FROM JACK POLYNOMIALS TO MINIMAL MODEL SPECTRA

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ABSTRACT. In this note, a deep connection between free field realisations of conformal field theories and symmetric polynomials is presented. We give a brief introduction into the necessary prerequisites of both free field realisations and symmetric polynomials, in particular Jack symmetric polynomials. Then we combine these two fields to classify the irreducible representations of the minimal model vertex operator algebras as an illuminating example of the power of these methods. While these results on the representation theory of the minimal models are all known, this note exploits the full power of Jack polynomials to present significant simplifications of the original proofs in the literature.

1. INTRODUCTION

Free field theories have long been of great interest to conformal field theory. Not only are they elegant tractable conformal field theories in their own right, but they are also a versatile tool for realising more complicated conformal field theories and making them tractable. The purpose of this note is to present, in a simple and familiar setting, a deep connection between free field theories and Jack symmetric polynomials. The symmetric polynomial methods will then be applied to the well known free field realisations of the Virasoro minimal models. However, it is important to stress that these methods work far more generally. We have simply chosen to discuss applications to the minimal models for pedagogical purposes.

A different example, where Jack symmetric polynomials have recently garnered a lot of attention, is the much celebrated AGT conjecture [1], which relates conformal field theories to the instanton calculus of Yang-Mills theories. The appearance of symmetric polynomials is due to the conformal field theories in question being resolved by Coulomb gas free field theories, just as the conformal field theories in this note are. Contrary to what was initially believed, it does not seem that Jack symmetric polynomials form the most natural basis for understanding the AGT conjecture. Rather, a generalisation of Jack symmetric polynomials seems to be needed [2].

The two main results discussed in this note are Theorems 5 and 6. Theorem 5, which is originally due to Mimachi and Yamada [3], gives elegant formulae for Virasoro singular vectors in Fock modules, while Theorem 6, which is originally due to Wang [4], determines the conformal highest weights of the irreducible representations of the minimal model vertex operator algebras (also called chiral algebras). The original proofs, impressive though they are, are rather complicated and this note gives novel, drastically shortened and streamlined proofs by using symmetric polynomials, their inner products and the specialisation map. These methods also have the advantage of being applicable in far greater generality, as is evidenced by the fact that they were developed in [5] while classifying the irreducible representations of certain logarithmic extensions of the minimal models. This formalism also generalises to the more involved case of admissible level $\widehat{\mathfrak{sl}}(2)$ theories [6].

We equate the minimal models with the simple vertex operator algebras obtained by taking the quotient of the universal Virasoro vertex operator algebras by their maximal ideals at special values of the central charge.[1] The representation theory of the minimal models can thus be obtained from that of the universal Virasoro vertex operator algebras. Minimal model representations are just the universal Virasoro vertex operator algebra representations that are annihilated by the maximal ideal. This elegant approach to classifying the representation theory of the minimal models seems to have first been considered by Feigin, Nakanishi and Ooguri [7] who applied it to a subset of the minimal models, because, in general, having full computational control over the maximal ideal is a very hard problem. However, free field realisations and symmetric polynomials are exactly the tools one needs to solve this problem and Theorem 6 extends the methods of annihilating ideals [7] to all minimal models.

[1] Universal means that we assume no relations on the defining field $T(z)$ other than those required by the axioms of vertex operator algebras.

This note is organised as follows. Section 2 gives an overview of the free boson, the simplest example of a free field theory, as well as vertex operators and screening operators. Section 3 introduces symmetric polynomials and, in particular, gives an overview of a one-parameter family of bases called the Jack symmetric polynomials. The properties of these Jack polynomials are what yield such explicit computational control that the representation theory of the minimal model representations can be classified. Section 3 ends with explicit formulae for singular vectors in terms of Jack polynomials. These formulae are originally due to Mimachi and Yamada [3], though we give the new, much simpler proof of [5]. In Section 4, the material of Sections 2 and 3 is combined to classify the highest weights of the irreducible minimal model representations, in a similar manner to the methods of Feigin, Nakanishi and Ooguri [7]. This is then used to prove the complete reducibility of the representation theory without recourse to the (perhaps less familiar) methods of Zhu [8].

Acknowledgements. The second author would like to thank Akihiro Tsuchiya for introducing him to the fascinating topics of free field realisations and symmetric polynomials. The second author would also like to thank James Lepowsky and Siddhartha Sahi for interesting discussions and Johannes Schmude for his extraordinary efforts in literature searching. Both authors thank Pierre Mathieu for interesting discussions and his strong encouragement to write this note. The first author's research is supported by the Australian Research Council Discovery Project DP1093910. The second author's research is supported by the Australian Research Council Discovery Early Career Researcher Award DE140101825.

2. The free boson

The free boson chiral algebra or Heisenberg vertex algebra is generated by a single field $a(z)$ which satisfies the operator product expansion

$$a(z)a(w) \sim \frac{1}{(z-w)^2}. \tag{2.1}$$

The Fourier expansion of the field $a(z)$ is

$$a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}, \tag{2.2}$$

thus the operator product expansion implies the following commutations relations:

$$[a_m, a_n] = m\delta_{m,-n}\mathbf{1}. \tag{2.3}$$

The Heisenberg Lie algebra $\mathfrak{H}$ is the infinite dimensional Lie algebra generated by the a_n and the central element $\mathbf{1}$. We identify the element $\mathbf{1}$ with the unit of the universal enveloping algebra $U(\mathfrak{H})$ of $\mathfrak{H}$ and assume that $\mathbf{1}$ acts as the identity on any $\mathfrak{H}$ representation.[2] The Heisenberg Lie algebra admits a triangular decomposition

$$\begin{aligned} \mathfrak{H} &= \mathfrak{H}_- \oplus \mathfrak{H}_0 \oplus \mathfrak{H}_+, & \mathfrak{H}_\pm &= \bigoplus_{n\geq 1} \mathbb{C}a_{\pm n}, \\ \mathfrak{H}_0 &= \mathbb{C}a_0 \oplus \mathbb{C}\mathbf{1}, & \mathfrak{H}_\geq &= \mathfrak{H}_0 \oplus \mathfrak{H}_+. \end{aligned} \tag{2.4}$$

The Verma modules $\mathcal{F}_\lambda$, $\lambda \in \mathbb{C}$, with respect to this decomposition are called Fock modules. They are generated by a highest weight vector $|\lambda\rangle$ on which $\mathfrak{H}_\geq$ acts by

$$a_n|\lambda\rangle = \lambda\delta_{n,0}|\lambda\rangle, \quad n \geq 0. \tag{2.5}$$

[Throughout this note, "kets" $|\lambda\rangle$ will be reserved for the highest weight vectors of Fock modules and λ will denote the highest weight.] The $\mathcal{F}_\lambda$ are then induced from $|\lambda\rangle$ by

$$\mathcal{F}_\lambda = U(\mathfrak{H}) \otimes_{U(\mathfrak{H}_\geq)} \mathbb{C}|\lambda\rangle. \tag{2.6}$$

[2] This is only a minor restriction, since a simple rescaling of the generators a_n allows one to have the central element act as multiplication by any non-zero number.

