.5.8) is

$$R = \begin{pmatrix} \frac{\partial P_\alpha}{\partial x^\beta} - \frac{\partial P_\beta}{\partial x^\alpha} & \frac{\partial P_\alpha}{\partial \xi_\delta} - \frac{\partial Q^\delta}{\partial x^\alpha} \\ \frac{\partial Q^\gamma}{\partial x^\beta} - \frac{\partial P_\beta}{\partial \xi_\gamma} & \frac{\partial Q^\gamma}{\partial \xi_\delta} - \frac{\partial Q^\delta}{\partial \xi_\gamma} \end{pmatrix}. \tag{3.5.9}$$

The $(2n) \times (2n)$-component real quantity (3.5.9) is a rank 2 covariant antisymmetric tensor (2-form) on $T'M$. It is an analogue of the electromagnetic tensor.

Substituting (3.5.4) into (3.5.9) we get an expression for curvature in terms of the eigenvector of the principal symbol

$$R = i \begin{pmatrix} [v^{(j)}_{x^\beta}]^* v^{(j)}_{x^\alpha} - [v^{(j)}_{x^\alpha}]^* v^{(j)}_{x^\beta} & [v^{(j)}_{\xi_\delta}]^* v^{(j)}_{x^\alpha} - [v^{(j)}_{x^\alpha}]^* v^{(j)}_{\xi_\delta} \\ [v^{(j)}_{x^\beta}]^* v^{(j)}_{\xi_\gamma} - [v^{(j)}_{\xi_\gamma}]^* v^{(j)}_{x^\beta} & [v^{(j)}_{\xi_\delta}]^* v^{(j)}_{\xi_\gamma} - [v^{(j)}_{\xi_\gamma}]^* v^{(j)}_{\xi_\delta} \end{pmatrix}. \tag{3.5.10}$$

Examination of formula (3.5.10) shows that, as expected, curvature is invariant under the gauge transformation (3.1.16), (3.1.17).

It is natural to take the trace of the upper right block in (3.5.8) which, in the notation (3.1.14), gives us

$$-i(\nabla_\alpha \nabla^\alpha - \nabla^\alpha \nabla_\alpha) = -i\{[v^{(j)}]^*, v^{(j)}\}. \tag{3.5.11}$$

Thus, we have shown that the RHS of formula (3.1.18) is the scalar curvature of our U(1) connection.

We end this section by proving, as promised in Section 3.1, formula (3.1.20) without referring to microlocal analysis. In the following arguments we use our standard notation for the orthogonal projections onto the eigenspaces of the principal symbol, i.e. we write $P^{(k)} := v^{(k)}[v^{(k)}]^*$. We have $\operatorname{tr}\{P^{(j)}, P^{(j)}\} = 0$ and $\sum_l P^{(l)} = I$ which implies

$$\begin{aligned} 0 &= \sum_{l,j} \operatorname{tr}(P^{(l)}\{P^{(j)}, P^{(j)}\}) \\ &= \sum_j \operatorname{tr}(P^{(j)}\{P^{(j)}, P^{(j)}\}) + \sum_{l,j:\ l\neq j} \operatorname{tr}(P^{(l)}\{P^{(j)}, P^{(j)}\}). \end{aligned} \tag{3.5.12}$$

But, according to formula (3.4.15), for $l \neq j$ we have

$$\operatorname{tr}(P^{(l)}\{P^{(j)}, P^{(j)}\}) = -\operatorname{tr}(P^{(j)}\{P^{(l)}, P^{(l)}\}),$$

so formula (3.5.12) can be rewritten as $\sum_j \operatorname{tr}(P^{(j)}\{P^{(j)}, P^{(j)}\}) = 0$. It remains only to note that, according to formula (3.4.17), $\operatorname{tr}(P^{(j)}\{P^{(j)}, P^{(j)}\}) = \{[v^{(j)}]^*, v^{(j)}\}$.

3.6 Singularity of the propagator at $t = 0$

Following the notation of [45], we denote by

$$\mathcal{F}_{\lambda\to t}[f(\lambda)] = \hat{f}(t) = \int e^{-it\lambda} f(\lambda)\, d\lambda$$

the one-dimensional Fourier transform and by

$$\mathcal{F}^{-1}_{t\to\lambda}[\hat{f}(t)] = f(\lambda) = (2\pi)^{-1} \int e^{it\lambda} \hat{f}(t)\, dt$$

its inverse.

Suppose that we have a Hamiltonian trajectory $(x^{(j)}(t;y,\eta), \xi^{(j)}(t;y,\eta))$ and a real number $T > 0$ such that $x^{(j)}(T;y,\eta) = y$. We will say in this case that we have a loop of length T originating from the point $y \in M$.

Remark 3.6.1. *There is no need to consider loops of negative length T because, given a $T > 0$, we have $x^{(j)}(T;y,\eta^+) = y$ for some $\eta^+ \in T'_yM$ if and only if we have $x^{(j)}(-T;y,\eta^-) = y$ for some $\eta^- \in T'_yM$. Indeed, it suffices to relate the $\eta^\pm$ in accordance with $\eta^\mp = \xi^{(j)}(\pm T;y,\eta^\pm)$.*

Denote by $\mathcal{T}^{(j)} \subset \mathbb{R}$ the set of lengths $T > 0$ of all possible loops generated by the Hamiltonian $h^{(j)}$. Here "all possible" refers to all possible starting points $(y,\eta) \in T'M$ of Hamiltonian trajectories. It is easy to see that $0 \notin \overline{\mathcal{T}^{(j)}}$. We put

$$\mathbf{T}^{(j)} := \begin{cases} \inf \mathcal{T}^{(j)} & \text{if} \quad \mathcal{T}^{(j)} \neq \emptyset, \\ +\infty & \text{if} \quad \mathcal{T}^{(j)} = \emptyset. \end{cases}$$

In the Riemannian case (i.e. the case when the Hamiltonian is a square root of a quadratic polynomial in ξ) it is known [41, 39] that there is a loop originating from every point of the manifold M and, moreover, there is an explicit estimate from above for the number $\mathbf{T}^{(j)}$. We are not aware of similar results for general Hamiltonians.

We also define $\mathbf{T} := \min\limits_{j=1,\dots,m^+} \mathbf{T}^{(j)}$.

Remark 3.6.2. *Note that negative eigenvalues of the principal symbol, i.e. Hamiltonians $h^{(j)}(x,\xi)$ with negative index $j = -1,\dots,-m^-$, do not affect the asymptotic formulae we are about to derive. This is because we are dealing with the case $\lambda \to +\infty$ rather than $\lambda \to -\infty$.*

Denote by

$$u(t,x,y) := \sum_k e^{-it\lambda_k} v_k(x)[v_k(y)]^* \tag{3.6.1}$$

the integral kernel of the propagator (3.1.1). The quantity (3.6.1) can be understood as a distribution in the variable $t \in \mathbb{R}$ depending on the parameters $x, y \in M$.

The main result of this section is the following

Lemma 3.6.1. *Let $\hat\rho : \mathbb{R} \to \mathbb{C}$ be an infinitely smooth function such that*

$$\operatorname{supp}\hat\rho \subset (-\mathbf{T},\mathbf{T}), \tag{3.6.2}$$

$$\hat\rho(0) = 1, \tag{3.6.3}$$

$$\hat\rho'(0) = 0. \tag{3.6.4}$$

Then, uniformly over $y \in M$, we have

$$\mathcal{F}^{-1}_{t\to\lambda}[\hat\rho(t)\operatorname{tr} u(t,y,y)] = n\,a(y)\,\lambda^{n-1} + (n-1)\,b(y)\,\lambda^{n-2} + O(\lambda^{n-3}) \tag{3.6.5}$$

as $\lambda \to +\infty$. The densities $a(y)$ and $b(y)$ appearing in the RHS of formula (3.6.5) are defined in accordance with formulae (3.1.21) and (3.1.22).

Proof Denote by $(S^*_yM)^{(j)}$ the $(n-1)$-dimensional unit cosphere in the cotangent fibre defined by the equation $h^{(j)}(y,\eta) = 1$ and denote by $d(S^*_yM)^{(j)}$ the surface area element on $(S^*_yM)^{(j)}$ defined by the condition $d\eta = d(S^*_yM)^{(j)}\,dh^{(j)}$. The latter means that we use spherical coordinates in the cotangent fibre with the Hamiltonian $h^{(j)}$ playing the role of the radial coordinate, see subsection 1.1.10 of

[45] for details. In particular, as explained in subsection 1.1.10 of [45], our surface area element $d(S^*_yM)^{(j)}$ is expressed via the Euclidean surface area element as

$$d(S^*_yM)^{(j)} = \Big(\sum_{\alpha=1}^{n} \big(h^{(j)}_{\eta_\alpha}(y,\eta)\big)^2 \Big)^{-1/2} \times \text{ Euclidean surface area element}\,.$$

Denote also $đ(S^*_yM)^{(j)} := (2\pi)^{-n}\, d(S^*_yM)^{(j)}$.

According to Corollary 4.1.5 from [45] we have uniformly over $y \in M$

$$\mathcal{F}^{-1}_{t\to\lambda}[\hat\rho(t)\,\mathrm{tr}\,u(t,y,y)] = \sum_{j=1}^{m^+} \big(c^{(j)}(y)\,\lambda^{n-1} + d^{(j)}(y)\,\lambda^{n-2} + e^{(j)}(y)\,\lambda^{n-2}\big) + O(\lambda^{n-3})\,, \quad (3.6.6)$$

where

$$c^{(j)}(y) = \int\limits_{(S^*_yM)^{(j)}} \mathrm{tr}\, u^{(j)}_0(0;y,\eta)\, đ(S^*_yM)^{(j)}\,, \quad (3.6.7)$$

$$d^{(j)}(y) = (n-1) \int\limits_{(S^*_yM)^{(j)}} \mathrm{tr}\left(-\,i\,\dot u^{(j)}_0(0;y,\eta) + \frac{i}{2}\big\{ u^{(j)}_0\big|_{t=0}, h^{(j)}\big\}(y,\eta)\right) đ(S^*_yM)^{(j)}\,, \quad (3.6.8)$$

$$e^{(j)}(y) = \int\limits_{(S^*_yM)^{(j)}} \mathrm{tr}[U^{(j)}(0)]_{\mathrm{sub}}(y,\eta)\, đ(S^*_yM)^{(j)}\,. \quad (3.6.9)$$

Here $u^{(j)}_0(t;y,\eta)$ is the principal symbol of the oscillatory integral (3.2.1) and $\dot u^{(j)}_0(t;y,\eta)$ is its time derivative. Note that in writing the term with the Poisson bracket in (3.6.8) we took account of the fact that Poisson brackets in [45] and in the thesis have opposite signs.

