Free fermions in classical and quantum integrable models

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Abstract

The aim of this thesis is to study the role of free fermions in certain classical and quantum integrable models. The classical models studied are the KP and BKP hierarchies of partial differential equations. We review the presence of free fermions in the theory of these hierarchies, a topic which was established in the series of papers [16], [17], [18], [19], [20], [21], [56], [78], [79] by the Kyoto school. The quantum models studied are descendents and relatives of the XYZ model, or its lattice equivalent, the eight-vertex model [2], [3], [4], [5]. We give a number of new results in the context of these models, revealing the presence of fermions in typical quantities such as Bethe eigenvectors, partition functions and scalar products. The appearance of classical fermions in these quantum mechanical objects has some powerful accompanying features.

1. Generally speaking, the fermions facilitate the calculation of the objects themselves. Often their presence causes complete factorization, and we study a number of partition functions and scalar products which fall under this category.

2. In special cases, the fermions allow us to prove that the object under consideration is a solution of a classical hierarchy. Since classical hierarchies and quantum models are both solved by a variant of inverse scattering, it is natural to expect that their solutions should be related. We hope that the results of this thesis indicate a deeper correspondence than that which is presently accepted in the literature.

The first two chapters of this thesis are introductory, and designed to fix the notations and concepts which appear later. Chapter 1 is based almost exclusively on [50], and reviews the polynomial solutions of the KP and BKP hierarchies via the fermionic approach. Chapter 2 describes the quantum inverse scattering method of the Leningrad school [61], in the setting of a generalized discrete model.

The last four chapters of the thesis contain new material. In chapter 3 we study the phase model of [9], [10], in parallel with the closely related i-boson model. We observe that the Bethe eigenvectors of these models admit a natural description in terms of charged and neutral fermions, respectively. This proves that the scalar products of these models are solutions of the KP and BKP hierarchies, respectively. We also derive generating functions for ordinary and strict plane partitions using the calculus of free fermions [35], [39]. In chapter 4 we consider the q-boson model of [8], which specializes to the models studied in chapter 3 in the respective limits

$q \to \infty$ and $q \to i$. We provide a description of the Bethe eigenvectors using the more complicated algebra of t-deformed fermions [51], [52]. This leads to a fermionic proof of a generating function for t-weighted plane partitions [36], [88].

Chapter 5 studies the trigonometric limit of the XYZ model, the XXZ model. We review the determinant expressions discovered in [48] and [81] for the partition function and scalar product, respectively. By writing these objects as expectation values of charged fermions we prove that they are solutions of the KP hierarchy [40], [41]. We also derive an explicit expression for the Bethe eigenvectors of the model. Chapter 6 considers the trigonometric Felderhof model [24], [29], [30], [31], which generalizes the free fermion point of the six-vertex model, and the elliptic Deguchi-Akutsu height model [25], which generalizes the free fermion point of the eight-vertex SOS model [4]. Motivated by the fermionic nature of these models, we give factorized expressions for their partition functions [37], [38].

Declaration

This is to certify that

1. This thesis comprises only my original work towards the PhD.

2. Due acknowledgement has been made in the text to all other material used.

3. This thesis is less than 100,000 words in length.

Michael Wheeler

Preface

This thesis contains material from the papers

1. A Caradoc, O Foda, M Wheeler, M Zuparic, *On the trigonometric Felderhof model with domain wall boundary conditions,* J. Stat. Mech. 0703:P010 (2007)

2. O Foda, M Wheeler, *BKP plane partitions,* JHEP01 (2007) 075

3. O Foda, M Wheeler, *Hall-Littlewood plane partitions and KP,* Int. Math. Res. Notices (2009), 2597–2619

4. O Foda, M Wheeler, M Zuparic, *Factorized domain wall partition functions in trigonometric vertex models,* J. Stat. Mech. (2007) P10016

5. O Foda, M Wheeler, M Zuparic, *Two elliptic height models with factorized domain wall partition functions,* J. Stat. Mech. (2008) P02001

6. O Foda, M Wheeler, M Zuparic, *On free fermions and plane partitions,* Journal of Algebra **321** (2009), 3249–3273

7. O Foda, M Wheeler, M Zuparic, *Domain wall partition functions and KP,* J. Stat. Mech. (2009) P03017

8. O Foda, M Wheeler, M Zuparic, *XXZ scalar products and KP,* Nucl. Phys. B **820** (2009), 649–663

which were written by the author (MW) and collaborators. The results and methods which have been selected from these papers are, in all instances, the original work of the author and his supervisor (OF).

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To my supervisor, Omar Foda, thank you for the countless hours you have devoted to my mathematical education. Needless to say, this thesis would not exist without your knowledge, insight, and great enthusiasm for science.

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To Felicia, or as I will always call you, Rola, you have been there since the beginning. Through all the times when my work has taken priority, you have been gracious and eternally patient. I would not have made it without you.

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Chapter 1

Free fermions in classical hierarchies

1.0 Introduction

The study of classical integrable models, in modern times at least, dates back to the work of C S Gardner, J M Greene, M D Kruskal and R M Miura in [44]. In this seminal paper, the authors discovered the classical inverse scattering method for solving the non-linear Korteweg-deVries (KdV) equation. The central idea behind this method is the observation that the KdV equation can be written as the compatibility condition for two linear partial differential equations, the Lax pair [64], effectively linearizing the problem. Soon afterwards, in [69] and [70], it was shown that solutions to the KdV equation give rise to infinitely many conserved quantities, proving that KdV is integrable in the sense of Liouville.

This chapter reviews the polynomial solutions of the KP hierarchy (which contains the KdV equation as a special case), and of the BKP hierarchy. The technique we employ stems from the collective works of M Sato, Y Sato and E Date, M Jimbo, M Kashiwara, T Miwa in the 1980s, which culminated in the beautiful paper [50]. It involves embedding the infinite dimensional Lie algebras A_∞ and B_∞ in the algebra of free fermions, and constructing a highest weight representation for the latter. The solutions of the bilinear equations of the KP and BKP hierarchies are recovered from the orbit of the highest weight vector under the corresponding transformation groups, GL_∞ and O_∞, respectively.

We have written this chapter in a manner which emphasizes the similarities between the theory of the KP and BKP hierarchies. The reader will find that every result which applies to one hierarchy has a parallel result in the context of the other. Except where otherwise indicated, the material of this chapter is taken from [50], and we have adopted most of the notations contained therein. Alternative references for the material on the KP hierarchy include chapter 9 of [1], [55] and the detailed book [71]. Introductory material on the BKP hierarchy can also be found in [18], [47], [76] and [91].

The synopsis of the first part of this chapter, on the KP hierarchy, is as follows. In section 1.1 we introduce the algebra of charged fermions Cl_ψ and construct a representation of this algebra on the Fock space $\mathcal{F}_\psi$. We write down the partition basis for $\mathcal{F}_\psi$ and define a bilinear form between this vector space and its dual. The Lie algebra A_∞ is given as a subalgebra of Cl_ψ, and we define the KP evolution operator as an element of the corresponding transformation group GL_∞. We conclude the section by calculating the form between the dual vacuum vector under time evolution, and a basis vector of $\mathcal{F}_\psi$. The result is a Schur polynomial, which collectively comprise a basis for the space of all polynomials.

In section 1.2 we start from the KP bilinear identity, which is an integral equation, and show that it gives rise to the infinitely many partial differential equations of the KP hierarchy. Using the basis of Schur polynomials, we write solutions of the KP hierarchy as forms between the dual vacuum under time evolution and special states $g_\psi|0\rangle$ in $\mathcal{F}_\psi$. We derive the necessary and sufficient condition on g_ψ to ensure KP solubility, and call it the charged fermionic bilinear identity (CFBI).

In section 1.3 we construct solutions of the CFBI. We prove that g_ψ satisfies the CFBI if and only if $g_\psi \in GL_\infty$. As an example, we write the partition basis vectors of $\mathcal{F}_\psi$ in this way, showing that every Schur polynomial is a solution of the KP hierarchy. There exists another perspective through which the CFBI can be solved. Expanding $g_\psi|0\rangle$ in the canonical basis of $\mathcal{F}_\psi$, we show that the CFBI is satisfied if and only if the expansion coefficients obey the KP Plücker relations. As an example, we give an explicit determinant solution of the KP Plücker relations.

The second part of this chapter, on the BKP hierarchy, has an almost identical structure. In section 1.4 we introduce the algebra of neutral fermions Cl_ϕ and construct a representation of this algebra on the Fock space $\mathcal{F}_\phi$. We give a basis for $\mathcal{F}_\phi$ and apply the previous bilinear form to this vector space and its dual. The Lie algebra B_∞ is given as a subalgebra of Cl_ϕ, and we define the BKP evolution operator as an element of the corresponding transformation group O_∞. We conclude the section by calculating the form between the dual vacuum vector under time evolution, and a basis vector of $\mathcal{F}_\phi$. The result is a Schur Q-polynomial, which collectively comprise a basis for the space of polynomials in odd variables.

In section 1.5 we start from the BKP bilinear identity, and show that it gives rise to the infinitely many partial differential equations of the BKP hierarchy. Using the basis of Schur Q-polynomials, we write solutions of the BKP hierarchy as forms between the dual vacuum under time evolution and special states $g_\phi|0\rangle$ in $\mathcal{F}_\phi$. We derive the necessary and sufficient condition on g_ϕ to ensure BKP solubility, and call it the neutral fermionic bilinear identity (NFBI).

In section 1.6 we construct solutions of the NFBI. We prove that g_ϕ satisfies the NFBI if and only if $g_\phi \in O_\infty$. As an example, we write the strict partition basis vectors of $\mathcal{F}_\phi$ in this way, showing that every Schur Q-polynomial is a solution of the BKP hierarchy. The NFBI can also be solved through another perspective. Expanding $g_\phi|0\rangle$ in the canonical basis of $\mathcal{F}_\phi$, we show that the NFBI is satisfied if and only if the expansion coefficients obey the BKP Plücker relations. As an example, we give an explicit Pfaffian solution of the BKP Plücker relations.

1.1 Charged fermions and related definitions

1.1.1 Charged fermions

Consider two infinite sets $\{\psi_m\}_{m\in\mathbb{Z}}$ and $\{\psi^*_m\}_{m\in\mathbb{Z}}$, where m is assumed to run over all integers. The elements in these sets are called *charged fermions*. Each fermion ψ_m is assigned positive charge $(+1)$, while each fermion ψ^*_m is assigned negative charge (-1). The charged fermions obey the anticommutation relations

$$\begin{aligned}
[\psi_m, \psi_n]_+ &= [\psi^*_m, \psi^*_n]_+ = 0 \\
[\psi_m, \psi^*_n]_+ &= \delta_{m,n}
\end{aligned} \tag{1.1.1}$$

for all $m, n \in \mathbb{Z}$, where we have defined the anticommutator $[a, b]_+ = ab + ba$. These equations contain as a special case $\psi_m^2 = {\psi^*_m}^2 = 0$ for all $m \in \mathbb{Z}$, which is a defining property of fermions.

1.1.2 Clifford algebra Cl_ψ

The *Clifford algebra* Cl_ψ is the associative algebra generated by 1 and the charged fermions $\{\psi_m\}_{m\in\mathbb{Z}}$ and $\{\psi^*_m\}_{m\in\mathbb{Z}}$, modulo the anticommutation relations (1.1.1). Considered as a vector space, Cl_ψ has the basis

$$\text{Basis}(Cl_\psi) = \Big\{1, \psi_{m_1} \dots \psi_{m_r}, \psi^*_{n_s} \dots \psi^*_{n_1}, \psi_{m_1} \dots \psi_{m_r} \psi^*_{n_s} \dots \psi^*_{n_1}\Big\} \tag{1.1.2}$$

where $\{m_1 > \cdots > m_r\}$ and $\{n_s < \cdots < n_1\}$ range over all integers, and the cardinalities of these sets take all values $r, s \geq 1$. The Clifford algebra Cl_ψ splits into the following direct sum of subalgebras

$$Cl_\psi = \bigoplus_{i\in\mathbb{Z}} Cl_\psi^{(i)} \tag{1.1.3}$$

where $Cl_\psi^{(i)}$ is the linear span of all monomials comprised of r positive $(+1)$ fermions and s negative (-1) fermions, such that $r - s = i$. In this chapter we are mainly interested in the subalgebra $Cl_\psi^{(0)}$ which has the basis

$$\text{Basis}\left(Cl_\psi^{(0)}\right) = \Big\{1, \psi_{m_1} \dots \psi_{m_r} \psi^*_{n_r} \dots \psi^*_{n_1}\Big\} \tag{1.1.4}$$

where $\{m_1 > \cdots > m_r\}$ and $\{n_r < \cdots < n_1\}$ range over all integers, and the cardinality of these sets takes all values $r \geq 1$.

1.1.3 Fock representations of Cl_ψ

We introduce a *vacuum vector* $|0\rangle$ and *dual vacuum vector* $\langle 0|$, and define actions of Cl_ψ on them by setting

$$\psi_m|0\rangle = \psi_n^*|0\rangle = 0, \quad \langle 0|\psi_m^* = \langle 0|\psi_n = 0 \tag{1.1.5}$$

for all integers $m < 0, n \geq 0$. The *Fock space* $\mathcal{F}_\psi$ and *dual Fock space* $\mathcal{F}_\psi^*$ are the vector spaces generated linearly by the action of Cl_ψ on $|0\rangle$ and $\langle 0|$, respectively. By virtue of the annihilation relations (1.1.5), they have the bases

$$\text{Basis}(\mathcal{F}_\psi) = \Big\{ |0\rangle, \psi_{m_1} \dots \psi_{m_r}|0\rangle, \psi_{n_s}^* \dots \psi_{n_1}^*|0\rangle, \psi_{m_1} \dots \psi_{m_r}\psi_{n_s}^* \dots \psi_{n_1}^*|0\rangle \Big\} \tag{1.1.6}$$

where $\{m_1 > \dots > m_r \geq 0\}$ and $\{n_s < \dots < n_1 < 0\}$ range over all non-negative and negative integers, respectively, and the cardinalities of these sets take all values $r, s \geq 1$, and

$$\text{Basis}(\mathcal{F}_\psi^*) = \Big\{ \langle 0|, \langle 0|\psi_{m_1} \dots \psi_{m_r}, \langle 0|\psi_{n_s}^* \dots \psi_{n_1}^*, \langle 0|\psi_{m_1} \dots \psi_{m_r}\psi_{n_s}^* \dots \psi_{n_1}^* \Big\} \tag{1.1.7}$$

where $\{0 > m_1 > \dots > m_r\}$ and $\{0 \leq n_s < \dots < n_1\}$ range over all negative and non-negative integers, respectively, and the cardinalities of these sets take all values $r, s \geq 1$. The representations of Cl_ψ on the vector spaces (1.1.6) and (1.1.7) are called the *Fock representations.*

Using the definition of the Clifford subalgebras (1.1.3), we decompose $\mathcal{F}_\psi$ and $\mathcal{F}_\psi^*$ into the following direct sums of subspaces

$$\mathcal{F}_\psi = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_\psi^{(i)}, \quad \mathcal{F}_\psi^* = \bigoplus_{i \in \mathbb{Z}} \mathcal{F}_\psi^{*(i)} \tag{1.1.8}$$

where $\mathcal{F}_\psi^{(i)}$ and $\mathcal{F}_\psi^{*(i)}$ are the subspaces generated linearly by the action of $Cl_\psi^{(i)}$ on $|0\rangle$ and $\langle 0|$, respectively. The bases of $\mathcal{F}_\psi^{(0)}$ and $\mathcal{F}_\psi^{*(0)}$ are given by

$$\text{Basis}\left(\mathcal{F}_\psi^{(0)}\right) = \Big\{ |0\rangle, \psi_{m_1} \dots \psi_{m_r}\psi_{n_r}^* \dots \psi_{n_1}^*|0\rangle \Big\} \tag{1.1.9}$$

where $\{m_1 > \dots > m_r \geq 0\}$ and $\{n_r < \dots < n_1 < 0\}$ range over all non-negative and negative integers, respectively, and the cardinality of these sets takes all values $r \geq 1$, and

$$\text{Basis}\left(\mathcal{F}_\psi^{*(0)}\right) = \Big\{ \langle 0|, \langle 0|\psi_{m_1} \dots \psi_{m_r}\psi_{n_r}^* \dots \psi_{n_1}^* \Big\} \tag{1.1.10}$$

where $\{0 > m_1 > \dots > m_r\}$ and $\{0 \leq n_r < \dots < n_1\}$ range over all negative and non-negative integers, respectively, and the cardinality of these sets takes all values $r \geq 1$.

Remark 1. In future sections we will occasionally exploit the notation

$$\psi_{\{m\}} = \psi_{m_1} \dots \psi_{m_r}, \qquad \psi^*_{\{n\}} = \psi^*_{n_s} \dots \psi^*_{n_1} \tag{1.1.11}$$

for all ordered sets $\{m\} = \{m_1 > \cdots > m_r\}$ and $\{n\} = \{n_s < \cdots < n_1\}$ with cardinality $r, s \geq 1$. We also define $\psi_{\{m\}} = \psi^*_{\{n\}} = 1$ when $\{m\}$ and $\{n\}$ are empty. For example, using this notation the basis (1.1.9) can be written as

$$\text{Basis}\left(\mathcal{F}^{(0)}_{\psi}\right) = \left\{\psi_{\{m\}}\psi^*_{\{n\}}|0\rangle \;\middle|\; \text{card}\{m\} = \text{card}\{n\}\right\} \tag{1.1.12}$$

where $\{m\}$ and $\{n\}$ range over all ordered sets of non-negative and negative integers, with the same cardinality. Similarly the basis (1.1.10) can be written as

$$\text{Basis}\left(\mathcal{F}^{*(0)}_{\psi}\right) = \left\{\langle 0|\psi_{\{m\}}\psi^*_{\{n\}} \;\middle|\; \text{card}\{m\} = \text{card}\{n\}\right\} \tag{1.1.13}$$

where $\{m\}$ and $\{n\}$ range over all ordered sets of negative and non-negative integers, with the same cardinality.

1.1.4 Partition basis

For all integers $l \geq 1$ we define the *charged vacua*

$$|l\rangle = \psi_{l-1} \dots \psi_0 |0\rangle, \quad |-l\rangle = \psi^*_{-l} \dots \psi^*_{-1}|0\rangle \tag{1.1.14}$$

and the *dual charged vacua*

$$\langle l| = \langle 0|\psi^*_0 \dots \psi^*_{l-1}, \quad \langle -l| = \langle 0|\psi_{-1} \dots \psi_{-l} \tag{1.1.15}$$

It is possible to express the elements of the bases (1.1.9) and (1.1.10) in terms of the charged vacua, as we see in the following lemma.

Lemma 1. The Fock subspaces $\mathcal{F}^{(0)}_{\psi}$ and $\mathcal{F}^{*(0)}_{\psi}$ have the equivalent bases

$$\text{Basis}\left(\mathcal{F}^{(0)}_{\psi}\right) = \left\{|0\rangle, \psi_{m_1} \dots \psi_{m_l}|-l\rangle\right\} \tag{1.1.16}$$

where $\{m_1 > \cdots > m_l > -l\}$ range over all integers greater than $-l$, and all values $l \geq 1$ are allowed, and

$$\text{Basis}\left(\mathcal{F}^{*(0)}_{\psi}\right) = \left\{\langle 0|, \langle -l|\psi^*_{n_l} \dots \psi^*_{n_1}\right\} \tag{1.1.17}$$

where $\{-l < n_l < \cdots < n_1\}$ range over all integers greater than $-l$, and all values $l \geq 1$ are allowed.

Proof. Firstly, we show that every element $\psi_{m_1} \dots \psi_{m_l}|-l\rangle$ of (1.1.16) is an element of (1.1.9), up to a minus sign. When $l = 1$ we have $\psi_{m_1}|-1\rangle = \psi_{m_1}\psi^*_{-1}|0\rangle$, which is clearly an element of (1.1.9). For $l > 1$, there exists some $r \geq 1$ such that

$$\psi_{m_1} \dots \psi_{m_l}|-l\rangle = \psi_{m_1} \dots \psi_{m_r}\psi_{m_{r+1}} \dots \psi_{m_l}\psi^*_{-l} \dots \psi^*_{-1}|0\rangle \tag{1.1.18}$$

with $\{m_1 > \dots > m_r \geq 0 > m_{r+1} > \dots > m_l\}$. Let $\{n_r < \dots < n_1\}$ be the set $\{-l < \dots < -1\}$ with the integers $\{m_l < \dots < m_{r+1}\}$ omitted. Using the anticommutation relations (1.1.1) and the fact that the string of fermions $\psi_{m_{r+1}} \dots \psi_{m_l}$ annihilates the vacuum $|0\rangle$, equation (1.1.18) becomes

$$\psi_{m_1} \dots \psi_{m_l}|-l\rangle = (-)^{\sum_{i=r+1}^{l}(m_i+i)}\psi_{m_1} \dots \psi_{m_r}\psi^*_{n_r} \dots \psi^*_{n_1}|0\rangle \tag{1.1.19}$$

where the right hand side is an element of (1.1.9), up to an irrelevant minus sign. Secondly, we show that every element $\psi_{m_1} \dots \psi_{m_r}\psi^*_{n_r} \dots \psi^*_{n_1}|0\rangle$ of (1.1.9) is an element of (1.1.16), up to a minus sign. Defining $-l = n_r$, we have

$$\psi_{m_1} \dots \psi_{m_r}\psi^*_{n_r} \dots \psi^*_{n_1}|0\rangle = \psi_{m_1} \dots \psi_{m_r}\psi^*_{n_r} \dots \psi^*_{n_1}\psi_{-1} \dots \psi_{-l}|-l\rangle \tag{1.1.20}$$

and we define $\{m_{r+1} > \dots > m_l\}$ to be the set $\{-1 > \dots > -l\}$ with the omission of $\{n_1 > \dots > n_r\}$. Using the anticommutation relations (1.1.1) and the fact that the string of fermions $\psi^*_{n_r} \dots \psi^*_{n_1}$ annihilates the charged vacuum $|-l\rangle$, equation (1.1.20) becomes

$$\psi_{m_1} \dots \psi_{m_r}\psi^*_{n_r} \dots \psi^*_{n_1}|0\rangle = (-)^{\sum_{i=1}^{r}(n_i+i)}\psi_{m_1} \dots \psi_{m_r}\psi_{m_{r+1}} \dots \psi_{m_l}|-l\rangle \tag{1.1.21}$$

where the right hand side is an element of (1.1.16), up to an irrelevant minus sign. This concludes the proof that the elements of the bases (1.1.9) and (1.1.16) are in one-to-one correspondence. The proof that the elements of the bases (1.1.10) and (1.1.17) are in one-to-one correspondence proceeds along completely analogous lines. □

A *partition* $\mu = \{\mu_1 \geq \dots \geq \mu_l > \mu_{l+1} = \dots = 0\}$ is a set of weakly decreasing non-negative integers, of which finitely many are non-zero. We refer to each integer μ_i as a *part* of μ, and to the sum of all parts

$$|\mu| = \sum_{i=1}^{l} \mu_i \tag{1.1.22}$$

as the *weight* of μ. The number l of non-zero parts is called the *length* of μ, and denoted by $\ell(\mu)$.

Every partition has a pictorial representation called a *Young diagram*. This is a collection of l rows of boxes, such that the i^{th} row is μ_i boxes long. Throughout the thesis, we will use the notation $[n, m]$ to denote a rectangular Young diagram with n rows and m columns. A partition μ will be said to satisfy $\mu \subseteq [n, m]$ if its Young diagram fits inside the rectangle $[n, m]$. For more information on the theory of partitions, we refer the reader to section 1 in chapter I of [65].

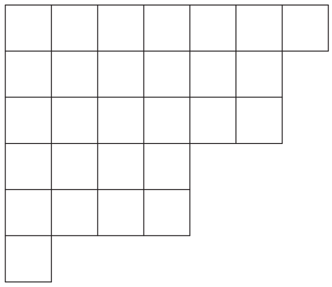

Figure 1.1: Young diagram of the partition $\mu = \{7, 6, 6, 4, 4, 1\}$. In this case $|\mu| = 7 + 6 + 6 + 4 + 4 + 1 = 28$ and $\ell(\mu) = 6$. This partition satisfies $\mu \subset [6, 7]$.

It is possible to match the elements of the bases (1.1.16) and (1.1.17) with partitions. We define $|0\rangle = |\emptyset)$ and $\langle 0| = (\emptyset|$, and for all sets of integers $\{m_1 > \cdots > m_l > -l\}$ we write

$$\psi_{m_1} \dots \psi_{m_l} |-l\rangle = |\mu_1, \dots, \mu_l) = |\mu), \quad \langle -l|\psi^*_{m_l} \dots \psi^*_{m_1} = (\mu_1, \dots, \mu_l| = (\mu| \tag{1.1.23}$$

where $\mu_i = m_i + i$ for all $1 \leq i \leq l$. Under this identification, we let $|\emptyset)$ and $(\emptyset|$ be copies of the *empty partition* $\emptyset$, and $|\mu)$ and $(\mu|$ be copies of the partition $\mu = \{\mu_1 \geq \cdots \geq \mu_l > 0\}$. This gives a one-to-one correspondence between the elements of the bases (1.1.16) and (1.1.17) and the elements of the set of all partitions. This will prove useful in later sections, when we encounter functions which are naturally indexed by partitions.

1.1.5 Charged fermion expectation values

For arbitrary $g \in Cl_\psi$ we define its *vacuum expectation value* $\langle g \rangle \in \mathbb{C}$ by

$$\langle g \rangle = \langle 0|g|0 \rangle \tag{1.1.24}$$

where it is assumed that $\langle 0|1|0\rangle = 1$ and

$$\langle 0|(g_1 + g_2)|0\rangle = \langle 0|g_1|0\rangle + \langle 0|g_2|0\rangle, \quad \langle 0|\kappa g_1|0\rangle = \kappa \langle 0|g_1|0\rangle \tag{1.1.25}$$

for all $g_1, g_2 \in Cl_\psi$ and $\kappa \in \mathbb{C}$. It is straightforward to show that the anticommutation relations (1.1.1), the annihilation actions (1.1.5) and the conditions (1.1.25) define the value of $\langle g \rangle$ completely and unambiguously.

Lemma 2. Let $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_r\}$ be two arbitrary sets of integers. We claim that

$$\langle \psi_{m_1} \ldots \psi_{m_r} \psi^*_{n_r} \ldots \psi^*_{n_1} \rangle = \det \Big(\langle \psi_{m_i} \psi^*_{n_j} \rangle \Big)_{1 \leq i,j \leq r} \tag{1.1.26}$$

This is a special case of *Wick's theorem*, see chapter 4 of [71].

Proof. The case $r = 1$ is trivial, since

$$\langle \psi_{m_1} \psi^*_{n_1} \rangle = \det \Big(\langle \psi_{m_i} \psi^*_{n_j} \rangle \Big)_{i,j=1} \tag{1.1.27}$$

Using this case as the basis for induction, we assume there exists $r \geq 2$ such that

$$\langle \psi_{m_1} \ldots \psi_{m_{r-1}} \psi^*_{n_{r-1}} \ldots \psi^*_{n_1} \rangle = \det \Big(\langle \psi_{m_i} \psi^*_{n_j} \rangle \Big)_{1 \leq i,j \leq r-1} \tag{1.1.28}$$

for all sets of integers $\{m_1, \ldots, m_{r-1}\}$ and $\{n_1, \ldots, n_{r-1}\}$. We define

$$\begin{aligned} I_1 &= \langle \psi_{m_1} \ldots \psi_{m_r} \psi^*_{n_r} \ldots \psi^*_{n_1} \rangle \\ I_2 &= \sum_{k=1}^{r} (-)^{k+1} \langle \psi_{m_1} \psi^*_{n_k} \rangle \langle \psi_{m_2} \ldots \psi_{m_r} \psi^*_{n_r} \ldots \widehat{\psi^*_{n_k}} \ldots \psi^*_{n_1} \rangle \end{aligned} \tag{1.1.29}$$

where $\{m_1, \ldots, m_r\}$ and $\{n_1, \ldots, n_r\}$ are arbitrary sets of integers, and $\widehat{\psi^*_{n_k}}$ means the omission of the indicated fermion. By the annihilation properties (1.1.5) of the fermions, $I_1 = I_2 = 0$ unless both $m_1 < 0$ and $m_1 = \{n_{k_1}, \ldots, n_{k_l}\}$ for some $1 \leq k_1 < \cdots < k_l \leq r$. When both these conditions are satisfied it is readily verified that

$$I_1 = I_2 = \sum_{k \in \{k_1, \ldots, k_l\}} (-)^{k+1} \langle \psi_{m_1} \psi^*_{n_k} \rangle \langle \psi_{m_2} \ldots \psi_{m_r} \psi^*_{n_r} \ldots \widehat{\psi^*_{n_k}} \ldots \psi^*_{n_1} \rangle \tag{1.1.30}$$

which proves that $I_1 = I_2$ for all m_1. Therefore we obtain

$$\begin{aligned} I_1 &= \sum_{k=1}^{r} (-)^{k+1} \langle \psi_{m_1} \psi^*_{n_k} \rangle \langle \psi_{m_2} \ldots \psi_{m_r} \psi^*_{n_r} \ldots \widehat{\psi^*_{n_k}} \ldots \psi^*_{n_1} \rangle \\ &= \sum_{k=1}^{r} (-)^{k+1} \langle \psi_{m_1} \psi^*_{n_k} \rangle \det \Big(\langle \psi_{m_i} \psi^*_{n_j} \rangle \Big)_{2 \leq i \leq r, j \neq k} = \det \Big(\langle \psi_{m_i} \psi^*_{n_j} \rangle \Big)_{1 \leq i,j \leq r} \end{aligned} \tag{1.1.31}$$

where we used the assumption (1.1.28) to produce the $(r-1) \times (r-1)$ determinant in the second line of (1.1.31). This completes the proof by induction. □

We define a *bilinear form* $\langle , \rangle$ which maps $\mathcal{F}_\psi^* \times \mathcal{F}_\psi \rightarrow \mathbb{C}$. Its action on the arbitrary vectors $\langle 0|g_1 \in \mathcal{F}_\psi^*$ and $g_2|0\rangle \in \mathcal{F}_\psi$ is given by

$$\Big\langle \langle 0|g_1, g_2|0\rangle \Big\rangle = \Big\langle g_1 g_2 \Big\rangle \tag{1.1.32}$$

Let $(\mu| = \langle -l|\psi_{m_l}^* \ldots \psi_{m_1}^*$ and $|\nu) = \psi_{n_1} \ldots \psi_{n_k}|-k\rangle$ be partition vectors. By a straightforward calculation, we obtain

$$\Big\langle (\mu|, |\nu) \Big\rangle = \langle -l|\psi_{m_l}^* \ldots \psi_{m_1}^* \psi_{n_1} \ldots \psi_{n_k}|-k\rangle = \delta_{k,l} \prod_{i=1}^{l} \delta_{m_i,n_i} = \delta_{\mu,\nu} \tag{1.1.33}$$

In other words, the bilinear form (1.1.32) induces orthonormality between the partition elements of $\mathcal{F}_\psi^{*(0)}$ and $\mathcal{F}_\psi^{(0)}$, which confirms that these vector spaces are genuinely dual.

1.1.6 Lie algebra A_∞

Let $A_\infty \subset Cl_\psi^{(0)}$ be the vector space whose elements $X \in A_\infty$ are given by

$$X = \sum_{i,j \in \mathbb{Z}} a_{i,j} : \psi_i \psi_j^* : + \kappa \tag{1.1.34}$$

where we have defined the normal ordering $: \psi_i \psi_j^* := \psi_i \psi_j^* - \langle 0|\psi_i \psi_j^*|0\rangle$, and where the coefficients satisfy $a_{i,j} = 0$ for $|i-j|$ sufficiently large, with $\kappa \in \mathbb{C}$.

Lemma 3. The vector space A_∞ becomes a Lie algebra when it is equipped with the commutator as Lie bracket.