The parameter λ is called the *Heisenberg weight*. As is well known, the $\mathcal{F}_\lambda$ are all irreducible. As a vector space,

$$\mathcal{F}_\lambda \cong U(\mathfrak{H}_-) = \mathbb{C}[a_{-1}, a_{-2}, \dots]. \tag{2.7}$$

As a representation over itself, the Heisenberg vertex algebra is identified with $\mathcal{F}_0$ and the operator state correspondence is given by

$$\left|0\right\rangle \longleftrightarrow \mathbf{1}, \qquad a_{-1}\left|0\right\rangle \longleftrightarrow a(z), \qquad a_{-n_1-1}\cdots a_{-n_i-1}\left|0\right\rangle \longleftrightarrow \; : \frac{\partial^{n_1}}{n_1!}a(z)\cdots\frac{\partial^{n_i}}{n_i!}a(z):. \tag{2.8}$$

The Heisenberg vertex algebra can be endowed with the structure of a vertex operator algebra by choosing an energy-momentum tensor. This choice is not unique; there is a one parameter family of choices:

$$T(z) = \frac{1}{2} : \alpha(z)^2 : + \frac{\alpha_0}{2}\partial a(z), \quad \alpha_0 \in \mathbb{C}. \tag{2.9}$$

The parameter α_0 determines the central charge of the energy momentum tensor:

$$c = 1 - 3\alpha_0^2. \tag{2.10}$$

The coefficients of the Fourier expansion of the energy momentum tensor are, by definition, the generators L_n of the Virasoro algebra. Formula (2.9) identifies the Virasoro generators with infinite sums of elements of the universal enveloping algebra $U(\mathfrak{H})$ of the Heisenberg Lie algebra:

$$\begin{aligned} T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \frac{1}{2}\sum_{n,m\in\mathbb{Z}} : a_m a_{n-m} : z^{-n-2} - \sum_{n\in\mathbb{Z}} \frac{\alpha_0}{2}(n+1)a_n z^{-n-2}, \\ L_n &= \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_m a_{n-m} : - \frac{\alpha_0}{2}(n+1)a_n. \end{aligned} \tag{2.11}$$

This identification gives an action of the Virasoro algebra on the Fock modules $\mathcal{F}_\lambda$. The Fock modules thus become Virasoro highest weight representations, that is,

$$L_n\left|\lambda\right\rangle = h_\lambda \delta_{n,0}\left|\lambda\right\rangle, \quad n \geq 0, \qquad h_\lambda = \frac{1}{2}\lambda(\lambda - \alpha_0). \tag{2.12}$$

Though the Fock modules are irreducible as Heisenberg representations, they need not be so as Virasoro representations.

The Heisenberg weights λ for which the conformal weight h_λ is 1 play a special role as we shall see below. These weights are roots of the degree 2 polynomial $h_\lambda - 1$ and we denote them by α_+, α_-. They satisfy the relations

$$\alpha_\pm = \frac{\alpha_0 \pm \sqrt{\alpha_0^2 + 8}}{2}, \qquad \alpha_+ + \alpha_- = \alpha_0, \qquad \alpha_+\alpha_- = -2. \tag{2.13}$$

Theorem 1 (Feigin-Fuchs [9])**.** *Let*

$$\alpha_{r,s} = \frac{1-r}{2}\alpha_+ + \frac{1-s}{2}\alpha_-, \quad r,s \in \mathbb{Z}. \tag{2.14}$$

(1) For $\alpha_+^2 \in \mathbb{C}^*$ (or equivalently for $\alpha_-^2 \in \mathbb{C}^*$), the Fock module $\mathcal{F}_\lambda$ is reducible as a Virasoro representation if $\lambda = \alpha_{r,s}$ for some $r,s \in \mathbb{Z}$, $rs > 0$.

(2) If α_+^2 is non-rational (or equivalently if α_-^2 is non-rational), then the Fock module $\mathcal{F}_\lambda$ is reducible as a Virasoro representation if and only if $\lambda = \alpha_{r,s}$ for some $r,s \in \mathbb{Z}$, $rs > 0$.

(3) If α_+^2 is positive rational (or equivalently if α_-^2 is positive rational), then the Fock module $\mathcal{F}_\lambda$ is reducible as a Virasoro representation if and only if $\lambda = \alpha_{r,s}$ for some $r,s \in \mathbb{Z}$.

We omit the corresponding result for negative rational $\alpha_\pm^2$ as the application to the minimal models does not require it. We remark that Feigin and Fuchs also determined the precise structure of Fock modules as Virasoro representations in [9]. For a comprehensive account of Virasoro representation theory, we recommend the book by Iohara and Koga [10].

The work of Feigin and Fuchs shows that one can realise the universal Virasoro vertex operator algebra at arbitrary central charge c as a vertex operator subalgebra of the Heisenberg vertex operator algebra. This free field realisation is called the *Coulomb gas* in the physics literature.

The Fock modules $\mathcal{F}_\lambda$ with $\lambda \neq 0$ can be given a "generalised" vertex algebra structure, that is, an operator state correspondence can also be defined for the states of $\mathcal{F}_\lambda$, though the operator product expansions of these fields are generally not local. The operators corresponding to the generating states $|\lambda\rangle \in \mathcal{F}_\lambda$ are called vertex operators in the physics literature. These should not be confused with the fields (called chiral fields) of the vertex operator algebra.

Before we can define vertex operators, we need to introduce an operator $\hat{a}$ whose commutation relations with the Heisenberg algebra are

$$[a_n, \hat{a}] = \delta_{n,0}\mathbf{1}, \qquad [\mathbf{1}, \hat{a}] = 0. \tag{2.15}$$

The exponential of $\hat{a}$ shifts weights, that is, for $\lambda, \mu \in \mathbb{C}$,

$$a_0 e^{\mu\hat{a}}|\lambda\rangle = \mu e^{\mu\hat{a}}|\lambda\rangle + e^{\mu\hat{a}}a_0|\lambda\rangle = (\mu+\lambda)e^{\mu\hat{a}}|\lambda\rangle. \tag{2.16}$$

We identify $e^{\mu\hat{a}}|\lambda\rangle = |\mu+\lambda\rangle$. Note that $e^{\mu\hat{a}}$ does not define a homomorphism of $\mathfrak{H}$ representations, since it does not commute with $\mathfrak{H}$.

The vertex operator $V_\lambda(z)$ corresponding to the state $|\lambda\rangle$ is

$$V_\lambda(z) = e^{\lambda\hat{a}} z^{\lambda a_0} \prod_{m\geq 1} \exp\left(\lambda \frac{a_{-m}}{m} z^m\right) \prod_{m\geq 1} \exp\left(-\lambda \frac{a_m}{m} z^{-m}\right). \tag{2.17}$$

The vertex operators are therefore linear maps

$$V_\lambda(z) : \mathcal{F}_\mu \to \mathcal{F}_{\mu+\lambda}[[z,z^{-1}]]z^{\lambda\mu}. \tag{2.18}$$

The $V_\lambda(z)$ are often defined as the "normally ordered exponentials" of a field

$$\phi(z) = \hat{a} + a_o \log z - \sum_{n\neq 0} \frac{a_n}{n} z^{-n}, \quad V_\lambda(z) =: e^{\lambda\phi(z)} :. \tag{2.19}$$

Clearly $\partial\phi(z) = a(z)$, which in turn implies that the operator product expansions of ϕ with itself and with a are

$$a(z)\phi(w) \sim \frac{1}{z-w}, \qquad \phi(z)\phi(w) \sim \log(z-w). \tag{2.20}$$

Using these operator product expansions, one can verify that the vertex operators V_λ are conformal primaries of conformal weight h_λ, that is, that

$$T(z)V_\lambda(w) \sim \frac{h_\lambda}{(z-w)^2}V_\lambda(w) + \frac{1}{z-w}\partial V_\lambda(w). \tag{2.21}$$

