Risk models with capital injections

by

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Abstract

One of the main issues in ruin theory is that existing formulae for continuous time models can only be applied to some special claim size distributions and the analytical expressions for other claim size distributions do not exist. This thesis addresses this issue by considering discrete time models as approximations to continuous time models, including the classical risk model, the Markov-modulated risk model and the classical risk model with dividends. It also shows that how these models are affected by the introduction of capital injections.

In Chapters 3 and 4 we construct a Gerber-Shiu function and use this to analyse the classical risk model with capital injections both analytically and probabilistically. Quantities such as the ultimate ruin probability and the joint density of the time of ruin and the number of claims until ruin are obtained by the inversion of the Laplace transform of our Gerber-Shiu function.

In Chapter 5 we develop a discrete time model to approximate the probability of ruin in infinite and finite time under the classical risk model with capital injections, and show that capital injections can lead to a reduction in the probability of ruin even when claim amounts follow a heavy-tailed distribution.

In Chapter 6 we extend our numerical algorithm from Chapter 5 to approximate the ultimate probability of ruin under a two-state Markov-modulated risk model with and without capital injections, and the density of the time of ruin under the same model with more than two states.

The final chapter investigates dividend strategies with capital injections. We examine the effect of capital injections on the barrier and threshold strategies and consider a reinsurance arrangement that covers any fall below a positive pre-determined surplus level, so that the insurance company may operate indefinitely.

To my dearest mum
and the memory of my beloved father

"We cannot change the wind, but set sail differently"

–Aristotle

Declaration

This is to certify that:

(i) the thesis comprises only my original work towards the PhD except where indicated in the preface,

(ii) due acknowledgement has been made in the text to all other material used,

(iii) the thesis is fewer than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Marjan Qazvini

Preface

Part of this thesis will appear as the following:

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Chapter 1

A survey of risk models

Classical ruin theory is motivated by the question of insolvency of an insurance company. It considers a simplified version of a real life insurance operation and examines the evolution of its funds over time. The theory assumes that the insurer starts with some non-negative amount of money. The inflow and outflow of cash include the premium income paid by policyholders and the claim expenses incurred by the insurer. Ruin theory is concerned with the level of an insurer's surplus. The initial goal of early researchers was to determine the probability of the insurer's surplus becoming negative; the event that we call ruin.

The aim of this chapter is to review some results in ruin theory. In the first section, we consider classical ruin theory in the continuous time case, then in Section 2 we provide analogues of results given in the first section in the discrete time case. The final section presents the Gerber-Shiu function and how it gives rise to a uniform treatment of ruin-related quantities.

1.1 The continuous time case

Throughout this chapter we consider a simple model for an insurer's surplus, which has only three components: initial surplus (or surplus at time zero), premiums received and claims paid. Thus, in this model we do not take into account investment income, tax and other expenses, yet we can gain insight using such a simple model.

The surplus process of an insurance company in continuous time is modelled by

$$U(t) = u + ct - S(t)$$

where u is the initial surplus, c is the rate of premium income per unit time, which is assumed to be received continuously, and $\{S(t)\}_{t\geq 0}$ is the aggregate claims process, defined by

$$S(t) = \sum_{i=1}^{N(t)} X_i$$

where $\{N(t)\}_{t\geq 0}$ is a Poisson process with Poisson parameter λ, and $N(t)$ denotes the number of claims that occur in the fixed time interval $[0, t]$. Further, $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables, where X_i represents the amount of the ith claim. Let $F = 1 - \bar{F}$ be the distribution function of X_1, with $F(0) = 0$, density function f, moment generating function M_X, and nth moment $E[X_1^n] = m_n$ with $m_1 := m$. Also, the process $\{S(t)\}_{t\geq 0}$ is a compound Poisson process with Poisson parameter λ. The positive loading condition is $c = (1+\theta)\lambda m$, where $\theta > 0$ is the premium loading factor.

We denote by $G(x, t) = \Pr(S(t) \leq x)$ the distribution function of the random variable $S(t)$ with density function

$$g(x,t) = \frac{\partial}{\partial x} G(x,t) = \sum_{n=1}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} f^{n*}(x) \tag{1.1}$$

for $x > 0$, where f^{n*} denotes the n-fold convolution of f with itself. See, for example, Bowers et al. (1997).

Let $\alpha(x)$ be a function defined for all $x \geq 0$. Then, its Laplace transform is defined as $\tilde{\alpha}(s) = \int_0^{\infty} e^{-sx}\alpha(x)dx$. The Laplace transform of $S(t)$ which is the same as the Laplace-Stieltjes transform of G is given by

$$\int_0^{\infty} e^{-sx} dG(x,t) = \exp\{\lambda t(\tilde{f}(s) - 1)\}. \tag{1.2}$$

See, for example, Panjer and Willmot (1992).

1.1.1 The probability of ultimate ruin

One of the questions considered in risk modelling is the probability that the surplus level drops below 0. We denote the time of ruin from initial surplus u by T_u and define it as $T_u = \inf\{t : U(t) < 0 \mid U(0) = u\}$ with $T_u = \infty$ if $U(t) \geq 0$ for all $t > 0$. The probability of ruin in infinite time is thus

$$\psi(u) = \Pr(T_u < \infty \mid U(0) = u) = 1 - \delta(u)$$

where $\delta(u)$ is the probability of survival from initial surplus u.

There are different approaches to evaluating the probability of ruin; some give exact values of $\psi(u)$ and some only give an approximation. We will review these techniques in the following theorems. We start by introducing the concept of the adjustment coefficient, denoted by R, which plays an important role in ruin theory and is a crude measure of risk for the surplus process $\{U(t)\}_{t\geq 0}$. The adjustment coefficient is the unique positive solution of the so-called Lundberg's equation, which is given by

$$\lambda + cr = \lambda M_X(r) \tag{1.3}$$

assuming M_X exists. Normally, the higher R is, the less risky the surplus process is.

Theorem 1.1. (Lundberg's inequality)
Lundberg's inequality states that the ultimate ruin probability is bounded above by

$$\psi(u) \leq \exp\{-Ru\}$$

assuming that R exists.

Proof. See, for example, Lundberg (1932). □

Martingale and induction approaches can be used to prove Lundberg's inequality (see, for example, Gerber, 1979 and Dickson, 2005). Willmot and Lin (1994) presented a Lundberg bound based on the tail of a compound geometric distribution. The Lundberg upper bound depends on the existence of R. Willmot (1994) derived an upper bound based on a *new worse than used* (NWU) distribution that can also be applied to heavy-tailed distributions for which R does not exist.

Theorem 1.2. If $B(x)$ is the distribution function of a non-negative random variable and $\bar{B}(x) = 1 - B(x)$, then $B(x)$ is NWU if $\bar{B}(x)\bar{B}(y) \leq \bar{B}(x+y)$ for $x \geq 0$, $y \geq 0$. Suppose $\bar{B}(x)$ also satisfies

$$\int_0^\infty \left(\bar{B}(x)\right)^{-1} dF(x) \leq c(x)(1+\theta),$$

where $c(x)$ is a non-decreasing function for $x \geq 0$, then

$$\psi(x) \leq c(x)\bar{B}(x), \qquad x \geq 0.$$

Proof. See Willmot (1994, Theorem 1 and Section 5). □

Dickson (1994a) introduced an upper bound for the ruin probability when the moment generating function M_X does not exist. His approach is based on a truncated moment generating function.

Theorem 1.3. Let t be a real positive number and K_t be the unique positive solution to

$$\int_0^t \exp\{K_t x\} f(x)\, dx = 1 + \theta.$$

Then, for $0 \leq u \leq t$,

$$\psi(u) \leq \exp\{-K_t u\} + \beta(t)$$

where $\beta(t) = (1 - P(t))/(1 + \theta - P(t))$ and

$$P(t) = \int_0^t (1 - F(x))\, dx/m \tag{1.4}$$

is an equilibrium distribution function.

Proof. See Dickson (1994a, Section 2). □

Remark 1.1. The function P is also known as the ladder hight distribution (or integrated tail distribution) of F.

Other references on upper bounds for the classical risk model include, for example, De Vylder and Goovaerts (1984), Cai and Wu (1997), and Cai and Garrido (1999).

Theorem 1.4. (Cramér's asymptotic formula)
Cramér's asymptotic formula states that

$$\psi(u) \sim Ce^{-Ru} \tag{1.5}$$

where R is the adjustment coefficient and

$$C = \frac{c/\lambda - m}{E[Xe^{RX}] - c/\lambda}.$$

Proof. See, for example, Gerber (1979). □

Remark 1.2. Cramér's asymptotic formula provides an estimate for the ruin probability when u is sufficiently large. This estimate is exact for exponentially distributed individual claim amounts.

An exact expression for the probability of ruin can be obtained for some distributions for which we can establish and solve an integro-differential equation for $\delta(u)$.

Conditioning on the time and the amount of the first claim we obtain

$$\delta(u) = \int_0^\infty \lambda e^{-\lambda t} \int_0^{u+ct} f(x)\delta(u + ct - x)\, dx\, dt \tag{1.6}$$

which can be manipulated to obtain an integro-differential equation.

Theorem 1.5. (Integro-differential equation)
The probability of survival satisfies

$$\frac{d}{du}\delta(u) = \frac{\lambda}{c}\delta(u) - \frac{\lambda}{c}\int_0^u f(x)\delta(u - x)\, dx \tag{1.7}$$

with $\delta(0) = 1 - \lambda m/c$.

Proof. See, for example, Gerber (1979). □

Remark 1.3. In the classical risk model, the probability of ruin for initial surplus 0 is independent of the distribution function of the individual claim amounts.

We can also express the ruin probability as satisfying a defective renewal equation.

Theorem 1.6. (Defective renewal equation)
For $u \geq 0$, the probability of ruin satisfies the following defective renewal equation:

$$\psi(u) = \psi(0)\int_0^u \psi(u - y)dP(y) + \psi(0)\bar{P}(u) \tag{1.8}$$

where $\bar{P}(u) = 1 - P(u)$ is the ladder height distribution from (1.4)

Proof. See, for example, Gerber (1979). □

By successive substitution, we can write equation (1.8) as the so-called convolution formula.

Theorem 1.7. (Beekman's convolution formula)
The ruin probability can be stated as the tail of a compound geometric distribution, i.e.

$$\psi(u) = \delta(0)\sum_{n=1}^{\infty}(\psi(0))^n \bar{P}^{n*}(u)$$

where $\bar{P}^{n*}$ is the n-fold convolution of $\bar{P}$ with itself.

Proof. See, for example, Beekman (1974). □

Using equation (1.7), we can create a differential equation for certain forms of f by eliminating the integral term. For example, when individual claim amounts are exponentially distributed, differentiation of equation (1.7) can lead to a second-order differential equation and when claim amounts have an Erlang(n) distribution, repeated differentiation of equation (1.7) gives rise to a higher order differential equation. These differential equations can then be solved by standard techniques. We can also solve equation (1.7) via Laplace transforms.

Theorem 1.8. The Laplace transform of the survival function is given by

$$\tilde{\delta}(s) = \frac{c\delta(0)}{cs - \lambda + \lambda\tilde{f}(s)}. \tag{1.9}$$

Proof. See, for example, Dickson (2005). □

Gerber et al. (1987) argued that the probability of ruin is a crude measure of stability, and that we need to know how serious the situation is if ruin occurs. The next section addresses this question.

1.1.2 The severity of ruin

Let $|U(T_u)|$ be the deficit at the time of ruin. We define the probability that ruin occurs and that the insurer's deficit at ruin, or severity of ruin, is at most y by

$$H_1(u,y) = \Pr(T_u < \infty \text{ and } |U(T_u)| \leq y \mid U(0) = u) = \int_0^\infty \int_0^y w(u,x,t)\,dx\,dt$$

where $w(u,x,t)$ is the (defective) joint density of $|U(T_u)|$ and T_u. We note that $\lim_{y\to\infty} H_1(u,y) = \psi(u)$, so that

$$\frac{H_1(u,y)}{\psi(u)} = \Pr(|U(T_u)| \leq y \mid T_u < \infty, U(0) = u)$$

is a proper distribution function. Hence, for a given initial surplus u, $H_1(u,\cdot)$ is a (defective) distribution with (defective) density

$$h_1(u,y) = \frac{\partial}{\partial y} H_1(u,y).$$

Lemma 1.1. When the initial surplus is 0,

$$h_1(0, y) = \frac{\lambda}{c}(1 - F(y)).$$

Proof. See, for example, Bowers et al. (1997). □

Using Lemma 1.1 and conditioning on the first occasion on which the surplus falls below its initial level u, we have the following result.

Theorem 1.9. The probability and severity of ruin function satisfies

$$H_1(u, y) = \int_0^u h_1(0, x) H_1(u - x, y)\, dx + \int_u^{u+y} h_1(0, x)\, dx.$$

Proof. See Gerber et al. (1987). □

Gerber et al. (1987) found an explicit expression for $h_1(u, y)$ when claim amounts follow a combination of exponential distributions and a combination of gamma distributions. Further, Dufresne and Gerber (1988a) defined the function $\psi(u, y)$, satisfied by $H_1(u, y) = \psi(u) - \psi(u, y)$, to be the probability that ruin occurs and that the deficit at the time of ruin exceeds y. They used this function to derive the probability and severity of ruin function for claim amounts distributed as a translation of a combination of exponential distributions.

1.1.3 The joint distribution of the surplus prior to ruin and the deficit at ruin

In this section, we consider the joint distribution of the surplus immediately prior to ruin and the deficit at ruin. First, we introduce the marginal distribution of the surplus immediately before ruin.

Let T_u^- be the time immediately prior to ruin, and let $U(T_u^-)$ denote the surplus level immediately prior to payment of the claim that causes ruin. Then, the probability that ruin occurs from initial surplus u and that the surplus immediately prior to ruin is at most x is

$$H_2(u, x) = \Pr(T_u < \infty \text{ and } U(T_u^-) \leq x \mid U(0) = u) = \int_0^\infty \int_0^\infty \int_0^x w(u, z, y, t)\, dz\, dy\, dt$$

where $w(u,z,y,t)$ is the (defective) joint density of $U(T_u^-)$, $|U(T_u)|$ and T_u. We note that $\lim_{x\to\infty} H_2(u,x) = \psi(u)$, so that $H_2(u,x)$ is a (defective) distribution function with the corresponding (defective) density

$$h_2(u,x) = \frac{\partial}{\partial x} H_2(u,x).$$

The function $H_2(u,x)$ is continuous at $u = x$ but is not differentiable – see Dickson (1992). Therefore, the two cases $u < x$ and $u > x$ must be considered separately.

We now define

$$\begin{aligned} H(u,x,y) &= \Pr(T_u < \infty, U(T_u^-) \le x, |U(T_u)| \le y \mid U(0) = u) \\ &= \int_0^\infty \int_0^y \int_0^x w(u,z,s,t)\, dz\, ds\, dt \end{aligned}$$

to be the (defective) joint distribution of the surplus immediately prior to ruin and the deficit at ruin with $h(u,x,y)$ being the (defective) joint density.

Dufresne and Gerber (1988b) considered the joint density of the surplus prior to ruin and the deficit at ruin in terms of the marginal density of the surplus prior to ruin as

$$h(u,x,y) = h_2(u,x)\frac{f(x+y)}{1-F(x)}. \tag{1.10}$$

They derived an explicit expression for $h(u,x,y)$ when claim amounts have an exponential distribution and a combination of exponential distributions.

Lemma 1.2. The (defective) joint density of $U(T_u^-)$ and $|U(T_u)|$ from initial surplus 0 is given by

$$h(0,x,y) = \frac{\lambda}{c} f(x+y). \tag{1.11}$$

Proof. See Dufresne and Gerber (1988b). □

Dickson (1992) derived an expression for $h(u,x,y)$.

Theorem 1.10. The (defective) joint density of $U(T_u^-)$ and $|U(T_u)|$ is given by

$$h(u,x,y) = \begin{cases} h(0,x,y)\frac{1-\psi(u)}{1-\psi(0)} & \text{if } u < x, \\ \\ h(0,x,y)\frac{\psi(u-x)-\psi(u)}{1-\psi(0)} & \text{if } u \ge x. \end{cases} \tag{1.12}$$

Proof. See Dickson (1992). □

Remark 1.4. Dickson (1992) used a duality argument to show that the distribution function of the surplus prior to ruin is the same as the distribution function of the deficit at ruin when $u = 0$, i.e. $H_2(0, x) = H_1(0, x)$.

Up until now we have discussed ruin-related quantities in an infinite time horizon. However, it is perhaps more realistic to look at the probability that ruin occurs before a fixed time t. The finite time ruin probability is our next topic.

1.1.4 The probability of ruin in finite time

We define

$$\psi(u,t) = \Pr(T_u \leq t \mid U(0) = u) = \int_0^t w(u,\tau)\,d\tau = 1 - \delta(u,t) \tag{1.13}$$

to be the finite time ruin probability, where $\delta(u, t)$ is the finite time survival probability and $w(u, t)$ is the defective density of the time of ruin given by

$$w(u,t) = \frac{\partial}{\partial t}\psi(u,t).$$

There are two main approaches to the analysis of the distribution of the time of ruin: (i) Prabhu's (1961) formula, and (ii) the Gerber-Shiu function that we will introduce later.

Prabhu (1961) provided an expression for the finite time survival probability.

Theorem 1.11. (Prabhu's formula)
The survival probability in finite time satisfies the partial integro-differential equation:

$$\frac{\partial}{\partial t}\delta(u,t) = c\frac{\partial}{\partial u}\delta(u,t) - \lambda\delta(u,t) + \lambda\int_0^u \delta(u-s,t)f(s)\,ds \tag{1.14}$$

and the solution to (1.14) is expressed as

$$\delta(u,t) = G(u+ct,t) - c\int_0^t g(u+cs,s)\delta(0,t-s)\,ds \tag{1.15}$$

where $g(u, t)$ is given by (1.1) and

$$\delta(0,t) = \frac{1}{ct}\int_0^{ct} G(x,t)\,dx. \tag{1.16}$$

Proof. See, for example, Prabhu (1961) or Seal (1974). □

Seal (1974) provided a probabilistic interpretation for (1.15) and calculated values of $\delta(u,t)$ numerically. Dickson (2007) used Prabhu's (1961) formula to present a general expression for the (defective) density of the time of ruin.

Theorem 1.12. When the initial surplus is u we have

$$\begin{aligned} w(u,t) &= \lambda e^{-\lambda t}\bar{F}(u+ct)+\lambda\int_0^{u+ct} g(u+ct-x,t)\bar{F}(x)\,dx \\ &\quad -c\int_0^t g(u+cs,s)w(0,t-s)\,ds. \end{aligned} \tag{1.17}$$

Proof. See Dickson (2007). □

Dickson (2007) has given probabilistic interpretation to formula (1.17) and showed that expression (1.17) can be extended to other ruin-related quantities in finite time using similar interpretations. Recently, Willmot (2015) applied a partial differential equation to study the (defective) joint distribution function of the time of ruin and the deficit at ruin. We define

$$W(u,y,t)=\Pr(T_u\le t,|U(T_u)|\le y\mid U(0)=u)$$

to be the (defective) joint distribution function of the time of ruin and the deficit at ruin. The following theorem is given by Willmot (2015).

Theorem 1.13. The function $W(u,y,t)$ satisfies the partial integro-differential equation

$$\begin{aligned} \frac{\partial}{\partial t}W(u,y,t) &= c\frac{\partial}{\partial u}W(u,y,t)-\lambda W(u,y,t)+\lambda\int_0^u W(u-x,y,t)f(x)\,dx \\ &\quad +\lambda\left(\bar{F}(u)-\bar{F}(u+y)\right) \end{aligned} \tag{1.18}$$

and the solution to (1.18) may be expressed as

$$\begin{aligned} W(u,y,t) &= e^{-\lambda t}\alpha_1(u+ct,y)-\alpha_1(u,y)+\int_0^{u+ct}\alpha_1(u+ct-x,y)g(x,t)\,dx \\ &\quad -c\int_0^t W(0,x,y)g(u+c(t-x),t-x)\,dx \end{aligned}$$

where

$$\alpha_1(u,y)=\frac{1}{\theta m}\int_0^u (1-\psi(u-x))\left(\bar{F}(x)-\bar{F}(x+y)\right)\,dx.$$

Proof. See Willmot (2015). □

One issue with Prabhu's (1961) formula is that it is expressed in terms of $g(x,t)$. Therefore, to find explicit solutions for $\delta(u,t)$, an explicit expression for $g(x,t)$ must exist. Otherwise, we can apply the numerical method provided by Seal (1974). Alternatively, we can approximate $g(x,t)$ by Panjer's (1981) recursion formula (see Section 1.2) and compute its values recursively. For example, Dickson and Waters (1992) considered a discrete time risk model and presented a numerical algorithm that can provide approximations to $W(u,y,t)$. Further, Dickson and Waters (2002) applied the algorithm in Dickson and Waters (1991) to approximate $w(u,t)$.

1.2 The discrete time case

Discrete time risk models are of interest to us, because we can use them to approximate risk models in continuous time. Analytical expressions for ruin-related quantities in infinite time exist, provided that we have the Laplace transform of claim amount distributions. Also, many of the expressions for ruin-related quantities in finite time are expressed in terms of $g(x,t)$. Therefore, to treat such expressions we need the functional form of $g(x,t)$ which does not always exist. In a discrete time model, we can calculate numerical values of the aggregate claims distribution and compute different ruin-related quantities. We now introduce a discrete time risk model.

The surplus process of an insurance company at time n, $n = 1,2,3,\dots$ is denoted by $U^d(n)$ and is defined by

$$U^d(n) = u + n - \sum_{i=1}^{n} Y_i$$

for $n = 1,2,3,\dots$, where $u = U^d(0)$ is the insurer's initial surplus, n is the total premium income up to time n – assuming that the insurer's premium income per unit time is 1. The insurer's aggregate claim amount in the ith time interval is denoted by Y_i and $\{Y_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables, each distributed on the non-negative integers, with $E[Y_1] < 1$, probability function $\{g(x)\}_{x=0}^{\infty}$ and distribution function G.

In our discrete time model there are two possible definitions of ruin. In the first definition, ruin occurs when the surplus falls below 0. In the second one, ruin occurs when the surplus falls to 0 or below 0. We note that under the latter definition, ruin does not occur at time 0 if $u = 0$. Dickson and Waters (1991, 1992) argued that the second definition of ruin gives rise to a better approximation to the continuous time model. Based on their definition of ruin, we denote the time of ruin from initial surplus u by T_u^d and define it as

$$T_u^d = \min\{n \geq 1 : U^d(n) \leq 0 \mid U^d(0) = u\}$$

with $T_u^d = \infty$ if $U^d(n) > 0$ for $n = 1, 2, 3, \ldots$. The probability of ultimate ruin is thus

$$\psi^d(u) = \Pr(T_u^d < \infty \mid U^d(0) = u) = 1 - \delta^d(u)$$

where $\delta^d(u)$ is the probability of survival from initial surplus u. Also, for an integer value of t, we define the finite time ruin probability as

$$\psi^d(u,t) = \Pr(T_u^d \leq t \mid U^d(0) = u).$$

Further, let $H^d(u,y)$ denote the probability and severity of ruin function for $u = 0, 1, 2, \ldots$, $y = 1, 2, 3, \ldots$. It is defined by

$$H^d(u,y) = \Pr(T_u^d < \infty \text{ and } |U^d(T_u^d)| < y \mid U^d(0) = u)$$

and the (defective) probability function of the severity of ruin for $u = 0, 1, 2, \ldots$ and $y = 0, 1, 2, \ldots$ is defined as

$$h^d(u,y) = \Pr(T_u^d < \infty \text{ and } |U^d(T_u^d)| = y \mid U^d(0) = u).$$

As we have described above, our interest in this discrete time model is because of its capacity in the approximation of the classical risk model. The approximation procedure involves both the rescaling of monetary units and time units. This means that in the first step we discretise the individual claim amount distributions using a suitable scaling parameter and then we change the time scale. If the scaling parameter is $\beta > 0$, for $i = 1, 2, 3, \ldots$, we can replace X_i by $X_{1,i}$ where $X_{1,i}$ is a discrete random variable distributed on $0, 1/\beta, 2/\beta, \ldots$. Then, we change the time scale. In particular, we change the Poisson parameter to $1/(1+\theta)\beta$ which means that our premium income per unit time is 1. Therefore, for example, $\psi^d(u\beta, (1+\theta)\beta t)$ and $h^d(u\beta, y\beta)$ provide approximations to $\psi(u,t)$ and $h_1(u,y)$. We would expect such approximation to be good if the interval between the time points at which we check the surplus is small. In this approximation, the larger β is, the better the approximation is.

There are a number of ways in which a continuous distribution, F, with $F(0) = 0$, might be discretised. See, for example, Panjer and Lutek (1983). One approach is through matching probabilities, proposed by De Vylder and Goovaerts (1988, Section 7).

Result 1.1. A discrete distribution with probability function $\{f(x)\}_{x=0}^{\infty}$ and distribution function $\mathcal{F}(x)$, can be created from a continuous distribution with distribution function F by

$$\mathcal{F}(x) = \sum_{j=0}^{x} f(j) = \int_{x}^{x+1} F(y)\,dy. \tag{1.19}$$

Such a discretisation procedure is mean preserving. See, for example, De Vylder and Goovaerts (1988) or Dickson (2005).

After we find the discretised version of the individual claim amount distribution we can apply Panjer's (1981) recursion formula to calculate the probability function of aggregate claims.

Theorem 1.14. (Panjer recursion formula)
If a counting distribution with probability function $\{p_n\}_{n=0}^{\infty}$ satisfies the recursion

$$p_n = p_{n-1}\Big(a + \frac{b}{n}\Big)$$

for $n = 1, 2, 3, \ldots$, where a and b are constants, and individual claims have probability function f, then the probability function of aggregate claims is given recursively by

$$g(x) = \frac{1}{1 - af(0)} \sum_{k=1}^{x} \Big(a + \frac{bk}{x}\Big) f(k) g(x-k) \tag{1.20}$$

with $g(0) = p_0 + \sum_{n=1}^{\infty} p_n f(0)^n$.

Proof. See Panjer (1981). □

Remark 1.5. When the counting distribution is Poisson with parameter λ, the Panjer recursion formula is given by

$$g(x) = \frac{\lambda}{x} \sum_{k=1}^{x} k f(k) g(x-k) \tag{1.21}$$

for $x = 1, 2, 3, \ldots$, with $g(0) = \exp\{\lambda(f(0) - 1)\}$.

Similar to the classical risk model we can define an upper bound for $\psi^d(u)$. The following theorem gives Lundberg's inequality for our discrete time model.

Theorem 1.15. (Lundberg's inequality)
The ultimate ruin probability satisfies

$$\psi^d(u) \leq e^{-R^d u} \tag{1.22}$$

where R^d is the adjustment coefficient and is the unique positive root of

$$E[\exp\{r(Y_1 - 1)\}] = 1. \tag{1.23}$$

Proof. See, for example, Bowers et al. (1997, Section 13.2) or Dickson (2005, Section 6.5). □

Gerber (1988) showed how a compound binomial model is analogous to the compound Poisson model of classical risk theory. Extending Gerber's (1988) results, Dickson (1994b) showed that the ultimate ruin probability for a compound binomial model gives a good approximation to the ultimate ruin probability in the classical continuous time compound Poisson model.

Theorem 1.16. The ultimate probability of survival is given by

$$\delta^d(u) = \delta^d(0) + \sum_{k=1}^{u} \delta^d(k)[1 - G(u-k)] \tag{1.24}$$

for $u = 1, 2, 3, \ldots$, with $\delta^d(0) = 1 - E[Y_1]$.

Proof. See Dickson (1994b). □

Dickson (1994b) derived an expression for $\delta^d(u)$ in terms of $h^d(0, y)$, which is interpreted as the probability that the surplus falls below its initial level for the first time and by amount y.

Theorem 1.17. The probability of survival can be written as

$$\delta^d(u) = \delta^d(0) + \sum_{y=1}^{u} h^d(0, u-y)\delta^d(y) \tag{1.25}$$

for $u = 1, 2, 3, \ldots$, with $\delta^d(0) = 1 - \sum_{y=0}^{\infty} h^d(0, y) = 1 - E[Y_1]$. Further, $\delta^d(0) = \theta/(1+\theta)$.

Proof. See Dickson (1994b). □

Remark 1.6. Dickson et al. (1995) pointed out that expression (1.25) is stable for recursive calculation, whereas expression (1.24) can be unstable for large values of u.

Dickson and Waters (1992) developed a recursive formula for the probability and severity of ruin function.

Theorem 1.18. The probability and severity of ruin function H^d satisfies

$$H^d(u+1,y) = g(0)^{-1}\left(H^d(u,y) - \sum_{j=1}^{u} g(j)H^d(u+1-j,y) + G(u) - G(u+y)\right) \tag{1.26}$$

for $u = 0, 1, 2, \ldots$, and $H^d(0,y) = \sum_{j=0}^{y-1}(1 - G(j))$.

Proof. See Dickson and Waters (1992). □

Explicit solutions for the finite time ruin probability are generally not available. Dickson and Waters (1991) adapted the recursive algorithm of De Vylder and Goovaerts (1988) to approximate the probability of ruin in finite time.

Theorem 1.19. The finite time ruin probability can be calculated recursively from

$$\psi^d(u,1) = \sum_{k=u+1}^{\infty} g(k)$$

and for $t > 1$,

$$\psi^d(u,t) = \psi^d(u,1) + \sum_{k=0}^{u} g(k)\psi^d(u+1-k,t-1). \tag{1.27}$$

Proof. See Dickson and Waters (1991). □

Remark 1.7. Dickson and Waters' (1991, 1992) algorithms result in time consuming calculations, particularly when u and t are large. De Vylder and Goovaerts (1988) presented a truncation procedure that could be applied to such algorithms. This procedure reduces the number of calculations involved by ignoring small probabilities, and a bound can be placed on the error that is introduced.

Dickson and Waters (2002) showed that the finite time ruin probability could be used to approximate the density of T_u at jh for some (small) $h > 0$ and $j = 1, 2, 3, \dots$, as:

$$w(u,t) \approx \frac{\psi^d(u, jh) - \psi^d(u,(j-1)h)}{h\psi^d(u)}. \tag{1.28}$$

In the next section, we investigate ruin theory by means of the famous Gerber-Shiu function.

1.3 Gerber-Shiu analysis

Up until now, we have looked at different ruin-related quantities individually. Gerber and Shiu (1998) introduced a function that provides a uniform treatment of these quantities. In this section, we review the classical risk model based on Gerber-Shiu analyses.

The Gerber-Shiu discounted penalty function is defined as

$$\begin{aligned}\phi_\delta(u) &= E\big[e^{-\delta T_u}\omega(U(T_u^-), |U(T_u)|)I(T_u < \infty) \mid U(0) = u\big] \\ &= \int_0^\infty e^{-\delta t}\int_0^\infty\int_0^\infty \omega(x,y)w(u,x,y,t)\,dx\,dy\,dt \end{aligned} \tag{1.29}$$

for $u \geq 0$, where δ is a non-negative parameter which can be considered either as the parameter of a Laplace transform or the force of interest, I is the indicator function, so that $I(A) = 1$ if the event A occurs and equals 0 otherwise and $\omega(x, y)$ is a non-negative penalty function, defined for $x \geq 0$ and $y > 0$. The function $\phi_\delta(u)$ represents different ruin-related quantities depending on the form taken by $\omega(x, y)$. For example, if $\omega(x, y) = 1$, then $\phi_\delta(u)$ gives the Laplace transform of the time of ruin, and $\phi_0(u)$ is the ruin probability $\psi(u)$. For $\omega(x, y) = I\{x \leq y_1\}I\{y \leq y_2\}$, $\phi_0(u)$ gives the (defective) joint distribution function of the surplus immediately prior to ruin $U(T_u^-)$ and the deficit at ruin $|U(T_u)|$. For $\omega(x, y) = x^n y^m$, $\phi_0(u)$ gives the joint moments of $U(T_u^-)$ and $|U(T_u)|$.

Central to Gerber-Shiu analysis is the equation which Gerber and Shiu (1998) called Lundberg's fundamental equation. It is given by

$$\delta + \lambda - cs = \lambda\tilde{f}(s). \tag{1.30}$$

Theorem 1.20. Lundberg's fundamental equation has a unique positive root, denoted $\rho \equiv \rho(\delta)$ with $\rho(\delta) = 0$ when $\delta = 0$. There may also be a negative root, denoted $-R \equiv -R(\delta)$.

Proof. See Gerber and Shiu (1998). □

Remark 1.8. When $\delta = 0$, equation (1.30) is equivalent to (1.3) and R is the adjustment coefficient.

We now state a theorem that shows the Gerber-Shiu function $\phi_\delta(u)$ satisfies a defective renewal equation.

Theorem 1.21. (Defective renewal equation)
For $\delta > 0$, $\phi_\delta(u)$ satisfies the following defective renewal equation

$$\phi_\delta(u) = \int_0^u \phi_\delta(u-x)a(x)\,dx + b(u) \tag{1.31}$$

with

$$a(x) = \frac{\lambda}{c}\int_x^\infty e^{-\rho(y-x)} f(y)\,dy$$

and

$$b(u) = \frac{\lambda}{c}\int_u^\infty e^{-\rho(x-u)}\int_x^\infty \omega(x, y-x) f(y)\,dy\,dx$$

where $\rho > 0$ is the unique positive solution of Lundberg's fundamental equation.

Proof. See Gerber and Shiu (1998). □

The discounted joint density of $U(T_u^-)$ and $|U(T_u)|$ for $u > 0$ is defined by

$$h_\delta(u, x, y) = \int_0^\infty e^{-\delta t} w(u, x, y, t)\,dt \tag{1.32}$$

and for $u = 0$ by

$$h_\delta(0, x, y) = \frac{\lambda}{c} e^{-\rho x} f(x+y). \tag{1.33}$$

Further, the discounted marginal probability density function of $U(T_u^-)$ for $u > 0$ is defined by

$$h_{\delta,2}(u, x) = \int_0^\infty h_\delta(u, x, y)\,dy$$

and for $u = 0$ by

$$h_{\delta,2}(0, x) = \frac{\lambda}{c} e^{-\rho x} \bar{F}(x). \tag{1.34}$$

Gerber and Shiu (1998) provided an equivalent expression to (1.10) when $\delta > 0$, which is given by

$$h_\delta(u, x, y) = h_{\delta,2}(u, x) \frac{f(x + y)}{1 - F(x)}.$$

We note that if we set $u = 0$ in equation (1.31), $\phi_\delta(0) = b(0)$, so

$$\frac{\lambda}{c} \int_0^\infty \int_0^\infty e^{-\rho x} \omega(x, y) f(x + y)\, dy\, dx = \int_0^\infty \int_0^\infty \omega(x, y) h_\delta(0, x, y)\, dy\, dx$$

by (1.29) and (1.32). Since this identity holds for an arbitrary function $\omega(x, y)$ we can simply recover equations (1.33) and (1.34). Using these results, we can write a defective renewal equation for $h_\delta(u, x, y)$ and $h_{\delta,2}(u, x)$.

Corollary 1.1. The defective renewal equations for $h_\delta(u, x, y)$ and $h_{\delta,2}(u, x)$, for $x, y, u > 0$ are, respectively, given by

$$h_\delta(u, x, y) = \int_0^u h_\delta(u - z, x, y) a(z)\, dz + \frac{\lambda}{c} e^{-\rho(x-u)} f(x + y) I(x > u)$$

and

$$h_\delta(u, x) = \int_0^u h_\delta(u - z, x) a(z)\, dz + \frac{\lambda}{c} e^{-\rho(x-u)} \bar{F}(x) I(x > u).$$

We can also write a defective renewal equation for $\phi_\delta(u)$ by probabilistic reasoning.

Theorem 1.22. The discounted penalty function can be written as

$$\begin{aligned}
\phi_\delta(u) &= \int_0^\infty \int_0^\infty \int_0^u e^{-\delta t} \phi_\delta(u - y) w(0, x, y, t)\, dy\, dx\, dt \\
&\quad + \int_0^\infty \int_0^\infty \int_u^\infty e^{-\delta t} \omega(x + u, y - u) w(0, x, y, t)\, dy\, dx\, dt \\
&= \int_0^\infty \int_0^u \phi_\delta(u - y) h_\delta(0, x, y)\, dy\, dx + \int_0^\infty \int_u^\infty \omega(x + u, y - u) h_\delta(0, x, y)\, dy\, dx.
\end{aligned}$$

Proof. See Gerber and Shiu (1998, formula 3.21). □

An alternative approach to presenting the Gerber-Shiu function has been introduced by Lin and Willmot (1999) where the general solution for $\phi_\delta(u)$ is expressed as the tail of a compound geometric distribution function. Considering $\delta > 0$ and $\omega(x,y) = 1$, (1.29) reduces to the Laplace transform of the time of ruin, denoted by $\bar{K}(u)$ and defined as

$$\bar{K}(u) = E[e^{-\delta T_u} I(T_u < \infty) \mid U(0) = u].$$

The next theorem gives an expression for $\bar{K}(u)$.

Theorem 1.23. The defective renewal equation for $\phi_\delta(u)$ can be expressed as

$$\phi_\delta(u) = \frac{1}{1+\beta}\int_0^u \phi_\delta(u-x)c(x)\,dx + \frac{1}{1+\beta}B(u) \tag{1.35}$$

where $(1+\beta)^{-1} = \frac{\lambda}{c}\int_0^\infty e^{-\rho y}\bar{F}(y)\,dy$, $c(x) = (1+\beta)a(x)$ and $B(u) = (1+\beta)b(u)$; accordingly the Laplace transform of the time of ruin satisfies

$$\bar{K}(u) = \frac{1}{1+\beta}\int_0^u \bar{K}(u-x)c(x)\,dx + \frac{1}{1+\beta}\bar{C}(u)$$

which can also be expressed as the tail of a compound geometric distribution as

$$\bar{K}(u) = \sum_{n=1}^{\infty} \frac{\beta}{1+\beta}\Big(\frac{1}{1+\beta}\Big)^n \bar{C}^{n*}(u) \tag{1.36}$$

where $\bar{C}^{n*}(u)$ is the tail of the n-fold convolution of $C(u) = \int_0^u c(x)\,dx$.

Proof. See Lin and Willmot (1999). □

Lin and Willmot (2000) developed recursive relations for the moments of the time of ruin, the surplus prior to ruin and the deficit at ruin. They provided examples in the cases of claim amounts with exponential, combinations of exponential and mixtures of Erlang distributions.

Theorem 1.24. (The nth moment of the time of ruin)
The nth moment of the time of ruin, given that ruin has occurred, is given by

$$E[T_u^n | T_u < \infty] = \frac{\psi_n(u)}{\psi(u)}$$

for $n = 1, 2, 3, \ldots$, with $\psi_0(u) = \psi(u)$, the probability of ruin, and $\psi_n(u)$ is given recursively by

$$\psi_n(u) = \frac{n}{\lambda m\theta}\bigg(\int_0^u \psi(u-x)\psi_{n-1}(x)\,dx + \delta(u)\int_0^\infty \psi_{n-1}(x)\,dx - \int_0^u \psi_{n-1}(x)\,dx\bigg).$$

Proof. See Lin and Willmot (2000). □

The problem of the moments of ruin-related quantities has also been considered by Albrecher and Boxma (2005a). Their approach is to find these moments by differentiating the Laplace transform of the respective functions.

Dickson and Willmot (2005) found an expression for the density of the time of ruin by inverting its Laplace transform through a Laplace transform relationship given in the next theorem.

Theorem 1.25. For two functions A and B, if

$$\int_0^\infty e^{-\rho t} A(t)\, dt = \int_0^\infty e^{-\delta t} B(t)\, dt$$

then

$$B(t) = ce^{-\lambda t} A(ct) + \int_0^{ct} \frac{x}{t} g(ct - x, t) A(x)\, dx$$

where ρ is the unique positive solution of Lundberg's fundamental equation.

Proof. See Dickson and Willmot (2005). □

Cheung et al. (2008) obtained general expressions for $w(u, y, t)$ when claim amounts follow a combination of exponential and mixed Erlang distributions. Using this result, Dickson (2008) derived the bivariate Laplace transform of the joint density of the time of ruin and the deficit at ruin and applied the Laplace transform relationship of Theorem 1.25 to invert the bivariate Laplace transform in the cases of individual claim amounts with Erlang(2) and a mixture of two exponential distributions. Dickson (2007, 2008) investigated the density of the time of ruin with two different approaches. The functional form of $w(u, t)$ in the case of exponential claim amount distribution in Dickson (2007) corresponds to the result had been obtained by Dickson et al. (2005). Further, Dickson (2008) pointed out that his approach could reproduce the result in Drekic and Willmot (2003).

When $\omega(x, y) = e^{-sx-zy}$, the Gerber-Shiu function represents the trivariate Laplace transform of the time of ruin, the surplus immediately prior to ruin and the deficit at ruin. Landriault and Willmot (2009) found an explicit expression for the (defective) joint distribution of T_u, $U(T_u^-)$ and $|U(T_u)|$ by inverting their trivariate Laplace transform. Landriault et al. (2011) extended the Gerber-Shiu function and applied Lagrange's expansion theorem to find the (defective) distribution of the number of claims until ruin. The extended Gerber-Shiu function is given by

$$\phi_{r,\delta}(u) = E[r^{N_{T_u}} e^{-\delta T_u} I(T_u < \infty) \mid U(0) = u] = \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} w(u, n, t)\, dt$$

where r is the parameter of a probability generating function, N_{T_u} is the number of claims until ruin (including the claim causing ruin), and $w(u, n, t)$ is the joint density of N_{T_u} and T_u, defined for $n = 1, 2, 3, \ldots$, and $t > 0$. From this we can find quantities such as the joint density of the time of ruin and the number of claims until ruin $w(u, n, t)$, the probability function of the number of claims until ruin $p(u, n)$, and the moments of the number of claims until ruin.

Dickson (2012) studied $\phi_{r,\delta}(u)$ under the classical risk model by applying probabilistic arguments from Prabhu (1961) to find $w(u, n, t)$ and demonstrated that there is a strong correlation between the number of claims until ruin and the time of ruin for the exponential claim amounts case.

Gerber-Shiu functions have been studied for a variety of risk models, for example by Li and Lu (2008) for a Markov-modulated risk model, discussed in the next chapter, and by Schmidli (2015) for a risk model with interest.

Chapter 2

Other risk models

In the previous chapter, we reviewed well-known discrete and continuous time risk models. In this chapter, we explain other risk processes which can be obtained by modifying the classical risk process. We start with the Markov-modulated model and then consider barrier models. In particular, we introduce models under which the surplus process is bounded by an upper value and/or cannot fall below a pre-determined level.

2.1 The Markov-modulated risk model

In the classical risk model the Poisson parameter and the distribution of individual claims are fixed throughout. It can be more realistic to relax this assumption and to let the arrival intensities and claim size distribution change. For the first time, this issue has been addressed by Ammeter (1948). He considered a model that starts each year with a new intensity which is independent of the past intensities. To make this model more realistic, we can assume that the arrival intensities are governed by a continuous time Markov chain on a finite state space which represents, for example, an environmental process. Discussions of risk models in a Markovian environment can be found in, for example, Reinhard (1984), Asmussen (1989), Grandell (1991) and Asmussen and Albrecher (2010); see also Neuts (1966) in the context of queuing theory.

2.1.1 The continuous time case

We denote the surplus of an insurance company in a continuous time model by $U(t)$ and define it as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

where $N(t)$ is the number of claims that have occurred up to time t. We assume that $\{J(t)\}_{t\geq 0}$ is a homogeneous, irreducible and aperiodic continuous time Markov process with finite state space $M = \{1, \ldots, m\}$, and intensity matrix

$$\begin{pmatrix} \alpha_{11} & \ldots & \alpha_{1m} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \ldots & \alpha_{mm} \end{pmatrix}$$

where $\alpha_{ii} = -\sum_{i=1, i\neq j}^{m} \alpha_{ij}$, for $i \in M$, and $\pi = (\pi_1, \ldots, \pi_m)$ is the unique stationary probability distribution of $\{J(t)\}_{t\geq 0}$, given by

$$\pi_i = \frac{(\lambda_i \eta_i)/\alpha_i}{\sum_{i=1}^{m} (\lambda_i \eta_i)/\alpha_i} \tag{2.1}$$

where η_i is the unique stationary probability distribution of the embedded Markov chain with transition probabilities $p_{ii} = 0$, $p_{ij} = \alpha_{ij}/\alpha_i$ and $\alpha_{ii} = -\alpha_i$; see Reinhard (1984, formula 4.2). In this model, at time t, claims occur according to a Poisson process with intensity λ_i if $J(t) = i$ and the corresponding claim amounts have distribution F_i with finite mean m_i. The initial surplus is u and c is the premium income per unit of time. We assume that c is fixed regardless of the state of the process and satisfies the positive loading condition (see, for example, Albrecher and Boxma, 2005a):

$$\sum_{i=1}^{m} \pi_i m_i < c \sum_{i=1}^{m} \frac{\pi_i}{\lambda_i}. \tag{2.2}$$

Define $T_u = \inf\{t : U(t) < 0 \mid U(0) = u\}$, with $T_u = \infty$ if $U(t) \geq 0$, for all $t \geq 0$ to be the time of ruin given initial surplus u. Then, the probability that ruin occurs in infinite time due to a claim in state j, given initial state i and initial surplus u, is defined by

$$\psi_{ij}(u) = \Pr(T_u < \infty, J(T_u) = j \mid U(0) = u, J(0) = i)$$

with $\delta_{ij}(u) = 1 - \psi_{ij}(u)$. Further, the probability that ruin occurs in infinite time given initial surplus u and initial environment state i is given by

$$\psi_i(u) = \Pr(T_u < \infty \mid U(0) = u, J(0) = i) = \sum_{j=1}^{m} \psi_{ij}(u)$$

and $\delta_i(u) = 1 - \psi_i(u)$. We denote by $\psi_i(u,t)$ the probability of ruin in finite time and define it by

$$\psi_i(u,t) = \Pr(T_u \leq t \mid U(0) = u, J(0) = i).$$

Also, define

$$H_{1,ij}(u,y) = \Pr(T_u < \infty \text{ and } |U(T_u)| \leq y, J(T_u) = j \mid U(0) = u, J(0) = i)$$

to be the probability that ruin occurs in state j and the deficit at ruin, or the severity of ruin, is at most y, given initial state i. Then

$$h_{1,ij}(u,y) = \frac{\partial}{\partial y} H_{1,ij}(u,y)$$

is its (defective) density, and

$$H_{1,i}(u,y) = \Pr(T_u < \infty \text{ and } |U(T_u)| \leq y \mid U(0) = u, J(0) = i)$$

is the probability that ruin occurs and the deficit at the time of ruin is at most y, given initial state i. Similarly, we have that $H_{1,i}(u,y) = \sum_{j=1}^{m} H_{1,ij}(u,y)$.

Main results

The Markov-modulated risk model has been investigated by many researchers. Reinhard (1984) considered a semi-Markov risk model and derived a system of integro-differential equations for the survival probabilities when claim amounts are exponentially distributed. Asmussen (1989) examined the ruin probability for the Markov-modulated risk model and obtained a Cramér-Lundberg approximation and a diffusion approximation for the ultimate ruin probability. Bäuerle (1996) considered the expected value of the time of ruin. Lu and Li (2005) solved the system of integro-differential equations derived by Reinhard (1984) through Laplace transform techniques. In their model, the surplus process is given by

$$U(t) = u + \mathcal{C}(t) - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0$$

where $\mathcal{C}(t)$ is the aggregate premium received during the interval $(0, t]$, defined as

$$\mathcal{C}(t) = \sum_{k=1}^{N_\epsilon(t)} c_{J(k-1)}\big(U(k) - U(k-1)\big) + c_{J(N_\epsilon(t))}\big(t - T(N_\epsilon(t))\big), \quad t \geq 0$$

where $N_\epsilon(t) = \sup\{n \in N : U(n) \leq t\}$ and $U(n)$ is the time at which the nth transition of the environment process occurs, $J(n)$ is the state of the environment after its nth transition, c_i is the premium rate given that at that time the process is in state i. The Laplace transform of the survival probability derived from the integro-differential equation in Reinhard (1984) is given in the following theorem.

Theorem 2.1. The Laplace transform of $\delta_i(u)$ satisfies the following equation:

$$\left[s - \frac{\lambda_i + \alpha_i}{c_i} + \frac{\lambda_i}{c_i}\tilde{f}_i(s)\right]\tilde{\delta}_i(s) + \frac{\alpha_i}{c_i}\sum_{k=1}^{m} p_{ik}\tilde{\delta}_k(s) = \tilde{\delta}_i(0)$$

for $i = 1, 2, \ldots, m$ or in matrix form $A(s)\tilde{\Delta}(s) = \Delta(0)$, where

$$A(s) = \begin{bmatrix} s - \frac{\lambda_1(1-\tilde{f}_1(s))+\alpha_1}{c_1} & & \\ & \ddots & \\ & & s - \frac{\lambda_m(1-\tilde{f}_m(s))+\alpha_m}{c_m} \end{bmatrix} + \begin{bmatrix} \frac{\alpha_1}{c_1} & & \\ & \ddots & \\ & & \frac{\alpha_m}{c_m} \end{bmatrix} P,$$

$\tilde{\Delta}(s) = (\tilde{\Delta}_1(s), \ldots, \tilde{\Delta}_m(s))^T$, $\Delta(0) = (\Delta_1(0), \ldots, \Delta_m(0))^T$ and $P = (p_{ij})_{i,j=1}^m$ is the transition probability matrix as defined above. Then, the solution to $\tilde{\Delta}(s)$ is

$$\tilde{\Delta}(s) = [A(s)]^{-1}\Delta(0)$$

where $\det[A(s)] = 0$ is the characteristic equation.

Proof. See Lu and Li (2005). □

Lu and Li (2005) found an explicit form for the survival probability in a two-state model for exponential claim sizes and when the claim amounts distributions belong to the K_n-family, meaning that the Laplace transform of f_i is in the form of $\tilde{f}_i(s) = y_{k-1}^{(i)}(s)/y_k^{(i)}(s)$, where $y_{k-1}^{(i)}(s)$ is a polynomial of degree $k-1$ or less, while $y_k^{(i)}(s)$ is a polynomial of degree k, satisfying $y_{k-1}(0) = y_k(0)$.

Lu (2006) studied the probability and severity of ruin function and provided solutions when claim amounts have exponential and a mixture of Erlang distributions by inverting the Laplace transform of the system of integro-differential equations developed by Snoussi (2002). Li and Lu (2008) analysed a Gerber-Shiu discounted penalty function and found explicit formulae when $u = 0$ and when the claim amounts distribution is from the K_n-family.

Define for $\delta \geq 0, u \geq 0$ and $i, j \in M$,

$$\phi_{i,j}(u) = E\big[e^{-\delta T_u}\omega(U(T_u^-), |U(T_u)|)I(T_u < \infty, J(T_u) = j) \mid U(0) = u, J(0) = i\big] \quad (2.3)$$

to be the Gerber-Shiu function if ruin is caused by a claim in state j, given initial surplus u and initial environment state i. Then, $\phi_i(u) = \sum_{j=1}^{m} \phi_{i,j}(u)$, is the expected discounted penalty function at ruin, given initial surplus u and initial state i. When $\delta = 0$ and $\omega(x, y) = 1$, $\phi_{i,j}(u)$ simplifies to the infinite time ruin probability. The integro-differential equation of the Gerber-Shiu function is given in the following theorem.

Theorem 2.2. For $i \in M$, $\phi_{i,j}(u)$ satisfies

$$c\phi_{i,i}'(u) = (\lambda_i + \delta)\phi_{i,i}(u) - \lambda_i\left(\int_0^u \phi_{i,i}(u - x)f_i(x)dx + \xi_i(u)\right) - \sum_{k=1}^{m} \alpha_{i,k}\phi_{k,i}(u)$$

and for $i \neq j$,

$$c\phi_{i,j}'(u) = (\lambda_i + \delta)\phi_{i,j}(u) - \lambda_i \int_0^u \phi_{i,j}(u - x)f_i(x)dx - \sum_{k=1}^{m} \alpha_{i,k}\phi_{k,j}(u)$$

where $\xi_i(u) = \int_u^\infty \omega(u, x - u)f_i(x)dx$.

Proof. See Li and Lu (2008). □

The next theorem gives the Laplace transform of $\phi_{i,j}(u)$.

Theorem 2.3. The Laplace transform of $\phi_{i,j}(u)$ for $i, j \in M$ is given by

$$\tilde{\phi}_{i,j}(s) = \left[s - \frac{\lambda_i + \delta}{c} + \frac{\lambda_i}{c}\tilde{f}_i(s)\right]^{-1}\left(\phi_{i,j}(0) - \frac{\lambda_i}{c}\tilde{\xi}_i(s)I(i = j) - \frac{1}{c}\sum_{k=1}^{m} \alpha_{i,k}\tilde{\phi}_{k,j}(s)\right)$$

or in matrix form by

$$\tilde{\Phi}(s) = [B(s)]^{-1}[\Phi(0) - \tilde{\Xi}(s)] = \frac{B^*(s)\Phi(0) - B^*(s)\tilde{\Xi}(s)}{\det[B(s)]} \quad (2.4)$$

where $\Phi(u) = (\phi_{i,j}(u))_{i,j=1}^m$, $\tilde{\Phi}(s) = (\tilde{\phi}_{i,j}(s))_{i,j=1}^m$ with $B^*(s)$ being the adjoint matrix of $B(s)$. Further,

$$\begin{aligned} S_i(s) &= s - \frac{\lambda_i + \delta}{c} + \frac{\lambda_i}{c}\tilde{f}_i(s), \\ A &= (\alpha_{ij})_{i,j=1}^m, \\ B(s) &= diag(S_1(s), S_2(s), \ldots, S_m(s)) + A/c, \\ \tilde{\Xi}(s) &= diag(\lambda_1\tilde{\xi}_1(s)/c, \lambda_2\tilde{\xi}_2(s)/c, \ldots, \lambda_m\tilde{\xi}_m(s)/c). \end{aligned}$$

Proof. See Li and Lu (2008). □

Li and Lu (2008) applied divided differences to solve equation (2.4) and found an expression for $\phi_i(0)$. The expected discounted penalty function in a semi-Markovian dependent risk model was investigated by Albrecher and Boxma (2005a). They considered an irreducible discrete time Markov chain that governs the transition between states. In this model, at each instant of a claim the Markov chain jumps to a new state and the claim amounts distribution depends on this new state. Ma et al. (2010) gave results on the duration of negative surplus for a two-state Markov-modulated risk model. Ng and Yang (2006) presented a closed form solution for the joint distribution of the surplus immediately before and after ruin when the initial surplus is zero and when the claim amounts are phase-type distributed.

In the next section, we look at the literature on Markov-modulated risk models in discrete time. We have explained in Chapter 1 that a discrete time model can be used to approximate a continuous time model. We can also extend this fact to the Markov-modulated risk model. For example, Cossette et al. (2004b) showed that the compound binomial model in a Markovian environment can approximate a risk model based on a particular Cox model and the marked Markov-modulated Poisson process.

2.1.2 The discrete time case

Suppose $\{J_n\}_{n \in N}$ is a homogeneous, irreducible and aperiodic Markov chain with a finite state space $M = \{1, \ldots, m\}$ and transition probability matrix

$$\begin{pmatrix} p_{11} & \cdots & p_{1m} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mm} \end{pmatrix}$$

where $p_{ij} = \Pr(J_n = j | J_{n-1} = i, J_k \quad k \leq n-1)$, for $i, j \in M$, and $\pi = (\pi_1, \ldots, \pi_m)$ is the unique stationary probability distribution. The insurer's surplus at time n, $n = 1, 2, 3, \ldots$ is denoted $U^d(n)$ and is defined by

$$U^d(n) = u + n - \sum_{i=1}^{n} Y_i,$$

where $u = U^d(0)$ is the insurer's initial surplus, or the surplus at time 0 and Y_i is the insurer's aggregate claim amount in the ith time interval. In the Markov-modulated model, the random variables Y_i are no longer independent. In particular, the distribution of $\{Y_n\}_{n=1}^{\infty}$ is influenced by the environmental Markov chain. The conditional joint distribution of Y_n and J_n given the previous state J_{n-1} is defined by

$$\begin{aligned} g_{ij}(x) &= \Pr(Y_n = x, J_n = j | J_{n-1} = i, J_k, Y_k, k \leq n-1) \\ &= p_{ij} g_j(x), \end{aligned} \tag{2.5}$$

where $g_i(x) = \sum_{j=1}^{m} g_{ij}(x)$, and $G_i(y) = \sum_{j=1}^{m} \sum_{x=0}^{y} g_{ij}(x)$ for $y = 0, 1, 2, \ldots$.

Remark 2.1. Reinhard and Snoussi (2000, 2001, 2002) defined (2.5) such that zero claims are only possible when the state prior to the occurrence of the claim is state 1. This condition, has been relaxed in Chen et al. (2014b) and throughout we assume $g_{ij}(x)$ is defined for $x = 0, 1, 2, \ldots$.

For all $i, j \in M$ we define

$$(\mu_n)_{ij} = \sum_{x=1}^{\infty} x^n g_{ij}(x) < \infty$$

to be nth moment of the aggregate claim amount in state j given initial state i, with $(\mu_1)_{ij} := \mu_{ij}$, and

$$(\mu_n)_i = \sum_{j=1}^{m} (\mu_n)_{ij} \tag{2.6}$$

with $(\mu_1)_i := \mu_i$. Also, we define the probability generating function of $g_{ij}(x)$ by $\tilde{g}_{ij}(s) = \sum_{x=0}^{\infty} s^x g_{ij}(x)$. In the discrete time model we assume that the insurer's premium income per unit time is 1, so that n is the total premium income up to time n.

Let T_u^d be the time of ruin given initial surplus u, and defined as $T_u^d = \min\{n \geq 1 : U^d(n) < 0 \mid U^d(0) = u\}$ with $T_u^d = \infty$ if $U^d(n) \geq 0$ for $n = 1, 2, 3, \ldots$.

Remark 2.2. We remark that the definition of ruin, here, is based on papers by, for example, Reinhard and Snoussi (2000, 2001, 2003) or Chen et al. (2014b), i.e. ruin occurs when the surplus falls below 0, which is different from the definition of ruin in Section 1.2, i.e. ruin occurs when the surplus falls to 0 or below 0.

The probability of ultimate ruin given initial surplus u and initial environment state i is given by

$$\psi_i^d(u) = \Pr(T_u^d < \infty \mid U^d(0) = u, J(0) = i) = 1 - \delta_i^d(u).$$

To make sure that ruin is not certain, we assume the positive loading condition holds, that is $\sum_{i=1}^m \pi_i \mu_i < 1$. See, for example, Chen et al. (2014b). Also, we define the probability that ruin occurs in state j and the insurer's deficit at ruin is y, given initial environment state i, as

$$h_{ij}^d(u, y) = \Pr(T_u^d < \infty, |U(T_u^d)| = y, J(T_u^d) = j \mid U^d(0) = u, J(0) = i)$$

and define

$$h_i^d(u, y) = \Pr(T_u^d < \infty, |U(T_u^d)| = y \mid U^d(0) = u, J(0) = i)$$

to be the probability that ruin occurs and that the deficit at ruin is y given initial state i. We then have $h_i^d(u, y) = \sum_{j=1}^m h_{ij}^d(u, y)$.

Main results

The ruin probability in the semi-Markov model has been considered by Reinhard and Snoussi (2000). They derived recursive formulae for the probability of ruin with the following restrictions:

$$\sum_{j=1}^m g_{1j}(0) > 1 \quad \text{and} \quad \sum_{j=1}^m g_{ij}(0) = 0, \quad \text{for} \quad i \neq 1.$$

That is, the claim size may be zero only in time periods starting from state 1. Assuming these restrictions, Reinhard and Snoussi (2001) established a recursive system to find the distribution of the surplus prior to ruin and in another paper, Reinhard and Snoussi (2002), they studied the problem of the probability of ruin and the deficit at ruin.

An extension to the compound binomial model is the compound Markov binomial model. This model is based on the Markov Bernoulli process with a dependency between the occurrence of claims. Cossette et al. (2003) investigated the finite time and infinite time ruin probability in a compound Markov binomial framework. They presented recursive formulae and found a Lundberg-type exponential bound for the ruin probability under the condition that $E[Y_1|J(0) = i] < 1$. Later, Cossette et al. (2004a) demonstrated that the infinite time ruin probability in the compound Markov binomial model can be expressed as the tail of a compound geometric distribution. Using this fact, they introduced a new upper bound for the ruin probability that can overcome the restrictions of the exponential bound they had found in their previous paper. Reinhard and Snoussi (2004) considered the probability of ruin and the deficit at ruin and relaxed their previous assumptions that zero claims can only happen in state 1. They developed a system of equations for the probability of ruin by a monotonically converging algorithm. Yang et al. (2009) derived an explicit expression for the discounted joint probability function of the surplus prior to ruin and the deficit at ruin for initial surplus 0 under a Markov-dependent model and introduced a generalised Lundberg's equation. Chen et al. (2014a) analysed a discrete semi-Markov risk model in the presence of an upper barrier and studied the dividend problem in a two-state and three-state model. Chen et al. (2014b) also considered the survival probability and found recursive formulae for the calculation of survival probability. They relaxed the restrictions in the model of Reinhard and Snoussi (2000) and provided two equations from which it is possible to find $\delta_i^d(0)$. These two equations are given in the following theorem.

Theorem 2.4. The survival probability can be calculated recursively by

$$\delta_i^d(k) = \begin{cases} \frac{1}{f_0}\left(h_k^{(i)} - \sum_{n=0}^{k-1} \delta_i^d(n) f_{k-n}\right) & \text{if} \quad f_0 \neq 0, \\ \frac{1}{f_1}\left(h_{k+1}^{(i)} - \sum_{n=0}^{k-1} \delta_i^d(n) f_{k+1-n}\right) & \text{if} \quad f_0 = 0 \quad \text{and} \quad f_1 \neq 0 \end{cases}$$

for $i = 1, 2$ and $k \in N$ where

$$\bar{g}_{ii}(1) = g_{ii}(1) - 1, \qquad \bar{g}_{ii}(k) = g_{ii}(k), \quad i = 1, 2, \ k \in N \setminus \{1\},$$

$$f_k = \sum_{n=0}^{k} [(\bar{g}_{11}(n)\bar{g}_{22}(k-n)) - g_{21}(n)g_{12}(k-n)],$$

$$h_k^{(1)} = e_1\bar{g}_{22}(k) - e_2 g_{12}(k), \quad k \in N,$$

and for $i = 1, 2$, $e_i = \sum_{j=1}^{2} g_{ij}(0)\delta_j^d(0)$. Assuming that $f_0 > 0$, values of $\delta_i^d(0)$ for $i = 1, 2$ can be found from

$$\delta_1^d(0)(g_{11}(0)p_{21} + g_{21}(0)p_{12}) + \delta_2^d(0)(g_{12}(0)p_{21} + g_{22}(0)p_{12}) = p_{12}(1 - \mu_2) + p_{21}(1 - \mu_1)$$

and

$$[g_{11}(0)(\tilde{g}_{22}(\rho) - \rho) - g_{21}(0)\tilde{g}_{12}(\rho)]\delta_1^d(0) = [g_{22}(0)\tilde{g}_{12}(\rho) - g_{12}(0)(\tilde{g}_{22}(\rho) - \rho)]\delta_2^d(0)$$

with $\rho \in (-1, 0)$ being the root of

$$(\tilde{g}_{11}(\rho) - \rho)(\tilde{g}_{22}(\rho) - \rho) - \tilde{g}_{21}(\rho)\tilde{g}_{12}(\rho) = 0.$$

Proof. See Chen et al. (2014b). □

The model presented by Chen et al. (2014b) covers the compound binomial model (with time-correlated claims) and the compound Markov binomial model (with time-correlated claims).

2.2 Barrier models

In this section, we look at the literature on surplus processes with barriers. We can modify a surplus process by putting a constraint on it. In a surplus process with an upper barrier, the surplus cannot move upwards without limit and with a lower barrier, it cannot fall below a fixed value. We now explain how the former can include dividend payments to an insurance company's shareholders and how the latter can lead to a reinsurance arrangement in the form of capital injections.

2.2.1 The dividend barrier

In the first chapter, we discussed some of the shortcomings of ruin theory. One of the defects in the assumptions of classical risk theory is that the surplus process is allowed to grow infinitely and that the sole objective of insurers is to minimise the ruin probability. However, in the real world, insurance companies seek to maximise profit as well. De Finetti (1957) tackled this problem by considering a situation in which as an insurer's surplus increases, some of it will be paid out as dividends to the insurance company's shareholders. He suggested that the expected future life time of the company or the expected discounted value of future dividends would represent more useful criteria than would the probability of ruin. See, for example, Bühlmann (1970). Dividends are distributed according to a certain strategy, such as the barrier, threshold, linear barrier, multilayer strategies, and so on. Under a barrier strategy, whenever the surplus attains an upper barrier, say level b, the premium income c is paid to shareholders as dividends until the next claim occurs, so that in this modified surplus process, the surplus never attains a level greater than b. Under a threshold strategy, whenever the surplus attains level b dividends are paid at a rate less than c until the surplus drops below b. Unlike the barrier strategy, under the threshold strategy, the surplus can grow above b. However, the rate of growth above b is lower than the rate of growth below b. According to a linear barrier strategy, the upper barrier is a straight line with intercept b and slope a with $a < c$. Under this strategy, whenever the surplus attains the upper barrier, i.e. $b + at$, dividends are distributed at rate $c - a$ until the next claim causes the surplus to fall below the linear barrier, so that the surplus never upcrosses $b + at$. A multilayer strategy is a strategy with multiple thresholds. The rate of dividend payments to shareholders depends on the level of the surplus. For example, whenever the surplus reaches level b_i, the dividend is paid out at rate d_i until the surplus crosses a threshold at which point the rate of dividend payment changes, possibly to 0.

Let the random variable D denote the sum of discounted dividends until ruin, with $V_n(u, b) = E[D^n]$. Much of the literature on dividends aims to find the optimal strategy such that $V_1(u, b)$ is maximised. Borch (1967, 1990) built on De Finetti's idea and studied this problem in the continuous time case; see also Borch et al. (2014). Bühlmann (1970) verified that if a barrier strategy is applied, $V_1(u, b)$ satisfies an integro-differential equation and found the optimal barrier b^* for claim amounts with an exponential distribution. Gerber (1974) investigated this problem under the linear barrier strategy. Gerber and Shiu (1998) found an expression for the expected present value of dividend payments until ruin, which is given by

$$V_1(u, b) = \frac{e^{\rho u} - \Psi(u)}{\rho e^{\rho b} - \Psi'(b)}, \quad 0 \le u \le b,$$

where ρ is the positive root of Lundberg's fundamental equation and $\Psi(u)$ is the Gerber-Shiu function for the classical risk model with $\omega(x, y) = e^{\rho u}$. They also showed that b^*

satisfies the following condition:

$$\frac{\partial^2}{\partial u^2} V(u, b^*)\Big|_{u=b^*} = 0.$$

Since the introduction of the Gerber-Shiu function, the study of different ruin-related quantities has attracted the attention of researchers. Lin et al. (2003) defined a Gerber-Shiu function by

$$\phi_{\delta,b}(u) = E\big[e^{-\delta T_{u,b}} \omega(U(T_{u,b}^-), |U(T_{u,b})|) \mid U(0) = u\big] \tag{2.7}$$

where $T_{u,b}$ is the time of ruin, $U(T_{u,b}^-)$ is the surplus prior to ruin and $|U(T_{u,b})|$ is the deficit at ruin for a surplus process with dividends. Unlike in the classical risk model, the Gerber-Shiu function for a risk model with dividends does not include $I(\cdot)$ as the occurrence of ruin is certain in such a model. Lin et al. (2003) derived an integro-differential equation for $\phi_{\delta,b}(u)$ and presented its solution as a linear combination of the Gerber-Shiu function without a dividend barrier and the solution of an associated homogeneous integro-differential equation, as follows.

Theorem 2.5. The Gerber-Shiu function (2.7) satisfies the integro-differential equation

$$\frac{\partial}{\partial u}\phi_{\delta,b}(u) = \frac{\lambda + \delta}{c}\phi_{\delta,b}(u) - \frac{\lambda}{c}\int_0^u f(x)\phi_{\delta,b}(u-y)dx - \frac{\lambda}{c}\int_u^\infty f(x)\omega(u, x-u)dx \tag{2.8}$$

for $0 \leq u \leq b$ and the boundary condition is $\frac{\partial}{\partial u}\phi_{\delta,b}(u)\big|_{u=b} = 0$. Further, the solution to (2.8) is given by

$$\phi_{\delta,b}(u) = \phi_\delta(u) - \frac{\phi_\delta'(b)}{v'(b)}v(u) \tag{2.9}$$

where $\phi_\delta(u)$ is given by (1.29) and $v(u)$ is the solution to the following homogeneous integro-differential equation:

$$\frac{\partial}{\partial u}v(u) = \frac{\lambda + \delta}{c}v(u) - \frac{\lambda}{c}\int_0^u f(x)v(u-x)dx.$$

Proof. See Lin et al. (2003). □

Assuming different forms of penalty function, Lin et al. (2003) found the Laplace transform of the time of ruin, the distribution of the surplus prior to ruin and moments of the deficit at ruin. Dickson and Waters (2004) investigated the problem of optimal dividend strategy in a discrete time risk model and showed that a discrete time risk model can be used as an approximation to the continuous time model. They also derived an integro-differential equation for $V_n(u,b)$.

Theorem 2.6. If a barrier strategy is applied, $V_n(u,b)$ satisfies the following integro-differential equation:

$$\frac{d}{du}V_n(u,b) = \frac{\lambda + n\delta}{c}V_n(u,b) - \frac{\lambda}{c}\int_0^u f(x)V_n(u-x,b)dx$$

with boundary conditions

$$\frac{d}{du}V_n(u,b)\Big|_{u=b} = nV_{n-1}(b,b)$$

for $n = 1, 2, 3, \ldots$, with $V_0(b,b) = E[D^0] = 1$.

Proof. See Dickson and Waters (2004). □

Dickson and Waters (2004) suggested an alternative approach to De Finetti's in which the shareholders are held responsible for the deficit at the time of ruin. Suppose $Y_{u,b}$ is the deficit at ruin from initial surplus u. Dickson and Waters' (2004) approach requires the maximisation of $L(u,b)$, given by

$$L(u,b) = V_1(u,b) - u - E[e^{-\delta T_{u,b}}Y_{u,b}].$$

Gerber et al. (2006b) studied Dickson and Waters' (2004) modification and compared the optimal barrier level obtained by maximising $V_1(u,b)$ and $L(u,b)$. Albrecher et al. (2005b) developed a partial integro-differential equation for the Gerber-Shiu function in the case when dividends are distributed according to a time-dependent linear barrier strategy. Gerber et al. (2006a) extended the work of Lin et al. (2003) and introduced the dividends-penalty identity through probabilistic reasoning, which is given by

$$\phi_{\delta,b}(u) = \phi_\delta(u) - \phi_\delta'(b)V_1(u,b). \tag{2.10}$$

Yuen et al. (2007) showed that the idea of Lin et al. (2003) also holds under a surplus process with interest – that is, the solution to an integro-differential equation for the Gerber-Shiu function can be expressed as the sum of the Gerber-Shiu function with no dividend barrier and another function which is independent of the penalty function.

Extending their work, Cai et al. (2009) investigated the Gerber-Shiu function for a risk model with two layers, Δ and b. Under this model, if the surplus process exceeds Δ, the amount of surplus in excess of Δ earns interest and when it exceeds b dividends are paid out at rate α to the shareholders of an insurance company. They found that the dividends-penalty identity holds for their model as well. Gerber and Yang (2010) provided an analytical interpretation for the dividends-penalty identity in equation (2.10).

The threshold strategy is another topic considered, for example, by Gerber and Shiu (2006) who derived expressions for the expected present value of dividend income to shareholders when claims have an exponential or a mixed exponential distribution. Dickson and Drekic (2006) considered the threshold strategy and found expressions for $V_1(u, b)$ by probabilistic reasoning. They derived formulae in terms of the joint density of the time of ruin and the deficit at ruin and they illustrated their applications for claim amount distributions which are subject to a particular factorisation. Their work was extended by Cheung et al. (2008) who derived a recursive expression for $V_n(u, b)$ by probabilistic reasoning. They found the optimal threshold, by considering both the maximisation of $V_1(u, b)$ and minimisation of the coefficient of variation of discounted dividends.

Other references on the dividend strategy include, for example, Li and Lu (2008) who considered the dividends-penalty identity under the Markov-modulated risk model. For a comprehensive survey on dividend strategies, see Avanzi (2009). Recently, Albrecher et al. (2011) studied a model under which dividends are paid periodically rather than continuously and the surplus process can be observed at random times, so that it covers the continuous time and discrete time models as limiting cases. They obtained explicit expressions for the Gerber-Shiu function and provided numerical examples to examine the effect of random observation times on different ruin-related quantities.

2.2.2 Capital injections

A modification to De Finetti's model was introduced by Borch (1990). He considered a lower limit on the surplus of an insurance company. In his model, new capital is injected after an unfavourable underwriting period. Borch (1990) pointed out that such an injection can be provided by a reinsurance arrangement. Pafumi (1998) suggested a reinsurance contract under which whenever the surplus is negative, the reinsurer makes the required payment to bring the surplus back to zero. Dickson and Waters (2004) considered the reinsurance contract proposed by Pafumi (1998) in the presence of a dividend barrier. They modified the surplus process by assuming that the initial capital is provided by shareholders who also purchase a reinsurance policy which provides them with the amount of the deficit each time that ruin occurs, so that the surplus at the time of ruin is then 0 and the insurance operation can continue from this surplus level. In such a modified process, the surplus moves indefinitely between 0 and b. Nie et al. (2011) studied a risk model under which the insurer's surplus starts at a level $u \geq k$, where $k > 0$ is a fixed constant. On any occasion that the surplus falls between the levels 0 and k from above k, a capital injection restores the surplus level to k. If the surplus falls below 0 from a level above k, ruin occurs. Nie et al. (2011) explained how the capital injections can be provided by reinsurance and how an insurer can reduce its ultimate ruin probability by effecting such insurance.

Suppose $T_{u,k}$ is the time of ruin from initial surplus u in the model with capital injections. Then

$$\psi_k(u) = \Pr(T_{u,k} < \infty \mid U(0) = u).$$

Theorem 2.7. When the initial surplus is $u \geq k$, the ultimate ruin probability is given by

$$\psi_k(u) = \psi(u-k) - H_1(u-k,k)(1-\psi_k(k)) \tag{2.11}$$

where

$$\psi_k(k) = \frac{\psi(0) - H_1(0,k)}{1 - H_1(0,k)}. \tag{2.12}$$

Proof. See Nie et al. (2011). □

Remark 2.3. For the surplus process with capital injections we use the same notation as for the classical risk model, but with a subscript k.

Scheer and Schmidli (2011) investigated a model with both an upper and a lower barrier. In their model, if the surplus falls below 0, capital is injected which not only covers the deficit at the time of ruin, but also provides additional capital C. They showed that the optimal strategy is of band type; that is a strategy under which dividends are paid according to the region where $U(t)$ is located – for example, a barrier strategy has two bands. Also, Scheer and Schmidli (2011) provided a method to determine the solution to their integro-differential equation and the unknown value of C numerically. Breuer and Badescu (2014) analysed the problem of capital injections in the Markov-additive risk model with phase-type claims. They generalised the Gerber-Shiu function by introducing the minimal risk reserve before ruin and presented numerical examples for some ruin-related quantities. Nie et al. (2015) considered the finite time ruin probability for their (2011) risk model with capital injections. Their approach is based on the number of capital injections and the times between capital injections. They obtained a general expression for the distribution function of $T_{u,k}$, but were only able to implement their results if the joint density $w(u, y, t)$ admits a particular factorisation.

In this thesis, we consider the problem of capital injections under the classical risk model, Markov-modulated risk model and risk models with dividends. We show how capital injections can be incorporated into these models and how they would change the underlying risk process. The rest of this thesis is organised as follows:

In Chapters 3 and 4 we apply a Gerber-Shiu discounted penalty function as a useful tool to analyse the classical risk model with capital injections. We illustrate how the Gerber-Shiu function can facilitate the derivation of the joint density of the time of ruin and the number of claims until ruin. In Chapter 3 we take an analytical approach and in Chapter 4 we demonstrate our results with probabilistic arguments. Both approaches have advantages and disadvantages. We will see that the probabilistic approach can be applied only to claim amounts distributions which are subject to a particular decomposition, whereas the analytical approach can be applied to a wider range of distributions.

In Chapter 5 we develop a discrete time model and introduce an algorithm that provides approximations to the continuous time model with capital injections. One aim is to examine the effect of capital injections when claim amounts follow a heavy-tailed distribution for which analytical expressions for ruin-related quantities do not exist. In such cases, we need numerical methods to compute quantities such as the probability of ruin. We investigate whether the introduction of the capital injections gives rise to a reduction in the probability of ruin.

In Chapter 6 we consider a discrete time model and build a numerical algorithm to approximate quantities such as the probability of ruin and the probability and severity of ruin for a two-state Markov-modulated model. We will also consider approximating the density of the time of ruin for a Markov-modulated model with more than two states. Then, we extend our results to a Markov-modulated model with capital injections.

In Chapter 7 we study dividend strategies with capital injections in a classical risk model. We consider a reinsurance contract that provides compensation on any occasion that the surplus falls below k, so that the company never goes out of business and dividends could be paid indefinitely. We also investigate a threshold strategy both by solving an inhomogeneous integro-differential equation directly and by probabilistic arguments. Finally, we demonstrate that a dividends-penalty identity holds for our model.

Chapter 3

Gerber-Shiu analysis: analytical approach

3.1 Introduction

In this chapter, we consider the classical risk model with capital injections studied by Nie et al. (2011, 2015). We construct a Gerber-Shiu function and show that whilst this tool is not efficient for finding the ultimate ruin probability and the joint distribution of the surplus immediately prior to ruin and the deficit at ruin, it provides an efficient way of studying ruin-related quantities in finite time. In particular, we find a general expression for the joint distribution of the time of ruin and the number of claims until ruin and find an extension of Prabhu's (1961) formula for the finite time survival probability in the classical risk model. We also consider the correlation coefficient between the time of ruin and the number of claims until ruin. We then illustrate our results in the case of claim sizes with exponential and Erlang(2) distributions and obtain some interesting identities. In particular, we generalise results from the classical risk model and prove the identity of two known formulae for that model in the case of exponentially distributed claims.

3.2 A Gerber-Shiu function

We now construct a Gerber-Shiu function that allows us to analyse the probability of ultimate ruin, the distribution of the surplus immediately prior to ruin, the distribution of the deficit at ruin, the density of the time of ruin and the probability function of the number of claims until ruin. We denote our Gerber-Shiu function by $\phi_{k,r,\delta}(u)$ and define it by

$$\phi_{k,r,\delta}(u) = E\left[r^{N_{T_{u,k}}} e^{-\delta T_{u,k}} \omega(U(T_{u,k}^-), |U(T_{u,k})|) I(T_{u,k} < \infty) \mid U(0) = u\right] \tag{3.1}$$

for $\delta \geq 0$, $0 < r \leq 1$ and $u \geq k$, where $T_{u,k}$ is the time of ruin, $N_{T_{u,k}}$ is the number of claims until ruin, $U(T_{u,k}^-)$ is the surplus immediately before ruin, $|U(T_{u,k})|$ is the deficit at ruin and $\omega(x, y)$ is a penalty function defined for $x \geq k$ and $y > 0$. As in Landriault et al. (2011), we interpret δ as the parameter of a Laplace transform and r as the parameter of a probability generating function. Further, $\phi_{k,r,\delta}(u)$ is defined to be 0 for $0 \leq u < k$.