Observe that the integrands in formulae (3.6.7) and (3.6.8) are positively homogeneous in η of degree 0, whereas the integrand in formula (3.6.9) is positively homogeneous in η of degree -1. In order to have the same degree of homogeneity,

we rewrite formula (3.6.9) in equivalent form

$$e^{(j)}(y) = \int\limits_{(S^*_y M)^{(j)}} \big(h^{(j)} \operatorname{tr}[U^{(j)}(0)]_{\mathrm{sub}}\big)(y,\eta)\, đ(S^*_y M)^{(j)}\,. \tag{3.6.10}$$

Switching from surface integrals to volume integrals with the help of formula (1.1.15) from [45], we rewrite formulae (3.6.7), (3.6.8) and (3.6.10) as

$$c^{(j)}(y) = n \int\limits_{h^{(j)}(y,\eta)<1} \operatorname{tr} u_0^{(j)}(0;y,\eta)\, đ\eta\,, \tag{3.6.11}$$

$$d^{(j)}(y) = n(n-1)\times \int\limits_{h^{(j)}(y,\eta)<1} \operatorname{tr}\left(-\,i\,\dot{u}_0^{(j)}(0;y,\eta) + \frac{i}{2}\big\{u_0^{(j)}\big|_{t=0}, h^{(j)}\big\}(y,\eta)\right) đ\eta\,, \tag{3.6.12}$$

$$e^{(j)}(y) = n \int\limits_{h^{(j)}(y,\eta)<1} \big(h^{(j)} \operatorname{tr}[U^{(j)}(0)]_{\mathrm{sub}}\big)(y,\eta)\, đ\eta\,. \tag{3.6.13}$$

Substituting formulae (3.1.10) and (3.1.12) into formulae (3.6.11) and (3.6.12) we get

$$c^{(j)}(y) = n \int\limits_{h^{(j)}(y,\eta)<1} đ\eta\,, \tag{3.6.14}$$

$$d^{(j)}(y) = -n(n-1)\times \int\limits_{h^{(j)}(y,\eta)<1} \left([v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\}\right)(y,\eta)\, đ\eta\,. \tag{3.6.15}$$

Substituting formula (3.1.18) into formula (3.6.13) we get

$$e^{(j)}(y) = -n\,i \int\limits_{h^{(j)}(y,\eta)<1} \big(h^{(j)}\{[v^{(j)}]^*, v^{(j)}\}\big)(y,\eta)\, đ\eta\,. \tag{3.6.16}$$

Substituting formulae (3.6.14)–(3.6.16) into formula (3.6.6) we arrive at (3.6.5). □

Remark 3.6.3. *The proof of Lemma 3.6.1 given above was based on the use*

of Corollary 4.1.5 from [45]. In the actual statement of Corollary 4.1.5 in [45] uniformity in $y \in M$ was not mentioned because the authors were dealing with a manifold with a boundary. Uniformity reappeared in the subsequent Theorem 4.2.1 which involved pseudodifferential cut-offs separating the point y from the boundary.

3.7 Mollified spectral asymptotics

Our spectral function $e(\lambda, x, x)$ was initially defined only for $\lambda > 0$, see formula (1.3.2). We extend the definition to the whole real line by setting

$$e(\lambda, x, x) := 0 \quad \text{for} \quad \lambda \leq 0.$$

Theorem 3.7.1. *Let $\rho : \mathbb{R} \to \mathbb{C}$ be a function from Schwartz space $\mathcal{S}(\mathbb{R})$ whose Fourier transform $\hat{\rho}$ satisfies conditions (3.6.2)–(3.6.4). Then, uniformly over $x \in M$, we have*

$$\int e(\lambda - \mu, x, x)\, \rho(\mu)\, d\mu = a(x)\, \lambda^n + b(x)\, \lambda^{n-1} + \begin{cases} O(\lambda^{n-2}) & \text{if} \quad n \geq 3, \\ O(\ln \lambda) & \text{if} \quad n = 2, \end{cases} \tag{3.7.1}$$

as $\lambda \to +\infty$. The densities $a(x)$ and $b(x)$ appearing in the RHS of formula (3.7.1) are defined in accordance with formulae (3.1.21) and (3.1.22).

Proof Denote by $e'(\lambda, x, x)$ the derivative, with respect to the spectral parameter, of the spectral function. Here "derivative" is understood in the sense of distributions. The explicit formula for $e'(\lambda, x, x)$ is

$$e'(\lambda, x, x) := \sum_{k=1}^{+\infty} \|v_k(x)\|^2\, \delta(\lambda - \lambda_k). \tag{3.7.2}$$

Formula (3.7.2) gives us

$$\int e'(\lambda-\mu,x,x)\,\rho(\mu)\,d\mu=\sum_{k=1}^{+\infty}\|v_k(x)\|^2\,\rho(\lambda-\lambda_k). \tag{3.7.3}$$

Formula (3.7.3) implies, in particular, that, uniformly over $x\in M$, we have

$$\int e'(\lambda-\mu,x,x)\,\rho(\mu)\,d\mu=O(|\lambda|^{-\infty})\quad\text{as}\quad\lambda\to-\infty\,, \tag{3.7.4}$$

where $O(|\lambda|^{-\infty})$ is shorthand for "tends to zero faster than any given inverse power of $|\lambda|$".

Formula (3.7.3) can also be rewritten as

$$\int e'(\lambda-\mu,x,x)\,\rho(\mu)\,d\mu=\mathcal{F}^{-1}_{t\to\lambda}[\hat\rho(t)\,\mathrm{tr}\,u(t,x,x)]-\sum_{k\le0}\|v_k(x)\|^2\,\rho(\lambda-\lambda_k)\,, \tag{3.7.5}$$

where the distribution $u(t,x,y)$ is defined in accordance with formula (3.6.1). Clearly, we have

$$\sum_{k\le0}\|v_k(x)\|^2\,\rho(\lambda-\lambda_k)=O(\lambda^{-\infty})\quad\text{as}\quad\lambda\to+\infty\,. \tag{3.7.6}$$

Formulae (3.7.5), (3.7.6) and Lemma 3.6.1 imply that, uniformly over $x\in M$, we have

$$\int e'(\lambda-\mu,x,x)\,\rho(\mu)\,d\mu=$$
$$n\,a(x)\,\lambda^{n-1}+(n-1)\,b(x)\,\lambda^{n-2}+O(\lambda^{n-3})\quad\text{as}\quad\lambda\to+\infty\,. \tag{3.7.7}$$

It remains to note that

$$\frac{d}{d\lambda}\int e(\lambda-\mu,x,x)\,\rho(\mu)\,d\mu=\int e'(\lambda-\mu,x,x)\,\rho(\mu)\,d\mu\,. \tag{3.7.8}$$

Formulae (3.7.8), (3.7.4) and (3.7.7) imply (3.7.1). □

Theorem 3.7.2. *Let $\rho : \mathbb{R} \to \mathbb{C}$ be a function from Schwartz space $\mathcal{S}(\mathbb{R})$ whose Fourier transform $\hat{\rho}$ satisfies conditions (3.6.2)–(3.6.4). Then we have*

$$\int N(\lambda - \mu)\, \rho(\mu)\, d\mu = a\, \lambda^n + b\, \lambda^{n-1} + \begin{cases} O(\lambda^{n-2}) & \text{if} \quad n \geq 3, \\ O(\ln \lambda) & \text{if} \quad n = 2, \end{cases} \tag{3.7.9}$$

as $\lambda \to +\infty$. The constants a and b appearing in the RHS of formula (3.7.9) are defined in accordance with formulae (3.1.6), (3.1.21), (3.1.7) and (3.1.22).

Proof Formula (3.7.9) follows from formula (3.7.1) by integration over M, see also formula (3.1.4). □

In stating Theorems 3.7.1 and 3.7.2 we assumed the mollifier ρ to be complex-valued. This was done for the sake of generality but may seem unnatural when mollifying real-valued functions $e(\lambda, x, x)$ and $N(\lambda)$. One can make our construction look more natural by dealing only with real-valued mollifiers ρ. Note that if the function ρ is real-valued and even then its Fourier transform $\hat{\rho}$ is also real-valued and even and, moreover, condition (3.6.4) is automatically satisfied.

3.8 Unmollified spectral asymptotics

In this section we derive asymptotic formulae for the spectral function $e(\lambda, x, x)$ and the counting function $N(\lambda)$ without mollification. The section is split into two subsections: in the first we derive one-term asymptotic formulae and in the second — two-term asymptotic formulae.

3.8.1 One-term spectral asymptotics

Theorem 3.8.1. *We have, uniformly over $x \in M$,*

$$e(\lambda, x, x) = a(x)\, \lambda^n + O(\lambda^{n-1}) \tag{3.8.1}$$

as $\lambda \to +\infty$.

Proof The result in question is an immediate consequence of formulae (3.7.8), (3.7.7) and Theorem 3.7.1 from the thesis and Corollary B.2.2 from [45]. □

Theorem 3.8.2. *We have*

$$N(\lambda) = a\lambda^n + O(\lambda^{n-1}) \tag{3.8.2}$$

as $\lambda \to +\infty$.

Proof Formula (3.8.2) follows from formula (3.8.1) by integration over M, see also formula (3.1.4). □

3.8.2 Two-term spectral asymptotics

Up till now, in Section 3.7 and subsection 3.8.1, our logic was to derive asymptotic formulae for the spectral function $e(\lambda, x, x)$ first and then obtain corresponding asymptotic formulae for the counting function $N(\lambda)$ by integration over M. Such an approach will not work for two-term asymptotics because the geometric conditions required for the existence of two-term asymptotics of $e(\lambda, x, x)$ and $N(\lambda)$ will be different: for $e(\lambda, x, x)$ the appropriate geometric conditions will be formulated in terms of *loops*, whereas for $N(\lambda)$ the appropriate geometric conditions will be formulated in terms of *periodic trajectories*.

Hence, in this subsection we deal with the spectral function $e(\lambda, x, x)$ and the counting function $N(\lambda)$ separately.

In what follows the point $y \in M$ is assumed to be fixed.

Denote by $\Pi_y^{(j)}$ the set of normalised ($h^{(j)}(y, \eta) = 1$) covectors η which serve as starting points for loops generated by the Hamiltonian $h^{(j)}$. Here "starting point" refers to the starting point of a Hamiltonian trajectory $(x^{(j)}(t; y, \eta), \xi^{(j)}(t; y, \eta))$ moving forward in time ($t > 0$), see also Remark 3.6.1.