In particular, for $h_{\alpha_\pm} = 1$,

$$T(z)V_{\alpha_\pm}(w) \sim \frac{1}{(z-w)^2}V_{\alpha_\pm}(w) + \frac{1}{z-w}\partial V_{\alpha_\pm}(w) = \partial_w \frac{V_{\alpha_\pm}(w)}{z-w}, \tag{2.22}$$

that is, the singular terms of these operator product expansions constitute total derivatives. Vertex operators with conformal weight 1 are called *screening operators* and were introduced by Dotsenko and Fateev [11]. The conformal weight being 1 implies that the residue

$$\mathcal{Q}_\pm = \frac{1}{2\pi\mathrm{i}} \oint V_{\alpha_\pm}(w)\mathrm{d}w \tag{2.23}$$

is a Virasoro homomorphism, because of (2.22). In other words,

$$[T(z), \mathcal{Q}_\pm] = \frac{1}{2\pi\mathrm{i}} \oint T(z)V_{\alpha_\pm}(w)\mathrm{d}w = 0. \tag{2.24}$$

This residue is, of course, only well defined when the exponent of $z^{\alpha_\pm a_0}$ is an integer. Thus, for $\alpha_\pm\mu \in \mathbb{Z}$, the residue of the vertex operator $V_{\alpha_\pm}(z)$ defines a Virasoro homomorphism

$$\mathcal{Q}_\pm : \mathcal{F}_\mu \to \mathcal{F}_{\mu+\alpha_\pm}. \tag{2.25}$$

A natural question that one can ask in this context is whether these residues can be generalised to obtain more Virasoro homomorphisms. The answer is "yes", at least for suitable Heisenberg weights μ. The solution to generalising these Virasoro homomorphisms lies in composing screening operators. The composition of n vertex operators V_{μ_i}, $i = 1,\dots,n$, is given by

$$\begin{aligned} &V_{\mu_1}(z_1)\cdots V_{\mu_n}(z_n) \\ &\quad = e^{\hat{a}\sum_{i=1}^n \mu_i} \prod_{1\le i<j\le n} (z_i - z_j)^{\mu_i\mu_j} \prod_{i=1}^n z_i^{\mu_i a_0} \prod_{m\ge1} \exp\left(\frac{a_{-m}}{m}\sum_{i=1}^n \mu_i z_i^m\right) \prod_{m\ge1} \exp\left(-\frac{a_m}{m}\sum_{i=1}^n \mu_i z_i^{-m}\right). \end{aligned} \tag{2.26}$$

This formula is derived by using the operator product expansions above or by using the commutation relations of the Heisenberg Lie algebra. If we set $\mu_i = \alpha_\pm$, $i = 1,\dots,n$, then the above formula simplifies to

$$\begin{aligned} &V_{\alpha_\pm}(z_1)\cdots V_{\alpha_\pm}(z_n) \\ &\quad = e^{n\alpha_\pm\hat{a}} \prod_{1\le i<j\le n} (z_i - z_j)^{\alpha_\pm^2} \prod_{i=1}^n z_i^{\alpha_\pm a_0} \prod_{m\ge1} \exp\left(\alpha_\pm\frac{a_{-m}}{m}\sum_{i=1}^n z_i^m\right) \prod_{m\ge1} \exp\left(-\alpha_\pm\frac{a_m}{m}\sum_{i=1}^n z_i^{-m}\right). \end{aligned} \tag{2.27}$$

We take the opportunity to introduce a family of symmetric polynomials, called power sums, to simplify notation:

$$\mathsf{p}_m(z) = \sum_{i=1}^n z_i^m, \qquad \overline{\mathsf{p}_m(z)} = \sum_{i=1}^n z_i^{-m}. \tag{2.28}$$

Up to a phase factor, which we suppress, the second factor of (2.26) can be rewritten as

$$\prod_{1\le i\ne j\le n} (z_i - z_j)^{\kappa_\pm}, \quad \kappa_\pm = \frac{\alpha_\pm^2}{2}. \tag{2.29}$$

If we evaluate the product of these n screening operators on a Fock module $\mathcal{F}_\mu$, then the a_0 generator acts by multiplication with μ and therefore

$$\begin{aligned} &V_{\alpha_\pm}(z_1)\cdots V_{\alpha_\pm}(z_n)\Big|_{\mathcal{F}_\mu} \\ &\quad = e^{n\alpha_\pm\hat{a}} \prod_{1\le i\ne j\le n} (z_i - z_j)^{\kappa_\pm} \prod_{i=1}^n z_i^{\alpha_\pm \mu} \prod_{m\ge1} \exp\left(\alpha_\pm\frac{a_{-m}}{m}\mathsf{p}_m(z)\right) \prod_{m\ge1} \exp\left(-\alpha_\pm\frac{a_m}{m}\overline{\mathsf{p}_m(z)}\right). \end{aligned} \tag{2.30}$$

Let

$$c_n(\kappa_\pm) = \frac{2\pi \mathrm{i}}{(n-1)!}\prod_{j=1}^{n-1}\frac{\Gamma(1+(j+1)\kappa_\pm)\Gamma(-j\kappa_\pm)}{\Gamma(\kappa_\pm+1)}. \tag{2.31}$$

Theorem 2 (Tsuchiya-Kanie [12])**.** *If $d(d+1)\kappa_+ \notin \mathbb{Z}$ and $d(n-d)\kappa_+ \notin \mathbb{Z}$, for all integers d satisfying $1 \le d \le n-1$, then for each Heisenberg weight $\alpha_{n,k}$, $k \in \mathbb{Z}$, there exists a cycle Δ_n such that*

$$\mathcal{Q}_+^{[n]} = \frac{1}{c_n(\kappa_+)}\int_{\Delta_n} V_{\alpha_+}(z_1)\cdots V_{\alpha_+}(z_n)\mathrm{d}z_1\cdots\mathrm{d}z_n \tag{2.32}$$

is a non-trivial Virasoro homomorphism

$$\mathcal{Q}_+^{[n]} : \mathcal{F}_{\alpha_{n,k}} \to \mathcal{F}_{\alpha_{-n,k}}. \tag{2.33}$$

Likewise, if $d(d+1)\kappa_- \notin \mathbb{Z}$ and $d(n-d)\kappa_- \notin \mathbb{Z}$, for all integers d satisfying $1 \le d \le n-1$, then for each Heisenberg weight $\alpha_{k,n}, k \in \mathbb{Z}$, there exists a cycle Δ_n such that

$$\mathcal{Q}_-^{[n]} = \frac{1}{c_n(\kappa_-)}\int_{\Delta_n} V_{\alpha_-}(z_1)\cdots V_{\alpha_-}(z_n)\mathrm{d}z_1\cdots\mathrm{d}z_n \tag{2.34}$$

is a non-trivial Virasoro homomorphism

$$\mathcal{Q}_-^{[n]} : \mathcal{F}_{\alpha_{k,n}} \to \mathcal{F}_{\alpha_{k,-n}}. \tag{2.35}$$

In particular,

(1) *if $k \geq 1$, then there exist vectors $v \in \mathcal{F}_{\alpha_{n,k}}$ and $w \in \mathcal{F}_{\alpha_{k,n}}$ such that*

$$\mathcal{Q}_+^{[n]} v = \left|\alpha_{-n,k}\right\rangle \in \mathcal{F}_{\alpha_{-n,k}}, \qquad \mathcal{Q}_-^{[n]} w = \left|\alpha_{k,-n}\right\rangle \in \mathcal{F}_{\alpha_{k,-n}}, \tag{2.36}$$

while $\left|\alpha_{n,k}\right\rangle$ and $\left|\alpha_{k,n}\right\rangle$ are annihilated by $\mathcal{Q}_+^{[n]}$ and $\mathcal{Q}_-^{[n]}$, respectively,

(2) *if $k \leq 0$, then*

$$\mathcal{Q}_+^{[n]} \left|\alpha_{n,k}\right\rangle \neq 0 \in \mathcal{F}_{\alpha_{-n,k}}, \qquad \mathcal{Q}_-^{[n]} \left|\alpha_{k,n}\right\rangle \neq 0 \in \mathcal{F}_{\alpha_{k,-n}}. \tag{2.37}$$

The explicit construction of the cycles Δ_n is rather subtle and we refer to [12] for the details. Intuitively, Δ_n can be thought of as n concentric circles about 0 that are pinched together at 1.