Proof. We prove the closure of A_∞ under commutation. Let X be given by (1.1.34) and define

$$X_1 = \sum_{i,j \in \mathbb{Z}} a_{i,j}^{(1)} : \psi_i \psi_j^* : + \kappa_1 \tag{1.1.35}$$

where $a_{i,j}^{(1)} = 0$ for $|i-j|$ sufficiently large. By a simple calculation, we obtain

$$[X, X_1] = \sum_{i,j \in \mathbb{Z}} a_{i,j}^{(2)} : \psi_i \psi_j^* : + \kappa_2 \tag{1.1.36}$$

where for all $i, j \in \mathbb{Z}$ we have defined

$$a_{i,j}^{(2)} = \sum_{k\in\mathbb{Z}} \left(a_{i,k} a_{k,j}^{(1)} - a_{i,k}^{(1)} a_{k,j} \right) \tag{1.1.37}$$

and where

$$\kappa_2 = \left(\sum_{i<0,j\geq 0} - \sum_{i\geq 0,j<0} \right) a_{i,j} a_{j,i}^{(1)} \tag{1.1.38}$$

Clearly the right hand side of equation (1.1.36) is also an element of A_∞, proving the closure of the set under commutation. □

1.1.7 A_∞ Heisenberg subalgebra

We define operators $H_m \in A_\infty$ which are given by

$$H_m = \sum_{i\in\mathbb{Z}} : \psi_i \psi_{i+m}^* : \tag{1.1.39}$$

for all $m \in \mathbb{Z}$. Together with the central element 1, the operators $\{H_m\}_{m\in\mathbb{Z}}$ generate a Heisenberg subalgebra. The closure of this set follows from the commutation relation

$$[H_m, H_n] = m\delta_{m+n,0} \tag{1.1.40}$$

for all $m, n \in \mathbb{Z}$.[1] The Heisenberg generators (1.1.39) also have simple commutation relations with the charged fermions (1.1.1), given by

$$[H_m, \psi_n] = \psi_{n-m}, \quad [H_m, \psi_n^*] = -\psi_{m+n}^* \tag{1.1.41}$$

for all $m, n \in \mathbb{Z}$.

1.1.8 KP evolution operators

We introduce the Hamiltonian

$$H\{t\} = \sum_{n=1}^{\infty} t_n H_n \tag{1.1.42}$$

[1] For a detailed proof of (1.1.40), see the exercise 5.1 in chapter 4 of [71]. Alternatively, one can notice that H_m, H_n are obtained from X, X_1 by setting $a_{i,j} = \delta_{i+m,j}$, $a_{i,j}^{(1)} = \delta_{i+n,j}$ and $\kappa = \kappa_1 = 0$. Substituting these values into (1.1.37) and (1.1.38), we obtain the desired commutator (1.1.40) by virtue of (1.1.36).

where $\{t\} = \{t_1, t_2, t_3, \ldots\}$ is an infinite set of free variables, and define the generating functions

$$\Psi(k) = \sum_{i\in\mathbb{Z}} \psi_i k^i, \quad \Psi^*(k) = \sum_{i\in\mathbb{Z}} \psi_i^* k^{-i} \tag{1.1.43}$$

where k is an indeterminate. Using these definitions and the equations (1.1.41), we obtain the commutation relations

$$[H\{t\}, \Psi(k)] = \sum_{n=1}^{\infty}\sum_{i\in\mathbb{Z}} [H_n, \psi_i] t_n k^i = \left(\sum_{n=1}^{\infty} t_n k^n\right) \Psi(k) \tag{1.1.44}$$

$$[H\{t\}, \Psi^*(k)] = \sum_{n=1}^{\infty}\sum_{i\in\mathbb{Z}} [H_n, \psi_i^*] t_n k^{-i} = -\left(\sum_{n=1}^{\infty} t_n k^n\right) \Psi^*(k) \tag{1.1.45}$$

which, in turn, imply that

$$e^{H\{t\}}\Psi(k) = \exp\left(\sum_{n=1}^{\infty} t_n k^n\right) \Psi(k) e^{H\{t\}} \tag{1.1.46}$$

$$e^{H\{t\}}\Psi^*(k) = \exp\left(-\sum_{n=1}^{\infty} t_n k^n\right) \Psi^*(k) e^{H\{t\}} \tag{1.1.47}$$

Following the terminology of [50], the operator $e^{H\{t\}}$ in this pair of equations is called a *KP evolution operator*. As we shall see, the KP evolution operator plays an essential role in constructing solutions of the KP hierarchy of partial differential equations.

1.1.9 Schur polynomials

For all $m \geq 0$, the *one-row Schur polynomial* $\chi_m\{t\}$ in the infinite set of variables $\{t\}$ is defined by

$$\chi_m\{t\} = \mathrm{Coeff}_{k^m}\left[\exp\left(\sum_{n=1}^{\infty} t_n k^n\right)\right] \tag{1.1.48}$$

where $\mathrm{Coeff}_{k^m}[f(k)]$ denotes the coefficient of k^m in the Taylor series expansion of $f(k)$. We also define $\chi_m\{t\} = 0$ when $m < 0$. From this definition, the *Schur polynomial* [2] $\chi_\mu\{t\}$ associated to the arbitrary partition $\mu = \{\mu_1 \geq \cdots \geq \mu_l \geq 0\}$ is given by

[2] The polynomials χ_μ are sometimes called *character polynomials,* in reference to their connection with the linear representations of the symmetric groups, see section 7 in chapter I of [65].

$$\chi_\mu\{t\} = \det\left(\chi_{(\mu_i - i + j)}\{t\}\right)_{1\le i,j\le l} \tag{1.1.49}$$

where $\chi_{(\mu_i-i+j)}\{t\}$ represents a one-row Schur polynomial. The following result maps the partition elements of $\mathcal{F}_\psi^{(0)}$ to their corresponding Schur polynomials.

Lemma 4. Let $|\mu) = |\mu_1, \ldots, \mu_l) = \psi_{m_1} \ldots \psi_{m_l}| - l\rangle$ be an element of the partition basis of $\mathcal{F}_\psi^{(0)}$, where $\mu_i = m_i + i$ for all $1 \le i \le l$. We claim that

$$\chi_\mu\{t\} = (\emptyset|e^{H\{t\}}|\mu) \tag{1.1.50}$$

Proof. Using the definition of the one-row Schur polynomials (1.1.48) in the commutation relation (1.1.46) and then extracting the coefficient of k^m from the resulting equation, we obtain

$$e^{H\{t\}}\psi_m = \left(\sum_{i=0}^{\infty} \psi_{(m-i)}\chi_i\{t\}\right) e^{H\{t\}} \tag{1.1.51}$$

By definition, we have

$$(\emptyset|e^{H\{t\}}|\mu) = \langle 0|e^{H\{t\}}\psi_{m_1}\ldots\psi_{m_l}| - l\rangle \tag{1.1.52}$$

and using the commutation relation(1.1.51) we move the evolution operator $e^{H\{t\}}$ in (1.1.52) towards the right, obtaining

$$(\emptyset|e^{H\{t\}}|\mu) = \sum_{i_1,\ldots,i_l=0}^{\infty} \left\langle \psi_{(m_1-i_1)}\ldots\psi_{(m_l-i_l)}\psi^*_{-l}\ldots\psi^*_{-1}\right\rangle \chi_{i_1}\{t\}\ldots\chi_{i_l}\{t\} \tag{1.1.53}$$

where we have used the fact that $e^{H\{t\}}| - l\rangle = | - l\rangle$. Applying lemma 2 to the previous vacuum expectation value, we find

$$(\emptyset|e^{H\{t\}}|\mu) = \sum_{i_1,\ldots,i_l=0}^{\infty} \det\left(\langle\psi_{(m_p-i_p)}\psi^*_{-q}\rangle\right)_{1\le p,q\le l} \chi_{i_1}\{t\}\ldots\chi_{i_l}\{t\} \tag{1.1.54}$$

Collecting the one-row Schur polynomial $\chi_{i_p}\{t\}$ as a factor multiplying the p^{th} row of the determinant, we obtain

$$(\emptyset|e^{H\{t\}}|\mu) = \det\left(\sum_{i_p=0}^{\infty}\langle\psi_{(m_p-i_p)}\psi^*_{-q}\rangle\chi_{i_p}\{t\}\right)_{1\le p,q\le l} \tag{1.1.55}$$

Using the anticommutation relations (1.1.1), the annihilation properties (1.1.5), and the fact that $\chi_m\{t\} = 0$ for all $m < 0$, equation (1.1.55) becomes

$$(\emptyset|e^{H\{t\}}|\mu) = \det\left(\sum_{i_p\in\mathbb{Z}} \delta_{i_p,(m_p+q)}\chi_{i_p}\{t\}\right)_{1\le p,q\le l} \tag{1.1.56}$$

Finally, using the Kronecker delta to truncate the sum occurring in the previous determinant, we obtain

$$(\emptyset|e^{H\{t\}}|\mu) = \det\Big(\chi_{(m_p+q)}\{t\}\Big)_{1\le p,q\le l} = \det\Big(\chi_{(\mu_p-p+q)}\{t\}\Big)_{1\le p,q\le l} \tag{1.1.57}$$

and the final determinant is the Schur polynomial $\chi_\mu\{t\}$. □

Lemma 5. Let $(\mu\pm 1) = \Big\{(\mu_1\pm 1)\ge\cdots\ge(\mu_l\pm 1)\ge 0\Big\}$ be a pair of partitions, and set $m_i = \mu_i - i$ for all $1\le i\le l$. We claim that

$$\chi_{(\mu+1)}\{t\} = \det\Big(\chi_{(\mu_i-i+j+1)}\{t\}\Big)_{1\le i,j\le l} = \langle -1|e^{H\{t\}}\psi_{m_1}\ldots\psi_{m_l}|-l-1\rangle \tag{1.1.58}$$

$$\chi_{(\mu-1)}\{t\} = \det\Big(\chi_{(\mu_i-i+j-1)}\{t\}\Big)_{1\le i,j\le l} = \langle 1|e^{H\{t\}}\psi_{m_1}\ldots\psi_{m_l}|-l+1\rangle \tag{1.1.59}$$

Proof. Using the commutation relation (1.1.51) we move the evolution operator $e^{H\{t\}}$ in (1.1.58) and (1.1.59) towards the right, obtaining

$$\chi_{(\mu+1)}\{t\} = \Big\langle\psi_{-1}\psi_{(m_1-i_1)}\ldots\psi_{(m_l-i_l)}\psi^*_{(-l-1)}\ldots\psi^*_{-1}\Big\rangle\chi_{i_1}\{t\}\ldots\chi_{i_l}\{t\} \tag{1.1.60}$$

$$\chi_{(\mu-1)}\{t\} = \Big\langle\psi^*_0\psi_{(m_1-i_1)}\ldots\psi_{(m_l-i_l)}\psi^*_{(-l+1)}\ldots\psi^*_{-1}\Big\rangle\chi_{i_1}\{t\}\ldots\chi_{i_l}\{t\} \tag{1.1.61}$$

where in both cases summation over all $\{0\le i_1,\ldots,i_l<\infty\}$ is implied. Using the anticommutation relations (1.1.1) to move ψ_{-1} and ψ^*_0 towards the right in their respective equations (1.1.60) and (1.1.61), and then applying lemma 2, we find

$$\chi_{(\mu+1)}\{t\} = \det\Big(\sum_{i_p\in\mathbb{Z}}\langle\psi_{(m_p-i_p)}\psi^*_{(-q-1)}\rangle\chi_{i_p}\{t\}\Big)_{1\le p,q\le l} \tag{1.1.62}$$

$$\chi_{(\mu-1)}\{t\} = \sum_{j=1}^{l}(-)^{j+1}\chi_{m_j}\{t\}\det\Big(\sum_{i_p\in\mathbb{Z}}\langle\psi_{(m_p-i_p)}\psi^*_{(-q+1)}\rangle\chi_{i_p}\{t\}\Big)_{p\ne j,2\le q\le l} \tag{1.1.63}$$

with $\chi_m\{t\} = 0$ for all $m < 0$. Finally, evaluating the expectation values within these determinants explicitly, we recover

$$\chi_{(\mu+1)}\{t\} = \det\Big(\chi_{(m_p+q+1)}\{t\}\Big)_{1\le p,q\le l} = \det\Big(\chi_{(\mu_p-p+q+1)}\{t\}\Big)_{1\le p,q\le l} \tag{1.1.64}$$

$$\begin{aligned}\chi_{(\mu-1)}\{t\} &= \sum_{j=1}^{l}(-)^{j+1}\chi_{m_j}\{t\}\det\Big(\chi_{(m_p+q-1)}\{t\}\Big)_{p\neq j,2\le q\le l} \\ &= \det\Big(\chi_{(m_p+q-1)}\{t\}\Big)_{1\le p,q\le l} = \det\Big(\chi_{(\mu_p-p+q-1)}\{t\}\Big)_{1\le p,q\le l}\end{aligned} \tag{1.1.65}$$

where the second line of (1.1.65) follows from the expansion of an $l \times l$ determinant down its first column. □

1.1.10 Schur functions

Following section 2 in chapter I of [65], the *complete symmetric function* $h_m\{x\}$ in the infinite set of variables $\{x\} = \{x_1, x_2, x_3, \ldots\}$ is defined as

$$h_m\{x\} = \text{Coeff}_{k^m}\left[\prod_{i=1}^{\infty}\frac{1}{1-x_i k}\right] \tag{1.1.66}$$

From this definition, the *Schur function*[3] $s_\mu\{x\}$ associated to the partition $\mu = \{\mu_1 \ge \cdots \ge \mu_l \ge 0\}$ is given by

$$s_\mu\{x\} = \det\Big(h_{(\mu_i-i+j)}\{x\}\Big)_{1\le i,j\le l} \tag{1.1.67}$$

Lemma 6. The Schur polynomial $\chi_\mu\{t\}$ and the Schur function $s_\mu\{x\}$ are equal under the change of variables $t_n = \frac{1}{n}\sum_{i=1}^{\infty} x_i^n$ for all $n \ge 1$.

Proof. Fixing $t_n = \frac{1}{n}\sum_{i=1}^{\infty} x_i^n$ for all $n \ge 1$, we obtain

$$\text{Coeff}_{k^m}\left[\exp\left(\sum_{n=1}^{\infty} t_n k^n\right)\right] = \text{Coeff}_{k^m}\left[\exp\left(\sum_{i=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}(x_i k)^n\right)\right] = \text{Coeff}_{k^m}\left[\prod_{i=1}^{\infty}\frac{1}{1-x_i k}\right] \tag{1.1.68}$$

implying that $\chi_m\{t\} = h_m\{x\}$ for all $m \ge 0$. Therefore under the prescribed change of variables we have

$$\det\Big(\chi_{(\mu_i-i+j)}\{t\}\Big)_{1\le i,j\le l} = \det\Big(h_{(\mu_i-i+j)}\{x\}\Big)_{1\le i,j\le l} \tag{1.1.69}$$

which completes the proof. □

[3] Notice that we use the words *polynomial* and *function* to distinguish between $\chi_\mu\{t\}$ and $s_\mu\{x\}$. Unlike $\chi_\mu\{t\}$, which always depends on finitely many of the variables $\{t\}$, the function $s_\mu\{x\}$ can depend on infinitely many variables $\{x\}$, and it would be improper to call it a polynomial.

It is well known that the Schur functions $s_\mu\{x\}$ comprise a basis for the ring of symmetric functions in $\{x\}$.[4] This fact, together with the equality of $\chi_\mu\{t\}$ and $s_\mu\{x\}$ under the previous change of variables, proves that the Schur polynomials $\chi_\mu\{t\}$ are a basis for the set of all polynomials in $\{t\}$.

Throughout the rest of the thesis, we will usually consider Schur functions $s_\mu\{x\}$ in a finite number of variables $\{x\} = \{x_1, \ldots, x_N\}$. These are obtained from the formulae (1.1.66), (1.1.67) by setting $x_n = 0$ for all $n > N$. In these cases, $s_\mu\{x\} = 0$ if $\ell(\mu) > N$, which is explained in section 3, chapter I of [65].

1.2 KP hierarchy

1.2.1 KP hierarchy in bilinear form

The *KP hierarchy* is an infinite set of partial differential equations in infinitely many independent variables $\{t\} = \{t_1, t_2, t_3, \ldots\}$. This hierarchy of differential equations actually derives from a single integral equation, called the *KP bilinear identity.* A function $\tau\{t\}$ which satisfies every differential equation in the hierarchy, or equivalently, satisfies the KP bilinear identity, is called a *KP τ-function.*

Define the shifted sets of variables

$$\{t \pm \epsilon_k\} = \left\{ t_1 \pm k^{-1}, t_2 \pm \frac{1}{2}k^{-2}, t_3 \pm \frac{1}{3}k^{-3}, \ldots \right\} \tag{1.2.1}$$

where k is a free parameter. Using this notation, $\tau\{t \pm \epsilon_k\}$ should be understood as a function with n^{th} argument $t_n \pm \frac{1}{n}k^{-n}$. The KP bilinear identity is the equation

$$\oint \exp\left(\sum_{n=1}^{\infty}(t_n - s_n)k^n\right)\tau\{t - \epsilon_k\}\tau\{s + \epsilon_k\}\frac{dk}{2\pi i} = 0 \tag{1.2.2}$$

where $\{t\} = \{t_1, t_2, t_3, \ldots\}$ and $\{s\} = \{s_1, s_2, s_3, \ldots\}$ are two infinite sets of variables, and the integration in the k-plane is taken around a small contour at $k = \infty$. Equivalently, this equation says that the sum of residues of the integrand in the k-plane is equal to zero.

In this thesis we will always assume that $\tau\{t\}$ is a polynomial in its variables. In this special case, $\tau\{t - \epsilon_k\}$ and $\tau\{s + \epsilon_k\}$ have singularities in k only at $k = 0$. This means that the integral (1.2.2) is equal to the coefficient of k^{-1} in the Laurent series of the integrand. Therefore, for polynomial τ-functions, the KP bilinear identity becomes

$$\text{Coeff}_{k^{-1}}\left[\exp\left(\sum_{n=1}^{\infty}(t_n - s_n)k^n\right)\tau\{t - \epsilon_k\}\tau\{s + \epsilon_k\}\right] = 0 \tag{1.2.3}$$

[4] See section 3 in chapter I of [65].

where $\text{Coeff}_{k^{-1}}[f(k)]$ denotes the coefficient of k^{-1} in the Laurent series of $f(k)$. We expose the infinitely many differential equations which underly the equation (1.2.3) by making the substitutions $\{t\} \rightarrow \{t-s\}$ and $\{s\} \rightarrow \{t+s\}$, giving

$$\text{Coeff}_{k^{-1}}\left[\exp\left(-2\sum_{n=1}^{\infty} s_n k^n\right)\tau\{t-s-\epsilon_k\}\tau\{t+s+\epsilon_k\}\right] = 0 \tag{1.2.4}$$

Using the definition (1.1.48), the exponential term in (1.2.4) can be replaced with a sum over one-row Schur polynomials, producing the equation

$$\text{Coeff}_{k^{-1}}\left[\sum_{m=0}^{\infty} \chi_m\{-2s\}k^m \tau\{t-s-\epsilon_k\}\tau\{t+s+\epsilon_k\}\right] = 0 \tag{1.2.5}$$

or equivalently,

$$\sum_{m=0}^{\infty} \chi_m\{-2s\}\text{Coeff}_{k^{-m-1}}\Big[\tau\{t-s-\epsilon_k\}\tau\{t+s+\epsilon_k\}\Big] = 0 \tag{1.2.6}$$

In order to progress from this last equation to an infinite set of differential equations, we need to introduce the notion of bilinear differential operators.

Definition 1. The *bilinear differential operators*[5] $\{D\} = \{D_1, D_2, D_3, \dots\}$ act on *ordered pairs* of functions $f\{t\}\cdot g\{t\}$. Letting $P\{D\}$ denote an arbitrary polynomial combination of these operators, we define

$$P\{D\}f\{t\}\cdot g\{t\} = P\{\partial_z\}\Big(f\{t+z\}g\{t-z\}\Big)_{\{z\}\rightarrow\{0\}} \tag{1.2.7}$$

where $\{\partial_z\} = \{\partial_{z_1}, \partial_{z_2}, \partial_{z_3}, \dots\}$ and $\{t\pm z\} = \{t_1 \pm z_1, t_2 \pm z_2, t_3 \pm z_3, \dots\}$, and where we have set $\{z\} \rightarrow \{0\}$ after differentiation.

Lemma 7.

$$\tau\{t-s-\epsilon_k\}\tau\{t+s+\epsilon_k\} = \exp\left(\sum_{n=1}^{\infty}(s_n + \frac{1}{n}k^{-n})D_n\right)\tau\{t\}\cdot\tau\{t\} \tag{1.2.8}$$

Proof. For an arbitrary function $f(z)$ and $\kappa \in \mathbb{C}$, we have the Taylor series identity

$$f(\kappa) = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!}\kappa^n = \Big(e^{\kappa\partial_z}f(z)\Big)_{z\rightarrow 0} \tag{1.2.9}$$

[5] The definition of these operators is due to R Hirota, see [47].

Extending this identity to infinitely many variables, we find

$$\begin{aligned}\tau\{t+s+\epsilon_k\}\tau\{t-s-\epsilon_k\} &= \exp\left(\sum_{n=1}^{\infty}\kappa_n\partial_{z_n}\right)\Big(\tau\{t+z\}\tau\{t-z\}\Big)_{\{z\}\to\{0\}} \\ &= \exp\left(\sum_{n=1}^{\infty}\kappa_n D_n\right)\tau\{t\}\cdot\tau\{t\} \qquad (1.2.10)\end{aligned}$$

where we have defined $\kappa_n = (s_n + \frac{1}{n}k^{-n})$ for all $n \geq 1$. The final line (1.2.10) follows from the definition (1.2.7) of the action of the operators $\{D\}$, and achieves the proof of (1.2.8). □

Returning to equation (1.2.6), we employ the result (1.2.8) to obtain

$$\sum_{m=0}^{\infty}\chi_m\{-2s\}\mathrm{Coeff}_{k^{-m-1}}\left[\exp\left(\sum_{n=1}^{\infty}(s_n+\frac{1}{n}k^{-n})D_n\right)\right]\tau\{t\}\cdot\tau\{t\}=0 \qquad (1.2.11)$$

Recalling the definition (1.1.48) of the one-row Schur polynomials once again, equation (1.2.11) becomes

$$\sum_{m=0}^{\infty}\chi_m\{-2s\}\chi_{(m+1)}\{\mathbb{D}\}\exp\left(\sum_{n=1}^{\infty}s_nD_n\right)\tau\{t\}\cdot\tau\{t\}=0 \qquad (1.2.12)$$

where we have defined the set of operators $\{\mathbb{D}\} = \{D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \ldots\}$. For all $\{1 \leq m_1 < \cdots < m_l\}$ and $\{n_1, \ldots, n_l \geq 1\}$ the coefficient of the monomial $s_{m_1}^{n_1} \ldots s_{m_l}^{n_l}$ on the left hand side of (1.2.12) must vanish, giving rise to infinitely many consistency equations, which are the differential equations of the KP hierarchy.

Example 1. Up to an irrelevant factor the coefficient of s_1^3 on the left hand side of (1.2.12) is equal to $\left(D_1^4 - 4D_1D_3 + 3D_2^2\right)\tau\{t\}\cdot\tau\{t\}$, implying that

$$\left(D_1^4 - 4D_1D_3 + 3D_2^2\right)\tau\{t\}\cdot\tau\{t\} = 0 \qquad (1.2.13)$$

which is the KP equation in bilinear form. Higher equations in the hierarchy are obtained from the coefficients of different monomials.

1.2.2 Charged fermion bilinear identity

We turn to constructing solutions of the KP bilinear identity (1.2.2) using the calculus of the charged fermions $\{\psi_m\}_{m\in\mathbb{Z}}$ and $\{\psi_m^*\}_{m\in\mathbb{Z}}$. The following result states that certain special vacuum expectation values are KP τ-functions.

Theorem 1. Let g_ψ be a finite element of $Cl_\psi^{(0)}$ and define

$$\tau\{t\} = \langle 0|e^{H\{t\}} g_\psi|0\rangle = \left\langle e^{H\{t\}} g_\psi \right\rangle \tag{1.2.14}$$

The polynomial $\tau\{t\}$ satisfies the KP bilinear identity (1.2.2) if and only if g_ψ satisfies the *charged fermion bilinear identity* (CFBI)

$$\sum_{i\in\mathbb{Z}} \psi_i g_\psi|0\rangle \otimes \psi_i^* g_\psi|0\rangle = 0 \tag{1.2.15}$$

Proof. We split the proof into three steps. In the first two steps we prove that if (1.2.15) holds, then $\tau\{t\}$ as given by (1.2.14) satisfies the KP bilinear identity (1.2.2). In the third step we prove the converse statement.

Step 1. Acting upon the left hand side of (1.2.15) with the tensored dual states $\langle 1|e^{H\{t\}} \otimes \langle -1|e^{H\{s\}}$ we have the result

$$\sum_{i\in\mathbb{Z}} \langle 1|e^{H\{t\}} \psi_i g_\psi|0\rangle\langle -1|e^{H\{s\}} \psi_i^* g_\psi|0\rangle = 0 \tag{1.2.16}$$

We convert the sum on the left hand side of (1.2.16) into a contour integral, using the generating functions (1.1.43) to write

$$\oint \langle 1|e^{H\{t\}} \Psi(k) g_\psi|0\rangle\langle -1|e^{H\{s\}} \Psi^*(k) g_\psi|0\rangle \frac{dk}{2\pi i k} = 0 \tag{1.2.17}$$

where the contour of integration surrounds the pole at $k = 0$. By virtue of the commutation relations (1.1.46) and (1.1.47), it is possible to switch the order of $e^{H\{t\}}$ and $\Psi(k)$, and likewise $e^{H\{s\}}$ and $\Psi^*(k)$ in (1.2.17), giving

$$\oint \exp\left(\sum_{n=1}^{\infty}(t_n - s_n)k^n\right) \langle 1|\Psi(k)e^{H\{t\}} g_\psi|0\rangle\langle -1|\Psi^*(k)e^{H\{s\}} g_\psi|0\rangle \frac{dk}{2\pi i k} = 0 \tag{1.2.18}$$

Step 2. (Lemma 8.) We propose the pair of identities

$$\langle 1|\Psi(k) = \langle 0|\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}k^{-n}H_n\Big) \tag{1.2.19}$$

$$\langle -1|\Psi^*(k) = k\langle 0|\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}k^{-n}H_n\Big) \tag{1.2.20}$$

Proof. (Lemma 8.) From the definitions of the dual charged vacua (1.1.15) and the generating functions (1.1.43), we obtain

$$\langle 1|\Psi(k) = \langle 0|\psi_0^* \sum_{i\in\mathbb{Z}} \psi_i k^i = \langle 0|\psi_0^* \sum_{i=0}^{\infty} \psi_{-i} k^{-i} \tag{1.2.21}$$

$$\langle -1|\Psi^*(k) = \langle 0|\psi_{-1} \sum_{i\in\mathbb{Z}} \psi_i^* k^{-i} = k\langle 0|\psi_{-1} \sum_{i=0}^{\infty} \psi_{i-1}^* k^{-i} \tag{1.2.22}$$

where the annihilation properties (1.1.5) have been used to truncate the sums. Rearranging the right hand sides of these equations, we find

$$\langle 1|\Psi(k) = \langle 0| + \sum_{l=1}^{\infty} (-k)^{-l} \langle -l|\psi_{-l+1}^* \dots \psi_0^* \tag{1.2.23}$$

$$\langle -1|\Psi^*(k) = k\left(\langle 0| + \sum_{i=1}^{\infty} k^{-i}\langle -1|\psi_{i-1}^*\right) \tag{1.2.24}$$

Now consider the expression (1.1.48) for the one-row Schur polynomial. When the variables $\{t\}$ are set to $t_n = \pm k^{-n}/n$ for all $n \geq 1$, this expression simplifies greatly. We obtain

$$\chi_m\{t\}\Big|_{t_n=-k^{-n}/n} = \begin{cases} (-k)^{-m}, & m \leq 1 \\ 0, & m \geq 2 \end{cases}, \quad \chi_m\{t\}\Big|_{t_n=k^{-n}/n} = k^{-m} \tag{1.2.25}$$

and substitute these formulae into the expression (1.1.49) for the Schur polynomial associated to $\mu = \{\mu_1 \geq \dots \geq \mu_l > 0\}$, giving

$$\chi_\mu\{t\}\Big|_{t_n=-k^{-n}/n} = (-k)^l \prod_{i=1}^{l} \delta_{\mu_i,1}, \quad \chi_\mu\{t\}\Big|_{t_n=k^{-n}/n} = k^{-\mu_1}\delta_{l,1} \tag{1.2.26}$$

Using the result of lemma 4, these equations become

$$\langle 0|\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}k^{-n}H_n\Big)\psi_{m_1}\dots\psi_{m_l}|-l\rangle = (-k)^{-l}\prod_{i=1}^{l}\delta_{m_i+i,1} \tag{1.2.27}$$

$$\langle 0|\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}k^{-n}H_n\Big)\psi_{m_1}\dots\psi_{m_l}|-l\rangle = k^{-m_1-1}\delta_{l,1} \tag{1.2.28}$$

where we have defined $|\mu\rangle = \psi_{m_1}\dots\psi_{m_l}|-l\rangle$ as usual. Finally, due to the orthonormality of partition vectors (1.1.33), we obtain

$$\langle 0|\exp\Big(-\sum_{n=1}^{\infty}\frac{1}{n}k^{-n}H_n\Big) = \langle 0| + \sum_{l=1}^{\infty}(-k)^{-l}\langle -l|\psi^*_{-l+1}\dots\psi^*_0 \tag{1.2.29}$$

$$\langle 0|\exp\Big(\sum_{n=1}^{\infty}\frac{1}{n}k^{-n}H_n\Big) = \langle 0| + \sum_{i=1}^{\infty}k^{-i}\langle -1|\psi^*_{i-1} \tag{1.2.30}$$

Comparing these equations with (1.2.23) and (1.2.24), we complete the proof of (1.2.19) and (1.2.20). □

Applying (1.2.19) and (1.2.20) to equation (1.2.18), we obtain

$$\oint \exp\left(\sum_{n=1}^{\infty}(t_n - s_n)k^n\right)\langle 0|e^{H\{t-\epsilon_k\}}g_\psi|0\rangle\langle 0|e^{H\{s+\epsilon_k\}}g_\psi|0\rangle\frac{dk}{2\pi i} = 0 \tag{1.2.31}$$

Note the disappearance of the factor of $\frac{1}{k}$ from the integrand of (1.2.18), which is due to the cancelling factor of k in (1.2.20). Equation (1.2.31) proves that if (1.2.15) holds, functions given by (1.2.14) satisfy the KP bilinear identity (1.2.2).

Step 3. For any finite $g_\psi \in Cl_\psi^{(0)}$ there exists an integer $l \geq 1$ and coefficients $\kappa_{\{m\},\{n\}}$ such that

$$\sum_{i\in\mathbb{Z}}\psi_i g_\psi|0\rangle\otimes\psi_i^* g_\psi|0\rangle = \sum_{\substack{\mathrm{card}\{m\}=l\\ \mathrm{card}\{n\}=l}}\kappa_{\{m\},\{n\}}\psi_{\{m\}}|-l+1\rangle\otimes\psi_{\{n\}}|-l-1\rangle \tag{1.2.32}$$

where the sum is over all sets of integers $\{m\} = \{m_1 > \cdots > m_l \geq -l+1\}$ and $\{n\} = \{n_1 > \cdots > n_l \geq -l-1\}$, of fixed cardinality l. Acting upon both sides of this equation with the tensored dual states $\langle 1|e^{H\{t\}}\otimes\langle -1|e^{H\{s\}}$ we find

$$\oint \exp\left(\sum_{n=1}^{\infty}(t_n - s_n)k^n\right)\langle 0|e^{H\{t-\epsilon_k\}}g_\psi|0\rangle\langle 0|e^{H\{s+\epsilon_k\}}g_\psi|0\rangle\frac{dk}{2\pi i} = \tag{1.2.33}$$
$$\sum_{\substack{\mathrm{card}\{m\}=l\\ \mathrm{card}\{n\}=l}}\kappa_{\{m\},\{n\}}\chi_{(\mu-1)}\{t\}\chi_{(\nu+1)}\{s\}$$

where the left hand side has already been derived in steps 1 and 2, and the right hand side follows from lemma 5, with $\mu_i = m_i + i$ and $\nu_i = n_i + i$. Assuming that $\tau\{t\}$ as given by (1.2.14) satisfies the KP bilinear identity, we thus obtain

$$\sum_{\substack{\mathrm{card}\{m\}=l\\ \mathrm{card}\{n\}=l}}\kappa_{\{m\},\{n\}}\chi_{(\mu-1)}\{t\}\chi_{(\nu+1)}\{s\} = 0 \tag{1.2.34}$$

which can only be true if all of the coefficients $\kappa_{\{m\},\{n\}} = 0$, since the Schur polynomials are linearly independent. Substituting this trivial value for the coefficients into (1.2.32), we recover the CFBI (1.2.15). This completes the proof of the converse statement. □

1.3 Solutions of the CFBI

1.3.1 Orbit of GL_∞

Theorem 2. Suppose g_ψ is a finite element of $Cl_\psi^{(0)}$. Then g_ψ solves the CFBI (1.2.15) if and only if

$$g_\psi|0\rangle = e^{X_1}\dots e^{X_l}|0\rangle \tag{1.3.1}$$

for some $\{X_1,\dots,X_l\} \in A_\infty$. In other words, the solution space of (1.2.15) is generated by the orbit of the Lie group

$$GL_\infty = \left\{ e^{X_1}\dots e^{X_l} \;\middle|\; X_i \in A_\infty \text{ for all } 1 \le i \le l \right\} \tag{1.3.2}$$

Proof. We split the proof into two steps. In the first step we prove the forward statement, in the second step we prove its converse.