The reason we are not interested in large negative t is that the refined Fourier Tauberian theorem we will be applying, Theorem B.5.1 from [45], does not require information regarding large negative t. And the underlying reason for the latter is the fact that the function we are studying, $e(\lambda, x, x)$ (and, later, $N(\lambda)$), is real-valued. The real-valuedness of the function $e(\lambda, x, x)$ implies that its Fourier transform, $\hat{e}(t, x, x)$, possesses the symmetry $\hat{e}(-t, x, x) = \overline{\hat{e}(t, x, x)}$.

The set $\Pi_y^{(j)}$ is a subset of the $(n-1)$-dimensional unit cosphere $(S_y^*M)^{(j)}$ and the latter is equipped with a natural Lebesgue measure, see proof of Lemma 3.6.1. It is known, see Lemma 1.8.2 in [45], that the set $\Pi_y^{(j)}$ is measurable.

Definition 3.8.1. *A point $y \in M$ is said to be* nonfocal *if for each $j = 1, \dots, m^+$ the set $\Pi_y^{(j)}$ has measure zero.*

With regards to the range of the index j in Definition 3.8.1, as well as in subsequent Definitions 3.8.2–3.8.4, see Remark 3.6.2.

We call a loop of length $T > 0$ *absolutely focused* if the function

$$|x^{(j)}(T; y, \eta) - y|^2$$

has an infinite order zero in the variable η, and we denote by $(\Pi_y^a)^{(j)}$ the set of normalised $(h^{(j)}(y, \eta) = 1)$ covectors η which serve as starting points for absolutely focused loops generated by the Hamiltonian $h^{(j)}$. It is known, see Lemma 1.8.3 in [45], that the set $(\Pi_y^a)^{(j)}$ is measurable and, moreover, the set $\Pi_y^{(j)} \setminus (\Pi_y^a)^{(j)}$ has measure zero. This allows us to reformulate Definition 3.8.1 as follows.

Definition 3.8.2. *A point $y \in M$ is said to be* nonfocal *if for each $j = 1, \dots, m^+$ the set $(\Pi_y^a)^{(j)}$ has measure zero.*

In practical applications it is easier to work with Definition 3.8.2 because the set $(\Pi_y^a)^{(j)}$ is usually much thinner than the set $\Pi_y^{(j)}$.

In order to derive a two-term asymptotic formula for the spectral function $e(\lambda, x, x)$ we need the following lemma (compare with Lemma 3.6.1).

Lemma 3.8.1. *Suppose that the point $y \in M$ is nonfocal. Then for any complex-valued function $\hat{\gamma} \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} \hat{\gamma} \subset (0, +\infty)$ we have*

$$\mathcal{F}^{-1}_{t\to\lambda}[\hat{\gamma}(t) \operatorname{tr} u(t, y, y)] = o(\lambda^{n-1}) \tag{3.8.3}$$

as $\lambda \to +\infty$.

Proof The result in question is a special case of Theorem 4.4.9 from [45]. □

The following theorem is our main result regarding the spectral function $e(\lambda, x, x)$.

Theorem 3.8.3. *If the point $x \in M$ is nonfocal then the spectral function $e(\lambda, x, x)$ admits the two-term asymptotic expansion (1.3.3) as $\lambda \to +\infty$.*

Proof The result in question is an immediate consequence of formulae (3.7.7), Theorem 3.7.1 and Lemma 3.8.1 from this part of the thesis and Theorem B.5.1 from [45]. □

We now deal with the counting function $N(\lambda)$.

Suppose that we have a Hamiltonian trajectory $(x^{(j)}(t; y, \eta), \xi^{(j)}(t; y, \eta))$ and a real number $T > 0$ such that $(x^{(j)}(T; y, \eta), \xi^{(j)}(T; y, \eta)) = (y, \eta)$. We will say in this case that we have a T-periodic trajectory originating from the point $(y, \eta) \in T'M$.

Denote by $(S^*M)^{(j)}$ the unit cosphere bundle, i.e. the $(2n-1)$-dimensional surface in the cotangent bundle defined by the equation $h^{(j)}(y, \eta) = 1$. The unit cosphere bundle is equipped with a natural Lebesgue measure: the $(2n-1)$-dimensional surface area element on $(S^*M)^{(j)}$ is $dy\, d(S^*_yM)^{(j)}$ where $d(S^*_yM)^{(j)}$ is the $(n-1)$-dimensional surface area element on the unit cosphere $(S^*_yM)^{(j)}$, see proof of Lemma 3.6.1.

Denote by $\Pi^{(j)}$ the set of points in $(S^*M)^{(j)}$ which serve as starting points for periodic trajectories generated by the Hamiltonian $h^{(j)}$. It is known, see Lemma 1.3.4 in [45], that the set $\Pi^{(j)}$ is measurable.

Definition 3.8.3. *We say that the nonperiodicity condition is fulfilled if for each* $j = 1, \dots, m^+$ *the set* $\Pi^{(j)}$ *has measure zero.*

We call a T-periodic trajectory *absolutely periodic* if the function

$$|x^{(j)}(T; y, \eta) - y|^2 + |\xi^{(j)}(T; y, \eta) - \eta|^2$$

has an infinite order zero in the variables (y, η), and we denote by $(\Pi^a)^{(j)}$ the set of points in $(S^*M)^{(j)}$ which serve as starting points for absolutely periodic trajectories generated by the Hamiltonian $h^{(j)}$. It is known, see Corollary 1.3.6 in [45], that the set $(\Pi^a)^{(j)}$ is measurable and, moreover, the set $\Pi^{(j)} \setminus (\Pi^a)^{(j)}$ has measure zero. This allows us to reformulate Definition 3.8.3 as follows.

Definition 3.8.4. *We say that the nonperiodicity condition is fulfilled if for each* $j = 1, \dots, m^+$ *the set* $(\Pi^a)^{(j)}$ *has measure zero.*

In practical applications it is easier to work with Definition 3.8.4 because the set $(\Pi^a)^{(j)}$ is usually much thinner than the set $\Pi^{(j)}$.

In order to derive a two-term asymptotic formula for the counting function $N(\lambda)$ we need the following lemma.

Lemma 3.8.2. *Suppose that the nonperiodicity condition is fulfilled. Then for any complex-valued function* $\hat{\gamma} \in C_0^\infty(\mathbb{R})$ *with* $\operatorname{supp} \hat{\gamma} \subset (0, +\infty)$ *we have*

$$\int_M \mathcal{F}^{-1}_{t\to\lambda}[\hat{\gamma}(t) \operatorname{tr} u(t, y, y)]\, dy = o(\lambda^{n-1}) \tag{3.8.4}$$

as $\lambda \to +\infty$.

Proof The result in question is a special case of Theorem 4.4.1 from [45]. □

The following theorem is our main result regarding the counting function $N(\lambda)$.

Theorem 3.8.4. *If the nonperiodicity condition is fulfilled then the counting function* $N(\lambda)$ *admits the two-term asymptotic expansion (3.1.5) as* $\lambda \to +\infty$.

Proof The result in question is an immediate consequence of formulae (3.1.4), (3.7.7), Theorem 3.7.1 and Lemma 3.8.2 from this chapter and Theorem B.5.1 from [45]. □

3.9 $\mathrm{U}(m)$ invariance of the second asymptotic coefficient

We prove in this section that the RHS of formula (3.1.22) is invariant under unitary transformation (3.1.23), (3.1.24) of our operator A. The arguments presented in this section bear some similarity to those from Section 3.5, the main difference being that the unitary matrix-function in question is now a function on the base manifold M rather than on $T'M$.

Fix a point $x \in M$ and an index j (index enumerating the eigenvalues and eigenvectors of the principal symbol) and consider the expression

$$\int\limits_{h^{(j)}(x,\xi)<1} \Big([v^{(j)}]^* A_{\mathrm{sub}} v^{(j)} - \frac{i}{2}\big\{[v^{(j)}]^*, A_1 - h^{(j)}, v^{(j)}\big\} + \frac{i}{n-1} h^{(j)} \big\{[v^{(j)}]^*, v^{(j)}\big\} \Big)(x,\xi)\, d\xi\,, \quad (3.9.1)$$

compare with (3.1.22). We will show that this expression is invariant under the transformation (3.1.23), (3.1.24).

The transformation (3.1.23), (3.1.24) induces the following transformation of the principal and subprincipal symbols of the operator A:

$$A_1 \mapsto R A_1 R^*, \quad (3.9.2)$$

$$A_{\mathrm{sub}} \mapsto R A_{\mathrm{sub}} R^* + \frac{i}{2}\left(R_{x^\alpha} (A_1)_{\xi_\alpha} R^* - R (A_1)_{\xi_\alpha} R^*_{x^\alpha}\right). \quad (3.9.3)$$

The eigenvalues of the principal symbol remain unchanged, whereas the eigenvectors transform as

$$v^{(j)} \mapsto Rv^{(j)}. \tag{3.9.4}$$

Substituting formulae (3.9.2)–(3.9.4) into the RHS of (3.9.1) we conclude that the increment of the expression (3.9.1) is

$$\int\limits_{h^{(j)}(x,\xi)<1} \Bigg(\frac{i}{2}[v^{(j)}]^* \left(R^* R_{x^\alpha}(A_1)_{\xi_\alpha} - (A_1)_{\xi_\alpha} R^*_{x^\alpha} R\right) v^{(j)} \\ - \frac{i}{2}\left([v^{(j)}]^* R^*_{x^\alpha} R(A_1 - h^{(j)}) v^{(j)}_{\xi_\alpha} - [v^{(j)}_{\xi_\alpha}]^* (A_1 - h^{(j)}) R^* R_{x^\alpha} v^{(j)}\right) \\ + \frac{i}{n-1} h^{(j)} \left([v^{(j)}]^* R^*_{x^\alpha} R v^{(j)}_{\xi_\alpha} - [v^{(j)}_{\xi_\alpha}]^* R^* R_{x^\alpha} v^{(j)}\right)\Bigg)(x,\xi)\, d\xi\,,$$

which can be rewritten as

$$-\frac{i}{2} \int\limits_{h^{(j)}(x,\xi)<1} \Bigg(h^{(j)}_{\xi_\alpha} \left([v^{(j)}]^* R^*_{x^\alpha} R v^{(j)} - [v^{(j)}]^* R^* R_{x^\alpha} v^{(j)}\right) \\ - \frac{2}{n-1} h^{(j)} \left([v^{(j)}]^* R^*_{x^\alpha} R v^{(j)}_{\xi_\alpha} - [v^{(j)}_{\xi_\alpha}]^* R^* R_{x^\alpha} v^{(j)}\right)\Bigg)(x,\xi)\, d\xi\,.$$