Depending on different forms taken by $\omega(x, y)$, we can obtain explicit results for quantities like the probability of ultimate ruin, the joint distribution of the surplus immediately prior to ruin and the deficit at ruin and the joint density of the time of ruin and the number of claims until ruin by inverting the Laplace transform of the Gerber-Shiu function. Therefore, we first need to derive the Laplace transform of the Gerber-Shiu function.

Theorem 3.1. The Laplace transform of the Gerber-Shiu function $\phi_{k,r,\delta}(u)$ satisfies

$$\begin{aligned} &\tilde{\phi}_{k,r,\delta}(s) \\ = \ & \frac{1}{cs - (\lambda + \delta) + \lambda r \tilde{f}(s)} \Bigg(c e^{-sk} \phi_{k,r,\delta}(k) - \lambda r \phi_{k,r,\delta}(k) \int_k^\infty e^{-su} \big(\bar{F}(u-k) - \bar{F}(u)\big)\, du \\ & -\lambda r \int_k^\infty e^{-su} \int_u^\infty f(x) \omega(u, x-u)\, dx\, du \Bigg). \end{aligned} \tag{3.2}$$

Further,

$$\phi_{k,r,\delta}(k) = \frac{\frac{\lambda r}{c} \int_k^\infty \int_u^\infty e^{-\rho(u-k)} f(x) \omega(u, x-u)\, dx\, du}{1 - \frac{\lambda r}{c} \int_k^\infty e^{-\rho(u-k)} \big(\bar{F}(u-k) - \bar{F}(u)\big)\, du}. \tag{3.3}$$

Proof. Using similar arguments to Landriault et al. (2011), for $u \geq k$ we can write

$$\begin{aligned} \phi_{k,r,\delta}(u) = \ & \int_0^\infty \lambda r e^{-\lambda t} e^{-\delta t} \int_0^{u+ct-k} f(x) \phi_{k,r,\delta}(u + ct - x)\, dx\, dt \\ & + \phi_{k,r,\delta}(k) \int_0^\infty \lambda r e^{-\lambda t} e^{-\delta t} \int_{u+ct-k}^{u+ct} f(x)\, dx\, dt \\ & + \int_0^\infty \lambda r e^{-\lambda t} e^{-\delta t} \int_{u+ct}^\infty f(x) \omega(u + ct, x - u - ct)\, dx\, dt. \end{aligned}$$

Setting $s = u + ct$ we get

$$\begin{aligned}\phi_{k,r,\delta}(u) &= \frac{\lambda r}{c}\int_u^{\infty} e^{-(\lambda+\delta)(s-u)/c}\left(\int_0^{s-k} f(x)\phi_{k,r,\delta}(s-x)\,dx + \phi_{k,r,\delta}(k)\int_{s-k}^{s} f(x)\,dx\right.\\ &\quad \left. + \int_s^{\infty} f(x)\omega(s, x-s)dx\right) ds \\ &= \frac{\lambda r}{c}\int_u^{\infty} e^{-(\lambda+\delta)(s-u)/c}\gamma(s)\,ds \qquad (3.4)\end{aligned}$$

where

$$\gamma(u) = \int_0^{u-k} f(x)\phi_{k,r,\delta}(u-x)\,dx + \int_{u-k}^{u} f(x)\phi_{k,r,\delta}(k)\,dx + \zeta(u) \qquad (3.5)$$

and

$$\zeta(u) = \int_u^{\infty} f(x)\omega(u, x-u)\,dx.$$

Using the operator T_s introduced by Dickson and Hipp (2001), and defined for an integrable function f as

$$T_s f(u) = \int_u^{\infty} e^{-s(y-u)} f(y)\,dy$$

we have

$$\phi_{k,r,\delta}(u) = \frac{\lambda r}{c} T_{\frac{\lambda+\delta}{c}}\gamma(u). \qquad (3.6)$$

Noting that $\phi_{k,r,\delta}(u) = 0$ for $0 \le u < k$ we have

$$T_s\phi_{k,r,\delta}(k) = \int_k^{\infty} e^{-s(x-k)}\phi_{k,r,\delta}(x)\,dx = e^{sk}\tilde{\phi}_{k,r,\delta}(s),$$

and similarly, $T_s\gamma(k) = e^{sk}\tilde{\gamma}(s)$. Applying the Dickson-Hipp operator to equation (3.6) we obtain

$$T_s\phi_{k,r,\delta}(k) = \frac{\lambda r}{c} T_{\frac{\lambda+\delta}{c}} T_s\gamma(k) = \frac{\lambda r}{c} T_s T_{\frac{\lambda+\delta}{c}}\gamma(k) = \frac{\lambda r}{c}\frac{T_{\frac{\lambda+\delta}{c}}\gamma(k) - T_s\gamma(k)}{s - \frac{\lambda+\delta}{c}}, \qquad (3.7)$$

where we have used properties of T_s given in Dickson and Hipp (2001). Further,

$$\begin{aligned}
T_s\gamma(k) &= \int_k^\infty e^{-s(u-k)}\gamma(u)\,du \\
&= e^{sk}\int_k^\infty e^{-su}\int_0^{u-k} f(x)\phi_{k,r,\delta}(u-x)\,dx\,du \\
&\quad +\phi_{k,r,\delta}(k)\int_k^\infty e^{-s(u-k)}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)\,du+\int_k^\infty e^{-s(u-k)}\zeta(u)\,du \\
&= e^{sk}\tilde{f}(s)\tilde{\phi}_{k,r,\delta}(s)+\phi_{k,r,\delta}(k)\int_k^\infty e^{-s(u-k)}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)\,du+T_s\zeta(k) \\
&= \tilde{f}(s)T_s\phi_{k,r,\delta}(k)+\phi_{k,r,\delta}(k)\int_k^\infty e^{-s(u-k)}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)\,du+T_s\zeta(k).
\end{aligned}$$

Substituting in (3.7) we obtain

$$\begin{aligned}
T_s\phi_{k,r,\delta}(k) &= \frac{\lambda r}{cs-\lambda-\delta}\Bigg(T_{\frac{\lambda+\delta}{c}}\gamma(k)-\tilde{f}(s)T_s\phi_{k,r,\delta}(k) \\
&\quad -\phi_{k,r,\delta}(k)\int_k^\infty e^{-s(u-k)}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)du-T_s\zeta(k)\Bigg)
\end{aligned}$$

which can be written in terms of Laplace transforms as

$$\begin{aligned}
e^{sk}\tilde{\phi}_{k,r,\delta}(s) &= \frac{\lambda r}{cs-\lambda-\delta}\Bigg(\frac{c}{\lambda r}\phi_{k,r,\delta}(k)-\tilde{f}(s)e^{sk}\tilde{\phi}_{k,r,\delta}(s) \\
&\quad -\phi_{k,r,\delta}(k)\int_k^\infty e^{-s(u-k)}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)\,du-\int_k^\infty e^{-s(u-k)}\zeta(u)\,du\Bigg).
\end{aligned}$$

Rearranging this identity we obtain formula (3.2).

To obtain formula (3.3) we first note from formula (41) of Landriault et al. (2011) that there exists $\rho\equiv\rho(\delta,r)$ which is the unique positive solution of $cs-\lambda-\delta+\lambda r\tilde{f}(s)=0$. Then, as ρ is a zero of the denominator of the right-hand side of (3.2), it must also be a zero of the numerator, giving formula (3.3). □

3.3 The probability of ultimate ruin

We now consider the ultimate ruin probability $\psi_k(u)$, for $u\geq k$. Nie et al. (2011) obtain expressions for this probability in the cases $u=k$ and $u>k$ by probabilistic arguments. We now show that their results can be obtained from formulae (3.2) and

(3.3). Setting $r = 1, \delta = 0$ and $\omega(x, y) = 1$ for $x \geq k$ and $y > 0$ in expression (3.1), we see that $\phi_{k,r,\delta}(u)$ reduces to $\psi_k(u)$. Our first result is easily obtained.

Theorem 3.2. When the initial surplus is k we have

$$\psi_k(k) = \frac{\frac{\lambda}{c}\int_k^\infty \bar{F}(u)\,du}{1 - \frac{\lambda}{c}\int_k^\infty \big(\bar{F}(u-k) - \bar{F}(u)\big)\,du} = \frac{\psi(0) - H_1(0,k)}{1 - H_1(0,k)}. \tag{3.8}$$

Proof. From Gerber and Shiu (1998) we know that $\rho = 0$ when $r = 1$ and $\delta = 0$. The result immediately follows by noting that $h_1(0, x) = \frac{\lambda}{c}\bar{F}(x)$ and $\int_k^\infty h_1(0, u)du = \psi(0) - H_1(0, k)$. □

Theorem 3.3. When the initial surplus is $u > k$ we have

$$\psi_k(u) = \psi(u-k) - H_1(u-k, k)[1 - \psi_k(k)]. \tag{3.9}$$

Proof. Using the fact that $\frac{c\delta(0)}{\tilde{\delta}(s)} = cs - \lambda + \lambda\tilde{f}(s)$ (see, for example, Dickson, 2005) equation (3.2) becomes

$$\begin{aligned}
\tilde{\psi}_k(s) &= \frac{\tilde{\delta}(s)}{c\delta(0)}\Bigg(ce^{-sk}\psi_k(k) - \lambda\psi_k(k)\int_k^\infty e^{-su}\big(\bar{F}(u-k) - \bar{F}(u)\big)\,du \\
&\quad -\lambda\int_k^\infty e^{-su}\int_u^\infty f(x)\,dx\,du\Bigg) \\
&= e^{-sk}\frac{\tilde{\delta}(s)}{\delta(0)}\psi_k(k) - \frac{\lambda\tilde{\delta}(s)}{c\delta(0)}\psi_k(k)\int_0^\infty e^{-s(u+k)}\big(\bar{F}(u) - \bar{F}(u+k)\big)\,du \\
&\quad -\frac{\lambda\tilde{\delta}(s)}{c\delta(0)}\int_0^\infty e^{-s(u+k)}\bar{F}(u+k)\,du \\
&= e^{-sk}\Bigg(\frac{\tilde{\delta}(s)}{\delta(0)}\psi_k(k) - \frac{\tilde{\delta}(s)}{\delta(0)}\psi_k(k)\int_0^\infty e^{-su}\big(h_1(0,u) - h_1(0,u+k)\big)\,du \\
&\quad -\frac{\tilde{\delta}(s)}{\delta(0)}\int_0^\infty e^{-su}h_1(0,u+k)\,du\Bigg).
\end{aligned} \tag{3.10}$$

We now apply the following results from Dickson (1998):

$$\tilde{h}_1(0,s) = \int_0^\infty e^{-su}h_1(0,u)\,du = 1 - \frac{\delta(0)}{s\tilde{\delta}(s)}$$

and

$$\tilde{H}_1(s,k) = \int_0^\infty e^{-su}H_1(u,k)\,du = \frac{\tilde{\Phi}(s,k)}{1 - \tilde{h}_1(0,s)} = \frac{\tilde{\Phi}(s,k)s\tilde{\delta}(s)}{\delta(0)}$$

where $\tilde{\Phi}(s,k) = \int_0^\infty e^{-su}\Phi(u,k)\,du$, and $\Phi(u,k) = \int_u^{u+k} h_1(0,x)\,dx$. Next, we can use properties of the Laplace transform of a derivative to write

$$\tilde{H}_1(s,k) = \frac{\tilde{\delta}(s)}{\delta(0)}\big(\tilde{\Phi}'(s,k) + H_1(0,k)\big) \tag{3.11}$$

with $\tilde{\Phi}'(s,k) = \int_0^\infty e^{-su}\Phi'(u,k)\,du = s\tilde{\Phi}(s,k) - H_1(0,k)$. Further,

$$\int_0^\infty e^{-su}\Phi'(u,k)\,du = \int_0^\infty e^{-su}\big(h_1(0,u+k) - h_1(0,u)\big)\,du.$$

Using these results we can write formula (3.10) as

$$\begin{aligned}
\tilde{\psi}_k(s) &= e^{-sk}\Bigg(\frac{\tilde{\delta}(s)}{\delta(0)}\psi_k(k) + \frac{\tilde{\delta}(s)}{\delta(0)}\psi_k(k)\tilde{\Phi}'(s,k) \\
&\quad -\frac{\tilde{\delta}(s)}{\delta(0)}\int_0^\infty e^{-su}\big(h_1(0,u+k) - h_1(0,u) + h_1(0,u)\big)\,du\Bigg) \\
&= e^{-sk}\Bigg(\frac{\tilde{\delta}(s)}{\delta(0)}\psi_k(k) + \frac{\tilde{\delta}(s)}{\delta(0)}\psi_k(k)\tilde{\Phi}'(s,k) - \frac{\tilde{\delta}(s)}{\delta(0)}\tilde{\Phi}'(s,k) - \frac{\tilde{\delta}(s)}{\delta(0)}\tilde{h}_1(0,s)\Bigg).
\end{aligned} \tag{3.12}$$

By rearranging formula (3.11) and substituting for $\tilde{\Phi}'(s,k)$ in (3.12) we get

$$\begin{aligned}
\tilde{\psi}_k(s) &= e^{-sk}\Bigg(\frac{\tilde{\delta}(s)}{\delta(0)}\psi_k(k) + \psi_k(k)\left[\tilde{H}_1(s,k) - \frac{\tilde{\delta}(s)}{\delta(0)}H_1(0,k)\right] \\
&\quad -\left[\tilde{H}_1(s,k) - \frac{\tilde{\delta}(s)}{\delta(0)}H_1(0,k)\right] - \frac{\tilde{\delta}(s)}{\delta(0)}\left[1 - \frac{\delta(0)}{s\tilde{\delta}(s)}\right]\Bigg) \\
&= e^{-sk}\Bigg(\frac{\tilde{\delta}(s)}{\delta(0)}\left[\psi_k(k) - \psi_k(k)H_1(0,k) + H_1(0,k) - 1 + \frac{\delta(0)}{s\tilde{\delta}(s)}\right] \\
&\quad -\tilde{H}_1(s,k)[1 - \psi_k(k)]\Bigg).
\end{aligned}$$

Inserting the right-hand side of equation (3.8) for $\psi_k(k)$ in the first square bracket gives

$$\begin{aligned}
\tilde{\psi}_k(s) &= e^{-sk}\left(\frac{\tilde{\delta}(s)}{\delta(0)}\left[-\delta(0) + \frac{\delta(0)}{s\tilde{\delta}(s)}\right] - \tilde{H}_1(s,k)[1 - \psi^k(k)]\right) \\
&= e^{-sk}[1/s - \tilde{\delta}(s)] - e^{-sk}\tilde{H}_1(s,k)[1 - \psi_k(k)].
\end{aligned}$$

Applying the shift property of the Laplace transform, i.e. $\int_k^\infty e^{-su}\alpha(u-k)\,du = e^{-sk}\tilde{\alpha}(s)$, we can invert $\tilde{\psi}_k(s)$ to get formula (3.9). □

3.4 The joint distribution of $U(T_{u,k}^-)$ and $|U(T_{u,k})|$

In this section we consider the joint distribution of the surplus immediately prior to ruin and the deficit at ruin, defined by

$$H_k(u,z,y) = \Pr(T_{u,k} < \infty, U(T_{u,k}^-) \le z, |U(T_{u,k})| \le y \mid U(0) = u)$$

with

$$H_{k,1}(u,y) = \Pr(T_{u,k} < \infty, |U(T_{u,k})| \le y \mid U(0) = u)$$

being the (defective) distribution of the deficit at the time of ruin and

$$H_{k,2}(u,z) = \Pr(T_{u,k} < \infty, U(T_{u,k}^-) \le z \mid U(0) = u)$$

being the (defective) distribution of the surplus immediately before ruin. Setting $r = 1$, $\delta = 0$ and $\omega(a,b) = I\{a \le x\}I\{b \le y\}$ we can write expression (3.1) as

$$\phi_{k,r,\delta}(u) = E[I(X \le x)I(Y \le y)I(T_{u,k} < \infty) \mid U(0) = u]$$

where $X = U(T_{u,k}^-)$ and $Y = |U(T_{u,k})|$. In this case, $\phi_{k,r,\delta}(u)$ reduces to $H_k(u,x,y)$.

Theorem 3.4. When the initial surplus is k we have

$$H_k(k,z,y) = \frac{\frac{\lambda}{c}\int_k^\infty \int_u^\infty f(x)I\{u \le z\}I\{x \le y+u\}\,dx\,du}{1 - \frac{\lambda}{c}\int_k^\infty \big(\bar{F}(u-k) - \bar{F}(u)\big)\,du} = \frac{\frac{\lambda}{c}\int_k^z \big(\bar{F}(u) - \bar{F}(u+y)\big)\,du}{1 - H_1(0,k)} \tag{3.13}$$

for $z \ge k$ and $y > 0$.

Proof. The proof follows from (3.3) since $\rho = 0$ when $r = 1$ and $\delta = 0$. □

Letting $z \to \infty$ in equation (3.13) yields

$$\begin{aligned} H_{k,1}(k,y) &= \frac{\int_k^\infty \big(h_1(0,u) - h_1(0,u+y)\big)\,du}{1 - H_1(0,k)} \\ &= \frac{H_1(0,k+y) - H_1(0,k)}{1 - H_1(0,k)} \end{aligned} \tag{3.14}$$

which is in agreement with Nie (2012). Also, letting $y \to \infty$ in equation (3.13) gives

$$H_{k,2}(k,z) = \frac{H_1(0,z) - H_1(0,k)}{1 - H_1(0,k)} \tag{3.15}$$

which is also in agreement with Nie (2012).

Theorem 3.5. When the initial surplus is $u > k$ we have

$$\begin{aligned}H_k(u,z,y) &= H_k(k,z,y)H_1(u-k,k) + \frac{\lambda\delta(u-k)}{c\delta(0)}\int_u^z \Big(\bar{F}(x) - \bar{F}(x+y)\Big) I(u \le z)\, dx \\ &\quad - \frac{\lambda}{c\delta(0)}\int_0^{(u\wedge z)-k} \big(\delta(u-k) - \delta(u-k-x)\big)\Big(\bar{F}(x+k) - \bar{F}(x+k+y)\Big)\, dx. \end{aligned} \tag{3.16}$$

Proof. We can write equation (3.2) as

$$\begin{aligned}\tilde{H}_k(s,z,y) &= \frac{\tilde{\delta}(s)}{c\delta(0)}\Bigg(ce^{-sk}H_k(k,z,y) - \lambda\int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)H_k(k,z,y)\, du \\ &\quad -\lambda\int_k^\infty e^{-su}\int_u^\infty f(x)I(u\le z)I(x\le u+y)\, dx\, du\Bigg) \\ &= \frac{\tilde{\delta}(s)}{c\delta(0)}\Bigg(ce^{-sk}H_k(k,z,y) - \lambda\int_0^\infty e^{-s(u+k)}\Big(\bar{F}(u) - \bar{F}(u+k)\Big)H_k(k,z,y)\, du \\ &\quad -\lambda\int_k^z e^{-su}\int_u^{u+y} f(x)\, dx\, du\Bigg). \end{aligned} \tag{3.17}$$

From Section 3.3 we know that $\Phi'(u,k) = \frac{\lambda}{c}\left(\bar{F}(u+k) - \bar{F}(u)\right)$. Therefore, we can rewrite (3.17) as

$$\begin{aligned}\tilde{H}_k(s,z,y) &= \frac{\tilde{\delta}(s)}{\delta(0)}e^{-sk}H_k(k,z,y) + \frac{\tilde{\delta}(s)}{\delta(0)}e^{-sk}H_k(k,z,y)\tilde{\Phi}'(s,k) \\ &\quad -\frac{\lambda\tilde{\delta}(s)}{c\delta(0)}\int_k^z e^{-su}\Big(\bar{F}(u) - \bar{F}(u+y)\Big)\, du. \end{aligned} \tag{3.18}$$

Rearranging formula (3.11) and substituting for $\tilde{\Phi}'(s,k)$ in (3.18) we have

$$\begin{aligned}\tilde{H}_k(s,z,y) &= \frac{\tilde{\delta}(s)}{\delta(0)}e^{-sk}H_k(k,z,y) + e^{-sk}H_k(k,z,y)\left[\tilde{H}_1(s,k) - \frac{\tilde{\delta}(s)}{\delta(0)}H_1(0,k)\right] \\ &\quad -\frac{\lambda\tilde{\delta}(s)}{c\delta(0)}\int_k^z e^{-su}\Big(\bar{F}(u) - \bar{F}(u+y)\Big)\, du \\ &= H_k(k,z,y)e^{-sk}\tilde{H}_1(s,k) + \frac{\tilde{\delta}(s)}{\delta(0)}H_k(k,z,y)e^{-sk}[1 - H_1(0,k)] \\ &\quad -\frac{\lambda\tilde{\delta}(s)}{c\delta(0)}\int_k^z e^{-su}\Big(\bar{F}(u) - \bar{F}(u+y)\Big)\, du. \end{aligned} \tag{3.19}$$

We can invert (3.19) by applying the shift property of the Laplace transform and noting that

$$\begin{aligned}
&\int_0^\infty e^{-s(u+k)} \int_0^u \delta(u-x)\Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)\,dx\,du \\
=\;& \int_k^\infty e^{-su} \int_0^{u-k} \delta(u-k-x)\Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)\,dx\,du.
\end{aligned}$$

We note that we need to consider separately the situations when $u \le z$ and $u > z$, since in the former situation ruin may or may not occur on the first drop below the initial level, but in the latter ruin may not occur with the surplus immediately prior to ruin being at most z. Therefore, after inserting for $H_k(k, z, y)$ the inverse of formula (3.19) is given by

$$\begin{aligned}
H_k(u,z,y) \;=\;& H_k(k,z,y)H_1(u-k,k) + \frac{\lambda\delta(u-k)}{c\delta(0)} \int_0^{z-k} \Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)\,dx \\
& -\frac{\lambda}{c\delta(0)} \int_0^{(u\wedge z)-k} \delta(u-k-x)\Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)\,dx
\end{aligned} \tag{3.20}$$

where $u \wedge z = \min(u, z)$. For $u \le z$, we can split the first integral so that formula (3.20) gives

$$\begin{aligned}
H_k(u,z,y) \;=\;& H_k(k,z,y)H_1(u-k,k) + \frac{\lambda\delta(u-k)}{c\delta(0)} \Bigg(\int_0^{u-k} \Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)\,dx \\
& + \int_{u-k}^{z-k} \Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)dx \Bigg) \\
& -\frac{\lambda}{c\delta(0)} \int_0^{u-k} \delta(u-k-x)\Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)\,dx \\
=\;& H_k(k,z,y)H_1(u-k,k) + \frac{\lambda\delta(u-k)}{c\delta(0)} \int_{u-k}^{z-k} \Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)dx \\
& +\frac{\lambda}{c\delta(0)} \int_0^{u-k} \big(\delta(u-k)-\delta(u-k-x)\big)\Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)\,dx
\end{aligned} \tag{3.21}$$

and for $u > z$, formula (3.20) yields

$$\begin{aligned}
H_k(u,z,y) \;=\;& H_k(k,z,y)H_1(u-k,k) \\
& +\frac{\lambda}{c\delta(0)} \int_0^{z-k} \big(\delta(u-k)-\delta(u-k-x)\big)\Big(\bar{F}(x+k)-\bar{F}(x+k+y)\Big)\,dx.
\end{aligned} \tag{3.22}$$

From identities (3.21) and (3.22), formula (3.16) follows. □

Letting $y \to \infty$, equation (3.21) becomes

$$\begin{aligned} H_{k,2}(u,z) &= H_{k,2}(k,z)H_1(u-k,k) + \frac{\lambda\delta(u-k)}{c\delta(0)}\int_{u-k}^{z-k}\bar{F}(x+k)\,dx \\ &+ \frac{\lambda}{c\delta(0)}\int_0^{u-k}\left(\psi(u-k-x)-\psi(u-k)\right)\bar{F}(x+k)\,dx \end{aligned}$$

and formula (3.22) yields

$$\begin{aligned} H_{k,2}(u,z) &= H_{k,2}(k,z)H_1(u-k,k) \\ &+ \frac{\lambda}{c\delta(0)}\int_0^{z-k}\left(\psi(u-x-k)-\psi(u-k)\right)\bar{F}(x+k)\,dx. \end{aligned}$$

which are in agreement with Nie (2012).

We can obtain $H_{k,1}(u,y)$ by inverting its Laplace transform. To find the Laplace transform of the (defective) distribution function of the deficit at ruin we let $z \to \infty$ in formula (3.19). Thus

$$\begin{aligned} \tilde{H}_{k,1}(s,y) &= e^{-sk}\Bigg(H_{k,1}(k,y)\tilde{H}_1(s,k) + \frac{\tilde{\delta}(s)}{\delta(0)}H_{k,1}(k,y)[1-H_1(0,k)] \\ &- \frac{\lambda\tilde{\delta}(s)}{c\delta(0)}\int_0^{\infty}e^{-su}\Big(\bar{F}(u+k)-\bar{F}(u+k+y)\Big)\,du\Bigg). \end{aligned}$$

We can modify formula (3.11) as

$$\tilde{H}_1(s,k+y) = \frac{\tilde{\delta}(s)}{\delta(0)}\left(\tilde{\Phi}'(s,k+y) + H_1(0,k+y)\right)$$

and

$$\tilde{\Phi}'(s,k+y) = \frac{\lambda}{c}\int_0^{\infty}e^{-su}\Big(\bar{F}(u+k+y)-\bar{F}(u)\Big)du.$$

Then, we rewrite formula (3.23), giving

$$\begin{aligned} \tilde{H}_{k,1}(s,y) &= e^{-sk}\Bigg(H_{k,1}(k,y)\tilde{H}_1(s,k) + \frac{\tilde{\delta}(s)}{\delta(0)}H_{k,1}(k,y)[1-H_1(0,k)] - \frac{\tilde{\delta}(s)}{\delta(0)}\tilde{\Phi}'(s,k) \\ &+ \frac{\tilde{\delta}(s)}{\delta(0)}\tilde{\Phi}'(s,k+y)\Bigg) \end{aligned}$$

and substituting for $\tilde{\Phi}'(s,k)$ and $\tilde{\Phi}'(s,k+y)$ yields

$$\begin{aligned}\tilde{H}_{k,1}(s,y) &= e^{-sk}\Bigg(H_{k,1}(k,y)\tilde{H}_1(s,k)+\frac{\tilde{\delta}(s)}{\delta(0)}H_{k,1}(k,y)[1-H_1(0,k)]\\ &\quad -\left[\tilde{H}_1(s,k)-\frac{\tilde{\delta}(s)}{\delta(0)}H_1(0,k)\right]+\left[\tilde{H}_1(s,k+y)-\frac{\tilde{\delta}(s)}{\delta(0)}H_1(0,k+y)\right]\Bigg)\\ &= e^{-sk}\Bigg(H_{k,1}(k,y)\tilde{H}_1(s,k)-\tilde{H}_1(s,k)+\tilde{H}_1(s,k+y)\\ &\quad +\frac{\tilde{\delta}(s)}{\delta(0)}H_{k,1}(k,y)[1-H_1(0,k)]-\frac{\tilde{\delta}(s)}{\delta(0)}[H_1(0,k+y)-H_1(0,k)]\Bigg).\end{aligned} \tag{3.23}$$

Replacing the second $H_{k,1}(k,y)$ in (3.23) by formula (3.14) and inverting $\tilde{H}_{k,1}(s,y)$ with respect to s, we obtain

$$H_{k,1}(u,y) = H_{k,1}(k,y)H_1(u-k,k) - H_1(u-k,k) + H_1(u-k,k+y) \tag{3.24}$$

which is in agreement with Nie (2012).

In this section, we have derived existing results by applying the Gerber-Shiu function. Although, Gerber-Shiu functions are very useful, they are not always the most efficient tools. However, we show in the next section that our Gerber-Shiu function can be used to obtain more general results relating to finite time ruin and in the next chapter, we see that probabilistic reasoning provides a much easier approach to finding the results of this section.

3.5 The joint density of $T_{u,k}$ and $N_{T_{u,k}}$

We now consider the joint density of the time of ruin and the number of claims until ruin by setting $0 < r < 1$, $\delta > 0$ and $\omega(x,y) = 1$ for $x \geq k$ and $y > 0$ in expression (3.1). (For convenience, we use the term joint density throughout when referring to two variables, even if one of the variables is discrete.) In this case the Gerber-Shiu function is

$$\phi_{k,r,\delta}(u) = E\left[r^{N_{T_{u,k}}}e^{-\delta T_{u,k}}I(T_{u,k}<\infty) \mid U(0)=u\right] = \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} w_k(u,n,t)\,dt$$

where $w_k(u,n,t)$ denotes the (defective) joint density of $N_{T_{u,k}}$ and $T_{u,k}$, given initial surplus u, defined for $n = 1, 2, 3, \ldots$, and $t > 0$ so that

$$p_k(u,n) = \int_0^\infty w_k(u,n,t)\,dt$$

is the probability mass function of the number of claims until ruin.

The next two results give expressions for the (defective) joint density of the number of claims until ruin and the time of ruin.

Theorem 3.6. Let the initial surplus be k. Then, the joint density of $N_{T_{k,k}}$ and $T_{k,k}$ is given by

$$w_k(k,1,t) = \lambda e^{-\lambda t}\bar{F}_{0,1}(ct) = \lambda e^{-\lambda t}\bar{F}(ct+k) \tag{3.25}$$

and for $n = 1, 2, 3, \ldots$,

$$\begin{aligned} & w_k(k,n+1,t) \\ = \; & \frac{\lambda^{n+1}}{c^n}\sum_{m=0}^{n}(-1)^m\binom{n}{m}e^{-\lambda t}\bar{F}_{m,n+1}(ct) \\ & + \sum_{j=0}^{n-1} e^{-\lambda t}\frac{\lambda^{n+1}}{c^{j+1}}\frac{t^{n-j-1}}{(n-j)!}\sum_{m=0}^{j}(-1)^m\binom{j}{m}\int_0^{ct} y f^{(n-j)*}(ct-y)\bar{F}_{m,j+1}(y)\,dy \end{aligned} \tag{3.26}$$

where for $n = 1, 2, 3, \ldots$ and $m = 0, 1, 2, \ldots, n-2$,

$$\bar{F}_{m,n}(u) = \int_0^u A^{(n-1-m)*}(x)B_k^{(m+1)*}(u-x)\,dx \tag{3.27}$$

with $A(x) = \bar{F}(x)$, $B_k(x) = \bar{F}(x+k)$, and $\bar{F}_{n-1,n}(u) = B_k^{n*}(u)$.

Proof. We start by rewriting formula (3.3) as

$$\phi_{k,r,\delta}(k) = \frac{\lambda r}{c}\int_k^\infty e^{-\rho(u-k)}\bar{F}(u)\,du\sum_{n=0}^{\infty}\left(\frac{\lambda r}{c}\int_k^\infty e^{-\rho(u-k)}\left(\bar{F}(u-k)-\bar{F}(u)\right)du\right)^n$$

and using the binomial expansion we obtain

$$\begin{aligned} \phi_{k,r,\delta}(k) \; = \; & \sum_{n=1}^{\infty}\frac{(\lambda r)^n}{c^n}\sum_{m=0}^{n-1}(-1)^m\binom{n-1}{m}\left(\int_0^\infty e^{-\rho u}\bar{F}(u)du\right)^{n-1-m} \\ & \times\left(\int_0^\infty e^{-\rho u}\bar{F}(u+k)du\right)^{m+1}, \end{aligned}$$

giving

$$\phi_{k,r,\delta}(k) = \sum_{n=1}^{\infty} \frac{(\lambda r)^n}{c^n} \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} \int_0^{\infty} e^{-\rho u} \bar{F}_{m,n}(u)\, du. \tag{3.28}$$

To invert formula (3.28) we use formula (44) of Landriault et al. (2011), i.e.

$$e^{-\rho u} = e^{-(\lambda+\delta)u/c} + \sum_{j=1}^{\infty} \frac{(\lambda r/c)^j}{j!} u \int_0^{\infty} (x+u)^{j-1}\, e^{-(\lambda+\delta)(x+u)/c}\, f^{j*}(x)\, dx.$$

Substituting for $e^{-\rho u}$ in (3.28) we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} r^n w_k(k,n,t) &= \sum_{n=1}^{\infty} \frac{(\lambda r)^n}{c^n} \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} \Bigg(e^{-\lambda t} c \bar{F}_{m,n}(ct) \\
&\quad + \sum_{j=1}^{\infty} e^{-\lambda t} \frac{(\lambda r)^j}{j!} t^{j-1} \int_0^{ct} y f^{j*}(ct-y) \bar{F}_{m,n}(y)\, dy \Bigg) \\
&= \sum_{n=1}^{\infty} \frac{(\lambda r)^n}{c^n} \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} e^{-\lambda t} c \bar{F}_{m,n}(ct) \\
&\quad + \sum_{n=1}^{\infty} r^{n+1} \sum_{j=0}^{n-1} e^{-\lambda t} \frac{\lambda^{n+1}}{c^{j+1}} \frac{t^{n-j-1}}{(n-j)!} \sum_{m=0}^{j} (-1)^m \binom{j}{m} \\
&\quad \times \int_0^{ct} y f^{(n-j)*}(ct-y) \bar{F}_{m,j+1}(y)\, dy.
\end{aligned} \tag{3.29}$$

Formulae (3.25) and (3.26) then follow by equating coefficients of powers of r in equation (3.29). □

We remark that if we set $r = 1$ in (3.29) then we obtain an expression for the density of the time of ruin, $w_k(k,t)$.

Theorem 3.7. When $u > k$,

$$w_k(u,1,t) = \lambda e^{-\lambda t} \bar{F}(u+ct) \tag{3.30}$$

and for $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} w_k(u, n+1, t) &= e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct-k} f^{n*}(u+ct-x-k)\lambda\bar{F}(x+k)\,dx \\ &+\int_0^t w_k(k, n, t-\tau)\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big)\,d\tau \\ &+\sum_{m=1}^{n-1}\int_0^t e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!}\int_0^{u+c\tau-k} f^{m*}(u+c\tau-k-x) \\ &\times\lambda\Big(\bar{F}(x)-\bar{F}(x+k)\Big)\,dx\,w_k(k, n-m, t-\tau)\,d\tau \\ &-c\sum_{m=1}^{n}\int_0^t e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!}f^{m*}(u+c\tau-k)w_k(k, n+1-m, t-\tau)\,d\tau \end{aligned} \tag{3.31}$$

with the usual convention that $\sum_{j=a}^{b} = 0$ if $b < a$.

Proof. We start by noting that

$$\tilde{\phi}_{k,r,\delta}(s) = \int_k^\infty e^{-su}\phi_{k,r,\delta}(u)\,du = \sum_{n=1}^\infty r^n\int_k^\infty e^{-su}\int_0^\infty e^{-\delta t}w_k(u, n, t)\,dt\,du.$$

Next, we rewrite formula (3.2) as

$$\begin{aligned} \tilde{\phi}_{k,r,\delta}(s) &= \frac{1}{\delta+\lambda-\lambda r\tilde{f}(s)-cs}\Bigg(-ce^{-sk}\phi_{k,r,\delta}(k) \\ &+\lambda r\phi_{k,r,\delta}(k)\int_k^\infty e^{-su}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)du+\lambda r\int_k^\infty e^{-su}\bar{F}(u)du\Bigg) \end{aligned} \tag{3.32}$$

and our approach is to invert first with respect to δ, and then with respect to s. Our derivation uses ideas in Panjer and Willmot (1992) and is based on the Laplace transform of $S(t)$; see also Willmot (2015). Our starting point is to define a function $A_r(u, t)$ whose Laplace transform with respect to u is $\tilde{A}_r(s, t)$, and its bivariate Laplace transform is given by

$$\tilde{\tilde{A}}_r(s, \delta) = \int_0^\infty e^{-\delta t}\tilde{A}_r(s, t)\,dt = \frac{1}{\delta+\lambda-\lambda r\tilde{f}(s)-cs}.$$

Inverting this expression with respect to δ gives $\tilde{A}_r(s, t) = \exp\{\lambda rt(\tilde{f}(s)-1)+\lambda rt-\lambda t+sct\}$. From the Laplace transform of $S(t)$ in formula (1.2) we deduce that $\tilde{A}_r(s, t)$

is the product of the Laplace transform of $S_r(t)$, where $S_r(t)$ is as $S(t)$ except that its Poisson parameter is λrt (and its density is denoted by $g_r(x,t)$), and the term $\exp\{\lambda t(r-1)+sct\}$. Hence we can write formula (3.32) as

$$\tilde{\phi}_{k,r,\delta}(s) = \tilde{\tilde{A}}_r(s,\delta)\Big(-ce^{-sk}\phi_{k,r,\delta}(k) + \lambda r\phi_{k,r,\delta}(k)\tilde{C}(s) + \lambda r\tilde{B}(s)\Big), \tag{3.33}$$

where $C(u) = (\bar{F}(u-k) - \bar{F}(u))I\{u > k\}$ and $B(u) = \bar{F}(u)I\{u > k\}$. Inverting equation (3.33) with respect to δ yields

$$\begin{aligned}
&\sum_{n=1}^{\infty} r^n \int_k^{\infty} e^{-su} w_k(u,n,t)\,du \\
= \ & -c\sum_{n=1}^{\infty} r^n e^{-sk} \int_0^t w_k(k,n,\tau)\exp\{\lambda r(t-\tau)(\tilde{f}(s)-1) + \lambda(t-\tau)(r-1) + sc(t-\tau)\}\,d\tau \\
&+\lambda r\tilde{C}(s)\sum_{n=1}^{\infty} r^n \int_0^t w_k(k,n,\tau)\exp\{\lambda r(t-\tau)(\tilde{f}(s)-1) + \lambda(t-\tau)(r-1) + sc(t-\tau)\}\,d\tau \\
&+\lambda r\tilde{B}(s)\exp\{\lambda rt(\tilde{f}(s)-1) + \lambda t(r-1) + sct\}.
\end{aligned}$$

Multiplying both sides by e^{-sct} gives

$$\begin{aligned}
&\sum_{n=1}^{\infty} r^n e^{-sct} \int_k^{\infty} e^{-su} w_k(u,n,t)\,du \\
= \ & -c\sum_{n=1}^{\infty} r^n e^{-sk} \int_0^t w_k(k,n,\tau)\exp\{\lambda r(t-\tau)(\tilde{f}(s)-1) + \lambda(t-\tau)(r-1) - sc\tau\}\,d\tau \\
&+\lambda r\tilde{C}(s)\sum_{n=1}^{\infty} r^n \int_0^t w_k(k,n,\tau)\exp\{\lambda r(t-\tau)(\tilde{f}(s)-1) + \lambda(t-\tau)(r-1) - sc\tau\}\,d\tau \\
&+\lambda r\tilde{B}(s)\exp\{\lambda rt(\tilde{f}(s)-1) + \lambda t(r-1)\}.
\end{aligned} \tag{3.34}$$

The left-hand side of equation (3.34) can be written as

$$\sum_{n=1}^{\infty} r^n \int_k^{\infty} e^{-s(u+ct)} w_k(u,n,t)\,du = \sum_{n=1}^{\infty} r^n \int_{ct+k}^{\infty} e^{-su} w_k(u-ct,n,t)\,du \tag{3.35}$$

and (minus) the first term on the right-hand side as

$$\begin{aligned}
&c\sum_{n=1}^{\infty} r^n e^{-sk} \int_0^t w_k(k,n,\tau)\exp\{\lambda r(t-\tau)(\tilde{f}(s)-1) + \lambda(t-\tau)(r-1) - sc\tau\}\,d\tau \\
= \ & c\sum_{n=1}^{\infty} r^n e^{-sk} \int_0^t w_k(k,n,\tau)\left(\tilde{g}_r(s,t-\tau) + e^{-\lambda r(t-\tau)}\right) e^{\lambda(t-\tau)(r-1)-sc\tau}\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&= c\sum_{n=1}^{\infty} r^n \int_0^t w_k(k,n,\tau) \int_0^{\infty} e^{-s(x+c\tau+k)} g_r(x,t-\tau) e^{\lambda(t-\tau)(r-1)}\,dx\,d\tau \\
&\quad + c\sum_{n=1}^{\infty} r^n \int_0^t w_k(k,n,\tau) e^{-s(c\tau+k)} e^{-\lambda(t-\tau)}\,d\tau \\
&= c\sum_{n=1}^{\infty} r^n \int_0^t w_k(k,n,\tau) \int_0^{\infty} e^{-s(x+c\tau+k)} \sum_{m=1}^{\infty} e^{-\lambda(t-\tau)} \frac{(\lambda r(t-\tau))^m}{m!} f^{m*}(x)\,dx\,d\tau \\
&\quad + \sum_{n=1}^{\infty} r^n \int_k^{ct+k} w_k(k,n,(u-k)/c) e^{-su} e^{-\lambda(t-(u-k)/c)}\,du \\
&= c\sum_{n=1}^{\infty} r^n \int_0^t w_k(k,n,\tau) \int_{c\tau+k}^{\infty} e^{-su} \sum_{m=1}^{\infty} e^{-\lambda(t-\tau)} \frac{(\lambda r(t-\tau))^m}{m!} f^{m*}(u-c\tau-k)\,du\,d\tau \\
&\quad + \sum_{n=1}^{\infty} r^n \int_k^{ct+k} w_k(k,n,(u-k)/c) e^{-su} e^{-\lambda(t-(u-k)/c)}\,du \\
&= c\sum_{n=1}^{\infty} r^n \int_k^{ct+k} e^{-su} \int_0^{(u-k)/c} w_k(k,n,\tau) \sum_{m=1}^{\infty} e^{-\lambda(t-\tau)} \frac{(\lambda r(t-\tau))^m}{m!} f^{m*}(u-c\tau-k)\,d\tau\,du \\
&\quad + c\sum_{n=1}^{\infty} r^n \int_{ct+k}^{\infty} e^{-su} \int_0^t w_k(k,n,\tau) \sum_{m=1}^{\infty} e^{-\lambda(t-\tau)} \frac{(\lambda r(t-\tau))^m}{m!} f^{m*}(u-c\tau-k)\,d\tau\,du \\
&\quad + \sum_{n=1}^{\infty} r^n \int_k^{ct+k} e^{-su} w_k(k,n,(u-k)/c) e^{-\lambda(t-(u-k)/c)}\,du.
\end{aligned}
$$

We can treat the second term of the right-hand side of equation (3.34) similarly to show that

$$
\begin{aligned}
&\lambda r\tilde{C}(s)\sum_{n=1}^{\infty} r^n \int_0^t w_k(k,n,\tau) \exp\{\lambda r(t-\tau)(\tilde{f}(s)-1) + \lambda(t-\tau)(r-1) - sc\tau\}\,d\tau \\
&= \lambda r\sum_{n=1}^{\infty} r^n \int_{ct}^{\infty} e^{-su} \int_0^t w_k(k,n,\tau) \int_0^{u-c\tau-k} \sum_{m=1}^{\infty} e^{-\lambda(t-\tau)} \frac{(\lambda r(t-\tau)^m}{m!} f^{m*}(x) \\
&\quad \times \big(\bar{F}(u-c\tau-k-x) - \bar{F}(u-c\tau-x)\big)\,dx\,d\tau\,du \\
&\quad + \lambda r\sum_{n=1}^{\infty} r^n \int_0^{ct} e^{-su} \int_0^{u/c} w_k(k,n,\tau) \int_0^{u-c\tau} \sum_{m=1}^{\infty} e^{-\lambda(t-\tau)} \frac{(\lambda r(t-\tau))^m}{m!} f^{m*}(x) \\
&\quad \times C(u-c\tau-x)\,dx\,d\tau\,du \\
&\quad + \lambda r\sum_{n=1}^{\infty} r^n \int_{ct}^{\infty} e^{-su} \int_0^t w_k(k,n,\tau) \\
&\quad \times \big(\bar{F}(u-c\tau-k) - \bar{F}(u-c\tau)\big) I(u-c\tau>k) e^{-\lambda(t-\tau)}\,d\tau\,du \\
&\quad + \lambda r\sum_{n=1}^{\infty} r^n \int_0^{ct} e^{-su} \int_0^{u/c} w_k(k,n,\tau) C(u-c\tau) e^{-\lambda(t-\tau)}\,d\tau\,du. \qquad (3.36)
\end{aligned}
$$

Finally, the third term on the right-hand side of equation (3.34) can be expressed as

$$\int_0^\infty e^{-su}\left(\lambda re^{-\lambda t}\bar{F}(u)I(u>k)+\lambda r\sum_{n=1}^\infty e^{-\lambda t}\frac{(\lambda rt)^n}{n!}\int_0^{u-k}\bar{F}(x+k)f^{n*}(u-x-k)\,dx\right)du.$$

By equating the coefficients of e^{-su} for $u > ct+k$ we obtain

$$\begin{aligned}
&\sum_{n=1}^\infty r^n w_k(u-ct,n,t)\\
=\;& \lambda re^{-\lambda t}\bar{F}(u)+\lambda r\sum_{n=1}^\infty e^{-\lambda t}\frac{(\lambda rt)^n}{n!}\int_0^{u-k}\bar{F}(x+k)f^{n*}(u-x-k)\,dx\\
&+\sum_{n=1}^\infty r^{n+1}\int_0^t w_k(k,n,\tau)\Big[\lambda e^{-\lambda(t-\tau)}\Big(\bar{F}(u-k-c\tau)-\bar{F}(u-c\tau)\Big)\\
&+\lambda\sum_{m=1}^\infty e^{-\lambda(t-\tau)}\frac{[\lambda r(t-\tau)]^m}{m!}\int_0^{u-c\tau-k}f^{m*}(x)\\
&\times\Big(\bar{F}(u-k-c\tau-x)-\bar{F}(u-c\tau-x)\Big)\,dx\Big]\,d\tau\\
&-c\sum_{n=1}^\infty r^n\sum_{m=1}^\infty\int_0^t w_k(k,n,\tau)e^{-\lambda(t-\tau)}\frac{[\lambda r(t-\tau)]^m}{m!}f^{m*}(u-c\tau-k)\,d\tau.
\end{aligned}$$

This results in

$$\begin{aligned}
&\sum_{n=1}^\infty r^n w_k(u,n,t)\\
=\;& \lambda re^{-\lambda t}\bar{F}(u+ct)+\lambda r\sum_{n=1}^\infty e^{-\lambda t}\frac{(\lambda rt)^n}{n!}\int_0^{u+ct-k}\bar{F}(x+k)f^{n*}(u+ct-x-k)\,dx\\
&+\sum_{n=1}^\infty r^{n+1}\int_0^t w_k(k,n,t-\tau)\Big[\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big)\\
&+\lambda\sum_{m=1}^\infty e^{-\lambda\tau}\frac{(\lambda r\tau)^m}{m!}\int_0^{u+c\tau-k}f^{m*}(x)\\
&\times\Big(\bar{F}(u+c\tau-k-x)-\bar{F}(u+c\tau-x)\Big)\,dx\Big]\,d\tau\\
&-c\sum_{n=1}^\infty r^n\sum_{m=1}^\infty\int_0^t w_k(k,n,t-\tau)e^{-\lambda\tau}\frac{(\lambda r\tau)^m}{m!}f^{m*}(u+c\tau-k)\,d\tau. \qquad (3.37)
\end{aligned}$$

To obtain $w_k(u,n,t)$ we proceed as follows. We rewrite expression (3.37) as

$$\begin{aligned}
&\sum_{n=1}^{\infty} r^n w_k(u,n,t) \\
= \;& \lambda r e^{-\lambda t}\bar{F}(u+ct) + \sum_{n=1}^{\infty} r^{n+1} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct-k} f^{n*}(u+ct-x-k)\lambda \bar{F}(x+k)\,dx \\
&+\sum_{n=1}^{\infty} r^{n+1}\int_0^t w_k(k,n,t-\tau)\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big)\,d\tau \\
&+\sum_{n=1}^{\infty} r^{n+1}\int_0^t w_k(k,n,t-\tau)\lambda\sum_{m=1}^{\infty} r^m e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!} \\
&\times\int_0^{u+c\tau-k} f^{m*}(x)\Big(\bar{F}(u+c\tau-k-x)-\bar{F}(u+c\tau-x)\Big)\,dx\,d\tau \\
&-c\sum_{n=1}^{\infty} r^n\sum_{m=1}^{\infty} r^m\int_0^t e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!} f^{m*}(u+c\tau-k)w_k(k,n,t-\tau)\,d\tau. \qquad (3.38)
\end{aligned}$$

Applying the Cauchy product $\sum_{n=1}^{\infty} a_n \sum_{m=1}^{\infty} b_m = \sum_{n=1}^{\infty}\sum_{m=1}^{n} a_{n+1-m}b_m$ to the double summations in (3.38) we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} r^n w_n^k(u,t) \\
= \;& \lambda r e^{-\lambda t}\bar{F}(u+ct) + \sum_{n=1}^{\infty} r^{n+1} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct-k} f^{n*}(u+ct-x-k)\lambda \bar{F}(x+k)\,dx \\
&+\sum_{n=1}^{\infty} r^{n+1}\int_0^t w_k(k,n,t-\tau)\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big)\,d\tau \\
&+\sum_{n=1}^{\infty} r^{n+2}\sum_{m=1}^{n}\int_0^t w_k(k,n+1-m,t-\tau)e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!} \\
&\times\int_0^{u+c\tau-k} f^{m*}(x)\lambda\Big(\bar{F}(u+c\tau-k-x)-\bar{F}(u+c\tau-x)\Big)\,dx\,d\tau \\
&-c\sum_{n=1}^{\infty} r^{n+1}\sum_{m=1}^{n}\int_0^t e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!} f^{m*}(u+c\tau-k)\,w_k(k,n+1-m,t-\tau)\,d\tau. \qquad (3.39)
\end{aligned}$$

Formulae (3.30) and (3.31) then follow by equating coefficients of powers of r in equation (3.39). □

Formula (3.31) generalises formula (8) of Dickson (2012) for the joint density of the time of ruin and the number of claims until ruin in the classical risk model – if we set $k = 0$ we recover the result for the classical risk model.

Further, we obtain the following result.

Theorem 3.8. For $u > k$, the density of $T_{u,k}$ is

$$\begin{aligned} w_k(u,t) &= \lambda e^{-\lambda t}\bar{F}(u+ct) + \lambda\int_0^{u+ct-k}\bar{F}(x+k)g(u+ct-x-k)\,dx \\ &\quad + \int_0^t w_k(k,t-\tau)\Big[\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big) \\ &\quad + \int_0^{u+c\tau-k} g(u+c\tau-k-x,\tau)\lambda\Big(\bar{F}(x)-\bar{F}(x+k)\Big)dx\Big]\,d\tau \\ &\quad -c\int_0^t w_k(k,t-\tau)g(u+c\tau-k,\tau)\,d\tau. \end{aligned} \tag{3.40}$$

Proof. Using formulae (3.30) and (3.31) we have

$$\begin{aligned} w_k(u,t) &= \sum_{n=0}^{\infty} w_k(u,n+1,t) \\ &= \lambda e^{-\lambda t}\bar{F}(u+ct) + \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct-k} f^{n*}(u+ct-x-k)\lambda\bar{F}(x+k)\,dx \\ &\quad + \sum_{n=1}^{\infty}\int_0^t w_k(k,n,t-\tau)\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big)\,d\tau \\ &\quad + \sum_{n=2}^{\infty}\sum_{m=1}^{n-1}\int_0^t w_k(k,n-m,t-\tau)e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!} \\ &\quad \times\int_0^{u+c\tau-k} f^{m*}(u+c\tau-k-x)\lambda\Big(\bar{F}(x)-\bar{F}(x+k)\Big)\,dx\,d\tau \\ &\quad -c\sum_{n=1}^{\infty}\sum_{m=1}^{n}\int_0^t w_k(k,n+1-m,t-\tau)e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!}f^{m*}(u+c\tau-k)\,d\tau. \end{aligned}$$

Reversing the Cauchy product gives

$$\begin{aligned} \sum_{n=0}^{\infty} w_k(u,n+1,t) &= \lambda e^{-\lambda t}\bar{F}(u+ct) \\ &\quad +\lambda\sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\int_0^{u+ct-k} f^{n*}(u+ct-x-k)\bar{F}(x+k)\,dx \\ &\quad +\sum_{n=1}^{\infty}\int_0^t w_k(k,n,t-\tau)\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big)\,d\tau \end{aligned}$$

$$
\begin{aligned}
&+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t w_k(k,n,t-\tau)e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!} \\
&\times\int_0^{u+c\tau-k} f^{m*}(u+c\tau-k-x)\lambda\Big(\bar{F}(x)-\bar{F}(x+k)\Big)\,dx\,d\tau \\
&-c\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\int_0^t e^{-\lambda\tau}\frac{(\lambda\tau)^m}{m!}f^{m*}(u+c\tau-k)w_k(k,n,t-\tau)\,d\tau. \\
= \;& \lambda e^{-\lambda t}\bar{F}(u+ct)+\lambda\int_0^{u+ct-k}\bar{F}(x+k)g(u+ct-x-k)\,dx \\
&+\sum_{n=1}^{\infty}\int_0^t w_k(k,n,t-\tau)\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big)\,d\tau \\
&+\sum_{n=1}^{\infty}\int_0^t w_k(k,n,t-\tau)\int_0^{u+c\tau-k} g(u+c\tau-k-x,\tau) \\
&\times\lambda\Big(\bar{F}(x)-\bar{F}(x+k)\Big)\,dx\,d\tau \\
&-c\sum_{n=1}^{\infty}\int_0^t w_k(k,n,t-\tau)g(u+c\tau-k,\tau)\,d\tau,
\end{aligned}
$$

and the result follows. □

Formula (3.40) generalises formula (5) of Dickson (2007) for the density of the time of ruin in the classical risk model. Once again, setting $k = 0$ gives the result for the classical risk model.

To understand formula (3.40), it is helpful to consider realisations of the surplus process that are representative of the different terms in the formula. Interpretation of components of formula (3.40) is similar to that given in Dickson (2007). We can interpret formula (3.40) as follows.

- The term $w_k(u,t)dt$ represents the probability that ruin occurs within the time period $(t, t+dt)$, from initial surplus u.

- The first term on the right-hand side of (3.40) represents the situation that there is no claim by time t and that ruin occurs within $(t, t+dt)$. For this to happen we require a claim in this interval exceeding the initial surplus and the total premium received up to time t. (Figure 3.1)

- In the second term, $g(u+ct-x-k,t)$ is the density associated with the situation that the surplus in a classical risk model is at level $x+k$ at time t. For ruin occurring in $(t, t + dt)$, we require a claim whose amount exceeds $x + k$. We note that the term $g(u + ct - x - k, t)$ refers to aggregate claims in a classical risk model, not the risk model with capital injections and therefore this term includes the possibility that the surplus is below k before time t. (Figure 3.2)

- We consider

$$g(u + c\tau - k - x, \tau)\lambda\Big(\bar{F}(x) - \bar{F}(x + k)\Big)w_k(k, t - \tau)$$

in the third term of (3.40). This term is associated with (i) an aggregate claim amount of $u + c\tau - k - x$ at time τ, resulting in a surplus of $x + k$ at time τ in a *classical* risk model, (ii) a claim whose amount is between x and $x + k$ at time τ (so that a capital injection occurs), and (iii) ruin occurring after a further time period of $t - \tau$ with the possibility of capital injections in this time period. We note that the term $g(u + c\tau - k - x, \tau)$ refers to aggregate claims in a classical risk model, not the risk model with capital injections, and this term includes the possibility that a realisation of a *classical* risk model which is at level $x + k$ at time τ has fallen below k prior to time τ. Figures 3.3 and 3.4 show realisations of the surplus process representing the second integral term, with a capital injection occurring at time τ. In these realisations, the surplus prior to time τ is always above k, and we distinguish between there being further capital injections after time τ (Figure 3.3) and no further capital injections (Figure 3.4). Figures 3.5 and 3.6 are as Figures 3.3 and 3.4 except that the surplus falls below k prior to time τ.

- Figures 3.7 and 3.8 show realisations of the surplus process representing the final term in formula (3.40). In these, τ is the last time at which the (classical) surplus process upcrosses through k, and we have distinguished between the cases where there are capital injections after time τ (Figure 3.7) and there are no capital injections after time τ (Figure 3.8). Only Figures 3.1, 3.3 and 3.4 show realisations of the surplus process with capital injections that result in ruin at time t. As in Prabhu's (1961) formula, the final term in formula (3.40) is a compensation term. Realisations like those in Figure 3.2 are compensated for by realisations such as that in Figure 3.8. Similarly, realisations such as those in Figures 3.5 and 3.6 are compensated for by realisations such as that in Figure 3.7. To see this, we note

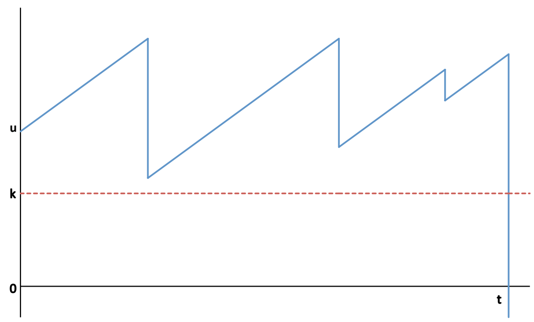

Figure 3.1: Classical surplus process always above k prior to t

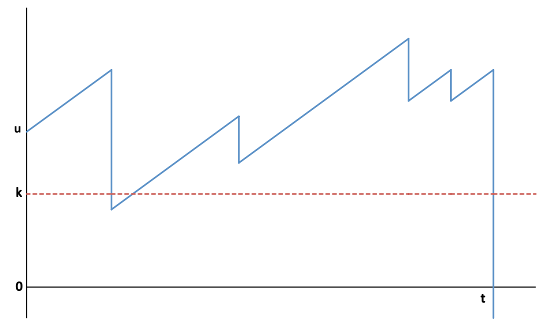

Figure 3.2: Classical surplus process below k prior to t

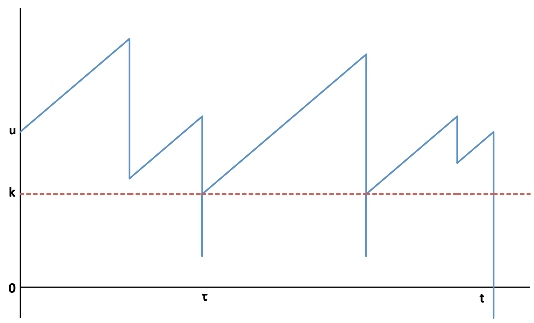

Figure 3.3: Classical surplus process above k prior to capital injection at τ; capital injected after τ

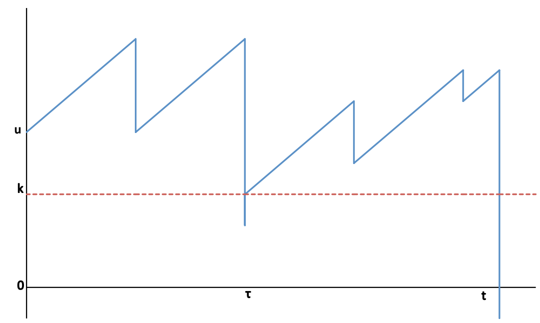

Figure 3.4: Classical surplus process above k prior to capital injection at τ; no capital injected after τ

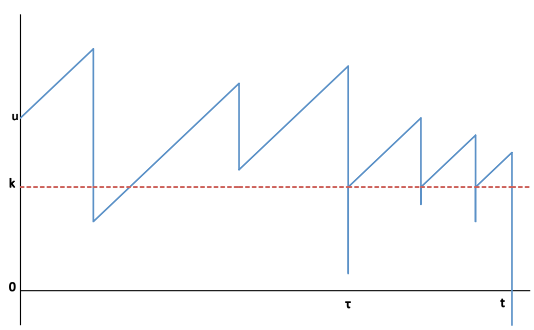

Figure 3.5: Classical surplus process below k prior to capital injection at τ; capital injected after τ

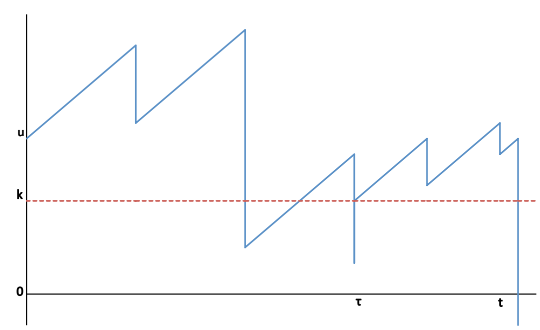

Figure 3.6: Classical surplus process below k prior to capital injection at τ; no capital injected after τ

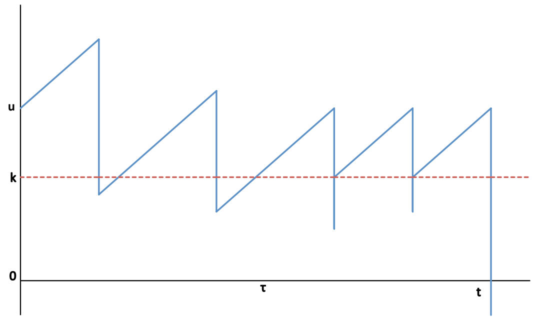

Figure 3.7: Classical surplus process upcrosses k at τ; ruin at t with capital injections between τ and t

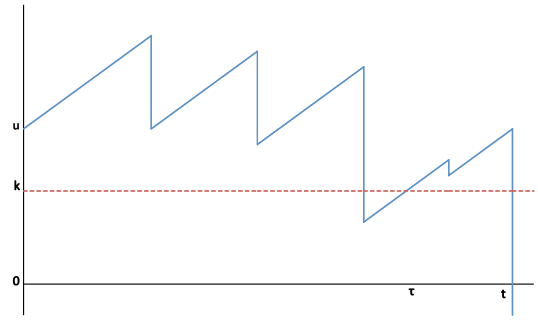

Figure 3.8: Classical surplus process upcrosses k at τ; ruin at t without capital injections between τ and t

that there is a last time (before time τ) at which the surplus process upcrosses through k in Figures 3.5 and 3.6.

From these figures it is not difficult to see that there exists a version of Prabhu's formula for the risk model with capital injections. Define $\delta_k(u,t) = \Pr(T_{u,k} > t \mid U(0) = u)$ to be the survival probability in finite time and $\tilde{\delta}_k(s,t)$ to be the Laplace transform of the survival probability in finite time with respect to u. A more general version of Prabhu's formula is given in the following result.

Theorem 3.9. The finite time survival probability is given by

$$\begin{aligned}\delta_k(u,t) &= G(u+ct-k,t) - c\int_0^t \delta_k(k,t-\tau)g(u+c\tau-k,\tau)\,d\tau \\ &\quad +\lambda\int_0^t \delta_k(k,t-\tau)\Bigg[e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k)-\bar{F}(u+c\tau)\Big) \\ &\quad +\int_0^{u+c\tau-k} g(x,\tau)\Big(\bar{F}(u+c\tau-k-x)-\bar{F}(u+c\tau-x)\Big)\,dx\Bigg]\,d\tau.\end{aligned} \tag{3.41}$$

Proof. We distinguish between the situations where there is a claim in the infinitesimal time interval $(0,h]$, or there is no claim during this interval. Thus

$$\begin{aligned}\delta_k(u,t+h) &= (1-\lambda h)\delta_k(u+ch,t) + \lambda h\Bigg(\int_0^{u+ch-k}\delta_k(u+ch-x,t)f(x)\,dx. \\ &\quad +\delta_k(k,t)\int_{u+ch-k}^{u+ch} f(x)\,dx\Bigg) + o(h).\end{aligned} \tag{3.42}$$

Noting that $\delta_k(u+ch,t) = \delta_k(u,t) + ch\frac{\partial}{\partial u}\delta_k(u,t) + o(h)$, we have

$$\begin{aligned}\delta_k(u,t+h) &= \delta_k(u,t) + ch\frac{\partial}{\partial u}\delta_k(u,t) - \lambda h\delta_k(u,t) \\ &\quad +\lambda h\int_0^{u+ch-k}\delta_k(u+ch-x,t)f(x)\,dx + o(h) \\ &\quad +\lambda h\delta_k(k,t)\Big(\bar{F}(u+ch-k) - \bar{F}(u+ch)\Big).\end{aligned}$$

Rearranging, dividing by h and letting $h \to 0$, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\delta_k(u,t) &= c\frac{\partial}{\partial u}\delta_k(u,t) - \lambda\delta_k(u,t) + \lambda\int_0^{u-k}\delta_k(u-x,t)f(x)\,dx \\ &\quad +\lambda\delta_k(k,t)\Big(\bar{F}(u-k) - \bar{F}(u)\Big).\end{aligned}$$

Taking the Laplace transform with respect to u results in

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{\delta}_k(s,t) &= cs\tilde{\delta}_k(s,t) - ce^{-sk}\delta_k(k,t) - \lambda\tilde{\delta}_k(s,t) \\ &\quad +\lambda\int_k^\infty e^{-su}\int_0^{u-k}\delta_k(u-x,t)f(x)\,dx\,du \\ &\quad +\lambda\delta_k(k,t)\int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\,du.\end{aligned}$$

Rearranging we have

$$\begin{aligned}&\frac{\partial}{\partial t}\tilde{\delta}_k(s,t) + (\lambda - cs - \lambda\tilde{f}(s))\tilde{\delta}_k(s,t) \\ = &-e^{-sk}c\delta_k(k,t) + \lambda\delta_k(k,t)\int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\,du.\end{aligned}$$

Solving this differential equation, yields

$$\begin{aligned}&\int_0^t \frac{\partial}{\partial \tau}e^{(\lambda-cs-\lambda\tilde{f}(s))\tau}\tilde{\delta}_k(s,\tau)\,d\tau = -c\int_0^t e^{-sk}\delta_k(k,\tau)e^{(\lambda-cs-\lambda\tilde{f}(s))\tau}\,d\tau \\ &+\lambda\int_0^t \delta_k(k,\tau)e^{(\lambda-cs-\lambda\tilde{f}(s))\tau}\int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\,du\,d\tau. \qquad (3.43)\end{aligned}$$

The left-hand side in (3.43) is $e^{(\lambda-cs-\lambda\tilde{f}(s))t}\tilde{\delta}_k(s,t) - \tilde{\delta}_k(s,0)$ where

$$\tilde{\delta}_k(s,0) = \int_k^\infty e^{-su}\delta_k(u,0)\,du = \frac{e^{-sk}}{s}.$$

Therefore,

$$\begin{aligned}&\tilde{\delta}_k(s,t) \\ = \;&\frac{e^{-sk}}{s}e^{cst+\lambda t(\tilde{f}(s)-1)} - ce^{-sk}\int_0^t \delta_k(k,\tau)\exp\{\lambda(t-\tau)(\tilde{f}(s)-1) + sc(t-\tau)\}\,d\tau \\ &+\lambda\int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\,du\int_0^t \delta_k(k,\tau)\exp\{\lambda(t-\tau)(\tilde{f}(s)-1) + sc(t-\tau)\}\,d\tau.\end{aligned}$$

Multiplying both sides by e^{-cst} gives

$$
\begin{aligned}
& e^{-cst}\int_k^\infty e^{-su}\delta_k(u,t)\,dt \\
= & \ \frac{e^{-sk}}{s}e^{\lambda t(\tilde{f}(s)-1)} - ce^{-sk}\int_0^t \delta_k(k,\tau)\exp\{\lambda(t-\tau)(\tilde{f}(s)-1)-sc\tau\}\,d\tau \\
& +\lambda\int_k^\infty e^{-su}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)\,du\int_0^t \delta_k(k,\tau)\exp\{\lambda(t-\tau)(\tilde{f}(s)-1)-sc\tau\}\,d\tau.
\end{aligned}
\tag{3.44}
$$

Applying the same argument as in Theorem 3.7 and using the Laplace transform of $S(t)$ in formula (1.2), equation (3.44) inverts with respect to s to

$$
\begin{aligned}
\delta_k(u-ct,t) \;=\;& G(u-k,t) - c\int_0^t \delta_k(k,\tau)g(u-c\tau-k,t-\tau)\,d\tau \\
& +\lambda\int_0^t \delta_k(k,\tau)\Bigg[e^{-\lambda(t-\tau)}\Big(\bar{F}(u-k-c\tau)-\bar{F}(u-c\tau)\Big) \\
& +\int_0^{u-k-c\tau} g(x,t-\tau)\Big(\bar{F}(u-k-c\tau-x)-\bar{F}(u-c\tau-x)\Big)\,dx\Bigg]\,d\tau
\end{aligned}
$$

and the result follows by replacing $u-ct$ by u. □

We remark that $\delta_k(k,t)$ can be obtained by numerical integration of formula (3.41). Alternatively, integrating $w_k(k,\tau)$ from 0 to t gives $\psi_k(k,t)$ from which we can find $\delta_k(k,t)$. Also, given the nature of the surplus process with capital injections, where the full amount of claims may not be paid by the insurer, it is rather remarkable that formulae for $w_k(u,t)$ and $\delta_k(u,t)$ exist in terms of G and g.

3.6 Examples

In this section, we apply the results of the previous section when the individual claim amount distributions are exponential and Erlang(2).

3.6.1 Exponential claims

We now consider the case when $\bar{F}(x)=e^{-\alpha x}$, where $x\geq 0$ with $\alpha>0$, and provide results for $w_k(u,n,t)$ for $u=k$ and $u>k$.

Result 3.1. When the initial surplus is k, for $n = 1, 2, 3, \ldots,$

$$w_k(k, n+1, t) = \lambda^{n+1} e^{-\alpha k-(\lambda+\alpha c)t} \sum_{j=0}^{n} \frac{(\alpha c)^{n-j} t^{2n-j}}{(n-j)!} (1-e^{-\alpha k})^j \frac{j+1}{(n+1)!} \tag{3.45}$$

and $w_k(k, 1, t) = \lambda e^{-\alpha k-(\lambda+\alpha c)t}$.

Derivation. We apply formula (3.26). For this we require $\bar{F}_{m,n}(x)$, which can be obtained from (3.27) with $A(x) = e^{-\alpha x}$ and $B_k(x) = e^{-\alpha(x+k)}$. To evaluate the integral we use the property of the Laplace transform of a convolution. Therefore we have

$$\tilde{A}(s)^{n-1-m} \tilde{B}(s)^{m+1} = e^{-\alpha k(m+1)} \left(\frac{1}{\alpha+s}\right)^n,$$

which inverts to

$$\bar{F}_{m,n}(u) = \frac{e^{-\alpha k(m+1)} e^{-\alpha u} u^{n-1}}{\Gamma(n)}.$$

Substituting $\bar{F}_{m,n}(u)$ in formula (3.26) gives

$$\begin{aligned} w_k(k, n+1, t) &= \frac{\lambda^{n+1}}{c^{n+1}} e^{-\lambda t} \sum_{m=0}^{n} (-1)^m \binom{n}{m} \frac{e^{-\alpha k(m+1)} e^{-\alpha ct} t^n c^{n+1}}{n!} \\ &\quad + \sum_{j=0}^{n-1} \frac{\lambda^{n+1}}{c^{j+1}} e^{-\lambda t} \frac{t^{n-j-1}}{(n-j)!} \int_0^{ct} \frac{y \alpha^{n-j} e^{-\alpha(ct-y)} (ct-y)^{n-j-1}}{(n-j-1)!} \\ &\quad \times \sum_{m=0}^{j} (-1)^m \binom{j}{m} \frac{e^{-\alpha k(m+1)} e^{-\alpha y} y^j}{j!} \, dy \\ &= \lambda^{n+1} e^{-\alpha k-(\lambda+\alpha c)t} \Bigg((1-e^{-\alpha k})^n \frac{t^n}{n!} \\ &\quad + \sum_{j=0}^{n-1} \frac{t^{n-j-1} \alpha^{n-j}}{c^{j+1} (n-j)!} (1-e^{-\alpha k})^j \int_0^{ct} \frac{y^{j+1} (ct-y)^{n-j-1}}{(n-j-1)! j!} \, dy \Bigg). \end{aligned}$$

As

$$\int_0^{ct} y^{j+1} (ct-y)^{n-j-1} \, dy = (ct)^{n+1} \frac{(j+1)! \, (n-j-1)!}{(n+1)!}$$

we obtain

$$\begin{aligned} w_k(k, n+1, t) &= \lambda^{n+1} e^{-\alpha k-(\lambda+\alpha c)t} \Bigg((1-e^{-\alpha k})^n \frac{t^n}{n!} \\ &\quad + \sum_{j=0}^{n-1} \frac{t^{2n-j} (\alpha c)^{n-j}}{(n-j)!} (1-e^{-\alpha k})^j \frac{j+1}{(n+1)!} \Bigg) \end{aligned} \tag{3.46}$$

and the result follows. Also, the formula for $w_k(k,1,t)$ follows immediately from formula (3.25). $\square$

Further, we can find $w_k(k,t)$ by summing $w_k(k,n,t)$, and if we do this we obtain

$$\begin{aligned} w_k(k,t) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{c^n} \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} e^{-\alpha k(1+m)} \frac{e^{-(\lambda+\alpha c)t}}{\Gamma(n)} c^n t^{n-1} \\ &\quad \times \left(1 + \sum_{j=1}^{\infty} \frac{(\alpha c\lambda)^j}{j!} t^{2j} \frac{\Gamma(n+1)}{\Gamma(n+1+j)}\right) \\ &= \lambda e^{-\alpha k-(\lambda+\alpha c)t} \sum_{n=1}^{\infty} \frac{(\lambda t(1-e^{-\alpha k}))^{n-1}}{(n-1)!} \, {}_0F_1(n+1;\lambda\alpha c t^2) \end{aligned} \tag{3.47}$$

where

$${}_0F_1(b;x) = \sum_{n=0}^{\infty} \frac{\Gamma(b)x^n}{\Gamma(b+n)n!}$$

is a hypergeometric function. We remark that formula (3.47) was obtained by Nie et al. (2015) using a different approach.

We next show that formula (3.40) leads to a more concise expression for $w_k(u,t)$ for $u > k$ than that obtained by Nie et al. (2015), then we show that the formulae are equivalent. The derivation of this equivalence is of interest not just for the risk model with capital injections, but also for the classical risk model as in the case $k = 0$, Nie et al.'s (2015) formula for $w_k(u,t)$ reduces to the expression for $w(u,t)$ that can be found in Drekic and Willmot (2003), whilst our new expression reduces to the formula for $w(u,t)$ first given in Dickson et al. (2005). Our approach is to adopt procedures in Dickson (2007).

Result 3.2. When $u > k$,

$$\begin{aligned} w_k(u,t) &= \lambda e^{-\alpha u-(\lambda+\alpha c)t} \sum_{n=0}^{\infty} \frac{(1-e^{-\alpha k})^n \lambda^n t^{n/2}}{(\alpha\lambda(u+ct-k))^{n/2}} \\ &\quad \times \left(I_n\left(\sqrt{4\alpha\lambda t(u+ct-k)}\right) - \frac{ct}{u+ct-k} I_{n+2}\left(\sqrt{4\alpha\lambda t(u+ct-k)}\right)\right) \end{aligned} \tag{3.48}$$

where

$$I_\nu(t) = \sum_{n=0}^{\infty} \frac{(t/2)^{2n+\nu}}{n!\,(n+\nu)!}$$

is the modified Bessel function of order ν.