Step 1. (Lemma 9.) Let $|u\rangle$ and $|v\rangle$ be arbitrary state vectors in $\mathcal{F}_\psi$, and let $g_\psi = e^{X_1}\dots e^{X_l}$ with each $X_i \in A_\infty$. We have

$$\sum_{i\in\mathbb{Z}} \psi_i g_\psi|u\rangle \otimes \psi_i^* g_\psi|v\rangle = \sum_{i\in\mathbb{Z}} g_\psi \psi_i|u\rangle \otimes g_\psi \psi_i^*|v\rangle \tag{1.3.3}$$

Proof. (Lemma 9.) For $m \ge 0$ and arbitrary $X \in A_\infty$, let $\mathcal{P}_m$ denote the proposition

$$(\psi_i \otimes \psi_i^*) \sum_{n=0}^{m} \binom{m}{n} X^n \otimes X^{m-n} = \sum_{n=0}^{m} \binom{m}{n} X^n \otimes X^{m-n} (\psi_i \otimes \psi_i^*) \tag{1.3.4}$$

where summation over all integers i is implied. The proposition $\mathcal{P}_0$ is trivial. Furthermore, letting $X \in A_\infty$ be given by (1.1.34), by direct calculation we obtain the commutation relations

$$[\psi_i, X] = -\sum_{j\in\mathbb{Z}} a_{j,i}\psi_j, \quad [\psi_i^*, X] = \sum_{j\in\mathbb{Z}} a_{i,j}\psi_j^* \tag{1.3.5}$$

Using these commutators in the left hand side of $\mathcal{P}_1$, we obtain

$$(\psi_i \otimes \psi_i^*)(1 \otimes X + X \otimes 1) = (1 \otimes X + X \otimes 1)(\psi_i \otimes \psi_i^*) + a_{i,j}\psi_i \otimes \psi_j^* - a_{j,i}\psi_j \otimes \psi_i^* \\ = (1 \otimes X + X \otimes 1)(\psi_i \otimes \psi_i^*) \tag{1.3.6}$$

where summation over all integers i, j is implied. This proves $\mathcal{P}_1$ is true. Now suppose $\mathcal{P}_m$ is true for some $m \geq 1$. Using the identity $\binom{m}{n} + \binom{m}{n-1} = \binom{m+1}{n}$ for all $1 \leq n \leq m$, we write

$$(\psi_i \otimes \psi_i^*) \sum_{n=0}^{m+1} \binom{m+1}{n} X^n \otimes X^{m+1-n} = \tag{1.3.7}$$
$$(\psi_i \otimes \psi_i^*) \left(\sum_{n=0}^{m} \binom{m}{n} X^n \otimes X^{m+1-n} + \sum_{n=1}^{m+1} \binom{m}{n-1} X^n \otimes X^{m+1-n} \right)$$

with summation implied over all integers i. Shifting the summation index of the second sum on the right hand side of (1.3.7), we obtain

$$(\psi_i \otimes \psi_i^*) \sum_{n=0}^{m+1} \binom{m+1}{n} X^n \otimes X^{m+1-n} = (\psi_i \otimes \psi_i^*) \sum_{n=0}^{m} \binom{m}{n} X^n \otimes X^{m-n}(1 \otimes X + X \otimes 1) \tag{1.3.8}$$

Now it is possible to use $\mathcal{P}_m$ and $\mathcal{P}_1$ in the right hand side of (1.3.8), to obtain

$$(\psi_i \otimes \psi_i^*) \sum_{n=0}^{m+1} \binom{m+1}{n} X^n \otimes X^{m+1-n} = \sum_{n=0}^{m} \binom{m}{n} X^n \otimes X^{m-n}(\psi_i \otimes \psi_i^*)(1 \otimes X + X \otimes 1) \\ = \sum_{n=0}^{m} \binom{m}{n} X^n \otimes X^{m-n}(1 \otimes X + X \otimes 1)(\psi_i \otimes \psi_i^*) \tag{1.3.9}$$

Finally, we use the reverse of (1.3.8) in the right hand side of (1.3.9), giving

$$(\psi_i \otimes \psi_i^*) \sum_{n=0}^{m+1} \binom{m+1}{n} X^n \otimes X^{m+1-n} = \sum_{n=0}^{m+1} \binom{m+1}{n} X^n \otimes X^{m+1-n}(\psi_i \otimes \psi_i^*) \tag{1.3.10}$$

Therefore $\mathcal{P}_m$ true $\implies$ $\mathcal{P}_{m+1}$ true, and the proposition (1.3.4) holds for all $m \geq 0$ by induction. By virtue of the proposition (1.3.4), for any $X \in A_\infty$ we have

$$
\begin{aligned}
(\psi_i \otimes \psi_i^*)(e^X \otimes e^X)|u\rangle \otimes |v\rangle =& (\psi_i \otimes \psi_i^*) \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^{m} \binom{m}{n} X^n \otimes X^{m-n} |u\rangle \otimes |v\rangle \\
=& \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^{m} \binom{m}{n} X^n \otimes X^{m-n} (\psi_i \otimes \psi_i^*)|u\rangle \otimes |v\rangle \\
=& (e^X \otimes e^X)(\psi_i \otimes \psi_i^*)|u\rangle \otimes |v\rangle
\end{aligned} \tag{1.3.11}
$$

Therefore we have proved that

$$
\sum_{i\in\mathbb{Z}} \psi_i e^X |u\rangle \otimes \psi_i^* e^X |v\rangle = \sum_{i\in\mathbb{Z}} e^X \psi_i |u\rangle \otimes e^X \psi_i^* |v\rangle \tag{1.3.12}
$$

for arbitrary $X \in A_\infty$. Using (1.3.12) l times successively, once for each e^{X_i} in g_ψ, we obtain equation (1.3.3). □

Corollary. Having established the validity of equation (1.3.3) we employ a particular case of it, namely when $|u\rangle = |v\rangle = |0\rangle$, which gives

$$
\sum_{i\in\mathbb{Z}} \psi_i g_\psi |0\rangle \otimes \psi_i^* g_\psi |0\rangle = \sum_{i\in\mathbb{Z}} g_\psi \psi_i |0\rangle \otimes g_\psi \psi_i^* |0\rangle = 0 \tag{1.3.13}
$$

where the final equality is due to the fact that for all $i \in \mathbb{Z}$ either $\psi_i|0\rangle = 0$ or $\psi_i^*|0\rangle = 0$. Equation (1.3.13) completes the proof that when $g_\psi|0\rangle$ is of the form (1.3.1), g_ψ satisfies the CFBI (1.2.15).

Step 2. (Lemma 10.) Let $g_\psi \in Cl_\psi^{(0)}$ satisfy the CFBI (1.2.15). Then for suitable $\{X_1, \ldots, X_l\} \in A_\infty$ we can write $g_\psi|0\rangle = e^{X_1} \ldots e^{X_l}|0\rangle$.

Proof. (Lemma 10.) We use the method of proof given in chapter 5 of [71]. Since $g_\psi|0\rangle \in \mathcal{F}_\psi^{(0)}$ we can expand it in terms of the basis (1.1.9) to give

$$
g_\psi|0\rangle = c_\emptyset|0\rangle + \sum_{m\geq 0, n<0} c_{m,n} \psi_m \psi_n^* |0\rangle + g_\psi^{(1)}|0\rangle \tag{1.3.14}
$$

for some suitable coefficients $c_\emptyset$ and $c_{m,n}$, and where all monomials within $g_\psi^{(1)} \in Cl_\psi^{(0)}$ consist of at least two positive (+1) fermions and two negative (−1) fermions.[6] From here, we need to consider the cases $c_\emptyset \neq 0$ and $c_\emptyset = 0$ separately.

Case 1. ($c_\emptyset \neq 0$) We define elements X_1, X_2 of A_∞ as follows

[6]Throughout the rest of the proof, we will always use $g_\psi^{(i)}$ to denote an element of $Cl_\psi^{(0)}$ with precisely this property.

$$X_1 = \log c_{\emptyset}, \quad X_2 = \sum_{m\geq 0, n<0} c^{(1)}_{m,n} \psi_m \psi^*_n \tag{1.3.15}$$

where $c^{(1)}_{m,n} = c_{m,n}/c_{\emptyset}$ for all $m \geq 0, n < 0$. We trivially obtain

$$e^{-X_1} g_{\psi}|0\rangle = |0\rangle + \sum_{m\geq 0, n<0} c^{(1)}_{m,n} \psi_m \psi^*_n |0\rangle + g^{(2)}_{\psi}|0\rangle \tag{1.3.16}$$

where we have defined $g^{(2)}_{\psi} = g^{(1)}_{\psi}/c_{\emptyset}$. Next, we act on equation (1.3.16) with the operator e^{-X_2}. Term by term we have

$$e^{-X_2}|0\rangle = |0\rangle - \sum_{m\geq 0, n<0} c^{(1)}_{m,n} \psi_m \psi^*_n |0\rangle + g^{(3)}_{\psi}|0\rangle \tag{1.3.17}$$

$$e^{-X_2} \sum_{m\geq 0, n<0} c^{(1)}_{m,n} \psi_m \psi^*_n |0\rangle = \sum_{m\geq 0, n<0} c^{(1)}_{m,n} \psi_m \psi^*_n |0\rangle + g^{(4)}_{\psi}|0\rangle \tag{1.3.18}$$

$$e^{-X_2} g^{(2)}_{\psi}|0\rangle = g^{(5)}_{\psi}|0\rangle \tag{1.3.19}$$

for some suitable $g^{(3)}_{\psi}, g^{(4)}_{\psi}, g^{(5)}_{\psi} \in Cl^{(0)}_{\psi}$. Combining these three results, we obtain

$$e^{-X_2} e^{-X_1} g_{\psi}|0\rangle = |0\rangle + g^{(6)}_{\psi}|0\rangle \tag{1.3.20}$$

where we have defined $g^{(6)}_{\psi} = g^{(3)}_{\psi} + g^{(4)}_{\psi} + g^{(5)}_{\psi}$. By virtue of equation (1.3.3) and the fact that g_{ψ} obeys the CFBI (1.2.15), we have

$$\begin{aligned} 0 &= \sum_{i\in\mathbb{Z}} e^{-X_2} e^{-X_1} \psi_i g_{\psi}|0\rangle \otimes e^{-X_2} e^{-X_1} \psi^*_i g_{\psi}|0\rangle \\ &= \sum_{i\in\mathbb{Z}} \psi_i e^{-X_2} e^{-X_1} g_{\psi}|0\rangle \otimes \psi^*_i e^{-X_2} e^{-X_1} g_{\psi}|0\rangle \end{aligned} \tag{1.3.21}$$

Substituting equation (1.3.20) for $e^{-X_2} e^{-X_1} g_{\psi}|0\rangle$ into the second line of (1.3.21) and using the annihilation properties (1.1.5), we find

$$\sum_{i\geq 0} \psi_i|0\rangle \otimes \psi^*_i g^{(6)}_{\psi}|0\rangle + \sum_{i<0} \psi_i g^{(6)}_{\psi}|0\rangle \otimes \psi^*_i|0\rangle + \sum_{i\in\mathbb{Z}} \psi_i g^{(6)}_{\psi}|0\rangle \otimes \psi^*_i g^{(6)}_{\psi}|0\rangle = 0 \tag{1.3.22}$$

We recall that all monomials within $g^{(6)}_{\psi} \in Cl^{(0)}_{\psi}$ consist of at least two positive (+1) fermions and two negative (−1) fermions. Therefore the left hand side of (1.3.22) vanishes if and only if

$$\psi_m g_\psi^{(6)}|0\rangle = \psi_n^* g_\psi^{(6)}|0\rangle = 0 \tag{1.3.23}$$

for all $m < 0, n \geq 0$. The only possible resolution of this equation is that $g_\psi^{(6)}|0\rangle = 0$. Substituting this value of $g_\psi^{(6)}|0\rangle$ into (1.3.20) we see that $e^{-X_2}e^{-X_1}g_\psi|0\rangle = |0\rangle$, or equivalently, $g_\psi|0\rangle = e^{X_1}e^{X_2}|0\rangle$. This completes the proof in the case $c_\emptyset \neq 0$.

Case 2. **($c_\emptyset = 0$)** We begin by stating an identity which we use in the proof. Fix two integers $p \geq 0, q < 0$ and two sets $\{m\} = \{m_1 > \cdots > m_r \geq 0\}$ and $\{n\} = \{n_1 < \cdots < n_r < 0\}$. The identity reads

$$e^{-\psi_p\psi_q^*}e^{\psi_q\psi_p^*}\psi_{\{m\}}\psi_{\{n\}}^*|0\rangle = \begin{cases} (-)^{i+j+r+1}\psi_{\{m\backslash m_i\}}\psi^*_{\{n\backslash n_j\}}|0\rangle, & p = m_i \\ & q = n_j \\ \psi_{\{m\}}\psi^*_{\{n\}}|0\rangle - \psi_p\psi_{\{m\}}\psi^*_{\{n\}}\psi_q^*|0\rangle, & p \notin \{m\} \\ & q \notin \{n\} \\ \psi_{\{m\}}\psi^*_{\{n\}}|0\rangle, & \text{otherwise} \end{cases} \tag{1.3.24}$$

where we have used the notation $\{m\backslash m_i\}, \{n\backslash n_j\}$ to denote the omission of the i^{th} and j^{th} elements from the sets $\{m\}$ and $\{n\}$, respectively. Returning to the proof, we observe that since $c_\emptyset = 0$ we can write

$$g_\psi|0\rangle = \sum_{\substack{\text{card}\{m\}=r \\ \text{card}\{n\}=r}} c_{\{m\},\{n\}}\psi_{\{m\}}\psi^*_{\{n\}}|0\rangle + g_\psi^{(7)}|0\rangle \tag{1.3.25}$$

where the sum is taken over all ordered sets of integers $\{m_1 > \cdots > m_r \geq 0\}$ and $\{n_1 < \cdots < n_r < 0\}$ of some fixed cardinality $r \geq 1$, and all monomials within $g_\psi^{(7)} \in Cl_\psi^{(0)}$ consist of at least $r+1$ positive $(+1)$ fermions and $r+1$ negative (-1) fermions. Let $c_{\{p\},\{q\}}$ be a particular non-zero coefficient in the sum (1.3.25), corresponding to the sets $\{p\} = \{p_1 > \cdots > p_r \geq 0\}$ and $\{q\} = \{q_1 < \cdots < q_r < 0\}$, and define

$$X_{2i-1} = -\psi_{q_i}\psi^*_{p_i}, \quad X_{2i} = \psi_{p_i}\psi^*_{q_i} \tag{1.3.26}$$

for all $1 \leq i \leq r$. Successively applying the identity (1.3.24) to $g_\psi|0\rangle$, we obtain

$$e^{-X_{2r}}e^{-X_{2r-1}}\ldots e^{-X_2}e^{-X_1}g_\psi|0\rangle = c_\emptyset^{(2)}|0\rangle + \sum_{m\geq 0, n<0} c^{(2)}_{m,n}\psi_m\psi_n^*|0\rangle + g_\psi^{(8)}|0\rangle \tag{1.3.27}$$

with $c_{\emptyset}^{(2)} = (-)^{r(r-1)/2} c_{\{p\},\{q\}}$ and the remaining coefficients $c_{m,n}^{(2)}$ suitably chosen, and where all monomials within $g_{\psi}^{(8)} \in Cl_{\psi}^{(0)}$ consist of at least two positive (+1) fermions and two negative (−1) fermions. Since $c_{\emptyset}^{(2)} \neq 0$, we can apply the procedure of case 1 to (1.3.27), ultimately obtaining

$$g_{\psi}|0\rangle = e^{X_1} e^{X_2} \dots e^{X_{2r+1}} e^{X_{2r+2}} |0\rangle \tag{1.3.28}$$

where we have defined

$$X_{2r+1} = \log c_{\emptyset}^{(2)}, \quad X_{2r+2} = \sum_{m \geq 0, n < 0} c_{m,n}^{(2)} / c_{\emptyset}^{(2)} \psi_m \psi_n^* \tag{1.3.29}$$

Since all $\{X_1, \dots, X_{2r+2}\} \in A_{\infty}$, equation (1.3.28) completes the proof in the $c_{\emptyset} = 0$ case. We have therefore proved lemma 10 which, in turn, finishes the proof of theorem 2. □

1.3.2 Schur polynomials and the orbit of GL_{∞}

Example 2. As a particular case of theorem 2, we show that every Schur polynomial (1.1.49) is a KP τ-function. Let $|\mu) = |\mu_1, \dots, \mu_l)$ be an arbitrary partition equal to the Fock space vector $\psi_{m_1} \dots \psi_{m_l} | -l\rangle$, where $m_i = \mu_i - i$ for all $1 \leq i \leq l$. Using equation (1.1.21) from the proof of lemma 1, we obtain

$$\chi_{\mu}\{t\} = \langle 0| e^{H\{t\}} \psi_{m_1} \dots \psi_{m_l} | -l\rangle = (-)^{\sum_{i=1}^{r} (n_i + i)} \left\langle e^{H\{t\}} \psi_{m_1} \dots \psi_{m_r} \psi_{n_r}^* \dots \psi_{n_1}^* \right\rangle \tag{1.3.30}$$

where $m_1 > \dots > m_r \geq 0 > m_{r+1} > \dots > m_l > -l$, and $\{n_r < \dots < n_1\}$ is the set $\{-l < \dots < -1\}$ with $\{m_l < \dots < m_{r+1}\}$ omitted. Defining

$$X_{2i-1} = -\psi_{n_i} \psi_{m_i}^*, \quad X_{2i} = \psi_{m_i} \psi_{n_i}^* \tag{1.3.31}$$

for all $1 \leq i \leq r$ and using the identity (1.3.24) from the last subsection, we obtain

$$e^{-X_{2r}} e^{-X_{2r-1}} \dots e^{-X_2} e^{-X_1} \psi_{m_1} \dots \psi_{m_r} \psi_{n_r}^* \dots \psi_{n_1}^* |0\rangle = (-)^{r(r-1)/2} |0\rangle \tag{1.3.32}$$

or equivalently,

$$\psi_{m_1} \dots \psi_{m_r} \psi_{n_r}^* \dots \psi_{n_1}^* |0\rangle = (-)^{r(r-1)/2} e^{X_1} e^{X_2} \dots e^{X_{2r-1}} e^{X_{2r}} |0\rangle \tag{1.3.33}$$

Substituting (1.3.33) into (1.3.30), we find

$$\chi_\mu\{t\} = (-)^{r+\sum_{i=1}^r n_i}\left\langle e^{H\{t\}}e^{X_1}\dots e^{X_{2r}}\right\rangle \tag{1.3.34}$$

Hence any Schur polynomial can be written as an expectation value of the form (1.2.14), with $g_\psi \in GL_\infty$. By theorem 2, the Schur polynomials are therefore τ-functions of the KP hierarchy.

1.3.3 KP Plücker relations

In this subsection we solve the CFBI (1.2.15) from another, more direct perspective. Let $g_\psi|0\rangle$ be a finite element of $\mathcal{F}_\psi^{(0)}$ expanded in terms of the basis (1.1.16). Since $g_\psi|0\rangle$ is finite, there exists some $l \geq 0$ and coefficients $c_{\{m\}}$ such that

$$g_\psi|0\rangle = \sum_{\mathrm{card}\{m\}=l} c_{\{m\}}\psi_{\{m\}}|-l\rangle \tag{1.3.35}$$

where the sum is over all sets of integers $\{m\} = \{m_1 > \cdots > m_l \geq -l\}$. Using this expansion of $g_\psi|0\rangle$, we obtain

$$\sum_{i\in\mathbb{Z}} \psi_i g_\psi|0\rangle \otimes \psi_i^* g_\psi|0\rangle = \sum_{i\in\mathbb{Z}} \sum_{\substack{\mathrm{card}\{m\}=l\\ \mathrm{card}\{n\}=l}} c_{\{m\}}c_{\{n\}}\psi_i\psi_{\{m\}}|-l\rangle \otimes \psi_i^*\psi_{\{n\}}|-l\rangle \tag{1.3.36}$$

where the second sum is over all sets of integers $\{m_1 > \cdots > m_l \geq -l\}$ and $\{n_1 > \cdots > n_l \geq -l\}$ of fixed cardinality l. Switching the order of the sums in (1.3.36) and using the annihilation properties (1.1.5) of the fermions, we find

$$\sum_{i\in\mathbb{Z}} \psi_i g_\psi|0\rangle \otimes \psi_i^* g_\psi|0\rangle = \sum_{\substack{\mathrm{card}\{m\}=l\\ \mathrm{card}\{n\}=l}} \sum_{i=1}^{l} (-)^{i-1} c_{\{m\}}c_{\{n\}}\psi_{\{n_i,m\}}|-l\rangle \otimes \psi_{\{n\backslash n_i\}}|-l\rangle \tag{1.3.37}$$

where $\psi_{\{n_i,m\}} = 0$ if $n_i \in \{m\}$. Changing the indexing sets of the first sum in (1.3.37), we obtain the equivalent expression

$$\sum_{i\in\mathbb{Z}} \psi_i g_\psi|0\rangle \otimes \psi_i^* g_\psi|0\rangle = \sum_{\substack{\mathrm{card}\{p\}=l+1\\ \mathrm{card}\{q\}=l-1}} \sum_{j=1}^{l+1} (-)^{j-1} c_{\{p\backslash p_j\}}c_{\{p_j,q\}}\psi_{\{p\}}|-l\rangle \otimes \psi_{\{q\}}|-l\rangle \tag{1.3.38}$$

with the first sum taken over all sets of integers $\{p_1 > \cdots > p_{l+1} \geq -l\}$ and $\{q_1 > \cdots > q_{l-1} \geq -l\}$, and where we have defined

$$c_{\{p_j,q\}} = (-)^{i-1} c_{\{q_1,\ldots,q_{i-1},p_j,q_i,\ldots,q_{l-1}\}} \tag{1.3.39}$$

if $q_{i-1} > p_j > q_i$ for some $1 \le i \le l$, and $c_{\{p_j,q\}} = 0$ if $p_j \in \{q\}$. The right hand side of (1.3.38) vanishes if and only if

$$\sum_{i=1}^{l+1} (-)^{i-1} c_{\{m \backslash m_i\}} c_{\{m_i,n\}} = 0 \tag{1.3.40}$$

for all sets $\{m_1 > \ldots > m_{l+1} \ge -l\}$ and $\{n_1 > \cdots > n_{l-1} \ge -l\}$. Collectively, these conditions are called the *KP Plücker relations*, and we summarize their significance with the following statement (which we have already proved).

Lemma 11. $g_\psi \in Cl_\psi^{(0)}$ satisfies the CFBI (1.2.15) if and only if the expansion coefficients (1.3.35) of $g_\psi|0\rangle$ obey the KP Plücker relations (1.3.40).

1.3.4 Determinant solution of KP Plücker relations

With the following result, we present a general determinant solution of the KP Plücker relations (1.3.40).

Lemma 12. To every set of integers $\{m\} = \{m_1, \ldots, m_l\}$ we associate the coefficient

$$c_{\{m\}} = \det \Big(c_{m_i,j} \Big)_{1 \le i,j \le l} = \begin{vmatrix} c_{m_1,1} & \cdots & c_{m_1,l} \\ \vdots & & \vdots \\ c_{m_l,1} & \cdots & c_{m_l,l} \end{vmatrix} \tag{1.3.41}$$

where the matrix entries $c_{i,j}$ are arbitrary constants. These coefficients satisfy the Plücker relations (1.3.40).

Proof. The proof is taken from the paper [73]. Define two ordered sets of integers $\{m\} = \{m_1 > \cdots > m_{l+1} \ge -l\}$ and $\{n\} = \{n_1 > \cdots > n_{l-1} \ge -l\}$, and from these sets construct a $2l \times 2l$ matrix M given by

$$M\Big(\{m\},\{n\}\Big) = \begin{pmatrix} c_{m_1,1} & \cdots & c_{m_1,l} & c_{m_1,1} & \cdots & c_{m_1,l} \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{m_{l+1},1} & \cdots & c_{m_{l+1},l} & c_{m_{l+1},1} & \cdots & c_{m_{l+1},l} \\ c_{n_1,1} & \cdots & c_{n_1,l} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{n_{l-1},1} & \cdots & c_{n_{l-1},l} & 0 & \cdots & 0 \end{pmatrix} \tag{1.3.42}$$

Using Laplace's formula for the expansion of $\det M$ yields

$$\det M = \sum_{i=1}^{l+1} (-)^{i-1} \begin{vmatrix} c_{m_1,1} & \cdots & c_{m_1,l} \\ \vdots & & \vdots \\ \widehat{c_{m_i,1}} & \cdots & \widehat{c_{m_i,l}} \\ \vdots & & \vdots \\ c_{m_{l+1},1} & \cdots & c_{m_{l+1},l} \end{vmatrix} \begin{vmatrix} c_{m_i,1} & \cdots & c_{m_i,l} \\ c_{n_1,1} & \cdots & c_{n_1,l} \\ \vdots & & \vdots \\ c_{n_{l-1},1} & \cdots & c_{n_{l-1},l} \end{vmatrix} \tag{1.3.43}$$

where the i^{th} row has been omitted from the first determinant, and inserted as the first row of the second determinant. Writing these determinants in terms of the coefficients (1.3.41), we obtain

$$\det M = \sum_{i=1}^{l+1} (-)^{i-1} c_{\{m\backslash m_i\}} c_{\{m_i,n\}} \tag{1.3.44}$$

which is precisely the left hand side of the KP Plücker relations (1.3.40). All that remains in the proof is to demonstrate that $\det M = 0$. This is achieved by subtracting the last l columns from the first l columns in $\det M$ to obtain an equivalent determinant $\det M^{(1)}$, where

$$M^{(1)}\Big(\{m\},\{n\}\Big) = \begin{pmatrix} 0 & \cdots & 0 & c_{m_1,1} & \cdots & c_{m_1,l} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & c_{m_{l+1},1} & \cdots & c_{m_{l+1},l} \\ c_{n_1,1} & \cdots & c_{n_1,l} & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ c_{n_{l-1},1} & \cdots & c_{n_{l-1},l} & 0 & \cdots & 0 \end{pmatrix} \tag{1.3.45}$$

Writing $\det M^{(1)} = \sum_{\sigma \in S_{2l}} \operatorname{sgn}(\sigma) \prod_{i=1}^{2l} M^{(1)}_{i,\sigma(i)}$ and using (1.3.45), we see that every product in this sum has at least one zero term, proving that $\det M = \det M^{(1)} = 0$. Hence we have shown that the sum (1.3.44) is equal to zero, proving that the coefficients (1.3.41) satisfy the KP Plücker relations (1.3.40). □

1.4 Neutral fermions and related definitions

1.4.1 Neutral fermions

Consider the infinite set $\{\phi_m\}_{m\in\mathbb{Z}}$, where m ranges over all integers. The elements in this set are called *neutral fermions*, and they are defined as linear combinations of the charged fermions (1.1.1). Explicitly, we set

$$\phi_m = \psi_m + (-)^m \psi^*_{-m} \tag{1.4.1}$$

for all $m \in \mathbb{Z}$.[7] From (1.4.1) and the charged fermion anticommutation relations (1.1.1), we see that the neutral fermions satisfy

$$[\phi_m, \phi_n]_+ = 2(-)^m \delta_{m+n,0} \tag{1.4.2}$$

for all $m, n \in \mathbb{Z}$. As a special case of the equations (1.4.2), we obtain

$$\phi_m^2 = \delta_{m,0} \tag{1.4.3}$$

for all $m \in \mathbb{Z}$. For our later convenience, we introduce a second set of fermions $\{\phi_m^*\}_{m\in\mathbb{Z}}$, defined as

$$\phi_m^* = (-)^m \phi_{-m} = \psi_m^* + (-)^m \psi_{-m} \tag{1.4.4}$$

for all $m \in \mathbb{Z}$. This second set of fermions is purely a relabelling of the first set.[8]

1.4.2 Clifford algebra Cl_ϕ

Let Cl_ϕ be the associative subalgebra of Cl_ψ generated by 1 and the neutral fermions $\{\phi_m\}_{m\in\mathbb{Z}}$, modulo the anticommutation relations (1.4.1). Considered as a vector space, Cl_ϕ has the basis

$$\text{Basis}\,(Cl_\phi) = \Big\{1, \phi_{m_1} \dots \phi_{m_r}\Big\} \tag{1.4.5}$$

where $\{m_1 > \cdots > m_r\}$ ranges over all integers, and the cardinality of this set takes all values $r \geq 1$. We decompose Cl_ϕ into the direct sum of subalgebras

$$Cl_\phi = \bigoplus_{i\in\{0,1\}} Cl_\phi^{(i)} \tag{1.4.6}$$

where $Cl_\phi^{(i)}$ is the linear span of all neutral fermion monomials of length i mod 2. In this chapter we are mainly interested in the subalgebra $Cl_\phi^{(0)}$, which has the basis

$$\text{Basis}\left(Cl_\phi^{(0)}\right) = \Big\{1, \phi_{m_1} \dots \phi_{m_{2r}}\Big\} \tag{1.4.7}$$

where $\{m_1 > \cdots > m_{2r}\}$ ranges over all integers, and the cardinality of this set takes all positive even values.

---

[7] In [50], the neutral fermions were defined as $\phi_m = \frac{1}{\sqrt{2}}(\psi_m + (-)^m \psi_{-m}^*)$. We have deliberately omitted the factor $\frac{1}{\sqrt{2}}$ from our definition, in order to obtain the correct normalization for polynomials which we later discuss.

[8] The neutral fermions ϕ_m^* are genuinely different from the second species defined in [50], which are given by $\hat{\phi}_m = \frac{i}{\sqrt{2}}(\psi_m - (-)^m \psi_{-m}^*)$.

1.4.3 Fock representation of Cl_ϕ

Using the annihilation properties (1.1.5) of the charged fermions, the action of Cl_ϕ on the vacuum $|0\rangle$ and dual vacuum $\langle 0|$ is given by

$$\phi_m|0\rangle = 0, \quad \langle 0|\phi_n = 0 \tag{1.4.8}$$

for all integers $m < 0, n > 0$. The Fock space $\mathcal{F}_\phi$ and dual Fock space $\mathcal{F}_\phi^*$ are the vector spaces generated linearly by the action of Cl_ϕ on $|0\rangle$ and $\langle 0|$, respectively. Due to the annihilation relations (1.4.8), they have the bases

$$\text{Basis}\left(\mathcal{F}_\phi\right) = \Big\{|0\rangle, \phi_{m_1}\dots\phi_{m_r}|0\rangle\Big\}, \quad \text{Basis}\left(\mathcal{F}_\phi^*\right) = \Big\{\langle 0|, \langle 0|\phi_{m_r}^*\dots\phi_{m_1}^*\Big\} \tag{1.4.9}$$

where in both cases $\{m_1 > \dots > m_r \geq 0\}$ ranges over all non-negative integers, and the cardinality of this set takes all values $r \geq 1$.