In view of the identity $R^*R = I$ the above expression can be further simplified, so that it reads now

$$i \int\limits_{h^{(j)}(x,\xi)<1} \Bigg(h^{(j)}_{\xi_\alpha} [v^{(j)}]^* R^* R_{x^\alpha} v^{(j)} \\ - \frac{1}{n-1} h^{(j)} \left([v^{(j)}]^* R^* R_{x^\alpha} v^{(j)}_{\xi_\alpha} + [v^{(j)}_{\xi_\alpha}]^* R^* R_{x^\alpha} v^{(j)}\right)\Bigg)(x,\xi)\, d\xi\,. \tag{3.9.5}$$

Denote

$$B_\alpha(x) := -iR^* R_{x^\alpha} \tag{3.9.6}$$

and observe that this set of matrices, enumerated by the tensor index α running through the values $1, \dots, n$, is Hermitian. Denote also $b_\alpha(x,\xi) := [v^{(j)}]^* B_\alpha v^{(j)}$ and observe that these b_α are positively homogeneous in ξ of degree 0. Then the

expression (3.9.5) can be rewritten as

$$-\int\limits_{h^{(j)}(x,\xi)<1} \left(h^{(j)}_{\xi_\alpha}\, b_\alpha - \frac{1}{n-1}\, h^{(j)}\, \frac{\partial b_\alpha}{\partial \xi_\alpha} \right)(x,\xi)\, d\xi\,. \tag{3.9.7}$$

Lemma 4.1.4 and formula (1.1.15) from [45] tell us that the expression (3.9.7) is zero.

3.10 Teleparallel connection

In this section we work under the additional assumptions (1.3.4), (1.3.5) and (1.3.6), i.e. we study a 2×2 matrix differential operator in dimension 3 with trace-free principal symbol. Our aim is to show that in this case the principal symbol generates additional geometric structures which allow us to reformulate the results of our spectral analysis in a much clearer geometric language.

Let us show first that the manifold M in this case is parallelizable. The principal symbol $A_1(x,\xi)$ is linear in ξ so it can be written as

$$A_1(x,\xi) = \sigma^\alpha(x)\, \xi_\alpha\,, \tag{3.10.1}$$

where $\sigma^\alpha(x)$, $\alpha = 1,2,3$, are some trace-free Hermitian 2×2 matrix-functions. Let us denote the elements of the matrices σ^α as $\sigma^\alpha{}_{\dot a b}$ where the dotted index, running through the values $\dot 1, \dot 2$, enumerates the rows and the undotted index, running through the values $1, 2$, enumerates the columns; this notation is taken from [14]. Put

$$V_1{}^\alpha(x) := \operatorname{Re} \sigma^\alpha{}_{\dot 1 2}(x), \quad V_2{}^\alpha(x) := -\operatorname{Im} \sigma^\alpha{}_{\dot 1 2}(x), \quad V_3{}^\alpha(x) := \sigma^\alpha{}_{\dot 1 1}(x). \tag{3.10.2}$$

Formula (3.10.2) defines a triple of smooth real vector fields $V_j(x)$, $j = 1,2,3$, on the manifold M. These vector fields are linearly independent at every point x of the manifold: this follows from the fact that $\det A_1(x,\xi) \neq 0$, $\forall (x,\xi) \in T'M$

(ellipticity). Thus, the triple of vector fields V_j is a *frame*. The existence of a frame means that the manifold M is parallelizable.

Conversely, given a frame V_j we uniquely recover the elliptic principal symbol $A_1(x,\xi)$ via formulae (3.10.1), (3.A.1) and (3.A.2). Thus, a principal symbol is equivalent to a frame.

It is easy to see that the frame elements V_j are orthonormal with respect to the metric (3.1.28). Moreover, the metric can be defined directly from the frame as

$$g^{\alpha\beta} = V_j{}^{\alpha}\, V_j{}^{\beta}\,, \tag{3.10.3}$$

where the repeated frame index j indicates summation over $j = 1, 2, 3$. The two definitions of the metric, (3.1.28) and (3.10.3), are equivalent.

Parallelizability implies orientability. Having chosen a particular orientation we define the Hodge star in the standard way. We will use the Hodge star later on in this section in order to simplify calculations involving the torsion tensor.

Note that the topological invariant $\mathbf{c}$ introduced in Section 3.1 in accordance with formula (3.1.33) can be equivalently (and more naturally) defined in terms of the frame as

$$\mathbf{c} := \operatorname{sgn} \det V_j{}^{\alpha}. \tag{3.10.4}$$

The crucial new geometric structure is the teleparallel connection. We already defined it in Section 3.1 in accordance with formula (3.1.34), i.e. via the principal symbol. This connection can be equivalently defined via the frame as follows. Suppose we have a vector v based at the point $y \in M$ and we want to construct a parallel vector u based at the point $x \in M$. We decompose the vector v with respect to the frame at the point y, $v = c^j V_j(y)$, and reassemble it with the same coefficients c^j at the point x, defining $u := c^j V_j(x)$.

We now define the covariant derivative corresponding to the teleparallel connection. Our teleparallel connection is a special case of an affine connection, so we are looking at a covariant derivative acting on vectors/covectors in the usual

manner

$$\nabla_\mu v^\alpha = \partial v^\alpha / \partial x^\mu + \Gamma^\alpha{}_{\mu\beta}\, v^\beta\,, \qquad \nabla_\mu w_\beta = \partial w_\beta / \partial x^\mu - \Gamma^\alpha{}_{\mu\beta}\, w_\alpha\,. \tag{3.10.5}$$

Of course, the above ∇ should not be confused with the ∇ from Section 3.5. The teleparallel connection coefficients are defined from the conditions

$$\nabla_\mu V_j{}^\alpha = 0\,, \tag{3.10.6}$$

where the V_j are elements of our frame. Formula (3.10.6) gives a system of 27 linear algebraic equations for the determination of 27 unknown connection coefficients. It is known (see, for example, formula (A2) in [9]), that the unique solution of this system is

$$\Gamma^\alpha{}_{\mu\beta} = V_k{}^\alpha(\partial \mathbf{V}_{k\beta}/\partial x^\mu)\,, \tag{3.10.7}$$

where

$$\mathbf{V}_{k\beta} := g_{\beta\gamma} V_k{}^\gamma\,. \tag{3.10.8}$$

The triple of covector fields $\mathbf{V}_k$, $k = 1, 2, 3$, is called the *coframe*. The frame and coframe uniquely determine each other via the relation

$$V_j{}^\alpha \mathbf{V}_{k\alpha} = \delta_{jk}. \tag{3.10.9}$$

One can check by performing explicit calculations that the teleparallel connection has the following two important properties:

- $\nabla_\alpha g_{\beta\gamma} = 0$, which means that the connection is metric compatible and
- $\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha = 0$, which means that the Riemann curvature tensor is zero.

The tensor characterising the "strength" of the teleparallel connection is not the Riemann curvature tensor but the torsion tensor (3.1.35). The teleparallel connection is, in a sense, the opposite of the more common Levi-Civita connection: the Levi-Civita connection has zero torsion but nonzero curvature, whereas the teleparallel connection has nonzero torsion but zero curvature. In Chapter 3 we distinguish these two affine connections by using different notation for connection coefficients: we write the teleparallel connection coefficients as $\Gamma^{\alpha}{}_{\beta\gamma}$ and the Levi-Civita connection coefficients (Christoffel symbols) as $\left\{ {\alpha \atop \beta\gamma} \right\}$, see formula (3.A.4).

Substituting (3.10.7) into (3.1.35) we arrive at the following explicit formula for the torsion tensor of the teleparallel connection

$$T = V_j \otimes d\mathbf{V}_j \,, \tag{3.10.10}$$

where the d stands for the exterior derivative. For the sake of clarity we rewrite formula (3.10.10) in more detailed form, retaining all tensor indices,

$$T^{\alpha}{}_{\beta\gamma} = V_j{}^{\alpha}(\partial \mathbf{V}_{j\gamma}/\partial x^{\beta} - \partial \mathbf{V}_{j\beta}/\partial x^{\gamma}) \,. \tag{3.10.11}$$

As always, the repeated index j appearing in formulae (3.10.10) and (3.10.11) indicates summation over $j = 1, 2, 3$.

As pointed out in Section 3.1, it is more convenient to work with the rank two tensor $\overset{*}{T}$ defined by formula (3.1.36) rather than with the rank three tensor T. Substituting (3.10.10) into (3.1.36) we get

$$\overset{*}{T} = V_j \otimes \operatorname{curl} \mathbf{V}_j \,, \tag{3.10.12}$$

where

$$(\operatorname{curl} \mathbf{V}_j)_{\beta} := (*d\mathbf{V}_j)_{\beta} = \frac{1}{2} \, (d\mathbf{V}_j)^{\gamma\delta} \, \varepsilon_{\gamma\delta\beta} \, \sqrt{\det g_{\mu\nu}} \,. \tag{3.10.13}$$

The remainder of this section is devoted to the proof of formula (3.1.37) expressing the scalar curvature of the U(1) connection via the torsion of the teleparallel

connection and the metric.

We fix an arbitrary point $Q \in T'M$ and prove formula (3.1.37) at this point. As the LHS and RHS of (3.1.37) are invariant under changes of local coordinates x, it is sufficient to prove formula (3.1.37) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the projection of the point Q onto the base manifold, $g_{\mu\nu}(0) = \delta_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the formula we are proving involves only first partial derivatives, we may assume, without loss of generality, that $g_{\mu\nu}(x) = \delta_{\mu\nu}$ for all x in some neighbourhood of the origin. Thus, it is sufficient to prove formula (3.1.37) for the case of Euclidean metric.