Derivation. The first two terms of equation (3.40) can be written as

$$\begin{aligned} & \lambda e^{-\lambda t}\bar{F}(u+ct) + \lambda \int_0^{u+ct-k} \bar{F}(u+ct-x) g(x,t)\, dx \\ = \; & \lambda e^{-\alpha u-(\lambda+\alpha c)t} + \lambda \int_0^{u+ct-k} e^{-\alpha(u+ct-x)} \sum_{n=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^n}{n!}\frac{\alpha^n x^{n-1} e^{-\alpha x}}{\Gamma(n)}\, dx \\ = \; & \lambda e^{-\alpha u-(\lambda+\alpha c)t} I_0\left(\sqrt{4\alpha\lambda t(u+ct-k)}\right). \end{aligned}$$

Next we consider the third term of equation (3.40). Inserting for $\bar{F}$ the first term in the bracket gives us

$$\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k) - \bar{F}(u+c\tau)\Big) = \lambda e^{-\lambda\tau}(1-e^{-\alpha k})e^{-\alpha(u+c\tau-k)}$$

and the second term in the bracket becomes

$$\begin{aligned} & \int_0^{u+c\tau-k} g(x,\tau)\lambda\Big(\bar{F}(u+c\tau-x-k) - \bar{F}(u+c\tau-x)\Big)\, dx \\ = \; & \lambda\sum_{n=1}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^n}{n!}\int_0^{u+c\tau-k}\frac{\alpha^n e^{-\alpha x}x^{n-1}}{(n-1)!}e^{-\alpha(u+c\tau-k-x)}(1-e^{-\alpha k})\, dx \\ = \; & \lambda\sum_{n=1}^{\infty} e^{-\lambda\tau}\frac{(\lambda\tau)^n}{n!}\frac{[\alpha(u+c\tau-k)]^n}{n!}e^{-\alpha(u+c\tau-k)}(1-e^{-\alpha k}). \end{aligned}$$

Further $w_k(k, t-\tau)$ comes from formula (3.47). Hence, switching τ and $t-\tau$, we can write the third term in (3.40) as

$$\begin{aligned} & \int_0^t w_k(k,t-\tau)\Big[\lambda e^{-\lambda\tau}\Big(\bar{F}(u+c\tau-k) - \bar{F}(u+c\tau)\Big) \\ & + \int_0^{u+c\tau-k} g(x,\tau)\lambda\Big(\bar{F}(u+c\tau-x-k) - \bar{F}(u+c\tau-x)\Big)\, dx\Big]\, d\tau \\ = \; & \lambda\int_0^t\sum_{r=0}^{\infty} e^{-\lambda(t-\tau)}\frac{(\lambda(t-\tau))^r}{r!}(1-e^{-\alpha k})\frac{\alpha^r e^{-\alpha(u+c(t-\tau)-k)}}{r!}(u+c(t-\tau)-k)^r \\ & \times\sum_{n=1}^{\infty}(1-e^{-\alpha k})^{n-1}e^{-\alpha k}\lambda^n\tau^{n-1} n e^{-\tau(\lambda+\alpha c)}\sum_{m=0}^{\infty}\frac{(\alpha c\lambda\tau^2)^m}{m!(n+m)!}\, d\tau \\ = \; & \lambda e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=1}^{\infty}(1-e^{-\alpha k})^n\lambda^n n\int_0^t \eta_1(\tau)\eta_2(t-\tau)\, d\tau \end{aligned}$$

where

$$\eta_1(t) = t^{n-1}\sum_{m=0}^{\infty}\frac{(\alpha c\lambda t^2)^m}{m!(n+m)!} = \frac{t^{-1}}{(\alpha c\lambda)^{n/2}} I_n\left(\sqrt{4\alpha c\lambda t^2}\right)$$

and

$$\eta_2(t) = \sum_{n=0}^{\infty} \frac{[\alpha\lambda t(u+ct-k)]^n}{n!\,n!} = I_0\left((4\alpha\lambda c)^{1/2}\sqrt{t^2+\frac{u-k}{c}t}\right).$$

We need two auxiliary results, the first of which was used in Dickson (2007). First, from Erdélyi (1954, page 201) we have that if

$$\phi(t) = \frac{t^{\nu/2}}{(t+\beta)^{\nu/2}} I_\nu\left(A\sqrt{t^2+\beta t}\right)$$

then

$$\tilde{\phi}(s) = \frac{A^\nu}{\sqrt{s^2-A^2}} \frac{1}{(s+\sqrt{s^2-A^2})^\nu} \exp\left\{\frac{\beta}{2}(s-\sqrt{s^2-A^2})\right\}. \tag{3.49}$$

Second, from Gradshteyn and Ryzhnik (2007, page 1117), we have that if

$$\varphi(t) = t^{-1} I_\nu(At),$$

then

$$\tilde{\varphi}(s) = \frac{A^\nu}{\nu}\left(s+\sqrt{s^2-A^2}\right)^{-\nu}.$$

Applying these results with $A = \sqrt{4\alpha\lambda c}$ and $B = (u-k)/c$, we find

$$\tilde{\eta}_1(s)\tilde{\eta}_2(s) = \frac{1}{\sqrt{s^2-4\alpha c\lambda}} \exp\left\{\frac{u-k}{2c}(s-\sqrt{s^2-4\alpha\lambda c})\right\} \frac{2^n}{n}\left(s+\sqrt{s^2-4\alpha\lambda c}\right)^{-n}$$

and using formula (3.49) this can be inverted to

$$\int_0^t \eta_1(\tau)\eta_2(t-\tau)\,d\tau = \frac{(\alpha\lambda c)^{-n/2}}{n} \frac{t^{n/2}}{(t+\frac{u-k}{c})^{n/2}} I_n\left(\sqrt{4\alpha\lambda t(u+ct-k)}\right).$$

The last term in equation (3.40) can be evaluated as

$$\begin{aligned}
& -c\int_0^t w_k(k,t-\tau)g(u+c\tau-k,\tau)\,d\tau \\
= & -c\int_0^t \sum_{r=1}^{\infty} e^{-\lambda(t-\tau)}\frac{(\lambda(t-\tau))^r}{r!}\frac{\alpha^r e^{-\alpha(u+c(t-\tau)-k)}}{(r-1)!}(u+c(t-\tau)-k)^{r-1} \\
& \times\sum_{n=1}^{\infty}(1-e^{-\alpha k})^{n-1}e^{-\alpha k}\lambda^n n\tau^{n-1}e^{-\tau(\lambda+\alpha c)}\sum_{m=0}^{\infty}\frac{(\alpha c\lambda\tau^2)^m}{m!\,(n+m)!}\,d\tau \\
= & -ce^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=1}^{\infty}(1-e^{-\alpha k})^{n-1}\lambda^n n\int_0^t \eta_1(\tau)\eta_3(t-\tau)\,d\tau,
\end{aligned}$$

where η_3 is given by

$$\begin{aligned}\eta_3(t) &= \frac{\alpha\lambda t}{\sqrt{\alpha\lambda t(u+ct-k)}} I_1\Big(\sqrt{4\alpha\lambda t(u+ct-k)}\Big) \\ &= \frac{\sqrt{\alpha\lambda t/c}}{\sqrt{\frac{u-k}{c}+t}} I_1\left(\sqrt{4\alpha\lambda c}\sqrt{\frac{u-k}{c}t+t^2}\right)\end{aligned}$$

and by (3.49) the Laplace transform of η_3 is found as

$$\tilde{\eta}_3(s) = \frac{\sqrt{\alpha\lambda/c}\sqrt{4\alpha\lambda c}}{\sqrt{s^2-4\alpha\lambda c}} \frac{1}{s+\sqrt{s^2-4\alpha\lambda c}} \exp\left\{\frac{u-k}{2c}(s-\sqrt{s^2-4\alpha\lambda c})\right\}.$$

Therefore, $\tilde{\eta}_1(s)\tilde{\eta}_3(s)$ inverts to

$$\int_0^t \eta_1(\tau)\eta_3(t-\tau)\,d\tau = \frac{\alpha\lambda}{n(\sqrt{\alpha\lambda c})^{n+1}}\left(\frac{ct}{ct+u-k}\right)^{(n+1)/2} I_{n+1}\left(\sqrt{4\alpha\lambda t(u+ct-k)}\right).$$

As a result we can write equation (3.40) as

$$\begin{aligned}& w_k(u,t) \\ =\; & \lambda e^{-\alpha u-(\lambda+\alpha c)t} I_0\left(\sqrt{4\alpha\lambda t(u+ct-k)}\right) \\ & +\lambda e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=1}^{\infty}\frac{(1-e^{-\alpha k})^n\lambda^n t^{n/2}}{[\alpha\lambda c(t+\frac{u-k}{c})]^{n/2}} I_n\left(\sqrt{4\alpha\lambda t(u+ct-k)}\right) \\ & -ce^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=1}^{\infty}\frac{(1-e^{-\alpha k})^{n-1}\alpha\lambda^{n+1}t^{(n+1)/2}}{[\alpha\lambda c(t+\frac{u-k}{c})]^{(n+1)/2}} I_{n+1}\left(\sqrt{4\alpha\lambda t(u+ct-k)}\right)\end{aligned} \tag{3.50}$$

and after simple algebra the result follows. □

Setting $k = 0$ in formula (3.48) yields formula (3.9) in Dickson et al. (2005) for the density of the time of ruin in the classical risk model.

Formula (3.48) is a simpler expression than that given for $w_k(u,t)$ by Nie et al. (2015) as it is based on infinite sums of Bessel functions, whereas the formula given by Nie et al. (2015) involves double infinite summation of Bessel functions. We now show formula (3.48) can be manipulated to obtain the expression given in Nie et al. (2015).

Result 3.3. We can express formula (3.48) as

$$\begin{aligned}w_k(u,t) &= e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n\sum_{j=1}^{\infty}\frac{[\alpha(u-k)]^{j-1}}{\Gamma(j)}(\lambda t)^{n+j}t^{-1} \\ &\quad \times\frac{1}{\Gamma(n+j)}\,{}_0F_1(n+j+1;\alpha\lambda ct^2).\end{aligned} \tag{3.51}$$

Derivation. Expanding the modified Bessel functions in formula (3.48) yields

$$\begin{aligned} w_k(u,t) &= \lambda e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n(\lambda t)^n\sum_{m=0}^{\infty}\frac{[\alpha\lambda t(u+ct-k)]^m}{m!\,(n+m)!} \\ &\quad -ce^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n(\lambda t)^{n+2}\alpha\sum_{m=0}^{\infty}\frac{[\alpha\lambda t(u+ct-k)]^m}{m!\,(m+n+2)!} \end{aligned}$$

and using the binomial expansion gives

$$\begin{aligned} & w_k(u,t) \\ = {} & \lambda e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n(\lambda t)^n\sum_{m=0}^{\infty}\frac{(\alpha\lambda t)^m}{m!\,(n+m)!}\sum_{j=0}^{m}\binom{m}{j}(u-k)^j(ct)^{m-j} \\ & -ce^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n(\lambda t)^{n+2}\sum_{m=0}^{\infty}\frac{\alpha(\alpha\lambda t)^m}{m!\,(m+n+2)!}\sum_{j=0}^{m}\binom{m}{j}(u-k)^j(ct)^{m-j}. \end{aligned}$$

Interchanging the order of the inner summations we find

$$\begin{aligned} & w_k(u,t) \\ = {} & \lambda e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n(\lambda t)^n\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\frac{(\alpha\lambda t)^m}{m!\,(n+m)!}\binom{m}{j}(u-k)^j(ct)^{m-j} \\ & -ce^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n(\lambda t)^{n+2}\sum_{j=0}^{\infty}\sum_{m=j}^{\infty}\frac{\alpha(\alpha\lambda t)^m}{m!\,(m+n+2)!}\binom{m}{j}(u-k)^j(ct)^{m-j} \end{aligned}$$

and changing the variables of summation we obtain

$$\begin{aligned} & w_k(u,t) \\ = {} & \lambda e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n(\lambda t)^n\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\frac{(\alpha\lambda t)^{m+j-1}(u-k)^{j-1}(ct)^m}{m!\,(j-1)!\,(n+m+j-1)!} \\ & -ce^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n(\lambda t)^{n+2}\sum_{j=1}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha(\alpha\lambda t)^{m+j-1}(u-k)^{j-1}(ct)^m}{m!\,(j-1)!\,(n+m+j+1)!} \\ = {} & e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n\sum_{j=1}^{\infty}\frac{[\alpha(u-k)]^{j-1}}{\Gamma(j)}(\lambda t)^{n+j}t^{-1} \\ & \times\left\{\sum_{m=0}^{\infty}\frac{(\alpha\lambda ct^2)^m}{m!\,(m+n+j-1)!}-\sum_{m=0}^{\infty}\frac{(\alpha\lambda ct^2)^{m+1}}{m!\,(m+n+j+1)!}\right\} \\ = {} & e^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}(1-e^{-\alpha k})^n\sum_{j=1}^{\infty}\frac{[\alpha(u-k)]^{j-1}}{\Gamma(j)}(\lambda t)^{n+j}t^{-1} \\ & \times\frac{1}{(t\sqrt{\alpha\lambda c})^{n+j-1}}\left\{I_{n+j-1}\left(2t\sqrt{\alpha\lambda c}\right)-I_{n+j+1}\left(2t\sqrt{\alpha\lambda c}\right)\right\}. \end{aligned}$$

From Abramowitz and Stegun (1972, formula 9.6.26) we have the following recursive relations

$$I_{\nu-1}(t) - I_{\nu+1}(t) = \frac{2\nu}{t} I_\nu(t).$$

Applying this identity we obtain

$$\begin{aligned} w_k(u,t) &= e^{-\alpha u-(\lambda+\alpha c)t} \sum_{n=0}^{\infty} (1-e^{-\alpha k})^n \sum_{j=1}^{\infty} \frac{[\alpha(u-k)]^{j-1}}{\Gamma(j)} (\lambda t)^{n+j} t^{-1} \\ &\quad \times \frac{n+j}{(t\sqrt{\alpha\lambda c})^{n+j}} I_{n+j}\left(2t\sqrt{\alpha\lambda c}\right) \end{aligned}$$

and using the identity

$$I_\nu(z) = \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu {}_0F_1(\nu+1; z^2/4) \tag{3.52}$$

the result follows. □

Setting $k = 0$ in formula (3.51) gives formula (2.7) of Drekic and Willmot (2003). Thus, our technique also establishes the identity of their formula with the comparatively simpler formula for $w(u,t)$ given in Dickson et al. (2005).

The next result gives the probability mass function of the number of claims until ruin.

Result 3.4. When the initial surplus is k, for $n = 1, 2, 3, \ldots,$

$$p_k(k, n+1) = e^{-\alpha k} \sum_{j=0}^{n} (1-e^{-\alpha k})^j \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{n-j} \left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+1} \frac{(2n-j)!\,(j+1)}{(n-j)!\,(n+1)!} \tag{3.53}$$

and $p_k(k,1) = \lambda e^{-\alpha k}/(\lambda+\alpha c)$.

Derivation. Integrating over t in formula (3.46) gives the required result. □

Setting $k = 0$ in formula (3.53) recovers formula (24) of Landriault et al. (2011). In the next result, we demonstrate that $p_k(k,n)$ is a defective probability function for $\alpha c > \lambda$ as discussed by Landriault et al. (2011).

Result 3.5. For $u = k$, the ultimate ruin probability is

$$\psi_k(k) = \frac{e^{-\alpha k}\lambda/(\alpha c)}{1-(1-e^{-\alpha k})\lambda/(\alpha c)} = \sum_{n=0}^{\infty} e^{-\alpha k}(1-e^{-\alpha k})^n \left(\frac{\lambda}{\alpha c}\right)^{n+1}. \tag{3.54}$$

Derivation.

$$\begin{aligned}
\psi_k(k) &= \sum_{n=1}^{\infty} p_k(k,n) \\
&= e^{-\alpha k} \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \lambda^n (1-e^{-\alpha k})^j \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{n-j-1} \left(\frac{\lambda}{\lambda+\alpha c}\right)^n \frac{(2n-j-2)!\,(j+1)}{(n-j-1)!\,n!} \\
&= e^{-\alpha k} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} (1-e^{-\alpha k})^j \left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+j+1} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^n \frac{(2n+j+1)!}{n!\,(n+j+1)!} \frac{j+1}{2n+j+1} \\
&= \sum_{j=0}^{\infty} e^{-\alpha k} (1-e^{-\alpha k})^j \left(\frac{\lambda}{\lambda+\alpha c}\right)^{j+1} \mathcal{B}_2^{j+1}\left(\frac{\lambda \alpha c}{(\lambda+\alpha c)^2}\right)
\end{aligned}$$

where $\mathcal{B}_t$ is the generalised binomial series given by

$$\mathcal{B}_t(z) = \sum_{k=0}^{\infty} \binom{tk+1}{k} \frac{1}{tk+1} z^k$$

with the property that

$$\mathcal{B}_t(z)^r = \sum_{k=0}^{\infty} \binom{tk+r}{k} \frac{r}{tk+r} z^k \tag{3.55}$$

and for $t = 2$, $\mathcal{B}_2(z) = \frac{1-\sqrt{1-4z}}{2z}$. See Graham et al. (1994). Therefore, it is easily seen that

$$\mathcal{B}_2\left(\frac{\lambda \alpha c}{(\lambda+\alpha c)^2}\right) = \frac{\lambda+\alpha c}{\alpha c},$$

giving

$$\sum_{n=1}^{\infty} p_k(k,n) = \sum_{j=0}^{\infty} e^{-\alpha k}(1-e^{-\alpha k})^j \left(\frac{\lambda}{\alpha c}\right)^{j+1}$$

which is the same as (3.54). □

The next result gives the probability mass function of the number of claims until ruin for $u > k$.

Result 3.6. When $u > k$, for $n = 2, 3, 4, \ldots$, $p_k(u, n)$ is given recursively by

$$\begin{aligned}
& p_k(u,n) \\
= \quad & \frac{\lambda^n \alpha^{n-1} e^{-\alpha u}}{(n-1)!^2} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{c^j (u-k)^{n-j-1} (n+j-1)!}{(\lambda+\alpha c)^{n+j}} \\
& + p_k(k, n-1) e^{-\alpha(u-k)} (1 - e^{-\alpha k}) \frac{\lambda}{\lambda + \alpha c} \\
& + \lambda \sum_{m=1}^{n-2} p_k(k, n-m-1) e^{-\alpha(u-k)} (1 - e^{-\alpha k}) \frac{(\alpha\lambda)^m}{m!^2} \sum_{j=0}^{m} \binom{m}{j} \frac{c^j (u-k)^{m-j} (m+j)!}{(\lambda+\alpha c)^{m+j+1}} \\
& - c \sum_{m=1}^{n-1} p_k(k, n-m) \frac{(\alpha\lambda)^m e^{-\alpha(u-k)}}{m!(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{c^j (u-k)^{m-j-1} (m+j)!}{(\lambda+\alpha c)^{m+j+1}}
\end{aligned} \tag{3.56}$$

and $p_k(u, 1) = \lambda e^{-\alpha u} / (\lambda + \alpha c)$.

Derivation. The formula for $p_k(u, 1)$ is obtained by integrating over t in (3.30). For $n = 2, 3, 4, \ldots$, we integrate over t in formula (3.31). Taking a term by term approach, the first term becomes

$$\begin{aligned}
& \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} \int_0^{u+ct-k} f^{n*}(x) \lambda \bar{F}(u + ct - x)\, dx\, dt \\
= \quad & \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^n}{n!} \int_0^{u+ct-k} \frac{\alpha^n e^{-\alpha x} x^{n-1}}{\Gamma(n)} \lambda e^{-\alpha(u+ct-x)}\, dx\, dt \\
= \quad & \lambda e^{-\alpha u} \frac{(\alpha\lambda)^n}{n! n!} \int_0^\infty e^{-(\lambda+\alpha c)t} t^n (u + ct - k)^n\, dt \\
= \quad & \lambda e^{-\alpha u} \frac{(\alpha\lambda)^n}{n!^2} \sum_{j=0}^{n} \binom{n}{j} \frac{c^j (u-k)^{n-j} (n+j)!}{(\lambda+\alpha c)^{n+j+1}}.
\end{aligned} \tag{3.57}$$

The second term in (3.31) can be written as

$$\begin{aligned}
& \int_0^\infty \int_0^t w_k(k, n, t-\tau) \lambda e^{-\lambda\tau} \Big(\bar{F}(u + c\tau - k) - \bar{F}(u + c\tau) \Big)\, d\tau\, dt \\
= \quad & \int_0^\infty \int_0^t w_k(k, n, t-\tau) \lambda e^{-\lambda\tau} (1 - e^{-\alpha k}) e^{-\alpha(u+c\tau-k)}\, d\tau\, dt \\
= \quad & p_k(k, n) e^{-\alpha(u-k)} (1 - e^{-\alpha k}) \frac{\lambda}{\lambda + \alpha c}
\end{aligned} \tag{3.58}$$

which can be obtained after switching the integrals and noting that

$p_k(k,n) = \int_0^\infty w_k(k,n,t)\,dt$. Then, we consider the third term. We have

$$\begin{aligned}
& \int_0^\infty \sum_{m=1}^{n-1} \int_0^t e^{-\lambda\tau} \frac{(\lambda\tau)^m}{m!} \int_0^{u+c\tau-k} f^{m*}(x) \\
& \times \lambda\Big(\bar{F}(u+c\tau-k-x) - \bar{F}(u+c\tau-x)\Big)\,dx\,w_k(k,n-m,t-\tau)\,d\tau\,dt \\
= \quad & \lambda \sum_{m=1}^{n-1} \int_0^\infty \int_0^t e^{-\lambda\tau} \frac{(\lambda\tau)^m}{m!\,m!} (1-e^{-\alpha k})\alpha^m e^{-\alpha(u+c\tau-k)}(u+c\tau-k)^m \\
& \times w_k(k,n-m,t-\tau)\,d\tau\,dt \\
= \quad & \lambda \sum_{m=1}^{n-1} p_k(k,n-m) e^{-\alpha(u-k)}(1-e^{-\alpha k}) \frac{(\alpha\lambda)^m}{m!^2} \sum_{j=0}^{m} \binom{m}{j} \frac{c^j (u-k)^{m-j}(m+j)!}{(\lambda+\alpha c)^{m+j+1}}.
\end{aligned} \tag{3.59}$$

The last term of formula (3.31) can be similarly evaluated, giving

$$\begin{aligned}
& -c \int_0^\infty \sum_{m=1}^{n} \int_0^t e^{-\lambda\tau} \frac{(\lambda\tau)^m}{m!} f^{m*}(u+c\tau-k) w_k(k,n+1-m,t-\tau)\,d\tau\,dt \\
= \quad & -c \sum_{m=1}^{n} p_k(k,n+1-m) \frac{(\alpha c)^m e^{-\alpha(u-k)}}{m!\,(m-1)!} \sum_{j=0}^{m-1} \binom{m-1}{j} \frac{c^j (u-k)^{m-j-1}(m+j)!}{(\lambda+\alpha c)^{m+j+1}}.
\end{aligned} \tag{3.60}$$

Combining (3.57), (3.58), (3.59) and (3.60) and changing $n+1$ to n, the result follows. □

Formula (3.56) generalises the formula for $p(u,n)$ in the classical risk model – if we set $k=0$ we obtain the result in Dickson (2012) for the classical risk model.

We conclude this section by plotting the probability function of the number of claims until ruin given that ruin occurs as it gives us a better insight as to the nature of $p_k(u,n)$. Figure 3.9 shows values of $p_k(15,n)/\psi_k(15)$ for $n = 1,2,\ldots,400$, when $\alpha = 1, c = 1.2, u = 15$ and $k = 2$, so that $\psi_k(15) = 0.04623$, which is in the range of practical interest. We observe that it is positively-skewed with the maximum probability being at $n = 33$. This feature is compatible to the plots of $p(u,n)/\psi(u)$ in Egídio dos Reis (2002) for the probability function of the number of claims until ruin given that ruin occurs in the classical risk model. In addition, the plot of $p_k(u,n)/\psi_k(u)$ exhibits a similar pattern to the graph of the density of the time of ruin as we would expect. See, for example, Nie et al. (2015) or Figure 5.5.

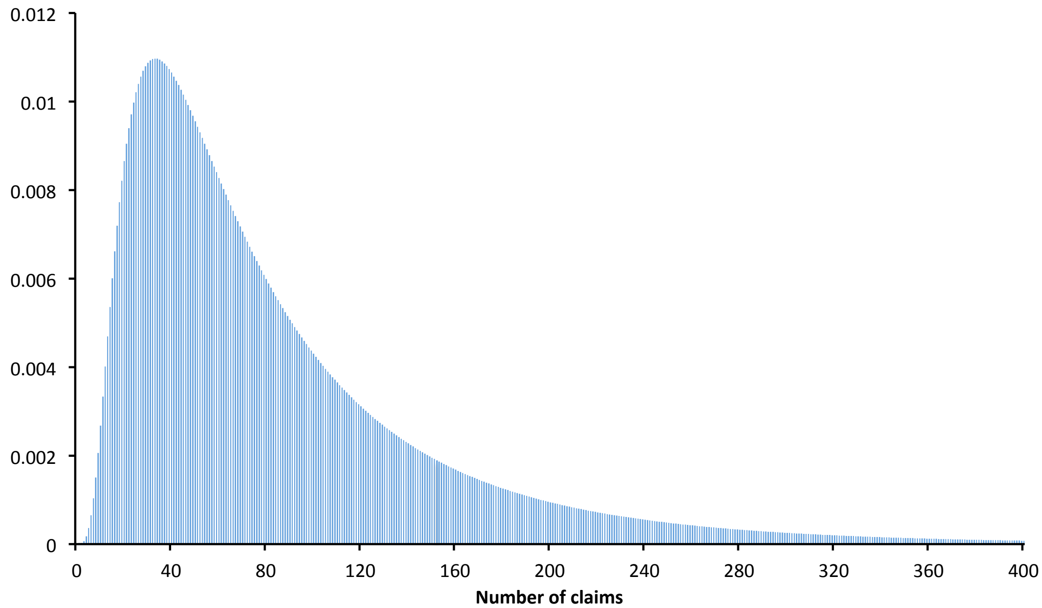

Figure 3.9: $p_k(15, n)/\psi_k(15)$, Exponential(1)

3.6.2 Erlang(2) claims

We now consider the case when individual claim amounts have an Erlang(2, α) distribution with $\bar{F}(x) = e^{-\alpha x}(1 + \alpha x)$, where $x \geq 0$ with $\alpha > 0$ and provide results for $w_k(u, n, t)$ for $u = k$ and $u > k$.

Result 3.7. When the initial surplus is k, for $n = 1, 2, 3, \ldots,$

$$w_k(k, n+1, t) = e^{-(\lambda+\alpha c)t} \sum_{j=0}^{n} \sum_{m=0}^{j} \binom{j}{m} \frac{a_k^m b_k^{j-m} \lambda^{n+1} (\alpha ct)^{2n-m-j+1} t^n}{(n-j)!\,(2n-m+1)!} \left(\frac{(1-a_k)}{\alpha ct}(2j-m+1) + (1-b_k)\frac{2j-m+2}{2n-m+2} \right) \tag{3.61}$$

and

$$w_k(k, 1, t) = \lambda e^{-(\lambda+\alpha c)t}[(1-a_k) + (1-b_k)\alpha ct] \tag{3.62}$$

where $a_k = 1 - e^{-\alpha k} - \alpha k e^{-\alpha k}$ and $b_k = 1 - e^{-\alpha k}$.

Derivation. We define

$$\begin{aligned}\bar{F}^1_n(t) &= \sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m\bar{F}_{m,n}(t)\\ &= \sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m\int_0^t A^{(n-1-m)*}(x)B_k^{(m+1)*}(t-x)\,dx\end{aligned}$$

and solve the integral by applying properties of the Laplace transform of a convolution as discussed in Result 3.2. Hence, the Laplace transform of $\bar{F}^1_n$ is given by

$$\begin{aligned}\tilde{\bar{F}}^1_n(s) &= \sum_{m=0}^{n-1}\binom{n-1}{m}(-1)^m\tilde{A}(s)^{n-1-m}\tilde{B}_k(s)^{m+1}\\ &= \tilde{B}_k(s)\big(\tilde{A}(s)-\tilde{B}_k(s)\big)^{n-1}\\ &= \left(\frac{e^{-\alpha k}}{\alpha+s}+\frac{\alpha k e^{-\alpha k}}{\alpha+s}+\frac{\alpha e^{-\alpha k}}{(\alpha+s)^2}\right)\\ &\quad\times\left(\frac{1}{\alpha+s}+\frac{\alpha}{(\alpha+s)^2}-\frac{e^{-\alpha k}}{\alpha+s}-\frac{\alpha e^{-\alpha k}}{(\alpha+s)^2}-\frac{\alpha k e^{-\alpha k}}{\alpha+s}\right)^{n-1}\\ &= \left(\frac{1-a_k}{\alpha+s}+\frac{\alpha(1-b_k)}{(\alpha+s)^2}\right)\left(\frac{a_k}{\alpha+s}+\frac{\alpha b_k}{(\alpha+s)^2}\right)^{n-1}\\ &= \left(\frac{1-a_k}{\alpha+s}+\frac{\alpha(1-b_k)}{(\alpha+s)^2}\right)\sum_{m=0}^{n-1}\binom{n-1}{m}\frac{a_k^m(\alpha b_k)^{n-1-m}}{(\alpha+s)^{2n-2-m}}\\ &= \sum_{m=0}^{n-1}\binom{n-1}{m}\frac{a_k^m b_k^{n-1-m}\alpha^{n-m}}{(\alpha+s)^{2n-m}}\left((1-a_k)\frac{\alpha+s}{\alpha}+(1-b_k)\right)\end{aligned}$$

which inverts to

$$\bar{F}^1_n(t) = e^{-\alpha t}\sum_{m=0}^{n-1}\binom{n-1}{m}a_k^m b_k^{n-1-m}\frac{\alpha^{n-m}t^{2n-m-1}}{\Gamma(2n-m-1)}\left(\frac{1-a_k}{\alpha t}+\frac{1-b_k}{2n-m-1}\right).$$

Writing formula (3.26) in terms of $\bar{F}^1_n$ gives

$$\begin{aligned}&w_k(k,n+1,t)\\ &= e^{-\lambda t}\frac{\lambda^{n+1}}{c^n}\bar{F}^1_{n+1}(ct)+\sum_{j=0}^{n-1}e^{-\lambda t}\frac{\lambda^{n+1}}{c^{j+1}}\frac{t^{n-j-1}}{(n-j)!}\int_0^{ct} yf^{(n-j)*}(ct-y)\bar{F}^1_{j+1}(y)\,dy\\ &= \lambda^{n+1}e^{-(\lambda+\alpha c)t}\sum_{m=0}^{n}\binom{n}{m}a_k^m b_k^{n-m}\frac{(\alpha ct)^{n-m}t^n}{(2n-m)!}\left((1-a_k)+(1-b_k)\frac{(\alpha ct)}{(2n-m+1)}\right)\\ &\quad+\sum_{j=0}^{n-1}e^{-\lambda t}\frac{\lambda^{n+1}}{c^{j+1}}\frac{t^{n-j-1}}{(n-j)!}\int_0^{ct} yf^{(n-j)*}(ct-y)\bar{F}^1_{j+1}(y)\,dy. \qquad (3.63)\end{aligned}$$

We first consider the integral by noting that the $(n-j)$-fold convolution of an Erlang$(2,\alpha)$ density with itself is the Erlang$(2(n-j),\alpha)$ density. Inserting for $\bar{F}^1_{j+1}$ we get

$$\begin{aligned}
&\int_0^{ct} y f^{(n-j)*}(ct-y)\bar{F}^1_{j+1}(y)\,dy \\
= \;& e^{-\alpha ct}\sum_{m=0}^{j}\binom{j}{m}a_k^m b_k^{j-m}\int_0^{ct}\frac{e^{\alpha y}y\alpha^{2(n-j)}(ct-y)^{2(n-j)-1}}{(2n-2j-1)!}\frac{\alpha^{j-m}e^{-\alpha y}y^{2j-m}}{(2j-m)!} \\
& \left((1-a_k)+(1-b_k)\frac{\alpha y}{2j-m+1}\right)dy.
\end{aligned}$$

The integral can be evaluated by using a property of the beta function, giving

$$\begin{aligned}
&\int_0^{ct} y f^{(n-j)*}(ct-y)\bar{F}^1_{j+1}(y)\,dy \\
= \;& e^{-\alpha ct}\sum_{m=0}^{j}\binom{j}{m}a_k^m b_k^{j-m}\frac{\alpha^{2n-m-j}(ct)^{2n-m+1}}{(2n-m+1)!} \\
& \left((1-a_k)(2j-m+1)+(1-b_k)\frac{\alpha ct(2j-m+2)}{2n-m+2}\right).
\end{aligned}$$

Inserting this in (3.63), formula (3.61) follows. We can obtain formula (3.62) by applying (3.25). □

To find the probability mass function of the number of claims until ruin we integrate over t in formula (3.62), giving

$$\begin{aligned}
p_k(k,1) &= \frac{\lambda(1-a_k)}{\lambda+\alpha c}+\frac{\alpha\lambda c(1-b_k)}{(\lambda+\alpha c)^2} \\
&= \frac{\lambda e^{-\alpha k}}{\lambda+\alpha c}\left(1+\alpha k+\frac{\alpha c}{\lambda+\alpha c}\right) \qquad (3.64)
\end{aligned}$$

where the last line is obtained after substituting for a_k and b_k. Similarly, integrating over t in formula (3.61), for $n=1,2,3,\ldots,$ yields

$$\begin{aligned}
&p_k(k,n+1) \\
= \;& \sum_{j=0}^{n}\sum_{m=0}^{j}\binom{j}{m}a_k^m b_k^{j-m}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n-m-j}\left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+1}\frac{(3n-m-j)!}{(n-j)!(2n-m+1)!} \\
& \left((1-a_k)(2j-m+1)+(1-b_k)\left(\frac{\alpha c}{\lambda+\alpha c}\right)\frac{(3n-m-j+1)(2j-m+2)}{(2n-m+2)}\right) \qquad (3.65)
\end{aligned}$$

which also holds for $n = 0$.

In the next result we find the ultimate ruin probability for $u = k$. Although this can be obtained from results in Nie et al. (2011) this new derivation is of some mathematical interest.

Result 3.8. For $u = k$, the ultimate ruin probability is

$$\psi_k(k) = \frac{(2e^{-\alpha k} + \alpha k e^{-\alpha k})\lambda/(\alpha c)}{1 - (2 - \alpha k e^{-\alpha k} - 2e^{-\alpha k})\lambda/(\alpha c)}. \tag{3.66}$$

Derivation. Summing over n in formula (3.65) yields

$$\begin{aligned}
\psi_k(k) &= \sum_{n=1}^{\infty} p_k(k,n) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{m=0}^{j} \binom{j}{m} a_k^m b_k^{j-m} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n-m-j} \left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+1} \frac{(3n-m-j)!}{(n-j)!(2n-m+1)!} \\
&\quad \left((1-a_k)(2j-m+1) + (1-b_k)\left(\frac{\alpha c}{\lambda+\alpha c}\right)\frac{(3n-m-j+1)(2j-m+2)}{(2n-m+2)}\right) \\
&= \sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\sum_{m=0}^{j} \binom{j}{m} a_k^m b_k^{j-m} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n-m-j} \left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+1} \frac{(3n-m-j)!}{(n-j)!(2n-m+1)!} \\
&\quad \left((1-a_k)(2j-m+1) + (1-b_k)\left(\frac{\alpha c}{\lambda+\alpha c}\right)\frac{(3n-m-j+1)(2j-m+2)}{(2n-m+2)}\right) \\
&= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{j} \binom{j}{m} a_k^m b_k^{j-m} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n+j-m} \left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+j+1} \frac{(3n-m+2j)!}{(2n+2j-m+1)!n!} \\
&\quad \left((1-a_k)(2j-m+1) + (1-b_k)\left(\frac{\alpha c}{\lambda+\alpha c}\right)\frac{(3n-m+2j+1)(2j-m+2)}{(2n+2j-m+2)}\right) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{j} \binom{j}{m} a_k^m b_k^{j-m} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n+j-m} \left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+j+1} \frac{(3n-m+2j)!}{(2n+2j-m+1)!n!} \\
&\quad \left((1-a_k)(2j-m+1) + (1-b_k)\left(\frac{\alpha c}{\lambda+\alpha c}\right)\frac{(3n-m+2j+1)(2j-m+2)}{(2n+2j-m+2)}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=m}^{\infty} \binom{j}{m} a_k^m b_k^{j-m} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n+j-m} \left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+j+1} \frac{(3n-m+2j)!}{(2n+2j-m+1)!n!} \\
&\quad \left((1-a_k)(2j-m+1) + (1-b_k)\left(\frac{\alpha c}{\lambda+\alpha c}\right)\frac{(3n-m+2j+1)(2j-m+2)}{(2n+2j-m+2)}\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \binom{m+j}{m} a_k^m b_k^{j} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n+j} \left(\frac{\lambda}{\lambda+\alpha c}\right)^{n+m+j+1} \frac{(3n+2j+m)!}{(2n+2j+m+1)!n!}
\end{aligned}$$

$$\left((1-a_k)(2j+m+1)+(1-b_k)\left(\frac{\alpha c}{\lambda+\alpha c}\right)\frac{(3n+2j+m+1)(2j+m+2)}{(2n+2j+m+2)}\right)$$

and by (3.55) this can be written as

$$\begin{aligned}\sum_{n=1}^{\infty} p_k(k,n) &= \sum_{m=0}^{\infty}\sum_{j=0}^{\infty}\binom{m+j}{m} a_k^m b_k^j \left(\frac{\alpha c}{\lambda+\alpha c}\right)^j \left(\frac{\lambda}{\lambda+\alpha c}\right)^{m+j+1} \mathcal{B}_3(z)^{2j+m+1} \\ &\quad \left((1-a_k)+(1-b_k)\frac{\alpha c \mathcal{B}_3(z)}{\lambda+\alpha c}\right) \\ &= \frac{\lambda \mathcal{B}_3(z)}{\lambda+\alpha c}\sum_{m=0}^{\infty}\left(\frac{\lambda a_k \mathcal{B}_3(z)}{\lambda+\alpha c}\right)^m \sum_{j=0}^{\infty}\binom{m+j}{m}\left(\frac{\lambda\alpha c b_k \mathcal{B}_3(z)^2}{(\lambda+\alpha c)^2}\right)^j \\ &\quad \left((1-a_k)+(1-b_k)\frac{\alpha c \mathcal{B}_3(z)}{\lambda+\alpha c}\right), \qquad (3.67)\end{aligned}$$

where $z=\frac{\lambda}{\lambda+\alpha c}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^2$.

Jain and Consul (1971) define the generalised negative binomial probability function for a random variable X as

$$\Pr(X=n)=\binom{\beta n+k}{n}\frac{k}{\beta n+k}p^n(1-p)^{n(\beta-1)+k},$$

for $n=0,1,2,\ldots$, where $k>0$, $0<p<1$ and $|\beta p|<1$. As $\sum_{n=0}^{\infty}\Pr(X=n)=1$,

$$(1-p)^{-k}=\sum_{n=0}^{\infty}\binom{\beta n+k}{n}\frac{k}{\beta n+k}p^n(1-p)^{n(\beta-1)}=\left(\mathcal{B}_\beta\left(p(1-p)^{\beta-1}\right)\right)^k,$$

we see that

$$(1-p)^{-1}=\mathcal{B}_\beta\left(p(1-p)^{\beta-1}\right).$$

Hence, $\mathcal{B}_3\left(\frac{\lambda}{\lambda+\alpha c}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^2\right)=\left(1-\frac{\lambda}{\lambda+\alpha c}\right)^{-1}$. Substituting for $\mathcal{B}_3(z)^2$ and $\mathcal{B}_3(z)$ in equation (3.67) yields

$$\sum_{n=1}^{\infty} p_k(k,n)=\frac{\lambda}{\alpha c}\sum_{m=0}^{\infty}\left(\frac{\lambda a_k}{\alpha c}\right)^m\sum_{j=0}^{\infty}\binom{m+j}{m}\left(\frac{\lambda b_k}{\alpha c}\right)^j\left[(1-a_k)+(1-b_k)\right].$$

The result follows by applying the identity (see Graham et al., 1994)

$$\frac{1}{(1-z)^{n+1}}=\sum_{k=0}^{\infty}\binom{n+k}{n}z^k$$

where $z=\lambda b_k/(\alpha c)$. □

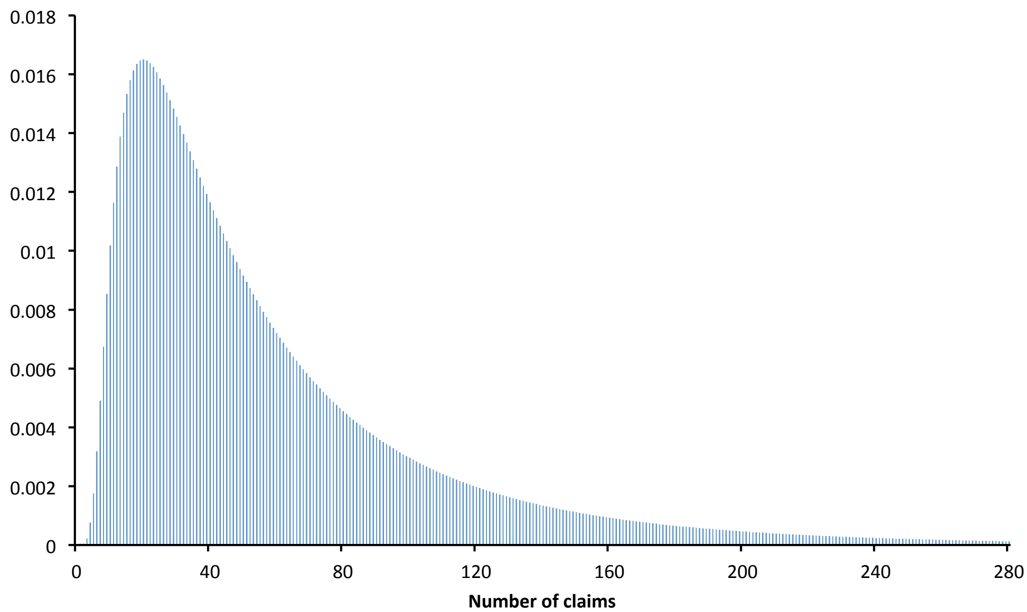

Figure 3.10: $p_k(9, n)/\psi_k(9)$, Erlang(2, 2)

For the case $u > k$ we can follow techniques in Dickson (2007), but the resulting equation is not suitable for computation as firstly, the expression involves triple and quadruple summations and secondly, we may not find a robust truncation point. However we can use software to numerically integrate formula (3.26) over t to find $p_k(u, n)$. Figure 3.10 illustrates the graph of the conditional probability function of the number of claims until ruin for Erlang(2, 2) claims with $\lambda = 1$ and $c = 1.2$. We choose $u = 9$ and $k = 2$, so that $\psi_k(9) = 0.04324$, which is in the range of practical interest. We observe that similar to Figure 3.9 the graph of $p_k(u, n)/\psi_k(u)$ is positively-skewed. In this case, the maximum probability is on the occurrence of the 20th claim. As we can see it is more likely that ruin occurs when $n < 30$ compared to the exponential case in Figure 3.9. In other words, $p_k(u, n)$ takes higher values in this range and ruin, if it occurs, occurs sooner for Erlang(2) claims than exponential claims. However, as pointed out by Egídio dos Reis (2002) for the classical risk model, our experience with $u = 9$ for exponential claims has shown that the plot of $p_k(u, n)/\psi_k(u)$ has the same shape as in Figure 3.10. Comparing Figure 3.10 with the figures in Egídio dos Reis (2002) we observe that the pattern of the probability mass function of the number of claims until ruin given that ruin occurs in the classical risk model with capital injections is similar to $p(u,n)/\psi(u)$ in the classical risk model.

3.7 Covariance between $T_{u,k}$ and $N_{T_{u,k}}$

In this section, we consider the covariance between the time of ruin and the number of claims until ruin given that ruin occurs which is given by

$$\mathrm{Cov}[N_{T_{u,k}}, T_{u,k}|T_{u,k} < \infty] = E[N_{T_{u,k}} T_{u,k}|T_{u,k} < \infty] - E[N_{T_{u,k}}|T_{u,k} < \infty]E[T_{u,k}|T_{u,k} < \infty].$$

First, we derive an expression for the nth moment of $T_{u,k}$, and then provide results without proof for the first moment of $N_{T_{u,k}}$ and the joint moment of $T_{u,k}$ and $N_{T_{u,k}}$ as the proofs are similar to those for the moments of $T_{u,k}$. We conclude this section by illustrating the application of our results in the case of individual claim amounts following an exponential distribution.

3.7.1 Moments of $T_{u,k}$ and $N_{T_{u,k}}$

Moments of the time of ruin are other ruin-related quantities that can be obtained from our Gerber-Shiu function. In this section, we adapt the recursive approach provided by Albrecher and Boxma (2005a) to derive moments of the time of ruin for our model. This problem is also considered by Nie et al. (2015). Their formulae for the moments of the ruin time are based on the joint density of the time of ruin and the deficit at ruin in the classical risk model and subject to the factorisation

$$w(u,y,t) = \sum_{i=1}^{m} h_i(u,t)\tau_i(y)$$

for some functions $\{h_i(u,t)\}_{i=1}^m$ and densities $\{\tau_i(y)\}_{i=1}^m$. This means that to implement their method we need the density of the time of ruin, whereas our recursive approach is based on the ultimate ruin probability which is simpler to calculate.

We set $r = 1$ and $\omega(x,y) = 1$ for $x \geq k$ and $y > 0$ in equation (3.1). In this case the Gerber-Shiu function is

$$\phi_{k,\delta}(u) = E\left[e^{-\delta T_{u,k}} I(T_{u,k} < \infty) \mid U(0) = u\right].$$

We now begin by denoting $\psi_{k,n}(u)$ to be the nth moment of the time of ruin in our risk model with capital injections, which is given by

$$\begin{aligned}\psi_{k,n}(u) &= E\left[T_{u,k}^n\, I(T_{u,k} < \infty)\right] \\ &= (-1)^n \frac{\partial^n}{\partial \delta^n}\phi_{k,\delta}(u)\Big|_{\delta=0}\end{aligned}$$

with $\psi_{k,0}(u) = \psi_k(u)$. Then, the nth moment of the time of ruin, given that ruin has occurred, is given by

$$E\left[T_{u,k}^n | T_{u,k} < \infty\right] = \frac{\psi_{k,n}(u)}{\psi_k(u)}.$$

The following result provides expressions for the moments of the time of ruin.

Theorem 3.10. The nth moment of the ruin time, $n = 1, 2, 3, \ldots$, is given recursively by

$$\begin{aligned}& E[T_{u,k}^n I(T_{u,k} < \infty)] \\ =\ & \frac{-n}{c\delta(0)} \int_0^{u-k} \delta(u-k-x) E[T_{x+k,k}^{n-1}\, I(T_{x+k,k} < \infty)]\, dx \\ & -\frac{\lambda}{c\delta(0)} E[T_{k,k}^n\, I(T_{k,k} < \infty)] \int_0^{u-k} \delta(u-k-x)\Big(\bar{F}(x) - \bar{F}(x+k)\Big)\, dx \\ & +\frac{\delta(u-k)}{\delta(0)} E[T_{k,k}^n\, I(T_{k,k} < \infty)] \end{aligned} \tag{3.68}$$

where

$$E[T_{k,k}^n\, I(T_{k,k} < \infty)] = \frac{n}{c(1 - H(0,k))} \int_k^\infty E[T_{u,k}^{n-1}\, I(T_{u,k} < \infty)]\, du. \tag{3.69}$$

Proof. We can rewrite formula (3.2) as

$$\begin{aligned}\tilde{\phi}_{k,\delta}(s) &= \frac{1}{\delta + \lambda - \lambda\tilde{f}(s) - cs}\Bigg(- ce^{-sk}\phi_{k,\delta}(k) \\ & +\lambda\phi_{k,\delta}(k) \int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\, du + \lambda \int_k^\infty e^{-su}\bar{F}(u)\, du\Bigg).\end{aligned}$$

The nth derivative of $\tilde{\phi}_{k,\delta}(s)$ with respect to δ by induction is found as

$$\begin{aligned}\frac{\partial^n}{\partial \delta^n}\tilde{\phi}_{k,\delta}(s) &= \frac{1}{cs - \delta - \lambda + \lambda\tilde{f}(s)}\Bigg(n\frac{\partial^{n-1}}{\partial \delta^{n-1}}\tilde{\phi}_{k,\delta}(s) + ce^{-sk}\frac{\partial^n}{\partial \delta^n}\phi_{k,\delta}(k) \\ & -\lambda\frac{\partial^n}{\partial \delta^n}\phi_{k,\delta}(k) \int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\, du\Bigg).\end{aligned} \tag{3.70}$$

Defining

$$\tilde{h}_n(s) = \frac{\partial^n}{\partial \delta^n}\tilde{\phi}_{k,\delta}(s)\Big|_{\delta=0}$$

and setting $\delta = 0$ in equation (3.70) we obtain

$$\begin{aligned}
\tilde{h}_n(s) &= \frac{1}{cs - \lambda + \lambda \tilde{f}(s)}\Bigg(n\tilde{h}_{n-1}(s) + (-1)^n c e^{-sk}\psi_{k,n}(k) \\
&\quad -(-1)^n \lambda \psi_{k,n}(k)\int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\,du\Bigg) \\
&= \frac{\tilde{\delta}(s)}{c\delta(0)}\Bigg(n\tilde{h}_{n-1}(s) + (-1)^n c e^{-sk}\psi_{k,n}(k) \\
&\quad -(-1)^n \lambda \psi_{k,n}(k)\int_k^\infty e^{-su}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\,du\Bigg). \qquad (3.71)
\end{aligned}$$

To invert $\tilde{h}_n(s)$ we note that

$$\tilde{h}_n(s) = \int_k^\infty e^{-su}(-1)^n \psi_{k,n}(u)\,du = (-1)^n \int_0^\infty e^{-s(u+k)}\psi_{k,n}(u+k)\,du.$$

Let $A_{n,k}(u) = \psi_{k,n}(u+k)$. Then, the first term on the right-hand side of equation (3.71) is

$$\frac{n(-1)^{n-1}\tilde{\delta}(s)}{c\delta(0)} e^{-sk}\tilde{A}_{n-1,k}(s)$$

which inverts to

$$\frac{n(-1)^{n-1}}{c\delta(0)}\int_0^{u-k} \psi_{k,n-1}(x+k)\delta(u-k-x)\,dx.$$

Using the shift property of the Laplace transform, the inverse of the second term on the right-hand side of (3.71) is $(-1)^n c\psi_{k,n}(k)\delta(u-k)/c\delta(0)$. Finally, the inverse of the last term in (3.71) is given by

$$\frac{(-1)^n \lambda}{c\delta(0)}\psi_{k,n}(k)\int_0^{u-k} \Big(\bar{F}(x) - \bar{F}(x+k)\Big)\delta(u-k-x)\,dx.$$

Combining these results, formula (3.68) follows.

To find formula (3.69) we apply the idea of Albrecher and Boxma (2005a, Lemma 6.1). We note that when $\delta = 0$, as $s = 0$ is a zero of the denominator of equation (3.70), it must also be a zero of the numerator. Thus

$$\frac{\partial^n}{\partial\delta^n}\phi_{k,\delta}(k)\Big|_{\delta=0}\left(\lambda\int_k^\infty e^{-su}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)\,du-ce^{-sk}\right)=n\frac{\partial^{n-1}}{\partial\delta^{n-1}}\tilde{\phi}_{k,\delta}(s)\Big|_{\delta=0}. \tag{3.72}$$

Letting $s\to 0$ in the bracket on the left-hand side of (3.72) gives

$$\lim_{s\to 0}\lambda\int_k^\infty e^{-su}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)\,du-ce^{-sk}=c[H_1(0,k)-1].$$

Rearranging (3.72) we obtain

$$\begin{aligned}\frac{\partial^n}{\partial\delta^n}\phi_{k,\delta}(k)\Big|_{\delta=0} &= \frac{-n}{c[1-H_1(0,k)]}\lim_{s\to 0}\int_k^\infty e^{-su}\frac{\partial^{n-1}}{\partial\delta^{n-1}}\phi_{k,\delta}(u)\,du\Big|_{\delta=0}\\ &= \frac{n(-1)^n}{c[1-H_1(0,k)]}\int_k^\infty \psi_{k,n-1}(u)\,du.\end{aligned}$$

□

Formulae (3.68) and (3.69) generalise results for the nth moment of the time of ruin in the classical risk model first provided by Lin and Willmot (2000), and then by Albrecher and Boxma (2005a). Indeed, their formulae can be obtained by setting $k=0$ in formulae (3.68) and (3.69).

We can obtain a simplified expression for the first moment of the time of ruin for $u=k$. Substituting for $\psi_k(u)$ from formula (3.9) in formula (3.69) yields

$$\begin{aligned}\psi_{k,1}(k) &= \frac{1}{c[1-H_1(0,k)]}\left(\int_k^\infty \psi(u-k)-\frac{1-\psi(0)}{1-H_1(0,k)}H_1(u-k,k)\,du\right)\\ &= \frac{E[L]}{c[1-H_1(0,k)]}-\frac{\delta(0)}{c[1-H_1(0,k)]^2}\int_0^\infty H_1(u,k)\,du \end{aligned}\tag{3.73}$$

where $E[L]=\int_0^\infty\psi(u)\,du=\frac{\lambda m_2}{2(c-\lambda m_1)}$ with m_1 and m_2 being the first and the second moments of the individual claim amount distribution. See, for example, Dickson (2005).

We now consider the moments of the number of claims until ruin by setting $\delta=0$, $\omega(x,y)=1$ for $x\geq k$ and $y>0$ in (3.1). In this case the Gerber-Shiu function is

$$\phi_{k,r}(u)=E\left[r^{N_{T_{u,k}}}I(T_{u,k}<\infty)\mid U(0)=u\right].$$

Taking the nth derivative of $\phi_{k,r}(u)$ with respect to r, then setting $r=1$ gives

$$\frac{\partial^n}{\partial r^n}\phi_{k,r}(u)\Big|_{r=1}=E[N_{T_{u,k}}(N_{T_{u,k}}-1)\ldots(N_{T_{u,k}}-n+1)I(T_{u,k}<\infty)].$$

The next two results give expressions for the first two factorial moments of the number of claims until ruin.

Theorem 3.11. When the initial surplus is $u > k$ we have

$$\begin{aligned}
& E[N_{T_{u,k}}\, I(T_{u,k} < \infty)] \\
= \; & \frac{\delta(u-k)}{\delta(0)} E[N_{T_{k,k}}\, I(T_{k,k} < \infty)] - \frac{\lambda}{c\delta(0)} \int_0^{u-k} \bar{F}(x+k)\delta(u-k-x)\, dx \\
& - \frac{\lambda}{c\delta(0)} \Big(\psi_k(k) + E[N_{T_{k,k}}\, I(T_{k,k} < \infty)] \Big) \int_0^{u-k} \Big(\bar{F}(x) - \bar{F}(x+k) \Big) \delta(u-k-x)\, dx \\
& - \frac{\lambda}{c\delta(0)} \int_0^{u-k} b(u-k-x)\psi_k(x+k)\, dx \qquad (3.74)
\end{aligned}$$

where $b(u) = \int_0^u \delta(u-x) f(x)\, dx$, and

$$E[N_{T_{k,k}}\, I(T_{k,k} < \infty)] = \frac{\psi_k(k) H_1(0,k) + \psi(0) - H_1(0,k) + \frac{\lambda}{c} \int_k^\infty \psi_k(u)\, du}{1 - H_1(0,k)}. \qquad (3.75)$$

Theorem 3.12. When $u > k$ we have

$$\begin{aligned}
& E[N_{T_{u,k}}(N_{T_{u,k}} - 1)\, I(T_{u,k} < \infty)] \\
= \; & \frac{\delta(u-k)}{\delta(0)} E[N_{T_{k,k}}(N_{T_{k,k}} - 1)\, I(T_{k,k} < \infty)] \\
& - \frac{2\lambda}{c\delta(0)} \int_0^{u-k} b(u-k-x) E[N_{T_{x+k,k}}\, I(T_{x+k,k} < \infty)]\, dx \\
& - \frac{\lambda}{c\delta(0)} \big(2E[N_{T_{k,k}}\, I(T_{k,k} < \infty)] + E[N_{T_{k,k}}(N_{T_{k,k}} - 1)\, I(T_{k,k} < \infty)] \big) \\
& \times \int_0^{u-k} \delta(u-k-x) \Big(\bar{F}(x) - \bar{F}(x+k) \Big)\, dx \qquad (3.76)
\end{aligned}$$

where $b(u) = \int_0^u \delta(u-x) f(x)\, dx$ and

$$\begin{aligned}
& E[N_{T_{k,k}}(N_{T_{k,k}} - 1)\, I(T_{k,k} < \infty)] \\
= \; & \frac{2E[N_{T_{k,k}}\, I(T_{k,k} < \infty)] H_1(0,k) + \frac{2\lambda}{c} \int_k^\infty E[N_{T_{u,k}}\, I(T_{u,k} < \infty)]\, du}{1 - H_1(0,k)}. \qquad (3.77)
\end{aligned}$$

We remark that if we set $k = 0$ in formulae (3.74), (3.75), (3.76) and (3.77), we can recover the results in the classical risk model without capital injections. See Dickson (2012).

Next we provide results for the joint moment of $T_{u,k}$ and $N_{T_{u,k}}$. Setting $\omega(x,y)=1$ in equation (3.1) and noting that

$$-\frac{\partial}{\partial r}\frac{\partial}{\partial \delta}\phi_{k,r,\delta}(u)\Big|_{r=1,\delta=0} = E[N_{T_{u,k}}T_{u,k}\, I(T_{u,k}<\infty)] \tag{3.78}$$

we can obtain the covariance between $N_{T_{u,k}}$ and $T_{u,k}$.

Theorem 3.13. When $u>k$ we have

$$\begin{aligned}
E[N_{T_{u,k}}T_{u,k}\, I(T_{u,k}<\infty)] \;=\;& \frac{\delta(u-k)}{\delta(0)}E[N_{T_{k,k}}T_{k,k}\, I(T_{k,k}<\infty)] \\
&-\frac{\lambda}{c\delta(0)}\Big(E[T_{k,k}\, I(T_{k,k}<\infty)] + E[N_{T_{k,k}}T_{k,k}\, I(T_{k,k}<\infty)]\Big) \\
&\times \int_0^{u-k} \delta(u-k-x)\Big(\bar{F}(x)-\bar{F}(x+k)\Big)\, dx \\
&-\frac{1}{c\delta(0)}\int_0^{u-k} \delta(u-k-x)E[N_{T_{x+k,k}}\, I(T_{x+k,k}<\infty)]\, dx \\
&-\frac{\lambda}{c\delta(0)}\int_0^{u-k} b(u-k-x)E[T_{x+k,k}\, I(T_{x+k,k}<\infty)]\, dx
\end{aligned}$$

where $b(u)=\int_0^u \delta(u-x)f(x)\,dx$ and

$$\begin{aligned}
&E[N_{T_{k,k}}T_{k,k}\, I(T_{k,k}<\infty)] \\
=\;& \frac{1}{1-H_1(0,k)}\Bigg(E[T_{k,k}\, I(T_{k,k}<\infty)]H_1(0,k) + \frac{1}{c}\int_k^\infty E[N_{T_{u,k}}\, I(T_{u,k}<\infty)]\, du \\
&+\frac{\lambda}{c}\int_k^\infty E[T_{u,k}\, I(T_{u,k}<\infty)]\, du\Bigg).
\end{aligned} \tag{3.79}$$

We show an application of our results in the next example.

Example 3.1. Suppose $F(x)=1-e^{-\alpha x}$, where $x\geq 0$ with $\alpha>0$, then using the above formulae and the following results

$$\psi(u)=\frac{\lambda}{\alpha c}e^{-(\alpha-\lambda/c)u}, \qquad \psi_k(k)=\frac{\lambda e^{-\alpha k}}{\alpha c-\lambda(1-e^{-\alpha k})},$$

$$\psi_k(u)=\frac{\lambda e^{-(\alpha-\lambda/c)(u-k)-\alpha k}}{\alpha c-\lambda(1-e^{-\alpha k})}, \qquad H_1(0,k)=\psi(0)(1-e^{-\alpha k}),$$

we obtain

- $$E[T_{k,k}\, I(T_{k,k}<\infty)] = \frac{\alpha c\lambda e^{-\alpha k}}{(\alpha c-\lambda)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^2};$$
- $$E[T_{k,k}^2\, I(T_{k,k}<\infty)] = \frac{2\alpha\lambda c e^{-\alpha k}\big(\alpha^2c^2-\lambda^2(1-e^{-\alpha k})\big)}{(\alpha c-\lambda)^3\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^3};$$
- $$E[T_{u,k}\, I(T_{u,k}<\infty)] = \frac{\lambda e^{-\alpha k}\Big(\alpha c^2+\lambda(u-k)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)\Big)e^{-(\alpha-\lambda/c)(u-k)}}{c(\alpha c-\lambda)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^2};$$
- $$E[T_{u,k}^2\, I(T_{u,k}<\infty)] = \frac{-2}{c\delta(0)}\int_0^{u-k} E[T_{x+k,k}\, I(T_{x+k,k}<\infty)]\delta(u-k-x)\,dx$$
  $$+\frac{\delta(u-k)}{\delta(0)}E[T_{k,k}^2\, I(T_{k,k}<\infty)]$$
  $$-\frac{\lambda}{c\delta(0)}E[T_{k,k}^2\, I(T_{k,k}<\infty)](1-e^{-\alpha k})\frac{1}{\alpha}\big(1-e^{-(\alpha-\lambda/c)(u-k)}\big);$$
- $$E[T_{u,k}|T_{u,k}<\infty] = \frac{\Big(\alpha c^2+\lambda(u-k)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)\Big)}{c(\alpha c-\lambda)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)};$$
- $$E[N_{T_{k,k}}\, I(T_{k,k}<\infty)] = \frac{(\alpha c)^2\lambda e^{-\alpha k}}{(\alpha c-\lambda)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^2};$$
- $$E[N_{T_{k,k}}(N_{T_{k,k}}-1)\, I(T_{k,k}<\infty)] = \frac{2(\alpha c)^3\lambda^2 e^{-\alpha k}\big(\alpha c(2-e^{-\alpha k})-2\lambda(1-e^{-\alpha k})\big)}{(\alpha c-\lambda)^3\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^3};$$
- $$E[N_{T_{u,k}}\, I(T_{u,k}<\infty)] = \frac{\alpha\lambda e^{-\alpha k}\Big(\alpha c^2+\lambda(u-k)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)\Big)e^{-(\alpha-\lambda/c)(u-k)}}{(\alpha c-\lambda)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^2};$$
- $$E[N_{T_{u,k}}|T_{u,k}<\infty] = \frac{\alpha\Big(\alpha c^2+\lambda(u-k)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)\Big)}{(\alpha c-\lambda)\big(\alpha c-\lambda(1-e^{-\alpha k})\big)};$$
- $$E[N_{T_{u,k}}(N_{T_{u,k}}-1)\, I(T_{u,k}<\infty)] = \frac{\delta(u-k)}{\delta(0)}E[N_{T_{k,k}}(N_{T_{k,k}}-1)\, I(T_{k,k}<\infty)]$$
  $$-\frac{2\lambda}{c\delta(0)}\int_0^{u-k}(1-e^{-(\alpha-\lambda/c)(u-k-x)})E[N_{T_{x+k,k}}\, I(T_{x+k,k}<\infty)]dx$$
  $$-\frac{\lambda}{c\delta(0)}\Big(2E[N_{T_{k,k}}\, I(T_{k,k}<\infty)]+E[N_{T_{k,k}}(N_{T_{k,k}}-1)\, I(T_{k,k}<\infty)]\Big)$$
  $$\times(1-e^{-\alpha k})\frac{1}{\alpha}\big(1-e^{-(\alpha-\lambda/c)(u-k)}\big);$$
- $$\mathrm{Var}[T_{k,k}|T_{k,k}<\infty] = E[T_{k,k}^2|T_{k,k}<\infty]-\big(E[T_{k,k}|T_{k,k}<\infty]\big)^2$$

$$= \frac{\alpha c\Big(\alpha c(\alpha c+\lambda)-2\lambda^2(1-e^{-\alpha k})\Big)}{(\alpha c-\lambda)^3\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^2};$$

- $\mathrm{Var}[N_{T_{k,k}}|T_{k,k}<\infty] = E[N^2_{T_{k,k}}|T_{k,k}<\infty] - \big(E[N_{T_{k,k}}|T_{k,k}<\infty]\big)^2$

$$= \frac{(\alpha c)^2\lambda\Big(\alpha^2c^2(2-e^{-\alpha k})-\alpha\lambda c(1-2e^{-\alpha k})-\lambda^2(1-e^{-\alpha k})\Big)}{(\alpha c-\lambda)^3\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^2};$$

- $E[N_{T_{k,k}}T_{k,k}\,I(T_{k,k}<\infty)] = \dfrac{(\alpha c)^2\lambda e^{-\alpha k}\Big(\alpha^2c^2+\alpha\lambda c(2-e^{-\alpha k})-3\lambda^2(1-e^{-\alpha k})\Big)}{(\alpha c-\lambda)^3\big(\alpha c-\lambda(1-e^{-\alpha k})\big)^3};$

- $$\begin{aligned}E[N_{T_{u,k}}T_{u,k}\,I(T_{u,k}<\infty)] &= \frac{\delta(u-k)}{\delta(0)}E[N_{T_{k,k}}T_{k,k}\,I(T_{k,k}<\infty)]\\ &-\frac{\lambda}{c\delta(0)}\Big(E[T_{k,k}\,I(T_{k,k}<\infty)]+E[N_{T_{k,k}}T_{k,k}\,I(T_{k,k}<\infty)]\Big)(1-e^{-\alpha k})\\ &\times\frac{1}{\alpha}\big(1-e^{-(\alpha-\lambda/c)(u-k)}\big)\\ &-\frac{1}{c\delta(0)}\int_0^{u-k}\delta(u-k-x)E[N_{T_{x+k,k}}\,I(T_{x+k,k}<\infty)]\,dx\\ &-\frac{\lambda}{c\delta(0)}\int_0^{u-k}(1-e^{-(\alpha-\lambda/c)(u-k-x)})E[T_{x+k,k}\,I(T_{x+k,k}<\infty)]\,dx;\end{aligned}$$

- $$\mathrm{Corr}[N_{T_{u,k}},T_{u,k}|T_{u,k}<\infty] = \frac{\mathrm{Cov}[N_{T_{u,k}},T_{u,k}|T_{u,k}<\infty]}{\sqrt{\mathrm{Var}[N_{T_{u,k}}|T_{u,k}<\infty]\,\mathrm{Var}[N_{T_{u,k}}|T_{u,k}<\infty]}}. \qquad (3.80)$$

The relationship between the first moment of $T_{u,k}$ and $N_{T_{u,k}}$ is the same as in the classical risk model without capital injections (see Dickson, 2012), i.e.

$$E[N_{T_{u,k}}|T_{u,k}<\infty] = \alpha c E[T_{u,k}|T_{u,k}<\infty].$$

Table 3.1 shows the values of correlation coefficient when $\alpha = 1$ and $\lambda = 1$ for different values of u, k and c. The key to Table 3.1 is as follows:

(1) denotes the correlation coefficient when $k = 0$,

(2) denotes the correlation coefficient when $k = 1$,

(3) denotes the correlation coefficient when $k = 3$,

(4) denotes the correlation coefficient when $k = 5$.

We note the following from Table 3.1.

(i) There is no significant difference between $\text{Corr}[N_{T_{u,k}}, T_{u,k} | T_{u,k} < \infty]$ in the classical risk model with and without capital injections. In both models, there is a strong positive correlation between the time of ruin and the number of claims until ruin which is unsurprising.

(ii) The greater the premium is, the smaller the correlation coefficient between $T_{u,k}$ and $N_{T_{u,k}}$ becomes.

(iii) As k increases, the correlation coefficient gets larger.

(iv) We cannot identify any relationship between the initial surplus and the correlation coefficient.

Table 3.1: Values of correlation coefficient

u		$c = 1.1$	$c = 1.3$	$c = 1.5$
5	(1)	0.998867	0.991552	0.980306
	(2)	0.998869	0.991595	0.980449
	(3)	0.998915	0.991982	0.981241
	(4)	0.998980	0.992360	0.981922
15	(1)	0.998868	0.991581	0.980469
	(2)	0.998868	0.991598	0.980528
	(3)	0.998887	0.991739	0.980821
	(4)	0.998917	0.991837	0.980970
25	(1)	0.998868	0.991588	0.980510
	(2)	0.998868	0.991599	0.980548
	(3)	0.998879	0.991684	0.980727
	(4)	0.998897	0.991736	0.980807

3.8 Concluding remarks

We saw in this chapter that our Gerber-Shiu analysis is not as helpful in dealing with infinite time ruin problems as it is with dealing with finite time ruin problems. For the latter it has led us to the rather surprising conclusions that both the density of the time of ruin and the finite time survival (or ruin) probability for our risk model with capital injections can be expressed in terms of the aggregate claims distribution for the classical risk model.

Our analysis in the case of exponentially distributed individual claims has extended existing results for the classical risk model, and has shown the connection between two known formulae for the density of the time of ruin in the classical risk model.

We found a recursive expression for $p_k(u, n)$ when claim amounts follow an exponential distribution but for Erlang(2) claims were only able to calculate $p_k(u, n)$ by numerical integration. In the next chapter, we show that an explicit expression for $p_k(u, n)$ can be obtained for both exponential and Erlang(2) distributions using probabilistic reasoning.

Unfortunately we could obtain relatively simple expressions for $w_k(u, t)$ only when individual claim amounts are exponentially distributed. However, by using numerical integration of formula (3.40), it should be possible to obtain values for $w_k(u, t)$ for other individual claim amount distributions. For example, Willmot (2015) gives an expression for $g(x, t)$ when the individual claim amount distribution is an infinite mixture of Erlangs. Alternatively, we might approximate $g(x, t)$ using Panjer's (1981) recursion formula. In Chapters 5 and 6 we discuss another method in more detail and introduce a discrete time model that can approximate quantities such as $w_k(u, t)$. Another approach to calculating $w_k(u, t)$ is the approximation techniques discussed by Seal (1974) who considers finite time non-ruin probabilities in the classical risk model.

Applying techniques in Albrecher and Boxma (2005a) we obtained moments of $T_{u,k}$ and $N_{T_{u,k}}$ as well as quantities like $\mathrm{Cov}(T_{u,k}, N_{T_{u,k}})$. We observed that for a range of values of u and k and different premium loading factors the correlation coefficient was (unsurprisingly) very close to 1 in all our scenarios.

Chapter 4

Gerber-Shiu analysis: probabilistic approach

4.1 Introduction

In this chapter, we introduce an alternative representation of the Gerber-Shiu function to analyse the risk model that we considered in Chapter 3. Our approach in Chapter 3 gave rise to a general recursive expression for the probability function of the number of claims until ruin. Here, we construct our Gerber-Shiu function using a probabilistic argument so that we can obtain explicit expressions for the probability function of the number of claims until ruin in the case of claim amounts with exponential and Erlang(2) distributions. However, the main disadvantage of this method is that to invert the Laplace transform of the Gerber-Shiu function, we need to specify the functional form of the individual claim amount distribution. More specifically, this approach can only be applied to distribution functions that admit a particular factorisation.

4.2 A Gerber-Shiu function

We denote by $w_k(u, x, y, n, t)$ the (defective) joint density of the surplus immediately prior to ruin (x), the deficit at ruin (y), the number of claims until ruin (n) and the time of ruin (t), given initial surplus u for our risk model with capital injections, defined for $n = 1, 2, 3, \ldots$, $x \geq k$, $y > 0$ and $t > 0$, with corresponding notation $w(u, x, y, n, t)$ for the classical risk model, defined for $n = 1, 2, 3, \ldots$, $x \geq 0$, $y > 0$ and $t > 0$. Further, let

$$w(u,y,n,t) = \int_0^\infty w(u,x,y,n,t)\,dx.$$

Then

$$\begin{aligned}
\phi_{k,r,\delta}(u) &= E\left[r^{N_{T_{u,k}}} e^{-\delta T_{u,k}} \omega(U(T_{u,k}^-), |U(T_{u,k})|)\, I(T_{u,k} < \infty) \mid U(0) = u\right] \\
&= \sum_{n=1}^\infty r^n \int_0^\infty e^{-\delta t} \int_k^\infty \int_0^\infty \omega(x,y) w_k(u,x,y,n,t)\,dy\,dx\,dt \qquad (4.1) \\
&= \sum_{n=1}^\infty r^n \int_0^\infty e^{-\delta t} \Big\{ \int_0^\infty \int_0^k w(u-k,x,y,n,t)\phi_{k,r,\delta}(k)\,dy\,dx \\
&\quad + \int_0^\infty \int_k^\infty \omega(x+k, y-k) w(u-k,x,y,n,t)\,dy\,dx \Big\}\,dt. \qquad (4.2)
\end{aligned}$$

From this expression, depending on the form taken by $\omega(x,y)$, we can obtain different ruin-related quantities.

4.3 The probability of ultimate ruin

We now consider the ultimate ruin probability $\psi_k(u)$, for $u \geq k$. In Theorem 3.3 we provided an expression for $\psi_k(u)$. In this section, we show that expression (3.9) can be obtained with a much simpler method.

Theorem 4.1. When the initial surplus is $u \geq k$ we have

$$\psi_k(u) = \psi(u-k) - H_1(u-k,k)[1-\psi_k(k)].$$

Proof. Setting $r = 1$, $\delta = 0$ and $\omega(x,y) = 1$ for $x \geq 0$ and $y > 0$ in equation (4.2), we obtain

$$\begin{aligned}
\psi_k(u) &= \int_0^\infty \int_0^\infty \int_0^k w(u-k,x,y,t)\psi_k(k)\,dy\,dx\,dt \\
&\quad + \int_0^\infty \int_0^\infty \int_k^\infty w(u-k,x,y,t)\,dy\,dx\,dt \\
&= \int_0^\infty \int_0^k w(u-k,y,t)\psi_k(k)\,dy\,dt + \int_0^\infty \int_k^\infty w(u-k,y,t)\,dy\,dt \qquad (4.3)
\end{aligned}$$

where $w(u, x, y, t)$ is the (defective) joint density of the surplus immediately prior to ruin, the deficit at ruin and the time of ruin and $w(u, y, t)$ is the (defective) joint density of the deficit at ruin and the time of ruin in the classical risk model as defined in Chapter 1. Hence, equation (4.3) can be written as

$$\psi_k(u) = H_1(u-k, k)\psi_k(k) + \psi(u-k) - H_1(u-k, k).$$

Setting $u = k$ and rearranging gives $\psi_k(k)$. □

4.4 The joint distribution of $U(T_{u,k}^-)$ and $|U(T_{u,k})|$

In this section, we provide an alternative proof to Theorem 3.5 which is based on the results in the classical risk model and does not require the Laplace transform inversion of the Gerber-Shiu function and consequently is more straightforward.

Theorem 4.2. When the initial surplus is $u \geq k$ we have

$$\begin{aligned} H_k(u, z, y_1) &= H_k(k, z, y_1)H_1(u-k, k) + \frac{\lambda\delta(u-k)}{c\delta(0)}\int_u^z \Big(\bar{F}(x) - \bar{F}(x+y_1)\Big) I(u \leq z)\, dx \\ &\quad - \frac{\lambda}{c\delta(0)}\int_0^{(u\wedge z)-k} \left(\delta(u-k) - \delta(u-k-x)\right)\Big(\bar{F}(x+k) - \bar{F}(x+k+y_1)\Big)\, dx. \end{aligned}$$

Proof. Setting $r = 1$, $\delta = 0$ and $\omega(x+k, y-k) = I\{x \leq z-k\}I\{y \leq y_1+k\}$ in equation (4.2) and noting that the occurrence of ruin in the classical model, from initial surplus level $u-k$ with the surplus prior to ruin $z-k$, and the severity of ruin y_1+k, implies ruin in the classical risk model with capital injections with initial surplus level u, the surplus prior to ruin z and the severity of ruin y_1, we get

$$\begin{aligned} H_k(u, z, y_1) &= \int_0^\infty \int_0^\infty \int_0^k w(u-k, x, y, t) H_k(k, z, y_1)\, dy\, dx\, dt \\ &\quad + \int_0^\infty \int_0^{z+k} \int_k^{y_1+k} w(u-k, x, y, t)\, dy\, dx\, dt \\ &= H_1(u-k, k)H_k(k, z, y_1) + \int_0^{z-k}\int_k^{y_1+k} h(u-k, x, y)\, dy\, dx. \end{aligned} \tag{4.4}$$

To proceed, we need to apply two formulae from Chapter 1. Recall that Dickson (1992) has shown for the classical risk model

$$h(u,z,y) = \begin{cases} h(0,z,y)\frac{1-\psi(u)}{1-\psi(0)} & \text{for} \quad u < z, \\ \\ h(0,z,y)\frac{\psi(u-z)-\psi(u)}{1-\psi(0)} & \text{for} \quad u \geq z. \end{cases}$$

Further, Dufresne and Gerber (1988b) have shown that $h(0,x,y) = (\lambda/c)f(x+y)$. Therefore, for $u > z$ we have

$$\begin{aligned} H_k(u,z,y_1) &= H_k(k,z,y_1)H_1(u-k,k) + \int_0^{z-k}\int_k^{y_1+k} h(0,x,y)\frac{\psi(u-x)-\psi(u)}{\delta(0)}\,dy\,dx \\ &= H_k(k,z,y_1)H_1(u-k,k) \\ &\quad + \frac{\lambda}{c\delta(0)}\int_0^{z-k}(\psi(u-k-x)-\psi(u-k))\left(\bar{F}(x+k)-\bar{F}(x+k+y_1)\right)dx. \end{aligned} \tag{4.5}$$

For $u \leq z$ we need to distinguish between the surplus level prior to ruin being below $u-k$ and above $u-k$. Hence formula (4.4) can be written as

$$\begin{aligned} H_k(u,z,y_1) &= H_1(u-k,k)H_k(k,z,y_1) + \int_0^{u-k}\int_k^{y_1+k} h(u-k,x,y)\,dy\,dx \\ &\quad + \int_{u-k}^{z-k}\int_k^{y_1+k} h(u-k,x,y)\,dy\,dx \\ &= H_1(u-k,k)H_k(k,z,y_1) + \int_{u-k}^{z-k}\int_k^{y_1+k} h(0,x,y)\frac{\delta(u-k)}{\delta(0)}\,dy\,dx \\ &\quad + \int_0^{u-k}\int_k^{y_1+k} h(0,z,y)\frac{\psi(u-k-x)-\psi(u-k)}{\delta(0)}\,dy\,dx \\ &= H_1(u-k,k)H_k(k,z,y_1) + \frac{\lambda\delta(u-k)}{c\delta(0)}\int_{u-k}^{z-k}\left(\bar{F}(x+k)-\bar{F}(x+y_1+k)\right)dx \\ &\quad + \frac{\lambda}{c\delta(0)}\int_0^{u-k}(\psi(u-k-x)-\psi(u-k))\left(\bar{F}(x+k)-\bar{F}(x+k+y_1)\right)dx. \end{aligned} \tag{4.6}$$

The result follows after combining (4.5) and (4.6). □

Now, if we let $y_1 \to \infty$ in (4.4), we can develop formulae (3.23) and (3.23). Thus for $u > z$, we have

$$\begin{aligned} H_{k,2}(u,z) &= H_1(u-k,k)H_{k,2}(k,z) \\ &\quad + \frac{\lambda}{c\delta(0)}\int_0^{z-k}(\psi(u-x-k)-\psi(u-k))\,\bar{F}(x+k)\,dx \end{aligned}$$

and for $u \leq z$,

$$\begin{aligned} H_{k,2}(u,z) &= H_1(u-k,k)H_{k,2}(k,z) + \frac{\lambda\delta(u-k)}{c\delta(0)} \int_{u-k}^{z-k} \bar{F}(x+k)\,dx \\ &\quad + \frac{\lambda}{c\delta(0)} \int_0^{u-k} (\psi(u-x-k) - \psi(u-k))\,\bar{F}(x+k)\,dx. \end{aligned}$$

Setting $u = k$, gives us formula (3.15).

Now to derive formula (3.24) we let $z \to \infty$ in (4.4), giving

$$\begin{aligned} H_{k,1}(u,y_1) &= H_1(u-k,k)H_{k,1}(k,y_1) + \int_k^{y_1+k} h_1(u-k,y)\,dy \\ &= H_1(u-k,k)H_{k,1}(k,y_1) + H_1(u-k,y_1+k) - H_1(u-k,k). \end{aligned}$$

Putting $u = k$, yields (3.14).

In the previous chapter, our approach to deriving the ruin-related quantities in infinite time involved the inversion of the Laplace transform of the Gerber-Shiu function which was not efficient. Here, our technique is based on probabilistic arguments which leads to the same results and is more efficient. In the next section, we apply the same technique to find the joint density of the time of ruin and the number of claims until ruin in finite time.

4.5 The joint density of $T_{u,k}$ and $N_{T_{u,k}}$

If we set $\delta > 0$, $0 < r < 1$ and $\omega(x,y) = 1$ for all x and y in equations (4.1) and (4.2), they respectively become

$$\begin{aligned} \phi_{k,r,\delta}(u) &= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \int_k^{\infty} \int_0^{\infty} w_k(u,x,y,n,t)\,dy\,dx\,dt \\ &= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t}\, w_k(u,n,t)\,dt = \sum_{n=1}^{\infty} r^n\, w_{k,\delta}(u,n) \end{aligned}$$

where $w_{k,\delta}(u,n) = \int_0^{\infty} e^{-\delta t}\, w_k(u,n,t)\,dt$ (with $w_\delta(u,n)$ similarly defined), and

$$\begin{aligned} \phi_{k,r,\delta}(u) &= \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \int_0^{\infty} \int_0^{k} w(u-k,x,y,n,t)\phi_{k,r,\delta}(k)\,dy\,dx\,dt \\ &\quad + \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} \int_0^{\infty} \int_k^{\infty} w(u-k,x,y,n,t)\,dy\,dx\,dt. \end{aligned}$$

Defining $w_\delta(u,y,n) = \int_0^\infty e^{-\delta t}\, w(u,y,n,t)\, dt$ and $\phi_{r,\delta}(u,y) = \sum_{n=1}^\infty r^n\, w_\delta(u,y,n)$ we get

$$\phi_{k,r,\delta}(u) = \phi_{k,r,\delta}(k) \int_0^k \phi_{r,\delta}(u-k,y)\, dy + \int_k^\infty \phi_{r,\delta}(u-k,y)\, dy. \tag{4.7}$$

In particular, when $u = k$ we have

$$\phi_{k,r,\delta}(k) = \frac{\int_k^\infty \phi_{r,\delta}(0,y)\, dy}{1 - \int_0^k \phi_{r,\delta}(0,y)\, dy}. \tag{4.8}$$

Next we apply these formulae to find the probability function of the number of claims until ruin in the case of claim amounts with exponential and Erlang(2) distributions.