Let us try and understand the implications of Theorem 2 a little better. Since $\left|\alpha_{n,k}\right\rangle$ is a Virasoro highest weight vector, then so is $\mathcal{Q}_+^{[n]}\left|\alpha_{n,k}\right\rangle$ by virtue of $\mathcal{Q}_+^{[n]}$ being a Virasoro homomorphism. Thus, whenever $k \leq 0$, the vectors $\mathcal{Q}_+^{[n]}\left|\alpha_{n,k}\right\rangle$ and $\mathcal{Q}_-^{[n]}\left|\alpha_{k,n}\right\rangle$ generate Virasoro subrepresentations in $\mathcal{F}_{\alpha_{-n,k}}$ and $\mathcal{F}_{\alpha_{k,-n}}$. Such Virasoro highest weight vectors are called singular vectors.

For $\mu_+ = \alpha_{n,k},\ \mu_- = \alpha_{k,n}$,

$$\prod_{1 \leq i \neq j \leq n} (z_i - z_j)^{\kappa_\pm} \prod_{i=1}^{n} z_i^{\alpha_\pm \mu_\pm} = \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{z_i}{z_j}\right)^{\kappa_\pm} \prod_{i=1}^{n} z_i^{k-1}, \tag{2.38}$$

where we have used the defining formulae (2.14) for $\alpha_{n,k}, \alpha_{k,n}$ and (2.29) for $\kappa_\pm$. For later use we define

$$G_n(z; \kappa_\pm^{-1}) = \prod_{1 \leq i \neq j \leq n} \left(1 - \frac{z_i}{z_j}\right)^{\kappa_\pm}. \tag{2.39}$$

This seemingly odd choice of $\kappa_\pm^{-1}$ in the definition of G_n is to make our notation in Section 3 conform with the standard conventions in the symmetric polynomials literature [13]. The constant $c_n(\kappa_\pm)$ in (2.31) normalises the cycles Δ_n in (2.32) and (2.34) such that

$$\frac{1}{c_n(\kappa_\pm)} \int_{\Delta_n} G_n(z : \kappa_\pm^{-1}) \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_j}{z_1 \cdots z_j} = 1. \tag{2.40}$$

Henceforth, we will therefore denote by $[\Delta_n]$ the homology class of the cycle Δ_n that has been rescaled by $c_n(\kappa_\pm)^{-1}$, the $\kappa_\pm$-dependence being left implicit, that is,

$$\int_{[\Delta_n]} = \frac{1}{c_n(\kappa_\pm)} \int_{\Delta_n}. \tag{2.41}$$

The conditions on $\kappa_\pm$ at the beginning of Theorem 2 ensure that $c_n(\kappa_\pm) \neq 0$. These conditions are met for the applications to the minimal models in this note. For a systematic discussion of how to regularise $[\Delta_n]$ when these conditions are not met, see [5, Sections 3.2–3.4].

By expanding the formulae for the Virasoro homomorphisms $\mathcal{Q}_\pm^{[n]}$ on $\mathcal{F}_{\alpha_{n,k}}$ and $\mathcal{F}_{\alpha_{k,n}}$, one sees that

$$\mathcal{Q}_\pm^{[n]} = e^{n\alpha_\pm \hat{a}} \int_{[\Delta_n]} G_n(z; \kappa_\pm^{-1}) \prod_{i=1}^{n} z_i^k \prod_{m \geq 1} \exp\left(\alpha_\pm \frac{a_{-m}}{m} \mathrm{p}_m(z)\right) \prod_{m \geq 1} \exp\left(-\alpha_\pm \frac{a_m}{m} \overline{\mathrm{p}_m(z)}\right) \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_n}{z_1 \cdots z_n}. \tag{2.42}$$

Apart from the multivalued function G_n, the integrand consists of an infinite sum of monomials in $U(\mathfrak{H}_-) \otimes U(\mathfrak{H}_+)$ where the coefficients are products of polynomials in either positive or negative powers of the z_i. It turns out that for symmetric polynomials f, g, the pairing

$$\langle f, g \rangle_n^{\kappa_\pm^{-1}} = \int_{[\Delta_n]} G_n(z; \kappa_\pm^{-1}) f(z_1, \ldots, z_n) \overline{g(z_1, \ldots, z_n)} \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_n}{z_1 \cdots z_n}, \tag{2.43}$$

where $\overline{g(z_1,\dots,z_n)} = g(z_1^{-1},\dots,z_n^{-1})$, defines a non-degenerate symmetric bilinear form on the ring of symmetric polynomials in n variables. Evaluating the Virasoro homomorphism $\mathcal{Q}_{\pm}^{[n]}$ in (2.42) therefore reduces to evaluating inner products of symmetric polynomials. As we shall see, this is a well known problem with an elegant solution.

3. Symmetric polynomials

For a comprehensive study of symmetric polynomials, we recommend the book by Macdonald [13]. Let Λ_n be the ring of symmetric polynomials with complex coefficients. As a commutative ring, Λ_n is generated by a number of interesting sets of polynomials including the elementary symmetric polynomials

$$\mathsf{e}_i(z) = \sum_{1\le j_1<\cdots<j_i\le n} z_{j_1}\cdots z_{j_i}, \quad i=1,\dots,n \tag{3.1}$$

and the power sums

$$\mathsf{p}_i(z) = \sum_{j=1}^{n} z_j^i, \quad i=1,\dots,n. \tag{3.2}$$

These polynomials are algebraically independent and generate Λ_n freely, that is,

$$\Lambda_n = \mathbb{C}[\mathsf{e}_1,\dots,\mathsf{e}_n] = \mathbb{C}[\mathsf{p}_1,\dots,\mathsf{p}_n]. \tag{3.3}$$

The ring Λ_n is clearly also a complex vector space and it is natural to look for convenient bases. One such basis is constructed from the power sums. Let $\lambda = (\lambda_1,\dots,\lambda_k)$, $k\ge 0$, be a partition of an integer with largest part $\lambda_1 \le n$ (we follow the convention of listing the parts in weakly descending order). Then, for all such λ, the

$$\mathsf{p}_\lambda(z) = \mathsf{p}_{\lambda_1}(z)\cdots\mathsf{p}_{\lambda_k}(z) \tag{3.4}$$

are linearly independent and form a basis of Λ_n. Another convenient basis of Λ_n is given by the monomial symmetric polynomials. Let $\mu = (\mu_1,\dots,\mu_n)$ be a partition of length $\ell(\mu)$ at most n (if the partition is shorter than n pad it with 0s at the end until it is length n). Then, the monomial symmetric polynomials are defined as

$$\mathsf{m}_\mu(z) = \sum_\tau z_1^{\tau_1}\cdots z_n^{\tau_n}, \quad \tau\in\{\text{all distinct permutations of } \mu\}. \tag{3.5}$$

We shall refer to these polynomials as the symmetric monomials for brevity.