Using the definition of the Clifford subalgebras (1.4.6), we decompose $\mathcal{F}_\phi$ and $\mathcal{F}_\phi^*$ into the following direct sums of subspaces

$$\mathcal{F}_\phi = \bigoplus_{i\in\{0,1\}} \mathcal{F}_\phi^{(i)}, \quad \mathcal{F}_\phi^* = \bigoplus_{i\in\{0,1\}} \mathcal{F}_\phi^{*(i)} \tag{1.4.10}$$

where $\mathcal{F}_\phi^{(i)}$ and $\mathcal{F}_\phi^{*(i)}$ are the subspaces generated linearly by the action of $Cl_\phi^{(i)}$ on $|0\rangle$ and $\langle 0|$, respectively. The bases of $\mathcal{F}_\phi^{(0)}$ and $\mathcal{F}_\phi^{*(0)}$ are given by

$$\text{Basis}\left(\mathcal{F}_\phi^{(0)}\right) = \Big\{|0\rangle, \phi_{m_1}\dots\phi_{m_{2r}}|0\rangle\Big\}, \quad \text{Basis}\left(\mathcal{F}_\phi^{*(0)}\right) = \Big\{\langle 0|, \langle 0|\phi_{m_{2r}}^*\dots\phi_{m_1}^*\Big\} \tag{1.4.11}$$

where in both cases $\{m_1 > \dots > m_{2r} \geq 0\}$ ranges over all non-negative integers, and the cardinality of this set takes all positive even values.

Remark 2. In future sections we will sometimes adopt the notation

$$\phi_{\{m\}} = \phi_{m_1}\dots\phi_{m_r}, \quad \phi_{\{m\}}^* = \phi_{m_r}^*\dots\phi_{m_1}^* \tag{1.4.12}$$

for all ordered sets $\{m\} = \{m_1 > \dots > m_r\}$ with cardinality $r \geq 1$. We also define $\phi_{\{m\}} = \phi_{\{m\}}^* = 1$ when $\{m\}$ is empty. For example, using this notation the bases (1.4.11) can be written as

$$\text{Basis}\left(\mathcal{F}_\phi^{(0)}\right) = \Big\{\phi_{\{m\}}|0\rangle \;\Big|\; \text{card}\{m\} \text{ even}\Big\} \tag{1.4.13}$$

$$\text{Basis}\left(\mathcal{F}_\phi^{*(0)}\right) = \Big\{\langle 0|\phi_{\{m\}}^* \;\Big|\; \text{card}\{m\} \text{ even}\Big\} \tag{1.4.14}$$

where $\{m\}$ ranges over all ordered, even-cardinality sets of non-negative integers.

1.4.4 Strict partitions

A *strict partition* $\tilde{\mu} = \{\mu_1 > \cdots > \mu_l > \mu_{l+1} = \cdots = 0\}$ is a set of non-negative integers whose non-zero elements are strictly decreasing and finite in number.[9] When represented as a Young diagram they consist of l rows of boxes, such that no two rows have the same length.

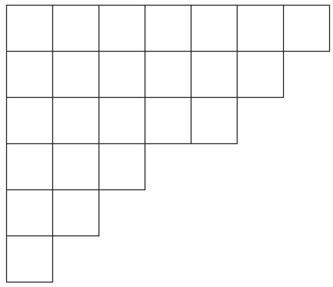

Figure 1.2: Young diagram of the strict partition $\tilde{\mu} = \{7, 6, 5, 3, 2, 1\}$. Every row of boxes has a different length.

The elements of the bases (1.4.11) can be matched with strict partitions. We define $|0\rangle = |\emptyset)$ and $\langle 0| = (\emptyset|$, and for all sets of integers $\{\mu_1 > \cdots > \mu_{2r} \geq 0\}$ we write

$$\begin{aligned} \phi_{\mu_1} \ldots \phi_{\mu_{2r}} |0\rangle &= |\mu_1, \ldots, \mu_{2r}) = |\tilde{\mu}) \\ \langle 0| \phi^*_{\mu_{2r}} \ldots \phi^*_{\mu_1} &= (\mu_1, \ldots, \mu_{2r}| = (\tilde{\mu}| \end{aligned} \tag{1.4.15}$$

We let $|\emptyset)$ and $(\emptyset|$ be copies of the empty partition $\emptyset$ as before, and $|\tilde{\mu})$ and $(\tilde{\mu}|$ be copies of the strict partition $\tilde{\mu} = \{\mu_1 > \cdots > \mu_{2r} \geq 0\}$. This gives a one-to-one correspondence between the elements of the bases (1.4.11) and the elements of the set of all strict partitions. Strict partitions of even length are paired to basis vectors with $\mu_{2r} > 0$, while strict partitions of odd length are paired to basis vectors with $\mu_{2r} = 0$. This correspondence is useful in later sections, when we encounter functions which are indexed by strict partitions.

1.4.5 Neutral fermion expectation values

Since $Cl_\phi \subset Cl_\psi$, the vacuum expectation value of $g_\phi \in Cl_\phi$ is inherited from the definition (1.1.24). The following result is the neutral fermion analogue of lemma 2.

Lemma 13. Let $\{m_1, \ldots, m_{2r}\}$ be an arbitrary set of integers, with even cardinality. We claim that

$$\langle \phi_{m_1} \ldots \phi_{m_{2r}} \rangle = \mathrm{Pf}\Big(\langle \phi_{m_i} \phi_{m_j} \rangle \Big)_{1 \leq i < j \leq 2r} \tag{1.4.16}$$

[9] Throughout the rest of this thesis we shall use a tilde to indicate that a partition is strict.

where Pf() denotes a Pfaffian.

Proof. The case $r = 1$ is trivial, since

$$\langle \phi_{m_1} \phi_{m_2} \rangle = \mathrm{Pf}\Big(\langle \phi_{m_i} \phi_{m_j} \rangle \Big)_{1 \leq i < j \leq 2} \tag{1.4.17}$$

Using this case as the basis for induction, we assume there exists $r \geq 2$ such that

$$\langle \phi_{m_1} \dots \phi_{m_{2r-2}} \rangle = \mathrm{Pf}\Big(\langle \phi_{m_i} \phi_{m_j} \rangle \Big)_{1 \leq i < j \leq 2r-2} \tag{1.4.18}$$

for all sets of integers $\{m_1, \dots, m_{2r-2}\}$. We define

$$\begin{aligned} I_1 &= \langle \phi_{m_1} \dots \phi_{m_{2r}} \rangle \\ I_2 &= \sum_{k=2}^{2r} (-)^k \langle \phi_{m_1} \phi_{m_k} \rangle \langle \phi_{m_2} \dots \widehat{\phi_{m_k}} \dots \phi_{m_{2r}} \rangle \end{aligned} \tag{1.4.19}$$

where $\{m_1, \dots, m_{2r}\}$ is an arbitrary set of integers, and $\widehat{\phi_{m_k}}$ means the omission of the indicated fermion. By the annihilation properties (1.4.8) of the fermions, $I_1 = I_2 = 0$ unless both $m_1 \leq 0$ and $m_1 = \{-m_{k_1}, \dots, -m_{k_l}\}$ for some integers $2 \leq k_1 < \dots < k_l \leq 2r$. When both these conditions are satisfied it is readily verified that

$$I_1 = I_2 = \sum_{k \in \{k_1, \dots, k_l\}} (-)^k \langle \phi_{m_1} \phi_{m_k} \rangle \langle \phi_{m_2} \dots \widehat{\phi_{m_k}} \dots \phi_{m_{2r}} \rangle \tag{1.4.20}$$

which proves that $I_1 = I_2$ for all m_1. Therefore we obtain

$$\begin{aligned} I_1 &= \sum_{k=2}^{2r} (-)^k \langle \phi_{m_1} \phi_{m_k} \rangle \langle \phi_{m_2} \dots \widehat{\phi_{m_k}} \dots \phi_{m_{2r}} \rangle \\ &= \sum_{k=2}^{2r} (-)^k \langle \phi_{m_1} \phi_{m_k} \rangle \mathrm{Pf}\Big(\langle \phi_{m_i} \phi_{m_j} \rangle \Big)_{\substack{2 \leq i < j \leq 2r \\ i,j \neq k}} = \mathrm{Pf}\Big(\langle \phi_{m_i} \phi_{m_j} \rangle \Big)_{1 \leq i < j \leq 2r} \end{aligned} \tag{1.4.21}$$

where we have used the assumption (1.4.18) to produce the $(2r-2) \times (2r-2)$ Pfaffian in the second line of (1.4.21). This completes the proof by induction. □

Since $\mathcal{F}_\phi \subset \mathcal{F}_\psi$ and $\mathcal{F}_\phi^* \subset \mathcal{F}_\psi^*$, we inherit a mapping $\mathcal{F}_\phi^* \times \mathcal{F}_\phi \to \mathbb{C}$ from the bilinear form (1.1.32). Calculating its action on the strict partition vectors $(\tilde{\mu}| = \langle 0 | \phi_{\mu_{2r}}^* \dots \phi_{\mu_1}^*$ and $|\tilde{\nu}) = \phi_{\nu_1} \dots \phi_{\nu_{2s}} |0\rangle$, we obtain

$$\Big\langle (\tilde{\mu}|, |\tilde{\nu}) \Big\rangle = \langle 0|\phi^*_{\mu_{2r}} \ldots \phi^*_{\mu_1} \phi_{\nu_1} \ldots \phi_{\nu_{2s}}|0\rangle = 2^{\ell(\tilde{\mu})} \delta_{r,s} \prod_{i=1}^{2r} \delta_{\mu_i,\nu_i} = 2^{\ell(\tilde{\mu})} \delta_{\tilde{\mu},\tilde{\nu}} \quad (1.4.22)$$

where $\ell(\tilde{\mu})$ denotes the length of $\tilde{\mu}$. In particular, $\ell(\tilde{\mu}) = 2r$ if $\mu_{2r} > 0$, and $\ell(\tilde{\mu}) = 2r - 1$ if $\mu_{2r} = 0$. The factors of 2 which appear in (1.4.22) are due to the anticommutation relations (1.4.2), and are characteristic of many calculations in the context of neutral fermions. In analogy with the earlier result (1.1.33), the bilinear form $\langle , \rangle$ induces orthogonality between the strict partition elements of $\mathcal{F}^{(0)}_{\phi}$ and $\mathcal{F}^{*(0)}_{\phi}$.

1.4.6 Lie algebra B_∞

Let $B_\infty \subset Cl^{(0)}_{\phi}$ be the vector space whose elements $Y \in B_\infty$ are of the form

$$Y = \sum_{i,j\in\mathbb{Z}} b_{i,j} : \phi_i \phi_j : + \kappa \quad (1.4.23)$$

where we have defined the normal ordering $: \phi_i \phi_j := \phi_i \phi_j - \langle 0|\phi_i \phi_j|0\rangle$, and where the coefficients satisfy $b_{i,j} = 0$ for $|i + j|$ sufficiently large, with $\kappa \in \mathbb{C}$.[10] Since $: \phi_i \phi_j := - : \phi_j \phi_i :$ and $: \phi_i \phi_i := 0$ for all $i, j \in \mathbb{Z}$, we can equivalently write the elements of this vector space as

$$Y = \sum_{i<j} \Big(b_{i,j} - b_{j,i} \Big) : \phi_i \phi_j : + \kappa \quad (1.4.24)$$

where the sum is now over all integers i, j such that $i < j$.

Lemma 14. The vector space B_∞ becomes a Lie algebra when it is equipped with the commutator as Lie bracket.

Proof. We prove the closure of B_∞ under commutation. Let Y be given by (1.4.23) and define

$$Y_1 = \sum_{i,j\in\mathbb{Z}} b^{(1)}_{i,j} : \phi_i \phi_j : + \kappa_1 \quad (1.4.25)$$

where $b^{(1)}_{i,j} = 0$ for $|i + j|$ sufficiently large. By direct calculation, we obtain

[10] In [50], B_∞ was defined as the subset of A_∞ which is invariant under a certain automorphism σ_0. The vector space (1.4.23) was denoted B'_∞ and shown to be isomorphic to $B_\infty \subset A_\infty$. Two different classes of τ-function were obtained for the BKP hierarchy, the first corresponding with B_∞, and the second with B'_∞. In this thesis we only consider the second class of τ-function, and have abbreviated $B'_\infty = B_\infty$ since there is no potential for confusion.

$$[Y, Y_1] = \sum_{i,j\in\mathbb{Z}} b^{(2)}_{i,j} : \phi_i \phi_j : +\kappa_2 \tag{1.4.26}$$

where for all $i, j \in \mathbb{Z}$ we have defined

$$b^{(2)}_{i,j} = 2 \sum_{k\in\mathbb{Z}} (-)^k \Big(b_{i,k} - b_{k,i} \Big) \Big(b^{(1)}_{-k,j} - b^{(1)}_{j,-k} \Big) \tag{1.4.27}$$

and where

$$\begin{aligned} \kappa_2 = {} & 2 \left(\sum_{k<0} - \sum_{k>0} \right) (-)^k \Big(b_{0,k} - b_{k,0} \Big) \Big(b^{(1)}_{-k,0} - b^{(1)}_{0,-k} \Big) \\ & + 4 \left(\sum_{i>0,j<0} - \sum_{i<0,j>0} \right) (-)^{i+j} \Big(b_{i,-j} b^{(1)}_{-i,j} + b^{(1)}_{i,-j} b_{j,-i} \Big) \end{aligned} \tag{1.4.28}$$

Clearly the right hand side of equation (1.4.26) is also an element of B_∞, proving the closure of the set under commutation. □

1.4.7 B_∞ Heisenberg subalgebra

Adopting the notation of exercise 14.15 in chapter 14 of [54], we define operators $\lambda_m \in B_\infty$ which are given by

$$\lambda_m = \frac{1}{4} \sum_{j\in\mathbb{Z}} (-)^j \phi_{(-j-m)} \phi_j \tag{1.4.29}$$

for all $m \in \tilde{\mathbb{Z}}$, where $\tilde{\mathbb{Z}} = \{\ldots, -3, -1, 1, 3, \ldots\}$ is the set of odd integers.[11] Together with the central element 1, the operators $\{\lambda_m\}_{m\in\tilde{\mathbb{Z}}}$ generate a Heisenberg subalgebra. The closure of this set follows from the commutation relation

$$[\lambda_m, \lambda_n] = \frac{m}{2} \delta_{m+n,0} \tag{1.4.30}$$

for all $m, n \in \tilde{\mathbb{Z}}$.[12] The Heisenberg generators (1.4.29) also have a simple commutation relation with the neutral fermions (1.4.1), given by

$$[\lambda_m, \phi_n] = \phi_{n-m} \tag{1.4.31}$$

for all $m \in \tilde{\mathbb{Z}}$ and $n \in \mathbb{Z}$.

[11] Throughout the remainder of this chapter, a tilde is also used to denote a set containing odd elements.

[12] The relation (1.4.30) can be proved by noticing that λ_m, λ_n are obtained from Y, Y_1 by setting $b_{i,j} = \frac{1}{4}(-)^j \delta_{i+j+m,0}$, $b^{(1)}_{i,j} = \frac{1}{4}(-)^j \delta_{i+j+n,0}$ and $\kappa = \kappa_1 = 0$. Substituting these values into (1.4.27) and (1.4.28), we obtain the desired commutator (1.4.30) by virtue of (1.4.26).

1.4.8 BKP evolution operators

We introduce the Hamiltonian

$$\lambda\{\tilde{t}\} = \sum_{n\in\tilde{\mathbb{N}}} t_n \lambda_n \tag{1.4.32}$$

where $\{\tilde{t}\} = \{t_1, t_3, t_5, \ldots\}$ is an infinite set of free variables with odd indices, and define the generating function

$$\Phi(k) = \sum_{i\in\mathbb{Z}} \phi_i k^i \tag{1.4.33}$$

where k is an indeterminate. Using these definitions and the equation (1.4.31), we obtain the commutation relation

$$[\lambda\{\tilde{t}\}, \Phi(k)] = \sum_{n\in\tilde{\mathbb{N}}} \sum_{i\in\mathbb{Z}} [\lambda_n, \phi_i] t_n k^i = \left(\sum_{n\in\tilde{\mathbb{N}}} t_n k^n \right) \Phi(k) \tag{1.4.34}$$

which, in turn, implies that

$$e^{\lambda\{\tilde{t}\}} \Phi(k) = \exp \left(\sum_{n\in\tilde{\mathbb{N}}} t_n k^n \right) \Phi(k) e^{\lambda\{\tilde{t}\}} \tag{1.4.35}$$

In analogy with the first part of the chapter, the operator $e^{\lambda\{\tilde{t}\}}$ is called a *BKP evolution operator*. It plays an essential role in constructing solutions of the BKP hierarchy of partial differential equations.

1.4.9 Schur Q-polynomials

For all $m \geq 0$, the *one-row Schur Q-polynomial* $\mathcal{Q}_m\{\tilde{t}\}$ in the infinite set of odd variables $\{\tilde{t}\}$ is defined by

$$\mathcal{Q}_m\{\tilde{t}\} = \text{Coeff}_{k^m} \left[\exp \left(\sum_{n\in\tilde{\mathbb{N}}} t_n k^n \right) \right] = \chi_m\{t_1, 0, t_3, 0, t_5, \ldots\} \tag{1.4.36}$$

where $\chi_m\{t_1, 0, t_3, 0, t_5, \ldots\}$ is the one-row Schur polynomial (1.1.48) with all even variables set to zero. We also define $\mathcal{Q}_m\{\tilde{t}\} = 0$ for all $m < 0$. From this, the *Schur*

Q-polynomial [13] $\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\}$ associated to the strict partition $\tilde{\mu} = \{\mu_1 > \cdots > \mu_{2r} \geq 0\}$ is given by

$$\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\} = \mathrm{Pf}\left(\mathcal{Q}_{\mu_i}\{\tilde{t}\}\mathcal{Q}_{\mu_j}\{\tilde{t}\} + 2\sum_{k=1}^{\mu_j}(-)^k\mathcal{Q}_{(\mu_i+k)}\{\tilde{t}\}\mathcal{Q}_{(\mu_j-k)}\{\tilde{t}\}\right)_{1\leq i<j\leq 2r} \tag{1.4.37}$$

where each $\mathcal{Q}_m\{\tilde{t}\}$ represents a one-row Schur Q-polynomial. The following result maps the strict partition elements of $\mathcal{F}_\phi^{(0)}$ to their corresponding Schur Q-polynomials.

Lemma 15. Let $|\tilde{\mu}) = |\mu_1, \ldots, \mu_{2r}) = \phi_{\mu_1}\ldots\phi_{\mu_{2r}}|0\rangle$ be an element of the strict partition basis of $\mathcal{F}_\phi^{(0)}$. We claim that

$$\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\} = (\emptyset|e^{\lambda\{\tilde{t}\}}|\tilde{\mu}) \tag{1.4.38}$$

This result may also be found in, for example, [76].

Proof. Using the definition of the one-row Schur Q-polynomials (1.4.36) in the commutation relation (1.4.35) and then extracting the coefficient of k^m from the resulting equation, we obtain

$$e^{\lambda\{\tilde{t}\}}\phi_m = \left(\sum_{i=0}^{\infty}\phi_{(m-i)}\mathcal{Q}_i\{\tilde{t}\}\right)e^{\lambda\{\tilde{t}\}} \tag{1.4.39}$$

By definition, we have

$$(\emptyset|e^{\lambda\{\tilde{t}\}}|\tilde{\mu}) = \langle 0|e^{\lambda\{\tilde{t}\}}\phi_{\mu_1}\ldots\phi_{\mu_{2r}}|0\rangle \tag{1.4.40}$$

and using the commutation relation (1.4.39) we move the evolution operator $e^{\lambda\{\tilde{t}\}}$ in (1.4.40) towards the right, obtaining

$$(\emptyset|e^{\lambda\{\tilde{t}\}}|\tilde{\mu}) = \sum_{i_1,\ldots,i_{2r}=0}^{\infty}\left\langle\phi_{(\mu_1-i_1)}\ldots\phi_{(\mu_{2r}-i_{2r})}\right\rangle\mathcal{Q}_{i_1}\{\tilde{t}\}\ldots\mathcal{Q}_{i_{2r}}\{\tilde{t}\} \tag{1.4.41}$$

where we have used the fact that $e^{\lambda\{\tilde{t}\}}|0\rangle = |0\rangle$. Applying lemma 13 to the previous vacuum expectation value, we find

[13] The Schur Q-polynomials were introduced in [80], which studied their connection with the projective representations of the symmetric and alternating groups.

$$(\emptyset|e^{\lambda\{\tilde{t}\}}|\tilde{\mu}) = \sum_{i_1,\dots,i_{2r}=0}^{\infty} \mathrm{Pf}\Big(\langle \phi_{(\mu_p-i_p)}\phi_{(\mu_q-i_q)}\rangle\Big)_{1\le p<q\le 2r} \mathcal{Q}_{i_1}\{\tilde{t}\}\dots\mathcal{Q}_{i_{2r}}\{\tilde{t}\} \tag{1.4.42}$$

Collecting the one-row Schur Q-polynomials inside the Pfaffian, we obtain

$$(\emptyset|e^{\lambda\{\tilde{t}\}}|\tilde{\mu}) = \mathrm{Pf}\left(\sum_{i_p=0}^{\infty}\sum_{i_q=0}^{\infty} \langle \phi_{(\mu_p-i_p)}\phi_{(\mu_q-i_q)}\rangle \mathcal{Q}_{i_p}\{\tilde{t}\}\mathcal{Q}_{i_q}\{\tilde{t}\}\right)_{1\le p<q\le 2r} \tag{1.4.43}$$

Using the annihilation properties (1.4.8) and defining $\mathcal{Q}_m\{\tilde{t}\} = 0$ for all $m < 0$, equation (1.4.43) becomes

$$(\emptyset|e^{\lambda\{\tilde{t}\}}|\tilde{\mu}) = \mathrm{Pf}\left(\sum_{i_p\in\mathbb{Z}}\sum_{i_q=0}^{\mu_q} \langle \phi_{(\mu_p-i_p)}\phi_{(\mu_q-i_q)}\rangle \mathcal{Q}_{i_p}\{\tilde{t}\}\mathcal{Q}_{i_q}\{\tilde{t}\}\right)_{1\le p<q\le 2r} \tag{1.4.44}$$

$$= \mathrm{Pf}\left(\mathcal{Q}_{\mu_p}\{\tilde{t}\}\mathcal{Q}_{\mu_q}\{\tilde{t}\} + \sum_{\substack{i_p\in\mathbb{Z}\\ 0\le i_q<\mu_q}} \langle \phi_{(\mu_p-i_p)}\phi_{(\mu_q-i_q)}\rangle \mathcal{Q}_{i_p}\{\tilde{t}\}\mathcal{Q}_{i_q}\{\tilde{t}\}\right)_{1\le p<q\le 2r}$$

We evaluate the vacuum expectation value inside the previous Pfaffian, using the anticommutation relations (1.4.2) and annihilation properties (1.4.8) to obtain

$$\langle \phi_{(\mu_p-i_p)}\phi_{(\mu_q-i_q)}\rangle = 2(-)^{\mu_p-i_p}\delta_{\mu_p-i_p,i_q-\mu_q} \tag{1.4.45}$$

for all $i_q < \mu_q$. Substituting this result into (1.4.44) and using the Kronecker delta to truncate the sum over i_p, we find

$$(\emptyset|e^{\lambda\{\tilde{t}\}}|\tilde{\mu}) = \mathrm{Pf}\left(\mathcal{Q}_{\mu_p}\{\tilde{t}\}\mathcal{Q}_{\mu_q}\{\tilde{t}\} + 2\sum_{i_q=0}^{\mu_q-1}(-)^{\mu_q-i_q}\mathcal{Q}_{(\mu_p+\mu_q-i_q)}\{\tilde{t}\}\mathcal{Q}_{i_q}\{\tilde{t}\}\right)_{1\le p<q\le 2r} \tag{1.4.46}$$

The Schur Q-polynomial (1.4.37) is recovered by making the change of indices $k = \mu_q - i_q$ in each entry of the Pfaffian (1.4.46). □

Corollary. Let $\{m\} = \{m_1 > \cdots > m_{2r+1} \ge 0\}$ be an ordered set of integers with odd cardinality. Given that $\langle \phi_0\phi_{n_1}\dots\phi_{n_{2r+1}}\rangle = \langle \phi_{n_1}\dots\phi_{n_{2r+1}}\phi_0\rangle$ for all sets of integers $\{n_1,\dots,n_{2r+1}\}$, we find

$$\langle 1|e^{\lambda\{\tilde{t}\}}\phi_{m_1}\dots\phi_{m_{2r+1}}|0\rangle = \langle 0|e^{\lambda\{\tilde{t}\}}\phi_{m_1}\dots\phi_{m_{2r+1}}\phi_0|0\rangle \tag{1.4.47}$$

where we have made use of the fact that $\langle 1| = \langle 0|\phi_0$. Using the result (1.4.38) from the previous lemma in the right hand side of (1.4.47), we obtain

$$\langle 1|e^{\lambda\{\tilde{t}\}}\phi_{m_1}\dots\phi_{m_{2r+1}}|0\rangle = \mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\} \tag{1.4.48}$$

where we have defined the strict partition

$$\tilde{\mu} = \begin{cases} \{m_1,\dots,m_{2r+1},0\}, & m_{2r+1} > 0 \\ \{m_1,\dots,m_{2r}\}, & m_{2r+1} = 0 \end{cases} \tag{1.4.49}$$

Equation (1.4.48) will prove useful in later calculations.

1.4.10 Schur Q-functions

Following section 2 in chapter III of [65], the function $q_m\{x\}$ in the infinite set of variables $\{x\}$ is defined as

$$q_m\{x\} = \mathrm{Coeff}_{k^m}\left[\prod_{i=1}^{\infty}\frac{1+x_i k}{1-x_i k}\right] \tag{1.4.50}$$

From this definition, the *Schur Q-function* $Q_{\tilde{\mu}}\{x\}$ associated to the strict partition $\tilde{\mu} = \{\mu_1 > \cdots > \mu_{2r} \geq 0\}$ is given by

$$Q_{\tilde{\mu}}\{x\} = \mathrm{Pf}\left(q_{\mu_i}\{x\}q_{\mu_j}\{x\} + 2\sum_{k=1}^{\mu_j}(-)^k q_{(\mu_i+k)}\{x\}q_{(\mu_j-k)}\{x\}\right)_{1\leq i<j\leq 2r} \tag{1.4.51}$$

Lemma 16. The Schur Q-polynomial $\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\}$ and the Schur Q-function $Q_{\tilde{\mu}}\{x\}$ are equal under the change of variables $t_n = \frac{2}{n}\sum_{i=1}^{\infty} x_i^n$ for all $n \in \tilde{\mathbb{N}}$.

Proof. Fixing $t_n = \frac{2}{n}\sum_{i=1}^{\infty} x_i^n$ for all $n \in \tilde{\mathbb{N}}$, we obtain

$$\mathrm{Coeff}_{k^m}\left[\exp\left(\sum_{n\in\tilde{\mathbb{N}}} t_n k^n\right)\right] = \mathrm{Coeff}_{k^m}\left[\exp\left(\sum_{i=1}^{\infty}\sum_{n\in\tilde{\mathbb{N}}}\frac{2}{n}(x_i k)^n\right)\right] = \mathrm{Coeff}_{k^m}\left[\prod_{i=1}^{\infty}\frac{1+x_i k}{1-x_i k}\right] \tag{1.4.52}$$

implying that $\mathcal{Q}_m\{\tilde{t}\} = q_m\{x\}$ for all $m \geq 0$. Comparing the definition (1.4.37) with (1.4.51), we immediately see that $\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\} = Q_{\tilde{\mu}}\{x\}$ under the prescribed change of variables. □

From section 8 in chapter III of [65], the Schur Q-functions $Q_{\tilde{\mu}}\{x\}$ comprise a basis for symmetric functions in $\{x\}$ which are independent of the even power sums $\frac{1}{2n}\sum_{i=1}^{\infty} x_i^{2n}$. This fact, together with the equality of $\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\}$ and $Q_{\tilde{\mu}}\{x\}$ under the previous change of variables, proves that the Schur Q-polynomials $\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\}$ are a basis for the set of all polynomials in the odd variables $\{\tilde{t}\}$.

In chapter 3 we will consider Schur Q-functions $Q_{\tilde{\mu}}\{x\}$ in finitely many variables $\{x\} = \{x_1, \ldots, x_N\}$. These are obtained from the formulae (1.4.50), (1.4.51) by setting $x_n = 0$ for all $n > N$. In these cases, $Q_{\tilde{\mu}}\{x\} = 0$ if $\ell(\tilde{\mu}) > N$.

1.5 BKP hierarchy

1.5.1 BKP hierarchy in bilinear form

The *BKP hierarchy* is an infinite set of partial differential equations in the independent variables $\{\tilde{t}\} = \{t_1, t_3, t_5, \ldots\}$. As in the case of the KP hierarchy, the BKP hierarchy derives from a single integral equation, called the *BKP bilinear identity.* A function $\tau\{\tilde{t}\}$ which satisfies every differential equation in the hierarchy, or equivalently, satisfies the BKP bilinear identity, is called a *BKP τ-function.*

Define the shifted sets of variables

$$\{\tilde{t} \pm 2\tilde{\epsilon}_k\} = \left\{t_1 \pm 2k^{-1}, t_3 \pm \frac{2}{3}k^{-3}, t_5 \pm \frac{2}{5}k^{-5}, \ldots\right\} \tag{1.5.1}$$

where k is a free parameter. The BKP bilinear identity is the equation

$$\oint \exp\left(\sum_{n\in\tilde{\mathbb{N}}}(t_n - s_n)k^n\right)\tau\{\tilde{t} - 2\tilde{\epsilon}_k\}\tau\{\tilde{s} + 2\tilde{\epsilon}_k\}\frac{dk}{2\pi ik} = \tau\{\tilde{t}\}\tau\{\tilde{s}\} \tag{1.5.2}$$

where $\{\tilde{t}\} = \{t_1, t_3, t_5, \ldots\}$ and $\{\tilde{s}\} = \{s_1, s_3, s_5, \ldots\}$ are two infinite sets of variables with odd subscripts, and the integration in the k-plane is taken around a small contour at $k = \infty$.

As in the case of the KP hierarchy, we will assume $\tau\{\tilde{t}\}$ is a polynomial in its variables. In this special case, $\tau\{\tilde{t} - 2\tilde{\epsilon}_k\}$ and $\tau\{\tilde{s} + 2\tilde{\epsilon}_k\}$ have singularities in k only at $k = 0$. Therefore, for polynomial τ-functions, the BKP bilinear identity becomes

$$\mathrm{Coeff}_1\left[\exp\left(\sum_{n\in\tilde{\mathbb{N}}}(t_n - s_n)k^n\right)\tau\{\tilde{t} - 2\tilde{\epsilon}_k\}\tau\{\tilde{s} + 2\tilde{\epsilon}_k\}\right] = \tau\{\tilde{t}\}\tau\{\tilde{s}\} \tag{1.5.3}$$

where $\mathrm{Coeff}_1[f(k)]$ denotes the coefficient of k^0 in the Laurent series of $f(k)$. We expose the infinitely many differential equations which underly the equation (1.5.3) by making the substitutions $\{\tilde{t}\} \to \{\tilde{t} - \tilde{s}\}$ and $\{\tilde{s}\} \to \{\tilde{t} + \tilde{s}\}$, giving

$$\text{Coeff}_1 \left[\exp\left(-2 \sum_{n\in\tilde{\mathbb{N}}} s_n k^n \right) \tau\{\tilde{t} - \tilde{s} - 2\tilde{\epsilon}_k\} \tau\{\tilde{t} + \tilde{s} + 2\tilde{\epsilon}_k\} \right] = \tau\{\tilde{t} - \tilde{s}\}\tau\{\tilde{t} + \tilde{s}\} \tag{1.5.4}$$

Using the definition (1.4.36), the exponential term in (1.5.4) can be replaced with a sum over one-row Schur Q-polynomials, producing the equation

$$\text{Coeff}_1 \left[\sum_{m=0}^{\infty} \mathcal{Q}_m\{-2\tilde{s}\} k^m \tau\{\tilde{t} - \tilde{s} - 2\tilde{\epsilon}_k\} \tau\{\tilde{t} + \tilde{s} + 2\tilde{\epsilon}_k\} \right] = \tau\{\tilde{t} - \tilde{s}\}\tau\{\tilde{t} + \tilde{s}\} \tag{1.5.5}$$

or equivalently,

$$\sum_{m=0}^{\infty} \mathcal{Q}_m\{-2\tilde{s}\} \text{Coeff}_{k^{-m}} \left[\tau\{\tilde{t} - \tilde{s} - 2\tilde{\epsilon}_k\} \tau\{\tilde{t} + \tilde{s} + 2\tilde{\epsilon}_k\} \right] = \tau\{\tilde{t} - \tilde{s}\}\tau\{\tilde{t} + \tilde{s}\} \tag{1.5.6}$$

In order to progress further we need the following result, which is a simple modification of lemma 7.