As both the LHS and RHS of (3.1.37) have the same degree of homogeneity in ξ, namely, -1, it is sufficient to prove formula (3.1.37) for ξ of norm 1. Moreover, by rotating our Cartesian coordinate system we can reduce the case of general ξ of norm 1 to the case

$$\xi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}. \tag{3.10.14}$$

There is one further simplification that can be made: we claim that it is sufficient to prove formula (3.1.37) for the case when

$$V_j{}^\alpha(0) = \mathbf{c}\delta_j{}^\alpha, \tag{3.10.15}$$

i.e. for the case when at the point $x = 0$ the elements of the frame are aligned with the coordinate axes; here $\mathbf{c} = \pm 1$ is the topological invariant defined in accordance with formula (3.1.33) or, equivalently, in accordance with formula (3.10.4). This claim follows from the observation that the LHS of formula (3.1.37) is invariant under rigid special unitary transformation of the column-function $v^+(x, \xi)$,

$$v^+ \mapsto Rv^+,$$

where "rigid" refers to the fact that the matrix $R \in \mathrm{SU}(2)$ is constant. Of course, the column-function Rv^+ is no longer an eigenvector of the original principal symbol, but a new principal symbol obtained from the old one by the rigid special

orthogonal transformation of the frame (3.A.27) with the 3×3 special orthogonal matrix O expressed in terms of the 2×2 special unitary matrix R in accordance with (3.A.28). One can always choose the special unitary matrix R so that at the point $x = 0$ the elements of the new frame are aligned with the coordinate axes (in fact, there are two possible choices of R which differ by sign). It remains only to note the well known fact that the tensor $\overset{*}{T}$ appearing in the RHS of formula (3.1.37) is also invariant under rigid special orthogonal transformation of the frame.

Having made all the simplifying assumptions listed above, we are now in a position to prove formula (3.1.37). We give the proof for the case

$$\mathbf{c} = +1\,. \tag{3.10.16}$$

There is no need to give a separate proof for the case $\mathbf{c} = -1$ as the two cases reduce to one another by means of the identity (3.1.20) and the observation that torsion (3.10.10) is invariant under inversion of the frame.

Let us calculate the RHS of (3.1.37) first. In view of (3.10.15) we have, in the linear approximation in x,

$$\begin{pmatrix} V_1{}^1(x) & V_1{}^2(x) & V_1{}^3(x) \\ V_2{}^1(x) & V_2{}^2(x) & V_2{}^3(x) \\ V_3{}^1(x) & V_3{}^2(x) & V_3{}^3(x) \end{pmatrix} = \begin{pmatrix} 1 & w^3(x) & -w^2(x) \\ -w^3(x) & 1 & w^1(x) \\ w^2(x) & -w^1(x) & 1 \end{pmatrix}, \tag{3.10.17}$$

where w is some smooth vector-function which vanishes at $x = 0$. Formula (3.10.17) is the standard formula for the linearisation of an orthogonal matrix about the identity; see also formula (10.1) in [6]. Note that in Cosserat elasticity literature the vector-function w is called the *vector of microrotations*. Substituting (3.10.17) into (3.10.12) we get, at $x = 0$,

$$\overset{*}{T}_{\alpha\beta} = \partial w_\beta / \partial x^\alpha - \delta_{\alpha\beta} \operatorname{div} w, \tag{3.10.18}$$

which is formula (10.5) from [6]. Here we freely lower and raise tensor indices using the fact that the metric is Euclidean (in the Euclidean case it does not matter whether a tensor index comes as a subscript or a superscript). Substituting (3.10.18) and (3.10.14) into the RHS of (3.1.37) we get, at our point $Q \in T'M$,

$$\frac{1}{2}\,\frac{\overset{*}{T}{}^{\alpha\beta}\xi_\alpha\xi_\beta}{(g^{\mu\nu}\xi_\mu\xi_\nu)^{3/2}} = -\frac{1}{2}(\partial w^1/\partial x^1 + \partial w^2/\partial x^2)\,. \tag{3.10.19}$$

Let us now calculate the LHS of (3.1.37). The equation for the eigenvector $v^+(x,\xi)$ of the principal symbol is

$$\begin{pmatrix} V_3{}^\alpha\xi_\alpha - \|\xi\| & (V_1 - iV_2)^\alpha\xi_\alpha \\ (V_1 + iV_2)^\alpha\xi_\alpha & -V_3{}^\alpha\xi_\alpha - \|\xi\| \end{pmatrix} \begin{pmatrix} v_1^+ \\ v_2^+ \end{pmatrix} = 0\,. \tag{3.10.20}$$

In view of (3.10.14), (3.10.15) and (3.10.16) the (normalised) solution of (3.10.20) at our point $Q \in T'M$ is

$$v^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Of course, our $v^+(x,\xi)$ is defined up to the gauge transformation (3.1.16), (3.1.17), however the LHS of (3.1.37) is invariant under this gauge transformation. We now perturb equation (3.10.20) about the point $Q \in T'M$, that is, about

$$x = 0, \qquad \xi = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$

making use of formula (3.10.17), which gives us the following equation for the increment δv^+ of the eigenvector $v^+(x,\xi)$ of the principal symbol:

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} \delta v_1^+ \\ \delta v_2^+ \end{pmatrix} + \begin{pmatrix} 0 & -w^2(x) - iw^1(x) \\ -w^2(x) + iw^1(x) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & \delta\xi_1 - i\delta\xi_2 \\ \delta\xi_1 + i\delta\xi_2 & -2\delta\xi_3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0,$$

or, equivalently,

$$\delta v_2^+ = \frac{1}{2}(-w^2(x) + iw^1(x) + \delta\xi_1 + i\delta\xi_2). \tag{3.10.21}$$

Formula (3.10.21) has to be supplemented by the normalisation condition $\|v^+(x,\xi)\| = 1$, which in its linearised form reads

$$\operatorname{Re}\delta v_1^+ = 0. \tag{3.10.22}$$

Formulae (3.10.22) and (3.10.21) define δv^+ modulo an arbitrary $\operatorname{Im}\delta v_1^+$, with this degree of freedom being associated with the gauge transformation (3.1.16), (3.1.17). Without loss of generality we may assume that the gauge is chosen so that

$$\operatorname{Im}\delta v_1^+ = 0. \tag{3.10.23}$$

Combining formulae (3.10.22), (3.10.23) and (3.10.21) we get

$$\delta v^+ = \frac{1}{2}\begin{pmatrix} 0 \\ -w^2(x) + iw^1(x) + \delta\xi_1 + i\delta\xi_2 \end{pmatrix}. \tag{3.10.24}$$

Recall that the w appearing in this formula is some smooth vector-function which vanishes at $x = 0$.

Differentiation of (3.10.24) gives us

$$\frac{\partial v^+}{\partial x^\alpha} = \frac{1}{2}\begin{pmatrix} 0 \\ -\partial w^2/\partial x^\alpha + i\partial w^1/\partial x^\alpha \end{pmatrix}, \tag{3.10.25}$$

$$\frac{\partial v^+}{\partial \xi_1} = \frac{1}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \frac{\partial v^+}{\partial \xi_2} = \frac{1}{2}\begin{pmatrix} 0 \\ i \end{pmatrix}, \qquad \frac{\partial v^+}{\partial \xi_3} = 0. \tag{3.10.26}$$

Formulae (3.10.25) and (3.10.26) imply that at our point $Q \in T'M$

$$-i\{[v^+]^*, v^+\} = -\frac{1}{2}(\partial w^1/\partial x^1 + \partial w^2/\partial x^2). \tag{3.10.27}$$

Comparing formulae (3.10.19) and (3.10.27) and recalling (3.10.16), we arrive at the required result (3.1.37).

We end this section by writing down an explicit self-contained formula for the trace of the tensor $\overset{*}{T}$. Note that according to formula (3.1.40), it is only the trace of $\overset{*}{T}$ that we need for our spectral asymptotics. Formulae (3.10.12) and (3.10.13) imply

$$\operatorname{tr}\overset{*}{T} = \sqrt{\det g^{\alpha\beta}}\,\big[\mathbf{V}_{j1}\,\partial\mathbf{V}_{j3}/\partial x^2 + \mathbf{V}_{j2}\,\partial\mathbf{V}_{j1}/\partial x^3 + \mathbf{V}_{j3}\,\partial\mathbf{V}_{j2}/\partial x^1 - \mathbf{V}_{j1}\,\partial\mathbf{V}_{j2}/\partial x^3 - \mathbf{V}_{j2}\,\partial\mathbf{V}_{j3}/\partial x^1 - \mathbf{V}_{j3}\,\partial\mathbf{V}_{j1}/\partial x^2\big]. \quad (3.10.28)$$

Here the coframe $\mathbf{V}_j$ is determined from the principal symbol $A_1(x,\xi)$ in accordance with formulae (3.10.1), (3.10.2) and (3.10.8) or (3.10.9), whereas the metric g is determined from the principal symbol $A_1(x,\xi)$ in accordance with formula (3.1.28) or (3.10.3).

3.11 Proof of Theorem 1.3.1

As Theorem 1.3.1 is an if and only if theorem, our proof comes in two parts.

3.11.1 Part 1 of the proof of Theorem 1.3.1

Let A be a massless Dirac operator on half-densities. We need to prove that a) the subprincipal symbol of this operator, $A_{\mathrm{sub}}(x)$, is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero.

As we have already established the formula for $b(x)$, see (3.1.40), this part of the proof of Theorem 1.3.1 reduces to proving that the explicit formula for the subprincipal symbol of the massless Dirac operator on half-densities is

$$A_{\mathrm{sub}}(x) = \frac{\mathbf{c}}{4}\left(\operatorname{tr}\overset{*}{T}(x)\right) I\,, \quad (3.11.1)$$

where I is the 2×2 identity matrix.

We give the proof of (3.11.1) for the case (3.10.16). There is no need to give a separate proof for the case $\mathbf{c} = -1$ as the two cases reduce to one another by inversion of the frame: the full symbol of the massless Dirac operator on half-densities changes sign under inversion of the frame and hence its subprincipal symbol changes sign under inversion of the frame, whereas torsion (3.10.10) is invariant under inversion of the frame.

We fix an arbitrary point $P \in M$ and prove the identity (3.11.1) at this point. As the LHS and RHS of (3.11.1) are invariant under changes of local coordinates x, it is sufficient to check the identity (3.11.1) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the point P, $g_{\mu\nu}(0) = \delta_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the identity we are proving involves only first partial derivatives, we may assume, without loss of generality, that $g_{\mu\nu}(x) = \delta_{\mu\nu}$ for all x in some neighbourhood of the origin. Furthermore, by rotating our Cartesian coordinate system we can achieve (3.10.15), which opens the way to the use, in the linear approximation in x, of formula (3.10.17).