As we can see from the power sums and the symmetric monomials, the set of partitions that label basis elements must be truncated once the weight $|\lambda| = \sum_i \lambda_i$ is greater than the number of variables. Specifically, there exist partitions with $\lambda_1 > n$, which are not allowed for the power sums, or $\ell(\lambda) > n$, which are not allowed for the symmetric monomials. This is why it is convenient to work in the limit of infinitely many variables:

$$\Lambda = \varprojlim_n \Lambda_n. \tag{3.6}$$

One can then easily recover the finite variable case by the projection

$$\begin{aligned} \gamma_n : \Lambda &\to \Lambda_n \\ z_j &\mapsto \begin{cases} z_j & 1\le j\le n \\ 0 & j>n \end{cases} \end{aligned} \tag{3.7}$$

that sets to 0 all but the first n variables. The power sums in infinitely many variables now generate Λ as their finite-variable versions did Λ_n. We continue to use (3.4) to define p_λ in the infinite-variable case.

$$\Lambda = \mathbb{C}[\mathsf{p}_1,\mathsf{p}_2,\mathsf{p}_3,\dots]. \tag{3.8}$$

The power sum and symmetric monomial bases of Λ are now labelled by all partitions of integers without restrictions on parts or lengths:

$$\Lambda = \bigoplus_{\lambda} \mathbb{C}\mathsf{p}_{\lambda} = \bigoplus_{\lambda} \mathbb{C}\mathsf{m}_{\lambda}. \tag{3.9}$$

The projection to the finite variable case is particularly easy in the symmetric monomial basis:

$$\begin{aligned} \gamma_n : \Lambda &\to \Lambda_n \\ \mathsf{m}_{\mu} &\mapsto \begin{cases} \mathsf{m}_{\mu}(z_1, \ldots, z_n) & \ell(\mu) \leq n \\ 0 & \text{else} \end{cases}. \end{aligned} \tag{3.10}$$

Recall that, by (2.7), the universal enveloping algebra $U(\mathfrak{H}_-)$ is also a polynomial algebra in infinitely many generators. Identifying these two algebras will be important below.

Proposition 3. *For $f, g \in \Lambda_n$ and $\kappa \in \mathbb{C}^*$ such that $d/\kappa \notin \mathbb{Z}$ for all integers satisfying $1 \leq d \leq n$, the bilinear form*

$$\langle f, g \rangle_n^{\kappa} = \int_{[\Delta_n]} G_n(z; \kappa) f(z_1, \ldots, z_n) \overline{g(z_1, \ldots, z_n)} \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_n}{z_1 \cdots z_n} \tag{3.11}$$

is

(1) symmetric,
(2) non-degenerate,
(3) graded: $\langle f, g \rangle_n^{\kappa} = 0$ if $\deg f \neq \deg g$.

Proposition 3 leads us to the basis of Λ_n that is most important for our purposes, the Jack polynomials $\mathsf{P}_{\lambda}^{\kappa}(z)$. These polynomials are characterised by two properties [13]:

(1) The Jack polynomials have upper triangular expansions in the basis of symmetric monomials with respect to the dominance ordering of partitions[3], that is,

$$\mathsf{P}_{\lambda}^{\kappa}(z) = \sum_{\lambda \geq \mu} u_{\lambda,\mu}(\kappa) \mathsf{m}_{\mu}(z), \tag{3.12}$$

where the $u_{\lambda,\mu}(\kappa) \in \mathbb{C}$ and $u_{\lambda,\lambda}(\kappa) = 1$.

(2) The Jack polynomials are mutually orthogonal:

$$\left\langle \mathsf{P}_{\lambda}^{\kappa}(z), \mathsf{P}_{\mu}^{\kappa}(z) \right\rangle_n^{\kappa} = 0, \quad \text{if } \lambda \neq \mu. \tag{3.13}$$

Since the dominance ordering of partitions is only a partial ordering, trying to determine the Jack polynomials by means of Gram-Schmidt orthogonalisation is an overdetermined problem. Showing that they exist is therefore non-trivial, see [13].

We prepare some notation regarding partitions. For a partition λ, let $s = (i, j) \in \lambda$ be a box in the Young tableau of λ, so that $i = 1, \ldots, \ell(\lambda)$ and $j = 1, \ldots, \lambda_i$. Then, the arm length, coarm length, leg length and coleg length are defined to be

$$a(s) = \lambda_i - j, \quad a'(s) = j - 1, \quad l(s) = \lambda'_j - i, \quad l'(s) = i - 1, \tag{3.14}$$

respectively, where λ' is the conjugate partition of λ, that is, the partition for which the columns and rows of the Young tableau have been exchanged.

Proposition 4.

(1) Jack polynomials exist (for all n and in the infinite variable limit).

[3] The dominance ordering is a partial ordering of partitions of equal weight defined by

$$\lambda \geq \mu \iff \sum_{i=1}^{k} \lambda_i \geq \sum_{i=1}^{k} \mu_i, \quad \forall k \geq 1.$$

(2) *The Jack polynomials satisfy the same projection formulae as the symmetric monomials:*

$$\gamma_n(\mathsf{P}^{\kappa}_{\lambda}(z)) = \begin{cases} \mathsf{P}^{\kappa}_{\lambda}(z_1,\dots,z_n) & \ell(\lambda)\le n \\ 0 & \text{else} \end{cases}. \tag{3.15}$$

(3) *For either a finite or infinite number of variables z_i, y_j,*

$$\begin{gathered} \prod_{i,j\ge 1}(1-z_iy_j)^{-1/\kappa} = \prod_{m\ge 1}\exp\left(\frac{1}{\kappa}\frac{\mathsf{p}_m(z)\mathsf{p}_m(y)}{m}\right) = \sum_{\lambda} b_\lambda(\kappa)\mathsf{P}^{\kappa}_{\lambda}(z)\mathsf{P}^{\kappa}_{\lambda}(y), \\ b_\lambda(\kappa) = \prod_{s\in\lambda}\frac{a(s)\kappa+l(s)+1}{(a(s)+1)\kappa+l(s)}. \end{gathered} \tag{3.16}$$

(4) *The norm of the mutually orthogonal Jack polynomials is*

$$\left\langle \mathsf{P}^{\kappa}_{\lambda}(z),\mathsf{P}^{\kappa}_{\lambda}(z)\right\rangle^{\kappa}_{n} = \prod_{s\in\lambda}\frac{(a(s)+1)\kappa+l(s)}{a(s)\kappa+l(s)+1}\,\frac{n+a'(s)\kappa-l'(s)}{n+(a'(s)+1)\kappa-l'(s)-1}. \tag{3.17}$$

(5) *For $X\in\mathbb{C}$, let $\Xi_X:\Lambda\to\mathbb{C}$ be the algebra homomorphism defined, in the power sum basis, by*

$$\Xi_X(\mathsf{p}_\lambda(y)) = X^{\ell(\lambda)}. \tag{3.18}$$

The map Ξ_X is called the specialisation map. *Then,*

$$\Xi_X\left(\mathsf{P}^{\kappa}_{\lambda}(y)\right) = \prod_{s\in\lambda}\frac{X+a'(s)\kappa-l'(s)}{a(s)\kappa+l(s)+1} \tag{3.19}$$

and

$$\prod_{i\ge 1}(1-z_i)^{-X/\kappa} = \prod_{m\ge 1}\exp\left(\frac{X}{\kappa}\frac{\mathsf{p}_m(z)}{m}\right) = \sum_{\lambda} b_\lambda(\kappa)\mathsf{P}^{\kappa}_{\lambda}(z)\,\Xi_X(\mathsf{P}^{\kappa}_{\lambda}(y)). \tag{3.20}$$

We stress that while this homomorphism applies to symmetric polynomials in any variables, we will only be applying it to those in the y variables.

(6) *Let $(m^n)=(m,\dots,m)$ be the partition consisting of n copies of m. Then,*

$$\gamma_n(\mathsf{P}^{\kappa}_{(m^n)}(z)) = \mathsf{P}^{\kappa}_{(m^n)}(z_1,\dots,z_n) = \mathsf{m}_{(m^n)}(z_1,\dots,z_n) = \prod_{i=1}^{n} z_i^m. \tag{3.21}$$

See [13] for proofs.