4.5.1 Exponential claims

We consider the case $\bar{F}(x) = e^{-\alpha x}$ for $x \geq 0$ with $\alpha > 0$. The next results give explicit expressions for $p_k(u,n)$.

Result 4.1. When the initial surplus is k, for $n = 1, 2, 3, \ldots,$

$$p_k(k,n) = e^{-\alpha k} \sum_{j=0}^{n-1} (1-e^{-\alpha k})^j \left(\frac{\alpha c}{\lambda + \alpha c}\right)^{n-j-1} \left(\frac{\lambda}{\lambda + \alpha c}\right)^n \frac{(j+1)\,(2n-j-2)!}{n!\,(n-j-1)!}. \tag{4.9}$$

Derivation. According to the results for the classical risk model in Dickson (2012, Section 7) we have

$$w(0,y,1,t) = \lambda e^{-\lambda t} f(ct+y) \tag{4.10}$$

and for $n = 2, 3, 4, \ldots,$

$$w(0,y,n,t) = \lambda \int_0^{ct} \frac{x}{ct}\, e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}\, f^{(n-1)*}(ct-x) f(x+y)\, dx. \tag{4.11}$$

Substituting for f, formula (4.10) can be written as

$$\begin{aligned} w(0,y,1,t) &= \lambda e^{-\lambda t}\, \alpha e^{-\alpha(ct+y)} \\ &= \lambda e^{-\lambda t} \bar{F}(ct)\, \alpha e^{-\alpha y} \\ &= w(0,1,t)\, \alpha e^{-\alpha y} \end{aligned} \tag{4.12}$$

where the last line in (4.12) comes from formula (4) of Dickson (2012). Also, we can write formula (4.11) as

$$
\begin{aligned}
w(0,y,n,t) &= \lambda \int_0^{ct} \frac{x}{ct}\, e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}\, e^{-\alpha(ct-x)}\, \frac{(ct-x)^{n-2}\,\alpha^{n-1}}{(n-2)!}\, \alpha e^{-\alpha(x+y)} dx \\
&= e^{-\lambda t}\frac{\lambda^n\, t^{n-1}}{(n-1)!}\int_0^{ct} \frac{x}{ct}\, e^{-\alpha(ct-x)}\frac{(ct-x)^{n-2}\,\alpha^{n-1}}{(n-2)!}\, e^{-\alpha x}\alpha e^{-\alpha y} dx \\
&= e^{-\lambda t}\frac{\lambda^n\, t^{n-1}}{(n-1)!}\int_0^{ct} \frac{x}{ct} f^{(n-1)*}(ct-x)\bar{F}(x)\, dx\, \alpha e^{-\alpha y} \\
&= w(0,n,t)\, \alpha e^{-\alpha y} \qquad (4.13)
\end{aligned}
$$

where the last line in (4.13) comes from formula (5) of Dickson (2012). Similarly,

$$
\begin{aligned}
w(u,y,1,t) &= \lambda\, e^{-\lambda t} f(u+ct+y) \\
&= w(u,1,t)\, \alpha e^{-\alpha y} \qquad (4.14)
\end{aligned}
$$

and for $n = 2, 3, 4, \dots,$

$$
\begin{aligned}
w(u,y,n,t) &= \lambda \int_0^{u+ct} e^{-\lambda t}\, \frac{(\lambda t)^{n-1}}{(n-1)!} f^{(n-1)*}(u+ct-x) f(x+y)\, dx \\
&\quad -c\sum_{r=1}^{n-1}\int_0^t e^{-\lambda s}\, \frac{(\lambda s)^r}{r!} f^{r*}(u+cs)\, w(0,y,n-r,t-s)\, ds \\
&= \Bigg(e^{-\lambda t}\, \frac{(\lambda t)^{n-1}}{(n-1)!}\int_0^{u+ct} f^{(n-1)*}(u+ct-x)\,\lambda\,\bar{F}(x)\, dx \\
&\quad -c\sum_{r=1}^{n-1}\int_0^t e^{-\lambda s}\, \frac{(\lambda s)^r}{r!} f^{r*}(u+cs)\, w(0,n-r,t-s)\, ds \Bigg) \alpha e^{-\alpha y} \\
&= w(u,n,t)\, \alpha e^{-\alpha y} \qquad (4.15)
\end{aligned}
$$

where the last line in (4.15) comes from formula (8) of Dickson (2012). Therefore, we can conclude that $w_{r,\delta}(u,y) = w_{r,\delta}(u)\, \alpha e^{-\alpha y}$. Hence formula (4.8) can be written as

$$
\phi_{k,r,\delta}(k) = \frac{\phi_{r,\delta}(0)\, e^{-\alpha k}}{1-\phi_{r,\delta}(0)\,(1-e^{-\alpha k})} = \sum_{i=0}^{\infty} e^{-\alpha k}(1-e^{-\alpha k})^i \phi_{r,\delta}(0)^{i+1} \qquad (4.16)
$$

where for $i = 0, 1, 2, \dots,$

$$
\phi_{r,\delta}(0)^{i+1} = \sum_{n=i+1}^{\infty} r^n \int_0^{\infty} e^{-\delta t} w^{(i+1)*}(0,n,t)\, dt
$$

and for $i = 1, 2, 3, \ldots,$

$$w^{(i+1)*}(0, n, t) = \sum_{j=1}^{n-1} \int_0^t w^{i*}(0, j, s)\, w(0, n-j, t-s)\, ds.$$

Inverting formula (4.16) with respect to δ yields

$$\sum_{n=1}^{\infty} r^n w_k(k, n, t) = \sum_{i=0}^{\infty} e^{-\alpha k}(1 - e^{-\alpha k})^i \sum_{n=i+1}^{\infty} r^n w^{(i+1)*}(0, n, t). \tag{4.17}$$

Applying the techniques in Nie et al. (2015) we can show that

$$\sum_{n=i+1}^{\infty} r^n\, w^{(i+1)*}(0, n, t) = (\lambda r)^{i+1}\, e^{-(\lambda+\alpha c)t} t^i\, (i+1) \sum_{n=0}^{\infty} \frac{(\alpha c \lambda r t^2)^n}{n!\,(n+i+1)!} \tag{4.18}$$

and inserting in equation (4.17) we obtain

$$\sum_{n=1}^{\infty} r^n\, w_k(k, n, t) = \sum_{n=1}^{\infty} (\lambda r)^n\, e^{-\alpha k}(1 - e^{-\alpha k})^{n-1}\, e^{-(\lambda+\alpha c)t}\, t^{n-1}\, n \sum_{j=0}^{\infty} \frac{(\alpha c \lambda r t^2)^j}{j!\,(j+n)!}.$$

Integrating over t yields the probability generating function of $N_{T_{k,k}}$, denoted by $\tilde{p}_{k,r}(k)$, as

$$\tilde{p}_{k,r}(k) = e^{-\alpha k} \sum_{n=1}^{\infty} (\lambda r)^n\, (1 - e^{-\alpha k})^{n-1}\, n \sum_{j=0}^{\infty} \frac{\Gamma(2j+n)}{j!\,\Gamma(n+1+j)}\, \frac{(\alpha c \lambda r)^j}{(\lambda + \alpha c)^{2j+n}} \tag{4.19}$$

and applying the identity

$$\sum_{n=1}^{\infty} \sum_{j=0}^{\infty} t_{n,j} = \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} t_{j+1, n-j-1},$$

we obtain

$$\tilde{p}_{k,r}(k) = e^{-\alpha k} \sum_{n=1}^{\infty} (\lambda r)^n \sum_{j=0}^{n-1} (1 - e^{-\alpha k})^j\, \frac{(j+1)\, \Gamma(2n-j-1)}{(n-j-1)!\Gamma(n+1)}\, \frac{(\alpha c)^{n-j-1}}{(\lambda + \alpha c)^{2n-j-1}}$$

from which the result follows. □

Formula (4.9) agrees with (3.53). Setting $k = 0$ in formula (4.9) yields the probability function of the number of claims until ruin from initial surplus 0 in the classical risk model. See, for example, Landriault et al. (2011).

Result 4.2. When $u > k$, for $n = 1, 2, 3, \dots,$

$$\begin{aligned} p_k(u,n) &= e^{-\alpha u}\sum_{i=0}^{n-1}\sum_{m=0}^{i}(1-e^{-\alpha k})^m \frac{(\alpha(u-k))^{i-m}}{(i-m)!}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^{n-i-1}\left(\frac{\lambda}{\lambda+\alpha c}\right)^{n} \\ &\quad \times \frac{(i+1)(2n-i-2)!}{n!\,(n-i-1)!}. \end{aligned} \tag{4.20}$$

Derivation. Formula (4.7) can be written as

$$\begin{aligned} \phi_{k,r,\delta}(u) &= \phi_{k,r,\delta}(k)\,\phi_{r,\delta}(u-k)\,(1-e^{-\alpha k}) + \phi_{r,\delta}(u-k)\,e^{-\alpha k} \\ &= \sum_{i=0}^{\infty} e^{-\alpha k}(1-e^{-\alpha k})^i\,\phi_{r,\delta}(0)^i\,\phi_{r,\delta}(u-k). \end{aligned} \tag{4.21}$$

To apply this expression we need an expression for $\phi_{r,\delta}(u)$ in terms of $\phi_{r,\delta}(0)$, and to obtain this we extend the approach in Dickson and Li (2010). We first note that

$$w(u,1,t) = \int_u^{\infty} w(0,y,1,t)\,dy = w(0,1,t)\,e^{-\alpha u} \tag{4.22}$$

and for $n = 2, 3, 4, \dots,$

$$\begin{aligned} & w(u,n,t) \\ = & \sum_{j=1}^{n-1}\int_0^t\int_0^u w(0,y,j,\tau)\,w(u-y,n-j,t-\tau)\,dy\,d\tau + \int_u^{\infty} w(0,y,n,t)\,dy \\ = & \sum_{j=1}^{n-1}\int_0^t\int_0^u w(0,j,\tau)\,\alpha e^{-\alpha y}\,w(u-y,n-j,t-\tau)\,dy\,d\tau + \int_u^{\infty} w(0,n,t)\,\alpha e^{-\alpha y}\,dy. \end{aligned} \tag{4.23}$$

Define

$$\tilde{\phi}_{r,\delta}(s) = \int_0^{\infty} e^{-su}\phi_{r,\delta}(u)\,du = \sum_{n=1}^{\infty} r^n \int_0^{\infty}\int_0^{\infty} e^{-su-\delta t} w(u,n,t)\,dt\,du = \sum_{n=1}^{\infty} r^n\,\tilde{w}_\delta(s,n).$$

Taking the bivariate Laplace transform of formula (4.22) gives

$$\tilde{w}_\delta(s,1) = w_\delta(0,1)\,\frac{1}{\alpha+s}$$

and the Laplace transform of formula (4.23) with respect to t is

$$w_\delta(u,n) = \sum_{j=1}^{n-1} w_\delta(0,j)\int_0^u \alpha e^{-\alpha y}\,w_\delta(u-y,n-j)\,dy + w_\delta(0,n)\,e^{-\alpha u}. \tag{4.24}$$

Now take the Laplace transform of formula (4.24) with respect to u, which results in

$$\tilde{w}_\delta(s,n) = \sum_{j=1}^{n-1} w_\delta(0,j)\,\tilde{w}_\delta(s,n-j)\,\frac{\alpha}{\alpha+s} + w_\delta(0,n)\,\frac{1}{\alpha+s}. \tag{4.25}$$

Multiplying formula (4.25) by r^n and summing over n, we have

$$\sum_{n=2}^{\infty} r^n\,\tilde{w}_\delta(s,n) = \sum_{n=2}^{\infty} r^n \sum_{j=1}^{n-1} w_\delta(0,j)\,\tilde{w}_\delta(s,n-j)\,\frac{\alpha}{\alpha+s} + \sum_{n=2}^{\infty} r^n w_\delta(0,n)\,\frac{1}{\alpha+s}. \tag{4.26}$$

Adding $r\tilde{w}_\delta(s,1)$ to both sides of (4.26) gives

$$\sum_{n=1}^{\infty} r^n \tilde{w}_\delta(s,n) = \sum_{n=2}^{\infty} r^n \sum_{j=1}^{n-1} w_\delta(0,j)\,\tilde{w}_\delta(s,n-j)\,\frac{\alpha}{\alpha+s} + \sum_{n=1}^{\infty} r^n\, w_\delta(0,n)\,\frac{1}{\alpha+s}$$

which can be written as

$$\begin{aligned}\tilde{\phi}_{r,\delta}(s) &= \phi_{r,\delta}(0)\,\tilde{\phi}_{r,\delta}(s)\,\frac{\alpha}{\alpha+s} + \phi_{r,\delta}(0)\,\frac{1}{\alpha+s} \\ &= \frac{\phi_{r,\delta}(0)\,\frac{1}{\alpha+s}}{1-\phi_{r,\delta}(0)\,\frac{\alpha}{\alpha+s}} = \frac{1}{\alpha}\sum_{n=1}^{\infty} \phi_{r,\delta}(0)^n \left(\frac{\alpha}{\alpha+s}\right)^n.\end{aligned}$$

Inverting $\tilde{\phi}_{r,\delta}(s)$ with respect to s yields

$$\phi_{r,\delta}(u) = \sum_{n=1}^{\infty} \phi_{r,\delta}(0)^n \frac{(\alpha u)^{n-1}\,e^{-\alpha u}}{(n-1)!}$$

and inserting this in equation (4.21) we get

$$\phi_{k,r,\delta}(u) = \sum_{i=0}^{\infty} e^{-\alpha u}(1-e^{-\alpha k})^i \sum_{n=1}^{\infty} \phi_{r,\delta}(0)^{i+n} \frac{[\alpha(u-k)]^{n-1}}{(n-1)!},$$

which by (4.18) inverts with respect to δ to

$$\sum_{i=0}^{\infty} e^{-\alpha u}(1-e^{-\alpha k})^i \sum_{n=1}^{\infty} (\lambda r)^{i+n} e^{-(\lambda+\alpha c)t}\, t^{i+n-1}\,(i+n) \sum_{j=0}^{\infty} \frac{(\alpha c\lambda r t^2)^j}{j!\,(j+n+i)!}\frac{[\alpha(u-k)]^{n-1}}{(n-1)!}.$$

Integrating with respect to t we find the probability generating function of $N_{T_{u,k}}$ as

$$\begin{aligned}\tilde{p}_{k,r}(u) &= \sum_{i=0}^{\infty} e^{-\alpha u}(1-e^{-\alpha k})^i \sum_{n=0}^{\infty} (\lambda r)^{i+n+1} \sum_{j=0}^{\infty} \frac{(\alpha c\lambda r)^j\,(i+n+1)}{j!\,(j+n+i+1)!}\frac{[\alpha(u-k)]^n}{n!} \\ &\quad \times \frac{(2j+i+n)!}{(\lambda+\alpha c)^{2j+i+n+1}}.\end{aligned}$$

We can then apply the identity

$$\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} t_{i,n,j} = \sum_{i=0}^{\infty}\sum_{n=0}^{i}\sum_{j=0}^{n} t_{j,n-j,i-n},$$

(see, for example, Graham et al., 1994, page 355) giving

$$\begin{aligned}\tilde{p}_{k,r}(u) &= \sum_{i=0}^{\infty} r^{i+1} e^{-\alpha u} \sum_{n=0}^{i}\sum_{j=0}^{n} (1-e^{-\alpha k})^j \frac{(\alpha(u-k))^{n-j}}{(n-j)!}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^{i-n}\left(\frac{\lambda}{\lambda+\alpha c}\right)^{i+1} \\ &\quad \times \frac{(n+1)\,(2i-n)!}{(i-n)!\,(i+1)!}.\end{aligned}$$

It follows that $p_k(u,1) = \lambda e^{-\alpha u}/(\lambda+\alpha c)$, and for $n = 2, 3, 4, \ldots,$

$$\begin{aligned}p_k(u,n) &= e^{-\alpha u} \sum_{i=0}^{n-1}\sum_{j=0}^{i} (1-e^{-\alpha k})^j \frac{(\alpha(u-k))^{i-j}}{(i-j)!}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^{n-i-1}\left(\frac{\lambda}{\lambda+\alpha c}\right)^{n} \\ &\quad \times \frac{(i+1)(2n-i-2)!}{n!(n-i-1)!} \qquad (4.27)\end{aligned}$$

which also holds for $n = 1$. □

We remark that setting $u = k$ in formula (4.27) yields formula (4.9). Further, setting $k = 0$ in formula (4.20) gives $p(u,n)$, the probability function of the number of claims until ruin in the classical model, as

$$p(u,n) = e^{-\alpha u} \sum_{i=0}^{n-1} \frac{(\alpha u)^i}{i!}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^{n-i-1}\left(\frac{\lambda}{\lambda+\alpha c}\right)^{n} \frac{(i+1)\,(2n-i-2)!}{n!\,(n-i-1)!}. \qquad (4.28)$$

We remark that this formula is new, and can also be obtained by manipulation of formula (23) of Landriault et al. (2011).

In this section, we have illustrated the application of the Gerber-Shiu function by finding the probability function of the number of claims until ruin through probabilistic reasoning. We saw that this approach enables us to find an explicit formula for $p_k(u,n)$, while our method in Chapter 3 gave rise to formula (3.56), which is a recursive expression.

In Chapter 3 we were not able to find a simple formula for $p_k(u,n)$ for claim amounts with an Erlang(2) distribution. In the next section, we show that with the approach explained in this chapter, we can find an explicit result for the probability function of the number of claims until ruin when the individual claim amount distribution is Erlang(2).

4.5.2 Erlang(2) claims

We now consider the situation when $\bar{F}(x) = e^{-\alpha x}(1+\alpha x)$, where $x \geq 0$ with $\alpha > 0$ and present results for $p_k(u,n)$.

Result 4.3. When the initial surplus is k, for $n = 1, 2, 3, \ldots,$

$$\begin{aligned} p_k(k,n) &= \sum_{m=0}^{n-1} \sum_{j=0}^{m} \binom{m}{j} a_k^j b_k^{m-j} \left(\frac{\lambda}{\lambda+\alpha c}\right)^n \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n-m-j-2} \\ &\quad \times \frac{(3n-m-j-3)!}{(2n-j-1)!\,(n-m-1)!} \Bigg((1-a_k)\,(2m-j+1) \\ &\quad + (1-b_k) \left(\frac{\alpha c}{\lambda+\alpha c}\right) \frac{(3n-m-j-2)\,(2m-j+2)}{(2n-j)} \Bigg). \end{aligned} \tag{4.29}$$

Derivation. Our proof is based on the technique in Dickson (2008). First we note that if we set $k = 0$ in formula (3.3) we obtain $\phi_{r,\rho}(0) = \frac{\lambda r}{c} \int_0^\infty \int_u^\infty e^{-\rho u} f(x) \omega(u, x-u)\, dx\, du$, which is the same as formula (2.26) of Gerber and Shiu (1998) when $r = 1$. Thus, with $\omega(x,y) = e^{-sy}$, formula (2) of Dickson (2008) can be generalised to

$$\sum_{n=1}^{\infty} r^n \int_0^\infty \int_0^\infty e^{-sy-\delta t}\, w(0,y,n,t)\, dy\, dt = \frac{\lambda r}{c} \int_0^\infty e^{-\rho t} \int_t^\infty e^{-s(y-t)}\, f(y)\, dy\, dt. \tag{4.30}$$

Inserting $f(x) = \alpha^2 x e^{-\alpha x}$ on the right-hand side of equation (4.30) we get

$$\frac{\lambda r}{c} \int_0^\infty e^{-\rho t} \int_t^\infty e^{-s(y-t)}\, \alpha^2 y e^{-\alpha y}\, dy\, dt = \frac{\lambda r}{c} \left(\frac{1}{\alpha+\rho} \frac{\alpha^2}{(\alpha+s)^2} + \frac{\alpha}{(\alpha+\rho)^2} \frac{\alpha}{\alpha+s} \right). \tag{4.31}$$

Inverting (4.30) with respect to s yields

$$\phi_{r,\delta}(0,y) = \sum_{n=1}^{\infty} r^n\, w_\delta(0,y,n) = \gamma_{r,\delta}(0)\, \alpha^2 y e^{-\alpha y} + \xi_{r,\delta}(0)\, \alpha e^{-\alpha y} \tag{4.32}$$

where

$$\gamma_{r,\delta}(0) \equiv \int_0^\infty e^{-\delta t}\, \gamma_r(0,t)\, dt = \frac{\lambda r}{c} \frac{1}{\alpha+\rho}$$

and

$$\xi_{r,\delta}(0) \equiv \int_0^\infty e^{-\delta t}\, \xi_r(0,t)\, dt = \frac{\lambda r}{c} \frac{\alpha}{(\alpha+\rho)^2}.$$

Further, from Cheung et al. (2008) we know that if $f(x) = \sum_{i=1}^{n} q_i \frac{\beta^i x^{i-1} \exp\{-\beta x\}}{\Gamma(i)}$ where $\sum_{i=1}^{n} q_i = 1$ and each $q_i \geq 0$ then

$$w(u, y, t) = \sum_{i=1}^{n} \eta_i(u, t) \sum_{k=n+1-i}^{n} q_k \frac{\beta^{k-n+i} y^{k-n+i-1} \exp\{-\beta y\}}{\Gamma(k-n+i)}.$$

Therefore, for $n = 2$ and $q_2 = 1$ we have the following identity for $u > 0$:

$$\phi_{r,\delta}(u, y) = \sum_{n=1}^{\infty} r^n \, w_\delta(u, y, n) = \gamma_{r,\delta}(u) \, \alpha^2 y e^{-\alpha y} + \xi_{r,\delta}(u) \, \alpha e^{-\alpha y}. \tag{4.33}$$

Inserting (4.32) in formula (4.8) yields

$$\begin{aligned} \phi_{k,r,\delta}(k) &= \Big(\gamma_{r,\delta}(0)(1-a_k) + \xi_{r,\delta}(0)(1-b_k)\Big) \sum_{n=0}^{\infty} \Big(a_k \gamma_{r,\delta}(0) + b_k \xi_{r,\delta}(0)\Big)^n \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{n}{j} a_k^j \, \gamma_{r,\delta}(0)^j \, b_k^{n-j} \, \xi_{r,\delta}(0)^{n-j} \Big(\gamma_{r,\delta}(0)(1-a_k) + \xi_{r,\delta}(0)(1-b_k)\Big) \end{aligned} \tag{4.34}$$

where $a_k = 1 - e^{-\alpha k} - \alpha k e^{-\alpha k}$ and $b_k = 1 - e^{-\alpha k}$. We consider the term

$$\gamma_{r,\delta}(0)^{j+1} \, \xi_{r,\delta}(0)^{n-j} = \frac{(\lambda r)^{n+1}}{c^{\,n+1}} \frac{\alpha^{n-j}}{(\alpha + \rho)^{2n-j+1}},$$

which is the Laplace transform with transform parameter ρ of

$$\left(\frac{\lambda r}{c}\right)^{n+1} \frac{(\alpha t)^{n-j} \, t^n \, e^{-\alpha t}}{(2n-j)!}$$

and by formula (44) of Landriault et al. (2011) is the Laplace transform with transform parameter δ of

$$\left(\frac{\lambda r}{c}\right)^{n+1} e^{-(\lambda + \alpha c)t} \sum_{m=0}^{\infty} \frac{(\lambda r t)^m \, (\alpha c t)^{n-j+2m} \, t^n \, c^{n+1} \, (2n-j+1)}{m! \, (2n-2m-j+1)!}. \tag{4.35}$$

Equation (4.35) generalises formula (10) of Dickson (2008). Similarly,

$$\gamma_{r,\delta}(0)^j \, \xi_{r,\delta}(0)^{n-j+1} = \frac{(\lambda r)^{n+1}}{c^{\,n+1}} \frac{\alpha^{n-j+1}}{(\alpha + \rho)^{2n-j+2}}$$

is the Laplace transform with transform parameter ρ of

$$\left(\frac{\lambda r}{c}\right)^{n+1} \frac{(\alpha t)^{n-j+1} \, t^n \, e^{-\alpha t}}{(2n-j+1)!}$$

and with the same method is the Laplace transform with transform parameter δ of

$$\left(\frac{\lambda r}{c}\right)^{n+1} e^{-(\lambda+\alpha c)t} \sum_{m=0}^{\infty} \frac{(\lambda r t)^m\,(\alpha c t)^{n-j+2m+1}\, t^n\, c^{n+1}\,(2n-j+2)}{m!\,(2n+2m-j+2)!} \tag{4.36}$$

which generalises formula (13) of Dickson (2008). Indeed, it covers the case when $r = 1$. Substituting equations (4.35) and (4.36) in (4.34) results in

$$\begin{aligned} \phi_{k,r}(k,t) &= e^{-(\lambda+\alpha c)t} \sum_{n=0}^{\infty}\sum_{j=0}^{n} \binom{n}{j} a_k^j b_k^{n-j} \frac{(\lambda r)^{n+1}(\alpha c t)^{n-j} t^n}{(2n-j)!} \sum_{m=0}^{\infty} \frac{(r\alpha^2 c^2 \lambda t^3)^m (2n-j+1)!}{m!\,(2n+2m-j+1)!} \\ &\quad \left((1-a_k) + \frac{(1-b_k)\alpha c t}{(2n-j+1)} \frac{(2n-j+2)}{(2n+2m-j+2)}\right). \end{aligned} \tag{4.37}$$

Integrating formula (4.37) over t yields

$$\begin{aligned} \tilde{p}_{k,r}(k) &= \sum_{n=1}^{\infty}\sum_{j=0}^{n-1} \binom{n-1}{j} a_k^j b_k^{n-1-j} \sum_{m=0}^{\infty} \left(\frac{\lambda r}{\lambda+\alpha c}\right)^{n+m} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2m+n-j-1} \\ &\quad \times \frac{(2n+3m-j-2)!}{m!\,(2n+2m-j-1)!} \Bigg((1-a_k)(2n-j-1) \\ &\quad + \frac{(1-b_k)\alpha c}{\lambda+\alpha c} \frac{(2n+3m-j-1)\,(2n-j)}{(2n+2m-j)}\Bigg). \end{aligned}$$

To find $p_k(k,n)$ we proceed as follows. Changing the order of the first two sums we get

$$\begin{aligned} \tilde{p}_{k,r}(k) &= \sum_{j=0}^{\infty}\sum_{n=j+1}^{\infty} \binom{n-1}{j} a_k^j b_k^{n-1-j} \sum_{m=0}^{\infty} \left(\frac{\lambda r}{\lambda+\alpha c}\right)^{n+m} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2m+n-j-1} \\ &\quad \times \frac{(2n+3m-j-2)!}{m!\,(2n+2m-j-1)!} \Bigg((1-a_k)(2n-j-1) \\ &\quad + \frac{(1-b_k)\alpha c}{\lambda+\alpha c} \frac{(2n+3m-j-1)\,(2n-j)}{(2n+2m-j)}\Bigg). \end{aligned}$$

Letting $i = n - j - 1$ we have

$$\begin{aligned} \tilde{p}_{k,r}(k) &= (1-a_k) \sum_{j=0}^{\infty}\sum_{n=0}^{\infty} \binom{n+j}{j} a_k^j b_k^{n} \sum_{m=0}^{\infty} \left(\frac{\lambda r}{\lambda+\alpha c}\right)^{n+m+j+1} \left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2m+n} \\ &\quad \times \frac{(2n+3m+j)!}{m!\,(2n+2m+j+1)!} \Bigg((1-a_k)(2n+j+1) \\ &\quad + \frac{(1-b_k)\alpha c}{\lambda+\alpha c} \frac{(2n+3m+j+1)\,(2n+j+2)}{(2n+2m+j+2)}\Bigg). \end{aligned}$$

Then, we apply the identity

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} t_{j,n,m} = \sum_{j=1}^{\infty}\sum_{n=0}^{j-1}\sum_{m=0}^{n} t_{m,n-m,j-n-1},$$

giving

$$\begin{aligned}\tilde{p}_{k,r}(k) &= \sum_{j=1}^{\infty}\sum_{n=0}^{j-1}\sum_{m=0}^{n}\binom{n}{m}a_k^m b_k^{n-m}\left(\frac{\lambda r}{\lambda+\alpha c}\right)^j\left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2j-n-m-2}\\ &\quad\times\frac{(3j-n-m-3)!}{(2j-m-1)!\,(j-n-1)!}\Bigg((1-a_k)(2n-m+1)\\ &\quad+\frac{(1-b_k)\alpha c}{\lambda+\alpha c}\frac{(3j-n-m-2)\,(2n-m+2)}{(2j-m)}\Bigg)\end{aligned}$$

which can be written as

$$\begin{aligned}\tilde{p}_{k,r}(k) &= \sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\sum_{j=0}^{m}\binom{m}{j}a_k^j b_k^{m-j}\left(\frac{\lambda r}{\lambda+\alpha c}\right)^n\left(\frac{\alpha c}{\lambda+\alpha c}\right)^{2n-m-j-2}\\ &\quad\times\frac{(3n-m-j-3)!}{(2n-j-1)!\,(n-m-1)!}\Bigg((1-a_k)(2m-j+1)\\ &\quad+\frac{(1-b_k)\alpha c}{\lambda+\alpha c}\frac{(3n-m-j-2)\,(2m-j+2)}{(2n-j)}\Bigg).\end{aligned}$$

This is the probability generating function of the number of claims until ruin from which formula (4.29) follows. □

Formula (4.29) is in agreement with formula (3.65) in Chapter 3.

Result 4.4. When $u > k$, with the convention that $\sum_{i=a}^{b} = 0$, if $b < a$, for $j = 1, 2, 3, \ldots,$

$$\begin{aligned}&p_k(u,j)\\ =\ & e^{-\alpha(u-k)}\sum_{n=0}^{j-2}\sum_{i=0}^{n}\sum_{m=0}^{i}\sum_{q=0}^{m}\binom{m}{q}\binom{n-m}{i-m}a_k^q b_k^{m-q}\frac{(\alpha(u-k))^{n+i-2m}}{(j-n-2)!}\\ &\times\left(\frac{\lambda(\alpha c)^2}{(\lambda+\alpha c)^3}\right)^j\left(\frac{\alpha c}{\lambda+\alpha c}\right)^{m-n-i-q-4}\\ &\Bigg[(1-a_k)\left(\frac{a_k}{(n+i-2m)!}+b_k\frac{\alpha(u-k)}{(n+i-2m+1)!}\right)\end{aligned}$$

$$
\times \frac{(2n+m-i-q+2)(3j+m-n-i-q-5)!}{(2j+m-i-q-2)!}
$$

$$
+\left(\frac{(a_k+b_k-2a_kb_k)}{(n+i-2m)!}+(1-b_k)b_k\frac{\alpha(u-k)}{(n+i-2m+1)!}\right)\left(\frac{\alpha c}{\lambda+\alpha c}\right)
$$

$$
\times \frac{(2n+m-i-q+3)(3j+m-n-i-q-4)!}{(2j+m-i-q-1)!}
$$

$$
\left.+\frac{(1-b_k)b_k}{(n-2m+i)!}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^2\frac{(2n+m-i-q+4)(3j+m-n-i-q-3)!}{(2j+m-i-q)!}\right]
$$

$$
+e^{-\alpha(u-k)}\sum_{n=0}^{j-1}\sum_{q=0}^{n}\binom{n}{q}\frac{(u-k)^{n+q}}{(n+q)!(j-n-1)!}\left(\frac{\lambda(\alpha c)^2}{(\lambda+\alpha c)^3}\right)^j\left(\frac{\lambda+\alpha c}{c}\right)^{n+q+2}
$$

$$
\left[\frac{(1-a_k)}{\alpha^2}\frac{(2n-q+1)(3j-n-q-3)!}{(2j-q-1)!}+\frac{(1-b_k)}{\alpha}\right.
$$

$$
\times\left(\frac{c}{\lambda+\alpha c}\right)\frac{(2n-q+2)(3j-n-q-2)!}{(2j-q)!}+(1-b_k)\frac{(u-k)}{\alpha(n+q+1)}
$$

$$
\left.\times\frac{(2n-q+1)(3j-n-q-3)!}{(2j-q-1)!}\right]. \tag{4.38}
$$

Derivation. Inserting expression (4.33) in (4.7) we obtain

$$
\begin{aligned}
\phi_{k,r,\delta}(u) &= \phi_{k,r,\delta}(k)\Big(a_k\,\gamma_{r,\delta}(u-k)+b_k\,\xi_{r,\delta}(u-k)\Big) \\
&\quad +(1-a_k)\gamma_{r,\delta}(u-k)+(1-b_k)\xi_{r,\delta}(u-k) \\
&= \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}a_k^j\,b_k^{n-j}\,\gamma_{r,\delta}(0)^j\,\xi_{r,\delta}(0)^{n-j}\Big((1-a_k)a_k\,\gamma_{r,\delta}(0)\,\gamma_{r,\delta}(u-k) \\
&\quad +(1-b_k)a_k\,b_k\,\xi_{r,\delta}(0)\,\gamma_{r,\delta}(u-k)+(1-a_k)b_k\,\gamma_{r,\delta}(0)\,\xi_{r,\delta}(u-k) \\
&\quad +(1-b_k)b_k\,\xi_{r,\delta}(0)\xi_{r,\delta}(u-k)+(1-a_k)\gamma_{r,\delta}(u-k)+(1-b_k)\xi_{r,\delta}(u-k)\Big).
\end{aligned} \tag{4.39}
$$

To find the joint density of the time of ruin and the number of claims until ruin we need to invert expression (4.39) with respect to δ. For the Laplace transform inversion of $\gamma_{r,\delta}(0)^j\xi_{r,\delta}(0)^{n-j}$ we can apply techniques in Result 4.3. To invert $\gamma_{r,\delta}(u-k)$ and $\xi_{r,\delta}(u-k)$ we note that in the classical risk model the occurrence of ruin from initial surplus u with a deficit of y at the first claim, is equivalent to ruin occurring from initial surplus 0 at the first claim, with a deficit at ruin of $u+y$. Hence

$$
w(u,y,1,t)=w(0,u+y,1,t). \tag{4.40}
$$

For $n = 2, 3, 4, \ldots$, we can modify equation (1) of Dickson (2008) as

$$w(u, y, n, t) = \sum_{j=1}^{n-1} \int_0^t \int_0^u w(0, x, j, \tau)\, w(u - x, y, n - j, t - \tau)\, dx\, d\tau + w(0, u + y, n, t).$$

Taking the Laplace transform of $w(u, y, n, t)$ with respect to t gives, $w_\delta(u, y, 1) = w_\delta(0, u + y, 1)$ and for $n = 2, 3, 4, \ldots$,

$$w_\delta(u, y, n) = \sum_{j=1}^{n-1} \int_0^u w_\delta(0, x, j) w_\delta(u - x, y, n - j)\, dx + w_\delta(0, u + y, n). \qquad (4.41)$$

Multiplying both sides of (4.41) by r^n and summing over n, we get

$$\sum_{n=2}^{\infty} r^n w_\delta(u, y, n) = \sum_{n=2}^{\infty} r^n \sum_{j=1}^{n-1} \int_0^u w_\delta(0, x, j)\, w_\delta(u - x, y, n - j)\, dx + \sum_{n=2}^{\infty} r^n\, w_\delta(0, u + y, n). \qquad (4.42)$$

Adding $r w_\delta(u, y, 1)$ to both sides of (4.42) gives

$$\sum_{n=1}^{\infty} r^n w_\delta(u, y, n) = \sum_{n=2}^{\infty} r^n \sum_{j=1}^{n-1} \int_0^u w_\delta(0, x, j) w_\delta(u - x, y, n - j)\, dx + \sum_{n=1}^{\infty} r^n\, w_\delta(0, u + y, n)$$

which can be written as

$$\phi_{r,\delta}(u, y) = \int_0^u \phi_{r,\delta}(0, x)\, \phi_{r,\delta}(u - x, y)\, dx + \phi_{r,\delta}(0, u + y). \qquad (4.43)$$

Inserting (4.32) and (4.33) on both sides of (4.43) yields

$$\begin{aligned} & \gamma_{r,\delta}(u)\alpha^2 y e^{-\alpha y} + \xi_{r,\delta}(u)\alpha e^{-\alpha y} \\ = & \int_0^u \Big(\gamma_{r,\delta}(0)\alpha^2 x e^{-\alpha x} + \xi_{r,\delta}(0)\alpha e^{-\alpha x}\Big)\Big(\gamma_{r,\delta}(u - x)\alpha^2 y e^{-\alpha y} + \xi_{r,\delta}(u - x)\alpha e^{-\alpha y}\Big) dx \\ & + \gamma_{r,\delta}(0)\, \alpha^2 (u + y) e^{-\alpha(u+y)} + \xi_{r,\delta}(0)\, \alpha e^{-\alpha(u+y)} \end{aligned}$$

and equating the coefficients of $\alpha^2 y e^{-\alpha y}$ and $\alpha e^{-\alpha y}$ on both sides, we find

$$\gamma_{r,\delta}(u) = \int_0^u \Big(\gamma_{r,\delta}(0)\, \alpha^2 x e^{-\alpha x} + \xi_{r,\delta}(0)\, \alpha e^{-\alpha x}\Big) \gamma_{r,\delta}(u - x)\, dx + \gamma_{r,\delta}(0) e^{-\alpha u} \qquad (4.44)$$

and

$$\begin{aligned} \xi_{r,\delta}(u) \;\; = \;\; & \int_0^u \Big(\gamma_{r,\delta}(0)\, \alpha^2 x e^{-\alpha x} + \xi_{r,\delta}(0)\, \alpha e^{-\alpha x}\Big) \xi_{r,\delta}(u - x)\, dx + \xi_{r,\delta}(0) e^{-\alpha u} \\ & + \gamma_{r,\delta}(0)\, \alpha u e^{-\alpha u}. \end{aligned} \qquad (4.45)$$

Define

$$\tilde{\gamma}_{r,\delta}(s) = \int_0^\infty \int_0^\infty e^{-su-\delta t}\,\gamma_r(u,t)\,dt\,du$$

and

$$\tilde{\xi}_{r,\delta}(s) = \int_0^\infty \int_0^\infty e^{-su-\delta t}\,\xi_r(u,t)\,dt\,du.$$

Taking the Laplace transform of (4.44) with respect to u gives

$$\tilde{\gamma}_{r,\delta}(s) = \left(\gamma_{r,\delta}(0)\frac{\alpha^2}{(\alpha+s)^2} + \xi_{r,\delta}(0)\frac{\alpha}{\alpha+s}\right)\tilde{\gamma}_{r,\delta}(s) + \gamma_{r,\delta}(0)\frac{1}{\alpha+s}.$$

Therefore, we have

$$\begin{aligned}
\tilde{\gamma}_{r,\delta}(s) &= \frac{\gamma_{r,\delta}(0)\frac{1}{\alpha+s}}{1-\gamma_{r,\delta}(0)\frac{\alpha^2}{(\alpha+s)^2}-\xi_{r,\delta}(0)\frac{\alpha}{\alpha+s}} \\
&= \gamma_{r,\delta}(0)\frac{1}{\alpha+s}\sum_{n=0}^{\infty}\left(\gamma_{r,\delta}(0)\frac{\alpha^2}{(\alpha+s)^2}+\xi_{r,\delta}(0)\frac{\alpha}{\alpha+s}\right)^n \\
&= \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\gamma_{r,\delta}(0)^{i+1}\,\xi_{r,\delta}(0)^{n-i}\left(\frac{\alpha}{\alpha+s}\right)^{n+i+1}.
\end{aligned}$$

Inverting $\tilde{\gamma}_{r,\delta}(s)$ with respect to s yields

$$\gamma_{r,\delta}(u) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\gamma_{r,\delta}(0)^{i+1}\,\xi_{r,\delta}(0)^{n-i}\,\frac{\alpha^{n+i+1}\,u^{n+i}\,e^{-\alpha u}}{(n+i)!} \tag{4.46}$$

and inverting $\gamma_{r,\delta}(u)$ with respect to δ gives

$$\begin{aligned}
\gamma_r(u,t) &= ce^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\left(\frac{\lambda r}{c}\right)^{n+1}\alpha^{2n}u^{n+i}(ct)^{2n-i}\frac{(2n-i+1)}{(n+i)!} \\
&\quad\times\sum_{q=0}^{\infty}\frac{(r\lambda\alpha^2c^2t^3)^q}{q!\,(2n+2q-i+1)!}.
\end{aligned} \tag{4.47}$$

Similarly, we can show that

$$\tilde{\xi}_{r,\delta}(s) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\gamma_{r,\delta}(0)^{i}\xi_{r,\delta}(0)^{n-i}\left(\frac{\alpha}{\alpha+s}\right)^{n+i+1}\left(\xi_{r,\delta}(0)+\gamma_{r,\delta}(0)\frac{\alpha}{\alpha+s}\right).$$

Inverting $\tilde{\xi}_{r,\delta}(s)$ with respect to s is

$$\xi_{r,\delta}(u) = \frac{1}{\alpha}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\gamma_{r,\delta}(0)^{i}\xi_{r,\delta}(0)^{n-i}\frac{\alpha^{n+i+1}u^{n+i}e^{-\alpha u}}{(n+i)!}\left(\xi_{r,\delta}(0)+\gamma_{r,\delta}(0)\frac{\alpha u}{(n+i+1)}\right) \tag{4.48}$$

and inverting $\xi_{r,\delta}(u)$ with respect to δ gives

$$\begin{aligned}\xi_r(u,t) &= ce^{-\alpha u-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\left(\frac{\lambda r}{c}\right)^{n+1}\alpha^{2n+1}u^{n+i}(ct)^{2n-i+1}\frac{1}{(n+i)!}\\ &\quad\times\sum_{q=0}^{\infty}\frac{(r\lambda\alpha^2c^2t^3)^q}{q!\,(2n+2q-i+1)!}\left(ct\frac{(2n-i+2)}{(2n+2q-i+2)}+u\frac{(2n-i+1)}{(n+i+1)}\right).\end{aligned} \tag{4.49}$$

Inserting (4.46) and (4.48) in equation (4.39) we get

$$\begin{aligned}&\phi_{k,r,\delta}(u)\\ =\ &\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}a_k^jb_k^{n-j}\gamma_{r,\delta}(0)^{i+j}\xi_{r,\delta}(0)^{n+m-i-j}\\ &\left(\frac{(1-a_k)}{\alpha}\,a_k\,\gamma_{r,\delta}(0)^2\,e_1(u-k)+\frac{(1-b_k)}{\alpha}\,a_k\,\gamma_{r,\delta}(0)\,\xi_{r,\delta}(0)\,e_1(u-k)\right.\\ &+\frac{(1-a_k)}{\alpha}\,b_k\,\gamma_{r,\delta}(0)\,\xi_{r,\delta}(0)\,e_1(u-k)+\frac{(1-a_k)}{\alpha}\,b_k\,\gamma_{r,\delta}(0)^2\,e_2(u-k)\\ &\left.+\frac{(1-b_k)}{\alpha}\,b_k\,\xi_{r,\delta}(0)^2\,e_1(u-k)+\frac{(1-b_k)}{\alpha}\,b_k\,\gamma_{r,\delta}(0)\,\xi_{r,\delta}(0)\,e_2(u-k)\right)\\ &+(1-a_k)\,\gamma_{r,\delta}(u-k)+(1-b_k)\,\xi_{r,\delta}(u-k)\end{aligned} \tag{4.50}$$

where e_1 and e_2 are the Erlang$(m+i+1,\alpha)$ and Erlang$(m+i+2,\alpha)$ probability density functions, respectively. Inverting $\phi_{k,r,\delta}(u)$ with respect to δ we find

$$\begin{aligned}&\sum_{n=1}^{\infty}r^nw_k(u,n,t)\\ =\ &e^{-(\lambda+\alpha c)t}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i}a_k^jb_k^{n-j}(\lambda rt)^{n+m+2}(\alpha ct)^{n+m-i-j}\\ &\times\sum_{q=0}^{\infty}\frac{(\lambda r\alpha^2c^2t^3)^q}{q!}\left(\frac{(1-a_k)}{\alpha}\frac{(2n+2m-i-j+2)}{t(2n+2m+2q-i-j+2)!}\Big[a_ke_1(u-k)+b_ke_2(u-k)\Big]\right.\\ &+c\frac{(2n+2m-i-j+3)}{(2n+2m+2q-i-j+3)!}\Big[(a_k+b_k-2a_kb_k)e_1(u-k)+(1-b_k)b_ke_2(u-k)\Big]\\ &\left.+c^2(1-b_k)\frac{b_k(\alpha t)(2n+2m-i-j+4)}{(2n+2m+2q-i-j+4)!}e_1(u-k)\right)\\ &+(1-a_k)\gamma_r(u-k,t)+(1-b_k)\xi_r(u-k,t),\end{aligned} \tag{4.51}$$

from which we can find the joint density of the time of ruin and the number of claims until ruin. Setting $r = 1$, in equation (4.51) we obtain the density of the time of ruin in the case of claim sizes with Erlang$(2,\alpha)$ distribution as derived by Nie et al. (2015). For computational purposes, (4.51) can be written in terms of Hypergeometric functions as (see, for example, Graham et al., 1994)

$$\begin{aligned}
& w_k(u,t) \\
= \; & \frac{(1-a_k)}{\alpha} e^{-(\lambda+\alpha c)t} \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i} a_k^j b_k^{n-j} \frac{(\lambda r t)^{n+m+2}(\alpha c t)^{n+m-i-j}}{t(2m+2n-i-j+1)!} \\
& \times \Big[a_k e_1(u-k) + b_k e_2(u-k)\Big] \\
& \times {}_0F_2\Big(\frac{2m+2n-i-j+3}{2}, \frac{2m+2n-i-j+4}{2}; \frac{\lambda r \alpha^2 c^2 t^3}{4}\Big) \\
& + \frac{1}{\alpha} e^{-(\lambda+\alpha c)t} \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i} a_k^j b_k^{n-j} \frac{(\lambda r t)^{n+m+2}(\alpha c t)^{n+m-i-j+1}}{t(2n+2m-i-j+2)!} \\
& \times \Big[(a_k + b_k - 2a_k b_k) e_1(u-k) + (1-b_k) b_k e_2(u-k)\Big] \\
& \times {}_0F_2\Big(\frac{2m+2n-i-j+4}{2}, \frac{2m+2n-i-j+5}{2}; \frac{\lambda r \alpha^2 c^2 t^3}{4}\Big) \\
& + \frac{(1-b_k)}{\alpha} e^{-(\lambda+\alpha c)t} \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i} a_k^j b_k^{n-j+1} \frac{(\lambda r t)^{n+m+2}(\alpha c t)^{m+n-i-j+2}}{t(2n+2m-i-j+3)!} \\
& \times {}_0F_2\Big(\frac{2m+2n-i-j+5}{2}, \frac{2m+2n-i-j+6}{2}; \frac{\lambda r \alpha^2 c^2 t^3}{4}\Big) e_1(u-k) \\
& + (1-a_k)\gamma_r(u-k,t) + (1-b_k)\xi_r(u-k,t).
\end{aligned}$$

Next, we find the probability generating function of the number of claims until ruin. For this we integrate formula (4.51) over t, giving

$$\begin{aligned}
& \tilde{p}_{k,r}(u) \\
= \; & e^{-\alpha(u-k)} \sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}\sum_{m=0}^{\infty}\sum_{i=0}^{m}\binom{m}{i} a_k^j b_k^{n-j} (\alpha(u-k))^{m+i} \left(\frac{\lambda r \alpha c}{(\lambda+\alpha c)^2}\right)^{n+m} \left(\frac{\lambda+\alpha c}{\alpha c}\right)^{i+j} \\
& \sum_{q=0}^{\infty} \left(\frac{\lambda r \alpha^2 c^2}{(\lambda+\alpha c)^3}\right)^{q} \left(\frac{\lambda r}{\lambda+\alpha c}\right)^{2} \Bigg[(1-a_k)\left(\frac{a_k}{(m+i)!} + b_k \frac{\alpha(u-k)}{(m+i+1)!}\right) \\
& \times \frac{(2m+2n+3q-i-j+1)!\,(2m+2n-i-j+2)}{q!\,(2q+2m+2n-i-j+2)!} \\
& + \left(\frac{(a_k+b_k-2a_k b_k)}{(m+i)!} + (1-b_k) b_k \frac{\alpha(u-k)}{(m+i+1)!}\right)\left(\frac{\alpha c}{\lambda+\alpha c}\right)
\end{aligned}$$

$$\times \frac{(2m+2n+3q-i-j+2)!\,(2m+2n-i-j+3)}{q!\,(2q+2m+2n-i-j+3)!}$$
$$\left.+\frac{(1-b_k)b_k}{(m+i)!}\left(\frac{\alpha c}{\lambda+\alpha c}\right)^2 \frac{(2m+2n+3q-i-j+3)!\,(2m+2n-i-j+4)}{q!\,(2q+2m+2n-i-j+4)!}\right]$$
$$+e^{-\alpha(u-k)}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\frac{\alpha^{2n}(u-k)^{n+i}}{(n+i)!}\left(\frac{\lambda r}{\lambda+\alpha c}\right)^{n+1}\left(\frac{c}{\lambda+\alpha c}\right)^{n-i}\sum_{q=0}^{\infty}\left(\frac{\lambda r\alpha^2c^2}{(\lambda+\alpha c)^3}\right)^q$$
$$\left[(1-a_k)(2n-i+1)\frac{(2n+3q-i)!}{q!\,(2q+2n-i+1)!}+(1-b_k)\alpha(2n-i+2)\left(\frac{c}{\lambda+\alpha c}\right)\right.$$
$$\left.\times\frac{(2n+3q-i+1)!}{q!\,(2q+2n-i+2)!}+(1-b_k)\frac{\alpha(u-k)\,(2n-i+1)}{(n+i+1)}\frac{(2n+3q-i)!}{q!\,(2q+2n-i+1)!}\right].$$

Then, we change the order of the first and the second pair of summations in the first three terms and the order of the first two summations in the last three terms, giving

$$\tilde{p}_{k,r}(u)$$
$$= e^{-\alpha(u-k)}\sum_{j=0}^{\infty}\sum_{n=j}^{\infty}\binom{n}{j}\sum_{i=0}^{\infty}\sum_{m=i}^{\infty}\binom{m}{i}a_k^jb_k^{n-j}(\alpha(u-k))^{m+i}\left(\frac{\lambda r\alpha c}{(\lambda+\alpha c)^2}\right)^{n+m}\left(\frac{\lambda+\alpha c}{\alpha c}\right)^{i+j}$$
$$\times\sum_{q=0}^{\infty}\left(\frac{\lambda r\alpha^2c^2}{(\lambda+\alpha c)^3}\right)^q\left(\frac{\lambda r}{\lambda+\alpha c}\right)^2\left[(1-a_k)\left(\frac{a_k}{(m+i)!}+b_k\frac{\alpha(u-k)}{(m+i+1)!}\right)\right.$$
$$\times\frac{(2m+2n+3q-i-j+1)!(2m+2n-i-j+2)}{q!\,(2q+2m+2n-i-j+2)!}+\left(\frac{(a_k+b_k-2a_kb_k)}{(m+i)!}\right.$$
$$\left.+(1-b_k)b_k\frac{\alpha(u-k)}{(m+i+1)!}\right)\left(\frac{\lambda+\alpha c}{\alpha c}\right)$$
$$\times\frac{(2m+2n+3q-i-j+2)!(2m+2n-i-j+3)}{q!\,(2q+2m+2n-i-j+3)!}$$
$$\left.+\frac{(1-b_k)b_k}{(m+i)!}\left(\frac{\lambda+\alpha c}{\alpha c}\right)^2\frac{(2m+2n+3q-i-j+3)!(2m+2n-i-j+4)}{q!\,(2q+2m+2n-i-j+4)!}\right]$$
$$+e^{-\alpha(u-k)}\sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}\frac{\alpha^{2n}(u-k)^{n+i}}{(n+i)!}\left(\frac{\lambda r}{\lambda+\alpha c}\right)^{n+1}\left(\frac{c}{\lambda+\alpha c}\right)^{n-i}\sum_{q=0}^{\infty}\left(\frac{\lambda r\alpha^2c^2}{(\lambda+\alpha c)^3}\right)^q$$
$$\left[(1-a_k)(2n-i+1)\frac{(2n+3q-i)!}{q!\,(2q+2n-i+1)!}+(1-b_k)\alpha(2n-i+2)\left(\frac{c}{\lambda+\alpha c}\right)\right.$$
$$\left.\times\frac{(2n+3q-i+1)!}{q!\,(2q+2n-i+2)!}+(1-b_k)\frac{\alpha(u-k)\,(2n-i+1)}{(n+i+1)}\frac{(2n+3q-i)!}{q!\,(2q+2n-i+1)!}\right].$$

Letting $l=n-j$ and $p=m-i$ in the first three terms and $z=n-i$ in the last three terms, we have

$$
\begin{aligned}
&\tilde{p}_{k,r}(u)\\
= \; & e^{-\alpha(u-k)}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+j}{j}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\binom{m+i}{i}a_k^j b_k^n(\alpha(u-k))^{m+2i}\left(\frac{\lambda r\alpha c}{(\lambda+\alpha c)^2}\right)^{n+m}\\
&\times\sum_{q=0}^{\infty}\left(\frac{\lambda r\alpha^2c^2}{(\lambda+\alpha c)^3}\right)^q\left(\frac{\lambda r}{\lambda+\alpha c}\right)^{i+j}\left[(1-a_k)\left(\frac{a_k}{(m+2i)!}+b_k\frac{\alpha(u-k)}{(m+2i+1)!}\right)\right.\\
&\left(\frac{\lambda r}{\lambda+\alpha c}\right)^2\frac{(2m+2n+3q+i+j+1)!\,(2m+2n+i+j+2)}{q!\,(2q+2m+2n+i+j+2)!}\\
&+\left(\frac{(a_k+b_k-2a_kb_k)}{(m+2i)!}+(1-b_k)b_k\frac{\alpha(u-k)}{(m+2i+1)!}\right)\left(\frac{\lambda r\alpha c}{(\lambda+\alpha c)^2}\right)\left(\frac{\lambda r}{\lambda+\alpha c}\right)\\
&\times\frac{(2m+2n+3q+i+j+2)!\,(2m+2n+i+j+3)}{q!\,(2q+2m+2n+i+j+3)!}\\
&\left.+\frac{(1-b_k)b_k}{(m+2i)!}\left(\frac{\lambda r\alpha c}{(\lambda+\alpha c)^2}\right)^2\frac{(2m+2n+3q+i+j+3)!\,(2n+2m+i+j+4)}{q!\,(2q+2n+2m+i+j+4)!}\right]\\
&+e^{-\alpha(u-k)}\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+i}{i}\frac{(\alpha)^{2n+2i}(u-k)^{n+2i}}{(n+2i)!}\\
&\times\sum_{q=0}^{\infty}\left(\frac{\lambda rc}{(\lambda+\alpha c)^2}\right)^n\left(\frac{\lambda r\alpha^2c^2}{(\lambda+\alpha c)^3}\right)^q\left(\frac{\lambda r}{\lambda+\alpha c}\right)^i\frac{(2n+3q+i)!}{q!\,(2q+2n+i+1)!}\\
&\left[(1-a_k)(2n+i+1)\left(\frac{\lambda r}{\lambda+\alpha c}\right)+(1-b_k)\alpha(2n+i+2)\left(\frac{\lambda rc}{(\lambda+\alpha c)^2}\right)\right.\\
&\left.\times\frac{(2n+3q+i+1)}{(2q+2n+i+2)}+(1-b_k)\frac{\alpha(u-k)(2n+i+1)}{(n+2i+1)}\left(\frac{\lambda r}{\lambda+\alpha c}\right)\right].
\end{aligned}
$$

Applying the following identity to the first three terms

$$\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{\infty}t_{j,n,i,m,q}=\sum_{j=1}^{\infty}\sum_{n=0}^{j-1}\sum_{i=0}^{n}\sum_{m=0}^{i}\sum_{q=0}^{m}t_{q,m-q,i-m,n-i,j-n-1}$$

and

$$\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}t_{i,n,q}=\sum_{i=1}^{\infty}\sum_{n=0}^{i-1}\sum_{q=0}^{n}t_{q,n-q,i-n-1}$$

to the last three terms yields

$$
\begin{aligned}
&\tilde{p}_{k,r}(u) \\
= \;& e^{-\alpha(u-k)} \sum_{j=1}^{\infty} r^{j+1} \sum_{n=0}^{j-1} \sum_{i=0}^{n} \sum_{m=0}^{i} \sum_{q=0}^{m} \binom{m}{q} \binom{n-m}{i-m} a_k^q b_k^{m-q} \frac{(\alpha(u-k))^{n+i-2m}}{(j-n-1)!} \\
& \times \left(\frac{\lambda(\alpha c)^2}{(\lambda+\alpha c)^3} \right)^j \left(\frac{\alpha c}{\lambda+\alpha c} \right)^{m-n-i-q} \left(\frac{\lambda}{\lambda+\alpha c} \right) \\
& \Bigg[(1-a_k) \left(\frac{a_k}{(n+i-2m)!} + b_k \frac{\alpha(u-k)}{(n+i-2m+1)!} \right) \\
& \times \left(\frac{\lambda+\alpha c}{\alpha c} \right)^2 \frac{(2n+m-i-q+2)(3j+m-n-i-q-2)!}{(m-i+2j-q)!} \\
& + \left(\frac{(a_k+b_k-2a_kb_k)}{(n+i-2m)!} + (1-b_k)b_k \frac{\alpha(u-k)}{(n+i-2m+1)!} \right) \left(\frac{\lambda+\alpha c}{\alpha c} \right) \\
& \times \frac{(2n+m-i-q+3)(3j+m-n-i-q-1)!}{(m-i+2j-q+1)!} \\
& + \frac{(1-b_k)b_k}{(n-2m+i)!} \frac{(2n+m-i-q+4)(3j+m-n-i-q)!}{(m-i+2j-q+2)!} \Bigg] \\
& + e^{-\alpha(u-k)} \sum_{i=1}^{\infty} r^i \sum_{n=0}^{i-1} \sum_{q=0}^{n} \binom{n}{q} \frac{\alpha^{2i-2}(u-k)^{n+q}}{(n+q)!(i-n-1)!} \left(\frac{\lambda c^2}{(\lambda+\alpha c)^3} \right)^i \left(\frac{\lambda+\alpha c}{c} \right)^{n+q+1} \\
& \Bigg[(1-a_k) \left(\frac{\lambda+\alpha c}{c} \right) \frac{(2n-q+1)(3i-n-q-3)!}{(2i-q-1)!} + (1-b_k)\alpha \\
& \times \frac{(2n-q+2)(3i-n-q-2)!}{(2i-q)!} + (1-b_k) \frac{\alpha(u-k)}{(n+q+1)} \left(\frac{\lambda+\alpha c}{c} \right) \\
& \times \frac{(2n-q+1)(3i-n-q-3)!}{(2i-q-1)!} \Bigg]. \qquad (4.52)
\end{aligned}
$$

Formula (4.38) follows by equating coefficients of powers of r in (4.52). □

We conclude this section by remarking that with the approach in Chapter 3, we could only calculate values of $p_k(u,n)$ by numerical integration of formula (3.26), whereas using the method of this chapter we are able to find a computationally tractable result for the probability mass function of the number of claims until ruin for claim amounts with the Erlang(2) distribution.

4.6 Concluding remarks

In this chapter, we have applied the probabilistic argument of the Gerber-Shiu function and obtained explicit results for ruin-related quantities in infinite and finite time. Similar to Chapter 3, we have found general expressions for quantities such as $\psi_k(u)$ and $H_k(u, x, y)$. Our techniques in Sections 4.3 and 4.4 were only based on existing results in the classical risk model and we did not need to invert the Laplace transform of the Gerber-Shiu function. Therefore, the method that we adopted here is more straightforward than in the previous chapter. Unlike in Chapter 3, we could not find a general expression for $w_k(u, n, t)$. Our analysis in this chapter can be extended to other distributions provided that they satisfy a particular decomposition. However, as we have seen it is difficult to obtain neat explicit solutions even for individual claim amount with an Erlang(2) distribution, we do not pursue further solutions.

In general, as explained before, it can be concluded that the method in this chapter is useful in finding ruin-related quantities in infinite time. By contrast, the method in the previous chapter is more helpful for quantities in finite time as the formula for $w_k(u, n, t)$ in Chapter 3 can be applied to a wider range of distributions.

Chapter 5

A discrete time risk model with capital injections

5.1 Introduction

Many formulae in risk theory are only applicable to specific claim amount distributions. For example, in Chapter 3 we saw that the formula for $w_k(u,t)$ can be implemented on the condition that an explicit expression for $g(x,t)$ exists. In this chapter, we explain an alternative way of finding values of $w_k(u,t)$ through an approximation method.

One purpose of this chapter is to present a numerical algorithm that can provide approximations to the finite time probability of ruin in the classical risk model with capital injections. Another purpose is to study the mechanism of such a risk model. Nie et al. (2011) show how capital injections can be provided by reinsurance and how an insurer can both reduce its ultimate ruin probability by effecting such reinsurance and release capital to other parts of its business when individual claim amounts follow an exponential and a mixed exponential distribution. In this chapter, we build a numerical algorithm that enables us to carry out the same analysis in the case of individual claim amounts following a heavy-tailed distribution for which analytical expressions for ruin probabilities do not exist.

5.2 Notation and definitions

In Chapter 3 we have explained that the underlying process for our risk model with capital injections is the classical risk model. Since our purpose in this chapter is to establish a risk model that can be used to approximate the classical risk model with capital injections, we assume that the underlying process is the surplus process that can be applied to approximate the classical risk model. In particular, we consider the discrete time risk model that we have described in Section 1.2. Recall that for $n = 1, 2, 3, \dots$ we define the insurer's surplus by

$$U^d(n) = u + n - \sum_{i=1}^{n} Y_i$$

where $u = U^d(0)$ is the initial surplus and Y_i is the insurer's aggregate claim amount in the ith time interval. We assume that the insurer's premium income per unit time is 1, so that n is total premium income up to time n. Also, T_u^d, $\psi^d(u)$, $\psi^d(u,t)$ and $H^d(u,y)$ are the time of ruin, the probability of ruin in infinite time, the probability of ruin in finite time and the probability and severity of ruin function, respectively. See Section 1.2 for details.

We now introduce our model. We consider a modified surplus process such that on any occasion the surplus falls below a positive integer k, but stays above 0 the deficit is recovered either internally through coinsurance or externally through a reinsurance arrangement and there is no capital injection when $u = k$. We say that ruin occurs when the surplus falls from above k to either 0 or below 0. For our risk model we use the same notation as in Section 1.2, but with a subscript k. Therefore, for example, $T_{u,k}^d$ is the time of ruin from initial surplus $u = k, k+1, \dots$ for the process modified by capital injections. The ultimate probability of ruin is thus

$$\psi_k^d(u) = \Pr(T_{u,k}^d < \infty \mid U^d(0) = u).$$

Further, for an integer value of t, we define the finite time ruin probability as

$$\psi_k^d(u,t) = \Pr(T_{u,k}^d \le t \mid U^d(0) = u).$$

In the following we build recursive formulae for $\psi_k^d(u)$ and $\psi_k^d(u,t)$ and explain how we can use these formulae to compute their values.

5.3 The probability of ultimate ruin

In this section, we use our discrete time risk model with capital injections to approximate the continuous time classical risk model with capital injections. To do this, we first provide a recursive expression for $\psi_k^d(u)$ and then explain how we can apply this to approximate $\psi_k(u)$ in the classical risk model with capital injections.

Theorem 5.1. When the initial surplus is $u = k, k+1, \dots,$ we have

$$\begin{aligned} \psi_k^d(u+1) &= g(0)^{-1}\Bigg(\psi_k^d(u) - \sum_{x=1}^{u+1-k} g(x)\psi_k^d(u+1-x) \\ &\qquad - \left(G(u) - G(u+1-k)\right)\psi_k^d(k) - 1 + G(u)\Bigg). \end{aligned} \tag{5.1}$$

Proof. Considering the aggregate claim amount, Y_1 in the first time period we have three situations:

(i) if $Y_1 = x$, $x = 0, 1, 2, \dots, u+1-k$, then the surplus at time 1 is $u+1-x$,

(ii) if $Y_1 = x$, $x = u+2-k, \dots, u$, then the surplus at time 1 is k, because whenever the surplus falls to a positive level below k capital is injected to restore the surplus to k,

(iii) if $Y_1 > u$, then $U^d(1) \leq 0$ and so ruin occurs at time 1.

Hence, for $u = k,\ k+1,\ \dots,$

$$\psi_k^d(u) = \sum_{x=0}^{u+1-k} g(x)\psi_k^d(u+1-x) + \sum_{x=u+2-k}^{u} g(x)\psi_k^d(k) + \sum_{x=u+1}^{\infty} g(x)$$

which can be written recursively as (5.1). □

A recursive expression like (5.1) needs an initial value which is given by the following result.

Theorem 5.2. When the initial surplus is k we have

$$\psi_k^d(k) = \frac{\psi^d(0) - H^d(0,k)}{1 - H^d(0,k)} \tag{5.2}$$

where $H^d(0,k) = \sum_{x=0}^{k-1}(1 - G(x))$.

Proof. Using the argument of conditioning on the amount of the first drop of the surplus process below its initial level we have

$$\psi_k^d(k) = \sum_{x=1}^{k-1} h^d(0,x)\psi_k^d(k) + \sum_{x=k}^{\infty} h^d(0,x),$$

where $h^d(0,x)$ is defined in Section 1.2. After rearranging, formula (5.2) follows. □

We can calculate values of $\psi_k^d(u)$ numerically to approximate $\psi_k(u)$. Our numerical procedure is based on the algorithm introduced by Dickson and Waters (1991, 1992). The idea is to rescale the time unit, so that the premium income is always 1 within the unit of time. For this, we assume that the the number of claims follows a Poisson distribution with parameter $1/[(1+\theta)\beta]$, where β is an integer-valued scaling factor. Then, we observe the surplus of the company and the aggregate claim amount at times $0, 1/\beta, 2/\beta, \ldots$. We note that in this algorithm we need to change both monetary units and time units. We can apply the method of De Vylder and Goovaerts (1988) to discretise the individual claim amount distributions and then we use Panjer's (1981) recursion formula to find values of the aggregate claim probability function. In the following we consider three distributions: exponential, Pareto and lognormal.

Applying Result 1.1 and rescaling with parameter β, we can find the discretised version of a scaled exponential distribution with parameter α by

$$f(0) = 1 - \frac{\beta}{\alpha}(1 - e^{-\alpha/\beta}) \tag{5.3}$$

and for $x = 1, 2, 3, \ldots$, by

$$f(x) = \frac{\beta}{\alpha} e^{-\alpha(1+x)/\beta}(e^{\alpha/\beta} - 1)^2. \tag{5.4}$$

The discretised version of a scaled Pareto distribution with parameters a and b is given by

$$f(0) = 1 - \frac{(\beta b)^a}{1-a}\left((1+\beta b)^{1-a} - (\beta b)^{1-a}\right) \tag{5.5}$$

and for $x = 1, 2, 3, \ldots$, by

$$f(x) = \frac{(\beta b)^a}{1-a}\left(-(\beta b + x + 1)^{1-a} + 2(\beta b + x)^{1-a} - (\beta b + x - 1)^{1-a}\right). \tag{5.6}$$

The discretised version of a scaled lognormal distribution with parameters μ and σ is

$$\begin{aligned}\mathcal{F}(x) &= (x+1)\Phi\left(\frac{\log(x+1)-(\mu+\log\beta)}{\sigma}\right) - x\,\Phi\left(\frac{\log x-(\mu+\log\beta)}{\sigma}\right) \\ &\quad - \exp\{\mu+\log\beta+\sigma^2/2\}\left(\Phi(L_2)-\Phi(L_1)\right) \end{aligned} \tag{5.7}$$

where $L_1 = \frac{\log x-(\mu+\log\beta)}{\sigma} - \sigma$ and $L_2 = \frac{\log(x+1)-(\mu+\log\beta)}{\sigma} - \sigma$. We can apply formula 26.2.17 in Abramowitz and Stegun (1972) to calculate Φ.

Further, we note that if $u\beta$, $k\beta$ and $y\beta$ are positive integers, then $\psi^d_{k\beta}(u\beta)$ and $H^d(u\beta, y\beta)$ give approximation to $\psi_k(u)$ and $H_1(u, y)$, respectively. See Dickson and Waters (1992, Section 1) or Dickson (2005, Section 7.9.2).

5.3.1 The premium for the reinsurance policy

We have explained that capital injections can be provided either as a reinsurance arrangement or as coinsurance. Here, we consider a reinsurance contract under which whenever the surplus falls between 0 and k, the reinsurance company provides payments. To evaluate the cost of such a contract, we first need to determine the expected value of total amounts of payments to be made by the reinsurer. Let $S_{u,k}$ denote the aggregate amount required to restore the process to level k given initial surplus u. Then, the expected value of all payments to be made by the reinsurance company is given by the following result.

Theorem 5.3. When the initial surplus is $u = k, k+1, \ldots,$ we have

$$E[S_{u,k}] = \sum_{y=0}^{k-1}(y + E[S_{k,k}])h^d(u-k, y) \tag{5.8}$$

where

$$E[S_{k,k}] = \frac{\sum_{y=0}^{k-1} yh^d(0,y)}{1-H^d(0,k)}. \tag{5.9}$$

Proof. We can write

$$E[S_{u,k}] = \sum_{y=0}^{k-1} yh^d(u-k,y) + E[S_{k,k}]\sum_{y=0}^{k-1} h^d(u-k,y)$$

by noting that the first term is the expected payment resulting from the first drop below k and the second term represents the expected value of the total payments that will happen after that from level k. Setting $u = k$ and rearranging yields (5.9). □

Using these results, we can calculate the reinsurance premium based on the expected value principle as $Q(u,k) = (1+\theta_R)E[S_{u,k}]$, where θ_R is the reinsurer's loading factor.

5.3.2 Numerical illustrations

One objective in the classical risk model with capital injections is to minimise the probability of ruin. A large value of k for the insurer is a guarantee that if a large claim causes the surplus to drop between 0 and k, it can restart from level k without incurring a huge loss. However, this means that the reinsurance policy will be expensive for the insurer and even unattractive to the reinsurer. On the other hand, a low level of k may not help the company to be on the safe side. Thus, there is a trade-off between k and the probability of ruin.

In this section, we consider three claim amount distributions:

(i) Exponential(1) with mean 1 and variance 1,

(ii) Pareto$(4,3)$ with mean 1 and variance 2,

(iii) Lognormal$(-0.69315, 1.17741)$ with mean 1 and variance 3.

Further, we assume $\lambda = 1$, $\theta = 0.2$, $\beta = 100$, $\theta_R = 0.6$ or 2. As in Nie et al. (2011) let U be the available funds (before scaling) that the insurer allocates to a portfolio $U = u + Q(u,k)$, where $u \leq U$ is the initial surplus and $Q(u,k)$ is the cost of reinsurance. Our goal is to find different combinations of u and k that minimise the probability of ruin subject to the constraint $U = u + (1+\theta_R)E[S_{u,k}]$ and see under which circumstances reinsurance is effective.

Figures 5.1, 5.2, 5.3 and 5.4 show the probability of ruin for different combinations of u and the corresponding k that minimise $\psi_k(u)$. The red dashed line represents the probability of ruin without capital injections from a pre-determined initial surplus U and the green solid line represents the probability of ruin with capital injections for surplus level u which is minimum and a unique k that satisfy the constraint $U = u + (1+\theta_R)E[S_{u,k}]$. We note that $\psi(u)$ is approximated by the method in Section 1.2 and $\psi_k(u)$ by the method in Section 5.3. Also, all numbers in these figures are expressed before scaling. Figures 5.1 and 5.2 illustrate the probability of ruin when claim amounts follow a lognormal distribution. In Figure 5.1 $\psi(15) = 0.225$, k is found subject to $15 = u + 1.6E[S_{u,k}]$ and reinsurance is always effective, that is $\psi_k(u) \leq \psi(U)$ for all u

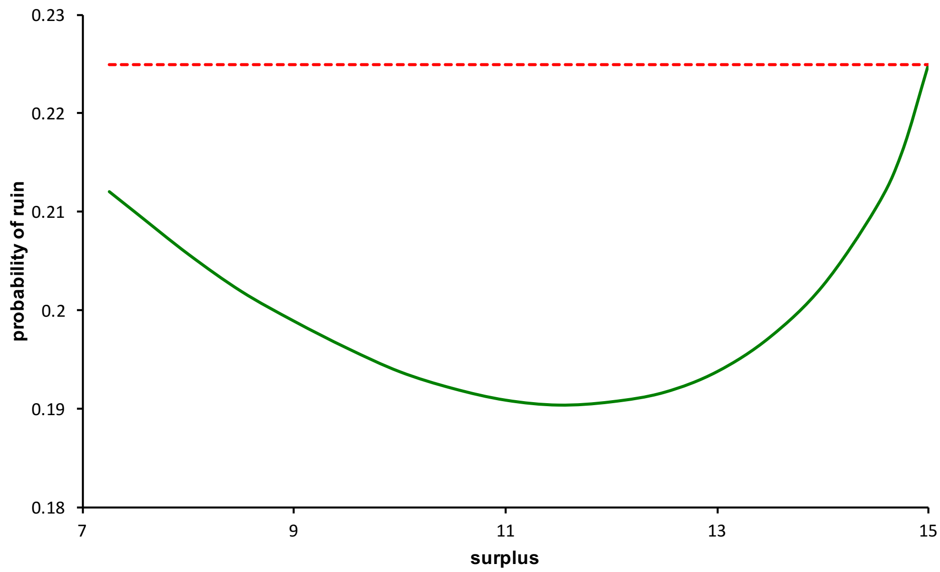

Figure 5.1: Claim amounts have a lognormal distribution with $U = 15$ and $\theta_R = 0.6$

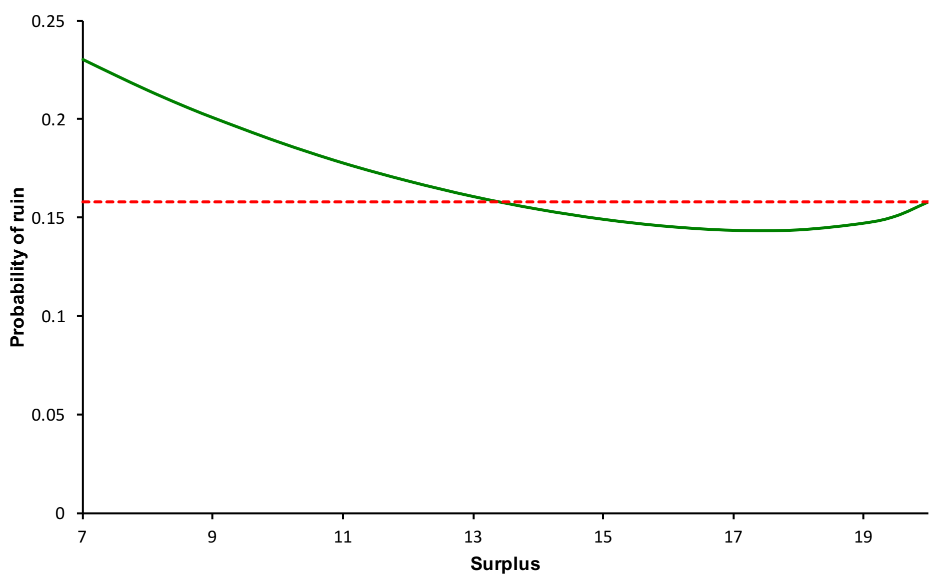

Figure 5.2: Claim amounts have a lognormal distribution with $U = 20$ and $\theta_R = 2$

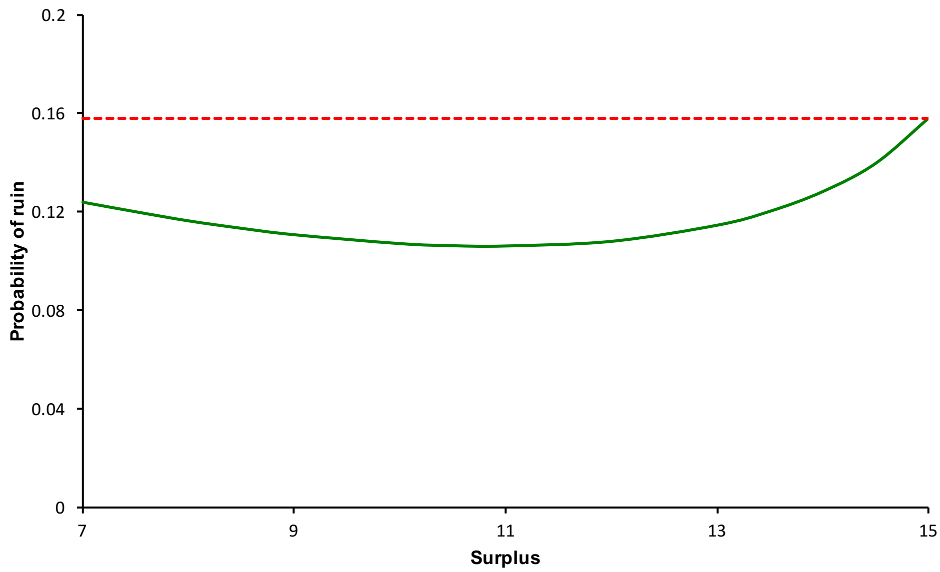

Figure 5.3: Claim amounts have a Pareto distribution with $U = 15$ and $\theta_R = 0.6$

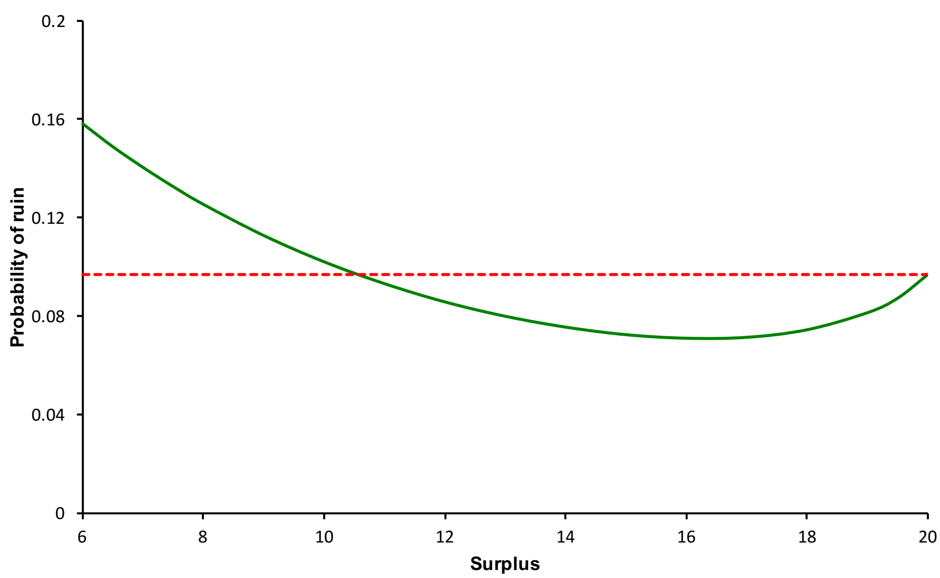

Figure 5.4: Claim amounts have a Pareto distribution with $U = 20$ and $\theta_R = 2$

and k, whereas in Figure 5.2 $\psi(20) = 0.1581$, k is found subject to $20 = u + 3E[S_{u,k}]$ and reinsurance is not always effective. As we can see for some combinations of u and k, $\psi_k(u)$ exceeds $\psi(U)$. In this figure, the minimum value of $\psi_k(u) = 0.1435$ is obtained when $u = 17.5$ and $k = 4.54$, which is equal to a reduction of about 9.23% in the probability of ruin. In Figure 5.1 when $u = 11.5$ and $k = 5.90$ the probability of ruin is reduced by 15.4%, which is considerably larger than 9.23%.