Substituting (3.10.17) into (3.A.1), we get, in the linear approximation in x,

$$\sigma^1 = \begin{pmatrix} w^2 & 1+iw^3 \\ 1-iw^3 & -w^2 \end{pmatrix} = \sigma_1\,,$$

$$\sigma^2 = \begin{pmatrix} -w^1 & -i+w^3 \\ i+w^3 & w^1 \end{pmatrix} = \sigma_2\,,$$

$$\sigma^3 = \begin{pmatrix} 1 & -iw^1-w^2 \\ iw^1-w^2 & -1 \end{pmatrix} = \sigma_3\,. \quad (3.11.2)$$

Recall that the w appearing in this formula is some smooth vector-function which vanishes at $x = 0$.

Substitution of (3.11.2) into (3.A.3) (which coincides with (3.A.30) because we assumed the metric to be Euclidean, $g_{\mu\nu}(x) = \delta_{\mu\nu}$) allows us to evaluate the full

symbol $A(x,\xi) = A_1(x,\xi) + A_0(x)$ of the massless Dirac operator on half-densities:

$$A_1(x,\xi) = \begin{pmatrix} \xi_3 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & -\xi_3 \end{pmatrix} + \begin{pmatrix} w^2\xi_1 - w^1\xi_2 & iw^3\xi_1 + w^3\xi_2 + (-iw^1 - w^2)\xi_3 \\ -iw^3\xi_1 + w^3\xi_2 + (iw^1 - w^2)\xi_3 & -w^2\xi_1 + w^1\xi_2 \end{pmatrix}, \quad (3.11.3)$$

$$A_0(0) = -\frac{i}{4} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \partial w^2/\partial x^1 & i\partial w^3/\partial x^1 \\ -i\partial w^3/\partial x^1 & -\partial w^2/\partial x^1 \end{pmatrix} + \dots . \quad (3.11.4)$$

Here formula (3.11.3) is written in the linear approximation in x, whereas formula (3.11.4) displays, for the sake of brevity, only one term out of nine (the one corresponding to $\alpha = \beta = 1$ in (3.A.3)) with the remaining eight terms concealed within the dots ....

Substituting (3.11.4) and (3.11.3) into (3.1.13), we get

$$A_{\mathrm{sub}}(0) = -\frac{1}{2} (\operatorname{div} w)\, I. \quad (3.11.5)$$

But, according to (3.10.18),

$$\operatorname{tr} \overset{*}{T}(0) = -2 \operatorname{div} w. \quad (3.11.6)$$

Formulae (3.11.5), (3.11.6) and (3.10.16) imply formula (3.11.1) at $x = 0$.

3.11.2 Part 2 of the proof of Theorem 1.3.1

Let A be an operator satisfying assumptions (1.3.4), (1.3.5) and (1.3.6) and such that a) the subprincipal symbol of this operator, $A_{\mathrm{sub}}(x)$, is proportional to the identity matrix and b) the second asymptotic coefficient of the spectral function, $b(x)$, is zero. We need to prove that A is a massless Dirac operator on half-densities.

As we have already established the formula for $b(x)$, see (3.1.40), we have, for our operator A, the identity (3.11.1). Let V_j be the frame corresponding to the principal symbol of the operator A, see formulae (3.10.1) and (3.10.2). Now, let B be the massless Dirac operator on half-densities corresponding to the same frame. Then the principal symbols of the operators A and B coincide. But the subprincipal symbols of the operators A and B coincide as well, as in both cases these are determined via the frame according to the same formula (3.11.1) (for the massless Dirac operator B this is the result from subsection 3.11.1). A first order differential operator is determined by its principal and subprincipal symbols, hence, $A = B$. $\square$

3.12 Spectral asymmetry

In this section we deal with the special case when the operator A is differential (as opposed to pseudodifferential). No assumptions are made regarding n, m or $\operatorname{tr} A_1$.

Our aim is to examine what happens when we change the sign of the operator. In other words, we compare the original operator A with the operator $\tilde{A} := -A$. In theoretical physics the transformation $A \mapsto -A$ would be interpreted as time reversal, see equation (3.1.3).

It is easy to see that for a differential operator the number m (number of equations in our system) has to be even and that the principal symbol has to have the same number of positive and negative eigenvalues. In the notation of Section 3.1 this fact can be expressed as $m = 2m^+ = 2m^-$.

It is also easy to see that the principal symbols of the two operators, A and $\tilde{A}$, and the eigenvalues and eigenvectors of the principal symbols are related as

$$A_1(x, \xi) = \tilde{A}_1(x, -\xi), \tag{3.12.1}$$

$$h^{(j)}(x,\xi) = \tilde{h}^{(j)}(x,-\xi), \tag{3.12.2}$$

$$v^{(j)}(x,\xi) = \tilde{v}^{(j)}(x,-\xi), \tag{3.12.3}$$

whereas the subprincipal symbols are related as

$$A_{\mathrm{sub}}(x) = -\tilde{A}_{\mathrm{sub}}(x). \tag{3.12.4}$$

Formulae (3.1.21), (3.1.22), (3.1.15), (3.1.14) and (3.12.1)–(3.12.4) imply

$$a(x) = \tilde{a}(x), \qquad b(x) = -\tilde{b}(x). \tag{3.12.5}$$

Substituting (3.12.5) into (3.1.6) and (3.1.7) we get

$$a = \tilde{a}, \qquad b = -\tilde{b}. \tag{3.12.6}$$

Formulae (3.1.5) and (3.12.6) imply that the spectrum of a generic first order differential operator is asymmetric about $\lambda = 0$. This phenomenon is known in differential geometry as *spectral asymmetry* [1, 2, 3, 4].

If we square our operator A and consider the spectral problem $A^2 v = \lambda^2 v$, then the terms $\pm b\lambda^{n-1}$ cancel out and the second asymptotic coefficient of the counting function (as well as the spectral function) of the operator A^2 turns to zero. This is in agreement with the known fact that for an even order semi-bounded matrix differential operator acting on a manifold without boundary the second asymptotic coefficient of the counting function is zero, see Section 6 of [52] and [42].

The case of the massless Dirac operator is special because, according to Theorem 1.3.1, the spectrum (as well as the spectral function) of this operator is asymptotically symmetric about $\lambda = 0$ in the two leading terms. However, despite this asymptotic symmetry, we believe that for a generic Riemannian 3-manifold the spectrum of the massless Dirac operator is asymmetric. In stating this belief we are in agreement with the discussion presented on page 1298 of [50]; note

that in the case of an odd-dimensional manifold the author of [50] refers to the massless Dirac operator as the *Pauli* operator. And, of course, our belief that for a generic Riemannian 3-manifold the spectrum of the massless Dirac operator is asymmetric is closely related to the fact that in dimension 3 the massless Dirac operator commutes with the operator of charge conjugation, see formulae (3.A.18) and (3.A.19).

3.13 Bibliographic review

To our knowledge, the first publication on the subject of two-term spectral asymptotics for systems was Ivrii's 1980 paper [26] in Section 2 of which the author stated, without proof, a formula for the second asymptotic coefficient of the counting function. In a subsequent 1982 paper [27] Ivrii acknowledged that the formula from [26] was incorrect and gave a new formula, labelled (0.6), followed by a "proof". In his 1984 Springer Lecture Notes [28] Ivrii acknowledged on page 226 that both his previous formulae for the second asymptotic coefficient were incorrect and stated, without proof, yet another formula.

Roughly at the same time Rozenblyum [40] also stated a formula for the second asymptotic coefficient of the counting function of a first order system.

The formulae from [26], [27] and [40] are fundamentally flawed because they are proportional to the subprincipal symbol. As our formulae (3.1.7) and (3.1.22) show, the second asymptotic coefficient of the counting function may be nonzero even when the subprincipal symbol is zero. This illustrates, yet again, the difference between scalar operators and systems.

The formula on page 226 of [28] gives an algorithm for the calculation of the correction term designed to take account of the effect described in the previous paragraph. This algorithm requires the evaluation of a limit of a complicated expression involving the integral, over the cotangent bundle, of the trace of the

symbol of the resolvent of the operator A constructed by means of pseudodifferential calculus. This algorithm was revisited in Ivrii's 1998 book, see formulae (4.3.39) and (4.2.25) in [29].

The next contributor to the subject was Safarov who, in his 1989 DSc Thesis [43], wrote down a formula for the second asymptotic coefficient of the counting function which was "almost" correct. This formula appears in [43] as formula (2.4). As explained in Section 3.1, Safarov lost only the curvature terms $-\frac{ni}{n-1}\int h^{(j)}\{[v^{(j)}]^*, v^{(j)}\}$. Safarov's DSc Thesis [43] provides arguments which are sufficiently detailed and we were able to identify the precise point (page 163) at which the mistake occurred.

In 1998 Nicoll rederived [35] Safarov's formula (3.1.10) for the principal symbols of the propagator, using a method slightly different from [43], but stopped short of calculating the second asymptotic coefficient of the counting function.

In 2007 Kamotski and Ruzhansky [30] performed an analysis of the propagator of a first order elliptic system based on the approach of Rozenblyum [40], but stopped short of calculating the second asymptotic coefficient of the counting function.

In 1984 Vassiliev considered systems in Section 6 of his paper [52]. However, that paper dealt with systems of a very special type: differential (as opposed to pseudodifferential) and of even (as opposed to odd) order. In this case the second asymptotic coefficients of the counting function and the spectral function vanish, provided the manifold does not have a boundary.

Appendix

3.A The massless Dirac operator

Let M be a 3-dimensional connected compact oriented manifold equipped with a Riemannian metric $g_{\alpha\beta}$, $\alpha, \beta = 1, 2, 3$ being the tensor indices. Note that we are more prescriptive in this appendix than in the main text of the Chapter 3: in the main text orientability and existence of a metric emerged as consequences of the way we stated the problem, whereas in this appendix they are *a priori* assumptions.

We work only in local coordinates with prescribed orientation.

It is known [49, 31] that a 3-dimensional oriented manifold is parallelizable, i.e. there exist smooth real vector fields V_j, $j = 1, 2, 3$, that are linearly independent at every point x of the manifold. (This fact is often referred to as *Steenrod's theorem*.) Each vector $V_j(x)$ has coordinate components $V_j{}^\alpha(x)$, $\alpha = 1, 2, 3$. Note that we use the Latin letter j for enumerating the vector fields (this is an *anholonomic* or *frame* index) and the Greek letter α for enumerating their components (this is a *holonomic* or *tensor* index). The triple of linearly independent vector fields V_j, $j = 1, 2, 3$, is called a *frame*. Without loss of generality we assume further on that the vector fields V_j are orthonormal with respect to our metric: this can always be achieved by means of the Gram–Schmidt process.