Armed with this knowledge of Jack polynomials, we can now explicitly evaluate the action of screening operators on Fock modules. Recall from equations (2.7) and (3.8) that both $U(\mathfrak{H}_-)$ and Λ are polynomial algebras in an infinite number of variables,

$$\mathbb{C}[a_{-1},a_{-2},\dots] = U(\mathfrak{H}_-)\cong\Lambda = \mathbb{C}[\mathsf{p}_1,\mathsf{p}_2,\dots], \tag{3.22}$$

and are therefore isomorphic. For $\delta\in\mathbb{C}$, we define the algebra isomorphism

$$\begin{aligned} \rho_\delta:\Lambda &\to U(\mathfrak{H}_-), \\ \mathsf{p}_n(y) &\mapsto \delta a_{-n}. \end{aligned} \tag{3.23}$$

As with the specialisation map, we will only be applying δ to polynomials in the y variables.

Theorem 5. *For $k\ge 0$, let $(k^n)=(k,\dots,k)$ be the partition consisting of n copies of k. Then, the Virasoro homomorphisms*

$$\mathcal{Q}^{[n]}_{+}:\mathcal{F}_{\alpha_{n,-k}}\to\mathcal{F}_{\alpha_{-n,-k}},\qquad \mathcal{Q}^{[n]}_{-}:\mathcal{F}_{\alpha_{-k,n}}\to\mathcal{F}_{\alpha_{-k,-n}} \tag{3.24}$$

map the vectors $|\alpha_{n,-k}\rangle$ *and* $|\alpha_{-k,n}\rangle$ *to the non-zero singular vectors*

$$\begin{aligned}\mathcal{Q}_+^{[n]}|\alpha_{n,-k}\rangle &= b_{(k^n)}(\kappa_+^{-1})\rho_{\frac{2}{\alpha_+}}\left(\mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(y)\right)|\alpha_{-n,-k}\rangle,\\ \mathcal{Q}_-^{[n]}|\alpha_{-k,n}\rangle &= b_{(k^n)}(\kappa_-^{-1})\rho_{\frac{2}{\alpha_-}}\left(\mathsf{P}_{(k^n)}^{\kappa_-^{-1}}(y)\right)|\alpha_{-k,-n}\rangle.\end{aligned} \tag{3.25}$$

Proof. We prove the formula for $\mathcal{Q}_+$. The one for $\mathcal{Q}_-$ follows similarly. The proof follows by direct evaluation using the theory of Jack polynomials:

$$\begin{aligned}\mathcal{Q}_+^{[n]}|\alpha_{n,-k}\rangle &= \int_{[\Delta_n]} G_n(z;\kappa_+^{-1})\prod_{i=1}^n z_i^{-k}\prod_{m\geq 1}\exp\left(\alpha_+\frac{\mathsf{p}_m(z)}{m}a_{-m}\right)|\alpha_{-n,-k}\rangle\frac{\mathrm{d}z_1\cdots\mathrm{d}z_n}{z_1\cdots z_n}\\ &\overset{1}{=}\left\langle \mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z),\prod_{m\geq 1}\exp\left(\alpha_+\frac{\mathsf{p}_m(z)}{m}a_{-m}\right)\right\rangle_n^{\kappa_+^{-1}}|\alpha_{-n,-k}\rangle\\ &=\left\langle \mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z),\rho_{\frac{2}{\alpha_+}}\left(\prod_{m\geq 1}\exp\left(\kappa_+\frac{\mathsf{p}_m(z)\mathsf{p}_m(y)}{m}\right)\right)\right\rangle_n^{\kappa_+^{-1}}|\alpha_{-n,-k}\rangle\\ &\overset{2}{=}\sum_\mu b_\mu(\kappa_+^{-1})\left\langle \mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z),\mathsf{P}_\mu^{\kappa_+^{-1}}(z)\right\rangle_n^{\kappa_+^{-1}}\rho_{\frac{2}{\alpha_+}}\left(\mathsf{P}_\lambda^{\kappa_+^{-1}}(y)\right)|\alpha_{-n,-k}\rangle\\ &\overset{3}{=} b_{(k^n)}(\kappa_+^{-1})\left\langle \mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z),\mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z)\right\rangle_n^{\kappa_+^{-1}}\rho_{\frac{2}{\alpha_+}}\left(\mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(y)\right)|\alpha_{-n,-k}\rangle\\ &\overset{4}{=} b_{(k^n)}(\kappa_+^{-1})\rho_{\frac{2}{\alpha_+}}\left(\mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(y)\right)|\alpha_{-n,-k}\rangle.\end{aligned} \tag{3.26}$$

Here we have used item (6) of Proposition 4 for $\overset{1}{=}$ to identify

$$\overline{\mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z)} = \prod_{i=1}^n z_i^{-k}; \tag{3.27}$$

item (3) of Proposition 4 for $\overset{2}{=}$ (remembering that the integration in the inner product is over the z variables); the orthogonality of Jack polynomials for $\overset{3}{=}$; and item (6) of Proposition 4 to see that

$$\mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z)\overline{\mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z)} = 1 \quad\Rightarrow\quad \left\langle \mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z),\mathsf{P}_{(k^n)}^{\kappa_+^{-1}}(z)\right\rangle_n^{\kappa_+^{-1}} = 1, \tag{3.28}$$

which justifies $\overset{4}{=}$. By direct evaluation of formula for $b_{(k^n)}(\kappa_+^{-1})$ in item (3) of Proposition 4, one sees that $b_{(k^n)}(\kappa_+^{-1})$ is a product of quotients of positive rational numbers and is therefore non-zero. ∎

This theorem is originally due to Mimachi and Yamada [3], though the much simpler and streamlined proof that we have presented here first appeared in [5, Proposition 3.24].

4. The minimal models and their representations

In Section 2, we constructed the universal Virasoro vertex operator algebra at central charge

$$c = 1 - 3\alpha_0^2, \quad \alpha_0 \in \mathbb{C}, \tag{4.1}$$

as a vertex operator subalgebra of the Heisenberg vertex operator algebra. At generic values of the central charge c (or equivalently, at generic values of α_0), the universal Virasoro vertex operator algebra is simple and contains no non-trivial ideals. However, there is a discrete set of central charges at which the universal Virasoro vertex operator algebra is not simple. The minimal model vertex operator algebras are the simple vertex operator algebras obtained, for these central charges, by taking the quotients of the universal vertex operator algebras by their maximal ideals.

Thus, the minimal model vertex operator algebras can be realised as subquotients of Heisenberg vertex operator algebras.

The minimal model central charges, that is, the central charges at which the universal Virasoro vertex operator algebras are non-simple, are precisely

$$c_{p_+,p_-} = c_{p_-,p_+} = 1 - 6\frac{(p_+ - p_-)^2}{p_+ p_-}, \tag{4.2}$$

where $p_+, p_- \geq 2$ are coprime integers [9]. We denote the minimal model vertex operator algebra of central charge c_{p_+,p_-} by $\mathsf{M}(p_+,p_-)$. To obtain these minimal model central charges for the Heisenberg algebra, we set

$$\alpha_0 = \alpha_+ + \alpha_-, \qquad \alpha_+ = \sqrt{\frac{2p_-}{p_+}}, \qquad \alpha_- = -\sqrt{\frac{2p_+}{p_-}}. \tag{4.3}$$

The parameters $\alpha_\pm$ are precisely the Heisenberg weights which, by formula (2.12), correspond to conformal weight 1. The $\kappa_\pm$ parameters introduced in Section 2 are thus,

$$\kappa_+ = \frac{\alpha_+^2}{2} = \frac{p_-}{p_+}, \qquad \kappa_- = \frac{\alpha_-^2}{2} = \frac{p_+}{p_-}. \tag{4.4}$$