Lemma 17.

$$\tau\{\tilde{t} - \tilde{s} - 2\tilde{\epsilon}_k\} \tau\{\tilde{t} + \tilde{s} + 2\tilde{\epsilon}_k\} = \exp\left(\sum_{n\in\tilde{\mathbb{N}}} (s_n + \frac{2}{n} k^{-n}) D_n \right) \tau\{\tilde{t}\} \cdot \tau\{\tilde{t}\} \tag{1.5.7}$$

Proof. We refer the reader to the proof of lemma 7. □

Returning to equation (1.5.6), we employ the result (1.5.7) to obtain

$$\sum_{m=0}^{\infty} \mathcal{Q}_m\{-2\tilde{s}\} \text{Coeff}_{k^{-m}} \left[\exp\left(\sum_{n\in\tilde{\mathbb{N}}} (s_n + \frac{2}{n} k^{-n}) D_n \right) \right] \tau\{\tilde{t}\} \cdot \tau\{\tilde{t}\} = \tau\{\tilde{t} - \tilde{s}\}\tau\{\tilde{t} + \tilde{s}\} \tag{1.5.8}$$

Recalling the definition (1.4.36) of the one-row Schur Q-polynomials once again, equation (1.5.8) becomes

$$\sum_{m=0}^{\infty} \mathcal{Q}_m\{-2\tilde{s}\} \mathcal{Q}_m\{2\tilde{\mathbb{D}}\} \exp\left(\sum_{n\in\tilde{\mathbb{N}}} s_n D_n \right) \tau\{\tilde{t}\} \cdot \tau\{\tilde{t}\} = \tau\{\tilde{t} - \tilde{s}\}\tau\{\tilde{t} + \tilde{s}\} \tag{1.5.9}$$

where we have defined the set of operators $\{\tilde{\mathbb{D}}\} = \{D_1, \frac{1}{3}D_3, \frac{1}{5}D_5, \ldots\}$. We notice that the right hand side of (1.5.9) can be expressed as

$$\tau\{\tilde{t}-\tilde{s}\}\tau\{\tilde{t}+\tilde{s}\} = \exp\left(\sum_{n\in\tilde{\mathbb{N}}} s_n D_n\right)\tau\{\tilde{t}\}\cdot\tau\{\tilde{t}\} \tag{1.5.10}$$

which cancels with the $m = 0$ term of the left hand side summation, yielding

$$\sum_{m=1}^{\infty} \mathcal{Q}_m\{-2\tilde{s}\}\mathcal{Q}_m\{2\tilde{\mathbb{D}}\}\exp\left(\sum_{n\in\tilde{\mathbb{N}}} s_n D_n\right)\tau\{\tilde{t}\}\cdot\tau\{\tilde{t}\} = 0 \tag{1.5.11}$$

For all odd $\{1 \leq m_1 < \cdots < m_l\}$ and $\{n_1,\ldots,n_l \geq 1\}$ the coefficient of the monomial $s_{m_1}^{n_1}\ldots s_{m_l}^{n_l}$ on the left hand side of (1.5.11) must vanish, giving rise to infinitely many consistency equations, which are the differential equations of the BKP hierarchy.

Example 3. Up to an irrelevant factor the coefficient of s_3^2 on the left hand side of (1.5.11) is equal to $(D_1^6 - 5D_1^3D_3 - 5D_3^2 + 9D_1D_5)\tau\{\tilde{t}\}\cdot\tau\{\tilde{t}\}$, implying that

$$(D_1^6 - 5D_1^3D_3 - 5D_3^2 + 9D_1D_5)\tau\{\tilde{t}\}\cdot\tau\{\tilde{t}\} = 0 \tag{1.5.12}$$

which is the BKP equation in bilinear form. Higher equations in the hierarchy are obtained from the coefficients of different monomials.

1.5.2 Neutral fermion bilinear identity

We now focus on constructing solutions of the BKP bilinear identity (1.5.2) using the calculus of the neutral fermions $\{\phi_m\}_{m\in\mathbb{Z}}$. The following result is the neutral fermion analogue of theorem 1.

Theorem 3. Let g_ϕ be a finite element of $Cl_\phi^{(0)}$ and define

$$\tau\{\tilde{t}\} = \langle 0|e^{\lambda\{\tilde{t}\}}g_\phi|0\rangle = \left\langle e^{\lambda\{\tilde{t}\}}g_\phi\right\rangle = \left\langle \phi_0 e^{\lambda\{\tilde{t}\}}g_\phi\phi_0\right\rangle \tag{1.5.13}$$

where the equality between these two expectation values follows from the fact that $\langle\phi_0\phi_{m_1}\ldots\phi_{m_{2r}}\phi_0\rangle = \langle\phi_{m_1}\ldots\phi_{m_{2r}}\rangle$ for all sets of integers $\{m_1,\ldots,m_{2r}\}$. The polynomial $\tau\{\tilde{t}\}$ satisfies the BKP bilinear identity (1.5.2) if and only if g_ϕ satisfies the *neutral fermion bilinear identity* (NFBI)

$$\sum_{i\in\mathbb{Z}} \phi_i g_\phi|0\rangle \otimes \phi_i^* g_\phi|0\rangle = g_\phi|1\rangle \otimes g_\phi|1\rangle \tag{1.5.14}$$

Proof. As we did in the proof of theorem 1, we split this proof into three steps. In the first two steps we prove that if (1.5.14) holds, then $\tau\{\tilde{t}\}$ as given by (1.5.13) satisfies the BKP bilinear identity (1.5.2). In the third step we prove the converse statement.

Step 1. Acting upon the left hand side of (1.5.14) with the tensored dual states $\langle 1|e^{\lambda\{\tilde{t}\}} \otimes \langle 1|e^{\lambda\{\tilde{s}\}}$ we have the result

$$\sum_{i\in\mathbb{Z}} \langle 1|e^{\lambda\{\tilde{t}\}}\phi_i g_\phi|0\rangle\langle 1|e^{\lambda\{\tilde{s}\}}\phi_i^* g_\phi|0\rangle = \tau\{\tilde{t}\}\tau\{\tilde{s}\} \tag{1.5.15}$$

We convert the sum on the left hand side of (1.5.15) into a contour integral, using the generating function (1.4.33) to write

$$\oint \langle 1|e^{\lambda\{\tilde{t}\}}\Phi(k)g_\phi|0\rangle\langle 1|e^{\lambda\{\tilde{s}\}}\Phi(-k)g_\phi|0\rangle\frac{dk}{2\pi ik} = \tau\{\tilde{t}\}\tau\{\tilde{s}\} \tag{1.5.16}$$

where the contour of integration surrounds the pole at $k = 0$. By virtue of the commutation relation (1.4.35), it is possible to switch the order of $e^{\lambda\{\tilde{t}\}}$ and $\Phi(k)$, and likewise $e^{\lambda\{\tilde{s}\}}$ and $\Phi(-k)$ in (1.5.16), giving

$$\oint e^{\sum_{n\in\tilde{\mathbb{N}}}(t_n-s_n)k^n}\langle 1|\Phi(k)e^{\lambda\{\tilde{t}\}}g_\phi|0\rangle\langle 1|\Phi(-k)e^{\lambda\{\tilde{s}\}}g_\phi|0\rangle\frac{dk}{2\pi ik} = \tau\{\tilde{t}\}\tau\{\tilde{s}\} \tag{1.5.17}$$

Step 2. (Lemma 18.) We propose the identity

$$\langle 1|\Phi(k) = \langle 0|\exp\Big(-\sum_{n\in\tilde{\mathbb{N}}}\frac{2}{n}k^{-n}\lambda_n\Big) \tag{1.5.18}$$

Proof. Using the fact that $\langle 1| = \langle 0|\phi_0$ and the definition of the generating function (1.4.33), we obtain

$$\langle 1|\Phi(k) = \langle 0|\phi_0\sum_{i\in\mathbb{Z}}\phi_i k^i = \langle 0| + \sum_{i=1}^{\infty}k^{-i}\langle 0|\phi_0\phi_{-i} \tag{1.5.19}$$

where we have used the identity $\phi_0^2 = 1$ and the annihilation properties (1.4.8) to truncate the sum. Now consider the expression (1.4.36) for the one-row Schur Q-polynomial. When the variables $\{\tilde{t}\}$ are set to $t_n = -2k^{-n}/n$ for all $n \in \tilde{\mathbb{N}}$, this expression simplifies greatly. We obtain

$$\mathcal{Q}_m\{\tilde{t}\}\Big|_{t_n=-2k^{-n}/n} = \begin{cases} 1, & m = 0 \\ 2(-k)^{-m}, & m \geq 1 \end{cases} \tag{1.5.20}$$

and substitute this formula into the expression (1.4.37) for the Schur Q-polynomial associated to $\tilde{\mu} = \{\mu_1 > \cdots > \mu_{2r} \geq 0\}$, giving

$$\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\}\Big|_{t_n=-2k^{-n}/n} = 2(-k)^{-\mu_1}\delta_{r,1}\delta_{\mu_2,0} \tag{1.5.21}$$

Using the result of lemma 15, this equation becomes

$$\langle 0|\exp\Big(-\sum_{n\in\tilde{\mathbb{N}}}\frac{2}{n}k^{-n}\lambda_n\Big)\phi_{\mu_1}\ldots\phi_{\mu_{2r}}|0\rangle = 2(-k)^{-\mu_1}\delta_{r,1}\delta_{\mu_2,0} \tag{1.5.22}$$

Finally, due to the orthogonality of strict partition vectors (1.4.22), we obtain

$$\langle 0|\exp\Big(-\sum_{n\in\tilde{\mathbb{N}}}\frac{2}{n}k^{-n}\lambda_n\Big) = \langle 0| + \sum_{i=1}^{\infty}(-k)^{-i}\langle 0|\phi_0^*\phi_i^* = \langle 0| + \sum_{i=1}^{\infty}k^{-i}\langle 0|\phi_0\phi_{-i} \tag{1.5.23}$$

Comparing this equation with (1.5.19), we complete the proof of (1.5.18). □

Applying (1.5.18) to equation (1.5.17), we obtain

$$\oint e^{\sum_{n\in\tilde{\mathbb{N}}}(t_n-s_n)k^n}\langle 0|e^{\lambda\{\tilde{t}-2\tilde{\epsilon}_k\}}g_\phi|0\rangle\langle 0|e^{\lambda\{\tilde{s}+2\tilde{\epsilon}_k\}}g_\phi|0\rangle\frac{dk}{2\pi ik} = \tau\{\tilde{t}\}\tau\{\tilde{s}\} \tag{1.5.24}$$

Equation (1.5.24) proves that if (1.5.14) holds, functions given by (1.5.13) satisfy the BKP bilinear identity (1.5.2).

Step 3. For any finite $g_\phi \in Cl_\phi^{(0)}$, there exist coefficients $\kappa_{\{m\},\{n\}}$ such that

$$\sum_{i\in\mathbb{Z}}\phi_i g_\phi|0\rangle\otimes\phi_i^* g_\phi|0\rangle - g_\phi|1\rangle\otimes g_\phi|1\rangle = \sum_{\{m\},\{n\}}\kappa_{\{m\},\{n\}}\phi_{\{m\}}|0\rangle\otimes\phi_{\{n\}}|0\rangle \tag{1.5.25}$$

where the sum is over all sets of integers $\{m\} = \{m_1 > \cdots > m_{2r+1} \geq 0\}$ and $\{n\} = \{n_1 > \cdots > n_{2s+1} \geq 0\}$, whose cardinalities can assume all odd values $(2r+1),(2s+1) \geq 1$. Acting upon both sides of this equation with the tensored dual states $\langle 1|e^{\lambda\{\tilde{t}\}} \otimes \langle 1|e^{\lambda\{\tilde{s}\}}$ we find

$$\oint e^{\sum_{n\in\tilde{\mathbb{N}}}(t_n-s_n)k^n}\langle 0|e^{\lambda\{\tilde{t}-2\tilde{\epsilon}_k\}}g_\phi|0\rangle\langle 0|e^{\lambda\{\tilde{s}+2\tilde{\epsilon}_k\}}g_\phi|0\rangle\frac{dk}{2\pi ik} - \langle 1|e^{\lambda\{\tilde{t}\}}g_\phi|1\rangle\langle 1|e^{\lambda\{\tilde{s}\}}g_\phi|1\rangle$$
$$= \sum_{\{m\},\{n\}}\kappa_{\{m\},\{n\}}\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\}\mathcal{Q}_{\tilde{\nu}}\{\tilde{s}\} \tag{1.5.26}$$

where the left hand side of (1.5.26) has already been derived in steps 1 and 2, and the right hand side follows from (1.4.48) with $\tilde{\mu}, \tilde{\nu}$ defined by (1.4.49). Assuming that $\tau\{\tilde{t}\}$ as given by (1.5.13) satisfies the BKP bilinear identity, we thus obtain

$$\sum_{\{m\},\{n\}} \kappa_{\{m\},\{n\}} \mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\} \mathcal{Q}_{\tilde{\nu}}\{\tilde{s}\} = 0 \tag{1.5.27}$$

which can only be true if all of the coefficients $\kappa_{\{m\},\{n\}} = 0$, since the Schur Q-polynomials are linearly independent. Substituting this trivial value for the coefficients into (1.5.25), we recover the NFBI (1.5.14). This completes the proof of the converse statement. □

1.6 Solutions of the NFBI

1.6.1 Orbit of O_∞

Theorem 4. Suppose g_ϕ is a finite element of $Cl_\phi^{(0)}$. Then g_ϕ solves the neutral fermionic bilinear identity (1.5.14) if and only if

$$g_\phi|0\rangle = e^{Y_1} \dots e^{Y_l}|0\rangle \tag{1.6.1}$$

for some $\{Y_1, \dots, Y_l\} \in B_\infty$. In other words, the solution space of (1.5.14) is generated by the orbit of the Lie group

$$O_\infty = \left\{ e^{Y_1} \dots e^{Y_l} \;\middle|\; Y_i \in B_\infty \text{ for all } 1 \leq i \leq l \right\} \tag{1.6.2}$$

Proof. As we did in the proof of theorem 2, we split this proof into two steps. In the first step we prove the forward statement, in the second step we prove its converse.

Step 1. (Lemma 19.) Let $|u\rangle$ and $|v\rangle$ be arbitrary state vectors in $\mathcal{F}_\phi$, and let $g = e^{Y_1} \dots e^{Y_l}$ with each $Y_i \in B_\infty$. We have

$$\sum_{i\in\mathbb{Z}} \phi_i g_\phi|u\rangle \otimes \phi_i^* g_\phi|v\rangle = \sum_{i\in\mathbb{Z}} g_\phi \phi_i|u\rangle \otimes g_\phi \phi_i^*|v\rangle \tag{1.6.3}$$

Proof. (Lemma 19.) For $m \geq 0$ and arbitrary $Y \in B_\infty$, let $\mathcal{P}_m$ denote the proposition

$$(\phi_i \otimes \phi_i^*) \sum_{n=0}^{m} \binom{m}{n} Y^n \otimes Y^{m-n} = \sum_{n=0}^{m} \binom{m}{n} Y^n \otimes Y^{m-n} (\phi_i \otimes \phi_i^*) \tag{1.6.4}$$

where summation over all integers i is implied. The proposition $\mathcal{P}_0$ is trivial. Furthermore, letting $Y \in B_\infty$ be given by (1.4.23), by direct calculation we obtain the commutation relations

$$[\phi_i, Y] = 2(-)^i \sum_{j\in\mathbb{Z}} (b_{-i,j} - b_{j,-i})\phi_j, \quad [\phi_i^*, Y] = 2\sum_{j\in\mathbb{Z}} (-)^j (b_{i,-j} - b_{-j,i})\phi_j^* \tag{1.6.5}$$

Using these commutators in the left hand side of $\mathcal{P}_1$, we obtain

$$\begin{aligned}(\phi_i \otimes \phi_i^*)(1 \otimes Y + Y \otimes 1) &= (1 \otimes Y + Y \otimes 1)(\phi_i \otimes \phi_i^*) \\ &\quad + 2(-)^j (b_{i,-j} - b_{-j,i})\phi_i \otimes \phi_j^* + 2(-)^i (b_{-i,j} - b_{j,-i})\phi_j \otimes \phi_i^* \\ &= (1 \otimes Y + Y \otimes 1)(\phi_i \otimes \phi_i^*)\end{aligned} \tag{1.6.6}$$

where summation over all integers i, j is implied. This proves $\mathcal{P}_1$ is true. Using the inductive procedure from the proof of lemma 9, we find that $\mathcal{P}_m$ is true for all $m \geq 0$.[14] By virtue of the proposition (1.6.4), for any $Y \in B_\infty$ we have

$$\begin{aligned}(\phi_i \otimes \phi_i^*)(e^Y \otimes e^Y)|u\rangle \otimes |v\rangle =&(\phi_i \otimes \phi_i^*) \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^{m} \binom{m}{n} Y^n \otimes Y^{m-n} |u\rangle \otimes |v\rangle \\ =& \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{n=0}^{m} \binom{m}{n} Y^n \otimes Y^{m-n} (\phi_i \otimes \phi_i^*)|u\rangle \otimes |v\rangle \\ =&(e^Y \otimes e^Y)(\phi_i \otimes \phi_i^*)|u\rangle \otimes |v\rangle\end{aligned} \tag{1.6.7}$$

Therefore we have proved that

$$\sum_{i\in\mathbb{Z}} \phi_i e^Y |u\rangle \otimes \phi_i^* e^Y |v\rangle = \sum_{i\in\mathbb{Z}} e^Y \phi_i |u\rangle \otimes e^Y \phi_i^* |v\rangle \tag{1.6.8}$$

for arbitrary $Y \in B_\infty$. Using (1.6.8) l times successively, once for each e^{Y_i} in g_ϕ, we prove (1.6.3). □

Corollary. Having established the validity of equation (1.6.3) we employ a particular case of it, namely when $|u\rangle = |v\rangle = |0\rangle$, which gives

$$\sum_{i\in\mathbb{Z}} \phi_i g_\phi |0\rangle \otimes \phi_i^* g_\phi |0\rangle = \sum_{i\in\mathbb{Z}} g_\phi \phi_i |0\rangle \otimes g_\phi \phi_i^* |0\rangle = g_\phi |1\rangle \otimes g_\phi |1\rangle \tag{1.6.9}$$

[14] Since the inductive basis $\mathcal{P}_1$ holds, the proof of (1.6.4) becomes immediate from the proof of (1.3.4) by substituting $\psi_i \to \phi_i$, $\psi_i^* \to \phi_i^*$, $X \to Y$.

where the final equality is due to the fact that for all $i \in \mathbb{Z}^\times$ either $\phi_i|0\rangle = 0$ or $\phi_i^*|0\rangle = 0$. Equation (1.6.9) completes the proof that when $g_\phi|0\rangle$ is of the form (1.6.1), g_ϕ satisfies the NFBI (1.5.14).

Step 2. (Lemma 20.) Let $g_\phi \in Cl_\phi^{(0)}$ satisfy the NFBI (1.5.14). Then for suitable $\{Y_1, \ldots, Y_l\} \in B_\infty$ we can write $g_\phi|0\rangle = e^{Y_1} \ldots e^{Y_l}|0\rangle$.

Proof. (Lemma 20.) The proof is analogous to the proof of lemma 10. Since $g_\phi|0\rangle \in \mathcal{F}_\phi^{(0)}$ we can expand it in terms of the basis (1.4.11), by writing

$$g_\phi|0\rangle = c_\emptyset|0\rangle + \sum_{m>n\geq 0} c_{m,n}\phi_m\phi_n|0\rangle + g_\phi^{(1)}|0\rangle \tag{1.6.10}$$

for some suitable coefficients $c_\emptyset$ and $c_{m,n}$, and where all monomials within $g_\phi^{(1)} \in Cl_\phi^{(0)}$ consist of at least four neutral fermions.[15] From here, we need to consider the cases $c_\emptyset \neq 0$ and $c_\emptyset = 0$ separately.

Case 1. ($c_\emptyset \neq 0$) We define the elements Y_1, Y_2 of B_∞ as follows

$$Y_1 = \log c_\emptyset, \quad Y_2 = \sum_{m>n\geq 0} c_{m,n}^{(1)}\phi_m\phi_n \tag{1.6.11}$$

where $c_{m,n}^{(1)} = c_{m,n}/c_\emptyset$ for all $m > n \geq 0$. We trivially obtain

$$e^{-Y_1}g_\phi|0\rangle = |0\rangle + \sum_{m>n\geq 0} c_{m,n}^{(1)}\phi_m\phi_n|0\rangle + g_\phi^{(2)}|0\rangle \tag{1.6.12}$$

where we have defined $g_\phi^{(2)} = g_\phi^{(1)}/c_\emptyset$. Next, we act on equation (1.6.12) with the operator e^{-Y_2}. Since $\left(\sum_{m>0} c_{m,0}^{(1)}\phi_m\phi_0\right)^2 = 0$, term by term we have

$$e^{-Y_2}|0\rangle = |0\rangle - \sum_{m>n\geq 0} c_{m,n}^{(1)}\phi_m\phi_n|0\rangle + g_\phi^{(3)}|0\rangle \tag{1.6.13}$$

$$e^{-Y_2}\sum_{m>n\geq 0} c_{m,n}^{(1)}\phi_m\phi_n|0\rangle = \sum_{m>n\geq 0} c_{m,n}^{(1)}\phi_m\phi_n|0\rangle + g_\phi^{(4)}|0\rangle \tag{1.6.14}$$

$$e^{-Y_2}g_\phi^{(2)}|0\rangle = g_\phi^{(5)}|0\rangle \tag{1.6.15}$$

for some suitable $g_\phi^{(3)}, g_\phi^{(4)}, g_\phi^{(5)} \in Cl_\phi^{(0)}$. Combining these three results, we obtain

[15]Throughout the rest of the proof, we will always use $g_\phi^{(i)}$ to denote an element of $Cl_\phi^{(0)}$ with precisely this property.

$$e^{-Y_2}e^{-Y_1}g_\phi|0\rangle = |0\rangle + g_\phi^{(6)}|0\rangle \tag{1.6.16}$$

where we have defined $g_\phi^{(6)} = g_\phi^{(3)} + g_\phi^{(4)} + g_\phi^{(5)}$. By virtue of equation (1.6.3) and the fact that g_ϕ obeys the NFBI (1.5.14), we have

$$\begin{aligned} e^{-Y_2}e^{-Y_1}g_\phi|1\rangle \otimes e^{-Y_2}e^{-Y_1}g_\phi|1\rangle &= \sum_{i\in\mathbb{Z}} e^{-Y_2}e^{-Y_1}\phi_i g_\phi|0\rangle \otimes e^{-Y_2}e^{-Y_1}\phi_i^* g_\phi|0\rangle \\ &= \sum_{i\in\mathbb{Z}} \phi_i e^{-Y_2}e^{-Y_1}g_\phi|0\rangle \otimes \phi_i^* e^{-Y_2}e^{-Y_1}g_\phi|0\rangle \end{aligned} \tag{1.6.17}$$

Substituting the expression (1.6.16) for $e^{-Y_2}e^{-Y_1}g_\phi|0\rangle$ into (1.6.17) and using the annihilation properties (1.4.8), we find

$$\begin{aligned} &\sum_{i\geq 0} \phi_i|0\rangle \otimes \phi_i^* g_\phi^{(6)}|0\rangle + \sum_{i\leq 0} \phi_i g_\phi^{(6)}|0\rangle \otimes \phi_i^*|0\rangle + \sum_{i\in\mathbb{Z}} \phi_i g_\phi^{(6)}|0\rangle \otimes \phi_i^* g_\phi^{(6)}|0\rangle \\ &= |1\rangle \otimes g_\phi^{(6)}|1\rangle + g_\phi^{(6)}|1\rangle \otimes |1\rangle + g_\phi^{(6)}|1\rangle \otimes g_\phi^{(6)}|1\rangle \end{aligned} \tag{1.6.18}$$

We recall that all monomials within $g_\phi^{(6)} \in Cl_\phi^{(0)}$ consist of at least four neutral fermions. Therefore the first two sums on the left hand side of (1.6.18) contain terms which do not appear in the rest of the equation. These terms vanish if and only if $\phi_i g_\phi^{(6)}|0\rangle = 0$ for all $i < 0$. The only possible resolution is that $g_\phi^{(6)}|0\rangle = 0$. Substituting this value for $g_\phi^{(6)}|0\rangle$ into (1.6.16) we see that $e^{-Y_2}e^{-Y_1}g_\phi|0\rangle = |0\rangle$, or equivalently, $g_\phi|0\rangle = e^{Y_1}e^{Y_2}|0\rangle$. This completes the proof in the case $c_\emptyset \neq 0$.

Case 2. ($c_\emptyset = 0$) We begin by stating two identities which we use in the proof. Fix two integers $p > q > 0$ and a set $\{m\} = \{m_1 > \cdots > m_{2r} \geq 0\}$. The first identity reads

$$e^{-\frac{1}{2}\phi_p\phi_q}e^{-\frac{1}{2}\phi_p^*\phi_q^*}\phi_{\{m\}}|0\rangle = \begin{cases} 2(-)^{i+j+1}\phi_{\{m\backslash m_i,m_j\}}|0\rangle, & p = m_i \\ & q = m_j \\ \phi_{\{m\}}|0\rangle + \frac{1}{2}\phi_p\phi_q\phi_{\{m\}}|0\rangle, & p \notin \{m\} \\ & q \notin \{m\} \\ \phi_{\{m\}}|0\rangle, & \text{otherwise} \end{cases} \tag{1.6.19}$$

where we have used the notation $\{m\backslash m_i, m_j\}$ to denote the omission of the i^{th} and j^{th} elements from the set $\{m\}$. The second identity reads

$$e^{-\frac{1}{2}\phi_p\phi_0}e^{-\phi_p^*\phi_0^*}\phi_{\{m\}}|0\rangle = \begin{cases} 2(-)^{i+1}\phi_{\{m\backslash m_i, m_{2r}\}}|0\rangle, & p = m_i \\ & m_{2r} = 0 \\ 2(-)^i\phi_{\{m\backslash m_i\}}\phi_0|0\rangle, & p = m_i \\ & m_{2r} > 0 \\ \phi_{\{m\}}|0\rangle + \frac{1}{2}\phi_p\phi_{\{m\backslash m_{2r}\}}|0\rangle, & p \notin \{m\} \\ & m_{2r} = 0 \\ \phi_{\{m\}}|0\rangle - \frac{1}{2}\phi_p\phi_{\{m\}}\phi_0|0\rangle, & p \notin \{m\} \\ & m_{2r} > 0 \end{cases} \tag{1.6.20}$$

where the meaning of the notations $\{m\backslash m_i, m_{2r}\}$, $\{m\backslash m_i\}$ and $\{m\backslash m_{2r}\}$ should be clear from previous explanations. Returning to the proof, we observe that since $c_\emptyset = 0$ we can write

$$g_\phi|0\rangle = \sum_{\text{card}\{m\}=2r} c_{\{m\}}\phi_{\{m\}}|0\rangle + g_\phi^{(7)}|0\rangle \tag{1.6.21}$$

where the sum is taken over all sets of integers $\{m_1 > \cdots > m_{2r} \geq 0\}$ of some fixed cardinality $2r \geq 2$, and all monomials within $g_\phi^{(7)} \in Cl_\phi^{(0)}$ consist of at least $2r + 2$ neutral fermions. Let $c_{\{p\}}$ be a particular non-zero coefficient in the sum (1.6.21), corresponding to the set $\{p\} = \{p_1 > \cdots > p_{2r} \geq 0\}$, and define

$$Y_{2i-1} = \frac{1}{2}\left(1 + \delta_{p_{2i},0}\right)\phi_{p_{2i-1}}^*\phi_{p_{2i}}^*, \quad Y_{2i} = \frac{1}{2}\phi_{p_{2i-1}}\phi_{p_{2i}} \tag{1.6.22}$$

for all $1 \leq i \leq r$. Successively applying the identities (1.6.19) and (1.6.20) to $g_\phi|0\rangle$, we obtain

$$e^{-Y_{2r}}e^{-Y_{2r-1}}\ldots e^{-Y_2}e^{-Y_1}g_\phi|0\rangle = c_\emptyset^{(2)}|0\rangle + \sum_{m>n\geq 0} c_{m,n}^{(2)}\phi_m\phi_n|0\rangle + g_\phi^{(8)}|0\rangle \tag{1.6.23}$$

with $c_\emptyset^{(2)} = 2^r c_{\{p\}}$ and the remaining coefficients $c_{m,n}^{(2)}$ suitably chosen, and where all monomials within $g_\phi^{(8)} \in Cl_\phi^{(0)}$ consist of at least four neutral fermions. Since $c_\emptyset^{(2)} \neq 0$, we can apply the procedure of case 1 to (1.6.23), ultimately obtaining

$$g_\phi|0\rangle = e^{Y_1}e^{Y_2}\ldots e^{Y_{2r+1}}e^{Y_{2r+2}}|0\rangle \tag{1.6.24}$$

where we have defined

$$Y_{2r+1} = \log c_{\emptyset}^{(2)}, \quad Y_{2r+2} = \sum_{m>n\geq 0} c_{m,n}^{(2)}/c_{\emptyset}^{(2)} \phi_m \phi_n |0\rangle \tag{1.6.25}$$

Since all $\{Y_1, \ldots, Y_{2r+2}\} \in B_\infty$, equation (1.6.24) completes the proof in the $c_\emptyset = 0$ case. We have therefore proved lemma 20 which, in turn, finishes the proof of theorem 4. □

1.6.2 Schur Q-polynomials and the orbit of O_∞

Example 4. As a particular case of theorem 4, we show that every Schur Q-polynomial (1.4.37) is a BKP τ-function. Let $|\tilde{\mu}) = |\mu_1, \ldots, \mu_{2r})$ be an arbitrary strict partition equal to the Fock space vector $\phi_{\mu_1} \ldots \phi_{\mu_{2r}}|0\rangle$. Recalling equation (1.4.40) from the proof of lemma 15, we have

$$\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\} = \left\langle e^{\lambda\{\tilde{t}\}} \phi_{\mu_1} \ldots \phi_{\mu_{2r}} \right\rangle \tag{1.6.26}$$

Defining

$$Y_{2i-1} = \frac{1}{2}(1 + \delta_{\mu_{2i},0}) \phi_{\mu_{2i-1}}^{*} \phi_{\mu_{2i}}^{*}, \quad Y_{2i} = \frac{1}{2} \phi_{\mu_{2i-1}} \phi_{\mu_{2i}} \tag{1.6.27}$$

for all $1 \leq i \leq r$ and using the identities (1.6.19) and (1.6.20) from the last subsection, we obtain

$$e^{-Y_{2r}} e^{-Y_{2r-1}} \ldots e^{-Y_2} e^{-Y_1} \phi_{\mu_1} \ldots \phi_{\mu_{2r}} |0\rangle = 2^r |0\rangle \tag{1.6.28}$$

or equivalently,

$$\phi_{\mu_1} \ldots \phi_{\mu_{2r}} |0\rangle = 2^r e^{Y_1} e^{Y_2} \ldots e^{Y_{2r-1}} e^{Y_{2r}} |0\rangle \tag{1.6.29}$$

Substituting (1.6.29) into (1.6.26), we find

$$\mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\} = 2^r \left\langle e^{\lambda\{\tilde{t}\}} e^{Y_1} \ldots e^{Y_{2r}} \right\rangle \tag{1.6.30}$$

Hence any Schur Q-polynomial can be written as an expectation value of the form (1.5.13), with $g_\phi \in O_\infty$. By theorem 4, the Schur Q-polynomials are therefore τ-functions of the BKP hierarchy [72], [91].