Figures 5.3 and 5.4 illustrate the probability of ruin when claim amounts have a Pareto distribution. Similar to the lognormal claim amounts, reinsurance is always effective in Figure 5.3 where $\psi(15) = 0.158$ and the probability of ruin is minimised subject to $15 = u + 1.6E[S_{u,k}]$. In this figure, the ruin probability is minimised when $u = 11$, $k = 5.97$ and the probability of ruin has decreased by 32.8% from 15.8% to 10.62%. However, in Figure 5.4 reinsurance is not always effective. In this figure, $\psi(20) = 0.097$ and the constraint is $20 = u + 3E[S_{u,k}]$. We can see that the minimum value of the probability of ruin is found for $u = 16.5$ and $k = 5.38$ where $\psi_k(u)$ decreases by 27.73% from 0.097 to 0.0701. We note that, although the means of the Pareto and lognormal distributions are the same, their variances are different and this has affected the shape of the graphs in Figures 5.1 and 5.3.

We remark that our findings for claim amounts with heavy-tailed distributions are compatible with the results in Nie et al. (2011). In other words, there are some combinations of u and k for which reinsurance is not always effective.

Up until now, we have looked at the optimal values of u and k from the insurer's perspective. However, as mentioned before, large values of k can be costly for the company and such reinsurance may not be provided by a reinsurance company. Now, we keep the level of k constant and consider a value of u to be optimal if it minimises $\psi_k(u)$. Our purpose is to see the effect of capital injections on a portfolio whose claim amounts follow a heavy-tailed distribution. The results are presented in Tables 5.1, 5.2 and 5.3. The key to these tables are as follows:

(1) approximate values of U, $\psi_k(u)$ and the optimal u with the condition $U = u + 1.2E[S_{u,k}]$,

(2) approximate values of U, $\psi_k(u)$ and the optimal u with the condition $U = u + 1.6E[S_{u,k}]$,

Table 5.1: Claim sizes have an exponential distribution

U	$\psi(U)$		$k=2$		$k=3$	
			u	$\psi_k(u)$	u	$\psi_k(u)$
16.88	5%	(1)	16.69	0.0351	16.47	0.0213
		(2)	16.63	0.0354	16.32	0.0219
		(3)	16.39	0.0369	15.72	0.0242
		(4)	16.63	0.0352	16.32	0.0216
18.22	4%	(1)	18.07	0.0279	17.89	0.0168
		(2)	18.02	0.0281	17.78	0.0171
		(3)	17.84	0.0290	17.33	0.0185
		(4)	18.02	0.0279	17.78	0.0170
19.95	3%	(1)	19.84	0.0207	19.71	0.0124
		(2)	19.80	0.0209	19.62	0.0126
		(3)	19.67	0.0213	19.31	0.0133
		(4)	19.80	0.0208	19.62	0.0125
22.38	2%	(1)	22.30	0.0138	22.22	0.0082
		(2)	22.28	0.0138	22.17	0.0083
		(3)	22.19	0.0140	21.97	0.0085
		(4)	22.28	0.0137	22.17	0.0082
26.54	1%	(1)	26.50	0.0068	26.46	0.0040
		(2)	26.49	0.0069	26.43	0.0041
		(3)	26.44	0.0069	26.34	0.0041
		(4)	26.49	0.0068	26.43	0.0040

(3) approximate values of U, $\psi_k(u)$ and the optimal u with the condition $U = u + 3E[S_{u,k}]$,

(4) exact values of U, $\psi_k(u)$ and the optimal u with the condition $U = u + 1.6E[S_{u,k}]$ for comparison with (2) – see Nie et al. (2011).

We consider three different situations to see how the results change with different premium assumptions. To examine the accuracy of our algorithm we compare the outputs in (2) with the exact results provided in Nie et al. (2011) in the case of exponential claims. The first column gives the value of the capital required to keep the ultimate probability of ruin at a pre-determined level. According to Tables 5.1, 5.2 and 5.3 the required U for the lognormal claims is higher than for the Pareto claims and the least capital is required for exponential claims. When the price of the reinsurance policy increases and the capital U is constant, less initial surplus u, can be allocated. This pattern can be observed for all three distributions but the effect is not very significant for lognormal claims and as U gets larger we cannot see any significant difference in initial surplus with different constraints. Moreover, the probability of ruin has not changed much when claims are lognormally distributed. If we look at different values of u and their corresponding probabilities for different levels of k, we notice that for

Table 5.2: Claim sizes have a lognormal distribution

U	$\psi(U)$		$k=2$		$k=3$	
			u	$\psi_k(u)$	u	$\psi_k(u)$
37.67	5%	(1)	37.61	0.0469	37.55	0.0443
		(2)	37.60	0.0469	37.52	0.0444
		(3)	37.54	0.0471	37.39	0.0447
41.31	4%	(1)	41.27	0.0375	41.22	0.0355
		(2)	41.25	0.0375	41.19	0.0355
		(3)	41.20	0.0377	41.09	0.0357
46.10	3%	(1)	46.07	0.0282	46.03	0.0267
		(2)	46.06	0.0282	46.01	0.0267
		(3)	46.02	0.0282	45.94	0.0268
53.03	2%	(1)	53.01	0.0188	52.98	0.0178
		(2)	53.00	0.0188	52.97	0.0178
		(3)	52.98	0.0188	52.92	0.0179
65.38	1%	(1)	65.37	0.0094	65.36	0.0090
		(2)	65.36	0.0094	65.35	0.0090
		(3)	65.35	0.0094	65.33	0.0090

Table 5.3: Claim sizes have a Pareto distribution

U	$\psi(U)$		$k=2$		$k=3$	
			u	$\psi_k(u)$	u	$\psi_k(u)$
27.01	5%	(1)	26.92	0.0442	26.82	0.0392
		(2)	26.89	0.0444	26.76	0.0394
		(3)	26.79	0.0448	26.53	0.0403
29.44	4%	(1)	29.37	0.0354	29.29	0.0314
		(2)	29.35	0.0354	29.24	0.0315
		(3)	29.27	0.0357	29.06	0.0320
32.61	3%	(1)	32.56	0.0265	32.50	0.0236
		(2)	32.54	0.0266	32.46	0.0236
		(3)	32.48	0.0267	32.33	0.0239
37.16	2%	(1)	37.12	0.0178	37.09	0.0158
		(2)	37.11	0.0178	37.06	0.0158
		(3)	37.08	0.0178	36.98	0.0160
45.25	1%	(1)	45.23	0.0089	45.21	0.0080
		(2)	45.22	0.0089	45.20	0.0080
		(3)	45.21	0.0089	45.16	0.0080

larger k we have less initial surplus available, but as U increases the difference between initial surplus for the same reinsurance policy is not noticeable.

Table 5.4: The released capital given a fixed probability of ruin

$\psi(U)$			U	u	$E[S_{u,k}]$	$R_{u,k}(1)$	$R_{u,k}(2)$	$R_{u,k}(3)$
5%	(*)	Exponential($k=2$)	16.88	14.53	0.2188	2.09	2.00	1.69
			16.88	14.56	0.2183	2.06	1.97	1.67
	(*)	Exponential($k=3$)	16.88	11.30	0.8	4.62	4.30	3.18
			16.88	11.36	0.7986	4.56	4.24	3.12
		Lognormal ($k=2$)	37.67	36.56	0.0456	1.06	1.04	0.97
		Lognormal ($k=3$)	37.67	35.58	0.1052	1.96	1.92	1.77
		Pareto ($k=2$)	27.01	25.59	0.0803	1.32	1.29	1.18
		Pareto ($k=3$)	27.01	24.18	0.2002	2.59	2.51	2.23
4%	(*)	Exponential($k=2$)	18.22	15.87	0.175	2.14	2.07	1.83
			18.22	15.90	0.1746	2.11	2.04	1.80
	(*)	Exponential($k=3$)	18.22	12.64	0.6438	4.81	4.55	3.65
			18.22	12.70	0.6388	4.75	4.50	3.60
		Lognormal ($k=2$)	41.31	40.20	0.0358	1.07	1.05	1.00
		Lognormal ($k=3$)	41.31	39.23	0.0823	1.98	1.95	1.83
		Pareto ($k=2$)	29.44	28.02	0.0633	1.34	1.32	1.23
		Pareto ($k=3$)	29.44	26.61	0.1574	2.64	2.58	2.36
3%	(*)	Exponential($k=2$)	19.95	17.60	0.1313	2.19	2.14	1.96
			19.95	17.63	0.1309	2.16	2.11	1.93
	(*)	Exponential($k=3$)	19.95	14.36	0.4875	5.01	4.81	4.13
			19.95	14.42	0.4796	4.95	4.76	4.09
		Lognormal ($k=2$)	46.10	44.99	0.0261	1.08	1.07	1.03
		Lognormal ($k=3$)	46.10	44.02	0.0600	2.01	1.98	1.90
		Pareto ($k=2$)	32.61	31.19	0.0465	1.36	1.35	1.28
		Pareto ($k=3$)	32.61	29.79	0.1153	2.68	2.64	2.47
2%	(*)	Exponential($k=2$)	22.38	20.03	0.0875	2.25	2.21	2.09
			22.38	20.06	0.0873	2.22	2.18	2.06
	(*)	Exponential($k=3$)	22.38	16.80	0.3188	5.20	5.07	4.62
			22.38	16.86	0.3193	5.14	5.01	4.56
		Lognormal ($k=2$)	53.03	51.93	0.0168	1.08	1.07	1.05
		Lognormal ($k=3$)	53.03	50.97	0.0385	2.01	2.00	1.94
		Pareto ($k=2$)	37.16	35.75	0.0301	1.37	1.36	1.32
		Pareto ($k=3$)	37.16	34.36	0.0744	2.71	2.68	2.58
1%	(*)	Exponential($k=2$)	26.54	24.19	0.0438	2.30	2.28	2.22
			26.54	24.22	0.0436	2.27	2.25	2.19
	(*)	Exponential($k=3$)	26.54	20.95	0.1625	5.40	5.33	5.10
			26.54	21.01	0.1599	5.34	5.27	5.05
		Lognormal ($k=2$)	65.38	64.29	0.0080	1.08	1.08	1.07
		Lognormal ($k=3$)	65.38	63.33	0.0181	2.03	2.02	2.00
		Pareto ($k=2$)	45.25	43.84	0.0142	1.39	1.39	1.37
		Pareto ($k=3$)	45.25	42.46	0.0350	2.75	2.73	2.69

Table 5.4 shows how a company can release its capital by not allowing the surplus to fall below a specified level. The key to Table 5.4 is as follows:

- $R_{u,k}(1) = U - u - 1.2E[S_{u,k}]$,
- $R_{u,k}(2) = U - u - 1.6E[S_{u,k}]$,
- $R_{u,k}(3) = U - u - 3E[S_{u,k}]$,
- $(*)$ exact values in the continuous time case – see Nie et al. (2011).

The outcomes of our algorithm for the exponential distribution are close to the exact values in the continuous time case. Table 5.4 illustrates that even if capital injections do not reduce the ruin probability considerably, they can still provide the company with the opportunity to release capital. As we can see the amount of released capital for our heavy-tailed claim distributions is not as much as for the exponential claim distribution. For example when $\theta_R = 0.6$ the percentage of released capital for $k = 2$ when claims are lognormally distributed ranges from 1.08 out of 65.38 available capital to 1.04 out of 37.67 available capital, i.e. 1.7% to 2.8% and for $k = 3$ ranges from 2.02 out of 65.38 to 1.92 out of 37.67, i.e. 3% to 5%. This amount for the Pareto claims distribution with $k = 2$ is from 3% to 5% and with $k = 3$ ranges from 6% to 9% which are obviously less than for the exponential claims distribution. The percentage of released capital for individual claim amounts with an exponential distribution is about 9% to 12% for $k = 2$ and 20% to 25% for $k = 3$.

5.4 The probability of ruin in finite time

In this section, we consider the probability of ruin in finite time and provide recursive formulae which we can use to approximate the finite time ruin probability in the classical risk model with capital injections. The method is based on the algorithm of Dickson and Waters (1991).

Theorem 5.4. For $u = k, k + 1, \ldots$, when $t = 1$,

$$\psi_k^d(u, 1) = \sum_{x=u+1}^{\infty} g(x) = 1 - G(u)$$

and for $t > 1$,

$$\begin{aligned} \psi_k^d(u, t) \quad = \quad & \sum_{x=0}^{u+1-k} g(x)\psi_k^d(u + 1 - x, t - 1) + (G(u) - G(u + 1 - k))\, \psi_k^d(k, t - 1) \\ & +1 - G(u). \end{aligned} \tag{5.10}$$

Proof. We start by considering $\psi_k^d(u, 1)$. If ruin occurs in the first time period, the aggregate claim amount must be greater than u so that in the first time period, the surplus is less than or equal to 0. Further, for the case $t > 1$ we note that if the aggregate claim amount is less than $u + 1 - k$, ruin can subsequently occur from level

$u+1-x$, if the aggregate claim amount is greater than $u+1-k$, but less than u, then ruin can subsequently occur from level k. Finally, if the aggregate claim amount exceeds u, ruin occurs. Hence, the expression for the probability of finite time ruin follows. □

The probability of ruin at time t can be calculated from the probability of ruin at times $t-1, t-2, \ldots, 1$, recursively. To do this, first we must find $\psi_k^d(w,1)$ for $w = k, k+1, \ldots, u+t-1$, then $\psi_k^d(w,2)$ for $w = k, k+1, \ldots, u+t-2$, etc. See Dickson and Waters (1991, Section 2). When we know the values of $\psi_k^d(u,t)$, we can apply the method of Dickson and Waters (2002, Section 6) to approximate the (defective) density of the time of ruin in the classical risk model with capital injections for $u \geq k$ and $j = 1, 2, \ldots, (1+\theta)\beta t$ by

$$(1+\theta)\beta\Big(\psi_k^d(u,j) - \psi_k^d(u,j-1)\Big) \tag{5.11}$$

where $\psi_k^d(u,j)$ is based on the values obtained from formula (5.10), i.e. after rescaling. Dividing formula (5.11) by $\psi_k^d(u)$ gives the approximation for the proper density of the time of ruin.

In Chapter 3 we found an explicit expression for the density of the time of ruin in the classical risk model with capital injections that can be implemented in the case of claim amounts for which $g(x,t)$ can be explicitly identified. Our interest here is to show that with our numerical method we can approximate the density of the time of ruin in the classical risk model with capital injections even for claim sizes following a heavy-tailed distribution. To examine the accuracy of our algorithm, we apply formula (3.48) and select a suitable truncation point to graph the conditional density of the ruin time for claim amounts that follow an exponential distribution. Figure 5.5 shows the exact and approximate density of the time of ruin, given that ruin occurs for $u = 10$, $\beta = 20$, $\theta = 0.2$ and $k = 1, 2, 3$ with u and t being expressed before scaling. As we can see, the exact and approximate densities are virtually indistinguishable. Now we apply our numerical procedure to plot the density of the time of ruin when claim amounts have Pareto and lognormal distributions. Figures 5.6 and 5.7 illustrate approximations to the density of the time of ruin for Pareto and lognormal claim distributions. We observe that when claim sizes have these heavy-tailed distributions, it is more likely that ruin occurs earlier in time compared to the exponential distribution. However, looking at Figures 5.5, 5.6 and 5.7 we can see that the likelihood of ruin occurring is higher for

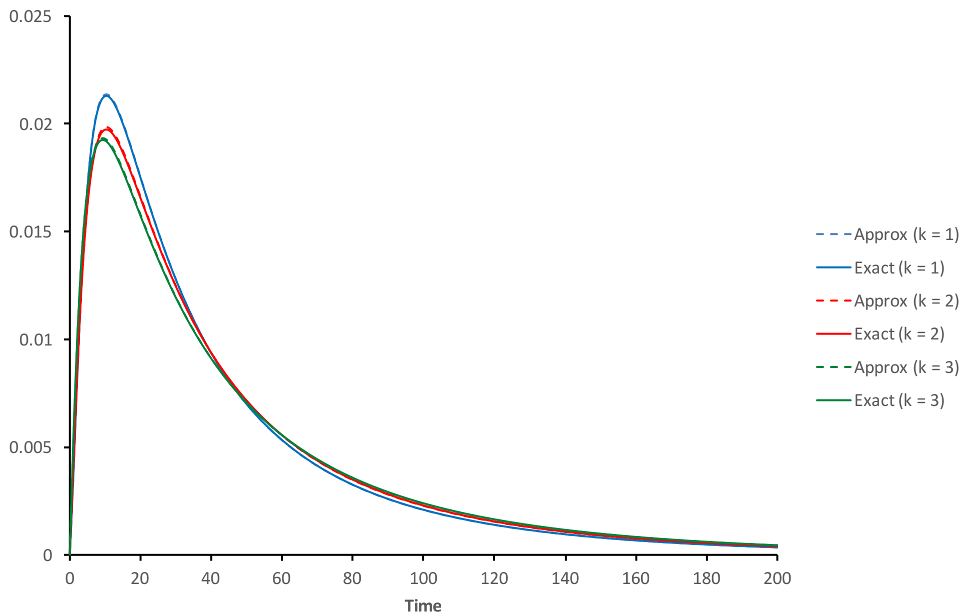

Figure 5.5: Exact and approximate densities of the time of ruin when claims have an exponential distribution

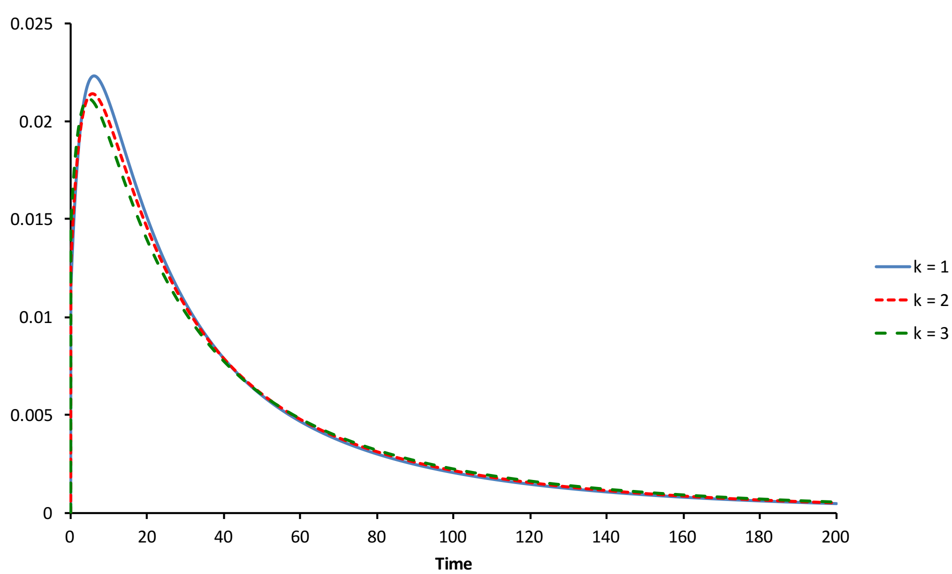

Figure 5.6: Approximations to the density of the time of ruin when claims have a Pareto distribution

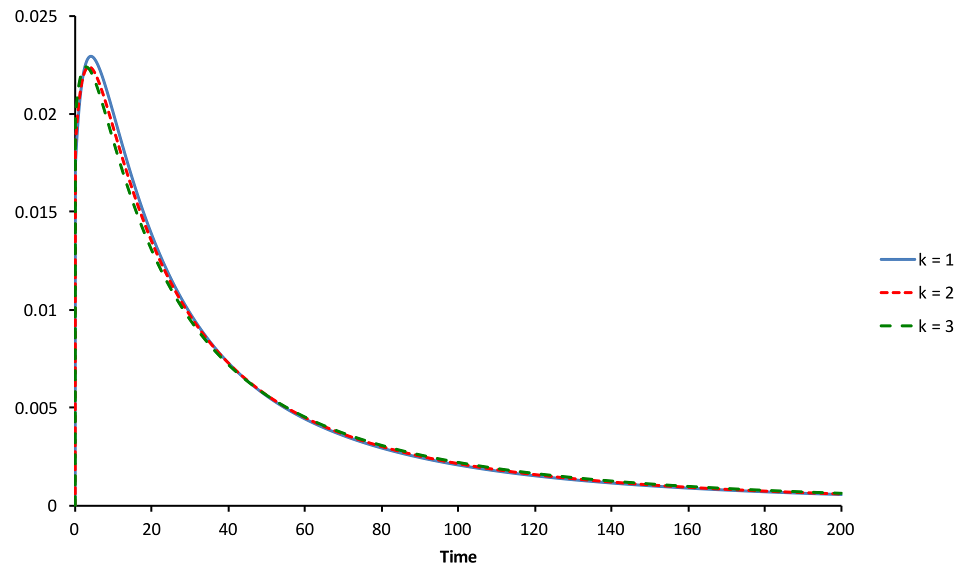

Figure 5.7: Approximations to the density of the time of ruin when claims have a lognormal distribution

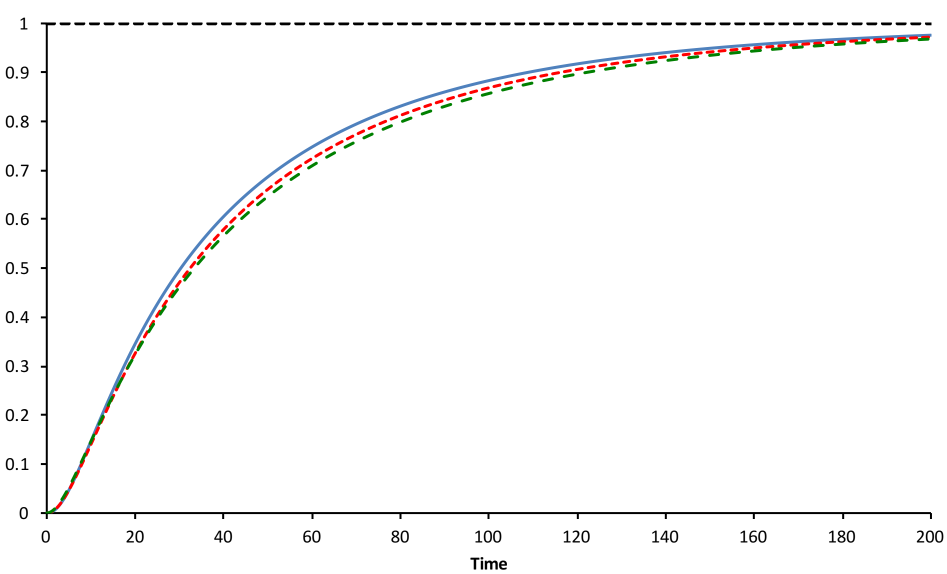

Figure 5.8: Approximations to the cumulative distribution of the time of ruin when claims have an exponential distribution

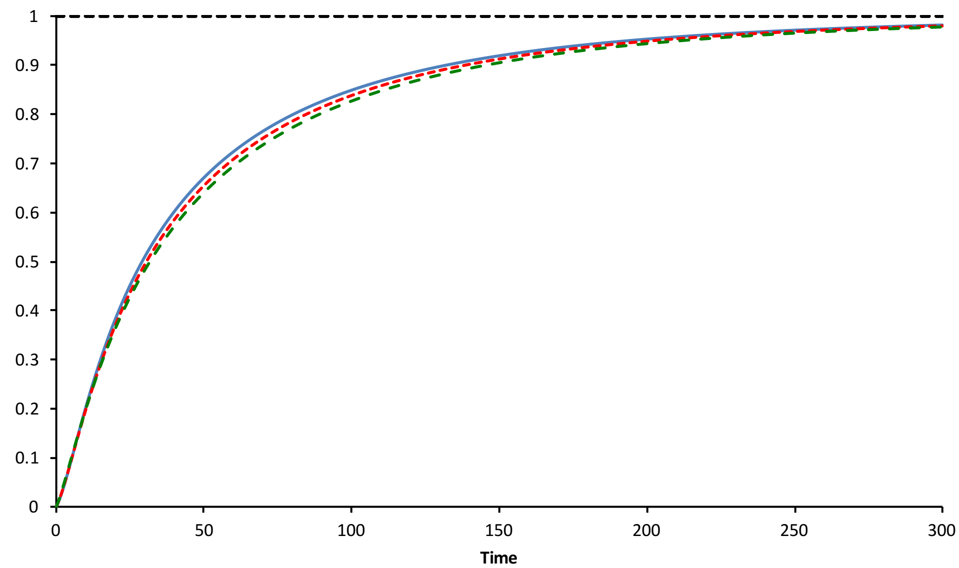

Figure 5.9: Approximations to the cumulative distribution of the time of ruin when claims have a Pareto distribution

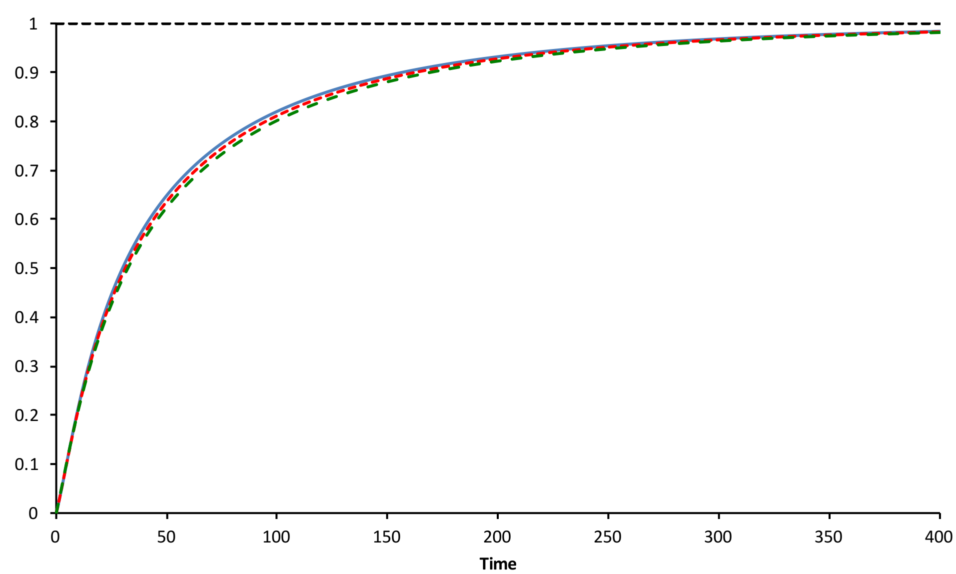

Figure 5.10: Approximations to the cumulative distribution of the time of ruin when claims have a lognormal distribution

Pareto and lognormal claim distributions than the exponential claim distribution for later time periods. However, the patterns are similar to those in Figure 5.5; the graph for $k = 1$ is located above $k = 2$ and $k = 3$. However, in the case of claim amounts with a Pareto distribution, it is not easy to distinguish between the graphs for $k = 2$ and $k = 3$ and in the case of claim amounts with a lognormal distribution, the difference is less clear for different levels of k. The density of the time of ruin has a fatter tail when claims are lognormally distributed compared to the claims with exponential and Pareto distributions. Further, we observe that the density of the time of ruin is positively-skewed in the risk model with capital injections for claim amounts with heavy-tailed distributions. This result is also noted by Dickson and Waters (2002) in the case of the classical risk model without capital injections.

Figures 5.8, 5.9 and 5.10 illustrate approximations to the cumulative distribution of the time of ruin for claims with exponential, Pareto and lognormal distributions, respectively. In Figure 5.8 the upper 5th percentile corresponds to $T_{u,k} = 150, 160$ and 168 for $k = 1, 2$ and 3, respectively. In the case of the lognormal distribution the upper 5th percentile is equivalent to $T_{u,k} = 237, 245$ and 254 for $k = 1, 2, 3$. When claim sizes have a Pareto distribution, the corresponding $T_{u,k}$ for the upper 5th percentile is 193 for $k = 1$, 202 for $k = 2$ and 211 for $k = 3$. We observe that the common feature of all these figures is that the the graph for $k = 1$ is over the graph for $k = 2$ and the graph for $k = 3$ is beneath the other two graphs, consequently the graph for $k = 2$ is between $k = 1$ and $k = 3$.

5.5 Concluding remarks

In this chapter, we have presented a numerical algorithm to study the classical risk model with capital injections. For this, we have introduced a discrete time risk model with capital injections and produced analogues of the results in Nie et al. (2011). We have applied our algorithm to approximate the finite time and infinite time ruin probability in the classical risk model with capital injections when claim amounts have Pareto and lognormal distributions. Comparing our results with Nie et al. (2011, 2015) we have demonstrated that our algorithm produces close approximations to the exact values. Further, we found that this model can lead to a reduction in the ultimate ruin probability when claim amounts have heavy-tailed distributions.

Chapter 6

The Markov-modulated risk model

6.1 Introduction

In this chapter, we study Markov-modulated risk models. In the continuous time model, the arrival intensities and the distribution of the individual claim amounts in different periods of time depend on a state process, representing, for example, different weather, economic, or environmental conditions, and therefore can be considered more flexible than the classical risk model. There is much research that considers different ruin-related quantities in the framework of the continuous time Markov-modulated model – see Chapter 2 and references therein. However, an issue is that, similar to the classical risk model, either explicit expressions for such quantities do not exist or they are complicated to obtain.

An aim of this chapter is to introduce a discrete time model that can be used to approximate ruin-related quantities in the continuous time Markov-modulated risk model. For this, we first adapt ideas of Reinhard and Snoussi (2002) and Chen et al. (2014b) to build recursive formulae for the probability of ruin and probability and severity of ruin in a discrete time model. Then, we use these formulae and by modifying Dickson and Waters' (1991, 1992) algorithm for the approximation of the classical risk model, create stable algorithms that can provide approximations to the probability of ruin and probability and severity of ruin in the continuous time Markov-modulated model. Following that we extend our algorithm from Chapter 5 to analyse the (defective) density of the time of ruin in an m-state Markov-modulated model. In the final section, we introduce capital injections to the continuous time Markov-modulated model and provide formulae for the probability of ruin. We then create algorithms to approximate the probability of ruin and the (defective) density of the time of ruin in the continuous time Markov-modulated model with capital injections.

6.2 Notation and definitions

In Section 2.1 we have introduced the continuous time Markov-modulated risk model. Throughout this chapter we adopt the same notation and definitions as in Section 2.1.

In this section, we consider a discrete time model that can be used to approximate the continuous time Markov-modulated model. Such a discrete time model has been explained in detail in Section 2.1.2. Using the same notation, the surplus of an insurance company at time $n = 1, 2, 3, \dots$ is modelled by

$$U^d(n) = u + n - \sum_{i=1}^{n} Y_i \tag{6.1}$$

where u is the insurer's initial surplus, n is the total premium income up to time n – assuming that the insurer's premium income per unit time is 1 – and Y_i denotes the insurer's aggregate claim amount in the ith time interval. Let $\{J_n\}_{n\in N}$ be a homogeneous, irreducible and aperiodic Markov chain with a finite state space $M = \{1, \dots, m\}$ and transition probabilities $p_{ij} = \Pr(J_n = j|J_{n-1} = i, J_k \;\; k \leq n-1)$, for $i, j, \in M$. The conditional joint distribution of Y_n and J_n given the previous state J_{n-1} is defined by

$$g_{ij}(x) = \Pr(Y_n = x, J_n = j|J_{n-1} = i, J_k, Y_k, k \leq n-1) = p_{ij} g_j(x)$$

where $g_i(x) = \sum_{j=1}^{m} g_{ij}(x)$, and $G_i(y) = \sum_{j=1}^{m} \sum_{x=0}^{y} g_{ij}(x)$ for $y = 0, 1, 2, \dots$. Further, $\tilde{g}_{ij}(s) = \sum_{x=0}^{\infty} s^x g_{ij}(x)$. For all $i, j \in M$ we define $(\mu_n)_{ij} = \sum_{x=1}^{\infty} x^n g_{ij}(x) < \infty$ to be nth moment of the aggregate claim amount in state j, given initial state i.

Let T_u^d be the time of ruin given initial surplus u, and define as $T_u^d = \min\{n \geq 1 : U^d(n) \leq 0 \mid U^d(0) = u\}$ with $T_u^d = \infty$ if $U^d(n) > 0$ for $n = 1, 2, 3, \dots$. We remark that in this chapter we adopt the same definition of ruin as in Chapter 5.

Denote by $\psi_i^d(u)$ the ultimate probability of ruin given initial surplus u and initial environment state i which is given by

$$\psi_i^d(u) = \Pr(T_u^d < \infty \mid U^d(0) = u, J(0) = i) = 1 - \delta_i^d(u)$$

where $\delta_i^d(u)$ is the probability of survival. Also, we denote by $\psi_i^d(u,t)$ the finite time probability of ruin given initial surplus u and initial environment state i which is given by

$$\psi_i^d(u,t) = \Pr(T_u^d \leq t \mid U^d(0) = u, J(0) = i).$$

Also, we define the probability that ruin occurs in state j and the insurer's deficit at ruin is at most y, given initial environment state i, as

$$H_{ij}^d(u,y) = \Pr(T_u^d < \infty, |U(T_u^d)| \leq y, J(T_u^d) = j \mid U^d(0) = u, J(0) = i)$$

with the probability mass function being $h_{ij}^d(u,y)$ and $h_i^d(u,y) = \sum_{j=1}^m h_{ij}^d(u,y)$. Further, $H_i^d(u,y) = \sum_{j=1}^m H_{ij}^d(u,y)$.

6.3 The probability of ruin and the probability and severity of ruin

In this section, we present recursive formulae for $\psi_i^d(u)$ and $h_i^d(u,y)$ when $m = 2$. As a recursive formula needs an initial value, we first derive expressions for $\psi_i^d(0)$ and $h_i^d(0,y)$ and then provide results which we can use to calculate the ultimate probability of ruin and the probability and severity of ruin in our discrete time model.

6.3.1 Starting values $\psi_i^d(0)$ and $h_{ij}^d(0,y)$

Chen et al. (2014b) have derived two equations that define the relationship between $\delta_1^d(0)$ and $\delta_2^d(0)$ under their definition of ruin. Here we develop the equivalent of their equations for our definition of ruin. The first equation is obtained by a different method, but for the second equation, we use the method of generating functions as in Chen et al. (2014b).

Theorem 6.1. When $m = 2$ and $u = 0$, $\delta_1^d(0)$ and $\delta_2^d(0)$ satisfy

$$p_{12}\delta_2^d(0) + p_{21}\delta_1^d(0) = p_{12}(1 - \mu_2) + p_{21}(1 - \mu_1) \tag{6.2}$$

and

$$(\tilde{g}_{11}(\rho)(\tilde{g}_{22}(\rho) - \rho) - \tilde{g}_{12}(\rho)\tilde{g}_{21}(\rho))\,\delta_1^d(0) = (\tilde{g}_{12}(\rho)\tilde{g}_{22}(\rho) - \tilde{g}_{12}(\rho)(\tilde{g}_{22}(\rho) - \rho))\,\delta_2^d(0), \tag{6.3}$$

where $\rho \in (0,1)$ is the solution to

$$L_1(s) = (\tilde{g}_{11}(s) - s)(\tilde{g}_{22}(s) - s) - \tilde{g}_{12}(s)\tilde{g}_{21}(s) = 0. \tag{6.4}$$

Proof. We begin with

$$\psi_i^d(u) = \sum_{j=1}^{2}\sum_{x=0}^{u} g_{ij}(x)\psi_j^d(u+1-x) + \sum_{j=1}^{2}\sum_{x=u+1}^{\infty} g_{ij}(x). \tag{6.5}$$

We assume that $\sum_{u=0}^{\infty}\psi_i^d(u)$ exists and discuss conditions under which this assumption holds in the Appendix.

Summing over u from 0 to ∞ in (6.5) gives

$$\sum_{u=0}^{\infty}\psi_i^d(u) = \sum_{j=1}^{2}\sum_{u=0}^{\infty}\sum_{x=0}^{u} g_{ij}(x)\psi_j^d(u+1-x) + \sum_{j=1}^{2}\sum_{u=0}^{\infty}\sum_{x=u+1}^{\infty} g_{ij}(x). \tag{6.6}$$

Setting $u+1=n$ in the first term on the right-hand side of (6.6) we get

$$\sum_{u=0}^{\infty}\psi_i^d(u) = \sum_{j=1}^{2}\sum_{n=1}^{\infty}\sum_{x=0}^{n-1} g_{ij}(x)\psi_j^d(n-x) + \sum_{j=1}^{2}\sum_{u=0}^{\infty}\sum_{x=u+1}^{\infty} g_{ij}(x).$$

Then, changing the order of summation yields

$$\begin{aligned}\sum_{u=0}^{\infty}\psi_i^d(u) &= \sum_{j=1}^{2}\left(\sum_{x=0}^{\infty} g_{ij}(x)\sum_{u=1}^{\infty}\psi_j^d(u) + \sum_{x=1}^{\infty} x g_{ij}(x)\right)\\ &= \sum_{j=1}^{2}\left(p_{ij}\sum_{u=1}^{\infty}\psi_j^d(u) + \mu_{ij}\right),\end{aligned}$$

which for $i=1$ can be written as

$$\psi_1^d(0) + \sum_{u=1}^{\infty}\psi_1^d(u) = p_{11}\sum_{u=1}^{\infty}\psi_1^d(u) + \mu_{11} + p_{12}\sum_{u=1}^{\infty}\psi_2^d(u) + \mu_{12} \tag{6.7}$$

and for $i=2$,

$$\psi_2^d(0) + \sum_{u=1}^{\infty}\psi_2^d(u) = p_{21}\sum_{u=1}^{\infty}\psi_1^d(u) + \mu_{21} + p_{22}\sum_{u=1}^{\infty}\psi_2^d(u) + \mu_{22}. \tag{6.8}$$

Rearranging (6.7) and (6.8) gives us

$$\begin{cases}\psi_1^d(0) + p_{12}\sum_{u=1}^{\infty}\psi_1^d(u) = \mu_1 + p_{12}\sum_{u=1}^{\infty}\psi_2^d(u),\\ \psi_2^d(0) + p_{21}\sum_{u=1}^{\infty}\psi_2^d(u) = \mu_2 + p_{21}\sum_{u=1}^{\infty}\psi_1^d(u)\end{cases}$$

and formula (6.2) follows.

We can build the second relationship between $\delta_1^d(0)$ and $\delta_2^d(0)$ using the method of generating functions similar to Chen et al. (2014b). After adjusting for our definition of ruin, we can rewrite formula (3) of their paper as

$$\delta_i^d(u) = \sum_{j=1}^{2}\sum_{x=0}^{u} g_{ij}(x)\delta_j^d(u+1-x). \tag{6.9}$$

Multiplying both sides by s^{u+1} and summing over u yields

$$\sum_{u=0}^{\infty} s^{u+1}\delta_i^d(u) = \sum_{j=1}^{2}\sum_{u=0}^{\infty} s^{u+1}\sum_{x=0}^{u} g_{ij}(x)\delta_j^d(u+1-x).$$

We define $\tilde{\delta}_i^d(s) = \sum_{x=0}^{\infty} s^x \delta_i^d(x)$. Setting $u+1=n$, we find that

$$\begin{aligned}
s\tilde{\delta}_i^d(s) &= \sum_{j=1}^{2}\sum_{n=1}^{\infty}\sum_{x=0}^{n-1} s^n g_{ij}(x)\delta_j^d(n-x) \\
&= \sum_{j=1}^{2}\sum_{n=1}^{\infty}\left(\sum_{x=0}^{n} s^n g_{ij}(x)\delta_j^d(n-x) - s^n g_{ij}(n)\delta_j^d(0)\right) \\
&= \sum_{j=1}^{2}\left(\sum_{n=1}^{\infty}\sum_{x=0}^{n} s^n g_{ij}(x)\delta_j^d(n-x) - \sum_{n=1}^{\infty} s^n g_{ij}(n)\delta_j^d(0)\right),
\end{aligned}$$

and by noting that

$$\tilde{g}_{ij}(s)\tilde{\delta}_j^d(s) = \sum_{n=0}^{\infty}\sum_{x=0}^{n} s^n g_{ij}(x)\delta_j^d(n-x)$$

we get

$$\begin{aligned}
s\tilde{\delta}_i^d(s) &= \sum_{j=1}^{2}\tilde{g}_{ij}(s)\tilde{\delta}_j^d(s) - \sum_{j=1}^{2}\tilde{g}_{ij}(s)\delta_j^d(0) \\
&= \sum_{j=1}^{2}\tilde{g}_{ij}(s)\tilde{\delta}_j^d(s) - e_i(s),
\end{aligned} \tag{6.10}$$

where $e_i(s) = \sum_{j=1}^{2}\tilde{g}_{ij}(s)\delta_j^d(0)$. We can rewrite (6.10) as a system of equations. Thus

$$\begin{cases}
\left(\tilde{g}_{11}(s) - s\right)\tilde{\delta}_1^d(s) + \tilde{g}_{12}(s)\tilde{\delta}_2^d(s) = e_1(s), \\
\tilde{g}_{21}(s)\tilde{\delta}_1^d(s) + \left(\tilde{g}_{22}(s) - s\right)\tilde{\delta}_2^d(s) = e_2(s).
\end{cases}$$

It follows that

$$((\tilde{g}_{11}(s)-s)(\tilde{g}_{22}(s)-s)-\tilde{g}_{12}(s)\tilde{g}_{21}(s))\,\tilde{\delta}_1^d(s)=e_1(s)(\tilde{g}_{22}(s)-s)-e_2(s)\tilde{g}_{12}(s). \tag{6.11}$$

Equation (6.11) is similar to equation (5) of Chen et al. (2014b). The only difference is the definitions of $e_1(s)$ and $e_2(s)$ on the right-hand side. In their paper, $e_i(s)$ is defined in terms of $g_{ij}(0)$, whereas here, it is defined in terms of $\tilde{g}_{ij}(s)$. To find a relationship between $\delta_1^d(0)$ and $\delta_2^d(0)$ we proceed as follows. Following Chen et al. (2014b), we can write equation (6.11) as $L_1(s)\tilde{\delta}_1^d(s)=L_2(s)$, so that

$$L_1(s)=(\tilde{g}_{11}(s)-s)(\tilde{g}_{22}(s)-s)-\tilde{g}_{12}(s)\tilde{g}_{21}(s). \tag{6.12}$$

Substituting $e_1(s)$ and $e_2(s)$ on the right-hand side of (6.11) yields

$$\begin{aligned} L_2(s) \;=\;& (\tilde{g}_{11}(s)(\tilde{g}_{22}(s)-s)-\tilde{g}_{12}(s)\tilde{g}_{21}(s))\,\delta_1^d(0) \\ &+(\tilde{g}_{12}(s)(\tilde{g}_{22}(s)-s)-\tilde{g}_{12}(s)\tilde{g}_{22}(s))\,\delta_2^d(0). \end{aligned} \tag{6.13}$$

Then, by noting that $\tilde{g}_{ij}(0)=g_{ij}(0)$, we have

$$L_2(0)=(g_{11}(0)g_{22}(0)-g_{12}(0)g_{21}(0))\,\delta_1^d(0)$$

and that $\tilde{g}_{ij}(1)=p_{ij}$, we get

$$L_2(1)=(p_{11}(p_{22}-1)-p_{12}p_{21})\,\delta_1^d(0)-p_{12}\delta_2^d(0).$$

We assume $L_2(0)>0$ which holds under the condition that $p_{11}p_{22}>p_{12}p_{21}$ – see case 2 in Chen et al. (2014b, page 210) – and applies to all our numerical examples that follow. Further, given that $L_2(1)<0$, we can conclude that there exists $\rho\in(0,1)$, which is the solution to $L_1(\rho)=0$ so that $L_2(\rho)=0$. Setting $L_2(\rho)=0$ in expression (6.13), we can find the second relationship between $\delta_1^d(0)$ and $\delta_2^d(0)$ which is given by (6.3). □

The next result gives starting values $h_{ij}^d(0,y)$.

Theorem 6.2. When $m=2$, the initial surplus is 0 and $y=0,1,2,\ldots$, $h_{11}^d(0,y)$ and $h_{21}^d(0,y)$ satisfy

$$p_{12}h_{21}^d(0,y)+p_{21}h_{11}^d(0,y)=p_{12}(p_{21}-G_{21}(y))+p_{21}(p_{11}-G_{11}(y)) \tag{6.14}$$

and

$$
\begin{aligned}
& h_{11}^d(0,y)\left(\tilde{g}_{11}(\rho)\left(\tilde{g}_{22}(\rho)-\rho\right)-\tilde{g}_{12}(\rho)\tilde{g}_{21}(\rho)\right)+\tilde{g}_{12}(\rho)\sum_{u=0}^{\infty}\rho^{u+1}g_{21}(u+1+y) \\
= \quad & h_{21}^d(0,y)\left(\tilde{g}_{12}(\rho)\tilde{g}_{22}(\rho)-\tilde{g}_{12}(\rho)\left(\tilde{g}_{22}(\rho)-\rho\right)\right)+\left(\tilde{g}_{22}(\rho)-\rho\right)\sum_{u=0}^{\infty}\rho^{u+1}g_{11}(u+1+y).
\end{aligned}
\tag{6.15}
$$

Further, $h_{22}^d(0,y)$ and $h_{12}^d(0,y)$ satisfy

$$
p_{12}h_{22}^d(0,y)+p_{21}h_{12}^d(0,y)=p_{12}\left(p_{22}-G_{22}(y)\right)+p_{21}\left(p_{12}-G_{12}(y)\right) \tag{6.16}
$$

and

$$
\begin{aligned}
& h_{12}^d(0,y)\left(\tilde{g}_{11}(\rho)\left(\tilde{g}_{22}(\rho)-\rho\right)-\tilde{g}_{12}(\rho)\tilde{g}_{21}(\rho)\right)+\tilde{g}_{12}(\rho)\sum_{u=0}^{\infty}\rho^{u+1}g_{22}(u+1+y) \\
= \quad & h_{22}^d(0,y)\left(\tilde{g}_{12}(\rho)\tilde{g}_{22}(\rho)-\tilde{g}_{12}(\rho)\left(\tilde{g}_{22}(\rho)-\rho\right)\right)+\left(\tilde{g}_{22}(\rho)-\rho\right)\sum_{u=0}^{\infty}\rho^{u+1}g_{12}(u+1+y),
\end{aligned}
\tag{6.17}
$$

provided that $p_{22}p_{11}>p_{12}p_{21}$ and

$$
\begin{aligned}
p_{12}A_{21}(1,y)+p_{21}A_{11}(1,y) \quad < \quad & \mid((p_{22}-1)p_{11}-p_{12}p_{21})\mid h_{11}^d(0,y) \\
& +\mid((p_{22}-1)p_{12}-p_{12}p_{22})\mid h_{21}^d(0,y),
\end{aligned}
\tag{6.18}
$$

where $A_{ij}(s,y)=\sum_{u=0}^{\infty}s^{u+1}g_{ij}(u+1+y)$ and ρ is the solution to equation (6.4).

Proof. We adapt ideas in Reinhard and Snoussi (2002) to our ruin definition and write

$$
h_{ij}^d(u,y)=\sum_{l=1}^{2}\sum_{x=0}^{u}g_{il}(x)h_{lj}^d(u+1-x,y)+g_{ij}(u+1+y). \tag{6.19}
$$

Assuming $\sum_{u=0}^{\infty}h_{ij}^d(u,y)$ exists (which it will if $\sum_{u=0}^{\infty}\psi_i^d(u)$ exists), summing over u yields

$$
\sum_{u=0}^{\infty}h_{ij}^d(u,y)=\sum_{l=1}^{2}\sum_{u=0}^{\infty}\sum_{x=0}^{u}g_{il}(x)h_{lj}^d(u+1-x,y)+\sum_{u=0}^{\infty}g_{ij}(u+1+y).
$$

Setting $u+1=n$, $u+1+y=x$ and changing the order of summation gives

$$\begin{aligned}\sum_{u=0}^{\infty} h_{ij}^{d}(u,y) &= \sum_{l=1}^{2}\sum_{n=1}^{\infty}\sum_{x=0}^{n-1} g_{il}(x)h_{lj}^{d}(n-x,y) + \sum_{x=1+y}^{\infty} g_{ij}(x) \\ &= \sum_{l=1}^{2}\sum_{x=0}^{\infty} g_{il}(x)\sum_{n=x+1}^{\infty} h_{lj}^{d}(n-x,y) + (p_{ij}-G_{ij}(y))\end{aligned}$$

which can be rewritten as

$$h_{ij}^{d}(0,y) + \sum_{u=1}^{\infty} h_{ij}^{d}(u,y) = \sum_{l=1}^{2} p_{il}\sum_{u=1}^{\infty} h_{lj}^{d}(u,y) + (p_{ij}-G_{ij}(y)).$$

When $i=j=1$, we have

$$h_{11}^{d}(0,y) + p_{12}\sum_{u=1}^{\infty} h_{11}^{d}(u,y) = p_{12}\sum_{u=1}^{\infty} h_{21}^{d}(u,y) + (p_{11}-G_{11}(y)), \tag{6.20}$$

and when $i=2$ and $j=1$, we have

$$h_{21}^{d}(0,y) + p_{21}\sum_{u=1}^{\infty} h_{21}^{d}(u,y) = p_{21}\sum_{u=1}^{\infty} h_{11}^{d}(u,y) + (p_{21}-G_{21}(y)). \tag{6.21}$$

Multiplying (6.20) by p_{21} and (6.21) by p_{12} and adding the resulting equations, we obtain the relationship between $h_{11}^{d}(0,y)$ and $h_{21}^{d}(0,y)$, which is given by (6.14). Similarly, we can derive equation (6.16) which shows the relationship between $h_{12}^{d}(0,y)$ and $h_{22}^{d}(0,y)$.

We apply the method of generating functions to build the second pair of equations. Multiplying formula (6.19) by s^{u+1} and summing over u yields

$$\sum_{u=0}^{\infty} s^{u+1}h_{ij}^{d}(u,y) = \sum_{l=1}^{2}\sum_{u=0}^{\infty}\sum_{x=0}^{u} s^{u+1}g_{il}(x)h_{lj}^{d}(u+1-x,y) + \sum_{u=0}^{\infty} s^{u+1}g_{ij}(u+1+y). \tag{6.22}$$

We define $\tilde{h}_{ij}^{d}(s,y) = \sum_{u=0}^{\infty} s^{u}h_{ij}^{d}(u,y)$, and set $n=u+1$ in the first term on the right-hand side of (6.22) to get

$$s\tilde{h}_{ij}^{d}(s,y) = \sum_{l=1}^{2}\sum_{n=1}^{\infty}\sum_{x=0}^{n-1} s^{n}g_{il}(x)h_{lj}^{d}(n-x,y) + A_{ij}(s,y).$$

Hence

$$s\tilde{h}_{ij}^{d}(s,y) = \sum_{l=1}^{2} \tilde{g}_{il}(s)\tilde{h}_{lj}^{d}(s,y) - e_{ij}(s,y) + A_{ij}(s,y) \tag{6.23}$$

where $e_{ij}(s,y) = \sum_{l=1}^{2} \tilde{g}_{il}(s) h^d_{lj}(0,y)$. We can write (6.23) for $i=1,2$ and $j=1$ as

$$\begin{cases} (\tilde{g}_{11}(s)-s)\tilde{h}^d_{11}(s,y) + \tilde{g}_{12}(s)\tilde{h}^d_{21}(s,y) = e_{11}(s,y) - A_{11}(s,y) \\ \tilde{g}_{21}(s)\tilde{h}^d_{11}(s,y) + (\tilde{g}_{22}(s)-s)\tilde{h}^d_{21}(s,y) = e_{21}(s,y) - A_{21}(s,y), \end{cases}$$

giving

$$\begin{aligned} & \left((\tilde{g}_{11}(s)-s)\left(\tilde{g}_{22}(s)-s\right) - \tilde{g}_{12}(s)\tilde{g}_{21}(s) \right) \tilde{h}^d_{11}(s,y) \\ = \;& (\tilde{g}_{22}(s)-s)\, e_{11}(s,y) - \tilde{g}_{12}(s) e_{21}(s,y) + \tilde{g}_{12}(s) A_{21}(s,y) - (\tilde{g}_{22}(s)-s)\, A_{11}(s,y). \end{aligned} \tag{6.24}$$

Similarly, for $i=1,2$ and $j=2$ we have

$$\begin{cases} (\tilde{g}_{11}(s)-s)\tilde{h}^d_{12}(s,y) + \tilde{g}_{12}(s)\tilde{h}^d_{22}(s,y) = e_{12}(s,y) - A_{12}(s,y) \\ \tilde{g}_{21}(s)\tilde{h}^d_{12}(s,y) + (\tilde{g}_{22}(s)-s)\tilde{h}^d_{22}(s,y) = e_{22}(s,y) - A_{22}(s,y), \end{cases}$$

giving

$$\begin{aligned} & \left((\tilde{g}_{11}(s)-s)\left(\tilde{g}_{22}(s)-s\right) - \tilde{g}_{12}(s)\tilde{g}_{21}(s) \right) \tilde{h}^d_{12}(s,y) \\ = \;& (\tilde{g}_{22}(s)-s)\, e_{12}(s,y) - \tilde{g}_{12}(s) e_{22}(s,y) + \tilde{g}_{12}(s) A_{22}(s,y) - (\tilde{g}_{22}(s)-s)\, A_{12}(s,y). \end{aligned} \tag{6.25}$$

Inserting for $e_{11}(s,y)$ and $e_{21}(s,y)$, we can write equation (6.24) as $L_1(s)\tilde{h}^d_{11}(s,y) = L_2^{(1)}(s)$, where $L_1(s)$ is given by (6.12), and

$$\begin{aligned} L_2^{(1)}(s) \;=\; & \left((\tilde{g}_{22}(s)-s)\, \tilde{g}_{11}(s) - \tilde{g}_{12}(s)\tilde{g}_{21}(s) \right) h^d_{11}(0,y) + \tilde{g}_{12}(s) A_{21}(s,y) \\ & + \left((\tilde{g}_{22}(s)-s)\, \tilde{g}_{12}(s) - \tilde{g}_{12}(s)\tilde{g}_{22}(s) \right) h^d_{21}(0,y) - (\tilde{g}_{22}(s)-s)\, A_{11}(s,y). \end{aligned}$$

Similarly, equation (6.25) can be written as $L_1(s)\tilde{h}^d_{12}(s,y) = L_2^{(2)}(s)$, where

$$\begin{aligned} L_2^{(2)}(s) \;=\; & \left((\tilde{g}_{22}(s)-s)\, \tilde{g}_{11}(s) - \tilde{g}_{12}(s)\tilde{g}_{21}(s) \right) h^d_{12}(0,y) + \tilde{g}_{12}(s) A_{22}(s,y) \\ & + \left((\tilde{g}_{22}(s)-s)\, \tilde{g}_{12}(s) - \tilde{g}_{12}(s)\tilde{g}_{22}(s) \right) h^d_{22}(0,y) - (\tilde{g}_{22}(s)-s)\, A_{12}(s,y). \end{aligned}$$

Setting $s=0$, and noting that $A_{ij}(0,y)=0$, we have

$$L_2^{(1)}(0) = \left(g_{22}(0) g_{11}(0) - g_{12}(0) g_{21}(0) \right) h^d_{11}(0,y).$$

Also, setting $s = 1$, and noting that $A_{ij}(1, y) = \sum_{u=0}^{\infty} g_{ij}(u + 1 + y) < p_{ij}$ gives

$$
\begin{aligned}
L_2^{(1)}(1) &= ((p_{22} - 1)\, p_{11} - p_{12}p_{21})\, h_{11}^d(0, y) + p_{12}A_{21}(1, y) \\
&\quad + ((p_{22} - 1)\, p_{12} - p_{12}p_{22})\, h_{21}^d(0, y) + p_{21}A_{11}(1, y).
\end{aligned}
$$

Assuming $L_2^{(1)}(0) > 0$ and $L_2^{(1)}(1) < 0$, which holds under the assumptions stated in the theorem and applies to all our numerical examples, we can conclude that there exists $\rho \in (0, 1)$ such that $L_1(\rho) = L_2^{(1)}(\rho) = 0$, and by the same argument that $L_1(\rho) = L_2^{(2)}(\rho) = 0$. Therefore, we can find the second pair of equations that define the relationship between $h_{11}(0, y), h_{21}(0, y)$ and $h_{12}(0, y), h_{22}(0, y)$. □

6.3.2 The probability of ultimate ruin

In this section, building on the idea of Chen et al. (2014b), we first develop two recursive formulae for the probability of ruin in our discrete time model when $m = 2$, then we show that as these formulae are unstable for computation we need to construct an algorithm in terms of $\{h_{ij}^d(0, y)\}_{y=0}^{\infty}$, $i, j = 1, 2$. Our approach is different from Chen et al. (2014b) in one major respect: we follow the definition of ruin in Chapter 5, i.e. ruin occurs when the surplus falls to or goes below zero. With our definition of ruin, equation (3) of Chen et al. (2014b) changes to

$$
\psi_i^d(u) = \sum_{j=1}^{2}\sum_{x=0}^{u} g_{ij}(x)\psi_j^d(u + 1 - x) + \sum_{j=1}^{2}\sum_{x=u+1}^{\infty} g_{ij}(x). \tag{6.26}
$$

We can write (6.26) for each state as

$$
\begin{aligned}
\psi_1^d(u + 1) &= f_0^{-1}\bigg(g_{12}(0)\psi_2^d(u) - g_{22}(0)\psi_1^d(u) \\
&\quad + \sum_{x=1}^{u} \psi_1^d(u + 1 - x)\,(g_{22}(0)g_{11}(x) - g_{12}(0)g_{21}(x)) \\
&\quad + \sum_{x=1}^{u} \psi_2^d(u + 1 - x)\,(g_{22}(0)g_{12}(x) - g_{12}(0)g_{22}(x)) \\
&\quad + g_{22}(0)\,(1 - G_1(u)) - g_{12}(0)\,(1 - G_2(u)) \bigg)
\end{aligned} \tag{6.27}
$$

for $u = 0, 1, 2, \ldots$, where $f_0 = g_{12}(0)g_{21}(0) - g_{11}(0)g_{22}(0)$ as defined by Chen et al. (2014b, Section 3) and

$$\begin{aligned}
\psi_2^d(u+1) &= f_0^{-1}\Bigg(g_{21}(0)\psi_1^d(u) - g_{11}(0)\psi_2^d(u) \\
&\quad + \sum_{x=1}^{u} \psi_1^d(u+1-x)\left(g_{11}(0)g_{21}(x) - g_{21}(0)g_{11}(x)\right) \\
&\quad + \sum_{x=1}^{u} \psi_2^d(u+1-x)\left(g_{11}(0)g_{22}(x) - g_{21}(0)g_{12}(x)\right) \\
&\quad + g_{11}(0)\left(1 - G_2(u)\right) - g_{21}(0)\left(1 - G_1(u)\right)\Bigg). \qquad (6.28)
\end{aligned}$$

Proceeding numerically with formulae (6.27) and (6.28) we have faced a problem of instability. This problem arises because these two formulae involve subtracting many terms. According to Panjer and Wang (1993, Section 11.5) this is a reason for a recursion scheme to be unstable. To solve this problem, we modify equation (6.5) of Dickson et al. (1995). The result is given in the following.

Theorem 6.3. When $m = 2$, for $u = 1, 2, 3, \ldots$, we have

$$\begin{aligned}
\psi_1^d(u) &= f^{-1}\Bigg(\left(1 - h_{22}^d(0,0)\right)\psi_1^d(0) + h_{12}^d(0,0)\psi_2^d(0) \\
&\quad + \sum_{x=1}^{u-1} \psi_1^d(u-x)\left(\left(1 - h_{22}^d(0,0)\right)h_{11}^d(0,x) + h_{12}^d(0,0)h_{21}^d(0,x)\right) \\
&\quad + \sum_{x=1}^{u-1} \psi_2^d(u-x)\left(\left(1 - h_{22}^d(0,0)\right)h_{12}^d(0,x) + h_{12}^d(0,0)h_{22}^d(0,x)\right) \\
&\quad - \left(1 - h_{22}^d(0,0)\right)\sum_{x=0}^{u-1} h_1^d(0,x) - h_{12}^d(0,0)\sum_{x=0}^{u-1} h_2^d(0,x)\Bigg) \qquad (6.29)
\end{aligned}$$

and

$$\begin{aligned}
\psi_2^d(u) &= f^{-1}\Bigg(\left(1 - h_{11}^d(0,0)\right)\psi_2^d(0) + h_{21}^d(0,0)\psi_1^d(0) \\
&\quad + \sum_{x=1}^{u-1} \psi_1^d(u-x)\left(\left(1 - h_{11}^d(0,0)\right)h_{21}^d(0,x) + h_{21}^d(0,0)h_{11}^d(0,x)\right) \\
&\quad + \sum_{x=1}^{u-1} \psi_2^d(u-x)\left(\left(1 - h_{11}^d(0,0)\right)h_{22}^d(0,x) + h_{21}^d(0,0)h_{12}^d(0,x)\right) \\
&\quad - \left(1 - h_{11}^d(0,0)\right)\sum_{x=0}^{u-1} h_2^d(0,x) - h_{21}^d(0,0)\sum_{x=0}^{u-1} h_1^d(0,x)\Bigg) \qquad (6.30)
\end{aligned}$$

where $f = \left(1 - h_{11}^d(0,0)\right)\left(1 - h_{22}^d(0,0)\right) - h_{12}^d(0,0)h_{21}^d(0,0)$.

Proof. We begin with

$$\psi_i^d(u) = \sum_{j=1}^{2}\sum_{x=0}^{u-1} h_{ij}^d(0,x)\psi_j^d(u-x) + \sum_{j=1}^{2}\sum_{x=u}^{\infty} h_{ij}^d(0,x). \tag{6.31}$$

By noting that $\psi_i^d(0) = \sum_{x=0}^{\infty} h_i^d(0,x)$ we can write (6.31) as

$$\begin{aligned}\psi_1^d(u) &= h_{11}^d(0,0)\psi_1^d(u) + h_{12}^d(0,0)\psi_2^d(u) + \sum_{x=1}^{u-1} h_{11}^d(0,x)\psi_1^d(u-x) \\ &\quad + \sum_{x=1}^{u-1} h_{12}^d(0,x)\psi_2^d(u-x) + \psi_1^d(0) - \sum_{x=0}^{u-1} h_1^d(0,x)\end{aligned} \tag{6.32}$$

and

$$\begin{aligned}\psi_2^d(u) &= h_{21}^d(0,0)\psi_1^d(u) + h_{22}^d(0,0)\psi_2^d(u) + \sum_{x=1}^{u-1} h_{21}^d(0,x)\psi_1^d(u-x) \\ &\quad + \sum_{x=1}^{u-1} h_{22}^d(0,x)\psi_2^d(u-x) + \psi_2^d(0) - \sum_{x=0}^{u-1} h_2^d(0,x).\end{aligned} \tag{6.33}$$

Rearranging (6.32) and (6.33), and solving a system of equations we can obtain (6.29) and (6.30), respectively. □

We have not experienced any problem of numerical instability with formulae (6.29) and (6.30) as discussed by Dickson et al. (1995) for the approximation of the classical risk model. These formulae can be easily applied provided that we know the values of $\psi_i^d(0)$ and $h_{ij}^d(0,0)$ for $i,j = 1,2$.

Unfortunately, we were not able to calculate the ultimate ruin probability for $m > 2$. This issue arises, because we need m equations to be able to find the initial values for an m-state model. Using formulae (6.5) and (6.9), we can only obtain two equations which are not sufficient for finding the initial values in a model with more than two states. In Section 6.5.2 we suggest a method that gives us an estimate for the ultimate ruin probability for a model with $m > 2$ states.

6.3.3 The probability and severity of ruin

In this section, we derive recursive formulae for the probability and severity of ruin function in our discrete time model when $m = 2$. For this, we can modify equation

(4.2) of Dickson et al. (1995), which can be used to approximate $H_1(u,y)$ in the classical risk model, and write expressions for the probability and severity of ruin function from which we can approximate $H_{1,i}(u,y)$ in the Markov-modulated risk model.

Theorem 6.4. For $m=2$, $u=1,2,3,\ldots$ and $y=0,1,2,\ldots$ we have

$$\begin{aligned} H_1^d(u,y) &= f^{-1}\Bigg(\left(1-h_{22}^d(0,0)\right)\left(H_1^d(0,u+y)-H_1^d(0,u)\right) \\ &+h_{12}^d(0,0)\left(H_2^d(0,u+y)-H_2^d(0,u)\right) \\ &+\sum_{x=1}^{u-1} H_1^d(u-x,y)\left(h_{11}^d(0,x)\left(1-h_{22}^d(0,0)\right)+h_{12}^d(0,0)h_{21}^d(0,x)\right) \\ &+\sum_{x=1}^{u-1} H_2^d(u-x,y)\left(h_{12}^d(0,x)\left(1-h_{22}^d(0,0)\right)+h_{12}^d(0,0)h_{22}^d(0,x)\right)\Bigg) \end{aligned} \tag{6.34}$$

and

$$\begin{aligned} H_2^d(u,y) &= f^{-1}\Bigg(h_{21}^d(0,0)\left(H_1^d(0,u+y)-H_1^d(0,u)\right) \\ &+\left(1-h_{11}^d(0,0)\right)\left(H_2^d(0,u+y)-H_2^d(0,u)\right) \\ &+\sum_{x=1}^{u-1} H_1^d(u-x,y)\left(\left(1-h_{11}^d(0,0)\right)h_{21}^d(0,x)+h_{21}^d(0,0)h_{11}^d(0,x)\right) \\ &+\sum_{x=1}^{u-1} H_2^d(u-x,y)\left(\left(1-h_{11}^d(0,0)\right)h_{22}^d(0,x)+h_{21}^d(0,0)h_{12}^d(0,x)\right)\Bigg) \end{aligned} \tag{6.35}$$

where $f=\left(1-h_{11}^d(0,0)\right)\left(1-h_{22}^d(0,0)\right)-h_{12}^d(0,0)h_{21}^d(0,0)$.

Proof. We start with

$$\begin{aligned} H_i^d(u,y) &= \sum_{j=1}^{2}\sum_{x=0}^{u-1} h_{ij}^d(0,x)H_j^d(u-x,y)+\sum_{j=1}^{2}\sum_{x=u}^{u+y-1} h_{ij}^d(0,x) \\ &= H_i^d(0,u+y)-H_i^d(0,u)+\sum_{j=1}^{2}\sum_{x=0}^{u-1} h_{ij}^d(0,x)H_j^d(u-x,y) \end{aligned} \tag{6.36}$$

where the last line is obtained by noting that $H_i^d(0,y)=\sum_{x=0}^{y-1} h_i^d(0,x)$. For $i=1$, (6.36) can be written as

$$\begin{aligned} H_1^d(u,y) &= H_1^d(0,u+y) - H_1^d(0,u) + h_{11}^d(0,0)H_1^d(u,y) + h_{12}^d(0,0)H_2^d(u,y) \\ &\quad + \sum_{x=1}^{u-1} h_{11}^d(0,x)H_1^d(u-x,y) + \sum_{x=1}^{u-1} h_{12}^d(0,x)H_2^d(u-x,y) \end{aligned}$$

and for $i = 2$ we have

$$\begin{aligned} H_2^d(u,y) &= H_2^d(0,u+y) - H_2^d(0,u) + h_{21}^d(0,0)H_1^d(u,y) + h_{22}^d(0,0)H_2^d(u,y) \\ &\quad + \sum_{x=1}^{u-1} h_{21}^d(0,x)H_1^d(u-x,y) + \sum_{x=1}^{u-1} h_{22}^d(0,x)H_2^d(u-x,y). \end{aligned}$$

After rearranging and solving a system of equations we obtain (6.34) and (6.35). □

In the next section, we develop algorithms to approximate the probability of ruin and probability and severity of ruin in the continuous time Markov-modulated model by applying equations (6.29), (6.30), (6.34) and (6.35).

6.4 Numerical illustrations

In this section, we extend our numerical algorithm in Chapter 5, which is based on the ideas of Dickson and Waters (1991, 1992), to approximate the probability of ruin and the probability and severity of ruin in the continuous time Markov-modulated model. Their idea is that if we split a time interval into a large set of small intervals we can use a discrete time model as an approximation to a continuous time model. Therefore, the application of our algorithms is based on rescaling of the time unit and the monetary unit. To discretise a continuous distribution we apply Result 1.1. For the time unit, without loss of generality we assume that premium income per unit of time is $c = 1$, and rescale the time period so that this assumption always holds. Therefore, if ct is the total premium income up to time t in the continuous time model, $t = 1/c\beta$ would be our time unit in the discrete time model, where β is the scaling factor for discretisation.

We consider two claim amount distributions: exponential with mean $1/\mu_i$, $i = 1, 2$, for which explicit results can be obtained in the continuous time case (and therefore we can compare our approximate values with the true values), and Pareto with parameters a_i and b_i for which we cannot find analytical formulae.

The first step is similar to our algorithm in the previous chapter. We need to discretise the claim size distributions – the discretised versions of scaled exponential and Pareto distributions are given by (5.4) and (5.5), respectively. Then, we can apply Panjer's (1981) recursion formula to calculate the aggregate claim amount distributions in states 1 and 2 given $\{p_{ij}\}_{i,j=1}^{2}$.

The next step is the computation of $\psi_1^d(u)$, $\psi_2^d(u)$ and their initial values. Equations (6.3), (6.15) and (6.17) are based on the probability generating functions of the aggregate claim amount with parameter ρ and in order to find ρ we need to solve $L_1(\rho) = 0$ from formula (6.4). In our model, where the distribution of the number of claims over $(0, t)$ in state j is Poisson with parameter $\lambda_j t$, we can define $\tilde{g}_{ij}$ by

$$\tilde{g}_{ij}(s) = p_{ij}\tilde{g}_j(s) = p_{ij} \exp\{\lambda_j t(\tilde{f}_j(s) - 1)\},$$

where $\tilde{f}_j(s) = \sum_{x=0}^{\infty} s^x f_j(x)$. The probability generating function of the claim amount distribution, has an explicit form in the case of the discretised exponential distribution. Therefore, we can calculate $\tilde{g}_{ij}(s)$ and substitute in (6.4) and solve $L_1(\rho) = 0$ to find ρ. However, the explicit form for the probability generating function of the discretised Pareto distribution does not exist, and we need to find ρ by numerical methods such as the Newton-Raphson method, where we find a sequence $\{\rho_n\}$ given by

$$\rho_{n+1} = \rho_n - \frac{L_1(\rho_n)}{L_1^{'}(\rho_n)}$$

where

$$L_1(\rho_n) = (\tilde{g}_{11}(\rho_n) - \rho_n)(\tilde{g}_{22}(\rho_n) - \rho_n) - \tilde{g}_{21}(\rho_n)\tilde{g}_{12}(\rho_n)$$

and

$$\begin{aligned} L_1^{'}(\rho_n) &= \tilde{g}_{11}^{'}(\rho_n)\tilde{g}_{22}(\rho_n) + \tilde{g}_{22}^{'}(\rho_n)\tilde{g}_{11}(\rho_n) - \Big(\tilde{g}_{12}^{'}(\rho_n)\tilde{g}_{21}(\rho_n) + \tilde{g}_{12}(\rho_n)\tilde{g}_{21}^{'}(\rho_n)\Big) \\ &\quad +2\rho_n - \Big(\tilde{g}_{11}(\rho_n) + \rho_n\tilde{g}_{11}^{'}(\rho_n) + \tilde{g}_{22}(\rho_n) + \rho\tilde{g}_{22}^{'}(\rho_n)\Big). \end{aligned}$$

Further, we have

$$\tilde{g}_{ij}^{'}(\rho_n) = p_{ij}\lambda_j t\tilde{f}_j^{'}(\rho_n) \exp\{\lambda_j t(\tilde{f}_j(\rho_n) - 1\}$$

and $\tilde{f}_j^{'}(\rho_n) = \sum_{x=1}^{\infty} x\rho_n^{x-1} f_j(x)$ for which we require to truncate the summation. Let L be the truncation point. Then, we choose L such that $\bar{\mathcal{F}}_i(L) = \sum_{i=L}^{\infty} f_i(x) < \epsilon$, where ϵ is a small strictly positive value.

As the sojourn times in states 1 and 2 are exponentially distributed with intensity rate α_i in the continuous time model and as our time intervals are very short, we can calculate the transition probability matrix as follows

$$\begin{pmatrix} e^{-\alpha_1/c\beta} & 1-e^{-\alpha_1/c\beta} \\ 1-e^{-\alpha_2/c\beta} & e^{-\alpha_2/c\beta} \end{pmatrix}.$$

In our numerical examples, we consider the situations in which α_1 takes values of $0.1, 0.3, 0.5, 0.7, 0.9$, and $\alpha_2 = 0.5$ is fixed. Our aim is to examine the impact of the length of stay on the probability of ruin in different states. Specifically, we will consider situations when either both states have equal expected aggregate claim amount $E[S_i]$, or one of them has greater $E[S_i]$. Without loss of generality, we assume that the arrival rate and the mean of the individual claims in state 1 is 1, i.e. $\lambda_1 = m_1 = 1$. Hence $E[S_1] = 1$, and we assume that $E[S_2]$ is either equal, greater or less than $E[S_1]$. Our numerical example is based on the following six cases for the continuous time model:

1. $E[S_1] = E[S_2]$: $\lambda_1 = 1, \lambda_2 = 2, \mu_1 = 1, \mu_2 = 2$,
2. $E[S_1] > E[S_2]$: $\lambda_1 = 1, \lambda_2 = 0.5, \mu_1 = 1, \mu_2 = 2$,
3. $E[S_1] < E[S_2]$: $\lambda_1 = 1, \lambda_2 = 2, \mu_1 = 1, \mu_2 = 0.5$,
4. $E[S_1] = E[S_2]$: $\lambda_1 = 1, \lambda_2 = 2, a_1 = 2, b_1 = 1, a_2 = 3, b_2 = 1$,
5. $E[S_1] > E[S_2]$: $\lambda_1 = 1, \lambda_2 = 0.5, a_1 = 2, b_1 = 1, a_2 = 3, b_2 = 1$,
6. $E[S_1] < E[S_2]$: $\lambda_1 = 1, \lambda_2 = 2, a_1 = 2, b_1 = 1, a_2 = 3, b_2 = 4$.

Further, we assume that the notional premium loading factor θ is 0.1 so that the positive loading condition given by (2.2) is satisfied.

Our experiments with different scaling factors show that, contrary to the classical risk model, $\beta = 20$ and $\beta = 50$ do not give satisfactory approximations and the approximate values of ruin probabilities are equal to the exact values only up to two decimal places. With $\beta = 100$ there are few cases that the approximations agree with the exact values to four decimal places. With $\beta = 300$, the approximation improves further. However, with $\beta = 500$ we did not observe significant improvement. Therefore, we set $\beta = 300$ throughout.

The last thing to consider is the truncation point. We need to test how robust the calculation of ρ is with respect to L. Table 6.1 shows the upper tail probability and the calculated ρ for different values of L in the case of exponential and Pareto distributions when $E[S_1] > E[S_2]$, i.e. Cases 2 and 5.

Table 6.1: Values of $\bar{\mathcal{F}}_1(L)$, $\bar{\mathcal{F}}_2(L)$ and ρ when $\beta = 300$

Scaled L	3000	6000	20000	65000
Exponential	(1) 4.5×10^{-5} (2) 2.1×10^{-9}	2.1×10^{-9} 4.3×10^{-18}	1.1×10^{-29} 1.3×10^{-58}	8.0×10^{-95} 6.4×10^{-189}
ρ	0.997108645688	0.997108645695	0.997108645695	0.997108645695
Pareto	(1) 8.0×10^{-3} (2) 7.5×10^{-4}	2.1×10^{-3} 1.1×10^{-4}	1.2×10^{-4} 2.3×10^{-6}	7.2×10^{-6} 4.5×10^{-8}
ρ	0.997284980997	0.997284981604	0.997284981604	0.997284981604

As we can observe, the calculated value of ρ is not highly sensitive to L and our experiments with the above values show that it does not impact approximations considerably. Since the choice of L affects the running time of our programmes, we set (scaled) $L = 3000$ in all our numerical examples.

6.4.1 Approximations to $\psi_1(u)$ and $\psi_2(u)$

Tables 6.2, 6.3 and 6.4 show exact and approximate values for the ultimate ruin probability with initial surplus u in the continuous time model when the individual claim amounts are exponentially distributed. We can apply the methods of Li and Lu (2008) for $u = 0$ to show that

$$\psi_{ii}(0) = \frac{\lambda_i \mu_i \left(c - \lambda_j \mu_j + c\rho^* \mu_j + \alpha_j \mu_i (1 + \rho^* \mu_j)\right)}{c(1 + \rho^* \mu_i)(c - \lambda_j \mu_j + c\rho^* \mu_j)} \tag{6.37}$$

and

$$\psi_{ij}(0) = \frac{\alpha_i \lambda_j \mu_j^2}{c(c - \lambda_j \mu_j + c\rho^* \mu_j)} \tag{6.38}$$

where $i, j = 1, 2$, and $i \neq j$ with ρ^* being the unique positive solution of the equation

$$\left(\rho^* - \frac{\alpha_2}{c} - \frac{\lambda_2}{c}(1 - \tilde{f}_2(\rho^*))\right)\left(\rho^* - \frac{\alpha_1}{c} - \frac{\lambda_1}{c}(1 - \tilde{f}_1(\rho^*))\right) - \frac{\alpha_1 \alpha_2}{c^2} = 0 \tag{6.39}$$

where $\tilde{f}_i(s)$ is the Laplace Transform of the claim size distribution and in the case of claim amounts following exponential distributions with parameter $1/\mu_i$, is $\tilde{f}_i(s) =$

$\mu_i^{-1}/(s+\mu_i^{-1})$. Equation (6.39) is, in fact, equation (13) of Lu and Li (2005) adjusted to our assumption of constant premium income regardless of the state being occupied.