Define Pauli matrices

$$\sigma^\alpha(x) := s^j \, V_j{}^\alpha(x) \,, \tag{3.A.1}$$

where

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_1 \,, \qquad s^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = s_2 \,, \qquad s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = s_3 \,. \tag{3.A.2}$$

In formula (3.A.1) summation is carried out over the repeated frame index $j = 1, 2, 3$, and $\alpha = 1, 2, 3$ is the free tensor index.

The massless Dirac operator is the matrix operator

$$W := -i\sigma^{\alpha}\left(\frac{\partial}{\partial x^{\alpha}} + \frac{1}{4}\sigma_{\beta}\left(\frac{\partial \sigma^{\beta}}{\partial x^{\alpha}} + \left\{ \begin{array}{c} \beta \\ \alpha\gamma \end{array} \right\}\sigma^{\gamma}\right)\right), \tag{3.A.3}$$

where summation is carried out over $\alpha, \beta, \gamma = 1, 2, 3$, and

$$\left\{ \begin{array}{c} \beta \\ \alpha\gamma \end{array} \right\} := \frac{1}{2}g^{\beta\delta}\left(\frac{\partial g_{\gamma\delta}}{\partial x^{\alpha}} + \frac{\partial g_{\alpha\delta}}{\partial x^{\gamma}} - \frac{\partial g_{\alpha\gamma}}{\partial x^{\delta}}\right) \tag{3.A.4}$$

are the Christoffel symbols. Here and throughout this appendix we raise and lower tensor indices using the metric. Note that we chose the letter "W" for denoting the massless Dirac operator because in theoretical physics literature it is often referred to as the *Weyl* operator.

Formula (3.A.3) is the formula from [14], only written in matrix notation (i.e. without spinor indices). Note that in the process of transcribing formulae from [14] into matrix notation we used the identity

$$\epsilon\sigma^{\alpha}\epsilon = (\sigma^{\alpha})^{T}, \tag{3.A.5}$$

$\alpha = 1, 2, 3$, where

$$\epsilon := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.A.6}$$

is the 'metric spinor'. The identity (3.A.5) gives a simple way of raising/lowering spinor indices in Pauli matrices in the non-relativistic ($\alpha \neq 0$) setting.

Physically, our massless Dirac operator (3.A.3) describes a single neutrino living in a 3-dimensional compact universe M. The eigenvalues of the massless Dirac operator are the energy levels.

Observe that the sign of $\det V_j{}^{\alpha}$ is preserved throughout the connected oriented manifold M. Having $\det V_j{}^{\alpha} > 0$ means that our frame has positive orientation (relative to the prescribed orientation of local coordinates) and $\det V_j{}^{\alpha} < 0$ means that our frame has negative orientation. Accordingly, we say that our massless Dirac operator (3.A.3) has positive/negative orientation depending on the sign

of $\det V_j{}^\alpha$. Of course, the transformation $W \mapsto -W$ changes the orientation of the massless Dirac operator.

The massless Dirac operator (3.A.3) acts on columns $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ of complex-valued scalar functions. In differential geometry this object is referred to as a (Weyl) spinor so as to emphasise the fact that v transforms in a particular way under transformation of the orthonormal frame V. However, as in our exposition the frame V is assumed to be chosen *a priori*, we can treat the components of the spinor as scalars. This issue will be revisited below when we state Property 4 of the massless Dirac operator.

We now list the main properties of the massless Dirac operator.

Property 1. The massless Dirac operator is invariant under changes of local coordinates x, i.e. it maps 2-columns of smooth scalar functions $M \to \mathbb{C}^2$ to 2-columns of smooth scalar functions $M \to \mathbb{C}^2$ regardless of the choice of local coordinates.

In order to establish this property we examine separately the two operators

$$\sigma^\alpha \frac{\partial}{\partial x^\alpha} \tag{3.A.7}$$

and

$$\sigma^\alpha \sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) \tag{3.A.8}$$

appearing in formula (3.A.3).

Let us act with the differential operator (3.A.7) on a 2-column $u : M \to \mathbb{C}^2$ of smooth scalar functions. Then $\frac{\partial u}{\partial x^\alpha}$ is a column-valued covector (i.e. pair of gradients), σ^α is a matrix-valued vector, so matrix multiplication combined with contraction in α gives a column-valued scalar. Thus, the operator (3.A.7) is invariant under changes of local coordinates.

As to the multiplication operator (3.A.8), its invariance follows from the observation that $\left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right)$ is a matrix-valued tensor.

Property 2. The massless Dirac operator is formally self-adjoint (symmetric) with respect to the inner product

$$\int_M v^* w \, \sqrt{\det g_{\alpha\beta}} \, dx \tag{3.A.9}$$

on 2-columns of smooth scalar functions $v, w : M \to \mathbb{C}^2$.

Indeed, the adjoint operator is

$$W^* = -i \frac{1}{\sqrt{\det g_{\kappa\lambda}}} \frac{\partial}{\partial x^\alpha} \sqrt{\det g_{\mu\nu}} \, \sigma^\alpha + \frac{i}{4} \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) \sigma_\beta \sigma^\alpha. \tag{3.A.10}$$

Comparing formulae (3.A.3) and (3.A.10) we see that in order to prove formal self-adjointness we need to show that

$$\left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) \sigma_\beta \sigma^\alpha + \sigma^\alpha \sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) = \frac{4}{\sqrt{\det g_{\kappa\lambda}}} \left(\frac{\partial}{\partial x^\alpha} \sqrt{\det g_{\mu\nu}} \, \sigma^\alpha \right). \tag{3.A.11}$$

We fix an arbitrary point $P \in M$ and prove the identity (3.A.11) at this point. In view of Property 1, it is sufficient to check the identity (3.A.11) in Riemann normal coordinates, i.e. local coordinates such that $x = 0$ corresponds to the point P, $g_{\mu\nu}(0) = \delta_{\mu\nu}$ and $\frac{\partial g_{\mu\nu}}{\partial x^\lambda}(0) = 0$. Moreover, as the identity we are proving involves only first partial derivatives, we may assume, without loss of generality, that $g_{\mu\nu}(x) = \delta_{\mu\nu}$ for all x in some neighbourhood of the origin. Thus, the problem has been reduced to proving that variable (i.e. dependent on x) Pauli matrices in Euclidean space satisfy the identity

$$\left(\frac{\partial \sigma^\beta}{\partial x^\alpha} \right) \sigma^\beta \sigma^\alpha + \sigma^\alpha \sigma^\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} \right) = 4 \left(\frac{\partial \sigma^\alpha}{\partial x^\alpha} \right). \tag{3.A.12}$$

Note that in (3.A.12) we made all the tensor indices upper, using the fact that the metric is Euclidean (in the Euclidean case it does not matter whether a tensor index comes as a subscript or a superscript). Of course, we still retain the convention of summation over repeated indices.

In order to prove (3.A.12) we recall the basic identity for Pauli matrices which in the Euclidean case reads

$$\sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu = 2I\delta^{\mu\nu}, \tag{3.A.13}$$

where I is the 2×2 identity matrix. (For a general metric one would have written the above formula with $g^{\mu\nu}$ instead of $\delta^{\mu\nu}$.) Formula (3.A.13) implies

$$\sigma^\mu\sigma^\mu = 3I, \tag{3.A.14}$$

$$\sigma^\mu\sigma^\kappa\sigma^\mu = -\sigma^\kappa, \tag{3.A.15}$$

$$\partial(\sigma^\mu\sigma^\nu + \sigma^\nu\sigma^\mu)/\partial x^\lambda = 0. \tag{3.A.16}$$

Using formulae (3.A.13)–(3.A.16) we get

$$\begin{aligned}
\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right)\sigma^\beta\sigma^\alpha + \sigma^\alpha\sigma^\beta\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right) &= -\sigma^\beta\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right)\sigma^\alpha - \sigma^\alpha\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right)\sigma^\beta \\
&= \sigma^\beta\sigma^\beta\left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right) + \left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right)\sigma^\beta\sigma^\beta + \sigma^\beta\left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right)\sigma^\beta + \sigma^\beta\left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right)\sigma^\beta \\
&\quad + \sigma^\beta\sigma^\alpha\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right) + \left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right)\sigma^\alpha\sigma^\beta \\
&= 3\left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right) + 3\left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right) - \left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right) - \left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right) \\
&\quad - \sigma^\alpha\sigma^\beta\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right) - \left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right)\sigma^\beta\sigma^\alpha + 2\delta^{\alpha\beta}\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right) + 2\delta^{\alpha\beta}\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right) \\
&= -\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right)\sigma^\beta\sigma^\alpha - \sigma^\alpha\sigma^\beta\left(\frac{\partial\sigma^\beta}{\partial x^\alpha}\right) + 8\left(\frac{\partial\sigma^\alpha}{\partial x^\alpha}\right).
\end{aligned} \tag{3.A.17}$$

Comparing the left- and right-hand sides of (3.A.17) we arrive at (3.A.12).

Property 3. The massless Dirac operator W commutes

$$\mathrm{C}(Wv) = W\mathrm{C}(v) \tag{3.A.18}$$

with the antilinear map

$$v \mapsto \mathrm{C}(v) := \epsilon\overline{v}. \tag{3.A.19}$$

Here the map (3.A.19) acts on columns $v = \begin{pmatrix} v_1 & v_2 \end{pmatrix}^T$ of complex-valued scalar

functions, with ϵ being the 'metric spinor' defined in accordance with (3.A.6). The commutativity property (3.A.18) follows from the explicit formula for the massless Dirac operator (3.A.3) and the identity $\epsilon\sigma^\alpha = -\overline{\sigma^\alpha}\epsilon$, $\alpha = 1, 2, 3$, the latter being a consequence of formula (3.A.5).

Formula (3.A.18) implies that v is an eigenfunction of the massless Dirac operator corresponding to an eigenvalue λ if and only if $\mathrm{C}(v)$ is an eigenfunction of the massless Dirac operator corresponding to the same eigenvalue λ. Hence, all eigenvalues of the massless Dirac operator have even multiplicity. Moreover, any eigenfunction v and its 'partner' $\mathrm{C}(v)$ make the same contribution to the spectral function (1.3.2) at every point x of the manifold M.

We do not use the commutativity property (3.A.18) of the massless Dirac operator in this chapter.

The antilinear operator (3.A.19) is, of course, the charge conjugation operator which we already encountered in Section 3.1, see formula (3.1.25). The difference between the arguments presented in this appendix and those in Section 3.1 is that in this appendix we deal with the differential operator, whereas in Section 3.1 we dealt with the principal symbol. This leads to opposite commutation properties: the charge conjugation operator commutes with the Weyl operator but it anticommutes with its principal symbol. The source of this difference is the i appearing in the RHS of formula (3.A.3).