1.6.3 BKP Plücker relations

In this subsection we solve the NFBI (1.5.14) from another, more direct perspective. Let $g_\phi|0\rangle$ and $g_\phi|1\rangle$ be corresponding finite elements of $\mathcal{F}_\phi^{(0)}$ and $\mathcal{F}_\phi^{(1)}$. Expanding $g_\phi|0\rangle$ in terms of the basis (1.4.11), there exist coefficients $c_{\{m\}}$ such that

$$g_\phi|0\rangle = \sum_{\{m\}} c_{\{m\}}\phi_{\{m\}}|0\rangle \tag{1.6.31}$$

where the sum is over all sets of integers $\{m\} = \{m_1 > \cdots > m_{2r} \geq 0\}$ whose cardinalities take all values $2r \geq 0$. Because $g_\phi|0\rangle$ is finite, $c_{\{m\}} = 0$ if card$\{m\}$ is sufficiently large. Similarly, we have

$$g_\phi|1\rangle = \sum_{\{m\}} c_{\{m\}}\phi_{\{m\}}|1\rangle \tag{1.6.32}$$

Using these expansions of $g_\phi|0\rangle$ and $g_\phi|1\rangle$, we obtain

$$\sum_{i\in\mathbb{Z}} \phi_i g_\phi|0\rangle \otimes \phi_i^* g_\phi|0\rangle - g_\phi|1\rangle \otimes g_\phi|1\rangle = \tag{1.6.33}$$

$$\sum_{\{m\},\{n\}} c_{\{m\}}c_{\{n\}} \left(\sum_{i\in\mathbb{Z}} \phi_i\phi_{\{m\}}|0\rangle \otimes \phi_i^*\phi_{\{n\}}|0\rangle - \phi_{\{m\}}\phi_0|0\rangle \otimes \phi_{\{n\}}\phi_0|0\rangle \right)$$

where the first sum is over all sets of integers $\{m\} = \{m_1 > \cdots > m_{2r} \geq 0\}$ and $\{n\} = \{n_1 > \cdots > n_{2s} \geq 0\}$, whose cardinalities take all even values $2r, 2s \geq 0$. Using the annihilation properties (1.4.8) of the fermions, we find

$$\sum_{i\in\mathbb{Z}} \phi_i g_\phi|0\rangle \otimes \phi_i^* g_\phi|0\rangle - g_\phi|1\rangle \otimes g_\phi|1\rangle = 2 \sum_{\{m\},\{n\}} c_{\{m\}}c_{\{n\}} \times \tag{1.6.34}$$

$$\left(\sum_{i=1}^{2r} (-)^{i-1} \phi_{\{m\backslash m_i\}}|0\rangle \otimes \phi_{\{m_i,n\}}|0\rangle + \sum_{j=1}^{2s} (-)^{j-1} \phi_{\{n_j,m\}}|0\rangle \otimes \phi_{\{n\backslash n_j\}}|0\rangle \right)$$

where we have defined $\phi_{\{m_i,n\}} = 0$ if $m_i \in \{n\}$ and $\phi_{\{n_j,m\}} = 0$ if $n_j \in \{m\}$. Changing the indexing sets of the first sum in (1.6.34), we obtain the equivalent expression

$$\sum_{i\in\mathbb{Z}} \phi_i g_\phi|0\rangle \otimes \phi_i^* g_\phi|0\rangle - g_\phi|1\rangle \otimes g_\phi|1\rangle = \tag{1.6.35}$$

$$2 \sum_{\{p\},\{q\}} \phi_{\{p\}}|0\rangle \otimes \phi_{\{q\}}|0\rangle \left(\sum_{i=1}^{2r-1} (-)^{i-1} c_{\{p\backslash p_i\}} c_{\{p_i,q\}} + \sum_{j=1}^{2s-1} (-)^{j-1} c_{\{q_j,p\}} c_{\{q\backslash q_j\}} \right)$$

where the sum is over all sets of integers $\{p\} = \{p_1 > \cdots > p_{2r-1} \geq 0\}$ and $\{q\} = \{q_1 > \cdots > q_{2s-1} \geq 0\}$ whose cardinalities take all odd values, and where we have defined

$$c_{\{p_i,q\}} = (-)^{j-1} c_{\{q_1,\ldots,q_{j-1},p_i,q_j,\ldots,q_{2s-1}\}} \tag{1.6.36}$$

if $q_{j-1} > p_i > q_j$ for some $1 \leq j \leq 2s$, and $c_{\{p_i,q\}} = 0$ if $p_i \in \{q\}$. A similar definition applies to $c_{\{q_j,p\}}$. The right hand side of (1.6.35) vanishes if and only if

$$\sum_{i=1}^{2r-1} (-)^i c_{\{m\backslash m_i\}} c_{\{m_i,n\}} + \sum_{j=1}^{2s-1} (-)^j c_{\{n_j,m\}} c_{\{n\backslash n_j\}} = 0 \tag{1.6.37}$$

for all sets $\{m_1 > \cdots > m_{2r-1} \geq 0\}$ and $\{n_1 > \cdots > n_{2s-1} \geq 0\}$. Collectively, these conditions are called the *BKP Plücker relations*, and we summarize their significance with the following statement (which we have already proved).

Lemma 21. $g_\phi \in Cl_\phi^{(0)}$ satisfies the NFBI (1.5.14) if and only if the expansion coefficients (1.6.31) of $g_\phi|0\rangle$ obey the BKP Plücker relations (1.6.37).

1.6.4 Pfaffian solution of BKP Plücker relations

With the following result, we present a general Pfaffian solution of the BKP Plücker relations (1.6.37).

Lemma 22. To every even-cardinality set of integers $\{m\} = \{m_1, \ldots, m_{2r}\}$ we associate the coefficient

$$c_{\{m\}} = \mathrm{Pf}\Big(c_{m_i,m_j}\Big)_{1\leq i<j\leq 2r} = |m_1, \ldots, m_{2r}| \tag{1.6.38}$$

where the matrix entries $c_{i,j}$ are arbitrary constants that satisfy the antisymmetry condition $c_{i,j} = -c_{j,i}$. These coefficients satisfy the BKP Plücker relations (1.6.37).

Proof. The proof is based on identity (2.97) in section 2.8 of [47]. Define two ordered sets of integers $\{m\} = \{m_1 > \cdots > m_{2r-1} \geq 0\}$ and $\{n\} = \{n_1 > \cdots > n_{2s-1} \geq 0\}$, with fixed odd cardinalities. By the definition of the coefficients (1.6.38), we obtain

$$\sum_{i=1}^{2r-1} (-)^i c_{\{m\backslash m_i\}} c_{\{m_i,n\}} = \sum_{i=1}^{2r-1} (-)^i |m_1, \ldots, \widehat{m_i}, \ldots, m_{2r-1}||m_i, n_1, \ldots, n_{2s-1}| \tag{1.6.39}$$

Expanding the second Pfaffian in (1.6.39), we have

$$\sum_{i=1}^{2r-1}(-)^i c_{\{m\backslash m_i\}}c_{\{m_i,n\}} = \sum_{i=1}^{2r-1}\sum_{j=1}^{2s-1}(-)^{i+j}|m_1,\ldots,\widehat{m_i},\ldots,m_{2r-1}||m_i,n_j||n_1,\ldots,\widehat{n_j},\ldots,n_{2s-1}| \tag{1.6.40}$$

Similarly, we find that

$$\sum_{j=1}^{2s-1}(-)^j c_{\{n_j,m\}}c_{\{n\backslash n_j\}} = \sum_{j=1}^{2s-1}(-)^j|n_j,m_1,\ldots,m_{2r-1}||n_1,\ldots,\widehat{n_j},\ldots,n_{2s-1}| \tag{1.6.41}$$

Expanding the first Pfaffian in (1.6.41), we obtain

$$\sum_{j=1}^{2s-1}(-)^j c_{\{n_j,m\}}c_{\{n\backslash n_j\}} = \sum_{j=1}^{2s-1}\sum_{i=1}^{2r-1}(-)^{j+i}|n_j,m_i||m_1,\ldots,\widehat{m_i},\ldots,m_{2r-1}||n_1,\ldots,\widehat{n_j},\ldots,n_{2s-1}| \tag{1.6.42}$$

Summing the equations (1.6.40) and (1.6.42) we observe that the right hand side of the resultant equation vanishes, where we have used the fact that $|m_i,n_j| = -|n_j,m_i|$ for all $1 \leq i \leq 2r-1$ and $1 \leq j \leq 2s-1$. This shows that the BKP Plücker relations (1.6.37) are satisfied. □

1.7 Conclusion

Before ending this chapter, we present a brief summary of the material that has been discussed. We especially wish to emphasize those results which find application in the remainder of the thesis.

The Clifford algebras Cl_ψ and Cl_ϕ provide a basic framework for the study of the KP and BKP hierarchies, respectively. We defined Fock representations of these algebras, and constructed partition (1.1.16) and strict partition (1.4.11) bases for the respective Fock subspaces $\mathcal{F}_\psi^{(0)}$ and $\mathcal{F}_\phi^{(0)}$. We demonstrated that the elements of these bases may be identified with Schur (1.1.49) and Schur Q-polynomials (1.4.37), via the respective equations (1.1.50) and (1.4.38). These polynomials, in turn, form a basis for the solution space of the KP and BKP hierarchies. We will continue to refer to the bases (1.1.16) and (1.4.11) and the polynomials (1.1.49) and (1.4.37) throughout the rest of the thesis, particularly in chapter 3, where they play a prominent role.

The KP and BKP bilinear identities, (1.2.2) and (1.5.2) respectively, contain all of the differential equations of their corresponding hierarchies. With theorems 1 and 3 we showed that the solutions of these bilinear identities, the τ-functions, are expressible as fermionic expectation values. The task of solving (1.2.2) and (1.5.2) was shown to be equivalent to solving the charged and neutral fermionic bilinear identities, (1.2.15) and (1.5.14), respectively.

Solutions of the CFBI and NFBI may be obtained from two different perspectives. The first perspective was demonstrated with theorems 2 and 4, where we showed that solutions of (1.2.15) and (1.5.14) are given by the orbit of the vacuum under GL_∞ and O_∞, respectively. The second perspective depends on finding solutions to the Plücker relations, (1.3.40) and (1.6.37). Explicit solutions of the Plücker relations were presented by the formulae (1.3.41) and (1.6.38), respectively. In the coming chapters we will encounter objects whose expansion coefficients are determinants or Pfaffians. By virtue of lemmas 12 and 22, we will thus be able to connect these objects with solutions of the KP and BKP hierarchies.

Chapter 2

Overview of quantum inverse scattering method

2.0 Introduction

In the 1970s the Leningrad school developed a quantum version of the technique which had been discovered in [44]. One of the first steps towards this quantization can be found in [92]. The technique itself became known as the quantum inverse scattering method and it was introduced in [28], where it was used in the context of integrable field-theoretical models. During the 1980s the method was extended to the discrete, lattice versions of these models. In this thesis we shall apply the quantum inverse scattering method to the descendents and relatives of another type of discrete model, the XYZ spin-$\frac{1}{2}$ chain, which is closely connected with the eight-vertex model of statistical mechanics [2], [3], [4], [5].

The purpose of this chapter is to provide a brief introduction to the quantum inverse scattering method, in a sufficiently general setting. All of the models that we study later are specializations of the generic model discussed here, and this chapter enables us to unify much of the notation and conventions used throughout the thesis. In section 2.1 we discuss the basic aspects common to all quantum integrable models which we study. These include the Hamiltonian $\mathcal{H}$ of the model, and its representation on a lattice of finitely many sites. The complete space of lattice states is denoted by $\mathcal{V}$, and the goal of the quantum inverse scattering method is to construct states within $\mathcal{V}$ which are eigenvectors of $\mathcal{H}$.

In the context of a quantum mechanical model, integrability means that $\mathcal{H}$ belongs to a family of commuting operators. The generating function of these operators is called the transfer matrix $t(u)$, and the quantum inverse scattering method is the technique through which $t(u)$ is constructed. In section 2.2, we review the quantum inverse scattering method for a generalized model. We introduce the R-matrix, which is a solution of the Yang-Baxter equation, as well as the L-matrix and monodromy matrix, whose entries are operators acting in $\mathcal{V}$. The commutation relations between these entries are given by the intertwining equations. We conclude by expressing the transfer matrix as the trace of the monodromy matrix. All of the objects defined in this section have graphical representations which we also include, since they provide a correspondence with the lattice models of statistical physics.

In section 2.3 we describe the algebraic Bethe Ansatz for calculating eigenvectors of the transfer matrix, and derive the system of equations necessary for the success of the Ansatz, known as the Bethe equations. We define the scalar product between two different Bethe eigenvectors, and discuss its graphical representation as a two-dimensional lattice. The definitions appearing in this section are essential to the remainder of the thesis, throughout which we focus on Bethe eigenvectors and scalar products across several different models.

The material presented in this chapter is fairly ubiquitous throughout the literature, and as such we do not adhere to any particular reference. Essentially we will provide gleanings from [27], [61], [85], which are three standard introductory works on the subject. For more information, the reader is referred to these sources.

2.1 Quantum integrable models

2.1.1 Discrete one-dimensional models and their space of states $\mathcal{V}$

In this thesis we will study several discrete one-dimensional quantum mechanical systems. The playing field for these systems is a one-dimensional integral lattice with finitely many sites. Particles are placed at each lattice site, and every unique way of assigning particles to the lattice is called a *configuration*. The system is allowed to adopt any state which is a linear combination of individual lattice configurations.

To place these concepts on a more mathematical foundation, it is convenient to use the language of state vectors. Let $M \geq 1$ denote the number of lattice sites, and associate a *local quantum space* $\mathcal{V}_i$ to the i^{th} site for all $1 \leq i \leq M$. These vector spaces have the basis

$$\text{Basis}(\mathcal{V}_i) = \left\{ |n\rangle_i \,\middle|\, n \in \mathfrak{N} \right\} \tag{2.1.1}$$

where n is a number which can take any value in the set $\mathfrak{N}$. The interpretation of $\mathfrak{N}$ and the state vectors $|n\rangle_i$ depends on the system under consideration.

In chapters 3 and 4 we study systems which obey *Bose-Einstein* statistics, and have no limit on the number of particles per site. For these systems, $\mathfrak{N} = \mathbb{N} \cup \{0\}$ and $|n\rangle_i$ represents the number of particles at the i^{th} lattice site. In this case the integers n are called *occupation numbers*. On the other hand, in chapters 5 and 6 we study systems which obey *Fermi-Dirac* statistics, and have one spin-$\frac{1}{2}$ particle per site. For these systems, $\mathfrak{N} = \{+\frac{1}{2}, -\frac{1}{2}\} = \{\uparrow, \downarrow\}$ and $|n\rangle_i$ represents the spin of a single particle at the i^{th} lattice site.

With the definition of each local space $\mathcal{V}_i$ fixed, we introduce the *global quantum space* $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_M$. This vector space has the basis

$$\text{Basis}(\mathcal{V}) = \Big\{ |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M \ \Big| \ n_1, \ldots, n_M \in \mathfrak{N} \Big\} \tag{2.1.2}$$

where the numbers $n_1, \ldots, n_M$ can take any value in the set $\mathfrak{N}$. Every basis vector $|n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M$ describes an individual lattice configuration. In the Bose-Einstein case, $|n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M$ represents a lattice configuration with n_i bosons at the i^{th} site for all $1 \leq i \leq M$. In the Fermi-Dirac case, $|n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M$ represents a lattice configuration of M fermions, such that the i^{th} fermion has intrinsic spin n_i for all $1 \leq i \leq M$. As mentioned previously, the system is allowed to adopt any state in $\mathcal{V}$.

2.1.2 Quantum algebras $\mathcal{A}_i$ and their representation on $\mathcal{V}_i$

In addition to the vector space (2.1.2), a quantum model is described by a set of commuting algebras $\mathcal{A}_1, \ldots, \mathcal{A}_M$. These algebras are in fact copies of a single algebra, with one copy assigned to each lattice site. For all $1 \leq i \leq M$, the algebra $\mathcal{A}_i$ has a representation on the local space $\mathcal{V}_i$, and from this we deduce the action of $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_M$ on $\mathcal{V}$.

Let us make some general remarks which categorize the algebras $\mathcal{A}_1, \ldots, \mathcal{A}_M$ for the models under our consideration. In all cases, $\mathcal{A}_i$ is generated by three elements $\{\mathfrak{a}_i^+, \mathfrak{a}_i^-, \mathfrak{a}_i^0\}$. When these generators act on basis vectors of $\mathcal{V}_i$, both $\mathfrak{a}_i^\pm$ produce a new state, while $\mathfrak{a}_i^0$ returns the original state. More specifically, we can say that

$$\mathfrak{a}_i^+ |n\rangle_i = a_i^+(n)|n+1\rangle_i, \quad \mathfrak{a}_i^- |n\rangle_i = a_i^-(n)|n-1\rangle_i, \quad \mathfrak{a}_i^0 |n\rangle_i = a_i^0(n)|n\rangle_i \tag{2.1.3}$$

for all $n \in \mathfrak{N}$ and some suitable constants $a_i^\pm(n), a_i^0(n)$. The interpretation of (2.1.3) is specific to the model at hand.

When the quantum system obeys Bose-Einstein statistics, $\mathfrak{a}_i^+/\mathfrak{a}_i^-$ play the role of creation/annihilation operators, adding/deleting particles from the i^{th} lattice site. In this case, $a_i^-(0) = 0$, and all states in $\mathcal{V}_i$ can be constructed from the action of $\mathfrak{a}_i^+$ on the vacuum state $|0\rangle_i$. When the system obeys Fermi-Dirac statistics, $\mathfrak{a}_i^+/\mathfrak{a}_i^-$ play the role of raising/lowering operators, raising/lowering the spin of the i^{th} particle. In this case, $a_i^+(\uparrow) = a_i^-(\downarrow) = 0$ and all states in $\mathcal{V}_i$ can be constructed from the action of $\mathfrak{a}_i^-$ on $|\uparrow\rangle_i$, or the action of $\mathfrak{a}_i^+$ on $|\downarrow\rangle_i$. For all physical systems, the state vector $|n\rangle_i$ is an eigenstate of the operator $\mathfrak{a}_i^0$.

2.1.3 Inner products

Let us define an inner product $\mathcal{I}_i$ on the local space $\mathcal{V}_i$. Suppose that $|m\rangle_i$ and $|n\rangle_i$ are two basis vectors of $\mathcal{V}_i$. The inner product $\mathcal{I}_i$ between these vectors is defined as

$$\mathcal{I}_i\Big(|m\rangle_i, |n\rangle_i\Big) = c_i(m)\delta_{m,n} \tag{2.1.4}$$

where $c_i(m)$ denotes a function of the discrete variable $m \in \mathfrak{N}$, which is specific to the model under consideration. Typically, this function is chosen such that the operators $\mathfrak{a}_i^+$ and $\mathfrak{a}_i^-$ are adjoint. Imposing this condition, we find that

$$\mathcal{I}_i\Big(\mathfrak{a}_i^+|m\rangle_i, |n\rangle_i\Big) = \mathcal{I}_i\Big(|m\rangle_i, \mathfrak{a}_i^-|n\rangle_i\Big) \implies a_i^+(m)c_i(n)\delta_{m+1,n} = a_i^-(n)c_i(m)\delta_{m,n-1} \tag{2.1.5}$$

which follows from the actions (2.1.3) of $\mathfrak{a}_i^\pm$ and the definition (2.1.4) of $\mathcal{I}_i$. Equation (2.1.5) is trivially satisfied if $m+1 \neq n$, while it leads to the constraint

$$a_i^+(m)c_i(m+1) = a_i^-(m+1)c_i(m) \tag{2.1.6}$$

on the function $c_i(m)$ in the case $m+1 = n$. The operator $\mathfrak{a}_i^0$ is self-adjoint without any further conditions imposed on the function $c_i(m)$, since

$$\mathcal{I}_i\Big(\mathfrak{a}_i^0|m\rangle_i, |n\rangle_i\Big) = a_i^0(m)c_i(m)\delta_{m,n} = a_i^0(n)c_i(m)\delta_{m,n} = \mathcal{I}_i\Big(|m\rangle_i, \mathfrak{a}_i^0|n\rangle_i\Big) \tag{2.1.7}$$

Now we construct an inner product $\mathcal{I}$ on the global space $\mathcal{V}$, using the definition (2.1.4) of the local inner products $\mathcal{I}_i$. Let $|m\rangle = |m_1\rangle_1 \otimes \cdots \otimes |m_M\rangle_M$ and $|n\rangle = |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M$ be basis vectors of $\mathcal{V}$. The inner product $\mathcal{I}$ between these vectors is defined as

$$\mathcal{I}\Big(|m\rangle, |n\rangle\Big) = \prod_{i=1}^{M} \mathcal{I}_i\Big(|m_i\rangle_i, |n_i\rangle_i\Big) = \prod_{i=1}^{M} c_i(m_i)\delta_{m_i,n_i} \tag{2.1.8}$$

That is, $\mathcal{I}$ induces orthogonality between the basis vectors of $\mathcal{V}$. The inner product between more general elements of $\mathcal{V}$ can be calculated from the assumption that $\mathcal{I}$ is bilinear.

2.1.4 Dual space of states $\mathcal{V}^*$

Rather than using the inner product notation adopted in the last subsection, a standard procedure is to introduce vector spaces which are dual to those already considered. To this end, let $\mathcal{V}_i^*$ denote the dual of $\mathcal{V}_i$. It has the basis

$$\text{Basis}(\mathcal{V}_i^*) = \Big\{\langle m|_i \;\Big|\; m \in \mathfrak{N}\Big\} \tag{2.1.9}$$

where the action of each dual state vector $\langle m|_i$ is given by

$$\langle m|_i() = \mathcal{I}_i\Big(|m\rangle_i, \Big) \tag{2.1.10}$$

The generators of $\mathcal{A}_i$ act on basis elements of $\mathcal{V}_i^*$ as follows

$$\langle m|_i \mathfrak{a}_i^+ = a_i^-(m)\langle m-1|_i, \quad \langle m|_i \mathfrak{a}_i^- = a_i^+(m)\langle m+1|_i, \quad \langle m|_i \mathfrak{a}_i^0 = a_i^0(m)\langle m|_i \tag{2.1.11}$$

By virtue of (2.1.3) and (2.1.11), the action (2.1.10) of the dual space $\mathcal{V}_i^*$, and the fact that $\mathfrak{a}_i^{\pm}$ are adjoint (whilst $\mathfrak{a}_i^0$ is self-adjoint), we recover the equations

$$\langle m|_i \mathfrak{a}_i^+ \Big(|n\rangle_i\Big) = \mathcal{I}_i\Big(\mathfrak{a}_i^-|m\rangle_i, |n\rangle_i\Big) = \mathcal{I}_i\Big(|m\rangle_i, \mathfrak{a}_i^+|n\rangle_i\Big) = \langle m|_i\Big(\mathfrak{a}_i^+|n\rangle_i\Big) \tag{2.1.12}$$

$$\langle m|_i \mathfrak{a}_i^- \Big(|n\rangle_i\Big) = \mathcal{I}_i\Big(\mathfrak{a}_i^+|m\rangle_i, |n\rangle_i\Big) = \mathcal{I}_i\Big(|m\rangle_i, \mathfrak{a}_i^-|n\rangle_i\Big) = \langle m|_i\Big(\mathfrak{a}_i^-|n\rangle_i\Big) \tag{2.1.13}$$

$$\langle m|_i \mathfrak{a}_i^0 \Big(|n\rangle_i\Big) = \mathcal{I}_i\Big(\mathfrak{a}_i^0|m\rangle_i, |n\rangle_i\Big) = \mathcal{I}_i\Big(|m\rangle_i, \mathfrak{a}_i^0|n\rangle_i\Big) = \langle m|_i\Big(\mathfrak{a}_i^0|n\rangle_i\Big) \tag{2.1.14}$$

These equations ensure that the quantities $\langle m|_i \mathfrak{a}_i^{\pm}|n\rangle_i, \langle m|_i \mathfrak{a}_i^0|n\rangle_i$ are well defined without specifying the direction in which the operators $\mathfrak{a}_i^{\pm}, \mathfrak{a}_i^0$ act. We refer to these quantities as *expectation values* of the operators $\mathfrak{a}_i^{\pm}, \mathfrak{a}_i^0$. More generally, expectation values of arbitrary elements of $\mathcal{A}_i$ are unambiguously defined.

These ideas can be extended to the dual $\mathcal{V}^*$ of the global vector space $\mathcal{V}$. Its basis is given by

$$\text{Basis}(\mathcal{V}^*) = \Big\{\langle m_1|_1 \otimes \cdots \otimes \langle m_M|_M \;\Big|\; m_1, \ldots, m_M \in \mathfrak{N}\Big\} \tag{2.1.15}$$

where the action of each dual state vector $\langle m| = \langle m_1|_1 \otimes \cdots \otimes \langle m_M|_M$ is given by

$$\langle m|() = \mathcal{I}\Big(|m\rangle, \Big) \tag{2.1.16}$$

By this definition, the inner product $\mathcal{I}(|m\rangle, |n\rangle)$ can be written as $\langle m|(|n\rangle)$, or more simply $\langle m|n\rangle$. Expectation values of arbitrary elements of $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_M$ remain well defined, regardless of the direction in which operators act.

2.1.5 Hamiltonian $\mathcal{H}$

The physical interactions of a quantum mechanical system are described by its *Hamiltonian* $\mathcal{H} \in \text{End}(\mathcal{V})$. In the models which we study, $\mathcal{H}$ is an algebraic combination of the operators $\mathfrak{a}_i^{\pm}, \mathfrak{a}_i^0$. It incorporates the interaction of the i^{th} lattice site with its nearest neighbours, the $(i-1)^{\text{th}}$ and $(i+1)^{\text{th}}$ lattice sites, for all $1 \leq i \leq M$. Periodicity is imposed, meaning that the 1^{st} and M^{th} sites are considered nearest neighbours.

To be more explicit, we give some examples of the types of Hamiltonians which we will encounter. In chapters 3 and 4 we will study models with Hamiltonians of the form

$$\mathcal{H} = \sum_{i=1}^{M} \left(\mathfrak{a}_i^+ \mathfrak{a}_{i+1}^- + \mathfrak{a}_i^- \mathfrak{a}_{i+1}^+ + \Delta \mathfrak{a}_i^0 \right) \tag{2.1.17}$$

while in chapters 5 and 6 we will study models with Hamiltonians of the form

$$\mathcal{H} = \sum_{i=1}^{M} \left(\mathfrak{a}_i^+ \mathfrak{a}_{i+1}^- + \mathfrak{a}_i^- \mathfrak{a}_{i+1}^+ + \Delta \mathfrak{a}_i^0 \mathfrak{a}_{i+1}^0 \right) \tag{2.1.18}$$

with Δ a constant, and where we assume the periodicity $\mathfrak{a}_{M+1}^{\pm} = \mathfrak{a}_1^{\pm}, \mathfrak{a}_{M+1}^0 = \mathfrak{a}_1^0$ in both (2.1.17) and (2.1.18).

An important goal in the study of a particular quantum mechanical model is to calculate the spectrum of its Hamiltonian $\mathcal{H}$. That is, one wishes to find state vectors $|\Psi\rangle \in \mathcal{V}$ which are eigenvectors of $\mathcal{H}$, satisfying

$$\mathcal{H}|\Psi\rangle = \mathcal{E}_{\Psi}|\Psi\rangle \tag{2.1.19}$$

and to compute the corresponding eigenvalues $\mathcal{E}_{\Psi}$. In accomplishing such a task, one is commonly said to have solved the model. All the models studied in this thesis are *exactly solvable*, meaning that their Hamiltonian $\mathcal{H}$ belongs to a family of commuting operators. The quantum inverse scattering method/algebraic Bethe Ansatz are techniques which diagonalize the entire family of commuting operators simultaneously. We shall devote the remainder of this chapter to describing these techniques.

2.2 Quantum inverse scattering method

2.2.1 R-matrix and Yang-Baxter equation

The quantum inverse scattering approach to solving a given quantum integrable model relies on an $n^2 \times n^2$ matrix called an *R-matrix*, where $n \geq 2$. The value of n and the entries of the R-matrix are specific to the model under consideration, however we can make three remarks which apply universally. **1.** The R-matrix depends on two parameters called *rapidities*, which we typically write as u and v,[1] **2.** The R-matrix is an element of $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b)$, where $\mathcal{V}_a$ and $\mathcal{V}_b$ are copies of $\mathbb{C}^n$, and are called *auxiliary vector spaces,* **3.** The R-matrix is a solution of the *Yang-Baxter equation,* which will be described in detail below.

In this thesis we will focus on the case $n = 2$, and models which have 4×4 R-matrices of the form

[1] In later chapters we will also use x and y for the rapidities. In those cases all the theory established here still applies, if one simply replaces u and v with x and y, respectively.

$$R_{ab}(u,v) = \begin{pmatrix} R^{++}_{++}(u,v) & R^{++}_{+-}(u,v) & R^{+-}_{++}(u,v) & R^{+-}_{+-}(u,v) \\ R^{++}_{-+}(u,v) & R^{++}_{--}(u,v) & R^{+-}_{-+}(u,v) & R^{+-}_{--}(u,v) \\ R^{-+}_{++}(u,v) & R^{-+}_{+-}(u,v) & R^{--}_{++}(u,v) & R^{--}_{+-}(u,v) \\ R^{-+}_{-+}(u,v) & R^{-+}_{--}(u,v) & R^{--}_{-+}(u,v) & R^{--}_{--}(u,v) \end{pmatrix}_{ab} \tag{2.2.1}$$

$$= \begin{pmatrix} a_+(u,v) & 0 & 0 & 0 \\ 0 & b_+(u,v) & c_+(u,v) & 0 \\ 0 & c_-(u,v) & b_-(u,v) & 0 \\ 0 & 0 & 0 & a_-(u,v) \end{pmatrix}_{ab}$$

where the entries are functions of the rapidities u, v which are specific to the model under consideration. We have placed the subscript ab on the R-matrix to denote the fact that it is an element of $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b)$, where $\mathcal{V}_a, \mathcal{V}_b$ are copies of $\mathbb{C}^2$. The R-matrices (2.2.1) are solutions of the *Yang-Baxter equation*

$$R_{ab}(u,v)R_{ac}(u,w)R_{bc}(v,w) = R_{bc}(v,w)R_{ac}(u,w)R_{ab}(u,v) \tag{2.2.2}$$

which is an identity acting in the tensor product $\mathcal{V}_a \otimes \mathcal{V}_b \otimes \mathcal{V}_c$ of three auxiliary spaces, for general values of the rapidities u, v, w. The Yang-Baxter equation is a strong restriction on the functions $a_\pm, b_\pm, c_\pm$ which are entries of the R-matrix (2.2.1). The full strength is revealed when (2.2.2) is written in component notation, giving

$$R^{i_1 k_1}_{i_2 k_2}(u,v) R^{k_1 j_1}_{i_3 k_3}(u,w) R^{k_2 j_2}_{k_3 j_3}(v,w) = R^{i_2 k_2}_{i_3 k_3}(v,w) R^{i_1 k_1}_{k_3 j_3}(u,w) R^{k_1 j_1}_{k_2 j_2}(u,v) \tag{2.2.3}$$

where all indices take values in $\{+1,-1\}$, with each of $\{i_1, i_2, i_3, j_1, j_2, j_3\}$ held fixed, while $\{k_1, k_2, k_3\}$ are summed. We see that (2.2.3) gives rise to 2^6 scalar equations involving the functions $a_\pm, b_\pm, c_\pm$, one corresponding to each configuration of the indices $\{i_1, i_2, i_3, j_1, j_2, j_3\}$.