For $u > 0$ we can apply the Gerber-Shiu function from Li and Lu (2008) given by formula (2.3) and write a system of differential equations that can be solved through the Laplace transform method after substituting for f. Define $\tilde{\psi}_{ij}(s)$ to be the Laplace transform of the probability of ruin in the continuous time Markov-modulated model. Then, we can show that

$$\tilde{\psi}_{ii}(s) = \frac{s\psi_{ii}(0) - \frac{\lambda_i}{c} + \frac{\psi_{ii}(0)}{\mu_i} - \left(\left(s\psi_{ji}(0) + \frac{\psi_{ji}(0)}{\mu_j}\right)\left(\frac{s\alpha_i}{c} + \frac{\alpha_i}{c\mu_i}\right)\right)/A_j(s)}{A_i(s) - \left(\left(\frac{s\alpha_j}{c} + \frac{\alpha_j}{c\mu_j}\right)\left(\frac{s\alpha_i}{c} + \frac{\alpha_i}{c\mu_i}\right)\right)/A_j(s)} \tag{6.40}$$

and

$$\tilde{\psi}_{ij}(s) = \frac{s\psi_{ij}(0) - \left(\frac{s\alpha_i}{c} + \frac{\alpha_i}{c\mu_i}\right)\tilde{\psi}_{jj}(s) + \psi_{ij}(0)/\mu_i}{A_i(s)} \tag{6.41}$$

where

$$A_i(s) = s^2 - s\left(\frac{\alpha_i + \lambda_i}{c} - \frac{1}{\mu_i}\right) - \frac{\alpha_i}{c\mu_i}. \tag{6.42}$$

Formulae (6.40) and (6.41) can be readily inverted with mathematical software.

The key for Tables 6.2 to 6.7 is as follows:

(1) denotes the approximation to $\psi_1(u)$,

(2) denotes the exact value of $\psi_1(u)$,

(3) denotes the ratio of the value in (1) to that in (2),

(4) denotes the approximation to $\psi_2(u)$,

(5) denotes the exact value of $\psi_2(u)$,

(6) denotes the ratio of the value in (4) to that in (5).

Table 6.2: Exponential distribution when $E[S_1] = E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 1.1$
0	(1)	0.90491	0.89968	0.89682	0.89523	0.89438
	(2)	0.90491	0.89967	0.89680	0.89520	0.89434
	(3)	1.00000	1.00001	1.00002	1.00003	1.00004
	(4)	0.92997	0.92477	0.92136	0.91899	0.91727
	(5)	0.92999	0.92479	0.92139	0.91901	0.91728
	(6)	0.99998	0.99998	0.99997	0.99998	0.99999
5	(1)	0.55287	0.51824	0.49457	0.47734	0.46423
	(2)	0.55288	0.51825	0.49458	0.47733	0.46421
	(3)	0.99998	0.99998	0.99998	1.00002	1.00004
	(4)	0.54820	0.51403	0.49076	0.47388	0.46106
	(5)	0.54820	0.51403	0.49076	0.47386	0.46103
	(6)	1.00000	1.00000	1.00000	1.00004	1.00007
10	(1)	0.33750	0.29784	0.27182	0.25346	0.23982
	(2)	0.33750	0.29784	0.27183	0.25345	0.23979
	(3)	1.00000	1.00000	0.99996	1.00004	1.00013
	(4)	0.33434	0.29514	0.26949	0.25141	0.23799
	(5)	0.33434	0.29515	0.26949	0.25139	0.23796
	(6)	1.00000	1.00003	1.00000	1.00008	1.00013
15	(1)	0.20602	0.17116	0.14938	0.13457	0.12387
	(2)	0.20602	0.17116	0.14939	0.13456	0.12385
	(3)	1.00000	1.00000	0.99993	1.00007	1.00016
	(4)	0.20408	0.16961	0.14810	0.13348	0.12293
	(5)	0.20408	0.16961	0.14810	0.13346	0.12290
	(6)	1.00000	1.00000	1.00000	1.00015	1.00024
20	(1)	0.12576	0.09836	0.08209	0.07145	0.06398
	(2)	0.12576	0.09836	0.08210	0.07144	0.06397
	(3)	1.00000	1.00000	0.99988	1.00014	1.00016
	(4)	0.12458	0.09747	0.08139	0.07087	0.06349
	(5)	0.12457	0.09746	0.08140	0.07085	0.06348
	(6)	1.00008	1.00010	0.99988	1.00028	1.00016
25	(1)	0.07677	0.05652	0.04512	0.03793	0.03305
	(2)	0.07676	0.05652	0.04513	0.03793	0.03304
	(3)	1.00013	1.00000	0.99978	1.00000	1.00030
	(4)	0.07605	0.05601	0.04473	0.03762	0.03280
	(5)	0.07604	0.05600	0.04474	0.03762	0.03279
	(6)	1.00013	1.00018	0.99978	1.00000	1.00030
30	(1)	0.04686	0.03248	0.02479	0.02014	0.01707
	(2)	0.04685	0.03247	0.02480	0.02014	0.01706
	(3)	1.00021	1.00031	0.99960	1.00000	1.00059
	(4)	0.04642	0.03219	0.02458	0.01998	0.01694
	(5)	0.04641	0.03218	0.02459	0.01997	0.01693
	(6)	1.00022	1.00031	0.99959	1.00050	1.00059
60	(1)	0.00242	0.00117	0.00068	0.00045	0.00032
	(2)	0.00241	0.00116	0.00069	0.00045	0.00032
	(3)	1.00415	1.00862	0.98551	1.00000	1.00000
	(4)	0.00240	0.00116	0.00068	0.00045	0.00032
	(5)	0.00238	0.00115	0.00069	0.00045	0.00032
	(6)	1.00840	1.00870	0.98551	1.00000	1.00000

Table 6.3: Exponential distribution when $E[S_1] > E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 0.9625$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 0.790625$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 0.6875$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 0.61875$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 0.569643$
0	(1)	0.92023	0.92661	0.92758	0.92723	0.92652
	(2)	0.92031	0.92673	0.92765	0.92719	0.92633
	(3)	0.99991	0.99987	0.99992	1.00004	1.00021
	(4)	0.85296	0.87965	0.89060	0.89640	0.89995
	(5)	0.85300	0.87970	0.89053	0.89616	0.89951
	(6)	0.99995	0.99994	1.00008	1.00027	1.00049
5	(1)	0.61221	0.62592	0.61928	0.60803	0.59608
	(2)	0.61247	0.62630	0.61939	0.60766	0.59504
	(3)	0.99958	0.99939	0.99982	1.00061	1.00175
	(4)	0.54953	0.57815	0.57972	0.57388	0.56586
	(5)	0.54973	0.57845	0.57970	0.57332	0.56458
	(6)	0.99964	0.99948	1.00003	1.00098	1.00227
10	(1)	0.40758	0.42361	0.41468	0.40028	0.38531
	(2)	0.40788	0.42406	0.41479	0.39979	0.38404
	(3)	0.99926	0.99894	0.99973	1.00123	1.00331
	(4)	0.36584	0.39128	0.38818	0.37779	0.36577
	(5)	0.36609	0.39166	0.38820	0.37719	0.36437
	(6)	0.99932	0.99903	0.99995	1.00159	1.00384
15	(1)	0.27134	0.28669	0.27768	0.26351	0.24908
	(2)	0.27163	0.28713	0.27777	0.26303	0.24786
	(3)	0.99893	0.99847	0.99968	1.00182	1.00492
	(4)	0.24356	0.26481	0.25993	0.24871	0.23644
	(5)	0.24380	0.26519	0.25997	0.24816	0.23517
	(6)	0.99902	0.99857	0.99985	1.00222	1.00540
20	(1)	0.18065	0.19403	0.18594	0.17348	0.16101
	(2)	0.18090	0.19442	0.18602	0.17306	0.15997
	(3)	0.99862	0.99799	0.99957	1.00243	1.00650
	(4)	0.16215	0.17922	0.17406	0.16373	0.15284
	(5)	0.16236	0.17956	0.17409	0.16327	0.15178
	(6)	0.99871	0.99811	0.99983	1.00282	1.00698
25	(1)	0.12027	0.13131	0.12451	0.11420	0.10408
	(2)	0.12047	0.13164	0.12457	0.11386	0.10325
	(3)	0.99834	0.99749	0.99952	1.00299	1.00804
	(4)	0.10795	0.12129	0.11655	0.10779	0.09880
	(5)	0.10813	0.12159	0.11658	0.10742	0.09796
	(6)	0.99834	0.99753	0.99974	1.00344	1.00857
30	(1)	0.08007	0.08887	0.08337	0.07518	0.06728
	(2)	0.08023	0.08914	0.08342	0.07491	0.06663
	(3)	0.99801	0.99697	0.99940	1.00360	1.00976
	(4)	0.07187	0.08209	0.07804	0.07096	0.06387
	(5)	0.07201	0.08233	0.07807	0.07067	0.06322
	(6)	0.99806	0.99708	0.99962	1.00410	1.01028
60	(1)	0.00698	0.00854	0.00752	0.00612	0.00491
	(2)	0.00700	0.00861	0.00751	0.00608	0.00482
	(3)	0.99714	0.99187	1.00133	1.00658	1.01867
	(4)	0.00626	0.00789	0.00704	0.00578	0.00466
	(5)	0.00628	0.00795	0.00703	0.00573	0.00457
	(6)	0.99682	0.99245	1.00142	1.00873	1.01969

Table 6.4: Exponential distribution when $E[S_1] < E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.65$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 2.3375$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 2.75$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 3.025$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 3.22143$
0	(1)	0.89881	0.88536	0.87870	0.87592	0.87521
	(2)	0.89869	0.88529	0.87868	0.87592	0.87523
	(3)	1.00013	1.00008	1.00002	1.00000	0.99998
	(4)	0.96109	0.94876	0.93948	0.93274	0.92785
	(5)	0.96108	0.94876	0.93950	0.93278	0.92790
	(6)	1.00001	1.00000	0.99998	0.99996	0.99995
5	(1)	0.70638	0.71981	0.71177	0.70512	0.70093
	(2)	0.70608	0.71968	0.71176	0.70518	0.70104
	(3)	1.00042	1.00018	1.00001	0.99991	0.99984
	(4)	0.82782	0.81297	0.79259	0.77682	0.76525
	(5)	0.82771	0.81292	0.79259	0.77687	0.76534
	(6)	1.00013	1.00006	1.00000	0.99994	0.99988
10	(1)	0.60098	0.61654	0.60085	0.58770	0.57856
	(2)	0.60061	0.61637	0.60083	0.58779	0.57871
	(3)	1.00062	1.00028	1.00003	0.99985	0.99974
	(4)	0.71205	0.69814	0.67009	0.64819	0.63220
	(5)	0.71184	0.69801	0.67002	0.64819	0.63227
	(6)	1.00030	1.00019	1.00010	1.00000	0.99989
15	(1)	0.51621	0.52944	0.50798	0.49038	0.47795
	(2)	0.51580	0.52924	0.50792	0.49044	0.47810
	(3)	1.00079	1.00038	1.00012	0.99988	0.99969
	(4)	0.61238	0.59962	0.56660	0.54093	0.52234
	(5)	0.61210	0.59941	0.56644	0.54085	0.52237
	(6)	1.00046	1.00035	1.00028	1.00015	0.99994
20	(1)	0.44387	0.45473	0.42951	0.40921	0.39487
	(2)	0.44346	0.45448	0.42940	0.40922	0.39500
	(3)	1.00092	1.00055	1.00026	0.99998	0.99967
	(4)	0.52664	0.51503	0.47914	0.45146	0.43161
	(5)	0.52632	0.51474	0.47888	0.45129	0.43156
	(6)	1.00061	1.00056	1.00054	1.00038	1.00012
25	(1)	0.38174	0.39058	0.36320	0.34151	0.32626
	(2)	0.38130	0.39028	0.36302	0.34146	0.32633
	(3)	1.00115	1.00077	1.00050	1.00015	0.99979
	(4)	0.45293	0.44241	0.40523	0.37684	0.35668
	(5)	0.45256	0.44203	0.40485	0.37656	0.35655
	(6)	1.00082	1.00086	1.00094	1.00074	1.00036
30	(1)	0.32830	0.33550	0.30716	0.28504	0.26959
	(2)	0.32787	0.33515	0.30690	0.28492	0.26961
	(3)	1.00131	1.00104	1.00085	1.00042	0.99993
	(4)	0.38953	0.38005	0.34277	0.31460	0.29479
	(5)	0.38914	0.37959	0.34226	0.31421	0.29457
	(6)	1.00100	1.00121	1.00149	1.00124	1.00075
60	(1)	0.13286	0.13502	0.11278	0.09673	0.08608
	(2)	0.13252	0.13441	0.11205	0.09616	0.08573
	(3)	1.00257	1.00454	1.00651	1.00593	1.00408
	(4)	0.15765	0.15308	0.12609	0.10704	0.09439
	(5)	0.15728	0.15223	0.12496	0.10604	0.09367
	(6)	1.00235	1.00558	1.00904	1.00943	1.00769

We note the following points about Tables 6.2, 6.3 and 6.4.

(i) In Tables 6.2 and 6.4 most of the approximations agree with the exact values up to four decimal places with the best results being obtained in Table 6.2 when the sojourn time is the same in both states, i.e. $\alpha_1 = \alpha_2 = 0.5$. The approximations in Table 6.3 are in agreement with the exact values up to three decimal places. In this table we get better approximations when $\alpha_1 < 0.9$.

(ii) The ratios of the approximate values to the exact values show that some of our approximations are overestimated and some are underestimated. For example, in Table 6.3 when $\alpha_1 = 0.7, 0.9$, the ruin probability is overestimated and the ratios are greater than one, whereas for $\alpha_1 = 0.1, 0.3, 0.5$, it is mostly underestimated. In Table 6.4, unlike in Table 6.3, if $\alpha_1 = 0.1, 0.3, 0.5$, the ruin probability is overestimated and if $\alpha_1 = 0.7, 0.9$, and $u = 0, 5, 10, 15$, all the approximations are underestimated. We cannot observe any pattern for the ratios with different values of u and α_1 in Table 6.2.

(iii) Generally, we observe that the approximation in the case of exponential distributions performs better for small values of u and α_1.

(iv) Regarding the relationship between $\psi_1(u)$ and $\psi_2(u)$ we can see that as u increases, $\psi_1(u)$ gets closer to $\psi_2(u)$. In Table 6.3, where $E[S_1] > E[S_2]$, values of $\psi_1(u)$ are always greater than $\psi_2(u)$. In Table 6.4, where $E[S_1] < E[S_2]$, $\psi_1(u)$ is always less than $\psi_2(u)$, but in Table 6.2, where $E[S_1] = E[S_2]$, we cannot identify any consistent pattern between $\psi_1(u)$ and $\psi_2(u)$ except that for a given value of u if $\psi_1(u) > \psi_2(u)$, it will hold across the table regardless of the mean of the sojourn time. In fact, we can see that the values of $\psi_1(u)$ and $\psi_2(u)$ are very close.

Tables 6.5, 6.6, and 6.7 show the approximate values of $\psi_i(u)$ with initial surplus u in the continuous time model for claim sizes with Pareto distributions. The Laplace transform of a Pareto distribution with parameters a_i and b_i is given by (see Nadarajah and Kotz, 2006)

$$\tilde{f}_i(s) = a_i(b_i s)^{a_i}\Gamma(-a_i, b_i s)e^{b_i s}.$$

Table 6.5: Pareto distribution when $E[S_1] = E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 1.1$
0	(1)	0.90486	0.89971	0.89703	0.89562	0.89492
	(2)	0.90485	0.89970	0.89701	0.89559	0.89488
	(3)	1.00001	1.00001	1.00002	1.00003	1.00004
	(4)	0.93026	0.92472	0.92115	0.91872	0.91697
	(5)	0.93028	0.92474	0.92118	0.91874	0.91699
	(6)	0.99998	0.99998	0.99997	0.99998	0.99998
5	(1)	0.70355	0.67201	0.64976	0.63324	0.62050
	(4)	0.70432	0.67279	0.65052	0.63397	0.62120
10	(1)	0.59768	0.55358	0.52211	0.49853	0.48019
	(4)	0.59668	0.55265	0.52126	0.49775	0.47947
15	(1)	0.52112	0.47026	0.43413	0.40712	0.38616
	(4)	0.51991	0.46914	0.43309	0.40616	0.38527
20	(1)	0.46135	0.40695	0.36866	0.34023	0.31827
	(4)	0.46020	0.40589	0.36768	0.33932	0.31744
25	(1)	0.41284	0.35686	0.31789	0.28919	0.26719
	(4)	0.41180	0.35591	0.31701	0.28839	0.26645
30	(1)	0.37250	0.31618	0.27743	0.24917	0.22765
	(4)	0.37158	0.31534	0.27667	0.24847	0.22701
60	(1)	0.22491	0.17636	0.14518	0.12360	0.10783
	(4)	0.22443	0.17596	0.14483	0.12329	0.10757

Table 6.6: Pareto distribution when $E[S_1] > E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 0.9625$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 0.790625$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 0.6875$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 0.61875$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 0.569643$
0	(1)	0.91660	0.92231	0.92389	0.92416	0.92396
	(2)	0.91668	0.92242	0.92394	0.92410	0.92375
	(3)	0.99991	0.99988	0.99995	1.00006	1.00023
	(4)	0.87111	0.88682	0.89430	0.89860	0.90137
	(5)	0.87116	0.88688	0.89425	0.89838	0.90095
	(6)	0.99994	0.99993	1.00006	1.00024	1.00047
5	(1)	0.73206	0.73109	0.72392	0.71549	0.70717
	(4)	0.70246	0.70751	0.70403	0.69816	0.69174
10	(1)	0.63301	0.62837	0.61737	0.60536	0.59376
	(4)	0.61139	0.61124	0.60304	0.59297	0.58281
15	(1)	0.55976	0.55261	0.53935	0.52535	0.51199
	(4)	0.54268	0.53914	0.52817	0.51575	0.50356
20	(1)	0.50151	0.49257	0.47793	0.46280	0.44851
	(4)	0.48747	0.48156	0.46885	0.45506	0.44176
25	(1)	0.45346	0.44324	0.42779	0.41208	0.39736
	(4)	0.44165	0.43403	0.42023	0.40568	0.39181
30	(1)	0.41291	0.40180	0.38590	0.36998	0.35516
	(4)	0.40281	0.39395	0.37951	0.36459	0.35051
60	(1)	0.25906	0.24654	0.23156	0.21736	0.20456
	(4)	0.25425	0.24291	0.22869	0.21500	0.20258

Table 6.7: Pareto distribution when $E[S_1] < E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.65$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 2.3375$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 2.75$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 3.025$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 3.22143$
0	(1)	0.89905	0.88658	0.88077	0.87845	0.87790
	(2)	0.89894	0.88651	0.88075	0.87845	0.87792
	(3)	1.00012	1.00008	1.00002	1.00000	0.99998
	(4)	0.95988	0.94672	0.93741	0.93094	0.92636
	(5)	0.95987	0.94672	0.93743	0.93098	0.92641
	(6)	1.00001	1.00000	0.99998	0.99996	0.99995
5	(1)	0.76919	0.76736	0.75968	0.75461	0.75168
	(4)	0.85178	0.83295	0.81604	0.80408	0.79570
10	(1)	0.69679	0.69378	0.68181	0.67334	0.66785
	(4)	0.77097	0.75055	0.72959	0.71469	0.70428
15	(1)	0.63713	0.63172	0.61635	0.60538	0.59808
	(4)	0.70277	0.68139	0.65771	0.64088	0.62916
20	(1)	0.58523	0.57739	0.55944	0.54664	0.53804
	(4)	0.64348	0.62131	0.59572	0.57758	0.56501
25	(1)	0.53922	0.52914	0.50925	0.49512	0.48560
	(4)	0.59118	0.56829	0.54135	0.52236	0.50925
30	(1)	0.49805	0.48597	0.46461	0.44955	0.43939
	(4)	0.54461	0.52104	0.49321	0.47369	0.46027
60	(1)	0.32095	0.30111	0.27719	0.26102	0.25025
	(4)	0.34680	0.32053	0.29253	0.27364	0.26096

Applying the method in Li and Lu (2008), we can show that

$$\psi_{ii}(0) = \frac{\lambda_i}{c(a_i-1)}\Big(b_i + \big(1 - a_i + \rho^* b_i + (a_i-1)a_i e^{\rho^* b_i}(\rho^* b_i)^{a_i}\Gamma(-a_i,\rho^* b_i)\big) \times \big(\alpha_j + \lambda_j - c\rho^* - a_j\lambda_j e^{\rho^* b_j}(\rho^* b_j)^{a_j}\Gamma(-a_j,\rho^* b_j)\big) / (\rho^* B_j(\rho^*))\Big) \quad (6.43)$$

and

$$\psi_{ij}(0) = \frac{\alpha_i\lambda_j\big(1 - a_j + \rho^* b_j + (a_j-1)a_j e^{\rho^* b_j}(\rho^* b_j)^{a_j}\Gamma(-a_j,\rho^* b_j)\big)}{c\rho^*(a_j-1)B_j(\rho^*)}, \quad (6.44)$$

where $B_j(\rho^*) = c\rho^* - \lambda_j + a_j\lambda_j e^{\rho^* b_j}(\rho^* b_j)^{a_j}\Gamma(-a_j,\rho^* b_j)$, $i \neq j$, and ρ^* can be found from formula (6.39). The key to Tables 6.5, 6.6 and 6.7 is the same as before.

In the classical risk model, the expression for the starting value of the ruin probability is independent of the individual claim amount distribution. Although this does not hold here as $\psi_{ij}(0)$ depends on the Laplace transform of the claim amount distribution in the continuous time case and on their probability generating function in the discrete time case, we can see that the initial values for exponential and Pareto distributions are fairly close for Cases 1 and 4, Cases 2 and 5 and Cases 3 and 6. We can identify a similar pattern between the approximate values of the ruin probability with claim amounts following Pareto distributions and claim amounts following exponential distributions. For example, in Table 6.6 when $\alpha_1 = 0.7, 0.9$ the ruin probability when $u = 0$ is overestimated, whereas for $\alpha_1 = 0.1, 0.3$ it is underestimated, which is different to Table 6.7 in which the ruin probability is overestimated for $\alpha_1 = 0.1, 0.3$, and underestimated for $\alpha_1 = 0.7, 0.9$. In addition, similar to Table 6.2 no particular pattern can be observed for Table 6.6.

This algorithm is a generalisation of that in Dickson and Waters (1991) for the approximation of the classical risk model. It can be concluded that our algorithm provides a reasonably close approximation to the exact values in the continuous time Markov-modulated model.

6.4.2 Approximations to $H_{1,1}(u, y)$ and $H_{1,2}(u, y)$

In this section, we consider numerical approximations to the probability and severity of ruin for the surplus levels $u = 0, 20, 60, 100$ in the continuous time model. The transition rates are $\alpha_1 = 0.1, 0.3, 0.5, 0.7, 0.9$ and $\alpha_2 = 0.5$ and we set the level of the deficit at ruin as $y = 1$ and $y = 3$. To calculate the exact starting values of the probability and severity of ruin we can apply the method of Li and Lu (2008). It is well-known that in the classical risk model when the individual claim amounts follow an exponential distribution, we can decompose the severity of ruin function into the probability of ruin and a function of y. Similarly, we can use this fact to find the initial values of the probability and severity of ruin in the case of the Markov-modulated model. Thus

$$H_{1,ij}(0, y) = \psi_{ij}(0)(1 - e^{-y/\mu_j}), \tag{6.45}$$

for $i, j = 1, 2$. Further, the exact values of the probability and severity of ruin for $u > 0$ are calculated by applying the Gerber-Shiu function in formula (2.3) similar to the probability of ruin. So, we can establish a system of differential equations and solve them through the Laplace transform method. Define $\tilde{H}_{1,ij}(s, y)$ to be the Laplace transform of the probability and severity of ruin function in the Markov-modulated model. We can show that

$$\begin{aligned} &\tilde{H}_{1,ii}(s, y) \\ = \quad & \frac{1}{A_i(s) - \left(\left(\frac{s\alpha_j}{c} + \frac{\alpha_j}{c\mu_j}\right)\left(\frac{s\alpha_i}{c} + \frac{\alpha_i}{c\mu_i}\right)\right)/A_j(s)} \Bigg\{ -\frac{\lambda_i}{c}\left(1 - e^{-y/\mu_i}\right) + sH_{1,ii}(0, y) \end{aligned}$$

$$+\frac{H_{1,ii}(0,y)}{\mu_i} - \left(\left(sH_{1,ji}(0,y) + \frac{H_{1,ji}(0,y)}{\mu_j}\right)\left(\frac{s\alpha_i}{c} + \frac{\alpha_i}{c\mu_i}\right)\right)/A_j(s)\Bigg\} \tag{6.46}$$

and

$$\tilde{H}_{1,ij}(s,y) = \frac{sH_{1,ij}(0,y) - \left(\frac{s\alpha_i}{c} + \frac{\alpha_i}{c\mu_i}\right)\tilde{H}_{1,jj}(s,y) + H_{1,ij}(0,y)/\mu_i}{A_i(s)} \tag{6.47}$$

where $A_i(s)$ is given by (6.42). Formulae (6.46) and (6.47) can be inverted easily with mathematical software.

The key for Tables 6.8 to 6.19 is as follows:

(1) denotes the approximation to $H_{1,1}(u,y)$,

(2) denotes the exact value of $H_{1,1}(u,y)$,

(3) denotes the ratio of the value in (1) to that in (2),

(4) denotes the approximation to $H_{1,2}(u,y)$,

(5) denotes the exact value of $H_{1,2}(u,y)$,

(6) denotes the ratio of the value in (4) to that in (5).

We note the following points about Tables 6.8 to 6.13.

(i) In all tables the approximation performs better for $y = 3$ than $y = 1$. This accords with the classical risk model as pointed out by Dickson and Waters (1992) that the approximation improves for higher values of y.

(ii) In Tables 6.10 and 6.11 the approximations for $\alpha_1 = 0.1, 0.3, 0.5$ are underestimated and for $\alpha_1 = 0.7, 0.9$ are overestimated when $u > 0$. This is in line with what we had observed for the ruin probability.

(iii) In Tables 6.12 and 6.13 all the approximate values for $\alpha_1 = 0.5, 0.7, 0.9$ are underestimated and in this case the approximation performs better than other cases.

Table 6.8: Exponential distribution, $y = 1$ and $E[S_1] = E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 1.1$
0	(1)	0.58538	0.60443	0.61999	0.63297	0.64398
	(2)	0.58587	0.60489	0.62045	0.63341	0.64441
	(3)	0.99916	0.99924	0.99926	0.99931	0.99933
	(4)	0.72919	0.73487	0.73947	0.74329	0.74650
	(5)	0.72998	0.73566	0.74027	0.74409	0.74730
	(6)	0.99892	0.99893	0.99892	0.99892	0.99893
20	(1)	0.08178	0.06689	0.05768	0.05146	0.04699
	(2)	0.08185	0.06695	0.05774	0.05150	0.04703
	(3)	0.99914	0.99910	0.99896	0.99922	0.99915
	(4)	0.08102	0.06629	0.05718	0.05104	0.04663
	(5)	0.08108	0.06634	0.05724	0.05108	0.04666
	(6)	0.99926	0.99925	0.99895	0.99922	0.99936
60	(1)	0.00158	0.00080	0.00048	0.00032	0.00024
	(2)	0.00157	0.00079	0.00049	0.00033	0.00024
	(3)	1.00637	1.01266	0.97959	0.96970	1.00000
	(4)	0.00156	0.00079	0.00048	0.00032	0.00024
	(5)	0.00155	0.00078	0.00048	0.00032	0.00024
	(6)	1.00645	1.01282	1.00000	1.00000	1.00000
100	(1)	0.00003	0.00000	0.00000	0.00000	0.00000
	(2)	0.00002	0.00000	0.00001	0.00000	0.00000
	(4)	0.00003	0.00000	0.00000	0.00000	0.00000
	(5)	0.00002	0.00000	0.00001	0.00000	0.00000

Table 6.9: Exponential distribution, $y = 3$ and $E[S_1] = E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 1.1$
0	(1)	0.86249	0.86209	0.86292	0.86428	0.86583
	(2)	0.86268	0.86224	0.86305	0.86437	0.86590
	(3)	0.99978	0.99983	0.99985	0.99990	0.99992
	(4)	0.91246	0.90935	0.90750	0.90633	0.90557
	(5)	0.91260	0.90948	0.90762	0.90645	0.90568
	(6)	0.99985	0.99986	0.99987	0.99987	0.99988
20	(1)	0.11995	0.09442	0.07918	0.06917	0.06213
	(2)	0.11997	0.09443	0.07920	0.06917	0.06212
	(3)	0.99983	0.99989	0.99975	1.00000	1.00161
	(4)	0.11883	0.09356	0.07850	0.06861	0.06165
	(5)	0.11885	0.09357	0.07852	0.06861	0.06165
	(6)	0.99983	0.99989	0.99975	1.00000	1.00000
60	(1)	0.00231	0.00112	0.00066	0.00044	0.00031
	(2)	0.00230	0.00111	0.00067	0.00044	0.00031
	(3)	1.00435	1.00901	0.98507	1.00000	1.00000
	(4)	0.00229	0.00111	0.00065	0.00043	0.00031
	(5)	0.00227	0.00110	0.00066	0.00043	0.00031
	(6)	1.00881	1.00909	0.98485	1.00000	1.00000
100	(1)	0.00004	0.00001	0.00000	0.00000	0.00000
	(2)	0.00003	0.00000	0.00002	0.00000	0.00000
	(4)	0.00004	0.00001	0.00000	0.00000	0.00000
	(5)	0.00003	0.00000	0.00002	0.00000	0.00000

Table 6.10: Exponential distribution, $y = 1$ and $E[S_1] > E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 0.9625$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 0.790625$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 0.6875$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 0.61875$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 0.569643$
0	(1)	0.58380	0.59470	0.60330	0.61136	0.61909
	(2)	0.58442	0.59543	0.60405	0.61207	0.61972
	(3)	0.99894	0.99877	0.99876	0.99884	0.99898
	(4)	0.58572	0.61291	0.62916	0.64126	0.65112
	(5)	0.58624	0.61356	0.62982	0.64188	0.65168
	(6)	0.99911	0.99894	0.99895	0.99903	0.99914
20	(1)	0.11439	0.12377	0.11969	0.11276	0.10571
	(2)	0.11467	0.12417	0.11990	0.11265	0.10518
	(3)	0.99756	0.99678	0.99825	1.00098	1.00504
	(4)	0.10268	0.11433	0.11204	0.10643	0.10034
	(5)	0.10292	0.11468	0.11221	0.10628	0.09980
	(6)	0.99767	0.99695	0.99849	1.00141	1.00541
60	(1)	0.00441	0.00545	0.00484	0.00398	0.00322
	(2)	0.00444	0.00550	0.00484	0.00396	0.00317
	(3)	0.99324	0.99091	1.00000	1.00505	1.01577
	(4)	0.00396	0.00503	0.00453	0.00375	0.00306
	(5)	0.00398	0.00508	0.00453	0.00373	0.00300
	(6)	0.99497	0.99016	1.00000	1.00536	1.02000
100	(1)	0.00017	0.00024	0.00020	0.00014	0.00010
	(2)	0.00017	0.00025	0.00019	0.00014	0.00010
	(3)	1.00000	0.96000	1.05263	1.00000	1.00000
	(4)	0.00015	0.00022	0.00018	0.00013	0.00009
	(5)	0.00015	0.00024	0.00018	0.00013	0.00009
	(6)	1.00000	0.91667	1.00000	1.00000	1.00000

Table 6.11: Exponential distribution, $y = 3$ and $E[S_1] > E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 0.9625$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 0.790625$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 0.6875$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 0.61875$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 0.569643$
0	(1)	0.87473	0.88217	0.88470	0.88605	0.88703
	(2)	0.87503	0.88255	0.88505	0.88631	0.88717
	(3)	0.99966	0.99957	0.99960	0.99971	0.99984
	(4)	0.81989	0.84734	0.85963	0.86685	0.87178
	(5)	0.82010	0.84760	0.85981	0.86689	0.87163
	(6)	0.99974	0.99969	0.99979	0.99995	1.00017
20	(1)	0.17167	0.18457	0.17709	0.16544	0.15376
	(2)	0.17195	0.18500	0.17723	0.16510	0.15283
	(3)	0.99837	0.99768	0.99921	1.00206	1.00609
	(4)	0.15409	0.17048	0.16577	0.15615	0.14596
	(5)	0.15434	0.17086	0.16586	0.15577	0.14500
	(6)	0.99838	0.99778	0.99946	1.00244	1.00662
60	(1)	0.00662	0.00812	0.00716	0.00584	0.00469
	(2)	0.00665	0.00819	0.00716	0.00580	0.00460
	(3)	0.99549	0.99145	1.00000	1.00690	1.01957
	(4)	0.00595	0.00750	0.00670	0.00551	0.00445
	(5)	0.00597	0.00756	0.00670	0.00547	0.00437
	(6)	0.99665	0.99206	1.00000	1.00731	1.01831
100	(1)	0.00026	0.00036	0.00029	0.00021	0.00014
	(2)	0.00026	0.00038	0.00028	0.00020	0.00014
	(3)	1.00000	0.94737	1.03571	1.05000	1.00000
	(4)	0.00023	0.00033	0.00027	0.00019	0.00014
	(5)	0.00023	0.00035	0.00026	0.00019	0.00013
	(6)	1.00000	0.94286	1.03846	1.00000	1.07692

Table 6.12: Exponential distribution, $y = 1$ and $E[S_1] < E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.65$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 2.3375$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 2.75$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 3.025$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 3.22143$
0	(1)	0.49229	0.44055	0.42038	0.40962	0.40299
	(2)	0.49267	0.44086	0.42067	0.40990	0.40327
	(3)	0.99923	0.99930	0.99931	0.99932	0.99931
	(4)	0.40556	0.38889	0.38119	0.37646	0.37325
	(5)	0.40606	0.38926	0.38151	0.37676	0.37354
	(6)	0.99877	0.99905	0.99916	0.99920	0.99922
20	(1)	0.18830	0.18459	0.17255	0.16364	0.15750
	(2)	0.18838	0.18471	0.17271	0.16382	0.15769
	(3)	0.98996	0.99935	0.99907	0.99890	0.99880
	(4)	0.22340	0.20903	0.19242	0.18045	0.17208
	(5)	0.22357	0.20921	0.19261	0.18066	0.17229
	(6)	0.99924	0.99914	0.99901	0.99884	0.99878
60	(1)	0.05634	0.05461	0.04502	0.03843	0.03416
	(2)	0.05629	0.05463	0.04507	0.03849	0.03423
	(3)	1.00089	0.99963	0.99889	0.99844	0.99796
	(4)	0.06684	0.06184	0.05020	0.04238	0.03732
	(5)	0.06681	0.06187	0.05026	0.04245	0.03740
	(6)	1.00045	0.99952	0.99881	0.99835	0.99786
100	(1)	0.01686	0.01616	0.01175	0.00903	0.00741
	(2)	0.01682	0.01616	0.01176	0.00905	0.00743
	(3)	1.00238	1.00000	0.99915	0.99779	0.99731
	(4)	0.02000	0.01830	0.01310	0.00995	0.00809
	(5)	0.01996	0.01830	0.01312	0.00998	0.00812
	(6)	1.00200	1.00000	0.99848	0.99699	0.99631

Table 6.13: Exponential distribution, $y = 3$ and $E[S_1] < E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.65$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 2.3375$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 2.75$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 3.025$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 3.22143$
0	(1)	0.79895	0.75472	0.73680	0.72761	0.72245
	(2)	0.79917	0.75496	0.73705	0.72787	0.72272
	(3)	0.99972	0.99968	0.99966	0.99964	0.99963
	(4)	0.76636	0.74824	0.73811	0.73138	0.72665
	(5)	0.76690	0.74865	0.73848	0.73173	0.72699
	(6)	0.99930	0.99945	0.99950	0.99952	0.99953
20	(1)	0.35466	0.35726	0.33614	0.31972	0.30824
	(2)	0.35643	0.35735	0.33632	0.31995	0.30851
	(3)	0.99503	0.99975	0.99946	0.99928	0.99912
	(4)	0.42078	0.40457	0.37483	0.35256	0.33676
	(5)	0.42085	0.40473	0.37507	0.35284	0.33707
	(6)	0.99983	0.99960	0.99936	0.99921	0.99908
60	(1)	0.10612	0.10570	0.08770	0.07508	0.06685
	(2)	0.10596	0.10568	0.08776	0.07518	0.06696
	(3)	1.00151	1.00019	0.99932	0.99867	0.99836
	(4)	0.12590	0.11970	0.09779	0.08280	0.07303
	(5)	0.12576	0.11970	0.09787	0.08291	0.07316
	(6)	1.00111	1.00000	0.99918	0.99867	0.99822
100	(1)	0.03175	0.03127	0.02288	0.01763	0.01450
	(2)	0.03166	0.03125	0.02290	0.01767	0.01453
	(3)	1.00284	1.00064	0.99913	0.99774	0.99794
	(4)	0.03767	0.03541	0.02551	0.01944	0.01584
	(5)	0.03758	0.03540	0.02554	0.01948	0.01588
	(6)	1.00240	1.00028	0.99883	0.99795	0.99748

(iv) Regarding the relationship between $H_{1,1}(u, y)$ and $H_{1,2}(u, y)$, we can see that similar to the probability of ruin, as the initial surplus increases the values for $H_{1,1}(u, y)$ and $H_{1,2}(u, y)$ get closer to each other. In Tables 6.8 to 6.11 the relationship between $H_{1,1}(u, y)$ and $H_{1,2}(u, y)$ is constant for a given value of u. This pattern holds in Table 6.12 when $y = 1$, but we cannot detect any consistent relationship when $y = 3$.

Tables 6.14 to 6.19 show approximations to the probability and severity of ruin when claim amounts follow Pareto(a_i, b_i) distributions. In this case using the method in Li and Lu (2008), the exact values of $H_{1,1}(0, y)$, and $H_{1,2}(0, y)$ can be obtained from

$$H_{1,ii}(0, y) = \frac{\lambda_i}{c}\tilde{w}_i(0) + \frac{\rho^* - \frac{\alpha_j}{c} - \frac{\lambda_j}{c}\left(1 - \tilde{f}_j(\rho^*)\right)}{1 - \frac{\lambda_j}{c\rho^*}\left(1 - \tilde{f}_j(\rho^*)\right)}\frac{\lambda_i}{c\rho^*}\left(\tilde{w}_i(\rho^*) - \tilde{w}_i(0)\right) \tag{6.48}$$

and

$$H_{1,ij}(0, y) = -\frac{\alpha_i\lambda_j\left(\tilde{w}_j(\rho^*) - \tilde{w}_j(0)\right)}{c^2\rho^*\left(1 - \frac{\lambda_j}{c\rho^*}\left(1 - \tilde{f}_j(\rho^*)\right)\right)}, \tag{6.49}$$

where $\tilde{f}_i(\rho^*) = a_i(b_i\rho^*)^{a_i}e^{b_i\rho^*}\Gamma(-a_i, b_i\rho^*)$, $\tilde{w}_i(\rho^*) = \int_0^\infty e^{-\rho^* u}\Big(F_i(u+y) - F_i(u)\Big)\,du$, and ρ^* is the unique positive solution to equation (6.39).

As we can see, all approximations for $u = 0$ are underestimated. In Tables 6.14 and 6.15 the relationship between $H_{1,1}(u, y)$ and $H_{1,2}(u, y)$ does not change for a given value of u, meaning that if $H_{1,1}(u, y) > H_{1,2}(u, y)$ for a given value of u, this relationship holds across the table. In Tables 6.16 and 6.17, where $E[S_1] > E[S_2]$, $H_{1,1}(u, y) > H_{1,2}(u, y)$. In Tables 6.18 and 6.19, where $E[S_1] < E[S_2]$, $H_{1,1}(u, y)$ is always less than $H_{1,2}(u, y)$. Overall, our algorithm performs reasonably well and provides good approximations to the exact values in the continuous time Markov-modulated model.

Table 6.14: Pareto distribution, $y = 1$ and $E[S_1] = E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 1.1$
0	(1)	0.47011	0.49533	0.51457	0.52980	0.54220
	(2)	0.47039	0.49560	0.51484	0.53007	0.54248
	(3)	0.99940	0.99946	0.99948	0.99949	0.99948
	(4)	0.60203	0.61284	0.62096	0.62729	0.63238
	(5)	0.60258	0.61340	0.62153	0.62787	0.63297
	(6)	0.99909	0.99909	0.99908	0.99908	0.99907
20	(1)	0.07934	0.07815	0.07706	0.07610	0.07528
	(4)	0.07920	0.07801	0.07693	0.07598	0.07516
60	(1)	0.02433	0.02046	0.01775	0.01577	0.01426
	(4)	0.02426	0.02039	0.01770	0.01572	0.01421
100	(1)	0.01097	0.00835	0.00672	0.00561	0.00482
	(4)	0.01094	0.00833	0.00670	0.00559	0.00480

Table 6.15: Pareto distribution, $y = 3$ and $E[S_1] = E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 1.1$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 1.1$
0	(1)	0.69467	0.71521	0.73073	0.74289	0.75270
	(2)	0.69479	0.71530	0.73080	0.74295	0.75275
	(3)	0.99983	0.99987	0.99990	0.99992	0.99993
	(4)	0.78721	0.79629	0.80311	0.80842	0.81268
	(5)	0.78740	0.79647	0.80329	0.80861	0.81287
	(6)	0.99976	0.99977	0.99978	0.99977	0.99977
20	(1)	0.15715	0.14883	0.14273	0.13806	0.13436
	(2)	0.15682	0.14852	0.14245	0.13779	0.13411
60	(1)	0.04923	0.03992	0.03378	0.02944	0.02623
	(4)	0.04909	0.03980	0.03367	0.02935	0.02614
100	(1)	0.02235	0.01645	0.01291	0.01060	0.00897
	(4)	0.02230	0.01641	0.01288	0.01057	0.00895

Table 6.16: Pareto distribution, $y = 1$ and $E[S_1] > E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 0.9625$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 0.790625$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 0.6875$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 0.61875$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 0.569643$
0	(1)	0.47610	0.50320	0.52022	0.53262	0.54246
	(2)	0.47648	0.50371	0.52078	0.53319	0.54302
	(3)	0.99920	0.99899	0.99892	0.99893	0.99897
	(4)	0.40959	0.45140	0.47894	0.49912	0.51490
	(5)	0.40979	0.45167	0.47922	0.49938	0.51511
	(6)	0.99951	0.99940	0.99942	0.99948	0.99959
20	(1)	0.08634	0.09111	0.09179	0.09120	0.09025
	(4)	0.08218	0.08767	0.08889	0.08870	0.08805
60	(1)	0.02902	0.02966	0.02869	0.02735	0.02602
	(4)	0.02818	0.02898	0.02814	0.02690	0.02563
100	(1)	0.01378	0.01371	0.01290	0.01197	0.01111
	(4)	0.01348	0.01347	0.01271	0.01182	0.01098

Table 6.17: Pareto distribution, $y = 3$ and $E[S_1] > E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 0.9625$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 0.790625$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 0.6875$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 0.61875$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 0.569643$
0	(1)	0.70405	0.72886	0.74278	0.75221	0.75938
	(2)	0.70427	0.72916	0.74308	0.75247	0.75956
	(3)	0.99969	0.99959	0.99960	0.99965	0.99976
	(4)	0.61814	0.66207	0.68865	0.70713	0.72106
	(5)	0.61824	0.66220	0.68873	0.70711	0.72092
	(6)	0.99984	0.99980	0.99988	1.00003	1.00019
20	(1)	0.17240	0.17801	0.17698	0.17414	0.17094
	(4)	0.16431	0.17147	0.17154	0.16949	0.16688
60	(1)	0.05905	0.05906	0.05641	0.05330	0.05032
	(4)	0.05736	0.05773	0.05534	0.05241	0.04958
100	(1)	0.02822	0.02749	0.02553	0.02349	0.02164
	(4)	0.02759	0.02701	0.02516	0.02319	0.02140

Table 6.18: Pareto distribution, $y = 1$ and $E[S_1] < E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.65$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 2.3375$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 2.75$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 3.025$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 3.22143$
0	(1)	0.39573	0.35490	0.33890	0.33102	0.32676
	(2)	0.03960	0.35510	0.33908	0.33119	0.32693
	(3)	0.99932	0.99944	0.99947	0.99949	0.99948
	(4)	0.40752	0.38036	0.36606	0.35725	0.35134
	(5)	0.40803	0.38072	0.36637	0.35753	0.35161
	(6)	0.99875	0.99905	0.99915	0.99922	0.99923
20	(1)	0.13030	0.12994	0.12349	0.11885	0.11575
	(4)	0.15088	0.14381	0.13434	0.12782	0.12341
60	(1)	0.05880	0.05813	0.05274	0.04895	0.04643
	(4)	0.06519	0.06272	0.05621	0.05172	0.04875
100	(1)	0.03030	0.02917	0.02537	0.02282	0.02118
	(4)	0.03310	0.03121	0.02686	0.02398	0.02212

Table 6.19: Pareto distribution, $y = 3$ and $E[S_1] < E[S_2]$

Unscaled u		$\alpha_1 = 0.1$ $\alpha_2 = 0.5$ $c = 1.65$	$\alpha_1 = 0.3$ $\alpha_2 = 0.5$ $c = 2.3375$	$\alpha_1 = 0.5$ $\alpha_2 = 0.5$ $c = 2.75$	$\alpha_1 = 0.7$ $\alpha_2 = 0.5$ $c = 3.025$	$\alpha_1 = 0.9$ $\alpha_2 = 0.5$ $c = 3.22143$
0	(1)	0.63841	0.60345	0.58913	0.58258	0.57965
	(2)	0.63855	0.60359	0.58927	0.58272	0.57979
	(3)	0.99978	0.99977	0.99976	0.99976	0.99976
	(4)	0.71220	0.68090	0.66275	0.65112	0.64320
	(5)	0.71266	0.68124	0.66306	0.65141	0.64347
	(6)	0.99935	0.99950	0.99953	0.99955	0.99958
20	(1)	0.26647	0.27051	0.25973	0.25157	0.24605
	(4)	0.30738	0.29875	0.28207	0.27018	0.26202
60	(1)	0.12113	0.12180	0.11165	0.10432	0.09941
	(4)	0.13419	0.13136	0.11896	0.11021	0.10434
100	(1)	0.06262	0.06126	0.05384	0.04876	0.04545
	(4)	0.06837	0.06553	0.05699	0.05122	0.04747

6.5 The probability of ruin in finite time

In this section, we consider the probability of ruin in finite time by extending our technique in Chapter 5 which is based on the ideas of Dickson and Waters (1991). First, we provide recursive formulae which can be used to approximate $\psi_i(u,t)$ for $i \in M$. Then, we explain how we can apply the modified truncation method of De Vylder and Goovaerts (1988, Section 5) to improve the computational efficiency of our numerical algorithm.

Theorem 6.5. For $u = 0, 1, 2, \dots$, when $t = 1$,

$$\psi_i^d(u,1) = \sum_{j=1}^{m} \sum_{x=u+1}^{\infty} g_{ij}(x) = 1 - G_i(u) \tag{6.50}$$

and for $t > 1$,

$$\psi_i^d(u,t) = \psi_i^d(u,1) + \sum_{j=1}^{m} \sum_{x=0}^{u} g_{ij}(x)\psi_j^d(u+1-x, t-1). \tag{6.51}$$

Proof. We first consider $\psi_i^d(u,1)$. For ruin to occur in the first time period, we require that the aggregate claim amount, Y_1, exceeds the initial surplus u. Hence (6.50) follows. For $t > 1$ we note that if ruin occurs at or before time t, then either

(i) $Y_1 > u$ so that ruin occurs at time 1, or

(ii) $Y_1 = x, x = 0, 1, 2, \dots, u$ and ruin occurs in the next $t-1$ time periods, from surplus level $u+1-x$ at time 1.

Thus, (6.51) follows. □

We can use formulae (6.50) and (6.51) to calculate finite time ruin probabilities, recursively. First we need to calculate $\psi_i^d(w,1)$ for $w = 0, 1, 2, \dots, u+t-1$ from (6.50), then using equation (6.51) we calculate $\psi_i^d(w,2)$ for $w = 0, 1, 2, \dots, u+t-2$. We continue this process until we reach to the point calculating $\psi_i^d(w,t)$ for $w = 0, 1, 2, \dots, u+t-\tau$, where $\tau = t$. This method requires a lot of time and computations, particularly when the values of u and t are large. Since many of the probabilities used in the calculations will be very small, we can reduce the number of calculations involved by modifying the truncation method of De Vylder and Goovaerts (1988, Section 5) which is based on ignoring small probabilities. Suppose $k_{i,1}$ is the least integer such that $G_i(k_{i,1}) \geq 1-\epsilon$, where ϵ is a small positive value. Then

$$g_{ij}^{\epsilon}(x) = \begin{cases} g_{ij}(x) & \text{for } \ x = 0,1,2,\ldots,k_{i,1} \\ 0 & \text{for } \ x = k_{i,1}+1, k_{i,1}+2,\ldots \end{cases}$$

and

$$\psi_i^{d\epsilon}(u,1) = \begin{cases} 1-G_i(u) & \text{for } \ u = 0,1,2,\ldots,k_{i,1} \\ 0 & \text{for } \ u = k_{i,1}+1, k_{i,1}+2,\ldots \end{cases} \tag{6.52}$$

Therefore, the upper truncation point in (6.50) is $k_{i,1}$ and in (6.51) is $\min(u, k_{i,1})$. For example, the aggregate claim amount distribution in state 1 is truncated at $k_{1,1}$ and in state 2 at $k_{2,1}$. Further, if $k_{i,t}$ is the least integer such that $\psi_i^d(k_{i,t}, t-1) \geq \epsilon$, we will calculate $\psi_j^d(u,t)$ for $u = 0,1,2,\ldots,k_{j,t}$. In other words, the lower truncation point in (6.51) is $\max(0, u+1-k_{j,t-1})$. Thus, we can evaluate the finite time ruin probability with the following expression

$$\psi_i^{d\epsilon}(u,t) = \psi_i^{d\epsilon}(u,1) + \sum_{j=1}^{m}\sum_{x=\mathcal{L}}^{\mathcal{U}} g_{ij}^{\epsilon}(x)\psi_j^{d\epsilon}(u+1-x, t-1) \tag{6.53}$$

where $\mathcal{L} = \max(0, u+1-k_{j,t-1})$ and $\mathcal{U} = \min(u, k_{i,1})$. We can demonstrate that similar to the classical risk model, the error introduced by using (6.52) and (6.53) in the Markov-modulated model can be bounded.

Theorem 6.6. For $t = 1$ we have

$$\psi_i^d(u,1) - \epsilon \leq \psi_i^{d\epsilon}(u,1) \leq \psi_i^d(u,1)$$

and for $t = 2,3,\ldots$ we have

$$\psi_i^d(u,t) - 2t\epsilon \leq \psi_i^{d\epsilon}(u,t) \leq \psi_i^d(u,t). \tag{6.54}$$

Proof. The proof is similar to De Vylder and Goovaerts (1988, Section 5) or Dickson and Waters (1991, Section 6) for their discrete time models.

When $t = 1$, $\psi_i^{d\epsilon}(u,1) = \psi_i^d(u,1)$ for $u \leq k_{i,1}$ and $\psi_i^{d\epsilon}(u,1) = 0$ for $u > k_{i,1}$, therefore $\psi_i^d(u,1) < \epsilon$.

Now assume $\psi_i^d(u,n) - 2n\epsilon \le \psi_i^{d\epsilon}(u,n)$ holds for $n = t$, then by induction we have

$$\begin{aligned}
\psi_i^d(u,t+1) - \psi_i^{d\epsilon}(u,t+1) &= \psi_i^d(u,1) + \sum_{j=1}^{m}\sum_{x=0}^{u} g_{ij}(x)\psi_j^d(u+1-x,t) \\
&\quad -\psi_i^{d\epsilon}(u,1) - \sum_{j=1}^{m}\sum_{x=0}^{u} g_{ij}^{\epsilon}(x)\psi_j^{d\epsilon}(u+1-x,t) \\
&\le \epsilon + \sum_{j=1}^{m}\sum_{x=0}^{u}\{g_{ij}(x) - g_{ij}^{\epsilon}(x)\}\psi_j^d(u+1-x,t) \\
&\quad + \sum_{j=1}^{m}\sum_{x=0}^{u} g_{ij}^{\epsilon}(x)\{\psi_j^d(u+1-x,t) - \psi_j^{d\epsilon}(u+1-x,t)\} \\
&\le \sum_{j=1}^{m}\sum_{x=0}^{u}\{g_{ij}(x) - g_{ij}^{\epsilon}(x)\} + 2t\epsilon\sum_{j=1}^{m}\sum_{x=0}^{u} g_{ij}^{\epsilon}(x) + \epsilon \\
&\le \epsilon + 2\epsilon t + \epsilon \le 2\epsilon(t+1).
\end{aligned}$$

Therefore, we can conclude that (6.54) holds true for $n = t + 1$. □

6.5.1 The density of the time of ruin for $m = 2$

The finite time ruin probability enables us to study the density of the time of ruin. In particular, it is easier to interpret the graph of a density function rather than a distribution function. Define

$$w_i(u,t) = \frac{\partial}{\partial t}\psi_i(u,t)$$

to be the (defective) density of the time of ruin in the continuous time Markov-modulated model. Adjusting the technique of Dickson and Waters (2002), we can approximate the (defective) density of the time of ruin in continuous time at $t = j/c\beta$ by

$$c\,\beta\left[\psi_i^d\Big(u, \frac{j}{c\beta}\Big) - \psi_i^d\Big(u, \frac{j-1}{c\beta}\Big)\right] \tag{6.55}$$

for $j = 1, 2, \ldots, c\beta t$, where $\psi_i^d(u,t)$ is calculated by formulae (6.50) and (6.51). Dividing (6.55) by $\psi_i^d(u)$ using the algorithm described in Section 6.4 gives us an approximation to the proper density of the time of ruin in the Markov-modulated model.

We now illustrate the application of (6.55) by considering the density of the time of ruin for six cases that we discussed in Section 6.4. In all figures $\alpha_1 = 0.1, \alpha_2 = 0.5, \beta =$

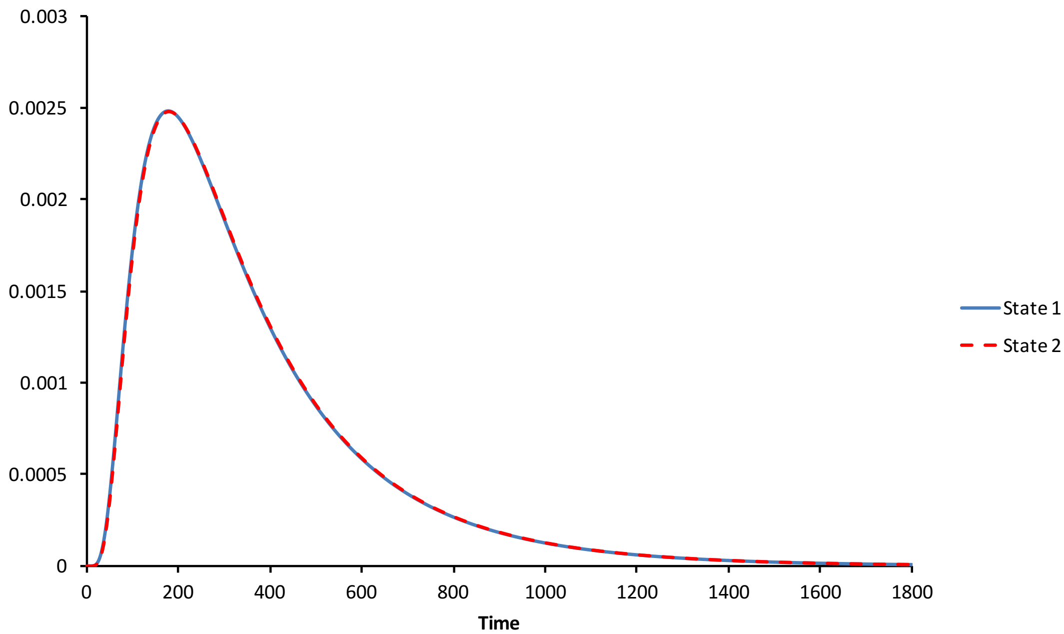

Figure 6.1: Case 1: $\psi_1(40) = 0.01744$ and $\psi_2(40) = 0.01727$

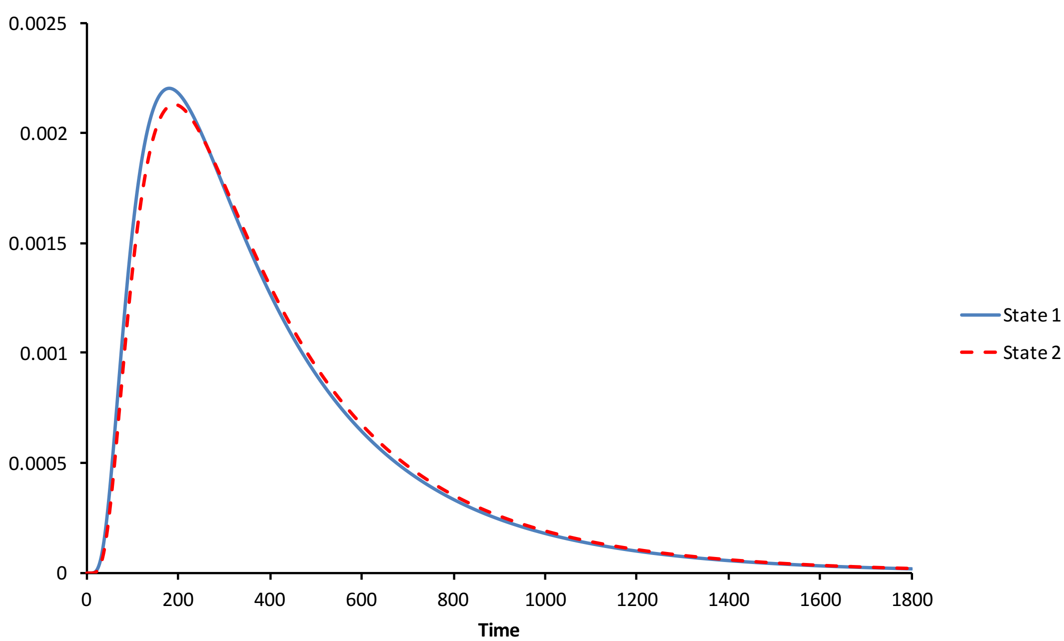

Figure 6.2: Case 2: $\psi_1(40) = 0.03418$ and $\psi_2(40) = 0.03071$

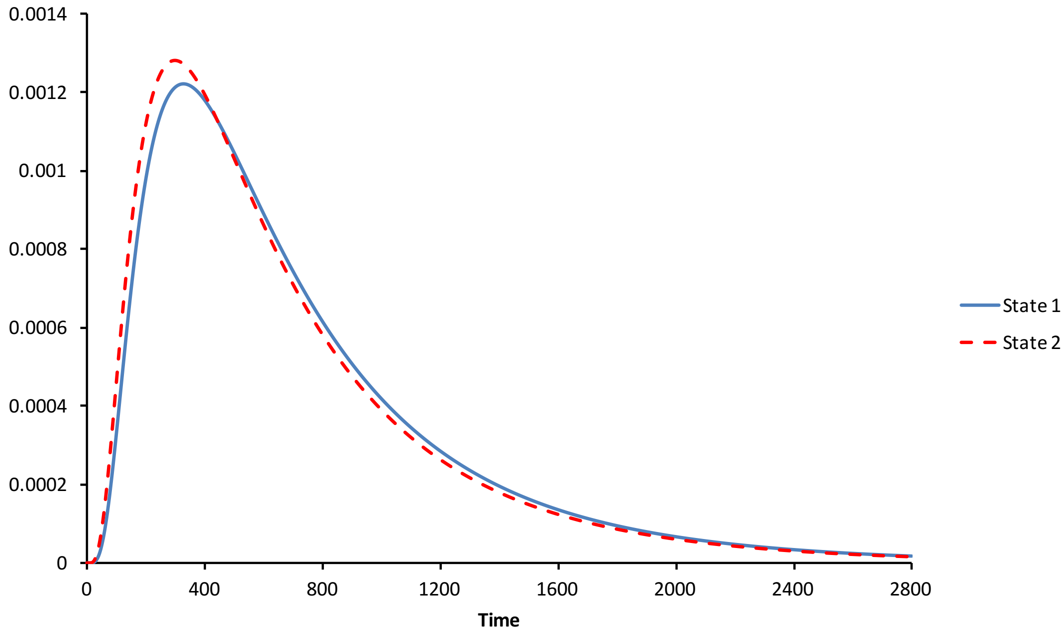

Figure 6.3: Case 3: $\psi_1(120) = 0.02304$ and $\psi_2(120) = 0.02721$

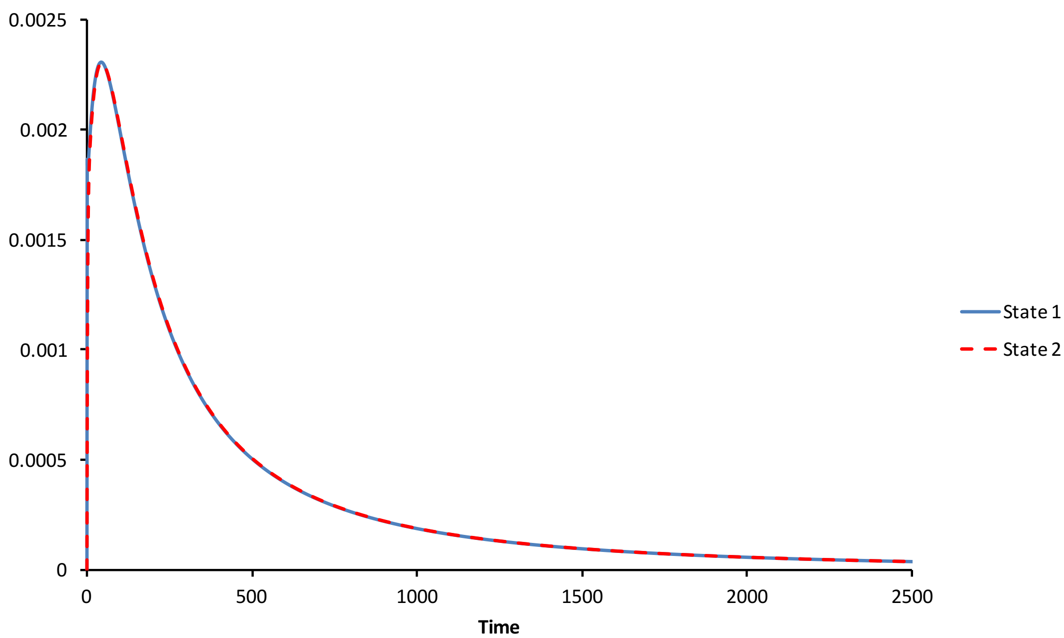

Figure 6.4: Case 4: $\psi_1(40) = 0.30885$ and $\psi_2(40) = 0.30812$

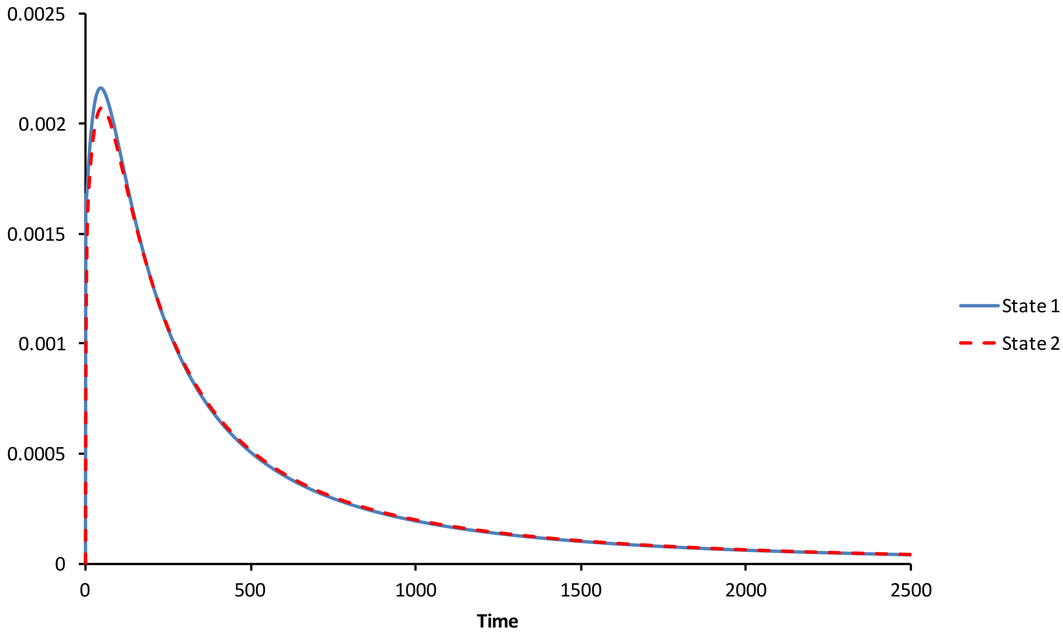

Figure 6.5: Case 5: $\psi_1(40) = 0.34427$ and $\psi_2(40) = 0.33677$

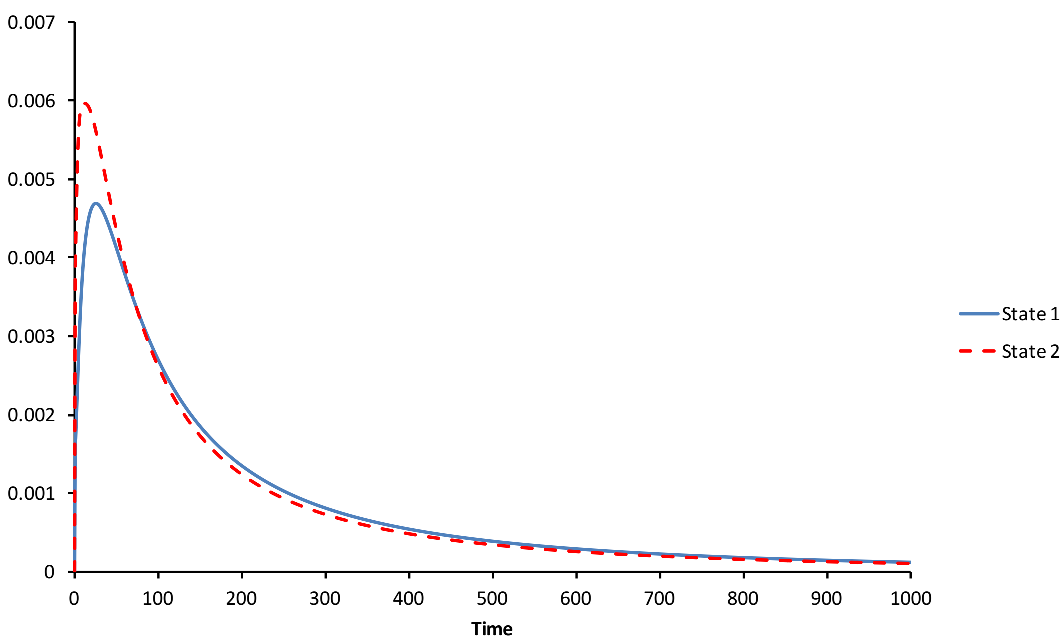

Figure 6.6: Case 6: $\psi_1(40) = 0.43292$ and $\psi_2(40) = 0.47021$

20, and $\theta = 0.1$. We remark that the accuracy of the approximations for the scaling factor 20 is only up to two decimal places. However, this precision is sufficient for our purpose here.

Figures 6.1, 6.2 and 6.3, show the proper density of the ruin time for exponential claim amounts. For these figures, we have chosen the initial surplus such that the ultimate ruin probability is in the range of practical interest. Figure 6.1 displays the density of the time of ruin, given that ruin occurs when $E[S_1] = E[S_2]$ for $u = 40$ (unscaled), where $\psi_1(40) = 0.01744$ and $\psi_2(40) = 0.01727$. In Section 6.4.1 we saw that when $E[S_1] = E[S_2]$, the ultimate ruin probabilities in states 1 and 2 are fairly close. Here, we observe that when $E[S_1] = E[S_2]$, the conditional density of the time of ruin in state 1 coincides with that in state 2. Figure 6.2 shows the situation when $E[S_1] > E[S_2]$, $\psi_1(40) = 0.03418$ and $\psi_2(40) = 0.03071$. We observe that for the time interval $t \in (170, 264)$ (unscaled) the density of the time of ruin, given that ruin occurs, in state 1 is located above that in state 2. As we can see in Figures 6.1 and 6.2 the maximum density is around $t = 200$ (unscaled). In Figure 6.3 we choose $u = 120$ (unscaled) with $\psi_1(120) = 0.02304$ and $\psi_2(120) = 0.02721$. This figure presents the density of the time of ruin, given that ruin occurs when $E[S_1] < E[S_2]$. We observe that the conditional density of the time of ruin in state 2 is located above the density in state 1. Therefore, we can conclude that the relationship between the density of the time of ruin in states 1 and 2 is the same as the relationship between $\psi_1(u)$ and $\psi_2(u)$. Figures 6.4, 6.5 and 6.6 illustrate the density of the time of ruin, given that ruin occurs when claim amounts follow Pareto distributions and initial surplus is 40 (unscaled). The pattern of these graphs is the same as the graphs for the exponential claim amounts. In other words, in Figure 6.4, where $E[S_1] = E[S_2]$, the conditional density of the time of ruin in both states is the same, in Figure 6.5 where $E[S_1] > E[S_2]$ the conditional density of the time of ruin in state 1 is above that in state 2, and in Figure 6.6 when $E[S_1] < E[S_2]$, the conditional density of the time of ruin in state 1 is below the density of the time of ruin in state 2. We can observe that the common feature of all these figures is that they are all positively-skewed and the skewness is heightened when claim amounts follow a heavy-tailed distribution.

6.5.2 The density of the time of ruin for $m > 2$

In Section 6.5 we provided an algorithm that can evaluate the probability of ruin in finite time in an m-state discrete time model. We then showed how we can apply this algorithm to approximate the density of the time of ruin when $m = 2$. In this section, we apply our numerical procedure to the case $m > 2$. There is nothing new about this algorithm except the estimation of the transition probabilities for which we suggest the following two methods and our experiments show both give very similar results:

(i) $p_{ii} = e^{-\alpha_{ii}/c\beta}$ and $p_{ij} = (1 - e^{-\alpha_{ii}/c\beta})\alpha_{ij}/\alpha_{ii}$,

(ii) $p_{ii} = 1 - \alpha_{ii}/c\beta$ and $p_{ij} = \alpha_{ij}/c\beta$.

We remark that (i) is in fact the method that we have already used in our numerical calculations for the case $m = 2$.

We consider a three-state model with the following intensity matrix

$$\begin{pmatrix} -0.6 & 0.2 & 0.4 \\ 0.1 & -0.3 & 0.2 \\ 0.5 & 0.3 & -0.8 \end{pmatrix}.$$

We calculate $c = 1.13548$ in the continuous time model so that formula (2.2) is satisfied and set $\beta = 20$. Figure 6.7 shows an approximation to the (defective) density of the time of ruin when arrival intensities are 1 in each state and claim amounts are exponentially distributed with means $m_1 = 1, m_2 = 0.5$ and $m_3 = 2$. Figure 6.8 illustrates the situation when claim amounts follow Pareto distributions with parameters $a_1 = 2, b_1 = 1, a_2 = 3, b_2 = 1$ and $a_3 = 2, b_3 = 1$ and arrival intensities $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. We observe that the common features of Figures 6.7 and 6.8 are that they are positively-skewed and the graph for the (defective) density of the time of ruin in state 3, where $E[S_3]$ is higher than the expected aggregate claim amount in the two other states, is located on the top and for state 2, where $E[S_2]$ is less than that in states 1 and 3, is at the bottom with the graph for state 1, where $E[S_3] > E[S_1] > E[S_2]$, being in the middle.

If we let the finite time period be sufficiently large, the graph of the cumulative distribution of the time of ruin can give us an approximation to the ultimate ruin probability for an m-state model. For example, our observation with the graphs of the

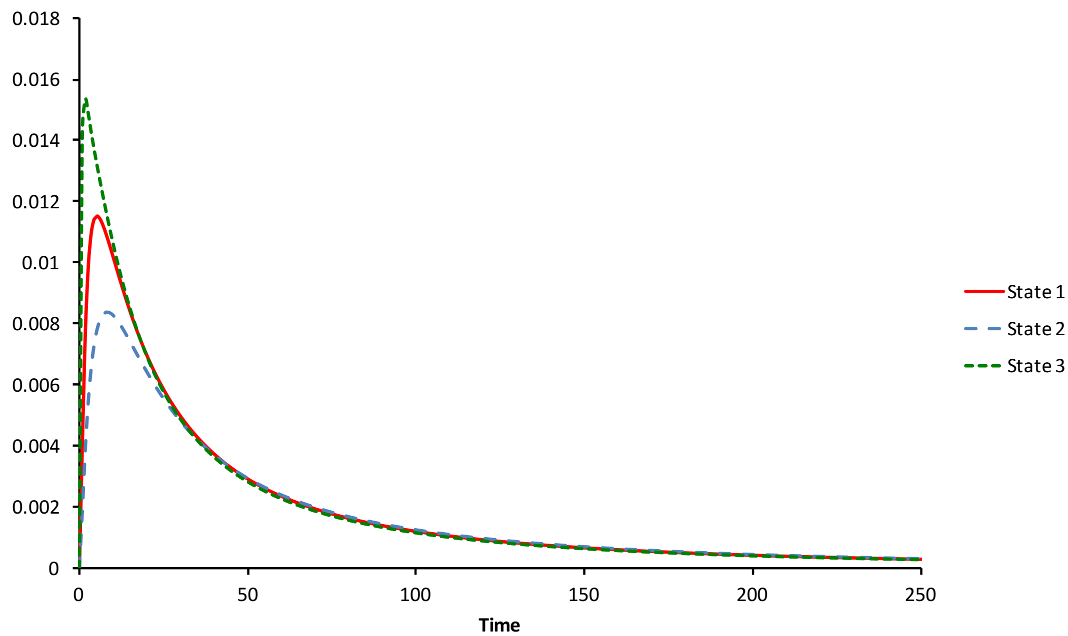

Figure 6.7: Exponential claims

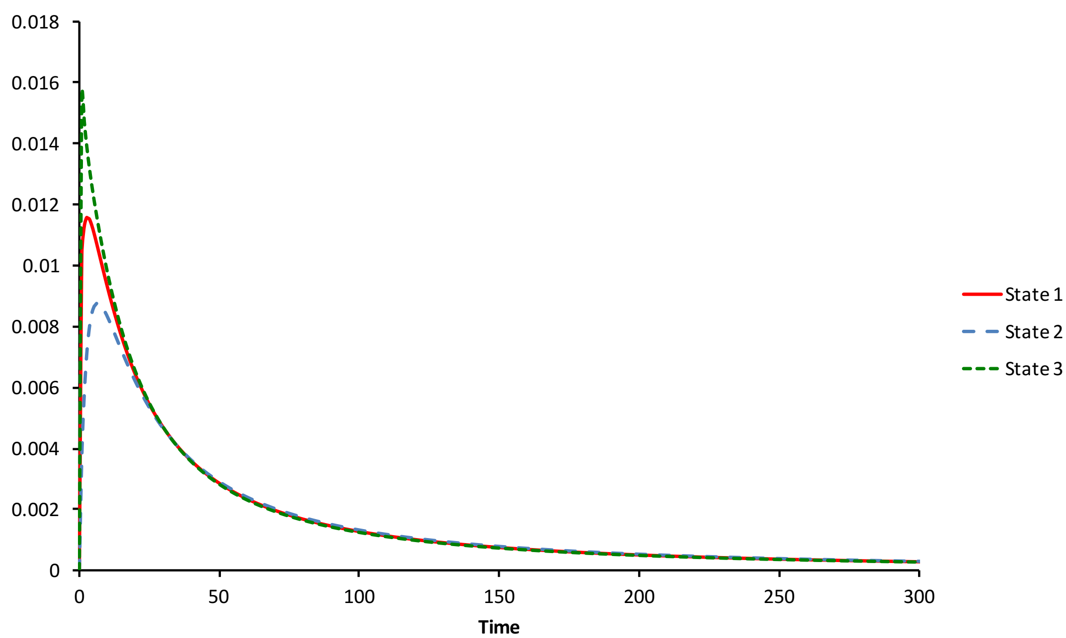

Figure 6.8: Pareto claims

cumulative distribution of the time of ruin for claim amounts with exponential distributions shows that for $u = 10$ (unscaled), $\psi_i(10)$ for $i = 1, 2, 3$ approaches $0.55864, 0.52368$ and 0.58057, respectively and when claim amounts follow Pareto distributions we can observe that for $u = 10$ (unscaled), $\psi_i(10)$ for $i = 1, 2, 3$ approaches $0.60752, 0.58292$ and 0.62175, respectively. This method can be extended easily to $m > 3$. However, we cannot verify the accuracy of the resulting $\psi_i(u)$ values, particularly, when claim amount distributions are Pareto.

We remark that Li et al. (2014) considered the density of the time of ruin in the continuous time Markov-modulated model. They derived a general expression for $w_i(u, t)$ and using numerical integration, implemented their formulae for $u = 0, 5$ when claim amounts in state 1 follow an exponential distribution and in state 2 follow an Erlang(2) distribution. As with the classical risk model, the formula for the (defective) density of the time of ruin in the Markov-modulated risk model, is expressed in terms of the density of the aggregate claim amount. However, in contrast to the classical risk model, the solution to the integro-differential equation that satisfies the distribution of the aggregate claim amount does not lead to a neat expression as formula (1.1). This issue arises as $G_{ij}(x)$ is a matrix under the Markov-modulated risk model. In general, it appears that our approach for the approximation of $w_i(u, t)$ is more straightforward than their numerical integration.

6.6 The Markov-modulated model with capital injections

In this section, we consider the Markov-modulated risk model with capital injections. First, we provide formulae for the ultimate ruin probability in the continuous time case. Then, we extend the algorithm used in Section 6.4 to approximate the probability of ruin in infinite time. Following that we discuss the finite time ruin probability and show how the algorithm in Section 6.5 can be modified to provide approximations to the density of the time of ruin in our model.

6.6.1 Notation and definitions

The underlying surplus process for a Markov-modulated risk model with capital injections is the Markov-modulated risk model which is defined by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

where $N(t)$ is the number of claims that have occurred up to time t. We assume that $\{J(t)\}_{t\geq 0}$ is a homogeneous, irreducible and aperiodic continuous time Markov process. See Section 2.1.1 for details.

Our risk model of interest is a Markov-modulated risk model modified by capital injections. Specifically, starting from initial surplus $u \geq k > 0$, if the surplus process falls between 0 and k a capital injection restores the surplus level to k, and ruin occurs only if the surplus process falls below 0 from a level above k. For the surplus process with capital injections we use the same notation as for the Markov-modulated risk model, but with a subscript k, so that, for example, $T_{u,k}$ denotes the time of ruin from initial surplus $u \geq k$ and $\psi_{k,i}(u)$ denotes the ultimate ruin probability given initial surplus u and initial environment state i, defined by

$$\psi_{k,i}(u) = \Pr(T_{u,k} < \infty \mid U(0) = u, J(0) = i) = 1 - \delta_{k,i}(u)$$

where $\delta_{k,i}(u)$ is the survival probability. Also,

$$\psi_{k,i}(u,t) = \Pr(T_{u,k} \leq t \mid U(0) = u, J(0) = i)$$

is the finite time ruin probability given initial surplus u and initial environment state i.