Property 4. This property has to do with a particular behaviour under SU(2) transformation. Let $R : M \to \mathrm{SU}(2)$ be an arbitrary smooth special unitary matrix-function. Let us introduce new Pauli matrices

$$\tilde{\sigma}^\alpha := R\sigma^\alpha R^* \tag{3.A.20}$$

and a new operator $\tilde{W}$ obtained by replacing the σ in (3.A.3) by $\tilde{\sigma}$. It turns out (and this is Property 4) that the two operators, $\tilde{W}$ and W, are related in exactly

the same way as the Pauli matrices, $\tilde{\sigma}$ and σ, that is,

$$\tilde{W} = RWR^*. \tag{3.A.21}$$

In order to prove formula (3.A.21) we write down the operator $\tilde{W}$ explicitly and rearrange terms:

$$\begin{aligned}\tilde{W} &:= -iR\sigma^\alpha R^* \left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4} R\sigma_\beta R^* \left(\frac{\partial (R\sigma^\beta R^*)}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} R\sigma^\gamma R^* \right) \right) \\ &= -iR\sigma^\alpha \frac{\partial}{\partial x^\alpha} R^* + iR\sigma^\alpha \frac{\partial R^*}{\partial x^\alpha} \\ &- \frac{i}{4} R\sigma^\alpha \sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \left\{ \begin{matrix} \beta \\ \alpha\gamma \end{matrix} \right\} \sigma^\gamma \right) R^* - \frac{i}{4} R\sigma^\alpha \sigma_\beta R^* \left(\frac{\partial R}{\partial x^\alpha} \sigma^\beta R^* + R\sigma^\beta \frac{\partial R^*}{\partial x^\alpha} \right) \\ &= RWR^* + iR\sigma^\alpha \frac{\partial R^*}{\partial x^\alpha} - \frac{i}{4} R\sigma^\alpha \sigma_\beta R^* \left(\frac{\partial R}{\partial x^\alpha} \sigma^\beta R^* + R\sigma^\beta \frac{\partial R^*}{\partial x^\alpha} \right)\end{aligned}$$

. Hence, proving (3.A.21) reduces to proving that

$$\sigma^\alpha \sigma_\beta R^* \left(\frac{\partial R}{\partial x^\alpha} \sigma^\beta R^* + R\sigma^\beta \frac{\partial R^*}{\partial x^\alpha} \right) = 4\sigma^\alpha \frac{\partial R^*}{\partial x^\alpha}\,. \tag{3.A.22}$$

In order to prove formula (3.A.22) it is sufficient to show that

$$\sigma_\beta R^* \frac{\partial R}{\partial x^\alpha} \sigma^\beta R^* + \sigma_\beta \sigma^\beta \frac{\partial R^*}{\partial x^\alpha} = 4 \frac{\partial R^*}{\partial x^\alpha}$$

which, in turn, in view of the identity $\sigma_\beta \sigma^\beta = 3I$ (we already used it in the special case of Euclidean metric, see formula (3.A.14)), is equivalent to proving that

$$\sigma_\beta R^* \frac{\partial R}{\partial x^\alpha} \sigma^\beta = \frac{\partial R^*}{\partial x^\alpha} R\,. \tag{3.A.23}$$

The fact that the matrix function R is special unitary implies that at every point x of the manifold M and for every index $\alpha = 1, 2, 3$ the matrix $R^* \frac{\partial R}{\partial x^\alpha}$ is trace-free anti-Hermitian, which, in view of the identity $\sigma_\beta \sigma^\gamma \sigma^\beta = -\sigma^\gamma$ (we already used it in the special case of Euclidean metric, see formula (3.A.15)), implies that

formula (3.A.23) can be equivalently rewritten as

$$-R^*\frac{\partial R}{\partial x^\alpha} = \frac{\partial R^*}{\partial x^\alpha}R\,. \tag{3.A.24}$$

But formula (3.A.24) is an immediate consequence of the identity $R^*R = I$.

Having proved Property 4, let us examine the geometric meaning of the transformation (3.A.20). Let us expand the new Pauli matrices $\tilde{\sigma}$ with respect to the basis (3.A.2):

$$\tilde{\sigma}^\alpha(x) = s^j\,\tilde{V}_j{}^\alpha(x). \tag{3.A.25}$$

Formulae (3.A.1), (3.A.25) and (3.A.20) give us the following identity relating the new vector fields $\tilde{V}^j$ and the old vector fields V^j:

$$Rs^kR^*V_k = s^j\,\tilde{V}_j\,. \tag{3.A.26}$$

Resolving (3.A.26) for $\tilde{V}_j$ we get

$$\tilde{V}_j = O_j{}^kV_k\,, \tag{3.A.27}$$

where the real scalars $O_j{}^k$ are given by the formula

$$O_j{}^k = \frac{1}{2}\,\mathrm{tr}(s_jRs^kR^*)\,. \tag{3.A.28}$$

Note that in writing formulae (3.A.26) and (3.A.27) we chose to hide the tensor index, i.e. we chose to hide the coordinate components of our vector fields. Say, formula (3.A.27) written in more detailed form reads $\tilde{V}_j{}^\alpha = O_j{}^kV_k{}^\alpha$.

The scalars (3.A.27) can be viewed as elements of a real 3×3 matrix-function O with the first index, j, enumerating rows and the second, k, enumerating columns. It is easy to check that this matrix-function O is special orthogonal. Hence, the new vector fields $\tilde{V}_j$ are orthonormal and have the same orientation as the old vector fields V_j. We have shown that the transformation (3.A.20) has the

geometric meaning of switching from our original oriented orthonormal frame V_j to a new oriented orthonormal frame $\tilde{V}_j$.

Formula (3.A.28) means that the special unitary matrix R is, effectively, a square root of the special orthogonal matrix O. It is easy to see that for a given matrix $O \in \mathrm{SO}(3)$ formula (3.A.28) defines the matrix $R \in \mathrm{SU}(2)$ uniquely up to sign. This observation allows us to view the issue of the geometric meaning of the transformation (3.A.20) the other way round: given a pair of orthonormal frames, V_j and $\tilde{V}_j$, with the same orientation, we can recover the special orthogonal matrix-function $O(x)$ from formula (3.A.27) and then attempt finding a smooth special unitary matrix-function $R(x)$ satisfying (3.A.28). Unfortunately, this may not always be possible due to topological obstructions. We can only guarantee the absence of topological obstructions when the two frames, V_j and $\tilde{V}_j$, are sufficiently close to each other, which is equivalent to saying that we can only guarantee the absence of topological obstructions when the special orthogonal matrix-function $O(x)$ is sufficiently close to the identity matrix for all $x \in M$.

We illustrate the possibility of a topological obstruction by means of an explicit example.

Example 3.A.1. *Consider the unit torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1, 2, 3$, of period 2π. The metric is assumed to be Euclidean. Define the orthonormal frame as*

$$V_1{}^\alpha = \begin{pmatrix} \cos k_3 x^3 \\ \sin k_3 x^3 \\ 0 \end{pmatrix}, \qquad V_2{}^\alpha = \begin{pmatrix} -\sin k_3 x^3 \\ \cos k_3 x^3 \\ 0 \end{pmatrix}, \qquad V_3{}^\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad (3.A.29)$$

where $k_3 \in \mathbb{Z}$ is a parameter. Let W be the massless Dirac operator corresponding to the frame (3.A.29) with some even k_3 and let $\tilde{W}$ be the massless Dirac operator corresponding to the frame (3.A.29) with some odd k_3. We claim that there does not exist a smooth matrix-function $R : \mathbb{T}^3 \to \mathrm{SU}(2)$ which would give (3.A.28), where $O(x)$ is the special orthogonal matrix-function defined by formula (3.A.27). To prove this, it is sufficient to show that the two operators, W and $\tilde{W}$, have

different spectra. Straightforward separation of variables shows that any half-even integer (positive or negative) is an eigenvalue of $\tilde{W}$ *but is not an eigenvalue of* W*. What happens in this example is that a special unitary matrix-function* $R(x)$ *satisfying (3.A.28) can be defined locally but not globally: if we try to construct* $R(x^3)$ *moving along the circumference of the torus* $x^3 \in (-\pi, \pi)$ *we end up with a discontinuity,* $\lim\limits_{x^3 \to -\pi^+} R(x^3) = -\lim\limits_{x^3 \to \pi^-} R(x^3)$.

In fact, one can generalise Example 3.A.1 by introducing rotations in three different directions, which leads to eight genuinely distinct parallelizations. See also [47] page 524.

Let us emphasise that the topological obstructions we were discussing have nothing to do with Stiefel–Whitney classes. We are working on a parallelizable manifold and the Stiefel–Whitney class of such a manifold is trivial. The topological issue at hand is that our parallelizable manifold may be equipped with different spin structures.

We say that two massless Dirac operators, W and $\tilde{W}$, are equivalent if there exists a smooth matrix-function $R : M \to \mathrm{SU}(2)$ such that the corresponding Pauli matrices, σ^α and $\tilde{\sigma}^\alpha$, are related in accordance with (3.A.20). In view of Property 4 (see formula (3.A.21)) all massless Dirac operators from the same equivalence class generate the same spectral function (1.3.2) and the same counting function (3.1.4), so for the purposes of this chapter viewing such operators as equivalent is most natural.

As explained above, there may be many distinct equivalence classes of massless Dirac operators, the difference between which is topological. Studying the spectral theoretic implications of these topological differences is beyond the scope of this thesis. The two-term asymptotics (1.3.3) and (3.1.5) derived in the main text of the Chapter 3 do not feel this topology.

In theoretical physics the SU(2) freedom involved in defining the massless Dirac operator is interpreted as a gauge degree of freedom. We do not adopt this point

of view (at least explicitly) in order to fit the massless Dirac operator into the standard spectral theoretic framework.

We defined the massless Dirac operator (3.A.3) as an operator acting on 2-columns of scalar functions, i.e. on 2-columns of quantities which do not change under changes of local coordinates. This necessitated the introduction of the density $\sqrt{\det g_{\alpha\beta}}$ in the formula (3.A.9) for the inner product. In spectral theory it is more common to work with half-densities. Hence, we introduce the operator

$$W_{1/2} := (\det g_{\kappa\lambda})^{1/4}\, W\, (\det g_{\mu\nu})^{-1/4} \tag{3.A.30}$$

which maps half-densities to half-densities. We call the operator (3.A.30) *the massless Dirac operator on half-densities*.

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