Example 1. Setting $\{i_1, i_2, i_3, j_1, j_2, j_3\} = \{-,+,-,-,-,+\}$ in (2.2.3), we recover the equation

$$\begin{aligned} R^{-+}_{+-}(u,v) R^{+-}_{-+}(u,w) R^{--}_{++}(v,w) + R^{--}_{++}(u,v) R^{--}_{--}(u,w) R^{+-}_{-+}(v,w) \\ = R^{+-}_{-+}(v,w) R^{--}_{++}(u,w) R^{--}_{--}(u,v) \end{aligned} \tag{2.2.4}$$

where some terms within the summation have vanished due to their corresponding R-matrix entries being zero. Substituting the functions which comprise the R-matrix entries into (2.2.4), we obtain

$$c_-(u,v)c_+(u,w)b_-(v,w) + b_-(u,v)a_-(u,w)c_+(v,w) = c_+(v,w)b_-(u,w)a_-(u,v) \tag{2.2.5}$$

2.2.2 Graphical representation of R-matrix

It is possible to represent the elements of the R-matrix (2.2.1) graphically, a procedure which leads to an elegant diagrammatic interpretation of the Yang-Baxter equation (2.2.3). This graphical correspondence is realized by matching each non-zero element of (2.2.1) with a *vertex*, as shown in figure 2.1.

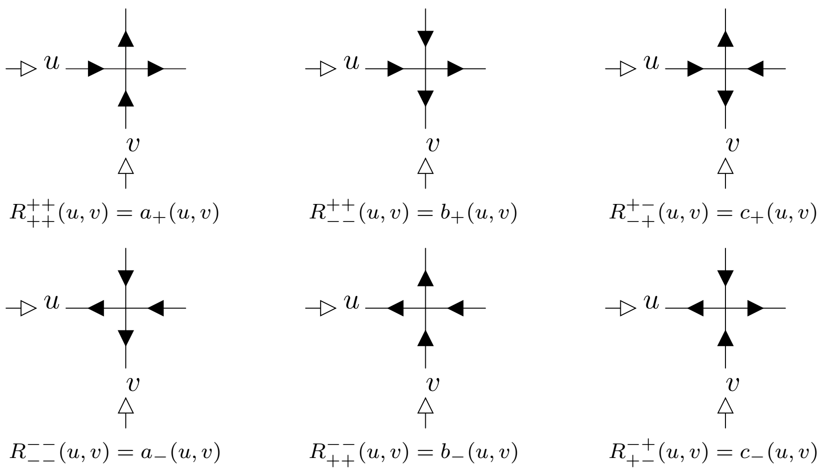

Figure 2.1: Six vertices of the generalized R-matrix. Each non-zero entry of (2.2.1) is matched with a vertex.

Each vertex in figure 2.1 is the intersection of a horizontal line with two black arrows attached, and a vertical line with two black arrows attached. The horizontal line is considered to have the rapidity u *flowing through it,* in the direction indicated by the horizontal white arrow. Similarly, the vertical line is considered to have the rapidity v flowing through it, in the direction indicated by the vertical white arrow. When a black arrow points in the direction of variable flow it is assigned the value $+1$, and conversely when a black arrow points opposite the direction of variable flow it is assigned the value -1.

In any given line, the black arrow nearest to the external white arrow is called *incoming*, since it precedes the intersection point of the vertex. The black arrow farthest from the external white arrow is called *outgoing*, since it succeeds the intersection point of the vertex. A line thus gives rise to an ordered pair of values $(i,j) \in \{+1,-1\}$, where i is the value assigned to the incoming arrow, and j is the value assigned to the outgoing arrow. The R-matrix element $R^{i_1 j_1}_{i_2 j_2}(u,v)$ is matched with the vertex having horizontal line values (i_1, j_1) and vertical line values (i_2, j_2).

Using these graphical definitions, the Yang-Baxter equation (2.2.3) may be written in the form shown in figure 2.2.

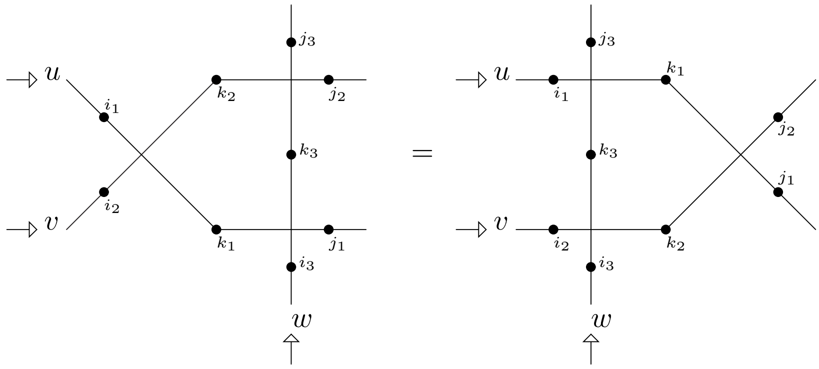

Figure 2.2: Graphical depiction of the Yang-Baxter equation.

Both sides of this equation should be interpreted as three conjoined vertices. There is a vertex at the intersection of the (u, v) lines, another at the intersection of the (u, w) lines, and yet another at the intersection of the (v, w) lines. The joining of these vertices is shorthand for multiplication of the three corresponding R-matrix elements. The absence of black arrows in this picture is a notational convenience, which requires some explanation.

Examining figure 2.2, we see that a conjoined trio of vertices possesses 6 *external* line segments, which have been labelled $\{i_1, i_2, i_3, j_1, j_2, j_3\}$. Each i or j represents an undisclosed black arrow, which is held fixed on both sides of the equation. By leaving these black arrows unspecified we can write (2.2.3) as a single equation, when in fact it implies 2^6 equations, one corresponding to each of the 2^6 external configurations.

The 3 *internal* line segments, which have been labelled $\{k_1, k_2, k_3\}$, have a different meaning. Each k is summed over a black arrow that points with the variable flow, and a black arrow that points opposite the variable flow. By omitting black arrows from these points we imply summation over 2^3 terms on each side of the equation. It should be noted that many terms in this summation vanish, since any vertex not shown in figure 2.1 is by definition equal to zero.

2.2.3 L-matrix and local intertwining equation

Another fundamental object in the quantum inverse scattering method is the $n \times n$ *L-matrix*. As before, the value of n and the entries of the L-matrix are specific to the model under consideration, but we can make three universal remarks. **1.** The L-matrix depends on a single rapidity u, **2.** The L-matrix is an element of $\mathrm{End}(\mathcal{V}_a)$, where $\mathcal{V}_a$ is a copy of $\mathbb{C}^n$, and its entries are elements of the m^{th} quantum algebra $\mathcal{A}_m$, **3.** The L-matrix satisfies the *local intertwining equation*, which will be described in detail below.

In this thesis we restrict our attention to models with 2×2 L-matrices of the form

$$L_{am}(u) = \begin{pmatrix} L_m^{++}(u) & L_m^{+-}(u) \\ L_m^{-+}(u) & L_m^{--}(u) \end{pmatrix}_a \tag{2.2.6}$$

where the entries depend on u and are elements of $\mathcal{A}_m$, specific to the model under consideration. We have placed the subscript a on the L-matrix to denote the fact that it is an element of $\mathrm{End}(\mathcal{V}_a)$, where $\mathcal{V}_a$ is a copy of $\mathbb{C}^2$. The L-matrix (2.2.6) satisfies the relation

$$R_{ab}(u,v)L_{am}(u)L_{bm}(v) = L_{bm}(v)L_{am}(u)R_{ab}(u,v) \tag{2.2.7}$$

which is an identity acting in the tensor product $\mathcal{V}_a \otimes \mathcal{V}_b$ of two auxiliary spaces, for general values of the rapidities u, v. The R-matrix $R_{ab}(u,v)$, given by (2.2.1), is said to *intertwine* the L-matrices $L_{am}(u)$ and $L_{bm}(v)$. For this reason we refer to (2.2.7) as the *local intertwining equation.* It is a local equation insofar as the entries of the L-matrices act only at the m^{th} site in the model. Writing (2.2.7) in component notation we obtain

$$R^{i_1 k_1}_{i_2 k_2}(u,v) L_m^{k_1 j_1}(u) L_m^{k_2 j_2}(v) = L_m^{i_2 k_2}(v) L_m^{i_1 k_1}(u) R^{k_1 j_1}_{k_2 j_2}(u,v) \tag{2.2.8}$$

where all indices take values in $\{+1,-1\}$, with each of $\{i_1, i_2, j_1, j_2\}$ held fixed, while $\{k_1, k_2\}$ are summed. Hence (2.2.8) gives rise to 2^4 commutation relations involving the operators $L_m^{++}(u), L_m^{+-}(u), L_m^{-+}(u), L_m^{--}(u)$, one corresponding to each configuration of the indices $\{i_1, i_2, j_1, j_2\}$.

Example 2. Setting $\{i_1, i_2, j_1, j_2\} = \{+,+,+,-\}$ in (2.2.7), we recover the commutation relation

$$R^{++}_{++}(u,v)L_m^{++}(u)L_m^{+-}(v) = L_m^{+-}(v)L_m^{++}(u)R^{++}_{--}(u,v) + L_m^{++}(v)L_m^{+-}(u)R^{-+}_{+-}(u,v) \tag{2.2.9}$$

where some terms within the summation have vanished due to their corresponding R-matrix entries being zero. Substituting the functions which comprise the R-matrix entries into (2.2.9), we obtain

$$a_+(u,v)L_m^{++}(u)L_m^{+-}(v) = b_+(u,v)L_m^{+-}(v)L_m^{++}(u) + c_-(u,v)L_m^{++}(v)L_m^{+-}(u) \tag{2.2.10}$$

Remark 1. Let us specialize, for the moment, to models obeying Bose-Einstein statistics. An important property of the L-matrix (2.2.6) is the action of its entries on the local vacuum states $|0\rangle_m$ and $\langle 0|_m$. In all of the bosonic models which we study, these vacuum states will be eigenvectors of $L_m^{++}(u)$ and $L_m^{--}(u)$, giving rise to the equations

$$L_m^{++}(u)|0\rangle_m = \alpha_m(u)|0\rangle_m, \quad \langle 0|_m L_m^{++}(u) = \alpha_m(u)\langle 0|_m \tag{2.2.11}$$
$$L_m^{--}(u)|0\rangle_m = \delta_m(u)|0\rangle_m, \quad \langle 0|_m L_m^{--}(u) = \delta_m(u)\langle 0|_m \tag{2.2.12}$$

where $\alpha_m(u)$ and $\delta_m(u)$ are functions of u which are specific to the model under consideration. Furthermore, $L_m^{+-}(u)$ and $L_m^{-+}(u)$ will play the role of creation and annihilation operators, giving rise to the equations

$$L_m^{+-}(u)|0\rangle_m \neq 0, \quad \langle 0|_m L_m^{+-}(u) = 0 \tag{2.2.13}$$
$$L_m^{-+}(u)|0\rangle_m = 0, \quad \langle 0|_m L_m^{-+}(u) \neq 0 \tag{2.2.14}$$

Analogous statements also apply to the models that we study which obey Fermi-Dirac statistics. In those cases, the equations (2.2.11)–(2.2.14) still hold, but with $|0\rangle_m, \langle 0|_m$ replaced by $|\uparrow\rangle_m, \langle\uparrow|_m$.

2.2.4 Graphical representation of L-matrix

The objects introduced in the previous subsection admit the following graphical description. We identify each element of the matrix $L_{am}(u)$ with a vertex, as shown in figure 2.3.

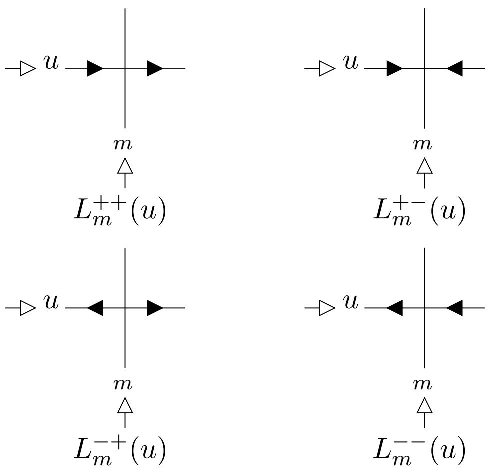

Figure 2.3: Four vertices of the generalized L-matrix. Each entry of (2.2.6) is matched with a vertex.

Each vertex in figure 2.3 is the intersection of a horizontal line with two black arrows attached, and a blank vertical line. The variable u flows through the horizontal line in the direction indicated, and the vertical line is marked with m, to indicate the m^{th} quantum space. On the horizontal line, when a black arrow points with the orientation it is assigned the value $+1$, and when a black arrow points against the orientation it is assigned the value -1. The vertical line has no values associated to it, and for the moment, it serves only to partition the horizontal line. The L-matrix element $L_m^{ij}(u)$ is matched with the vertex having horizontal line values (i, j).

Using these graphical definitions, the local intertwining equation (2.2.8) may be written in the form shown in figure 2.4.

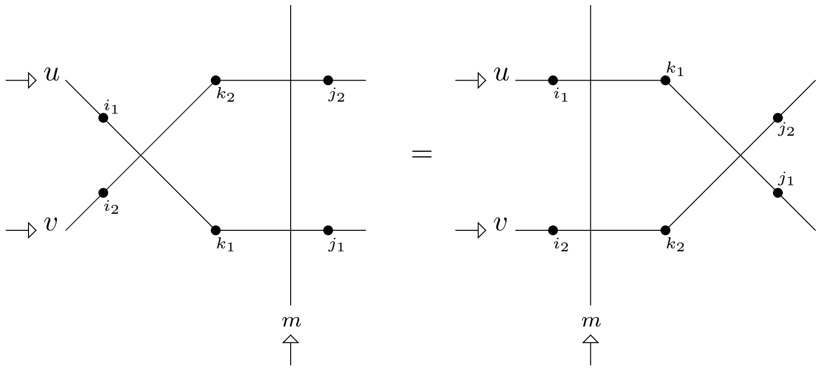

Figure 2.4: Graphical depiction of the local intertwining equation.

Both sides of this equation should be interpreted as three conjoined vertices. There is an R-matrix vertex at the intersection of the (u, v) lines, and L-matrix vertices at the remaining two intersection points. The joining of these vertices is shorthand for multiplication of the three corresponding matrix elements. In this multiplication, the L-matrix elements are ordered from the one closest the vertical white arrow to the one farthest.

We label the horizontal external line segments by $\{i_1, i_2, j_1, j_2\}$. Each i or j represents an undisclosed black arrow, which is held fixed on both sides of the equation. The internal horizontal line segments have been labelled $\{k_1, k_2\}$. Each k is summed over a black arrow that points with the orientation, and a black arrow that points against the orientation.

2.2.5 Monodromy matrix and global intertwining equation

The *monodromy matrix* is defined as an M-fold product of L-matrices, where the product is taken over each site of the model. It is given explicitly by[2]

[2]In Chapter 5 we will define the monodromy matrix as $T_a(u) = L_{a1}(u) \dots L_{aM}(u)$. This reversal of the quantum space ordering is merely a convenience and the results of this chapter hold for either definition.

$$T_a(u) = L_{aM}(u)\dots L_{a1}(u) = \overleftarrow{\prod_{m=1}^{M}} L_{am}(u) \tag{2.2.15}$$

where $L_{am}(u)$ is the L-matrix (2.2.6) of the model, acting at the m^{th} site. For the models under our consideration, which have 2×2 L-matrices, the monodromy matrix has the form

$$T_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a \tag{2.2.16}$$

where we have placed the subscript a on the monodromy matrix to denote the fact that it is an element of $\text{End}(\mathcal{V}_a)$, with $\mathcal{V}_a$ a copy of $\mathbb{C}^2$.

The entries $A(u), B(u), C(u), D(u)$ in (2.2.16) are the operators which result from performing the multiplication (2.2.15) of L-matrices. They are dependent on u, since each L-matrix in the product depends on u, and they are sums of 2^{M-1} monomials. These monomials, in turn, are products of M local operators, one acting at each site of the model. Therefore, in general, the entries of the monodromy matrix are complicated elements of $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_M$. For notational convenience, it is conventional to suppress the $\text{End}(\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_M)$ dependence of these entries.

Lemma 1. The monodromy matrix satisfies the equation

$$R_{ab}(u,v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u,v) \tag{2.2.17}$$

which is an identity acting in the tensor product $\mathcal{V}_a \otimes \mathcal{V}_b$ of two auxiliary spaces, for general values of the rapidities u, v. We refer to this as the *global intertwining equation*, since the R-matrix $R_{ab}(u,v)$ now intertwines the monodromy matrices $T_a(u), T_b(v)$ which act over all sites in the model.

Proof. The result (2.2.17) is a corollary of the local intertwining relation (2.2.7). Using the definition (2.2.15) of the monodromy matrix, we write

$$\begin{aligned} R_{ab}(u,v)T_a(u)T_b(v) &= R_{ab}(u,v)\left(\overleftarrow{\prod_{m=1}^{M}} L_{am}(u)\right)\left(\overleftarrow{\prod_{m=1}^{M}} L_{bm}(v)\right) \\ &= R_{ab}(u,v)\overleftarrow{\prod_{m=1}^{M}}\Big(L_{am}(u)L_{bm}(v)\Big) \end{aligned} \tag{2.2.18}$$

where we have changed the ordering of the L-matrices by commuting those which act in different spaces. Applying (2.2.7) M times successively to the right hand side of (2.2.18), we obtain

$$
\begin{aligned}
R_{ab}(u,v)T_a(u)T_b(v) &= \overleftarrow{\prod_{m=1}^{M}} \Big(L_{bm}(v)L_{am}(u)\Big) R_{ab}(u,v) \qquad (2.2.19)\\
&= \left(\overleftarrow{\prod_{m=1}^{M}} L_{bm}(v)\right)\left(\overleftarrow{\prod_{m=1}^{M}} L_{am}(u)\right) R_{ab}(u,v)\\
&= T_b(v)T_a(u)R_{ab}(u,v)
\end{aligned}
$$

where we have restored the original ordering of the L-matrices to complete the proof. □

As we have done with earlier equations, we can write (2.2.17) in component notation, obtaining

$$
R^{i_1k_1}_{i_2k_2}(u,v)T^{k_1j_1}(u)T^{k_2j_2}(v) = T^{i_2k_2}(v)T^{i_1k_1}(u)R^{k_1j_1}_{k_2j_2}(u,v) \qquad (2.2.20)
$$

where all indices take values in $\{+1,-1\}$, with each of $\{i_1,i_2,j_1,j_2\}$ held fixed, while $\{k_1,k_2\}$ are summed. Hence (2.2.20) gives rise to 2^4 commutation relations involving the operators $A(u), B(u), C(u), D(u)$, one corresponding to each configuration of the indices $\{i_1,i_2,j_1,j_2\}$.

Example 3. We will list four of the commutation relations contained in (2.2.20), since they are used later in this chapter. Setting $\{i_1,i_2,j_1,j_2\} = \{+,+,-,-\}$ in (2.2.20), we obtain

$$
\begin{aligned}
&R^{++}_{++}(u,v)T^{+-}(u)T^{+-}(v) = T^{+-}(v)T^{+-}(u)R^{--}_{--}(u,v)\\
&\implies a_+(u,v)B(u)B(v) = a_-(u,v)B(v)B(u) \qquad (2.2.21)
\end{aligned}
$$

Setting $\{i_1,i_2,j_1,j_2\} = \{-,-,+,+\}$ in (2.2.20), we obtain

$$
\begin{aligned}
&R^{--}_{--}(u,v)T^{-+}(u)T^{-+}(v) = T^{-+}(v)T^{-+}(u)R^{++}_{++}(u,v)\\
&\implies a_-(u,v)C(u)C(v) = a_+(u,v)C(v)C(u) \qquad (2.2.22)
\end{aligned}
$$

Setting $\{i_1,i_2,j_1,j_2\} = \{+,+,-,+\}$ in (2.2.20), we obtain

$$
\begin{aligned}
&R^{++}_{++}(u,v)T^{+-}(u)T^{++}(v) = T^{+-}(v)T^{++}(u)R^{+-}_{-+}(u,v) + T^{++}(v)T^{+-}(u)R^{--}_{++}(u,v)\\
&\implies a_+(u,v)B(u)A(v) = c_+(u,v)B(v)A(u) + b_-(u,v)A(v)B(u) \qquad (2.2.23)
\end{aligned}
$$

Finally, setting $\{i_1,i_2,j_1,j_2\} = \{-,+,-,-\}$ in (2.2.20), we obtain

$$
\begin{aligned}
&R^{-+}_{+-}(u,v)T^{+-}(u)T^{--}(v) + R^{--}_{++}(u,v)T^{--}(u)T^{+-}(v) = T^{+-}(v)T^{--}(u)R^{--}_{--}(u,v)\\
&\implies c_-(u,v)B(u)D(v) + b_-(u,v)D(u)B(v) = a_-(u,v)B(v)D(u) \qquad (2.2.24)
\end{aligned}
$$

Lemma 2. Let us consider, once again, models which obey Bose-Einstein statistics. Defining the global vacua

$$|0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_M, \quad \langle 0| = \langle 0|_1 \otimes \cdots \otimes \langle 0|_M \tag{2.2.25}$$

we find that these states are eigenvectors of the operators $A(u)$ and $D(u)$, giving rise to the equations

$$A(u)|0\rangle = \alpha(u)|0\rangle = \prod_{m=1}^{M} \alpha_m(u)|0\rangle, \quad \langle 0|A(u) = \alpha(u)\langle 0| = \prod_{m=1}^{M} \alpha_m(u)\langle 0| \tag{2.2.26}$$

$$D(u)|0\rangle = \delta(u)|0\rangle = \prod_{m=1}^{M} \delta_m(u)|0\rangle, \quad \langle 0|D(u) = \delta(u)\langle 0| = \prod_{m=1}^{M} \delta_m(u)\langle 0| \tag{2.2.27}$$

where $\alpha_m(u)$ and $\delta_m(u)$ are the eigenvalues as defined in (2.2.11) and (2.2.12), respectively. In addition, $B(u)$ and $C(u)$ play the role of global creation and annihilation operators, giving rise to the equations

$$B(u)|0\rangle \neq 0, \quad \langle 0|B(u) = 0 \tag{2.2.28}$$

$$C(u)|0\rangle = 0, \quad \langle 0|C(u) \neq 0 \tag{2.2.29}$$

Similar equations apply to models with Fermi-Dirac statistics, by replacing the global vacua $|0\rangle, \langle 0|$ in (2.2.26)–(2.2.29) with the spin-up states

$$|\Uparrow\rangle = |\uparrow\rangle_1 \otimes \cdots \otimes |\uparrow\rangle_M, \quad \langle \Uparrow| = \langle \uparrow|_1 \otimes \cdots \otimes \langle \uparrow|_M \tag{2.2.30}$$

Proof. These equations follow immediately from the definition of the monodromy matrix (2.2.15), and from the actions (2.2.11)–(2.2.14) of the L-matrix entries on the local vacuum states. For example,

$$\begin{aligned} A(u)|0\rangle &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} L_M^{++}|0\rangle_M & L_M^{+-}|0\rangle_M \\ L_M^{-+}|0\rangle_M & L_M^{--}|0\rangle_M \end{pmatrix} \cdots \begin{pmatrix} L_1^{++}|0\rangle_1 & L_1^{+-}|0\rangle_1 \\ L_1^{-+}|0\rangle_1 & L_1^{--}|0\rangle_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_M|0\rangle_M & L_M^{+-}|0\rangle_M \\ 0 & \delta_M|0\rangle_M \end{pmatrix} \cdots \begin{pmatrix} \alpha_1|0\rangle_1 & L_1^{+-}|0\rangle_1 \\ 0 & \delta_1|0\rangle_1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \prod_{m=1}^{M} \alpha_m(u)|0\rangle_1 \otimes \cdots \otimes |0\rangle_M \end{aligned} \tag{2.2.31}$$

The remaining equations are proved analogously. □

2.2.6 Graphical representation of monodromy matrix

Using the previous diagrammatic conventions for L-matrix elements, we identify each element of the matrix $T_a(u)$ with a *string of vertices*, as shown in figure 2.5.

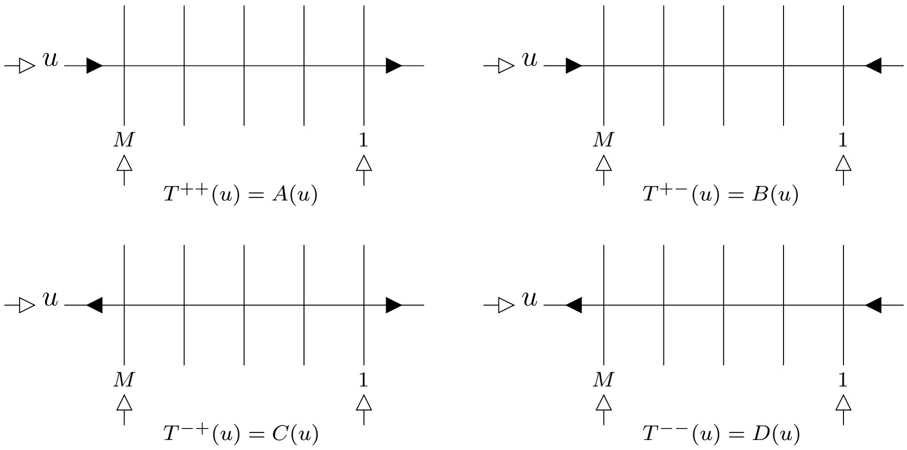

Figure 2.5: Four vertex-strings of the monodromy matrix. Each entry of (2.2.16) is matched with a string of L-matrix vertices.

These diagrams are interpreted as M conjoined L-matrix vertices which represent, from left to right, the multiplication of the corresponding L-matrix elements. The horizontal internal line segments are summed over black arrows that point with the orientation, and black arrows that point against the orientation, so each string of vertices implies a sum over 2^{M-1} terms. The monodromy matrix element $T^{ij}(u)$ is matched with the string of vertices having external horizontal line values (i, j).

The global intertwining equation (2.2.20) has the diagrammatic form

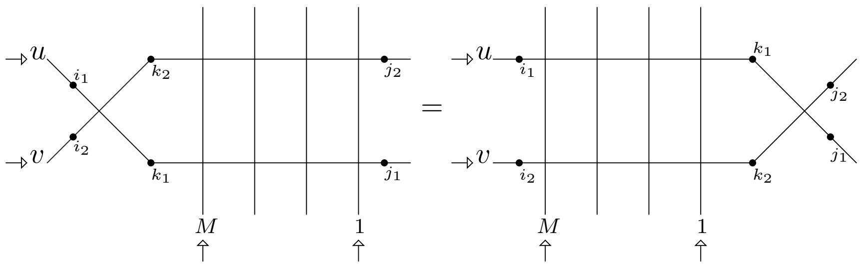

Figure 2.6: Graphical depiction of the global intertwining equation. This follows by repeated application of the local intertwining equation, which moves the R-matrix vertex attached on the left until it emerges from the right.

Both sides of this equation contain an R-matrix vertex at the intersection of the (u, v) lines, and L-matrix vertices at the remaining $2M$ intersection points. The joining of these vertices is shorthand for multiplication of the corresponding matrix elements. In this multiplication, the L-matrix vertices are ordered from those closest the vertical white arrows to those farthest.

We write the external line segments as $\{i_1, i_2, j_1, j_2\}$. Each i or j represents an undisclosed black arrow, which is held fixed on both sides of the equation. For simplicity we have not labelled the horizontal internal line segments, which are summed over black arrows that point with the orientation, and black arrows that point against the orientation.

2.2.7 Transfer matrix and quantum trace identities

In this subsection we describe the reconstruction of the Hamiltonian $\mathcal{H}$ in terms of a set of commuting operators. This reconstruction allows us to study the spectrum of $\mathcal{H}$ jointly with another operator, the *transfer matrix.*

Lemma 3. Define the transfer matrix $t(u)$ as the trace of the monodromy matrix, taken in the auxiliary space $\mathcal{V}_a$. In other words, let

$$t(u) = \mathrm{tr}_a T_a(u) = A(u) + D(u) \tag{2.2.32}$$

Then for all $u, v \in \mathbb{C}$ the transfer matrices $t(u), t(v)$ satisfy

$$[t(u), t(v)] = 0 \tag{2.2.33}$$

Proof. Consider the global intertwining equation (2.2.17). Multiplying this equation from the left by the inverse of the R-matrix (2.2.1) and taking the trace over $\mathcal{V}_a$ and $\mathcal{V}_b$, we obtain

$$\begin{aligned} \mathrm{tr}_a T_a(u) \mathrm{tr}_b T_b(v) &= \mathrm{tr}_a \mathrm{tr}_b \Big(R_{ab}^{-1}(u,v) T_b(v) T_a(u) R_{ab}(u,v) \Big) \\ &= \mathrm{tr}_a \mathrm{tr}_b \Big(T_b(v) T_a(u) R_{ab}(u,v) R_{ab}^{-1}(u,v) \Big) \\ &= \mathrm{tr}_b T_b(v) \mathrm{tr}_a T_a(u) \end{aligned} \tag{2.2.34}$$

where the second line of (2.2.34) follows from the cyclicity of the trace. Recalling the definition (2.2.32) of the transfer matrix, the final line of (2.2.34) completes the proof.

□

The transfer matrix can be viewed as a generating function of the conserved quantities of a given quantum integrable model. To see this, one expands $t(u)$ in powers of u or e^u (depending on the particular model) to obtain

$$t(u) = \sum_n \mathfrak{t}_n u^n, \quad \text{or} \quad t(u) = \sum_n \mathfrak{t}_n e^{nu} \tag{2.2.35}$$

where each $\mathfrak{t}_n$ is an element of $\mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_M$. Substituting the generating function (2.2.35) into the commutation relation (2.2.33), it follows that the quantities $\mathfrak{t}_m, \mathfrak{t}_n$ satisfy

$$[\mathfrak{t}_m, \mathfrak{t}_n] = 0, \quad \text{for all } m, n \tag{2.2.36}$$

meaning that they are in involution. An *integrable* quantum model is one whose Hamiltonian $\mathcal{H}$ may be expressed as an algebraic combination of the operators $\mathfrak{t}_n$, via a *quantum trace identity.* Whilst the specifics of this reconstruction depend on the model being studied, we always recover the equation

$$[\mathcal{H}, t(u)] = 0, \quad \text{for all } u \in \mathbb{C} \tag{2.2.37}$$

as a consequence. This leads us to the following important result.

Lemma 4. Let $|\Psi\rangle \in \mathcal{V}$ be an eigenvector of $\mathcal{H}$ with non-degenerate eigenvalue $\mathcal{E}_\Psi$. Then $|\Psi\rangle$ is also an eigenvector of $t(u)$.

Proof. Using the commutation relation (2.2.37), we have

$$\mathcal{H}t(u)|\Psi\rangle = t(u)\mathcal{H}|\Psi\rangle = \mathcal{E}_\Psi t(u)|\Psi\rangle \tag{2.2.38}$$

Hence $t(u)|\Psi\rangle$ is an eigenstate of $\mathcal{H}$ with eigenvalue $\mathcal{E}_\Psi$. Since this eigenvalue is non-degenerate, it follows that $t(u)|\Psi\rangle = \tau_\Psi(u)|\Psi\rangle$ for some scalar function $\tau_\Psi(u)$. This proves that $|\Psi\rangle$ is also an eigenvector of $t(u)$.

□

By virtue of lemma 4, we see that all non-degenerate eigenvectors of $\mathcal{H}$ are also eigenvectors of $t(u)$. Hence the non-degenerate spectrum of $\mathcal{H}$ can be recovered by studying the spectrum of $t(u)$. This will be the focus of the next section.

2.3 Algebraic Bethe Ansatz

2.3.1 Construction of the Bethe eigenvectors

Our goal in this subsection is to find vectors $|\Psi\rangle \in \mathcal{V}$ and $\langle\Psi| \in \mathcal{V}^*$ such that

$$t(u)|\Psi\rangle = \tau_\Psi(u)|\Psi\rangle, \quad \langle\Psi|t(u) = \tau_\Psi(u)\langle\Psi| \tag{2.3.1}$$

where $t(u)$ is the transfer matrix (2.2.32) and $\tau_\Psi(u)$ are suitable scalar functions. This goal is achieved by making an educated guess at the form of the eigenvectors, known as the *algebraic Bethe Ansatz.* In the following theorem, the eigenvectors $|\Psi\rangle$ and $\langle\Psi|$ are constructed using the off-diagonal entries of the monodromy matrix (2.2.16). The action of $t(u) = A(u) + D(u)$ on the proposed eigenvectors can then be calculated using the commutation relations (2.2.20).