The ideal $\mathsf{I}(p_+,p_-)$ of the universal Virasoro vertex operator algebra of central charge c_{p_+,p_-} is generated by a singular vector of conformal weight $(p_+-1)(p_--1)$ [9]. By using the screening operator formalism of Section 2, in particular Theorem 5, we can realise this singular vector using the screening operator $\mathcal{Q}_+(z) = V_{\alpha_+}(z)$ or $\mathcal{Q}_-(z) = V_{\alpha_-}(z)$. Writing

$$\mathcal{Q}_\pm^{[n]} = \int_{[\Delta_n]} V_{\alpha_\pm}(z_1)\cdots V_{\alpha_\pm}(z_n)\mathrm{d}z_1\cdots\mathrm{d}z_n, \tag{4.5}$$

we deduce that the singular vector in $\mathcal{F}_0$ which generates the ideal of the universal Virasoro vertex operator algebra, sitting inside the Heisenberg vertex operator algebra, is given by

$$\begin{aligned}
\mathcal{Q}_+^{[p_+-1]}\left|(1-p_+)\alpha_+\right\rangle &= b_{((p_--1)^{p_+-1})}(\kappa_+^{-1})\rho_{\frac{2}{\alpha_+}}\left(\mathsf{P}^{\kappa_+^{-1}}_{((p_--1)^{p_+-1})}(y)\right)\left|0\right\rangle, \\
\mathcal{Q}_-^{[p_--1]}\left|(1-p_-)\alpha_-\right\rangle &= b_{((p_+-1)^{p_--1})}(\kappa_-^{-1})\rho_{\frac{2}{\alpha_-}}\left(\mathsf{P}^{\kappa_-^{-1}}_{((p_+-1)^{p_--1})}(y)\right)\left|0\right\rangle.
\end{aligned} \tag{4.6}$$

The above equations are obtained directly from Theorem 5, the first by choosing $n = p_+ - 1,\ k = 1 - p_-$ and the second by choosing $n = p_- - 1,\ k = 1 - p_+$. For a given conformal weight, the Virasoro singular vectors of a Fock module are unique up to rescaling [14], if they exist, so the two vectors in (4.6) are proportional to each other.

As a final demonstration of the power of combining the screening operator and symmetric polynomial formalisms, we will classify the representations of the minimal model vertex operator algebras. Since the universal Virasoro vertex operator algebras are subalgebras of the Heisenberg vertex operator algebras, the Fock modules $\mathcal{F}_\mu$ are representations of the universal Virasoro vertex operator algebras for any $\mu \in \mathbb{C}$. However, the Virasoro representation generated from $\left|\mu\right\rangle$ can only be a representation of $\mathsf{M}(p_+,p_-)$ if each field corresponding to a vector in the ideal $\mathsf{I}(p_+,p_-)$ acts trivially. Moreover, any irreducible highest weight representation of $\mathsf{M}(p_+,p_-)$ must be realisable as a subquotient of a Fock module as, for any conformal weight, there exists a Fock module whose generating vector has that conformal weight, by (2.12).

Theorem 6. *Let*

$$h_{r,s} = \frac{(rp_- - sp_+)^2 - (p_+ - p_-)^2}{4p_+p_-}. \tag{4.7}$$

Up to isomorphism, there are exactly $\frac{1}{2}(p_+-1)(p_--1)$ inequivalent irreducible $\mathsf{M}(p_+,p_-)$ representations. They are given by the irreducible representations of the Virasoro algebra generated by highest weight vectors of *conformal weight*

$$h_{r,s}, \quad 1 \le r \le p_+ - 1,\ 1 \le s \le p_- - 1,\ rp_- + sp_+ \le p_+ p_-. \tag{4.8}$$

Proof. We only prove that the above list of irreducible representations of the Virasoro algebra is an upper bound on the set of inequivalent irreducible $\mathsf{M}(p_+, p_-)$ representations. In order to show that the list is saturated, one can then either construct all these representations by, for example, the coset construction [15, 16], by quantum hamiltonian reduction [17], or use Zhu's algebra [4, 8], this being the associative algebra of zero modes of the fields of the vertex operator algebra acting on highest weight vectors.

Consider the singular vector

$$\left|\chi\right\rangle = \mathcal{Q}_+^{[p_+-1]}\left|(1-p_+)\alpha_+\right\rangle \in \mathcal{F}_0 \tag{4.9}$$

of equation (4.6). The corresponding field is obtained by integrating $p_+ - 1$ vertex operators V_{α_+} over $[\Delta_{p_+-1}]$ about the vertex operator $V_{(1-p_+)\alpha_+}$, with $[\Delta_{p_+-1}]$ centred about the argument of $V_{(1-p_+)\alpha_+}$:

$$\chi(w) = \int_{[\Delta_{p_+-1}]} \mathcal{Q}_+(z_1+w)\cdots\mathcal{Q}_+(z_{p_+-1}+w)V_{(1-p_+)\alpha_+}(w)\mathrm{d}z_1\cdots\mathrm{d}z_{p_+-1}. \tag{4.10}$$

The vector $\left|\chi\right\rangle$ is an element of the ideal $\mathsf{I}(p_+, p_-)$, so the field $\chi(w)$ must therefore act trivially on any $\mathsf{M}(p_+, p_-)$ representation. Consequently,

$$\left\langle\mu\middle|\chi(w)\middle|\mu\right\rangle = 0, \tag{4.11}$$

where $\left|\mu\right\rangle$ is the highest weight vector of $\mathcal{F}_\mu$ and $\left\langle\mu\right|$ its dual which satisfies

$$\left\langle\mu\middle|\mu\right\rangle = 1, \qquad \left\langle\mu\right|a_n = \left\langle\mu\right|\delta_{n,0}\mu, \quad n \le 0. \tag{4.12}$$

We can evaluate $\left\langle\mu\middle|\chi(w)\middle|\mu\right\rangle$ using the theory of Jack polynomials. Applying formula (2.26) to simplify the composition of vertex operators in the definition of $\chi(w)$, we see that

$$\begin{aligned}
&\mathcal{Q}_+(z_1+w)\cdots\mathcal{Q}_+(z_{p_+-1}+w)V_{(1-p_+)\alpha_+}(w) \\
&\qquad = G_{p_+-1}(z;\kappa_+^{-1})\prod_{i=1}^{p_+-1}(z_i+w)^{\alpha_+a_0}\cdot w^{(1-p_+)\alpha_+a_0} \\
&\qquad\quad \times \prod_{m\ge1}\exp\left(\alpha_+\frac{a_{-m}}{m}\left(\mathsf{p}_m(z_1+w,\dots,z_{p_+-1}+w)+(1-p_+)w^m\right)\right) \\
&\qquad\quad \times \prod_{m\ge1}\exp\left(-\alpha_+\frac{a_m}{m}\left(\overline{\mathsf{p}_m(z_1+w,\dots,z_{p_+-1}+w)}+(1-p_+)w^{-m}\right)\right).
\end{aligned} \tag{4.13}$$