Our aim in this section is to develop a discrete time risk model which can be used to approximate the continuous time Markov-modulated risk model with capital injections. For this, we consider the discrete time surplus process that can be used to approximate the continuous time Markov-modulated risk model as the underlying surplus process. Therefore, for $n = 1, 2, 3, \ldots$ we define the insurer's surplus by

$$U^d(n) = u + n - \sum_{i=1}^{n} Y_i$$

where u is the insurer's initial surplus and Y_i is the insurer's aggregate claim amount in the ith time interval. We assume $\{J_n\}_{n\in N}$ is a homogeneous, irreducible and aperiodic

Markov chain with a finite space $M = \{1, \dots, m\}$ and transition probabilities $\{p_{ij}\}_{i,j=1}^m$. See Section 6.2 for details. To approximate the continuous time Markov-modulated risk model with capital injections, we consider this discrete time risk model with capital injections. Using the same notation as in Section 6.2 with a subscript k, we denote by $T_{u,k}^d$ the time of ruin from initial surplus $u = k, k+1, \dots$ for the process modified by capital injections. Thus

$$\psi_{k,i}^d(u) = \Pr(T_{u,k}^d < \infty \mid U^d(0) = u, J(0) = i) = 1 - \delta_{k,i}^d(u)$$

is the ultimate probability of ruin given initial surplus u and initial environment state i. Further,

$$\psi_{k,i}^d(u,t) = \Pr(T_{u,k}^d \leq t \mid U^d(0) = u, J(0) = i)$$

is defined to be the finite time probability of ruin given initial surplus u and initial environment state i. In our model ruin occurs when the surplus falls to 0 or below 0, and capital is injected when the surplus goes below a positive integer k, but stays above 0.

6.6.2 The probability of ultimate ruin in continuous time

In Chapters 3 and 4 we have derived expressions for the probability of ultimate ruin in the classical risk model with capital injections. Here, we extend our results to the Markov-modulated risk model with capital injections.

Theorem 6.7. When the initial surplus is k,

$$\psi_{k,i}(k) = \frac{\sum_{j=1,i\neq j}^m H_{1,ij}(0,k)\psi_{k,j}(k) + \psi_i(0) - H_{1,i}(0,k)}{1 - H_{1,ii}(0,k)} \tag{6.56}$$

and when $u > k$,

$$\begin{aligned}\psi_{k,i}(u) &= \psi_i(u-k) - H_{1,ii}(u-k,k)\,(1 - \psi_{k,i}(k)) \\ &\quad - \sum_{j=1,i\neq j}^m H_{1,ij}(u-k,k)\,(1 - \psi_{k,j}(k))\,. \end{aligned} \tag{6.57}$$

Proof. We start with the situation when $u = k$. Conditioning on the amount of the first drop below the initial surplus level we have

$$\begin{aligned}\psi_{k,i}(k) &= \sum_{j=1}^{m}\int_0^k h_{1,ij}(0,y)\psi_{k,j}(k)\,dy + \int_k^\infty h_{1,i}(0,y)\,dy \\ &= \sum_{j=1}^{m} H_{1,ij}(0,k)\psi_{k,j}(k) + \psi_i(0) - H_{1,i}(0,k)\end{aligned}$$

which can also be written as (6.56). Using the same argument for $u > k$, we can write

$$\begin{aligned}\psi_{k,i}(u) &= \sum_{j=1}^{m}\int_0^k h_{1,ij}(u-k,y)\psi_{k,j}(k)\,dy + \int_k^\infty h_{1,i}(u-k,y)\,dy \\ &= \sum_{j=1}^{m} H_{1,ij}(u-k,k)\psi_{k,j}(k) + \psi_i(u-k) - H_{1,i}(u-k,k)\end{aligned}$$

which can be expressed as (6.57). □

Setting $m = 1$ in (6.56) and (6.57), we can recover the results in the classical risk model with capital injections given by Theorem 3.3. We can find analytical expressions for (6.56) and (6.57) when individual claim sizes follow distributions for which expressions for $\psi_i(u)$ and $h_{1,i}(u)$ exist. However, in the case of heavy-tailed claim distributions there is no closed form for these functions and we need a method that gives approximation to $\psi_{k,i}(u)$.

6.6.3 The probability of ultimate ruin in discrete time

We now provide recursive formulae for $\psi^d_{k,i}(u)$, when $i = 1,2$ that can be used to approximate $\psi_{k,i}(u)$ in the Markov-modulated risk model with capital injections when $m = 2$.

Theorem 6.8. When $m = 2$, for $u = k, k+1, \ldots,$ and $i = 1$,

$$\begin{aligned}\psi^d_{k,1}(u) &= f^{-1}\Bigg(\sum_{x=1}^{u-k-1}\psi^d_{k,1}(u-x)\left(h^d_{11}(0,x)\left(1-h^d_{22}(0,0)\right)+h^d_{12}(0,0)h^d_{21}(0,x)\right) \\ &\quad+\sum_{x=1}^{u-k-1}\psi^d_{k,2}(u-x)\left(h^d_{12}(0,x)\left(1-h^d_{22}(0,0)\right)+h^d_{12}(0,0)h^d_{22}(0,x)\right) \\ &\quad+\left(1-h^d_{22}(0,0)\right)\left(H^d_{11}(0,u)-H^d_{11}(0,u-k)\right)\psi^d_{k,1}(k) \\ &\quad+h^d_{12}(0,0)\left(H^d_{21}(0,u)-H^d_{21}(0,u-k)\right)\psi^d_{k,1}(k)\end{aligned}$$

$$+\left(1-h_{22}^d(0,0)\right)\left(H_{12}^d(0,u)-H_{12}^d(0,u-k)\right)\psi_{k,2}^d(k)$$
$$+h_{12}^d(0,0)\left(H_{22}^d(0,u)-H_{22}^d(0,u-k)\right)\psi_{k,2}^d(k)$$
$$+\left(1-h_{22}^d(0,0)\right)\left(\psi_1^d(0)-H_1^d(0,u)\right)+h_{12}^d(0,0)\left(\psi_2^d(0)-H_2^d(0,u)\right)\Bigg) \tag{6.58}$$

and for $i=2$,

$$\begin{aligned}\psi_{k,2}^d(u) &= f^{-1}\Bigg(\sum_{x=1}^{u-k-1}\psi_{k,1}^d(u-x)\left(h_{11}^d(0,x)h_{21}^d(0,0)+h_{21}^d(0,x)\left(1-h_{11}^d(0,0)\right)\right)\\
&+\sum_{x=1}^{u-k-1}\psi_{k,2}^d(u-x)\left(h_{12}^d(0,x)h_{21}^d(0,0)+h_{22}^d(0,x)\left(1-h_{11}^d(0,0)\right)\right)\\
&+h_{21}^d(0,0)\left(H_{11}^d(0,u)-H_{11}^d(0,u-k)\right)\psi_{k,1}^d(k)\\
&+\left(1-h_{11}^d(0,0)\right)\left(H_{21}^d(0,u)-H_{21}^d(0,u-k)\right)\psi_{k,1}^d(k)\\
&+h_{21}^d(0,0)\left(H_{12}^d(0,u)-H_{12}^d(0,u-k)\right)\psi_{k,2}^d(k)\\
&+\left(1-h_{11}^d(0,0)\right)\left(H_{22}^d(0,u)-H_{22}^d(0,u-k)\right)\psi_{k,2}^d(k)\\
&+h_{21}^d(0,0)\left(\psi_1^d(0)-H_1^d(0,u)\right)+\left(1-h_{11}^d(0,0)\right)\left(\psi_2^d(0)-H_2^d(0,u)\right)\Bigg)\end{aligned} \tag{6.59}$$

where $f=\left(1-h_{11}^d(0,0)\right)\left(1-h_{22}^d(0,0)\right)-h_{12}^d(0,0)h_{21}^d(0,0)$ with

$$\psi_{k,1}^d(k)=\frac{\psi_1^d(0)-H_1^d(0,k)+H_{12}^d(0,k)\psi_{k,2}^d(k)}{1-H_{11}^d(0,k)} \tag{6.60}$$

and

$$\psi_{k,2}^d(k)=\frac{\psi_2^d(0)-H_2^d(0,k)+H_{21}^d(0,k)\psi_{k,1}^d(k)}{1-H_{22}^d(0,k)}. \tag{6.61}$$

Proof. We start with the situation when $u=k$. In this case, either a claim causes the surplus to fall between 0 and k, so the capital is injected in order to bring it back to level k and subsequently ruin occurs from level k, or a claim results in ruin by causing the surplus to fall to 0 or below 0. Thus

$$\psi_{k,i}^d(k)=\sum_{j=1}^{2}\sum_{x=0}^{k-1}h_{ij}^d(0,x)\psi_{k,j}^d(k)+\sum_{j=1}^{2}\sum_{x=k}^{\infty}h_{ij}^d(0,x). \tag{6.62}$$

Rearranging (6.62) yields (6.60) and (6.61).

We can now write an expression for $\psi^d_{k,i}(u)$ by noting that on the first occasion that the surplus falls below its initial level u, we have three situations:

(i) the surplus falls to $u-x$ such that it stays above k and ruin subsequently occurs from this surplus level,

(ii) the surplus falls below k but remains above 0 so that capital is injected and ruin occurs from level k,

(iii) ruin occurs when the surplus falls to either 0 or below 0.

Hence, we have

$$\begin{aligned}\psi^d_{k,i}(u) &= \sum_{j=1}^{2}\sum_{x=0}^{u-k-1} h^d_{ij}(0,x)\psi^d_{k,j}(u-x) + \sum_{j=1}^{2}\sum_{x=u-k}^{u-1} h^d_{ij}(0,x)\psi^d_{k,j}(k) + \sum_{j=1}^{2}\sum_{x=u}^{\infty} h^d_{ij}(0,x) \\ &= \sum_{j=1}^{2}\sum_{x=0}^{u-k-1} h^d_{ij}(0,x)\psi^d_{k,j}(u-x) + \sum_{j=1}^{2}\left(H^d_{ij}(0,u) - H^d_{ij}(0,u-k)\right)\psi^d_{k,j}(k) \\ &\quad + \psi^d_i(0) - H^d_i(0,u). \end{aligned} \tag{6.63}$$

For $i=1$, we have

$$\begin{aligned}\psi^d_{k,1}(u) &= h^d_{11}(0,0)\psi^d_{k,1}(u) + h^d_{12}(0,0)\psi^d_{k,2}(u) + \sum_{x=1}^{u-k-1} h^d_{11}(0,x)\psi^d_{k,1}(u-x) \\ &\quad + \sum_{x=1}^{u-k-1} h^d_{12}(0,x)\psi^d_{k,2}(u-x) + \left(H^d_{11}(0,u) - H^d_{11}(0,u-k)\right)\psi^d_{k,1}(k) \\ &\quad + \left(H^d_{12}(0,u) - H^d_{12}(0,u-k)\right)\psi^d_{k,2}(k) + \psi^d_1(0) - H^d_1(0,u) \end{aligned} \tag{6.64}$$

and for $i=2$,

$$\begin{aligned}\psi^d_{k,2}(u) &= h^d_{21}(0,0)\psi^d_{k,1}(u) + h^d_{22}(0,0)\psi^d_{k,2}(u) + \sum_{x=1}^{u-k-1} h^d_{21}(0,x)\psi^d_{k,1}(u-x) \\ &\quad + \sum_{x=1}^{u-k-1} h^d_{22}(0,x)\psi^d_{k,2}(u-x) + \left(H^d_{21}(0,u) - H^d_{21}(0,u-k)\right)\psi^d_{k,1}(k) \\ &\quad + \left(H^d_{22}(0,u) - H^d_{22}(0,u-k)\right)\psi^d_{k,2}(k) + \psi^d_2(0) - H^d_2(0,u). \end{aligned} \tag{6.65}$$

Rearranging (6.64) and (6.65) and solving a system of equations we obtain (6.58) and (6.59), respectively. □

We can readily evaluate $\psi_{k,1}^d(u)$ and $\psi_{k,2}^d(u)$ if we know the values of $h_{ij}^d(0,x)$. We remark that if we set $k=0$ in (6.63) we can recover equation (6.31).

6.6.4 Numerical illustrations

Tables 6.20, 6.21 and 6.22 show exact and approximate values of $\psi_{k,1}(u)$ and $\psi_{k,2}(u)$ when individual claim amounts follow distributions of Cases 1, 2 and 3 in Section 6.4. We can apply Theorem 6.7 and the methods in Sections 6.4.1 and 6.4.2 to calculate exact values of $\psi_{k,1}(u)$ and $\psi_{k,2}(u)$. In all tables, $\alpha_1 = 0.1, \alpha_2 = 0.5$, and $\theta = 0.1$. The key for these tables is as follows:

(1) denotes the exact value of $\psi_{k,1}(u)$ and $\psi_{k,2}(u)$,

(2) denotes the approximation to $\psi_{k,1}(u)$ and $\psi_{k,2}(u)$ with $\beta = 300$,

(3) denotes the approximation to $\psi_{k,1}(u)$ and $\psi_{k,2}(u)$ with $\beta = 100$,

(4) denotes the approximation to $\psi_{k,1}(u)$ and $\psi_{k,2}(u)$ with $\beta = 20$.

We note the following points about Tables 6.20, 6.21 and 6.22.

(i) As β gets larger, approximations improve. We commented in Section 6.4 that the best approximations were obtained when $\beta = 300$.

(ii) We get a better approximation as u increases. This applies to all scaling factors and is in line with what we have observed in Section 6.4.

(iii) Some of the approximate values are overestimated and some are underestimated and unlike in Section 6.4 there is no general rule about how this can happen.

(iv) In general, for $\beta = 300$, the approximations are close to the exact values.

Table 6.20: Exponential distribution when $E[S_1] = E[S_2]$

Unscaled		$k=1$		$k=2$		$k=3$	
u		$\psi_{k,1}(u)$	$\psi_{k,2}(u)$	$\psi_{k,1}(u)$	$\psi_{k,2}(u)$	$\psi_{k,1}(u)$	$\psi_{k,2}(u)$
5	(1)	0.51762	0.51244	0.40086	0.39489	0.24428	0.23833
	(2)	0.51789	0.51208	0.40158	0.39444	0.24529	0.23753
	(3)	0.51828	0.51251	0.40273	0.39563	0.24685	0.23911
	(4)	0.52061	0.51502	0.40951	0.40264	0.25607	0.24848
10	(1)	0.31597	0.31299	0.24467	0.24233	0.14907	0.14761
	(2)	0.31608	0.31310	0.24501	0.24267	0.14954	0.14807
	(3)	0.31631	0.31334	0.24571	0.24337	0.15048	0.14901
	(4)	0.31770	0.31475	0.24982	0.24748	0.15611	0.15460
20	(1)	0.11774	0.11664	0.09117	0.09032	0.05555	0.05502
	(2)	0.11778	0.11667	0.09130	0.09044	0.05572	0.05520
	(3)	0.11786	0.11675	0.09155	0.09069	0.05607	0.05554
	(4)	0.11834	0.11724	0.09305	0.09220	0.05815	0.05761
40	(1)	0.01635	0.01620	0.01266	0.01254	0.00771	0.00764
	(2)	0.01635	0.01620	0.01268	0.01256	0.00774	0.00766
	(3)	0.01636	0.01621	0.01271	0.01259	0.00778	0.00771
	(4)	0.01642	0.01627	0.01291	0.01279	0.00807	0.00799
120	(1)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	(2)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	(3)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	(4)	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 6.21: Exponential distribution when $E[S_1] > E[S_2]$

Unscaled		$k=1$		$k=2$		$k=3$	
u		$\psi_{k,1}(u)$	$\psi_{k,2}(u)$	$\psi_{k,1}(u)$	$\psi_{k,2}(u)$	$\psi_{k,1}(u)$	$\psi_{k,2}(u)$
5	(1)	0.58243	0.52276	0.47302	0.42454	0.30490	0.27355
	(2)	0.58280	0.52145	0.47427	0.42259	0.30756	0.27103
	(3)	0.58264	0.52137	0.47496	0.42329	0.30905	0.27245
	(4)	0.58165	0.52092	0.47888	0.42729	0.31771	0.28074
10	(1)	0.38787	0.34814	0.31502	0.28274	0.20305	0.18225
	(2)	0.38770	0.34799	0.31515	0.28286	0.20350	0.18263
	(3)	0.38734	0.34770	0.31541	0.28312	0.20437	0.18343
	(4)	0.38520	0.34601	0.31682	0.28458	0.20941	0.18808
20	(1)	0.17202	0.15440	0.13971	0.12540	0.09005	0.08083
	(2)	0.17183	0.15424	0.13968	0.12537	0.09019	0.08096
	(3)	0.17145	0.15391	0.13961	0.12533	0.09046	0.08120
	(4)	0.16919	0.15198	0.13915	0.12500	0.09197	0.08262
40	(1)	0.03384	0.03037	0.02748	0.02467	0.01771	0.01590
	(2)	0.03375	0.03030	0.02744	0.02463	0.01772	0.01590
	(3)	0.03359	0.03015	0.02735	0.02455	0.01772	0.01591
	(4)	0.32640	0.02932	0.02685	0.02412	0.01774	0.01594
120	(1)	0.00005	0.00005	0.00004	0.00004	0.00003	0.00002
	(2)	0.00005	0.00004	0.00004	0.00004	0.00003	0.00002
	(3)	0.00005	0.00004	0.00004	0.00004	0.00003	0.00002
	(4)	0.00005	0.00004	0.00004	0.00003	0.00002	0.00002

Table 6.22: Exponential distribution when $E[S_1] < E[S_2]$

Unscaled		$k=1$		$k=2$		$k=3$	
u		$\psi_{k,1}(u)$	$\psi_{k,2}(u)$	$\psi_{k,1}(u)$	$\psi_{k,2}(u)$	$\psi_{k,1}(u)$	$\psi_{k,2}(u)$
5	(1)	0.69657	0.81802	0.66574	0.78578	0.61216	0.72808
	(2)	0.69674	0.82063	0.66595	0.79031	0.61303	0.73617
	(3)	0.69746	0.82092	0.66692	0.79085	0.61438	0.73712
	(4)	0.70181	0.82268	0.67270	0.79405	0.62236	0.74272
10	(1)	0.59344	0.70349	0.56968	0.67572	0.52733	0.62603
	(2)	0.59384	0.70396	0.57016	0.67645	0.52795	0.62718
	(3)	0.59465	0.70445	0.57116	0.67718	0.52926	0.62831
	(4)	0.59957	0.70746	0.57716	0.68154	0.53707	0.63497
20	(1)	0.43825	0.52014	0.42094	0.49961	0.38998	0.46286
	(2)	0.43869	0.52050	0.42145	0.50004	0.39059	0.46343
	(3)	0.43958	0.52120	0.42246	0.50091	0.39181	0.46457
	(4)	0.44494	0.52545	0.42855	0.50609	0.39911	0.47133
40	(1)	0.23957	0.28434	0.23011	0.27311	0.21318	0.25302
	(2)	0.23996	0.28471	0.23053	0.27352	0.21366	0.25350
	(3)	0.24076	0.28546	0.23138	0.27435	0.21460	0.25445
	(4)	0.24558	0.29002	0.23654	0.27934	0.22029	0.26015
120	(1)	0.02139	0.02539	0.02055	0.02439	0.01904	0.02259
	(2)	0.02148	0.02549	0.02064	0.02449	0.01913	0.02270
	(3)	0.02167	0.02569	0.02082	0.02469	0.01931	0.02290
	(4)	0.02279	0.02692	0.02195	0.02593	0.02045	0.02415

6.6.5 The probability of ruin in finite time

We take a similar approach to Section 6.5 to approximate $\psi_{k,i}(u,t)$ in the Markov-modulated risk model with capital injections. We then discuss the truncation method of De Vylder and Goovaerts (1988) to improve the computational efficiency of our numerical algorithm.

Theorem 6.9. When $t=1$ and for $u=k,k+1,\ldots,$

$$\psi^d_{k,i}(u,1)=\sum_{j=1}^{m}\sum_{x=u+1}^{\infty}g_{ij}(x)=1-G_i(u), \tag{6.66}$$

and when $t>1$,

$$\begin{aligned}\psi^d_{k,i}(u,t) &= \psi^d_{k,i}(u,1)+\sum_{j=1}^{m}\sum_{x=0}^{u-k}g_{ij}(x)\psi^d_{k,j}(u+1-x,t-1)\\ &\quad+\sum_{j=1}^{m}\sum_{x=u-k+1}^{u}g_{ij}(x)\psi^d_{k,j}(k,t-1).\end{aligned} \tag{6.67}$$

Proof. We first consider the case $\psi^d_{k,i}(u,1)$. For ruin to occur within the first time period, we require that the aggregate claim amount Y_1 exceeds the initial surplus u.

Hence (6.66) follows. Next we consider the case $t > 1$. Ruin occurs at or before time t under the following situations:

(i) $Y_1 > u$, so that ruin occurs at time 1,

(ii) $Y_1 = x$, $x = 0, 1, 2, \ldots, u-k$ and ruin occurs in the next $t-1$ time periods, from surplus level $u + 1 - x$ at time 1,

(iii) $Y_1 = x$, $x = u-k+1, \ldots, u$ and ruin occurs in the next $t-1$ periods, from surplus level k at time 1.

Therefore, (6.67) follows. □

We can use (6.66) and (6.67) to evaluate $\psi^d_{k,i}(w,t)$ by first calculating $\psi^d_{k,i}(w,1)$ for $w = k, k+1, \ldots, u+t-1$. Then, we need to find $\psi^d_{k,i}(w,2)$ for $w = k, k+1, \ldots, u+t-2$, and finally $\psi^d_{k,i}(w,t)$ for $w = k, k+1, \ldots, u+t-\tau$, where $\tau = t$. As stated before, this is computationally intensive and time-consuming. To solve this problem, we can modify De Vylder and Goovaerts' (1988) algorithm for the approximation of the classical risk model and truncate the summations by ignoring small values of $g_{ij}(x)$ and $\psi^d_{k,j}(u+1-x, t-1)$. The summation in (6.66) can be truncated at $k_{i,1}$, where $k_{i,1}$ is the least integer such that $G_i(k_{i,1}) \geq 1 - \epsilon$, $\epsilon > 0$. The second term in (6.67) is calculated only for values that $\max(0, u+1-k_{j,t-1}) \leq x \leq \min(u-k, k_{i,1})$ and the third term for $u - k + 1 \leq x \leq \min(u, k_{i,1})$. Thus

$$\psi^{\epsilon d}_{k,i}(u,1) = 1 - G^{\epsilon}_i(u), \quad u = k, k+1, \ldots, k_{i,1},$$

and for $t > 1$

$$\begin{aligned}\psi^{\epsilon d}_{k,i}(u,t) &= \psi^{\epsilon d}_{k,i}(u,1) + \sum_{j=1}^{m} \sum_{x=\mathcal{L}}^{\mathcal{U}} g^{\epsilon}_{ij}(x) \psi^{\epsilon d}_{k,j}(u+1-x, t-1) \\ &\quad + \sum_{j=1}^{m} \sum_{x=u-k+1}^{\min(u,k_{i,1})} g^{\epsilon}_{ij}(x) \psi^{\epsilon d}_{k,j}(k, t-1) \end{aligned} \tag{6.68}$$

where $\mathcal{L} = \max(0, u+1-k_{i,t-1})$ and $\mathcal{U} = \min(u-k, k_{i,1})$.

6.6.6 The density of the time of ruin

In this section, our aim is to illustrate the shape of the density of the time of ruin under the Markov-modulated model with capital injections. We can apply the technique described in Section 6.5.1 and use formulae (6.66) and (6.67) to approximate the (defective) density of the time of ruin in continuous time at $t = j/c\beta$ by

$$c\beta\left[\psi^d_{k,i}\left(u, \frac{j}{c\beta}\right) - \psi^d_{k,i}\left(u, \frac{j-1}{c\beta}\right)\right] \tag{6.69}$$

for $j = 1, 2, \ldots, c\beta t$. Dividing (6.69) by $\psi^d_{k,i}(u)$ gives approximations to the proper density of the time of ruin.

Figures 6.9 to 6.14 show the approximate density of the time of ruin, given that ruin occurs, for $\alpha_1 = 0.1, \alpha_2 = 0.5, \beta = 20$ and $\theta = 0.1$. Figure 6.9 presents the situation when claim amounts follow exponential distributions (Case 1), for $u = 40$ (unscaled). As we can see, the conditional density of the time of ruin in state 1 and state 2 is not distinguishable. This feature can also be observed in Figure 6.12 which shows the density of the time of ruin, given that ruin occurs, when claim amounts follow Pareto distributions (Case 4) and $u = 40$ (unscaled). In both these figures the expected value of aggregate claim amount in state 1 and 2 is the same. Figures 6.10 and 6.13 display the conditional density of $T_{u,k}$ for Cases 2 and 5, respectively when $u = 40$ (unscaled). In both these figures, the expected aggregate claim amount in state 1 is greater than the expected aggregate claim amount in state 2 and we can see that the conditional density of the time of ruin in state 1 is located above the conditional density of the time of ruin in state 2. However, this is not quite clear in Figure 6.13 when claim amounts follow Pareto distributions. Similarly, when the expected aggregate claim amount in state 1 is less than the expected aggregate claim amount in state 2 we can see that the conditional density of the time of ruin in state 2 is above that in state 1. Cases 3 and 6 are illustrated by Figures 6.11 and 6.14, respectively. Unlike in Figure 6.13, this is obvious in Figure 6.14 where the dashed lines representing the density in state 2 are above the smooth lines representing the density in state 1.

The consistent feature of these figures is that they are all positively-skewed. This feature has been observed under the Markov-modulated model in Section 6.5.1 and the classical risk model with capital injections in Section 5.4 as well. Also, we can see that the density of the time of ruin for $k = 1$ is located above the density for $k = 2$ and $k = 3$ and the density of the time of ruin for $k = 2$ is above the density for $k = 3$.

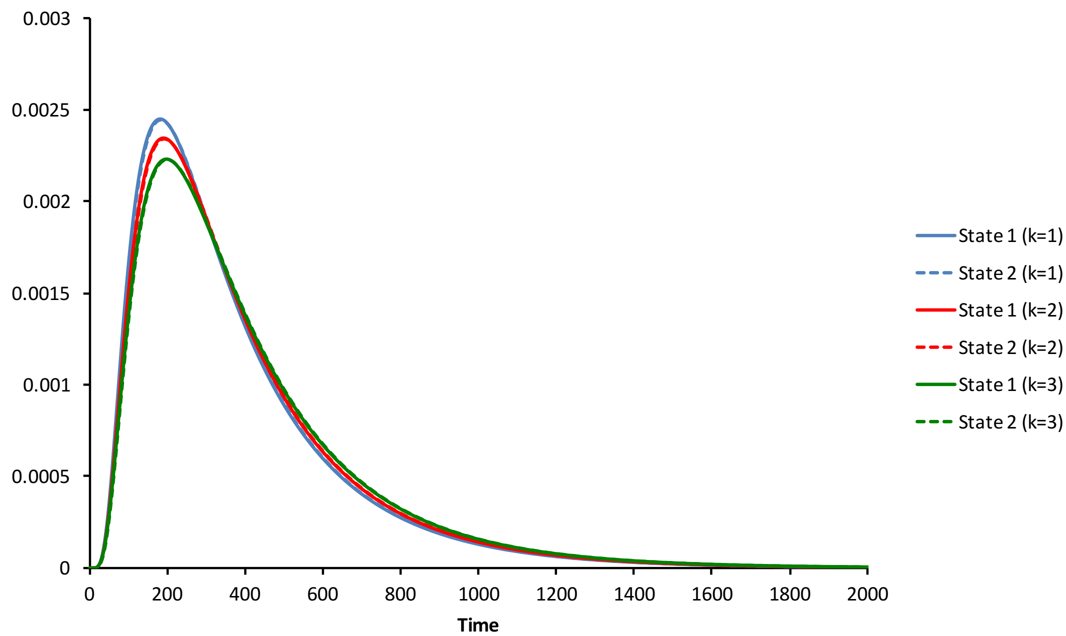

Figure 6.9: Exponential claim amounts when $E[S_1] = E[S_2]$

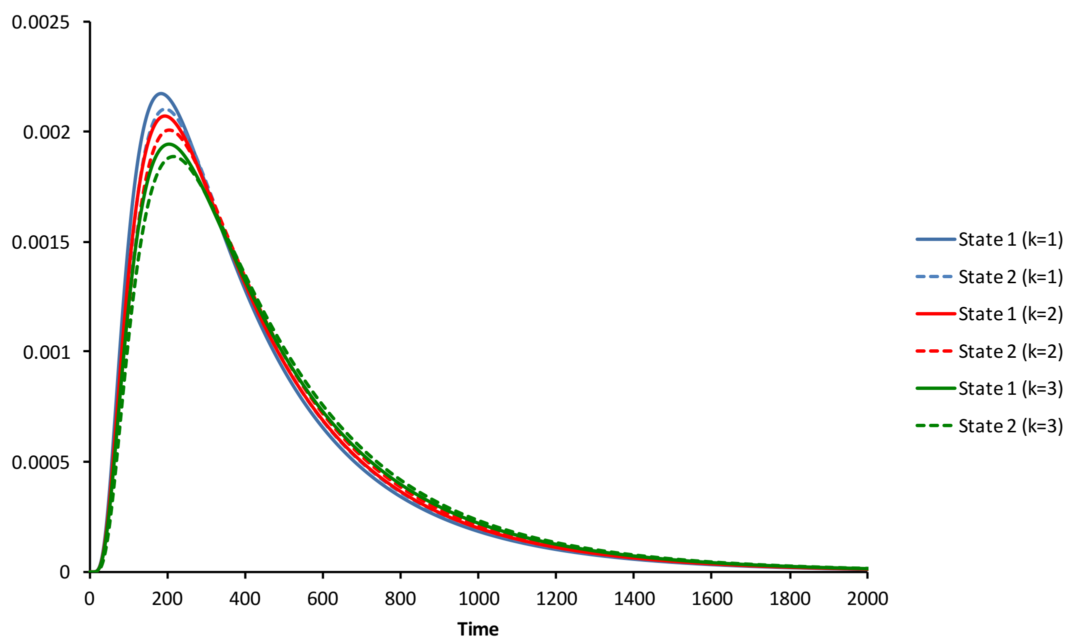

Figure 6.10: Exponential claim amounts when $E[S_1] > E[S_2]$

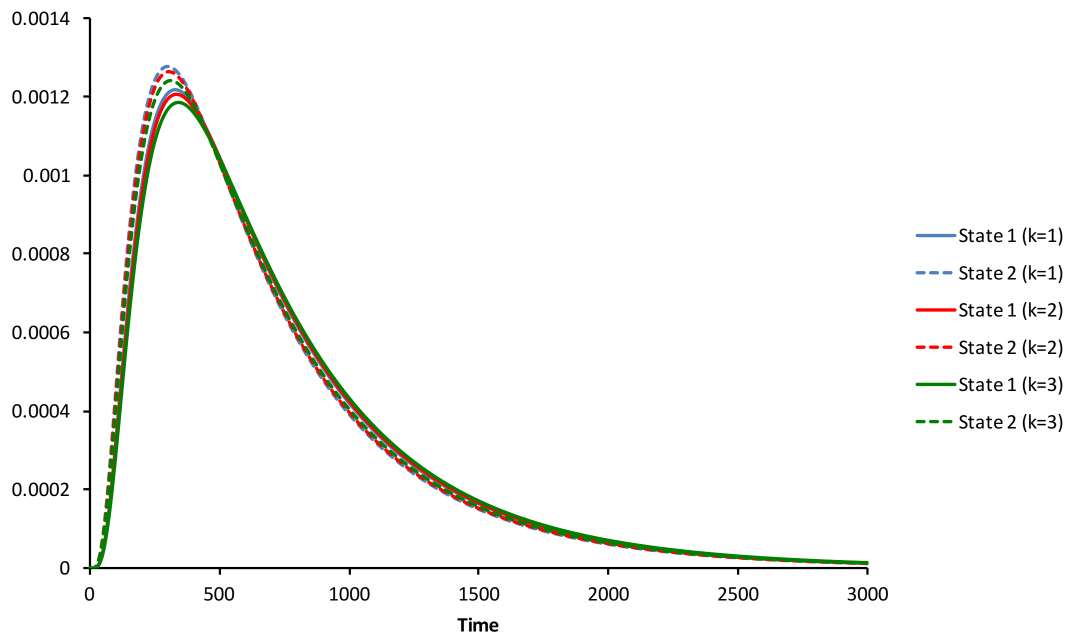

Figure 6.11: Exponential claim amounts when $E[S_1] < E[S_2]$

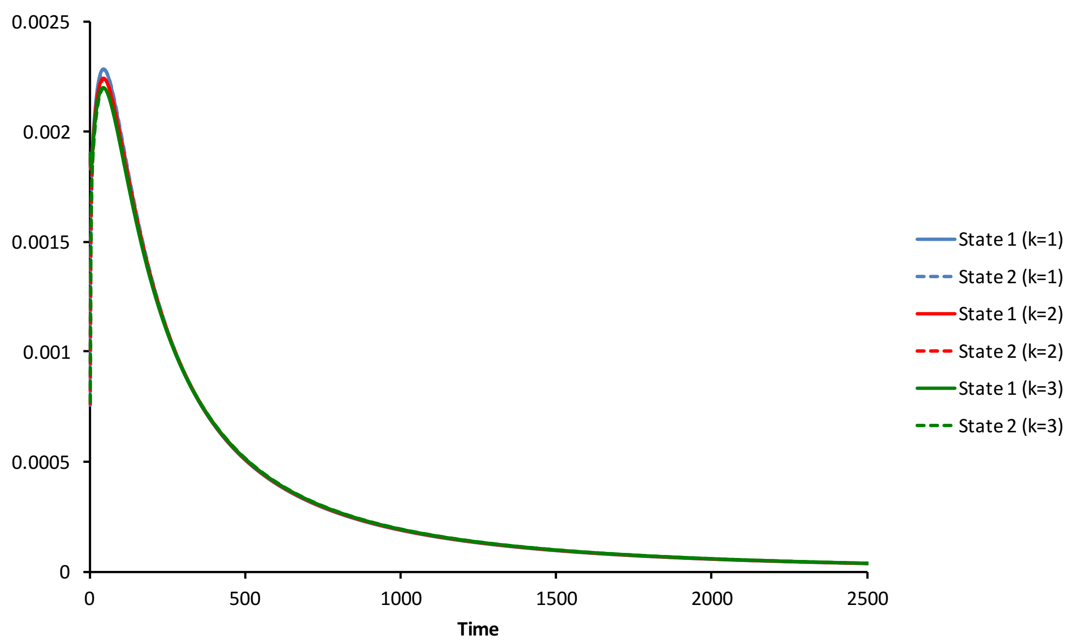

Figure 6.12: Pareto claim amounts when $E[S_1] = E[S_2]$

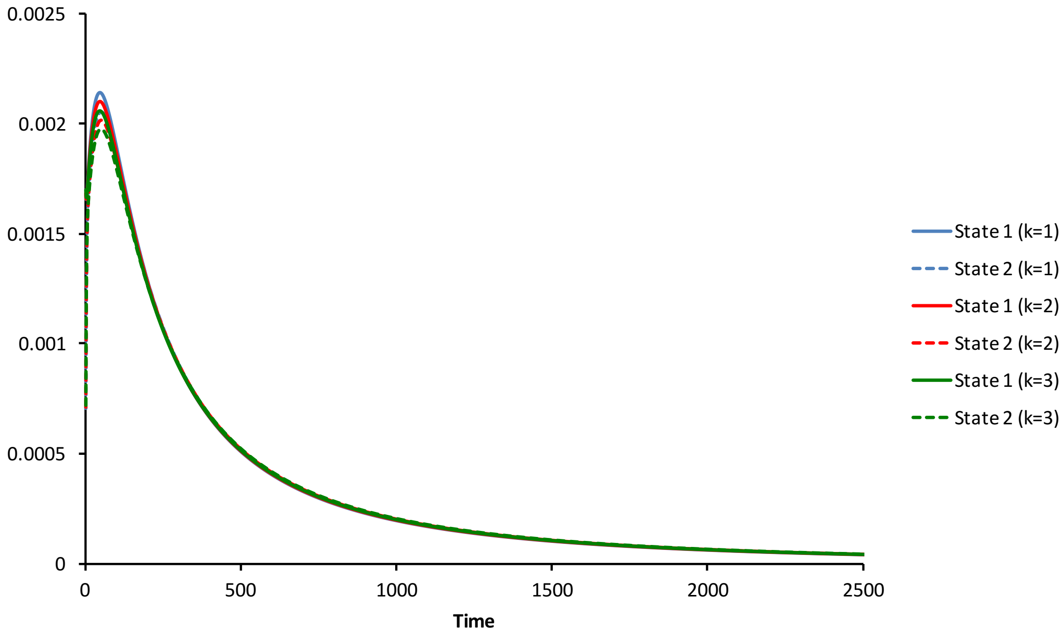

Figure 6.13: Pareto claim amounts when $E[S_1] > E[S_2]$

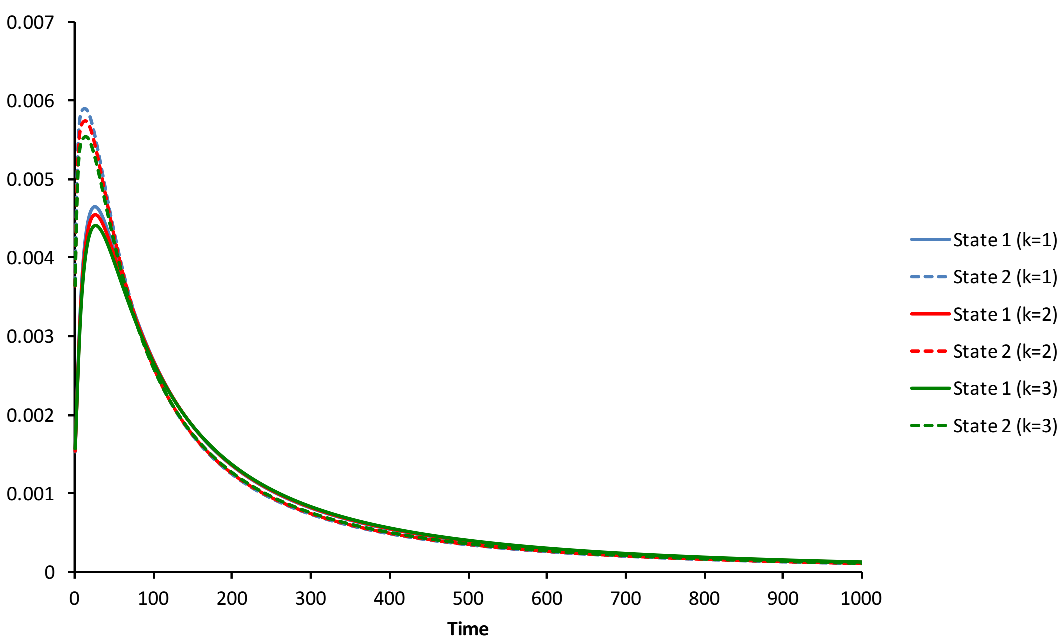

Figure 6.14: Pareto claim amounts when $E[S_1] < E[S_2]$

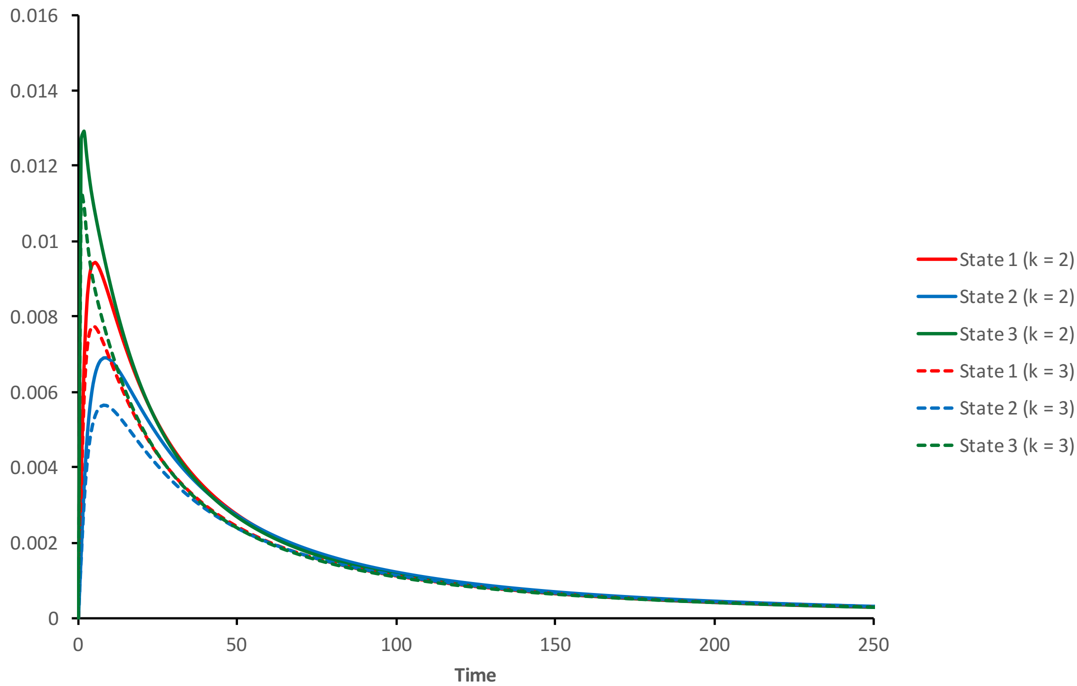

Figure 6.15: Exponential claims

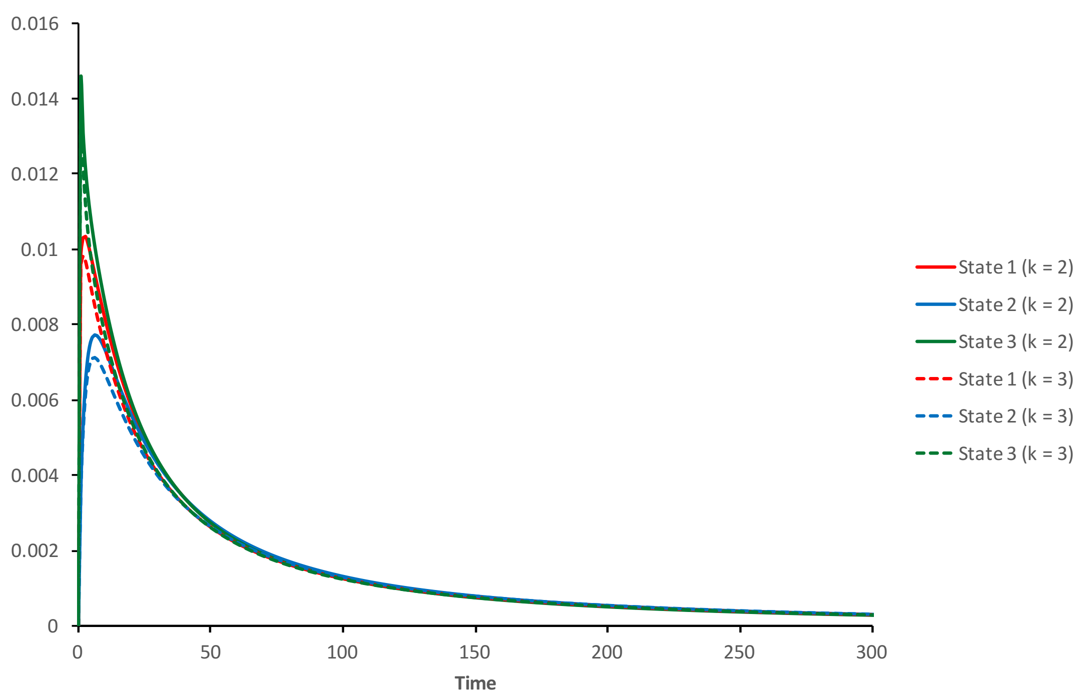

Figure 6.16: Pareto claims

This feature appears in all figures, although for Cases $4, 5$ and 6 when individual claim amounts follow Pareto distributions, it is not quite obvious.

Figures 6.15 and 6.16 show the (defective) density of the time of ruin in a three-state Markov-modulated model with capital injections for claim amounts following exponential and Pareto distributions, respectively when $u = 10$ (unscaled). We use the same parameters as in Section 6.5.2. As we can see, the graphs exhibit the same pattern as in Section 6.5.2, i.e. the density of the time of ruin in state 3 is located above the density of the time of ruin in state 1 and the density of the time of ruin in state 1 is above the density in state 2, which is not surprising as the expected aggregate claim amount in state 3 is greater than that in state 1 and the expected aggregate claim amount in state 1 is greater than that in state 2. Further, the densities follow the same order as in Section 5.4. We observe that the density of the time of ruin when $k = 2$ is above the density of the time of ruin when $k = 3$. As with the density of the time of ruin in a two-state Markov-modulated model, the density of the time of ruin in Figures 6.15 and 6.16 is positively-skewed.

6.7 Concluding remarks

In this chapter, we have extended our numerical algorithm in Chapter 5, which is based on the ideas of Dickson and Waters (1991, 1992), to the Markov-modulated risk model with and without capital injections. We have developed a discrete time version of the Markov-modulated model that can be used to approximate some ruin-related quantities in the continuous time Markov-modulated risk model. We have considered two distributions for individual claim amounts: exponential and Pareto. Comparing our results with exact values when claim amounts are exponentially distributed, we found that our algorithm can produce approximate values which are close to the exact values. In Section 6.5.2 we have graphed the density of the time of ruin for $m = 3$ and explained that this method can be extended to $m > 3$. It may also be possible to approximate the ultimate ruin probability by looking at the cumulative distribution of the time of ruin. In the final section, we incorporated capital injections into our model and showed how our modified algorithm can approximate the probability of ruin in infinite time when $m = 2$ and probability of ruin in finite time when $m > 2$.

Chapter 7

Dividend strategies with capital injections

7.1 Introduction

In this chapter, we consider the question of dividend strategies. Our aim is to examine the impact of capital injections on dividend payments to shareholders of an insurance company. We begin with the situation introduced by De Finneti (1957) and find the optimal level of surplus at which dividends start being distributed among shareholders under our model, by maximising the expected discounted value of dividend payments. Then, we propose a reinsurance arrangement under which any fall below the level k is compensated by this contract, meaning that the idea of capital injections is extended to any situation when the surplus falls below k, and as a result the insurance company may do business indefinitely. We illustrate the application of our results for individual claim amounts with exponential and mixed exponential distributions. Following that we discuss a threshold strategy and find the optimal value of the dividend barrier using two methods. In the final section, we demonstrate that the dividends-penalty identity given by Lin et al. (2003) holds for a model with capital injections.

7.2 Barrier strategy

In this section, we apply the idea of De Finneti (1957) to our risk model with capital injections.

We define $V_k(u,b)$ to be the expected present value at force of interest δ of dividends payable to shareholders prior to ruin whenever the surplus attains level b in the presence of capital injections. Let τ denote the time at which the surplus would reach b if there were no claims, so that $u + c\tau = b$. Thus, by considering whether or not a claim occurs before time τ for $k \le u < b$, we have

$$\begin{aligned} V_k(u,b) &= e^{-(\lambda+\delta)\tau}\, V_k(b,b) + \int_0^\tau \lambda e^{-(\lambda+\delta)t} \int_0^{u+ct-k} f(x)\, V_k(u+ct-x)\, dx\, dt \\ &\quad + \int_0^\tau \lambda e^{-(\lambda+\delta)t} \int_{u+ct-k}^{u+ct} f(x)\, V_k(k,b)\, dx\, dt. \end{aligned}$$

Substituting $s = u + ct$ we obtain

$$\begin{aligned} V_k(u,b) &= e^{-(\lambda+\delta)(b-u)/c}\, V_k(b,b) + \frac{\lambda}{c}\int_u^b e^{-(\lambda+\delta)(s-u)/c} \int_0^{s-k} f(x)\, V_k(s-x,b)\, dx\, ds \\ &\quad + \frac{\lambda}{c}\int_u^b e^{-(\lambda+\delta)(s-u)/c} \int_{s-k}^{s} f(x) V_k(k,b)\, dx\, ds, \end{aligned} \tag{7.1}$$

and then differentiating with respect to u we get

$$\begin{aligned} \frac{\partial}{\partial u} V_k(u,b) &= \frac{\lambda+\delta}{c} V_k(u,b) - \frac{\lambda}{c}\int_0^{u-k} f(x) V_k(u-x,b)\, dx \\ &\quad - \frac{\lambda}{c}\Big(\bar{F}(u-k) - \bar{F}(u)\Big) V_k(k,b). \end{aligned} \tag{7.2}$$

Similarly, by considering dividend payments before and after the first claim, we have

$$\begin{aligned} V_k(b,b) &= \int_0^\infty \lambda e^{-(\lambda+\delta)t} c\, \bar{s}_{\overline{t}|}\, dt + \int_0^\infty \lambda e^{-(\lambda+\delta)t} \int_0^{b-k} f(x) V_k(b-x,b)\, dx\, dt \\ &\quad + \int_0^\infty \lambda e^{-(\lambda+\delta)t} \int_{b-k}^{b} f(x)\, V_k(k,b)\, dx\, dt \end{aligned} \tag{7.3}$$

where $\bar{s}_{\overline{t}|} = (e^{\delta t} - 1)/\delta$ is the accumulated amount at time t of payments at rate 1 per unit time at force of interest δ per unit time. Integrating out in equation (7.3) we obtain

$$\begin{aligned} V_k(b,b) &= \frac{c}{\lambda+\delta} + \frac{\lambda}{\lambda+\delta}\int_0^{b-k} f(x) V_k(b-x,b)\, dx \\ &\quad + \frac{\lambda}{\lambda+\delta}\Big(\bar{F}(b-k) - \bar{F}(b)\Big) V_k(k,b). \end{aligned} \tag{7.4}$$

From equation (7.2) we find that

$$\begin{aligned} \frac{c}{\lambda+\delta}\frac{\partial}{\partial u} V_k(u,b)\Big|_{u=b} &= V_k(b,b) - \frac{\lambda}{\lambda+\delta}\int_0^{b-k} f(x) V_k(b-x,b)\, dx \\ &\quad - \frac{\lambda}{\lambda+\delta}\Big(\bar{F}(b-k) - \bar{F}(b)\Big) V_k(k,b) \end{aligned}$$

which, together with equation (7.3), gives the boundary condition

$$\frac{\partial}{\partial u} V_k(u,b)\Big|_{u=b} = 1. \tag{7.5}$$

In the following results, we find an expression for $V_k(u,b)$ in the case of exponential and mixed exponential claim amounts.

Result 7.1. When $F(x) = 1 - e^{-\alpha x}$, $x \geq 0$, with $\alpha > 0$,

$$V_k(u,b) = \frac{D_1(k)e^{\rho u} - D_2(k)e^{-Ru}}{D_1(k)\rho e^{\rho b} + D_2(k)Re^{-Rb}} \tag{7.6}$$

with

$$\begin{aligned} D_1(k) &= c(\alpha+\rho)e^{(\alpha-R)k} - \lambda(e^{\alpha k}-1)e^{-Rk}, \\ D_2(k) &= c(\alpha-R)e^{(\alpha+\rho)k} - \lambda(e^{\alpha k}-1)e^{\rho k}, \end{aligned} \tag{7.7}$$

where $\rho \equiv \rho(\delta) > 0$ and $-R \equiv -R(\delta) < 0$ are the roots of Lundberg's fundamental equation, given by

$$s^2 + \left(\alpha - \frac{\lambda+\delta}{c}\right)s - \frac{\alpha\delta}{c} = 0. \tag{7.8}$$

Derivation. Substituting for f, we write equation (7.2) as

$$\begin{aligned} \frac{\partial}{\partial u} V_k(u,b) &= \frac{\lambda+\delta}{c} V_k(u,b) - \frac{\lambda}{c}\int_k^u \alpha e^{-\alpha(u-x)} V_k(x,b)\,dx \\ &\quad - \frac{\lambda}{c} e^{-\alpha u}(e^{\alpha k}-1)V_k(k,b), \end{aligned} \tag{7.9}$$

and differentiation of equation (7.9) yields

$$\frac{\partial^2}{\partial u^2} V_k(u,b) + \left(\alpha - \frac{\lambda+\delta}{c}\right)\frac{\partial}{\partial u}V_k(u,b) - \frac{\alpha\delta}{c}V_k(u,b) = 0. \tag{7.10}$$

The general solution of equation (7.10) is given by

$$V_k(u,b) = \gamma_1 e^{\rho u} + \gamma_2 e^{-Ru} \tag{7.11}$$

and from (7.5) the boundary condition is

$$\gamma_1 \rho e^{\rho b} - \gamma_2 R e^{-Rb} = 1.$$

We can now insert the functional form (7.11) of $V_k(u,b)$ into equation (7.9), giving

$$\begin{aligned}
&\gamma_1\rho e^{\rho u}-\gamma_2 Re^{-Ru}\\
=\;&\frac{\lambda+\delta}{c}\left(\gamma_1 e^{\rho u}+\gamma_2 e^{-Ru}\right)-\frac{\alpha\lambda}{c}e^{-\alpha u}\int_k^u\left(\gamma_1 e^{(\alpha+\rho)x}+\gamma_2 e^{(\alpha-R)x}\right)dx\\
&-\frac{\lambda}{c}e^{-\alpha u}(e^{\alpha k}-1)\left(\gamma_1 e^{\rho k}+\gamma_2 e^{-Rk}\right). \qquad (7.12)
\end{aligned}$$

Rearranging this identity we obtain

$$\begin{aligned}
&\gamma_1 e^{\rho u}\left(\rho-\frac{\lambda+\delta}{c}+\frac{\alpha\lambda}{c}\frac{1}{\alpha+\rho}\right)-\gamma_2 e^{-Ru}\left(R+\frac{\lambda+\delta}{c}-\frac{\alpha\lambda}{c}\frac{1}{\alpha-R}\right)\\
&+e^{-\alpha u}\left[\frac{\lambda}{c}(e^{\alpha k}-1)\left(\gamma_1 e^{\rho k}+\gamma_2 e^{-Rk}\right)-\frac{\alpha\lambda}{c}\left(\frac{\gamma_1}{\alpha+\rho}e^{(\alpha+\rho)k}+\frac{\gamma_2}{\alpha-R}e^{(\alpha-R)k}\right)\right]=0.
\end{aligned}$$

Since

$$\rho-\frac{\lambda+\delta}{c}+\frac{\alpha\lambda}{c}\frac{1}{\alpha+\rho}=\frac{1}{\alpha+\rho}\left(\rho^2+\left(\alpha-\frac{\lambda+\delta}{c}\right)\rho-\frac{\alpha\delta}{c}\right)=0$$

and

$$R+\frac{\lambda+\delta}{c}-\frac{\alpha\lambda}{c}\frac{1}{\alpha-R}=\frac{-1}{\alpha-R}\left(R^2-\left(\alpha-\frac{\lambda+\delta}{c}\right)R-\frac{\alpha\delta}{c}\right)=0$$

by equation (7.8), the coefficients of $e^{\rho u}$ and e^{-Ru} are 0; consequently, we have

$$\frac{\gamma_1}{\gamma_2}=\frac{-(e^{\alpha k}-1)e^{-Rk}+e^{(\alpha-R)k}\alpha/(\alpha-R)}{(e^{\alpha k}-1)e^{\rho k}-e^{(\alpha+\rho)k}\alpha/(\alpha+\rho)}.$$

By noting that $(\alpha+\rho)(\alpha-R)=\alpha\lambda/c$ (see Dickson, 2005) we can write this as

$$\frac{\gamma_1}{\gamma_2}=\frac{c(\alpha+\rho)e^{(\alpha-R)k}-\lambda(e^{\alpha k}-1)e^{-Rk}}{\lambda(e^{\alpha k}-1)e^{\rho k}-c(\alpha-R)e^{(\alpha+\rho)k}}=-\frac{D_1(k)}{D_2(k)}. \qquad (7.13)$$

Using the boundary condition, formula (7.6) follows. □

To find the optimal values of b and k, we differentiate (7.6) once with respect to b and once with respect to k and then equate to zero. Hence we need to solve the following system of equations:

$$\begin{cases}
\left(D_1(k)\rho^2 e^{\rho b}-D_2(k)R^2 e^{-Rb}\right)\left(D_1(k)e^{\rho u}-D_2(k)e^{-Ru}\right)=0\\
\left(D_1'(k)e^{\rho u}-D_2'(k)e^{-Ru}\right)\left(D_1(k)\rho e^{\rho b}+D_2(k)Re^{-Rb}\right)\\
-\left(D_1'(k)\rho e^{\rho b}+D_2'(k)Re^{-Rb}\right)\left(D_1(k)e^{\rho u}-D_2(k)e^{-Ru}\right)=0
\end{cases}$$

so,

$$\left(D_1(k)\rho e^{\rho b}+D_2(k)Re^{-Rb}\right)\left(D_1(k)\rho^2 e^{\rho b}-D_2(k)R^2e^{-Rb}\right)$$
$$\times\left(D_1^{'}(k)e^{\rho u}-D_2^{'}(k)e^{-Ru}\right)=0. \tag{7.14}$$

To find the optimal value of b from (7.14) we have

$$D_1(k)\rho^2 e^{\rho b}-D_2(k)R^2e^{-Rb}=0. \tag{7.15}$$

Thus

$$b^*=\frac{1}{\rho+R}\log\frac{R^2D_2(k)}{\rho^2D_1(k)}. \tag{7.16}$$

Further, as

$$\begin{aligned}D_1^{'}(k) &= c(\alpha+\rho)(\alpha-R)e^{(\alpha-R)k}+\lambda Re^{-Rk}(e^{\alpha k}-1)-\lambda\alpha e^{(\alpha-R)k}\\ &= \lambda Re^{-Rk}(e^{\alpha k}-1),\end{aligned}$$

and $k>0$, $D_1(k)$ is an increasing function of k. Also, as

$$\begin{aligned}D_2^{'}(k) &= c(\alpha-R)(\alpha+\rho)e^{(\alpha+\rho)k}-\lambda\rho e^{\rho k}(e^{\alpha k}-1)-\lambda\alpha e^{(\alpha+\rho)k}\\ &= -\lambda\rho e^{\rho k}(e^{\alpha k}-1),\end{aligned}$$

$D_2(k)$ is a decreasing function of k and $D_2(k)<0$ for $k\geq\frac{-1}{\alpha}\log\frac{\lambda-c(\alpha-R)}{\lambda}$. Therefore, expression (7.16) shows the relationship between b^* and k that can lead to the maximisation of $V_k(u,b)$ on the condition that

$$0<k<\frac{-1}{\alpha}\log\frac{\lambda-c(\alpha-R)}{\lambda}. \tag{7.17}$$

We note that the numerator of (7.6) is independent of b. Therefore, we can get the same result as (7.16) if we minimise the denominator of (7.6). If we set $k=0$, (7.16) gives the optimal value of b in Bühlmann (1970), who considered the classical risk model with dividends, where he has shown that the first moment of discounted value of dividend payments can be factorised by $V_1(u,b)=C(b)h(u)$. (Recall from Chapter 1 that $V_1(u,b)$ is the expected present value at force of interest δ of dividends payable to the shareholders prior to ruin whenever the surplus attains level b in the classical risk model without capital injections.) We can see that such a factorisation applies to our model as well.

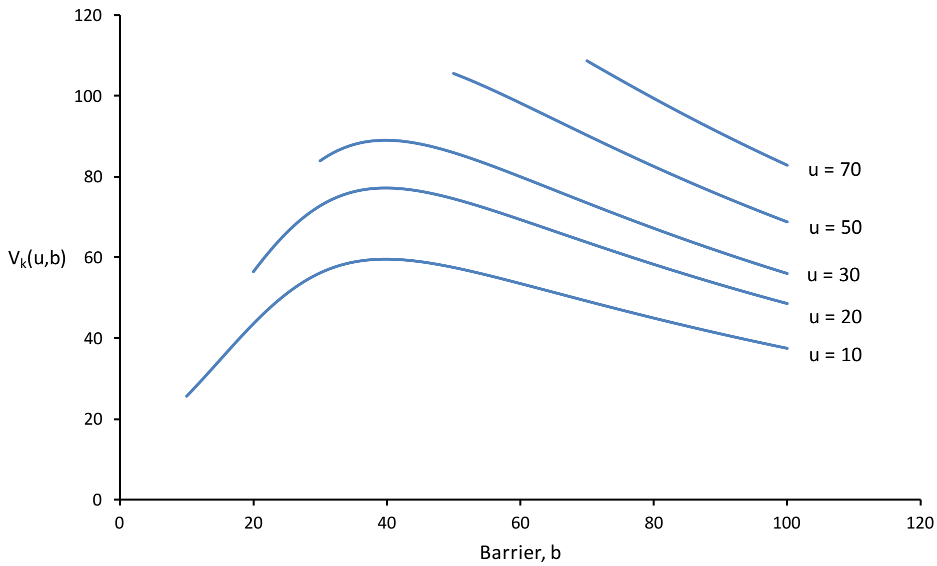

Figure 7.1: $V_k(u, b)$ for $k = 2$, exponential claims

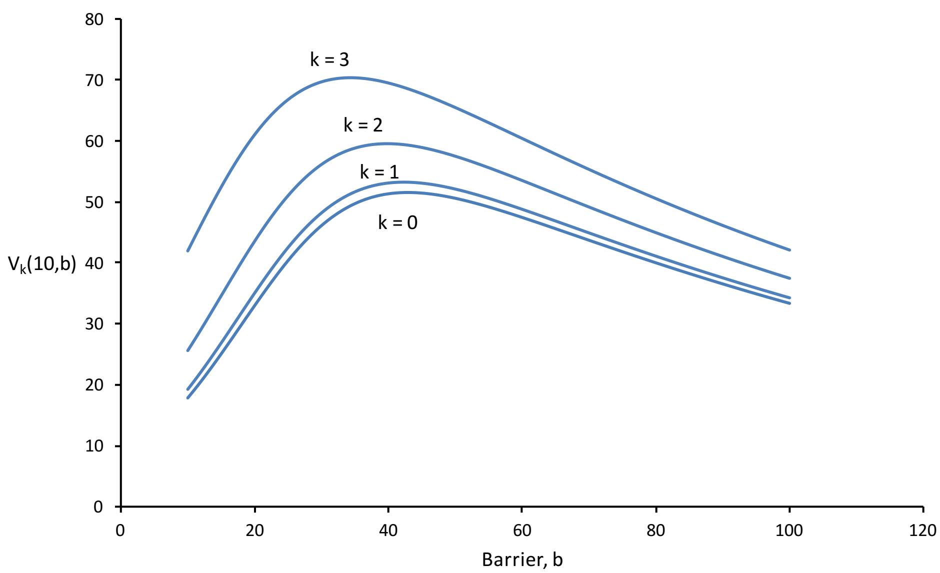

Figure 7.2: $V_k(10, b)$, exponential claims

From equation (7.16) we can conclude that the optimal value of b is independent of u. Now, to see the impact of capital injections on the optimal value of b we differentiate identity (7.16) with respect to k. Then

$$\frac{d}{dk}b^* = \frac{1}{\rho + R}\frac{D_2'(k)D_1(k) - D_1'(k)D_2(k)}{D_1(k)D_2(k)}.$$

We know that $D_2'(k)D_1(k) < 0$ and $D_1'(k)D_2(k) > 0$. Therefore, we can conclude that the optimal value of b is a decreasing function of k, which is reasonable, since as k increases, the likelihood of the surplus falling below 0 decreases and the insurance company is more confident to start paying dividends earlier, and the earlier the insurer starts distributing dividends, the higher $V_k(u, b)$ is.

Figures 7.1 and 7.2 illustrate the values of $V_k(u, b)$ when $\alpha = 1, \lambda = 100, c = 110$ and $\delta = 0.1$ for different values of u and k. From (7.17) we require that $0 < k < 4.70$. The fact that the optimal value of b is independent of u can be observed from Figure 7.1. Also, in Figure 7.2 as k increases, the optimal value of b decreases as we would expect.

Result 7.2. When $f(x) = p\alpha e^{-\alpha x} + q\beta e^{-\beta x}$, where $p + q = 1, 0 < p < 1$ and $\alpha < \beta$, we have

$$\begin{aligned} V_k(u,b) &= \frac{1}{L(b)}\Big(\left(A_2(k)B_3(k) - A_3(k)B_2(k)\right) e^{\rho u} \\ &\quad + \left(A_3(k)B_1(k) - A_1(k)B_3(k)\right) e^{-R_1 u} + \left(A_1(k)B_2(k) - A_2(k)B_1(k)\right) e^{-R_2 u}\Big) \end{aligned} \tag{7.18}$$

where

$$\begin{aligned} L(b) &= \rho e^{\rho b}\left(A_2(k)B_3(k) - A_3(k)B_2(k)\right) + R_1 e^{-R_1 b}\left(A_1(k)B_3(k) - A_3(k)B_1(k)\right) \\ &\quad + R_2 e^{-R_2 b}\left(A_2(k)B_1(k) - A_1(k)B_2(k)\right) \end{aligned} \tag{7.19}$$

and

$$\begin{aligned} A_1(k) &= \frac{\alpha}{\alpha+\rho}e^{(\alpha+\rho)k} - (e^{\alpha k} - 1)e^{\rho k}, \\ A_2(k) &= \frac{\alpha}{\alpha - R_1}e^{(\alpha - R_1)k} - (e^{\alpha k} - 1)e^{-R_1 k}, \\ A_3(k) &= \frac{\alpha}{\alpha - R_2}e^{(\alpha - R_2)k} - (e^{\alpha k} - 1)e^{-R_2 k}, \end{aligned}$$

$$\begin{aligned}
B_1(k) &= \frac{\beta}{\beta+\rho}e^{(\beta+\rho)k} - (e^{\beta k}-1)e^{\rho k}, \\
B_2(k) &= \frac{\beta}{\beta-R_1}e^{(\beta-R_1)k} - (e^{\beta k}-1)e^{-R_1 k}, \\
B_3(k) &= \frac{\beta}{\beta-R_2}e^{(\beta-R_2)k} - (e^{\beta k}-1)e^{-R_2 k},
\end{aligned} \tag{7.20}$$

where $\rho \equiv \rho(\delta) > 0$, $-R_i \equiv -R_i(\delta), i = 1, 2$ are the solutions to Lundberg's fundamental equation (see Gerber et al., 2006b, formula 7.13), given by

$$s^3 + \left(\alpha+\beta-\frac{\lambda+\delta}{c}\right)s^2 + \left(\alpha\beta+(p\alpha+q\beta)\frac{\lambda}{c}-(\alpha+\beta)\frac{\lambda+\delta}{c}\right)s - \frac{\alpha\beta\delta}{c} = 0. \tag{7.21}$$

Derivation. Substituting for f, we write equation (7.2) as

$$\begin{aligned}
\frac{\partial}{\partial u}V_k(u,b) &= \frac{\lambda+\delta}{c}V_k(u,b) - \frac{\lambda}{c}\int_k^u \left(p\alpha e^{-\alpha(u-x)} + q\beta e^{-\beta(u-x)}\right) V_k(x,b)\,dx \\
&\quad -\frac{\lambda}{c}\left(pe^{-\alpha u}(e^{\alpha k}-1) + qe^{-\beta u}(e^{\beta k}-1)\right)V_k(k,b).
\end{aligned} \tag{7.22}$$

Next, we apply the operator $\left(\frac{\partial}{\partial u}+\alpha\right)\left(\frac{\partial}{\partial u}+\beta\right)$ to both sides of equation (7.22). To do this, we differentiate equation (7.22) with respect to u. Hence

$$\begin{aligned}
\frac{\partial^2}{\partial u^2}V_k(u,b) &= \frac{\lambda+\delta}{c}\frac{\partial}{\partial u}V_k(u,b) - \frac{\lambda}{c}(p\alpha+q\beta)V_k(u,b) \\
&\quad +\frac{\lambda}{c}\int_k^u \left(p\alpha^2 e^{-\alpha(u-x)} + q\beta^2 e^{-\beta(u-x)}\right) V_k(x,b)\,dx \\
&\quad +\frac{\lambda}{c}\left(p\alpha e^{-\alpha u}(e^{\alpha k}-1) + q\beta e^{-\beta u}(e^{\beta k}-1)\right)V_k(k,b).
\end{aligned} \tag{7.23}$$

Differentiation of equation (7.23) with respect to u gives

$$\begin{aligned}
\frac{\partial^3}{\partial u^3}V_k(u,b) &= \frac{\lambda+\delta}{c}\frac{\partial^2}{\partial u^2}V_k(u,b) - \frac{\lambda}{c}(p\alpha+q\beta)\frac{\partial}{\partial u}V_k(u,b) + \frac{\lambda}{c}(p\alpha^2+q\beta^2)V_k(u,b) \\
&\quad -\frac{\lambda}{c}\int_k^u \left(p\alpha^3 e^{-\alpha(u-x)} + q\beta^3 e^{-\beta(u-x)}\right) V_k(x,b)\,dx \\
&\quad -\frac{\lambda}{c}\left(p\alpha^2 e^{-\alpha u}(e^{\alpha k}-1) + q\beta^2 e^{-\beta u}(e^{\beta k}-1)\right)V_k(k,b),
\end{aligned} \tag{7.24}$$

and hence we get

$$\begin{aligned}
&\frac{\partial^3}{\partial u^3}V_k(u,b) + \left(\alpha+\beta-\frac{\lambda+\delta}{c}\right)\frac{\partial^2}{\partial u^2}V_k(u,b) \\
&+\left(\alpha\beta+(p\alpha+q\beta)\frac{\lambda}{c}-(\alpha+\beta)\frac{\lambda+\delta}{c}\right)\frac{\partial}{\partial u}V_k(u,b) - \frac{\alpha\beta\delta}{c}V_k(u,b) = 0.
\end{aligned} \tag{7.25}$$

Thus, the general solution of equation (7.25) is of the form

$$V_k(u,b) = \eta_0 e^{\rho u} + \eta_1 e^{-R_1 u} + \eta_2 e^{-R_2 u}. \tag{7.26}$$

Further, from equation (7.5) the first boundary condition is given by

$$\eta_0 \rho e^{\rho b} - \eta_1 R_1 e^{-R_1 b} - \eta_2 R_2 e^{-R_2 b} = 1. \tag{7.27}$$

We can now insert the functional form (7.26) of $V_k(u,b)$ into equation (7.22), giving

$$\begin{aligned}
&\eta_0 \rho e^{\rho u} - \eta_1 R_1 e^{-R_1 u} - \eta_2 R_2 e^{-R_2 u} = \frac{\lambda+\delta}{c}\left(\eta_0 e^{\rho u} + \eta_1 e^{-R_1 u} + \eta_2 e^{-R_2 u}\right) \\
&-\frac{\lambda}{c}\int_k^u \left(p\alpha e^{-\alpha(u-x)} + q\beta e^{-\beta(u-x)}\right)\left(\eta_0 e^{\rho x} + \eta_1 e^{-R_1 x} + \eta_2 e^{-R_2 x}\right) dx \\
&-\frac{\lambda}{c}\left(pe^{-\alpha u}(e^{\alpha k}-1) + qe^{-\beta u}(e^{\beta k}-1)\right)\left(\eta_0 e^{\rho k} + \eta_1 e^{-R_1 k} + \eta_2 e^{-R_2 k}\right).
\end{aligned}$$

Rearranging, we obtain

$$\begin{aligned}
&\eta_0 e^{\rho u}\left(\rho - \frac{\lambda+\delta}{c} + \frac{\alpha\lambda}{c}\frac{p}{\alpha+\rho} + \frac{\beta\lambda}{c}\frac{q}{\beta+\rho}\right) \\
&-\eta_1 e^{R_1 u}\left(R_1 + \frac{\lambda+\delta}{c} - \frac{\alpha\lambda}{c}\frac{p}{\alpha-R_1} - \frac{\beta\lambda}{c}\frac{q}{\beta-R_1}\right) \\
&-\eta_2 e^{-R_2 u}\left(R_2 + \frac{\lambda+\delta}{c} - \frac{\alpha\lambda}{c}\frac{p}{\alpha-R_2} - \frac{\beta\lambda}{c}\frac{q}{\beta-R_2}\right) \\
= \;& \frac{\lambda}{c}e^{-\alpha u}\left[\frac{\eta_0}{\alpha+\rho}p\alpha e^{(\alpha+\rho)k} + \frac{\eta_1}{\alpha-R_1}p\alpha e^{(\alpha-R_1)k} + \frac{\eta_2}{\alpha-R_2}p\alpha e^{(\alpha-R_2)k}\right. \\
&\left.-p(e^{\alpha k}-1)\left(\eta_0 e^{\rho k} + \eta_1 e^{-R_1 k} + \eta_2 e^{-R_2 k}\right)\right] + \frac{\lambda}{c}e^{-\beta u}\left[\frac{\eta_0}{\beta+\rho}q\beta e^{(\beta+\rho)k}\right. \\
&\left.+\frac{\eta_1}{\beta-R_1}q\beta e^{(\beta-R_1)k} + \frac{\eta_2}{\beta-R_2}q\beta e^{(\beta-R_2)k} - q(e^{\beta k}-1)\left(\eta_0 e^{\rho k} + \eta_1 e^{-R_1 k} + \eta_2 e^{-R_2 k}\right)\right].
\end{aligned}$$

The coefficient of $e^{\rho u}$ is zero, since

$$\begin{aligned}
&\rho - \frac{\lambda+\delta}{c} + \frac{\alpha\lambda}{c}\frac{p}{\alpha+\rho} + \frac{\beta\lambda}{c}\frac{q}{\beta+\rho} = \frac{1}{(\alpha+\rho)(\beta+\rho)} \\
&\times\left[\rho^3 + \left(\alpha+\beta-\frac{\lambda+\delta}{c}\right)\rho^2 + \left(\alpha\beta + (p\alpha+q\beta)\frac{\lambda}{c} - (\alpha+\beta)\frac{\lambda+\delta}{c}\right)\rho - \frac{\alpha\beta\delta}{c}\right] = 0
\end{aligned}$$

by equation (7.21). Similarly, we can show that the coefficients of $e^{-R_1 u}$ and $e^{-R_2 u}$ are also zero. Consequently, the coefficients of $e^{-\alpha u}$ and $e^{-\beta u}$ are zero and we can write

$$\begin{aligned}
A_1(k)\eta_0 + A_2(k)\eta_1 + A_3(k)\eta_2 &= 0, \\
B_1(k)\eta_0 + B_2(k)\eta_1 + B_3(k)\eta_2 &= 0,
\end{aligned} \tag{7.28}$$

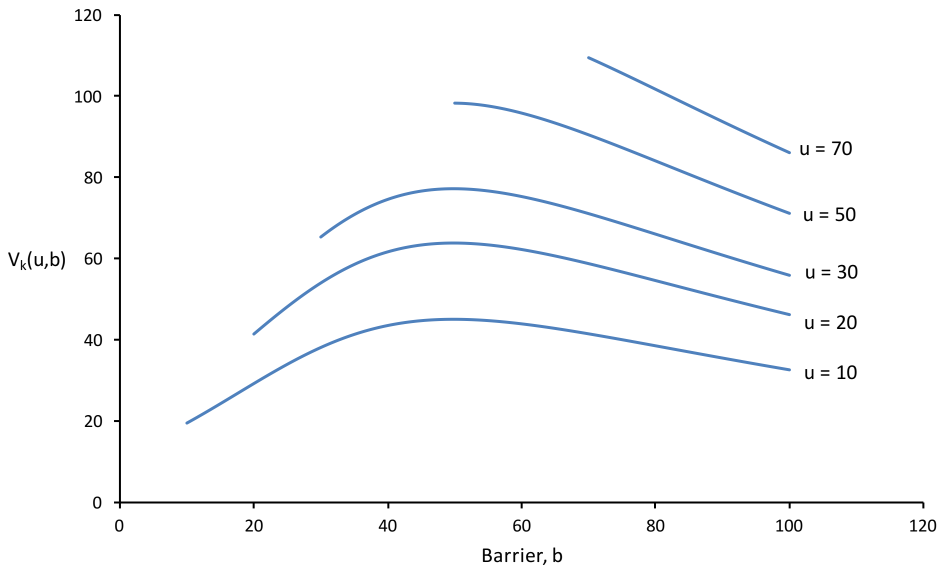

Figure 7.3: $V_k(u, b)$ for $k = 2$, mixed exponential claims

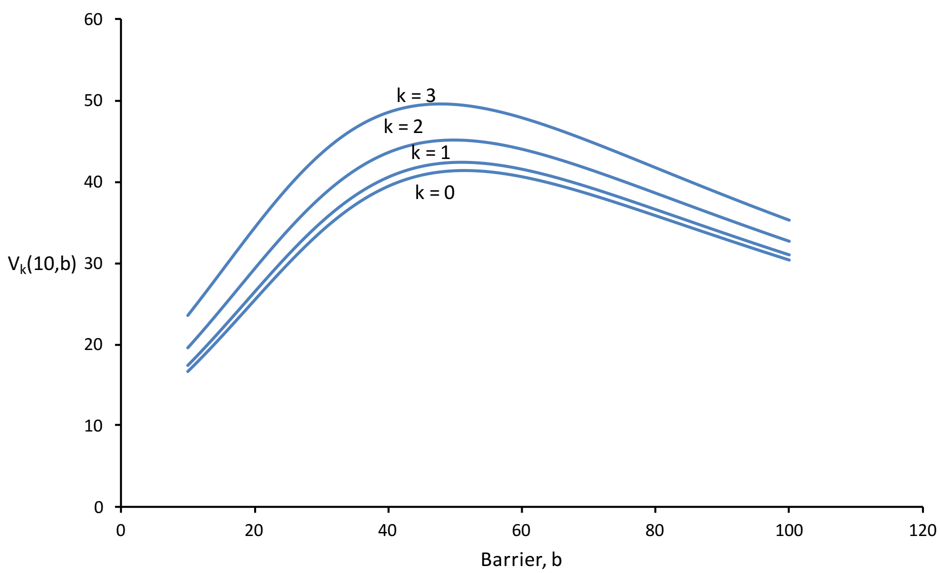

Figure 7.4: $V_k(10, b)$, mixed exponential claims

where $A_i(k)$ and $B_i(k)$ for $i = 1, 2, 3$ are given by (7.20). We can find η_0, η_1 and η_2 from equations (7.28) and the boundary condition in equation (7.27) and hence the result follows. □

Conditions (7.28) are a special case of equation (7.8) of Gerber et al. (2006b). We can see that similar to the exponential claim amounts case, as the denominator is independent of u, it is possible to maximise $V_k(u, b)$ by minimising the denominator. Therefore, the optimal barrier b satisfies the relationship

$$\frac{\rho^2 B_3(k)e^{\rho b^*} - R_1R_2B_1(k)e^{-R_2b^*}}{R_1^2B_3(k)e^{-R_1b^*} - R_2^2B_2(k)e^{-R_2k}} = \frac{A_1(k)B_3(k) - A_3(k)B_1(k)}{A_2(k)B_3(k) - A_3(k)B_2(k)}. \tag{7.29}$$

Now we consider the function $f(x) = \frac{1}{3}(\frac{1}{2}e^{-x/2}) + \frac{2}{3}(2e^{-2x})$ with $\lambda = 100, \delta = 0.1$ and $c = 110$. Substituting in (7.21) we get $\rho = 0.00884516, -R_1 = 0.0670988, -R_2 = 1.53175$.