The exponentials of Heisenberg generators a_m, $m \ne 0$, in (4.13) annihilate $\left\langle\mu\right|$ and $\left|\mu\right\rangle$. Thus,

$$\begin{aligned}
\left\langle\mu\middle|\chi(w)\middle|\mu\right\rangle &= \int_{[\Delta_{p_+-1}]}\left\langle\mu\middle|\mathcal{Q}_+(z_1+w)\cdots\mathcal{Q}_+(z_{p_+-1}+w)V_{(1-p_+)\alpha_+}(w)\middle|\mu\right\rangle\mathrm{d}z_1\cdots\mathrm{d}z_{p_+-1} \\
&= \int_{[\Delta_{p_+-1}]} G_{p_+-1}(z;\kappa_+^{-1})\prod_{i=1}^{p_+-1} z_i^{1-p_-}\cdot\prod_{i=1}^{p_+-1}(z_i+w)^{\alpha_+\mu}\cdot w^{(1-p_+)\alpha_+\mu}\frac{\mathrm{d}z_1\cdots z_{p_+-1}}{z_1\cdots z_{p_+-1}} \\
&= \int_{[\Delta_{p_+-1}]} G_{p_+-1}(z;\kappa_+^{-1})\prod_{i=1}^{p_+-1} z_i^{-(p_--1)}\cdot\prod_{i=1}^{p_+-1}\left(1+\frac{z_i}{w}\right)^{\alpha_+\mu}\frac{\mathrm{d}z_1\cdots\mathrm{d}z_{p_+-1}}{z_1\cdots z_{p_+-1}} \\
&= \left\langle \mathsf{P}^{\kappa_+^{-1}}_{((p_--1)^{p_+-1})}(z), \prod_{i=1}^{p_+-1}\left(1+\frac{z_i}{w}\right)^{\alpha_+\mu}\right\rangle^{\kappa_+^{-1}}_{p_+-1} \\
&= (-w)^{-(p_+-1)(p_--1)}\, b_{((p_--1)^{p_+-1})}(\kappa_+^{-1})\ \Xi_{\alpha_-\mu}\left(\mathsf{P}^{\kappa_+^{-1}}_{((p_--1)^{p_+-1})}(y)\right) \\
&= (-w)^{-(p_+-1)(p_--1)}\prod_{s\in(p_--1)^{p_+-1}}\frac{\alpha_-\mu + a'(s)/\kappa_+ - l'(s)}{(a(s)+1)/\kappa_+ + l(s)}
\end{aligned}$$

$$= (-w)^{-(p_+-1)(p_--1)} \prod_{i=1}^{p_+-1} \prod_{j=1}^{p_--1} \frac{\alpha_-\mu + (j-1)/\kappa_+ + 1 - i}{(p_- - j)/\kappa_+ + p_+ - 1 - i}, \tag{4.14}$$

where we have evaluated the inner product using item (5) of Proposition 4. Clearly, the denominator of the above product is non-singular, since $\kappa_+ = p_-/p_+$ is a positive rational number. Therefore, $\langle \mu | \chi(w) | \mu \rangle = 0$ whenever

$$0 = \prod_{i=1}^{p_+-1} \prod_{j=1}^{p_--1} (\alpha_-\mu + (j-1)/\kappa_+ + 1 - i) = C \prod_{i=1}^{p_+-1} \prod_{j=1}^{p_--1} (\mu - \alpha_{i,j}), \tag{4.15}$$

where C is a non-zero constant. We group the (i,j)-factor with the $(p_+ - i, p_- - j)$-factor:

$$(\mu - \alpha_{i,j}) \cdot \left(\mu - \alpha_{p_+-i,p_--j}\right) = 2h_\mu - 2h_{i,j}. \tag{4.16}$$

Thus,

$$\langle \mu | \chi(w) | \mu \rangle = 0 \iff \prod_{(r,s)} (h_\mu - h_{r,s}) = 0, \tag{4.17}$$

where the index (r,s) runs over all $1 \le r \le p_+ - 1$ and $1 \le s \le p_- - 1$, with $rp_- + sp_- < p_+p_-$. The above constraints imply that the conformal highest weight of an $\mathsf{M}(p_+, p_-)$ representation must be a root of the polynomial

$$f(h) = \prod_{(r,s)} (h - h_{r,s}), \tag{4.18}$$

that is, it must be equal to $h_{r,s}$ for some $1 \le r \le p_+ - 1$ and $1 \le s \le p_- - 1$, with $rp_- + sp_- < p_+p_-$. ∎

Showing that the representation theory of $\mathsf{M}(p_+, p_-)$ is completely reducible and that it can be used to construct rational conformal field theories requires only a little more work. The Virasoro Verma module of conformal weight $h_{r,s}$, where $1 \le r \le p_+ - 1$ and $1 \le s \le p_- - 1$, contains a maximal subrepresentation generated by two independent singular vectors of conformal weights $h' = h_{r,s} + rs$ and $h'' = h_{r,s} + (p_+ - r)(p_- - s)$ [9]. However, neither h' nor h'' are roots of (4.18). So the $\mathsf{M}(p_+, p_-)$ representation of conformal weight $h_{r,s}$ must be isomorphic to the irreducible quotient of theVirasoro Verma module of conformal weight $h_{r,s}$. This also implies that there exists no non-trivial extensions between irreducible representations with distinct conformal weights. In [18, Prop. 7.5], it was shown that the irreducible Virasoro representation of conformal weight $h_{r,s}$ admits no self extensions (as representations of the Virasoro algebra). Thus, neither do the irreducible $\mathsf{M}(p_+, p_-)$ representations. This proves that irreducible $\mathsf{M}(p_+, p_-)$ representations do not admit any non-trivial extensions and that therefore the representation theory of $\mathsf{M}(p_+, p_-)$ is completely reducible.

Corollary 7. *The Virasoro minimal model vertex operator algebras are rational, that is, they admit only a finite number of inequivalent irreducible representations and all representations are completely reducible.*

References

[1] L Alday, D Gaiotto and Y Tachikawa. Liouville correlation functions from four-dimensional gauge theories. *Lett. Math. Phys.*, 91:167–197, 2010.

[2] A Morozov and A Smirnov. Finalizing the proof of AGT relations with the help of the generalized Jack polynomials. *Lett. Math. Phys.*, 104:586–612, 2014.

[3] K Mimachi and Y Yamada. Singular vectors of the Virasoro algebra in terms of Jack symmetric polynomials. *Comm. Math. Phys.*, 174:447–455, 1995.

[4] W Wang. Rationality of Virasoro vertex operator algebras. *Int. Math. Res. Not.*, 7:197–211, 1993.

[5] A Tsuchiya and S Wood. On the extended W-algebra of type sl_2 at positive rational level. *Int. Math. Res. Not. to appear*, 2014. arXiv:1302.6435 [hep-th].

[6] D Ridout and S Wood. In preparation.

[7] B Feigin, T Nakanishi, and H Ooguri. The annihilating ideals of minimal models. *Int. J. Mod. Phys.*, A7S1A:217–238, 1992.

[8] Y Zhu. Modular invariance of characters of vertex operator algebras. *J. Amer. Math. Soc.*, 9:237–302, 1996.

[9] B Feigin and D Fuchs. Representations of the Virasoro algebra. *Adv. Stud. Contemp. Math.*, 7:465–554, 1990.

[10] K Iohara and Y Koga. *Representation theory of the Virasoro algebra*. Springer Monographs in Mathematics. Springer-Verlag, London, 2011.

[11] V Dotsenko and V Fateev. Conformal algebra and multipoint correlation functions in 2D statistical models. *Nucl. Phys. B*, 240:312–348, 1984.

[12] A Tsuchiya and Y Kanie. Fock space representations of the Virasoro algebra – intertwining operators. *Publ. Res. Inst. Math. Sci*, 22:259–327, 1986.

[13] I Macdonald. *Symmetric functions and Hall polynomials*. Oxford University Press, New York, 1995.

[14] A Astashkevich. On the structure of Verma modules over Virasoro and Neveu-Schwarz algebras. *Comm. Math. Phys.*, 186:531–562, 1997. arXiv:hep-th/9511032.

[15] P Goddard, A Kent, and D Olive. Virasoro algebras and coset space models. *Phys. Lett. B*, 152:88–92, 1985.

[16] V Kac and M Wakimoto. Modular invariant representations of infinite-dimensional Lie algebras and superalgebras. *Proc. Natl. Acad. Sci. USA*, 85:4956–4960, 1988.

[17] B Feigin and E Frenkel. Quantization of the Drinfeld-Sokolov reduction. *Phys. Lett. B*, 246:75–81, 1990.

[18] K Kytölä and D Ridout. On staggered indecomposable Virasoro modules. *J. Math. Phys.*, 50:123503, 2009. arXiv:0905.0108 [math-ph].

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