Figures 7.3 and 7.4 show values of $V_k(u, b)$ for different values of b when $k = 2$ is fixed in Figure 7.3 and $u = 10$ is fixed in Figure 7.4. We observe from Figure 7.3 that the optimal value of b is independent of u and as the initial surplus increases, the expected present value of dividend payments rises. This is consistent with Figure 7.1. However, unlike in Figure 7.2 we can see in Figure 7.4 that as k increases, the reduction in the optimal value of b is not very evident.

7.3 A reinsurance arrangement

In this section, we assume that the shareholders input u, and when the surplus drops either below k, or below 0, a compensation which is equal to the amount of the fall below k is provided by a reinsurance contract which has been purchased by shareholders, so that the surplus is restored to k. The insurance company can then continue from this surplus level, and the operation from the time of the fall is independent of the past, so that each time a claim causes the surplus to fall below k, after the capital injection, the surplus goes back to k and this time point is a renewal point of the new process. The surplus process is now moving indefinitely between k and b.

Let $Y_{u,b,k}$ be the deficit at ruin and let $T_{u,b,k}$ be the time of ruin in our model, assuming the initial surplus is u. We define $R_k(u, b) = E[e^{-\delta T_{u,b,k}}Y_{u,b,k}]$ to be the expected present value of the deficit at ruin and $\tilde{R}_k(u, b) = E[e^{-\delta T_{u,b,k}}]$ to be the expected present value of a unit payable at the time of ruin at force of interest δ. Also, let $T_{u,b}$ and $Y_{u,b}$ be the time of ruin and the deficit at the time of ruin in the classical model without capital injections when the surplus is bounded by the upper level b, respectively. Our purpose is to examine how our reinsurance contract affects the expected present value of net income to the shareholders. So, let $\tilde{V}_k(u,b)$ denote the expected present value of dividends only. Then, by noting that $V_1(u-k,b-k)$ is the expected present value at force of interest δ of dividends payable to the shareholders prior to ruin whenever the surplus attains level b in the classical risk model without capital injections, we can write

$$\tilde{V}_k(u,b) = V_1(u-k,b-k) + E[e^{-\delta T_{u-k,b-k}}]\tilde{V}_k(k,b). \tag{7.30}$$

Hence

$$\tilde{V}_k(k,b) = \frac{V_1(0,b-k)}{1-E[e^{-\delta T_{0,b-k}}]}. \tag{7.31}$$

Further, let $\tilde{W}_k(u,b)$ denote the expected present value of payments to be made by the reinsurance company when the surplus falls below k. Then

$$\tilde{W}_k(u,b) = E[e^{-\delta T_{u-k,b-k}}Y_{u-k,b-k}] + E[e^{-\delta T_{u-k,b-k}}]\tilde{W}_k(k,b) \tag{7.32}$$

which gives

$$\tilde{W}_k(k,b) = \frac{E[e^{-\delta T_{0,b-k}}Y_{0,b-k}]}{1-E[e^{-\delta T_{0,b-k}}]}. \tag{7.33}$$

Defining $N_k(u,b)$ to be the expected present value of net income to the shareholders, our strategy is to find the value of b which maximises

$$N_k(u,b) = \tilde{V}_k(u,b) - (1+\theta_R)\tilde{W}_k(u,b) - u. \tag{7.34}$$

We already know expressions for $V_k(u,b)$ and we can use formulae (44) of Dickson and Waters (2004) for $n=1$ to find $E[e^{-\delta T_{u,b}}Y_{u,b}]$ and $E[e^{-\delta T_{u,b}}]$ for exponentially distributed claim amounts. Therefore, it is straightforward to calculate the expected present value of net income for exponential claims. However, in the case of claim amounts following a mixed exponential distribution, we first derive expressions for $E[e^{-\delta T_{u,b,k}}Y_{u,b,k}]$ and $E[e^{-\delta T_{u,b,k}}]$. Then, setting $k=0$, gives the results for $E[e^{-\delta T_{u,b}}Y_{u,b}]$ and $E[e^{-\delta T_{u,b}}]$.

By considering the time and the amount of the first claim, and whether or not the first claim occurs before time τ, standard arguments – see, for example, Dickson and Waters (2004) – lead to

$$\begin{aligned}\frac{\partial}{\partial u}R_k(u,b) &= \frac{\lambda+\delta}{c}R_k(u,b) - \frac{\lambda}{c}\int_0^{u-k} f(y)R_k(u-y,b)\,dy \\ &\quad -\frac{\lambda}{c}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)R_k(k,b) - \frac{\lambda}{c}\int_u^{\infty}(y-u)f(y)\,dy.\end{aligned} \tag{7.35}$$

For $u = b$ we get

$$\begin{aligned}R_k(b,b) &= \int_0^{\infty}\lambda e^{-(\lambda+\delta)t}\Bigg(\int_0^{b-k} f(y)R_k(b-y,b)\,dy + \int_{b-k}^{b} f(y)R_k(k,b)\,dy \\ &\quad + \int_b^{\infty}(y-b)f(y)\,dy\Bigg)\,dt \\ &= \frac{\lambda}{\lambda+\delta}\Bigg(\int_0^{b-k} f(y)R_k(b-y,b)\,dy + \int_{b-k}^{b} f(y)R_k(k,b)\,dy \\ &\quad + \int_b^{\infty}(y-b)f(y)\,dy\Bigg).\end{aligned} \tag{7.36}$$

Setting $u = b$ in (7.35) yields

$$\begin{aligned}\frac{\partial}{\partial u}R_k(u,b)\Big|_{u=b} &= \frac{\lambda+\delta}{c}R_k(b,b) - \frac{\lambda}{c}\int_0^{b-k} f(y)R_k(b-y,b)\,dy \\ &\quad -\frac{\lambda}{c}\int_{b-k}^{b} f(y)R_k(k,b)\,dy - \frac{\lambda}{c}\int_b^{\infty}(y-b)f(y)\,dy\end{aligned} \tag{7.37}$$

and we can find the boundary condition as $\frac{\partial}{\partial u}R_k(u,b)\Big|_{u=b} = 0$.

Result 7.3. When $f(x) = p\alpha e^{-\alpha x} + q\beta e^{-\beta x}$ for $p + q = 1, 0 < p < 1$ and $\alpha < \beta$,

$$\begin{aligned}R_k(u,b) &= \frac{1}{\alpha\beta L(b)}\Bigg(\Big(R_1 e^{-R_1 b}(\beta B_3(k)-\alpha A_3(k)) + R_2 e^{-R_2 b}(\alpha A_2(k)-\beta B_2(k))\Big)e^{\rho u} \\ &\quad + \Big(\rho e^{\rho b}(\beta B_3(k)-\alpha A_3(k)) + R_2 e^{-R_2 b}(\beta B_1(k)-\alpha A_1(k))\Big)e^{-R_1 u} \\ &\quad + \Big(\rho e^{\rho b}(\alpha A_2(k)-\beta B_2(k)) + R_1 e^{-R_1 b}(\alpha A_1(k)-\beta B_1(k))\Big)e^{-R_2 u}\Bigg)\end{aligned} \tag{7.38}$$

where $L(b)$ is given by (7.19) and $A_i(k)$ and $B_i(k)$ for $i = 1, 2, 3$ are given by (7.20). Further, ρ, R_1 and R_2 are the roots of Lundberg's fundamental equation given by (7.21).

Derivation. Substituting for f, we can write (7.35) as

$$\begin{aligned}\frac{\partial}{\partial u}R_k(u,b) &= \frac{\lambda+\delta}{c}R_k(u,b) - \frac{\lambda}{c}\int_k^u \left(p\alpha e^{-\alpha(u-y)} + q\beta e^{-\beta(u-y)}\right) R_k(y,b)\,dy \\ &\quad -\frac{\lambda}{c}\left(pe^{-\alpha u}(e^k-1) + qe^{-\beta u}(e^{\beta k}-1)\right) R_k(k,b) \\ &\quad -\frac{\lambda}{c}\left(p\frac{e^{-\alpha u}}{\alpha} + q\frac{e^{-\beta u}}{\beta}\right). \end{aligned} \tag{7.39}$$

From this, it is straightforward to obtain

$$\begin{aligned}&\frac{\partial^3}{\partial u^3}R_k(u,b) + \left(\alpha+\beta-\frac{\lambda+\delta}{c}\right)\frac{\partial^2}{\partial u^2}R_k(u,b) \\ &+\left(\alpha\beta + (p\alpha+q\beta)\frac{\lambda}{c} - (\alpha+\beta)\frac{\lambda+\delta}{c}\right)\frac{\partial}{\partial u}R_k(u,b) - \frac{\alpha\beta\delta}{c}R_k(u,b) = 0.\end{aligned} \tag{7.40}$$

Thus, the general solution of equation (7.40) is of the form

$$R_k(u,b) = \zeta_0 e^{\rho u} + \zeta_1 e^{-R_u} + \zeta_2 e^{-R_2 u} \tag{7.41}$$

with the boundary condition from formula (7.37) being

$$\zeta_0 \rho e^{\rho b} - \zeta_1 R_1 e^{-R_1 b} - \zeta_2 R_2 e^{-R_2 b} = 0. \tag{7.42}$$

We now insert the functional form (7.41) of $R_k(u,b)$ into equation (7.39), giving

$$\begin{aligned}&\zeta_0\rho e^{\rho u} - \zeta_1 R_1 e^{-R_1 u} - \zeta_2 R_2 e^{-R_2 u} = \frac{\lambda+\delta}{c}\left(\zeta_0 e^{\rho u} + \zeta_1 e^{-R_1 u} + \zeta_2 e^{-R_2 u}\right) \\ &-\frac{\lambda}{c}\int_k^u \left(p\alpha e^{-\alpha(u-y)} + q\beta e^{-\beta(u-y)}\right)\left(\zeta_0 e^{\rho y} + \zeta_1 e^{-R_1 y} + \zeta_2 e^{-R_2 y}\right)\,dy \\ &-\frac{\lambda}{c}\left(pe^{-\alpha u}(e^{\alpha k}-1) + qe^{-\beta u}(e^{\beta k}-1)\right)\left(\zeta_0 e^{\rho k} + \zeta_1 e^{-R_1 k} + \zeta_2 e^{-R_2 k}\right) \\ &-\frac{\lambda}{c}\left(p\frac{e^{-\alpha u}}{\alpha} + q\frac{e^{-\beta u}}{\beta}\right).\end{aligned} \tag{7.43}$$

Rearranging, we obtain

$$\begin{aligned}&\zeta_0 e^{\rho u}\left(\rho - \frac{\lambda+\delta}{c} + \frac{\alpha\lambda}{c}\frac{p}{\alpha+\rho} + \frac{\beta\lambda}{c}\frac{q}{\beta+\rho}\right) \\ &-\zeta_1 e^{-R_1 u}\left(R_1 + \frac{\lambda+\delta}{c} - \frac{\alpha\lambda}{c}\frac{p}{\alpha-R_1} - \frac{\beta\lambda}{c}\frac{q}{\beta-R_1}\right) \\ &-\zeta_2 e^{-R_2 u}\left(R_2 + \frac{\lambda+\delta}{c} - \frac{\alpha\lambda}{c}\frac{p}{\alpha-R_2} - \frac{\beta\lambda}{c}\frac{q}{\beta-R_2}\right)\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{\lambda}{c}e^{-\alpha u}\Big[\frac{\zeta_0}{\alpha+\rho}p\alpha e^{(\alpha+\rho)k}+\frac{\zeta_1}{\alpha-R_1}p\alpha e^{(\alpha-R_1)k}+\frac{\zeta_2}{\alpha-R_2}p\alpha e^{(\alpha-R_2)k} \\
& -p(e^{\alpha k}-1)\big(\zeta_0 e^{\rho k}+\zeta_1 e^{-R_1 k}+\zeta_2 e^{-R_2 k}\big)-\frac{p}{\alpha}\Big]+\frac{\lambda}{c}e^{-\beta u}\Big[\frac{\zeta_0}{\beta+\rho}q\beta e^{(\beta+\rho)k} \\
& +\frac{\zeta_1}{\beta-R_1}q\beta e^{(\beta-R_1)k}+\frac{\zeta_2}{\beta-R_2}q\beta e^{(\beta-R_2)k}-q(e^{\beta k}-1)\left(\zeta_0 e^{\rho k}+\zeta_1 e^{-R_1 k}+\zeta_2 e^{-R_2 k}\right)-\frac{q}{\beta}\Big].
\end{aligned}
$$

Since the coefficients of $e^{\rho u}, e^{-R_1 u}$ and $e^{-R_2 u}$ are zero, we see that the coefficients of $e^{-\alpha u}$ and $e^{-\beta u}$ are zero, and find the following conditions

$$
\begin{aligned}
\zeta_0 A_1(k)+\zeta_1 A_2(k)+\zeta_2 A_3(k)-\frac{1}{\alpha}&=0, \\
\zeta_0 B_1(k)+\zeta_1 B_2(k)+\zeta_2 B_3(k)-\frac{1}{\beta}&=0. \qquad (7.44)
\end{aligned}
$$

Using (7.42) and (7.44), the result follows. □

In the case of exponential claim amounts, we can use the memoryless property of the exponential distribution and derive $E[e^{-\delta T_{u,b,k}}]$. For the mixed exponential claims we need to find $E[e^{-\delta T_{u,b,k}}]$ directly. Therefore, by the argument of conditioning on the time and the amount of the first claim we find the analogues of formula (7.35) as

$$
\begin{aligned}
\frac{\partial}{\partial u}\tilde{R}_k(u,b) \quad = \quad & \frac{\lambda+\delta}{c}\tilde{R}_k(u,b)-\frac{\lambda}{c}\int_0^{u-k} f(y)\tilde{R}_k(u-y,b)\,dy \\
& -\frac{\lambda}{c}\left(\bar{F}(u-k)-\bar{F}(u)\right)\tilde{R}_k(k,b)-\frac{\lambda}{c}\int_u^{\infty} f(y)\,dy \qquad (7.45)
\end{aligned}
$$

with the boundary condition being $\frac{\partial}{\partial u}\tilde{R}_k(u,b)\Big|_{u=b}=0$. Now, we can insert f in equation (7.45) and follow the same technique as before to obtain the following differential equation as

$$
\begin{aligned}
&\frac{\partial^3}{\partial u^3}\tilde{R}_k(u,b)=\left(\alpha+\beta-\frac{\lambda+\delta}{c}\right)\frac{\partial^2}{\partial u^2}\tilde{R}_k(u,b) \\
&+\left(\alpha\beta+(p\alpha+q\beta)\frac{\lambda}{c}-(\alpha+\beta)\frac{\lambda+\delta}{c}\right)\frac{\partial}{\partial u}\tilde{R}_k(u,b)-\frac{\alpha\beta\delta}{c}\tilde{R}_k(u,b)=0.
\end{aligned}
$$

The general solution to this equation is given by

$$
\tilde{R}_k(u,b)=\varsigma_0 e^{\rho u}+\varsigma_1 e^{-R_1 u}+\varsigma_2 e^{-R_2 u} \qquad (7.46)
$$

and the boundary condition is $\varsigma_0\rho e^{\rho b}-\varsigma_1 R_1 e^{-R_1 b}-\varsigma_2 R_2 e^{-R_2 b}=0$. Inserting the functional form (7.46) of $\tilde{R}_k(u,b)$ into equation (7.45), after simplifications we obtain two other conditions as

$$\begin{aligned}\varsigma_0 A_1(k) + \varsigma_1 A_2(k) + \varsigma_2 A_3(k) &= 1, \\ \varsigma_0 B_1(k) + \varsigma_1 A_2(k) + \varsigma_2 A_3(k) &= 1,\end{aligned}$$

where $A_i(k)$ and $B_i(k)$ for $i = 1, 2, 3$ are given by (7.20). Applying the boundary condition, it follows that

$$\begin{aligned}\tilde{R}_k(u,b) &= \frac{1}{L(b)}\Bigg(\left(R_1 e^{-R_1 b}(B_3(k) - A_3(k)) + R_2 e^{-R_2}(A_2(k) - B_2(k))\right) e^{\rho u} \\ &\quad + \left(\rho e^{\rho b}(B_3(k) - A_3(k)) + R_2 e^{-R_2 b}(B_1(k) - A_1(k))\right) e^{-R_1 u} \\ &\quad + \left(\rho e^{\rho b}(A_2(k) - B_2(k)) + R_1 e^{-R_1 b}(A_1(k) - B_1(k))\right)\Bigg) \qquad (7.47)\end{aligned}$$

where $L(b)$ is given by (7.19) and ρ, R_1 and R_2 are the roots of Lundberg's fundamental equation given by (7.21).

Setting $k = 0$ in formulae (7.18), (7.38) and (7.47) gives expressions for $V_1(u, b)$, $E[e^{-\delta T_{u,b}} Y_{u,b}]$ and $E[e^{-\delta T_{u,b}}]$, respectively, when claim amounts follow a mixed exponential distribution.

We now illustrate the applications of these formulae for claim amounts with exponential and mixed exponential distributions.

7.3.1 Exponential claims

Table 7.1 presents the results for the situation when claim amounts follow an exponential distribution with mean $1, \delta = 0.1, \lambda = 100$.

The key to Tables 7.1 and 7.2 is as follows:

(1) capital is injected when the surplus falls below $k = 1$,

(2) capital is injected when the surplus falls below $k = 2$,

(3) capital is injected when the surplus falls below $k = 3$,

$\tilde{b}$ is the optimal barrier that maximises (7.34), $\tilde{V}_k$ is the expected present value of total dividends, $\tilde{W}_k$ is the expected present value of total payments to be made by the reinsurance company, and N_k is expected present value of net income.

Table 7.1: Exponential claims with $c = 110$

u	θ_R		$\tilde{b}$	$\tilde{V}_k$	$\tilde{W}_k$	N_k
10	0.2	(1)	15.81	129.96	30.68	83.15
		(2)	16.81	129.22	30.91	82.13
		(3)	17.81	128.54	31.20	81.10
15		(1)	15.81	134.34	30.13	83.19
		(2)	16.81	133.39	30.17	82.19
		(3)	17.81	132.47	30.23	81.18
20		(1)	20.00	124.41	18.4103	82.32
		(2)	20.00	126.42	20.6333	81.66
		(3)	20.00	128.73	23.17	80.93
10	0.30	(1)	18.38	120.53	23.0956	80.51
		(2)	19.38	119.85	23.3730	79.46
		(3)	20.38	119.21	23.6988	78.40
15		(1)	18.38	124.59	22.2935	80.61
		(2)	19.38	123.71	22.3874	79.60
		(3)	20.38	122.86	22.5115	78.59
20		(1)	20.00	124.41	18.4103	80.48
		(2)	20.00	126.42	20.6333	79.60
		(3)	20.38	127.44	22.1672	78.62

We can observe the following from Table 7.1.

(i) As long as $u < b$ the optimal value of b is independent of u, otherwise $u = b$.

(ii) When $u < b$, as u increases, the expected present value of dividend payments and the expected present value of net income both increase, but the expected present value of compensations made by the reinsurer decreases. This is compatible with our expectations; since when u increases, it is less likely that the surplus falls below k.

(iii) As k increases, the cost of the reinsurance contract goes up and the optimal value of b increases. Also, provided that $u < b$ the expected present value of dividend payments and the expected present value of net income both decrease.

(iv) As θ_R increases, i.e. the reinsurance policy becomes more expensive, the optimal value of b increases, and the expected present value of dividend payments, compensations made by the reinsurer, and the expected present value of net income all decrease.

In general, the expected present value of net income is always positive, despite the fact that the reinsurance is expensive.

Table 7.2: Mixed exponential claims with $c = 110$

u	θ_R		$\tilde{b}$	$\tilde{V}_k$	$\tilde{W}_k$	N_k
10	0.20	(1)	19.56	139.10	42.1660	78.50
		(2)	20.56	138.46	42.5008	77.46
		(3)	21.56	137.87	42.8817	76.41
15		(1)	19.56	142.90	41.0912	78.59
		(2)	20.56	142.06	41.2342	77.58
		(3)	21.56	141.26	41.4110	76.57
20		(1)	20.00	145.73	39.2840	78.59
		(2)	20.56	146.58	40.8190	77.60
		(3)	21.56	145.61	40.8456	76.60
10	0.30	(1)	23.06	126.96	32.4163	74.82
		(2)	24.06	126.38	32.7956	73.75
		(3)	25.06	125.85	33.2177	72.66
15		(1)	23.06	130.43	31.0750	75.94
		(2)	24.06	129.67	31.2767	74.01
		(3)	25.06	128.94	31.5096	72.98
20		(1)	23.06	134.71	30.4732	75.09
		(2)	24.06	133.80	30.5447	74.09
		(3)	25.06	132.91	30.6391	73.08

7.3.2 Mixed exponential claims

Table 7.2 shows the results for the situation when individual claim amounts follow the density function $f(x) = \frac{1}{3}(\frac{1}{2}e^{-x/2}) + \frac{2}{3}(2e^{-2x})$ with $\lambda = 100, \delta = 0.1$ and $c = 110$.

We can observe the following from Table 7.2.

(i) As long as $u < b$ the optimal value of b is independent of u, otherwise $u = b$.

(ii) As u increases, the expected present value of dividends and the expected present value of net income both increase, but $\tilde{W}_k$ decreases.

(iii) As k increases, the optimal barrier b increases, but the expected present value of dividend payments and the expected present value of net income both decrease, and $\tilde{W}_k$ increases slightly.

(iv) When θ_R increases, the optimal barrier b increases, but the expected present value of dividend payments, the compensation to be made by the reinsurer, and the expected present value of net income all decrease.

(v) In general the results for exponential and mixed exponential claims follow a similar pattern. However, the expected present value of payments to be made by the reinsurer in the case of mixed exponential claims is greater than in the case of exponential claims. Consequently, the expected present value of net income under mixed exponential claims is less than that under exponential claims.

7.4 Threshold strategy

In this section, we consider the situation in which when the surplus exceeds level b only part of the premium is paid to the shareholders. In other words, the insurer pays dividends at rate $\hat{c} < c$ and receives premium at rate $c^* = c - \hat{c}$. See Section 2.2.1. We study the threshold dividend strategy by means of two methods. First we consider the case of exponential claim amounts and derive an integro-differential equation for the expected present value of dividend income to the shareholders of an insurance company which is denoted by $\mathcal{V}_k(u,b)$ and solve this equation by techniques in Gerber and Shiu (2006). We then show how a probabilistic approach, which is based on the idea in Dickson and Drekic (2006), can lead us to the same expression more easily. Finally, we apply this method to claim amounts with a mixed exponential distribution.

When $k < u < b$, no dividend is payable and the premium is received at rate c. Therefore, by considering whether or not a claim occurs before time τ, replacing $V_k(u,b)$ by $\mathcal{V}_k(u,b)$, we can obtain the equivalent equation (7.2) for $\mathcal{V}_k(u,b)$ as before. When $u > b$ we have three situations:

(i) if the amount of the first claim, x, at time $t > 0$ is less than $u + c^*t - k$, then the total value of the dividends at time t is $\hat{c}\bar{s}_{\overline{t}|} + \mathcal{V}_k(u + c^*t - x, b)$.

(ii) if the amount of the first claim, x, at time $t > 0$ is such that $u + c^*t - k \le x < u + c^*t$, then the total value of the dividends at time t is $\hat{c}\bar{s}_{\overline{t}|} + \mathcal{V}_k(k,b)$.

(iii) if the amount of the first claim, x, at time $t > 0$ exceeds $u + c^*t$, then the total value of the dividends at time t is $\hat{c}\bar{s}_{\overline{t}|}$.

Hence

$$\begin{aligned}\mathcal{V}_k(u,b) &= \int_0^\infty \lambda e^{-(\lambda+\delta)t} \int_0^{u+c^*t-k} f(x)\mathcal{V}_k(u + c^*t - x, b)\, dx\, dt \\ &\quad + \int_0^\infty \lambda e^{-(\lambda+\delta)t} \int_{u+c^*t-k}^{u+c^*t} f(x)\mathcal{V}_k(k,b)\, dx\, dt \\ &\quad + \int_0^\infty \lambda e^{-(\lambda+\delta)t}\hat{c}\bar{s}_{\overline{t}|}\, dt.\end{aligned}$$

Setting $u + c^*t = s$ we have

$$\begin{aligned}\mathcal{V}_k(u,b) &= \frac{\hat{c}}{\lambda+\delta} + \frac{\lambda}{c^*}\int_u^\infty e^{-(\lambda+\delta)(s-u)/c^*}\int_0^{s-k} f(x)\mathcal{V}_k(s-x,b)\,dx\,ds \\ &\quad + \frac{\lambda}{c^*}\int_u^\infty e^{-(\lambda+\delta)(s-u)/c^*}\int_{s-k}^{s} f(x)\mathcal{V}_k(k,b)\,dx\,ds.\end{aligned}$$

Differentiating with respect to u we obtain

$$\begin{aligned}\frac{\partial}{\partial u}\mathcal{V}_k(u,b) &= \frac{\lambda+\delta}{c^*}\mathcal{V}_k(u,b) - \frac{\hat{c}}{c^*} - \frac{\lambda}{c^*}\int_0^{u-k} f(x)\mathcal{V}_k(u-x,b)\,dx \\ &\quad - \frac{\lambda}{c^*}\int_{u-k}^{u} f(x)\mathcal{V}_k(k,b)\,dx. \qquad (7.48)\end{aligned}$$

Further, according to Gerber and Shiu (2006), $\frac{\partial}{\partial u}\mathcal{V}_k(u,b)$ is not necessarily continuous at $u = b$. After replacing $V_k(u,b)$ by $\mathcal{V}_k(u,b)$ in (7.2), from the resulting equation and (7.48) we can find the condition

$$c^*\frac{\partial}{\partial u}\mathcal{V}_k(u,b)\Big|_{u=b^+} = c\frac{\partial}{\partial u}\mathcal{V}_k(u,b)\Big|_{u=b^-} - \hat{c} \qquad (7.49)$$

which is reduced to the boundary condition in the barrier dividend strategy if $c^* = c - \hat{c} = 0$.

The next result gives an expression for $\mathcal{V}_k(u,b)$ in the case of claim amounts with an exponential distribution.

Result 7.4. When $F(x) = 1 - e^{-\alpha x}$, $x \geq 0$, with $\alpha > 0$, for $k \leq u < b$ we have

$$\mathcal{V}_k(u,b) = \frac{\hat{c}\hat{R}}{\delta\alpha L(b)}\left(D_1(k)e^{\rho u} - D_2(k)e^{-Ru}\right) \qquad (7.50)$$

where

$$\begin{aligned}L(b) &= D_1(k)e^{\rho b} - D_2(k)e^{-Rb} - (\alpha-\hat{R})e^{-\alpha b}\left(\frac{D_1(k)}{\alpha+\rho}\left(e^{(\alpha+\rho)b} - e^{(\alpha+\rho)k}\right)\right. \\ &\quad \left. - \frac{D_2(k)}{\alpha-R}\left(e^{(\alpha-R)b} - e^{(\alpha-R)k}\right)\right) - \frac{(\alpha-\hat{R})}{\alpha}(e^{\alpha k}-1)e^{-\alpha b}\left(D_1(k)e^{\rho k} - D_2(k)e^{-Rk}\right), \qquad (7.51)\end{aligned}$$

$D_1(k)$ and $D_2(k)$ are given by (7.7), and $-\hat{R} \equiv -\hat{R}(\delta)$ is the negative root of Lundberg's fundamental equation, given by (7.8) with premium $c^* = c - \hat{c}$. Also, $\rho \equiv \rho(\delta)$ and $-R \equiv -R(\delta)$ are the roots of Lundberg's fundamental equation with premium c.

Further, for $u \geq b$, we have

$$\mathcal{V}_k(u,b) = \frac{\hat{c}}{\delta}\left(1 - e^{-\hat{R}(u-b)}\right) + e^{-\hat{R}(u-b)}\mathcal{V}_k(b,b). \qquad (7.52)$$

Derivation. For $k \leq u < b$ we can simply substitute f in equation (7.2) and differentiate with respect to u. The general solution of the resulting equation is given by

$$\mathcal{V}_k(u,b) = \gamma_1 e^{\rho u} + \gamma_2 e^{-Ru}. \tag{7.53}$$

For $u \geq b$ we can write (7.48) as

$$\begin{aligned}\frac{\partial}{\partial u}\mathcal{V}_k(u,b) &= \frac{\lambda+\delta}{c^*}\mathcal{V}_k(u,b) - \frac{\hat{c}}{c^*} - \frac{\lambda}{c^*}\int_k^u \alpha e^{-\alpha(u-x)}\mathcal{V}_k(x,b)\,dx \\ &\quad -\frac{\lambda}{c^*}e^{-\alpha u}\left(e^{\alpha k}-1\right)\mathcal{V}_k(k,b).\end{aligned} \tag{7.54}$$

Differentiation of equation (7.54) yields

$$\frac{\partial^2}{\partial u^2}\mathcal{V}_k(u,b) + \left(\alpha - \frac{\lambda+\delta}{c^*}\right)\frac{\partial}{\partial u}\mathcal{V}_k(u,b) - \frac{\alpha\delta}{c^*}\mathcal{V}_k(u,b) + \frac{\alpha\hat{c}}{c^*} = 0$$

with the solution being given by

$$\mathcal{V}_k(u,b) = \tau_1 e^{\hat{\rho}u} + \tau_2 e^{-\hat{R}u} + C \tag{7.55}$$

where C is the particular solution and $\hat{\rho} \equiv \hat{\rho}(\delta)$ and $-\hat{R} \equiv -\hat{R}(\delta)$ are the roots of Lundberg's fundamental equation with premium c^*. From ideas in Gerber and Shiu (2006), $\lim_{u\to\infty}\mathcal{V}_k(u,b) = \hat{c}/\delta$, which is the present value of a perpetuity with continuous payments at rate $\hat{c}$. This leads to $\tau_1 = 0$ and $C = \hat{c}/\delta$ in (7.55). Thus

$$\mathcal{V}_k(u,b) = \tau_2 e^{-\hat{R}u} + \frac{\hat{c}}{\delta}. \tag{7.56}$$

We can now insert the functional form (7.53) and (7.56) of $\mathcal{V}_k(u,b)$ into equation (7.54), giving

$$\begin{aligned}&-\tau_2\hat{R}e^{-\hat{R}u} \\ =\;& \frac{\lambda+\delta}{c^*}\left(\tau_2 e^{-\hat{R}u} + \frac{\hat{c}}{\delta}\right) - \frac{\hat{c}}{c^*} - \frac{\alpha\lambda}{c^*}e^{-\alpha u}\int_k^b \left(\gamma_1 e^{(\alpha+\rho)x} + \gamma_2 e^{(\alpha-R)x}\right)\,dx \\ &-\frac{\lambda}{c^*}\int_b^u \alpha e^{-\alpha(u-x)}\left(\tau_2 e^{-\hat{R}x} + \frac{\hat{c}}{\delta}\right)\,dx - \frac{\lambda}{c^*}e^{-\alpha u}(e^{\alpha k}-1)\left(\gamma_1 e^{\rho k} + \gamma_2 e^{-Rk}\right).\end{aligned}$$

Rearranging gives

$$\begin{aligned}&-\tau_2 e^{-\hat{R}u}\left(\hat{R} + \frac{\lambda+\delta}{c^*} - \frac{\alpha\lambda}{c}\frac{1}{\alpha-\hat{R}}\right) = e^{-\alpha u}\Big[\frac{\alpha\lambda}{c^*}\frac{\tau_2}{\alpha-\hat{R}}e^{(\alpha-\hat{R})b} + \frac{\lambda}{c^*}\frac{\hat{c}}{\delta}e^{\alpha b} \\ &-\frac{\alpha\lambda}{c^*}\frac{\gamma_1}{\alpha+\rho}\left(e^{(\alpha+\rho)b} - e^{(\alpha+\rho)k}\right) - \frac{\alpha\lambda}{c^*}\frac{\gamma_2}{\alpha-R}\left(e^{(\alpha-R)b} - e^{(\alpha-R)k}\right) \\ &-\frac{\lambda}{c^*}(e^{\alpha k}-1)\left(\gamma_1 e^{\rho k} + \gamma_2 e^{-Rk}\right)\Big].\end{aligned}$$

As the coefficient of $e^{-\hat{R}u}$ is 0 we have

$$\gamma_1\left(\frac{\alpha}{\alpha+\rho}\left(e^{(\alpha+\rho)b}-e^{(\alpha+\rho)k}\right)+(e^{\alpha k}-1)e^{\rho k}\right)$$
$$+\gamma_2\left(\frac{\alpha}{\alpha-R}\left(e^{(\alpha-R)b}-e^{(\alpha-R)k}\right)+(e^{\alpha k}-1)e^{-Rk}\right)=\frac{\hat{c}}{\delta}e^{\alpha b}+\frac{\alpha}{\alpha-\hat{R}}\tau_2 e^{(\alpha-\hat{R})b}. \tag{7.57}$$

From (7.13) we know that $\gamma_1=-\frac{D_1(k)}{D_2(k)}\gamma_2$. Therefore (7.57) can be rewritten as

$$\gamma_2\Big(\frac{\alpha}{\alpha-R}\left(e^{(\alpha-R)b}-e^{(\alpha-R)k}\right)+(e^{\alpha k}-1)e^{-Rk}-\frac{D_1(k)}{D_2(k)}(e^{\alpha k}-1)e^{\rho k}$$
$$-\frac{D_1(k)}{D_2(k)}\frac{\alpha}{\alpha+\rho}\left(e^{(\alpha+\rho)b}-e^{(\alpha+\rho)k}\right)\Big)=\frac{\hat{c}}{\delta}e^{\alpha b}+\frac{\alpha\tau_2}{\alpha-\hat{R}}e^{(\alpha-\hat{R})b}. \tag{7.58}$$

From the continuity condition in (7.49) we require that

$$\mathcal{V}_k(b^-,b)=\mathcal{V}_k(b^+,b) \tag{7.59}$$

where $\mathcal{V}_k(b^-,b)$ from (7.53) is given by

$$\mathcal{V}_k(b^-,b)=\gamma_1 e^{\rho b}+\gamma_2 e^{-Rb}=\gamma_2\left(e^{-Rb}-\frac{D_1(k)}{D_2(k)}e^{\rho b}\right)$$

and $\mathcal{V}_k(b^+,b)$ from (7.56) by

$$\mathcal{V}_k(b^+,b)=\frac{\hat{c}}{\delta}+\tau_2 e^{-\hat{R}b}.$$

Using (7.58) and (7.59), formula (7.50) follows. For $u\geq b$, noting that the right-hand side of (7.59) is $\mathcal{V}_k(b,b)$ from (7.56) we can find

$$\tau_2=e^{\hat{R}b}(\mathcal{V}_k(b,b)-\hat{c}/\delta).$$

Thus

$$\mathcal{V}_k(u,b)=\frac{\hat{c}}{\delta}(1-e^{-\hat{R}(u-b)})+e^{-\hat{R}(u-b)}\mathcal{V}_k(b,b)$$

which is (7.52). □

We remark that setting $k=0$ in formulae (7.50) and (7.52), recovers results in Gerber and Shiu (2006) and Dickson and Drekic (2006) in the classical risk model with dividends.

We can maximise $\mathcal{V}_k(u,b)$ by minimising $L(b)$. Hence the solution to

$$\frac{\partial}{\partial b}L(b) = D_1(k)\rho e^{\rho b} + D_2(k)Re^{-Rb} + (\alpha - \hat{R})e^{-\alpha b}\left[\frac{\alpha D_1(k)}{\alpha+\rho}\left(e^{(\alpha+\rho)b} - e^{(\alpha+\rho)k}\right)\right.$$
$$-\frac{\alpha D_2(k)}{\alpha - R}\left(e^{(\alpha-R)b} - e^{(\alpha-R)k}\right) - D_1(k)e^{(\alpha+\rho)b} + D_2(k)e^{(\alpha-R)b}$$
$$\left.+(e^{\alpha k}-1)\left(D_1(k)e^{\rho k} - D_2(k)e^{-Rk}\right)\right] = 0 \tag{7.60}$$

gives the optimal value of b.

7.5 Threshold strategy: alternative approach

In this section, we adapt the ideas of Dickson and Drekic (2006) to find an expression for $\mathcal{V}_k(u,b)$ under the threshold dividend strategy.

Suppose dividends are payable at rate $\hat{c}$ when the surplus level is above level b, with $c^* = c - \hat{c} > \lambda m_1$, and that if ruin occurs, no further dividends are payable. Let $\hat{T}_u$ be the time of ruin and $\hat{w}(u,y,t)$ be the joint density of the deficit at ruin and the time of ruin for a classical surplus process with initial surplus u and premium rate c^*. We define $T_k(u,b)$ to be the expected present value of a unit payable when $u=b$ for the first time. See Gerber (1979, page 147). For $k \le u \le b$, dividends will be payable only if the surplus process reaches b without ruin first occurring. Thus

$$\mathcal{V}_k(u,b) = T_k(u,b)\mathcal{V}_k(b,b). \tag{7.61}$$

For $u \ge b$, dividends are payable immediately at rate $\hat{c}$ until the first time when the surplus falls below b (an event which may not occur). Conditioning on the time and the amount of the first fall below b we have two situations:

(i) the fall below b is less than $b-k$, so that the surplus process restarts from level $u-b$, or

(ii) the fall below b exceeds $b-k$, but is still less than b and the surplus process restarts from level k.

As the time of the fall below b is identical in distribution to $\hat{T}_{u-b}$, we can write

$$\begin{aligned}
\mathcal{V}_k(u,b) &= \hat{c}E\left[\bar{a}_{\overline{\hat{T}_{u-b}|}}\right] + \int_0^\infty e^{-\delta t} \int_0^{b-k} \hat{w}(u-b,y,t)\mathcal{V}_k(b-y,b)\,dy\,dt \\
&\quad + \int_0^\infty e^{-\delta t} \int_{b-k}^{b} \hat{w}(u-b,y,t)\mathcal{V}_k(k,b)\,dy\,dt \\
&= \frac{\hat{c}}{\delta}\left(1 - E\left[e^{-\delta \hat{T}_{u-b}}\right]\right) + \mathcal{V}_k(b,b)\int_0^\infty e^{-\delta t} \int_0^{b-k} \hat{w}(u-b,y,t)T_k(b-y,b)\,dy\,dt \\
&\quad + T_k(k,b)\mathcal{V}_k(b,b)\int_0^\infty e^{-\delta t} \int_{b-k}^{b} \hat{w}(u-b,y,t)\,dy\,dt. \qquad (7.62)
\end{aligned}$$

For $u = b$, equation (7.62) is written as

$$\begin{aligned}
\mathcal{V}_k(b,b) &= \frac{\hat{c}}{\delta}\left(1 - E\left[e^{-\delta \hat{T}_0}\right]\right) + \mathcal{V}_k(b,b)\int_0^\infty e^{-\delta t} \int_0^{b-k} \hat{w}(0,y,t)T_k(b-y,b)\,dy\,dt \\
&\quad + T_k(k,b)\mathcal{V}_k(b,b)\int_0^\infty e^{-\delta t} \int_{b-k}^{b} \hat{w}(0,y,t)\,dy\,dt
\end{aligned}$$

which gives

$$\mathcal{V}_k(b,b) = \frac{(\hat{c}/\delta)E\left[e^{-\delta \hat{T}_0}\right]}{1 - \int_0^\infty e^{-\delta t}\int_0^{b-k} \hat{w}(0,y,t)T_k(b-y,b)\,dy\,dt - T_k(k,b)\int_0^\infty e^{-\delta t}\int_{b-k}^{b} \hat{w}(0,y,t)\,dy\,dt}.$$

Substituting in (7.61) for $k \leq u \leq b$ we obtain

$$\mathcal{V}_k(u,b) = \frac{(\hat{c}/\delta)E\left[e^{-\delta \hat{T}_0}\right]T_k(u,b)}{1 - \int_0^\infty e^{-\delta t}\int_0^{b-k} \hat{w}(0,y,t)T_k(b-y,b)\,dy\,dt - T_k(k,b)\int_0^\infty e^{-\delta t}\int_{b-k}^{b} \hat{w}(0,y,t)\,dy\,dt}. \qquad (7.63)$$

Next, we derive an expression for $T_k(u,b)$. By considering whether or not a claim occurs before time τ for $k \leq u < b$, we have

$$\begin{aligned}
T_k(u,b) &= e^{-(\lambda+\delta)\tau}T_k(b,b) + \int_0^\tau \lambda e^{-(\lambda+\delta)t} \int_0^{u+ct-k} f(x)T_k(u+ct-x,b)\,dx\,dt \\
&\quad + \int_0^\tau \lambda e^{-(\lambda+\delta)t} \int_{u+ct-k}^{u+ct} f(x)T_k(k,b)\,dx\,dt.
\end{aligned}$$

Setting $s = u + ct$ and differentiating with respect to u we find an integro-differential equation for $T_k(u,b)$ as

$$\begin{aligned}\frac{\partial}{\partial u}T_k(u,b) &= \frac{\lambda+\delta}{c}T_k(u,b) - \frac{\lambda}{c}\int_0^{u-k} f(x)T_k(u-x,b)\,dx \\ &\quad -\frac{\lambda}{c}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)T_k(k,b). \qquad (7.64)\end{aligned}$$

We now introduce an auxiliary function h_k, which is independent of b and satisfies the integro-differential equation

$$\begin{aligned}\frac{\partial}{\partial u}h_k(u) &= \frac{\lambda+\delta}{c}h_k(u) - \frac{\lambda}{c}\int_0^{u-k} f(x)h_k(u-x)\,dx \\ &\quad -\frac{\lambda}{c}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)h_k(k). \qquad (7.65)\end{aligned}$$

Comparing (7.64) and (7.65) we can write $T_k(u,b) = C(b)h_k(u)$. By noting that $T_k(b,b) = 1$ we have

$$T_k(u,b) = \frac{h_k(u)}{h_k(b)}. \qquad (7.66)$$

The next result shows the application of (7.63) and (7.66) in the case of claim amounts with an exponential distribution.

Result 7.5. When $F(x) = 1 - e^{-\alpha x}$, with $\alpha > 0$, for $k \le u \le b$, we have

$$\mathcal{V}_k(u,b) = \frac{\hat{c}\hat{R}}{\alpha\delta L(b)}\big(D_1(k)e^{\rho u} - D_2(k)e^{-Ru}\big) \qquad (7.67)$$

where

$$\begin{aligned}&L(b)\\ =\; &\big(D_1(k)e^{\rho b} - D_2(k)e^{-Rb}\big) - (\alpha-\hat{R})e^{-\alpha b}\left(\frac{D_1(k)}{\alpha+\rho}\left(e^{(\alpha+\rho)b} - e^{(\alpha+\rho)k}\right)\right.\\ &\left.-\frac{D_2(k)}{\alpha-R}\left(e^{(\alpha-R)b} - e^{(\alpha-R)k}\right)\right) - (1-\hat{R}/\alpha)e^{-\alpha b}(e^{\alpha k}-1)(D_1(k)e^{\rho k} - D_2(k)e^{-Rk}),\end{aligned}$$

$D_1(k)$ and $D_2(k)$ are given by (7.7), and $-\hat{R} \equiv -\hat{R}(\delta)$ is the negative root of Lundberg's fundamental equation, given by (7.8) with premium $c^* = c - \hat{c}$. Also, $\rho \equiv \rho(\delta)$ and $-R \equiv -R(\delta)$ are the roots of Lundberg's fundamental equation with premium c.

Further, for $u \ge b$, we have

$$\mathcal{V}_k(u,b) = \frac{\hat{c}}{\delta}(1 - e^{-\hat{R}(u-b)}) + e^{-\hat{R}(u-b)}\mathcal{V}^k(b,b). \qquad (7.68)$$

Derivation. Substituting f in (7.65) we have

$$\frac{\partial}{\partial u}h_k(u) = \frac{\lambda+\delta}{c}h_k(u) - \frac{\lambda}{c}\int_k^u \alpha e^{-\alpha(u-x)}h_k(x)\,dx - \frac{\lambda}{c}e^{-\alpha u}(e^{\alpha k}-1)h_k(k). \quad (7.69)$$

Differentiating with respect to u we get

$$\frac{\partial^2}{\partial u^2}h_k(u) + \left(\alpha - \frac{\lambda+\delta}{c}\right)\frac{\partial}{\partial u}h_k(u) - \frac{\alpha\delta}{c}h_k(u) = 0 \quad (7.70)$$

with the general solution being $h_k(u) = \sigma_1 e^{\rho u} + \sigma_2 e^{-Ru}$. Then, we insert the general solution in equation (7.69) and simplify to get

$$\frac{\sigma_1}{\sigma_2} = \frac{c(\alpha+\rho)e^{(\alpha-R)k} - \lambda(e^{\alpha k}-1)e^{-Rk}}{\lambda(e^{\alpha k}-1)e^{\rho k} - c(\alpha-R)e^{(\alpha+\rho)k}} = -\frac{D_1(k)}{D_2(k)}.$$

It follows that

$$h_k(u) = C\left(D_1(k)e^{\rho u} - D_2(k)e^{-Ru}\right)$$

where C is an arbitrary constant. See Gerber (1979). Thus $T_k(u,b)$ is given by

$$T_k(u,b) = \frac{D_1(k)e^{\rho u} - D_2(k)e^{-Ru}}{D_1(k)e^{\rho b} - D_2(k)e^{-Rb}} = \frac{h_k(u)}{h_k(b)}. \quad (7.71)$$

Hence, for $k \le u \le b$, we get

$$\mathcal{V}_k(u,b) = \frac{(\hat{c}/\delta)\left(1 - E\left[e^{-\delta\hat{T}_0}\right]\right)h_k(u)}{h_k(b) - \int_0^\infty e^{-\delta t}\int_0^{b-k}\hat{w}(0,y,t)h_k(b-y)\,dy\,dt - h_k(k)\int_0^\infty e^{-\delta t}\int_{b-k}^b \hat{w}(0,y,t)\,dy\,dt}. \quad (7.72)$$

Considering first the numerator of (7.72), we have (see, for example, Gerber and Shiu, 1998, or Dickson and Drekic, 2006)

$$E[e^{-\delta\hat{T}_0}] = 1 - \hat{R}/\alpha \quad (7.73)$$

where $-\hat{R} < 0$ is the negative solution of Lundberg's fundamental equation with premium rate c^*. Next, let $L(b)$ denote the denominator of (7.72). Then by noting that

$\hat{w}(u,y,t) = \hat{w}(u,t)\alpha e^{-\alpha y}$ we have

$$\begin{aligned} L(b) &= h_k(b) - \int_0^\infty e^{-\delta t} \int_0^{b-k} \hat{w}(0,y,t) h_k(b-y)\, dy\, dt \\ &\quad -h_k(k) \int_0^\infty e^{-\delta t} \int_{b-k}^{b} \hat{w}(0,y,t)\, dy\, dt \\ &= h_k(b) - \int_0^\infty e^{-\delta t} \hat{w}(0,t) \int_k^b \alpha e^{-\alpha(b-y)} h_k(y)\, dy\, dt \\ &\quad -h_k(k) \int_0^\infty e^{-\delta t} \hat{w}(0,t) \int_{b-k}^{b} \alpha e^{-\alpha y}\, dy\, dt. \end{aligned} \tag{7.74}$$

Now we consider the inner integral of the second term in (7.74):

$$\begin{aligned} &\int_k^b \alpha e^{-\alpha(b-y)} \left(D_1(k)e^{\rho y} - D_2(k)e^{-Ry}\right) dy \\ &= \alpha e^{-\alpha b} \left(\frac{D_1(k)}{\alpha+\rho} \left(e^{(\alpha+\rho)b} - e^{(\alpha+\rho)k}\right) - \frac{D_2(k)}{\alpha - R} \left(e^{(\alpha-R)b} - e^{(\alpha-R)k}\right)\right). \end{aligned}$$

Therefore, $L(b)$ is given by

$$\begin{aligned} &L(b) \\ =\ &\left(D_1(k)e^{\rho b} - D_2(k)e^{-Rb}\right) - (\alpha - \hat{R})e^{-\alpha b}\left(\frac{D_1(k)}{\alpha+\rho} \left(e^{(\alpha+\rho)b} - e^{(\alpha+\rho)k}\right) \right. \\ &\left. -\frac{D_2(k)}{\alpha - R} \left(e^{(\alpha-R)b} - e^{(\alpha-R)k}\right)\right) - (1-\hat{R}/\alpha)e^{-\alpha b}(e^{\alpha k}-1)(D_1(k)e^{\rho k} - D_2(k)e^{-Rk}). \end{aligned}$$

It follows that for $k \le u \le b$,

$$\mathcal{V}_k(u,b) = \frac{\hat{c}\hat{R}}{\alpha \delta L(b)} \left(D_1(k)e^{\rho u} - D_2(k)e^{-Ru}\right) \tag{7.75}$$

which is (7.67). Further, for $u > b$, we insert for $\hat{w}(u,y,t)$ and $T_k(u,b)$ in (7.62), giving

$$\begin{aligned} \mathcal{V}_k(u,b) &= \frac{\hat{c}}{\delta}\left(1 - E\left[e^{-\delta \hat{T}_{u-k}}\right]\right) \\ &\quad +\mathcal{V}_k(b,b)\left(\int_0^\infty e^{-\delta t}\hat{w}(u-b,t) \int_k^b \alpha e^{-\alpha(b-y)} \frac{h_k(y)}{h_k(b)}\, dy\, dt \right. \\ &\quad \left. +\frac{h_k(k)}{h_k(b)} \int_0^\infty e^{-\delta t}\hat{w}(u-b,t) \int_{b-k}^{b} \alpha e^{-\alpha y}\, dy\, dt\right). \end{aligned} \tag{7.76}$$

From Dickson and Drekic (2006) we know

$$E[e^{-\delta \hat{T}_{u-k}}] = \int_0^\infty e^{-\delta t}\hat{w}(u-b,t)\, dt = (1-\hat{R}/\alpha)e^{-\hat{R}(u-b)}. \tag{7.77}$$

Substituting in (7.76) we get

$$\begin{aligned}\mathcal{V}_k(u,b) &= \frac{\hat{c}}{\delta}\left(1-(1-\hat{R}/\alpha)e^{-\hat{R}(u-b)}\right)\\ &\quad+\mathcal{V}_k(b,b)(1-\hat{R}/\alpha)e^{-\hat{R}(u-b)}\left(\int_k^b \alpha e^{-\alpha(b-y)}\frac{h_k(y)}{h_k(b)}\,dy\,dt\right.\\ &\quad\left.+\frac{h_k(k)}{h_k(b)}\int_{b-k}^b \alpha e^{-\alpha y}\,dy\,dt\right). \end{aligned} \tag{7.78}$$

For $u=b$ we can write (7.78) as

$$\mathcal{V}_k(b,b)=\frac{\hat{c}\hat{R}}{\delta\alpha}+\mathcal{V}_k(b,b)(1-\hat{R}/\alpha)\left(\int_k^b \alpha e^{-\alpha(b-y)}\frac{h_k(y)}{h_k(b)}\,dy\,dt+\frac{h_k(k)}{h_k(b)}\int_{b-k}^b \alpha e^{-\alpha y}\,dy\,dt\right).$$

Thus, (7.78) is written as

$$\begin{aligned}\mathcal{V}_k(u,b) &= \frac{\hat{c}}{\delta}\left(1-(1-\hat{R}/\alpha)e^{-\hat{R}(u-b)}\right)+e^{-\hat{R}(u-b)}\left(V_k(b,b)-\frac{\hat{c}\hat{R}}{\delta\alpha}\right)\\ &= \frac{\hat{c}}{\delta}\left(1-e^{-\hat{R}(u-b)}\right)+e^{-\hat{R}(u-b)}\mathcal{V}_k(b,b) \end{aligned} \tag{7.79}$$

which is (7.68). □

We can find the optimal value of b numerically by solving $\frac{\partial}{\partial b}L(b)=0$.

Formulae (7.67) and (7.68) correspond to expressions (7.50) and (7.52) in Result 7.4 where we have solved the integro-differential equation directly. This method can be readily applied to other distributions. For example, in the case of mixed exponential distribution we can apply (7.62) and (7.63) if we know the expressions for $E[e^{-\delta\hat{T}_0}]$, $\hat{w}(u,y,t)$ and $T_k(u,b)$ which can be expressed in terms of h_k. The first two of these terms can be obtained from Dickson and Drekic (2006). For h_k we substitute $f(x)=p\alpha e^{-\alpha x}+q\beta e^{-\beta x}$, where $p>0, q>0$ and $p+q=1$ in (7.65) to get

$$\begin{aligned}\frac{\partial}{\partial u}h_k(u) &= \frac{\lambda+\delta}{c}h_k(u)-\frac{\lambda}{c}\int_k^u\left(p\alpha e^{-\alpha(u-x)}+q\beta e^{-\beta(u-x)}\right)h_k(x)\,dx\\ &\quad-\frac{\lambda}{c}\left(pe^{-\alpha u}(e^{\alpha k}-1)+qe^{-\beta u}(e^{\beta k}-1)\right)h_k(k). \end{aligned} \tag{7.80}$$

Taking the second and third derivatives of h_k and applying $\frac{\partial^3}{\partial u^3}h_k(u)+(\alpha+\beta)\frac{\partial^2}{\partial u^2}h_k(u)+\alpha\beta\frac{\partial}{\partial u}h_k(u)$, after simplifications we obtain

$$\begin{aligned}&\frac{\partial^3}{\partial u^3}h_k(u)+\left(\alpha+\beta-\frac{\lambda+\delta}{c}\right)\frac{\partial^2}{\partial u^2}h_k(u)+\left(\alpha\beta+(p\alpha+q\beta)\frac{\lambda}{c}\right.\\ &\left.-(\alpha+\beta)\frac{\lambda+\delta}{c}\right)\frac{\partial}{\partial u}h_k(u)-\frac{\alpha\beta\delta}{c}=0. \end{aligned} \tag{7.81}$$

Therefore,

$$\frac{h_k(u)}{h_k(b)} = \varrho_0 e^{\rho u} + \varrho_1 e^{-R_1 u} + \varrho_2 e^{-R_2 u} \tag{7.82}$$

with the boundary condition

$$\varrho_0 e^{\rho b} + \varrho_1 e^{-R_1 b} + \varrho_2 e^{-R_2 b} = 1$$

where $\rho > 0, -R_1 < 0$ and $-R_2 < 0$ are the solutions of Lundberg's fundamental equation in (7.21). Inserting the functional form $h_k(u)/h_k(b)$ into equation (7.80) yields

$$\begin{aligned}
&\varrho_0 \rho e^{\rho u} - \varrho_1 R_1 e^{-R_1 u} - \varrho_2 R_2 e^{-R_2 u} = \frac{\lambda + \delta}{c} \left(\varrho_0 e^{\rho u} + \varrho_1 e^{-R_1 u} + \varrho_2 e^{-R_2 u} \right) \\
&-\frac{\lambda}{c} \int_k^u \left(p\alpha e^{-\alpha(u-x)} + q\beta e^{-\beta(u-x)} \right) \left(\varrho_0 e^{\rho x} + \varrho_1 e^{-R_1 x} + \varrho_2 e^{-R_2 x} \right) dx \\
&-\frac{\lambda}{c} \left(p e^{-\alpha u} (e^{\alpha k} - 1) + q e^{-\beta u} (e^{\beta k} - 1) \right) \left(\varrho_0 e^{\rho k} + \varrho_1 e^{-R_1 k} + \varrho_2 e^{-R_2 k} \right).
\end{aligned}$$

We can solve for ϱ_0, ϱ_1 and ϱ_2 using the boundary condition and the following equalities:

$$\begin{aligned}
\varrho_0 A_1(k) + \varrho_1 A_2(k) + \varrho_2 A_3(k) &= 0, \\
\varrho_0 B_1(k) + \varrho_1 B_2(k) + \varrho_2 B_3(k) &= 0,
\end{aligned}$$

where $A_i(k)$ and $B_i(k)$ for $i = 1, 2, 3$ are given by (7.20).

As in Chapter 4 although the probabilistic argument is a more straightforward approach, it can only be applied to distributions which are subject to a particular factorisation.

7.6 Dividends-penalty identity

In Chapter 2 we have reviewed literature on the dividends-penalty identity and explained that such an identity holds under a variety of risk models. In this section, we verify that a dividends-penalty identity holds in our classical risk model with capital injections as defined in Section 7.2.

We now establish a Gerber-Shiu function for our risk model by applying the idea of Lin et al. (2003). Define the Gerber-Shiu function for a surplus process with dividends and capital injections by

$$\phi_{b,k,\delta}(u) = E[e^{-\delta T_{u,b,k}} \omega(U(T^-_{u,b,k}), |U(T_{u,b,k})|) \mid U(0) = u]$$

where $T_{u,b,k}$ is the time of ruin, $\omega(x,y)$ is a penalty function defined for $x \geq k$ and $y > 0$, $U(T^-_{u,b,k})$ is the surplus immediately prior to ruin and $|U(T_{u,b,k})|$ is the deficit at ruin from initial surplus u. We note that as ruin is certain under a risk model with dividends our Gerber-Shiu function does not contain $I(T_{u,b,k} < \infty)$.

Theorem 7.1. The function $\phi_{b,k,\delta}(u)$ satisfies

$$\phi_{b,k,\delta}(u) = \phi_{k,\delta}(u) - \phi'_{k,\delta}(b)V_k(u,b). \tag{7.83}$$

Proof. Conditioning on the time and the amount of the first claim and noting that ruin can occur before time τ (where $u + c\tau = b$) we can write

$$\begin{aligned}
\phi_{b,k,\delta}(u) &= \int_0^{\tau} \lambda e^{-(\lambda+\delta)t} \int_0^{u+ct-k} f(x)\phi_{b,k,\delta}(u+ct-x)\,dx\,dt \\
&+ \int_0^{\tau} \lambda e^{-(\lambda+\delta)t} \int_{u+ct-k}^{u+ct} f(x)\phi_{b,k,\delta}(k)\,dx\,dt \\
&+ \int_0^{\tau} \lambda e^{-(\lambda+\delta)t} \int_{u+ct}^{\infty} f(x)\omega(u+ct, x-u-ct)\,dx\,dt \\
&+ \int_{\tau}^{\infty} \lambda e^{-(\lambda+\delta)t} \int_0^{b-k} f(x)\phi_{b,k,\delta}(b-x)\,dx\,dt \\
&+ \int_{\tau}^{\infty} \lambda e^{-(\lambda+\delta)t} \int_{b-k}^{b} f(x)\phi_{b,k,\delta}(k)\,dx\,dt \\
&+ \int_{\tau}^{\infty} \lambda e^{-(\lambda+\delta)t} \int_b^{\infty} f(x)\omega(b, x-b)\,dx\,dt.
\end{aligned}$$

Setting $s = u + ct$ and differentiating with respect to u we obtain

$$\begin{aligned}
\frac{\partial}{\partial u}\phi_{b,k,\delta}(u) &= \frac{\lambda+\delta}{c}\phi_{b,k,\delta}(u) - \frac{\lambda}{c}\int_0^{u-k} f(x)\phi_{b,k,\delta}(u-x)\,dx \\
&- \frac{\lambda}{c}\Big(\bar{F}(u-k) - \bar{F}(u)\Big)\phi_{b,k,\delta}(k) - \frac{\lambda}{c}\int_u^{\infty} f(x)\omega(u, x-u)\,dx.
\end{aligned} \tag{7.84}$$

For $u = b$ we have

$$\begin{aligned}
\phi_{b,k,\delta}(b) &= \int_0^{\infty} \lambda e^{-(\lambda+\delta)t}\Bigg(\int_0^{b-k} f(x)\phi_{b,k,\delta}(b-x)\,dx + \int_{b-k}^{b} f(x)\phi_{b,k,\delta}(k)\,dx \\
&+ \int_b^{\infty} f(x)\omega(b, x-b)\,dx\Bigg)\,dt \\
&= \frac{\lambda}{\lambda+\delta}\Bigg(\int_0^{b-k} f(x)\phi_{b,k,\delta}(b-x)\,dx + \int_{b-k}^{b} f(x)\phi_{b,k,\delta}(k)\,dx \\
&+ \int_b^{\infty} f(x)\omega(b, x-b)\,dx\Bigg).
\end{aligned}$$

Also, from (7.84) we have

$$\begin{aligned}\frac{\partial}{\partial u}\phi_{b,k,\delta}(u)\Big|_{u=b} &= \frac{\lambda+\delta}{c}\phi_{b,k,\delta}(b)-\frac{\lambda}{c}\int_0^{b-k}f(x)\phi_{b,k,\delta}(b-x)\,dx\\&\quad-\frac{\lambda}{c}\Big(\bar{F}(b-k)-\bar{F}(b)\Big)\phi_{b,k,\delta}(k)-\frac{\lambda}{c}\int_b^{\infty}f(x)\omega(b,x-b)\,dx.\end{aligned}$$

Therefore, we find the boundary condition as

$$\frac{\partial}{\partial u}\phi_{b,k,\delta}(u)\Big|_{u=b}=0. \tag{7.85}$$

Our approach to solving (7.84) is similar to Lin et al. (2003) and Gerber et al. (2006a). In other words, we set up a dividends-penalty identity for our model. For this we write the solution to equation (7.84) as a linear combination of the Gerber-Shiu function when there is no dividend barrier and an auxiliary equation. Thus we write

$$\begin{aligned}\phi_{b,k,\delta}(u) &= \phi_{k,\delta}(u)+\eta h_k(u)\\&= \phi_{k,\delta}(u)-\phi'_{k,\delta}(b)\frac{h_k(u)}{h'_k(b)}\end{aligned} \tag{7.86}$$

where η can be found by applying the boundary condition from (7.85) and $\phi_{k,\delta}(u)$ is the solution to the integro-differential equation for the Gerber-Shiu function under the classical risk model with capital injections which can be obtained by differentiating (3.4) with respect to u, giving

$$\begin{aligned}\frac{\partial}{\partial u}\phi_{k,\delta}(u) &= \frac{\lambda+\delta}{c}\phi_{k,\delta}(u)-\frac{\lambda}{c}\int_0^{u-k}f(x)\phi_{k,\delta}(u-x)\,dx-\frac{\lambda}{c}\Big(\bar{F}(u-k)-\bar{F}(u)\Big)\phi_{k,\delta}(k)\\&\quad-\frac{\lambda}{c}\int_u^{\infty}f(x)\omega(u,x-u)\,dx.\end{aligned} \tag{7.87}$$

Further $h_k(u)$ is the solution to (7.65). If we compare (7.2) and (7.65) we can see that these two expressions are proportional. Applying an idea of Bühlmann (1970) we can write

$$V_k(u,b)=C(b)h_k(u). \tag{7.88}$$

To find $C(b)$ we substitute (7.88) in (7.4), giving

$$\begin{aligned}C(b)h_k(b) &= \frac{c}{\lambda+\delta}+\frac{\lambda}{\lambda+\delta}C(b)\int_0^{b-k}f(x)h_k(b-x)\,dx\\&\quad+\frac{\lambda}{\lambda+\delta}\Big(\bar{F}(b-k)-\bar{F}(b)\Big)C(b)h_k(k).\end{aligned}$$

Rearranging yields

$$\begin{aligned}
C(b) &= \frac{c/(\lambda+\delta)}{h_k(b) - \frac{\lambda}{\lambda+\delta}\int_0^{b-k} f(x)h_k(b-x)\,dx - \frac{\lambda}{\lambda+\delta}\Big(\bar{F}(b-k) - \bar{F}(b)\Big)h_k(k)} \\
&= \frac{1}{h_k'(b)}. \qquad (7.89)
\end{aligned}$$

It follows that

$$V_k(u,b) = \frac{h_k(u)}{h_k'(b)}. \qquad (7.90)$$

Hence (7.86) can be written as (7.83). □

7.7 Concluding remarks

In this chapter, we have considered the classical risk model modified by capital injections. We have found the optimal barrier b according to De Finneti (1957) and observed that in the case of both exponential and mixed exponential claim amounts, the optimal value of b is independent of u, which is a common feature in our model and the classical risk model. We have then considered a reinsurance contract that enables the company to operate indefinitely. Our numerical analysis shows that although such a contract is expensive, the expected present value of net income to the shareholders is positive. Also, we have found the optimal dividend barrier for our model under the threshold strategy by using probabilistic arguments and direct solution of an inhomogeneous integro-differential equation. Finally, we have verified that a dividends-penalty identity holds for our model.

Chapter 8

Conclusion

We have considered one way that capital injections can be incorporated into risk models such as the classical risk model, the Markov-modulated risk model and the classical risk model with dividends.

Our analysis under the classical risk model with capital injections was based on the well-known Gerber-Shiu function. We have found different ruin-related quantities including the ultimate ruin probability, the (defective) joint distribution of the surplus immediately prior to ruin and the deficit at ruin, the joint (defective) density of the time of ruin and the number of claims until ruin, and the covariance between the time of ruin and the number of claims until ruin. We have shown that although our Gerber-Shiu function is a useful tool to study ruin-related quantities in finite time, it is not an efficient way to derive such quantities in infinite time. We have obtained recursive and explicit expressions in the case of claim amounts following exponential and Erlang(2) distributions and pointed out that for other claim amount distributions either explicit expressions do not exist or they are difficult to obtain. To address this issue we proposed an approximation method which is based on the discretisation of the classical risk process and created a numerical algorithm that can approximate the probability of ruin in infinite and finite time under the classical risk model with capital injections in the case of claim amounts following heavy-tailed distributions.

As with the classical risk model, it is difficult to obtain computationally tractable expressions for ruin-related quantities under the Markov-modulated risk model. We have tackled this issue by developing a discrete time risk model that can provide approximation to the continuous time Markov-modulated model. Our numerical algorithms approximate the probability of ruin and the probability and severity of ruin function for a two-state model and the (defective) density of the time of ruin for an m-state model. Also, we have briefly discussed how the approximation to the cumulative distribution function of the time of ruin can give an approximation to the ultimate ruin probability in an m-state model when $m > 2$. We have also shown how we can modify our algorithms to approximate the density of the time of ruin in an m-state Markov-modulated risk model with capital injections.

Insurance companies distribute parts of their surplus under different strategies among their shareholders, but what happens when they are certain that their surplus will not go below a certain level? We have considered the advantage of this situation to the shareholders by maximising the expected present value of net income allowing for capital injections provided by a reinsurance arrangement. We have observed that under our assumptions, the expected present value of net income to the shareholders is always positive, and although the cost of such a reinsurance contract can be high, dividends may be payable to the shareholders indefinitely. We have also considered the problem of the optimal dividend level under barrier and threshold strategies in our risk model and verified that the dividends-penalty identity holds for a risk model with dividends and capital injections.

There is scope for study of a number of questions based on this thesis. For example: (1) how some of the results of this research would change if we allow for capital injections in the form of co-insurance within an insurance company through a simulation study? (2) in the context of ruin probability, can we find optimal risk-sharing arrangements for a two (or more) risk processes? (3) can we extend the results of this study to dependent claim amounts involving a copula function? (4) is there an alternative way to approximate the continuous time Markov-modulated risk model?

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Appendix

In this section, our purpose is to show conditions under which $\sum_{x=0}^{\infty} \psi_i^d(x)$ exists for the discrete time Markov-modulated model of Chapter 6 when $m = 2$.

We first consider the situation that the net premium condition holds in both states 1 and 2 – see Section 2.1.2. So, we are assuming $E[Y_1|J(1) = 1] < 1$ and $E[Y_1|J(1) = 2] < 1$. Noting that under this condition the adjustment coefficient exists in both states, we can apply the method in Cossette et al. (2004a) to show that $\psi_i^d(u) \leq e^{-R^* u}$ where $R^* = \min(R_1, R_2)$ and R_1 and R_2 are the adjustment coefficients in states 1 and 2, respectively. Therefore, in this case, $\sum_{x=0}^{\infty} \psi_i(x) < \infty$.

Now we consider the situation that the adjustment coefficient does not exist in both states.

Let $\{{}^dL_n\}_{n=0}^{\infty}$ be the aggregate loss process in our Markov-modulated model, and let ${}^dL_i$ be a discrete random variable representing the maximum aggregate loss. Denote the distribution function of ${}^dL_i$ by $\delta_i^d(u)$, with $\psi_i^d(u) = 1 - \delta_i^d(u)$. Then, the first moment of the maximum aggregate loss is given by

$$E[{}^dL_i] = \sum_{u=0}^{\infty} \psi_i^d(u) = \sum_{u=0}^{\infty} \Pr({}^dL_i > u).$$

Therefore, we can conclude that $\sum_{u=0}^{\infty} \psi_i^d(u) < \infty$, if $E[{}^dL_i]$ exists. The next result gives an expression for $E[{}^dL_i]$ and is motivated by the ideas of Dickson and Waters (1992, Section 3).

Theorem. For $i = 1, 2$, the first moment of the maximum aggregate loss is given by

$$E[{}^dL_i] = -J_i^{'}(1)$$

where

$$J_1(s) = \frac{1 + (1 - s^{-1})\big(\psi_1^d(0)\tilde{g}_{11}(s) + \psi_2^d(0)\tilde{g}_{12}(s)\big) - \tilde{g}_1(s) + s^{-1}\tilde{g}_{12}(s)J_2(s)}{1 - s^{-1}\tilde{g}_{11}(s)} \tag{A.1}$$

and

$$J_2(s) \quad = \quad \frac{1+(1-s^{-1})\big(\psi_1^d(0)\tilde{g}_{21}(s)+\psi_2^d(0)\tilde{g}_{22}(s)\big)-\tilde{g}_2(s)+s^{-1}\tilde{g}_{21}(s)J_1(s)}{1-s^{-1}\tilde{g}_{22}(s)}. \tag{A.2}$$

Proof. Our starting point is equation (6.26). We can rewrite it as

$$\psi_i^d(u)=\sum_{j=1}^{2} g_{ij}(0)\psi_j^d(u+1)+\sum_{j=1}^{2}\sum_{x=1}^{u} g_{ij}(x)\psi_j^d(u+1-x)+\sum_{j=1}^{2}\sum_{x=u+1}^{\infty} g_{ij}(x).$$

We define

$$\begin{aligned} d_i(0) &= \psi_i^d(0) \\ &= \sum_{j=1}^{2} g_{ij}(0)\psi_j^d(1)+\sum_{j=1}^{2}\sum_{x=1}^{u} g_{ij}(x)\psi_j^d(1-x)+\sum_{j=1}^{2}\sum_{x=u+1}^{\infty} g_{ij}(x), \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} d_i(u) &= \psi_i^d(u)-\psi_i^d(u-1) \\ &= \sum_{j=1}^{2} g_{ij}(0)\psi_j^d(u+1)+\sum_{j=1}^{2}\sum_{x=1}^{u} g_{ij}(x)\psi_j^d(u+1-x)+1-G_i(u) \\ &\quad -\sum_{j=1}^{2} g_{ij}(0)\psi_j^d(u)-\sum_{j=1}^{2}\sum_{x=1}^{u-1} g_{ij}(x)\psi_j^d(u-x)-1+G_i(u-1). \end{aligned} \tag{A.4}$$

Then, by noting that $\psi_i^d(1)=d_i(1)+d_i(0)$, equation (A.4) can be written as

$$d_i(u)=\sum_{j=1}^{2} g_{ij}(0)d_j(u+1)+\sum_{j=1}^{2} g_{ij}(u)d_j(0)+\sum_{j=1}^{2}\sum_{x=1}^{u} g_{ij}(x)d_j(u+1-x)-g_i(u). \tag{A.5}$$

Further, we define $J_i(s)=d_i(0)+\sum_{n=1}^{\infty} s^n d_i(n)$. Then, by (A.3) and with the usual convention that $\sum_{j=a}^{b}=0$ when $b>a$, we have

$$\begin{aligned} J_i(s) &= \sum_{j=1}^{2} g_{ij}(0)\big(d_j(1)+d_j(0)\big)+1-g_i(0)+\sum_{j=1}^{2}\sum_{n=1}^{\infty} s^n\Bigg(g_{ij}(0)d_j(n+1) \\ &\quad +g_{ij}(n)d_j(0)-g_i(n)+\sum_{x=1}^{n} g_{ij}(x)d_j(n+1-x)\Bigg) \end{aligned}$$

$$= \sum_{j=1}^{2} g_{ij}(0)\big(d_j(1)+d_j(0)\big)+1-g_i(0)+\sum_{j=1}^{2}\big(\tilde{g}_{ij}(s)-g_{ij}(0)\big)d_j(0)$$
$$+s^{-1}\sum_{j=1}^{2} g_{ij}(0)\big(J_j(s)-sd_j(1)-d_j(0)\big)-(\tilde{g}_i(s)-g_i(0))$$
$$+s^{-1}\sum_{j=1}^{2}\big(\tilde{g}_{ij}(s)-g_{ij}(0)\big)\big(J_j(s)-d_j(0)\big). \quad \text{(A.6)}$$

Rearranging (A.6), formulae (A.1) and (A.2) follow. □

Our aim is to find an expression for $E[{}^dL_i]$. For this we note that

$$J_i^{'}(s)=\sum_{x=1}^{\infty} xs^{x-1}\big(\psi_i^d(x)-\psi_i^d(x-1)\big),$$

and therefore $J_i^{'}(1)=-\sum_{x=0}^{\infty}\psi_i^d(x)=-E[{}^dL_i]$. Taking the derivatives of (A.1) and (A.2), and setting $s=1$, and noting that $\tilde{g}_{ij}^{'}(1)=\mu_{ij}$, gives us the following system of equations

$$\begin{cases} p_{12}J_1^{'}(1)=p_{11}d_1(0)+p_{12}d_2(0)-\mu_1+p_{12}J_2^{'}(1),\\ p_{21}J_2^{'}(1)=p_{21}d_1(0)+p_{22}d_2(0)-\mu_2+p_{21}J_1^{'}(1). \end{cases} \quad \text{(A.7)}$$

In the next step we multiply the first equation of (A.7) by p_{21} and the second one by p_{12} and add the resulting equations together so that $J_1^{'}(s)$ and $J_2^{'}(s)$ are eliminated. Hence

$$p_{21}d_1(0)+p_{12}d_2(0)=p_{12}\mu_2+p_{21}\mu_1,$$

which is the same as equation (6.2).

Taking the second derivative of (A.1) and (A.2), and noting that $\tilde{g}_{ij}^{''}(1)=(\mu_2)_{ij}-\mu_{ij}$, yields

$$\begin{cases} p_{12}^2J_1^{''}(1)=2d_1(0)(\mu_{11}-p_{11})+2p_{12}d_2(0)(\mu_1-1)+2p_{12}(\mu_1-1)J_2^{'}(1)\\ +p_{12}^2J_2^{''}(1)+\mu_1(1+p_{11}-2\mu_{11})-(\mu_2)_1\,p_{12},\\ \\ p_{21}^2J_2^{''}(1)=2p_{21}d_1(0)(\mu_2-1)+2d_2(0)(\mu_{22}-p_{22})+2p_{21}(\mu_2-1)J_1^{'}(1)\\ +p_{21}^2J_1^{''}(1)+\mu_2(1+p_{22}-2\mu_{22})-(\mu_2)_2\,p_{21}. \end{cases} \quad \text{(A.8)}$$

Multiplying the first equation in (A.8) by p_{21}^2 and the second one by p_{12}^2, then adding the resulting equations gives rise to the elimination of $J_1^{''}(s)$ and $J_2^{''}(s)$ and we get a relationship between $J_1^{'}(1)$ and $J_2^{'}(1)$ as

$$\begin{aligned} & 2p_{21}p_{12}^2(1-\mu_2)J_1^{'}(1) + 2d_1(0)\big[p_{21}p_{12}^2(1-\mu_2) - p_{21}^2(\mu_{11}-p_{11})\big] \\ & +2p_{12}p_{21}^2(1-\mu_1)J_2^{'}(1) + 2d_2(0)\big[p_{12}p_{21}^2(1-\mu_1) - p_{12}^2(\mu_{22}-p_{22})\big] \\ = \; & \mu_2 p_{12}^2\big[1+p_{22}-2\mu_{22}\big] + \mu_1 p_{21}^2\big[1+p_{11}-2\mu_{11}\big] - (\mu_2)_1\, p_{12}p_{21}^2 - (\mu_2)_2\, p_{21}p_{12}^2 \end{aligned}$$

that together with one of the equations in (A.7) gives $E[^dL_1]$ and $E[^dL_2]$ as follows

$$\begin{aligned} E[^dL_1] \; = \; & \frac{-1}{2p_{21}p_{12}^2(\mu_2-1) + 2p_{12}p_{21}^2(\mu_1-1)}\Big\{2\psi_1^d(0)\big[p_{21}^2(p_{11}-\mu_{11}) \\ & +p_{12}^2p_{21}(1-\mu_2) + p_{11}p_{21}^2(\mu_1-1)\big] + 2\psi_2^d(0)\big[p_{12}p_{21}^2(1-\mu_1) \\ & +p_{12}^2(p_{22}-\mu_{22}) + p_{12}p_{21}^2(\mu_1-1)\big] + \mu_1 p_{21}^2(1+p_{11}-2\mu_{11}) \\ & +\mu_2 p_{12}^2(1+p_{22}-2\mu_{22}) - p_{12}p_{21}((\mu_2)_1\, p_{21} + (\mu_2)_2\, p_{12}) - 2\mu_1 p_{21}^2(\mu_1-1)\Big\}, \end{aligned} \tag{A.9}$$

and

$$\begin{aligned} E[^dL_2] \; = \; & \frac{-1}{2p_{21}p_{12}^2(\mu_2-1) + 2p_{12}p_{21}^2(\mu_1-1)}\Big\{2\psi_1^d(0)\big[p_{21}^2(p_{11}-\mu_{11}) \\ & +p_{12}^2p_{21}(1-\mu_2) - p_{11}p_{21}p_{12}(\mu_2-1)\big] + 2\psi_2^d(0)\big[p_{12}p_{21}^2(1-\mu_1) \\ & +p_{12}^2(p_{22}-\mu_{22}) - p_{21}p_{12}^2(\mu_2-1)\big] - \mu_1 p_{21}^2(1+p_{11}-2\mu_{11}) \\ & -\mu_2 p_{12}^2(1+p_{22}-2\mu_{22}) + p_{12}p_{21}((\mu_2)_1\, p_{21} + (\mu_2)_2\, p_{12}) + 2\mu_1 p_{21}p_{12}(\mu_2-1)\Big\}. \end{aligned} \tag{A.10}$$

We recall that $(\mu_n)_i$ represents the nth moment of the aggregate claim amounts given initial state i which is given by

$$(\mu_n)_i = \sum_{j=1}^{2}(\mu_n)_{ij} = \sum_{j=1}^{2} p_{ij}(\mu_n)_j.$$

From (A.9) and (A.10) we can conclude that $E[^dL_1]$ and $E[^dL_2]$ exist on the condition that (i) denominators in (A.9) and (A.10) are not zero and (ii) moments, i.e. μ_i for $i = 1, 2$ exist.