Theorem 1. Suppose that the entries of the R-matrix (2.2.1) satisfy[3]

$$a_{\pm}(u,v) = a(u,v), \quad b_{\pm}(u,v) = -b_{\pm}(v,u), \quad c_{\pm}(u,v) = c(u,v) = c(v,u) \tag{2.3.2}$$

Then the state vectors[4]

$$|\Psi\rangle = B(v_1)\dots B(v_N)|0\rangle = \overrightarrow{\prod_{n=1}^{N}} B(v_n)|0\rangle \tag{2.3.3}$$

$$\langle\Psi| = \langle 0|C(v_N)\dots C(v_1) = \langle 0|\overleftarrow{\prod_{n=1}^{N}} C(v_n) \tag{2.3.4}$$

are solutions of the eigenvector equations (2.3.1), with eigenvalues $\tau_{\Psi}(u)$ given by

$$\tau_{\Psi}(u) = \alpha(u)\prod_{n=1}^{N}\frac{a(v_n,u)}{b_{-}(v_n,u)} + \delta(u)\prod_{n=1}^{N}\frac{a(u,v_n)}{b_{-}(u,v_n)} \tag{2.3.5}$$

provided that the variables $\{v_1,\dots,v_N\}$ satisfy the system of coupled equations

$$\alpha(v_i)\prod_{\substack{n\neq i\\ n=1}}^{N} a(v_n,v_i) + (-)^N\delta(v_i)\prod_{\substack{n\neq i\\ n=1}}^{N} a(v_i,v_n) = 0 \tag{2.3.6}$$

for all $1 \le i \le N$. We refer to (2.3.3) and (2.3.4) as the *Bethe eigenvectors* of the model, and to the constraints (2.3.6) as the *Bethe equations.*

Proof. Following the procedure given in chapter VII of [61], we shall prove the theorem for (2.3.3), but omit the proof for (2.3.4) as it is very similar. Acting on the vector (2.3.3) with the transfer matrix $t(u)$, we obtain

$$t(u)|\Psi\rangle = \Big(A(u)+D(u)\Big)|\Psi\rangle = A(u)\prod_{n=1}^{N}B(v_n)|0\rangle + D(u)\prod_{n=1}^{N}B(v_n)|0\rangle \tag{2.3.7}$$

where the products in (2.3.7) are left unordered, because when $a_{+}(v_i,v_j) = a_{-}(v_i,v_j)$ the commutation relation (2.2.21) yields

$$[B(v_i),B(v_j)] = 0, \quad \text{for all } 1 \le i,j \le N \tag{2.3.8}$$

[3] The models studied in chapters 3,4,5 have R-matrices which obey (2.3.2). In chapter 6, we study models which require a slightly separate treatment.

[4] Throughout the theorem, we specialize to models with Bose-Einstein statistics. The results obtained apply equally to fermionic models, by replacing all instances of $|0\rangle, \langle 0|$ with $|\Uparrow\rangle, \langle\Uparrow|$.

In order to show that $|\Psi\rangle$ is an eigenvector of $t(u)$, we must calculate the two terms on the right hand side of (2.3.7). To this end, let $\mathcal{P}_N$ denote the proposition

$$A(u)\prod_{n=1}^{N}B(v_n)|0\rangle = \alpha(u)\prod_{n=1}^{N}\Big[\frac{a(v_n,u)}{b_-(v_n,u)}B(v_n)\Big]|0\rangle - \sum_{i=1}^{N}\alpha(v_i)\frac{c(v_i,u)}{b_-(v_i,u)}B(u)\prod_{\substack{n\neq i\\ n=1}}^{N}\Big[\frac{a(v_n,v_i)}{b_-(v_n,v_i)}B(v_n)\Big]|0\rangle \tag{2.3.9}$$

which we will prove for general $N \geq 1$. To begin, we use the commutation relation (2.2.23) with $u \to v_1, v \to u$ and equation (2.2.26) to calculate

$$A(u)B(v_1)|0\rangle = \alpha(u)\frac{a(v_1,u)}{b_-(v_1,u)}B(v_1)|0\rangle - \alpha(v_1)\frac{c(v_1,u)}{b_-(v_1,u)}B(u)|0\rangle \tag{2.3.10}$$

where we have used the fact that $a_+(v_1,u) = a(v_1,u), c_+(v_1,u) = c(v_1,u)$. This establishes that $\mathcal{P}_1$ is true. Now suppose $\mathcal{P}_{m-1}$ is true for some integer $m \geq 2$. Once again, using the commutation relation (2.2.23) we find that

$$A(u)\prod_{n=1}^{m}B(v_n)|0\rangle = \frac{a(v_1,u)}{b_-(v_1,u)}B(v_1)A(u)\prod_{n=2}^{m}B(v_n)|0\rangle - \frac{c(v_1,u)}{b_-(v_1,u)}B(u)A(v_1)\prod_{n=2}^{m}B(v_n)|0\rangle \tag{2.3.11}$$

Since $\mathcal{P}_{m-1}$ holds, we are able to explicitly calculate the terms on the right hand side of (2.3.11). Substituting $\mathcal{P}_{m-1}$ into (2.3.11) we obtain

$$\begin{aligned} A(u)\prod_{n=1}^{m}B(v_n)|0\rangle = {} & \frac{a(v_1,u)}{b_-(v_1,u)}B(v_1)\alpha(u)\prod_{n=2}^{m}\Big[\frac{a(v_n,u)}{b_-(v_n,u)}B(v_n)\Big]|0\rangle \\ & - \frac{a(v_1,u)}{b_-(v_1,u)}B(v_1)\sum_{i=2}^{m}\alpha(v_i)\frac{c(v_i,u)}{b_-(v_i,u)}B(u)\prod_{\substack{n\neq i\\ n=2}}^{m}\Big[\frac{a(v_n,v_i)}{b_-(v_n,v_i)}B(v_n)\Big]|0\rangle \\ & - \frac{c(v_1,u)}{b_-(v_1,u)}B(u)\alpha(v_1)\prod_{n=2}^{m}\Big[\frac{a(v_n,v_1)}{b_-(v_n,v_1)}B(v_n)\Big]|0\rangle \\ & + \frac{c(v_1,u)}{b_-(v_1,u)}B(u)\sum_{i=2}^{m}\alpha(v_i)\frac{c(v_i,v_1)}{b_-(v_i,v_1)}B(v_1)\prod_{\substack{n\neq i\\ n=2}}^{m}\Big[\frac{a(v_n,v_i)}{b_-(v_n,v_i)}B(v_n)\Big]|0\rangle \end{aligned} \tag{2.3.12}$$

Now consider the single Yang-Baxter equation (2.2.5), as given in example 1. Recalling the assumptions (2.3.2) and setting $u \to v_1, v \to v_i, w \to u$ in this equation, it becomes

$$-b_-(v_i,v_1)a(v_1,u)c(v_i,u)+c(v_i,v_1)c(v_1,u)b_-(v_i,u)=c(v_i,u)b_-(v_1,u)a(v_1,v_i) \tag{2.3.13}$$

where we have used $b_-(v_1,v_i)=-b_-(v_i,v_1),\ c(v_1,v_i)=c(v_i,v_1)$ to change the order of v_1,v_i on the left hand side. By virtue of (2.3.13) we are able to combine the second and fourth term on the right hand side of (2.3.12), which yields

$$\begin{aligned}A(u)\prod_{n=1}^{m}B(v_n)|0\rangle &= \alpha(u)\prod_{n=1}^{m}\Big[\frac{a(v_n,u)}{b_-(v_n,u)}B(v_n)\Big]|0\rangle \\ &- \alpha(v_1)\frac{c(v_1,u)}{b_-(v_1,u)}B(u)\prod_{n=2}^{m}\Big[\frac{a(v_n,v_1)}{b_-(v_n,v_1)}B(v_n)\Big]|0\rangle \\ &- \sum_{i=2}^{m}\alpha(v_i)\frac{c(v_i,u)}{b_-(v_i,u)}B(u)\prod_{\substack{n\neq i\\ n=1}}^{m}\Big[\frac{a(v_n,v_i)}{b_-(v_n,v_i)}B(v_n)\Big]|0\rangle\end{aligned} \tag{2.3.14}$$

proving that $\mathcal{P}_m$ is true. Therefore by induction $\mathcal{P}_N$ is true for arbitrary $N\geq 1$. By analogous arguments, which use the commutation relation (2.2.24) and the equation (2.2.27), we are also able to show that

$$\begin{aligned}D(u)\prod_{n=1}^{N}B(v_n)|0\rangle &= \delta(u)\prod_{n=1}^{N}\Big[\frac{a(u,v_n)}{b_-(u,v_n)}B(v_n)\Big]|0\rangle \\ &- \sum_{i=1}^{N}\delta(v_i)\frac{c(u,v_i)}{b_-(u,v_i)}B(u)\prod_{\substack{n\neq i\\ n=1}}^{N}\Big[\frac{a(v_i,v_n)}{b_-(v_i,v_n)}B(v_n)\Big]|0\rangle\end{aligned} \tag{2.3.15}$$

for arbitrary $N\geq 1$. Summing the equations (2.3.9) and (2.3.15), we find that $|\Psi\rangle$ is an eigenvector of $t(u)$ if and only if their sub-leading terms cancel via the equations

$$\alpha(v_i)\frac{c(v_i,u)}{b_-(v_i,u)}\prod_{\substack{n\neq i\\ n=1}}^{N}\frac{a(v_n,v_i)}{b_-(v_n,v_i)}+\delta(v_i)\frac{c(u,v_i)}{b_-(u,v_i)}\prod_{\substack{n\neq i\\ n=1}}^{N}\frac{a(v_i,v_n)}{b_-(v_i,v_n)}=0 \tag{2.3.16}$$

for all $1\leq i\leq N$. Cancelling factors which are common to both terms in (2.3.16), we obtain the Bethe equations (2.3.6). Furthermore, summing the leading terms in (2.3.9) and (2.3.15), we recover the eigenvalue (2.3.5).

□

2.3.2 Scalar product

When studying a quantum integrable model, aside from calculating the spectrum of its Hamiltonian $\mathcal{H}$, another important problem is the calculation of its *scalar product* $S_N(\{u\}_N, \{v\}_N)$. The scalar product is a function of the $2N$ variables $\{u\}_N = \{u_1, \ldots, u_N\}$, $\{v\}_N = \{v_1, \ldots, v_N\}$ given by

$$S_N\Big(\{u\}_N, \{v\}_N\Big) = \langle 0| \overleftarrow{\prod_{m=1}^{N}} C(u_m) \overrightarrow{\prod_{n=1}^{N}} B(v_n)|0\rangle \tag{2.3.17}$$

for Bose-Einstein models, and by

$$S_N\Big(\{u\}_N, \{v\}_N\Big) = \langle \Uparrow | \overleftarrow{\prod_{m=1}^{N}} C(u_m) \overrightarrow{\prod_{n=1}^{N}} B(v_n)| \Uparrow\rangle \tag{2.3.18}$$

for Fermi-Dirac models. For simplicity, in the remainder of this section we specialize to the former case, though our treatment may be equally applied to the latter.

In general the variables $\{u\}_N$ and $\{v\}_N$ are kept free. Specializing to the case where $\{u\}_N$ and $\{v\}_N$ are solutions of the Bethe equations, the scalar product expresses the action of a dual eigenstate $\langle \Psi_u|$ on another eigenstate $|\Psi_v\rangle$. If $\{u\}_N$ and $\{v\}_N$ are different solutions of the Bethe equations, this action is trivially zero. This follows from the fact that

$$\tau_{\Psi_u}(z) S_N\Big(\{u\}_N, \{v\}_N\Big) = \langle 0| \overleftarrow{\prod_{m=1}^{N}} C(u_m) t(z) \overrightarrow{\prod_{n=1}^{N}} B(v_n)|0\rangle = \tau_{\Psi_v}(z) S_N\Big(\{u\}_N, \{v\}_N\Big) \tag{2.3.19}$$

and since the eigenvalues $\tau_{\Psi_u}(z), \tau_{\Psi_v}(z)$ in (2.3.19) are assumed to be different, the only possible resolution is that $S_N(\{u\}_N, \{v\}_N) = 0$. On the other hand, when $\{u\}_N$ and $\{v\}_N$ are equal to the *same* solution of the Bethe equations, $S_N(\{v\}_N, \{v\}_N)$ is non-zero and used in the normalization of other physical entities, such as correlation functions.

In this thesis we will calculate the scalar product in a variety of different models.[5] A universal technique for achieving this is to use the commutation relations (2.2.20) and the annihilation rules (2.2.28), (2.2.29) to manipulate the operators appearing in (2.3.17). Unfortunately, this is a complicated approach which generally does not lead to a compact expression. Our approach will be to refrain from the commutation relations (2.2.20) as much as possible, preferring simpler techniques which pertain to each individual model.

[5] In chapters 3 and 4 we will calculate $S_N(\{u\}_N, \{v\}_N)$ when the variables $\{u\}_N, \{v\}_N$ are free. In chapters 5 and 6 we will consider an intermediate case, when one set of variables $\{u\}_N$ is free whilst the other, $\{v\}_N$, satisfies the Bethe equations.

2.3.3 Graphical representation of scalar product

In this subsection we specialize to models whose basis vectors are orthonormal. That is, we consider models which satisfy

$$\mathcal{I}\Big(|m\rangle, |n\rangle\Big) = \prod_{i=1}^{M} \delta_{m_i,n_i} \tag{2.3.20}$$

for all basis vectors $|m\rangle = |m_1\rangle_1 \otimes \cdots \otimes |m_M\rangle_M$ and $|n\rangle = |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M$ in $\mathcal{V}$. Comparing with equation (2.1.8), this corresponds to the case $c_i(m_i) = 1$ for all $1 \leq i \leq M$ and $m_i \in \mathfrak{N}$. For such models, the scalar product $S_N(\{u\}_N, \{v\}_N)$ can be graphically represented as a lattice. In order to demonstrate this, we prepare some notations.

For all $0 \leq i \leq N$, let β_i, γ_i denote vectors $(\beta_{i,1}, \ldots, \beta_{i,M}), (\gamma_{i,1}, \ldots, \gamma_{i,M}) \in \mathfrak{N}^M$ and define

$$\begin{aligned} |\beta_i\rangle &= |\beta_{i,1}\rangle_1 \otimes \cdots \otimes |\beta_{i,M}\rangle_M, \quad \langle\beta_i| = \langle\beta_{i,1}|_1 \otimes \cdots \otimes \langle\beta_{i,M}|_M \\ |\gamma_i\rangle &= |\gamma_{i,1}\rangle_1 \otimes \cdots \otimes |\gamma_{i,M}\rangle_M, \quad \langle\gamma_i| = \langle\gamma_{i,1}|_1 \otimes \cdots \otimes \langle\gamma_{i,M}|_M \end{aligned} \tag{2.3.21}$$

to be their corresponding states in $\mathcal{V}$ and $\mathcal{V}^*$. For all $0 \leq i, j \leq N$, we insert the complete sets of states $\sum_{\gamma_i} |\gamma_i\rangle\langle\gamma_i|$ and $\sum_{\beta_j} |\beta_j\rangle\langle\beta_j|$ into (2.3.17), yielding

$$S_N\Big(\{u\}_N, \{v\}_N\Big) = \sum_{\substack{\gamma_0,\ldots,\gamma_N \\ \beta_0,\ldots,\beta_N}} \delta_{\gamma_N,\vec{0}} \overleftarrow{\prod_{i=1}^{N}} \langle\gamma_i|C(u_i)|\gamma_{i-1}\rangle \delta_{\gamma_0,\beta_0} \overrightarrow{\prod_{j=1}^{N}} \langle\beta_{j-1}|B(v_j)|\beta_j\rangle \delta_{\beta_N,\vec{0}} \tag{2.3.22}$$

where a term in the sum is equal to zero unless $\beta_N = \gamma_N = \vec{0}$ and $\beta_0 = \gamma_0$. Using the diagrammatic conventions discussed earlier in the chapter we identify each term in (2.3.22) with a string of vertices, as shown in the following figures.

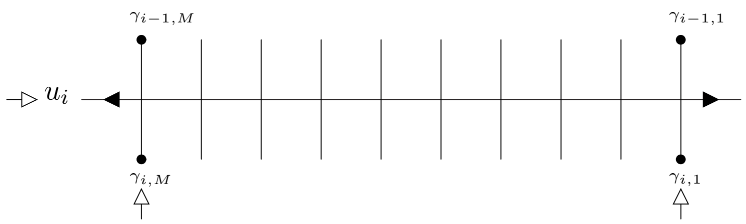

Figure 2.7: Vertex string for $\langle\gamma_i|C(u_i)|\gamma_{i-1}\rangle$.

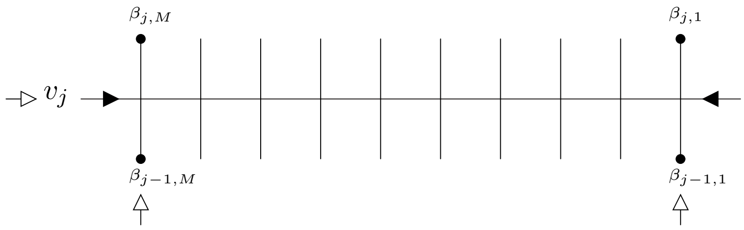

Figure 2.8: Vertex string for $\langle\beta_{j-1}|B(v_j)|\beta_j\rangle$.

Attaching these strings of vertices along identified indices, we arrive at a lattice representation of $S_N(\{u\}_N, \{v\}_N)$ as shown in figure 2.9.

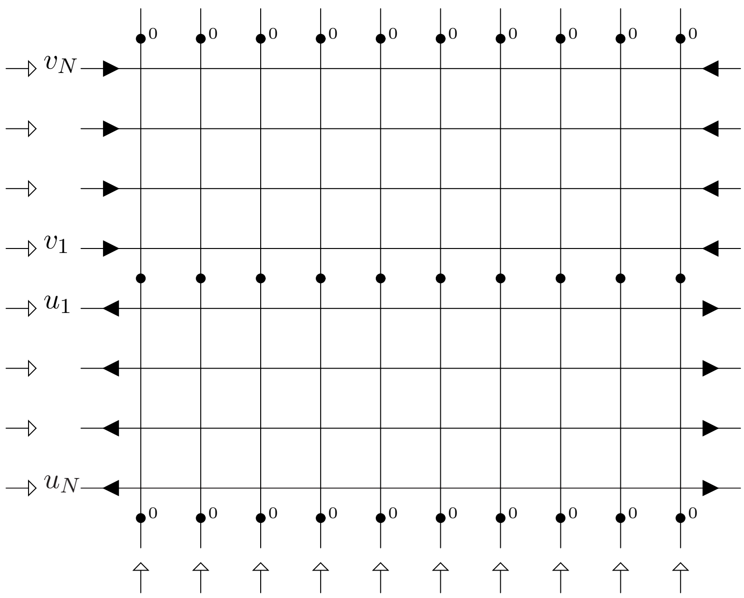

Figure 2.9: Graphical depiction of the scalar product.

The horizontal lines in this lattice should be interpreted as strings of L-matrix vertices, corresponding to monodromy matrix elements. The lowest N horizontal lines, through which the variables u_i are flowing, have external line values $(-1, +1)$. Therefore, they represent the monodromy matrix elements $\langle\gamma_i|T^{-+}(u_i)|\gamma_{i-1}\rangle = \langle\gamma_i|C(u_i)|\gamma_{i-1}\rangle$. The highest N horizontal lines, through which the variables v_j are flowing, have external line values $(+1, -1)$. Therefore, they represent the monodromy matrix elements $\langle\beta_{j-1}|T^{+-}(v_j)|\beta_j\rangle = \langle\beta_{j-1}|B(v_j)|\beta_j\rangle$. All horizontal internal line segments are summed over black arrows that point with the orientation, and black arrows that point against the orientation.

The vertical external line segments have been frozen to zero, to represent the fact that the vectors $\beta_N = \gamma_N = \vec{0}$. Conversely, the vertical internal line segments are summed over all elements of the set $\mathfrak{N}$. For simplicity, we have omitted all internal labels.

2.4 Conclusion

The definitions presented in this chapter will be used ubiquitously throughout the rest of the thesis. We shall consider a number of quantum integrable models, and in each case we list their quantum algebras $\mathcal{A}_1, \ldots, \mathcal{A}_M$ and define representations of these algebras on the vector space $\mathcal{V}$. We also state the Hamiltonian $\mathcal{H}$ of each model under our consideration, but this is mainly for completeness and it is *not* our aim to study their spectra in any detail.

The most essential material in this chapter is the description of the quantum inverse scattering method/algebraic Bethe Ansatz. Indeed, we shall apply these techniques to every model that we encounter, listing its R-matrix and L-matrix, and constructing its Bethe eigenvectors. The graphical conventions of this chapter are also prevalent throughout the remainder of our work, particularly in chapter 5 and 6.

Finally, let us remark that scalar products are of key interest in our later studies. We will calculate the scalar product of almost every model under our consideration, studying their role as τ-functions in chapters 3,5 and as generating functions of plane partitions in chapters 3,4.

Chapter 3

Bosonic models and plane partitions

3.0 Introduction

In this chapter we study two closely related quantum integrable models which are solvable by the algebraic Bethe Ansatz. We will discuss the relationship of these models with the hierarchies discussed in chapter 1, and with plane partitions, which are classical combinatorial objects. Similarly to chapter 1, every result in the context of the first model is mirrored by an analogous result in the context of the second, and accordingly we have split this chapter into two parallel parts.

The first part of the chapter considers the phase model, which was introduced in [9] and subsequently studied in [10] as the limiting case $q \rightarrow \infty$ of the more general q-boson model. In section 3.1 we give the space of states $\mathcal{V}$ of the model, the Hamiltonian $\mathcal{H}$, and review the construction of its eigenvectors using the algebraic Bethe Ansatz.

An explicit expression for the phase model Bethe eigenvectors was found by N M Bogoliubov in [7]. To perform this calculation, Bogoliubov defined a simple correspondence between the basis vectors of $\mathcal{V}$ and partitions. Under this correspondence, the Bethe eigenvectors can be written as sums over partitions which are weighted by Schur functions. In section 3.2 we reproduce these results while appealing to the charged fermion calculus discussed in chapter 1. We map the basis vectors of $\mathcal{V}$ to partitions in the Fock space $\mathcal{F}_\psi$ and calculate the image of the Bethe eigenvectors under this map. At the level of the vector space $\mathcal{V}$, a Bethe eigenvector is constructed by the action of B-operators acting on the vacuum state. We find that at the level of the Fock space $\mathcal{F}_\psi$, a Bethe eigenvector lies in the orbit of the Fock vacuum under GL_∞.

The connection of the phase model with plane partitions was also discovered in [7]. Using the diagrammatic interpretation of the monodromy matrix operators, as we described in chapter 2, Bogoliubov was able to show that the phase model scalar product is a generating function for plane partitions within a box of size

$N \times N \times M$. In section 3.3 we give another proof of this result, taking the perspective of A Okounkov and N Reshetikhin in [74], where it was observed that the diagonal slices of an arbitrary plane partition form a sequence of interlacing partitions. The proof involves showing that monodromy matrix operators act on a partition state to generate a sum of partitions which interlace with the original. We end the section by writing the phase model scalar product in the form of a KP τ-function, that is, as an expectation value of charged fermionic operators.

In section 3.4 we study the phase model as $M \to \infty$, which is the infinite lattice limit. We prove that in this limit the action of a B-operator on an arbitrary vector in $\mathcal{V}$ is equivalent to the action of a KP half-vertex operator on the image state in $\mathcal{F}_\psi$. This is achieved by showing that KP half-vertex operators act on a partition state in $\mathcal{F}_\psi$ to generate a sum of interlacing partitions. Working at the level of fermionic operators, the infinite lattice scalar product is readily evaluated. In fact, because KP half-vertex operators have simple commutation relations, we find that the scalar product factorizes into product form. We thus obtain a fermionic construction of MacMahon's generating function for plane partitions, first proposed in [75] and explained in detail in [39].

In the second part of the chapter we repeat the calculations of the first part, but in the context of the $q \to i$ limit of the q-boson model, where $i = \sqrt{-1}$. For brevity we call this the i-boson model, and to the best of our knowledge it has not been studied in the literature, in its own right. In section 3.5 we give the space of states $\tilde{\mathcal{V}}$ of the model, the Hamiltonian $\tilde{\mathcal{H}}$, and describe the construction of its eigenvectors using the algebraic Bethe Ansatz.[1]

After this, our attention turns to evaluating the Bethe eigenvectors of the i-boson model. There exists a correspondence between the basis vectors of $\tilde{\mathcal{V}}$ and strict partitions. Under this correspondence, a Bethe eigenvector can be written as a sum over strict partitions which are weighted by Schur Q-functions. In section 3.6 we make these notions precise by using the neutral fermion calculus discussed in chapter 1. We map basis vectors of $\tilde{\mathcal{V}}$ to strict partitions in the Fock space $\mathcal{F}_\phi$, and calculate the image of the Bethe eigenvectors under this map. At the level of the vector space $\tilde{\mathcal{V}}$, a Bethe eigenvector is constructed by the action of B-operators acting on the vacuum state. We find that at the level of the Fock space $\mathcal{F}_\phi$, a Bethe eigenvector lies in the orbit of the Fock vacuum under O_∞.

In section 3.7 we establish a connection between the i-boson model and strict plane partitions. Specifically, we prove that the scalar product of the i-boson model is a generating function for strict plane partitions within a box of size $N \times N \times M$. This time around, the diagonal slices of an arbitrary strict plane partition form a sequence of interlacing strict partitions. Hence the proof involves showing that monodromy matrix operators act on a strict partition state to generate a sum of strict partitions which interlace with the original. We end the section by writing the i-boson model scalar product in the form of a BKP τ-function, that is, as an

[1] In the second part of the chapter, we distinguish all operators and spaces from their direct counterparts in the first part by use of a tilde.

expectation value of neutral fermionic operators.

Finally, in section 3.8 we study the i-boson model as $M \to \infty$. We prove that in this limit the action of a B-operator on an arbitrary vector in $\tilde{\mathcal{V}}$ is equivalent to the action of a BKP half-vertex operator on the image state in $\mathcal{F}_\phi$. This is achieved by showing that BKP half-vertex operators act on a strict partition state in $\mathcal{F}_\phi$ to generate a sum of interlacing strict partitions. As in the case of section 3.4, the BKP half-vertex operators have simple commutation relations, meaning that the scalar product factorizes into product form. We thus obtain a fermionic construction of the generating function for strict plane partitions, which first appeared in [35] and later in [39].

The essence of this chapter is the observation that free fermions appear in the Bethe eigenvectors of two bosonic integrable models. We hope that this result is indicative of a deeper correspondence between the concerned classical and quantum models.

3.1 Phase model

3.1.1 Space of states $\mathcal{V}$ and inner product $\mathcal{I}$

Following the procedure outlined in the previous chapter, we construct a vector space $\mathcal{V}$ which provides the framework for study of the phase model. Consider a one-dimensional integral lattice, consisting of $M+1$ sites. A configuration of this lattice corresponds to placing $n_i \geq 0$ bosons at the i^{th} site, for all $0 \leq i \leq M$. The vector space $\mathcal{V}$ is defined as the linear span of all lattice configurations.

Mathematically, we represent a configuration by a tensor product of state vectors $|n_i\rangle_i$, where n_i is the i^{th} occupation number in that particular configuration. It follows that $\mathcal{V}$ has the basis

$$\text{Basis}\,(\mathcal{V}) = \Big\{ |n\rangle = |n_0\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M \Big\} \tag{3.1.1}$$

where $\{n_0, n_1, \ldots, n_M\}$ range over all non-negative integers. The inner product $\mathcal{I}$ between two basis vectors $|m\rangle = |m_0\rangle_0 \otimes \cdots \otimes |m_M\rangle_M$ and $|n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ is defined as

$$\mathcal{I}\Big(|m\rangle, |n\rangle\Big) = \prod_{i=0}^{M} \delta_{m_i,n_i} \tag{3.1.2}$$

which corresponds to setting $c_i(m_i) = 1$ for all $0 \leq i \leq M$, $m_i \geq 0$ in equation (2.1.8). We also define a space of states $\mathcal{V}^*$ that is dual to $\mathcal{V}$. We write

$$\text{Basis}\,(\mathcal{V}^*) = \Big\{ \langle m| = \langle m_0|_0 \otimes \langle m_1|_1 \otimes \cdots \otimes \langle m_M|_M \Big\} \tag{3.1.3}$$

where once again $\{m_0, m_1, \ldots, m_M\}$ range over all non-negative integers. The action of a basis vector $\langle m| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M \in \mathcal{V}^*$ on $|n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M \in \mathcal{V}$ is given by

$$\langle m|n\rangle = \mathcal{I}\Big(|m\rangle, |n\rangle\Big) \tag{3.1.4}$$

Both the vector spaces (3.1.1) and (3.1.3) are infinite-dimensional, since there is no upper bound enforced on the occupation numbers m_i and n_i.

3.1.2 Phase algebra

The phase algebra is defined in [7], [10]. It is generated by $\{\phi, \phi^\dagger, \mathcal{N}, \pi\}$ which satisfy the relations

$$[\phi, \phi^\dagger] = \pi, \quad [\mathcal{N}, \phi] = -\phi, \quad [\mathcal{N}, \phi^\dagger] = \phi^\dagger, \quad \phi\pi = \pi\phi^\dagger = 0 \tag{3.1.5}$$

This algebra is the $q \rightarrow \infty$ case of the q-boson algebra (4.1.4), discussed in the next chapter. We will consider $M+1$ copies of the phase algebra, generated by $\{\phi_0, \phi_0^\dagger, \mathcal{N}_0, \pi_0\}$ through to $\{\phi_M, \phi_M^\dagger, \mathcal{N}_M, \pi_M\}$.[2] Employing the language of chapter 2, we denote these algebras by $\mathcal{A}_0, \ldots, \mathcal{A}_M$ with $\mathfrak{a}_i^+ = \phi_i^\dagger, \mathfrak{a}_i^- = \phi_i$ and where $\mathcal{N}_i, \pi_i$ are both of type $\mathfrak{a}_i^0$. Different copies of the phase algebra are assumed to commute, giving rise to the equations

$$[\phi_i, \phi_j^\dagger] = \delta_{i,j}\pi_i, \quad [\mathcal{N}_i, \phi_j] = -\delta_{i,j}\phi_i, \quad [\mathcal{N}_i, \phi_j^\dagger] = \delta_{i,j}\phi_i^\dagger, \quad \phi_i\pi_i = \pi_i\phi_i^\dagger = 0 \tag{3.1.6}$$

for all $0 \leq i, j \leq M$.

3.1.3 Representations of phase algebras

Following [7], [10], we fix representations of the phase algebras on the vector space $\mathcal{V}$. We begin with a synopsis of the role played by each operator. Firstly ϕ_i is an annihilation operator, removing particles from the i^{th} lattice site. Conversely $\phi_i^\dagger$ is a creation operator, adding particles to the i^{th} lattice site. The operator $\mathcal{N}_i$ counts, but does not change, the number of particles at the i^{th} lattice site. Finally π_i is a *vacuum projector*, leaving the i^{th} site unchanged if it is unoccupied by particles, but annihilating any non-empty i^{th} state vector.

Let us more precisely define these representations of $\mathcal{A}_0, \ldots, \mathcal{A}_M$ on $\mathcal{V}$. The operator ϕ_i acts on the i^{th} state vector in a basis element. If the i^{th} occupation number is zero ϕ_i annihilates the basis element, otherwise it lowers the i^{th} occupation number by one. This is described by the equation

[2] The generators ϕ_i should not be confused with the neutral fermions of chapter 1.

$$\phi_i|n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \begin{cases} 0, & n_i = 0 \\ \\ |n_0\rangle_0 \otimes \cdots \otimes |n_i - 1\rangle_i \otimes \cdots \otimes |n_M\rangle_M, & n_i \geq 1 \end{cases} \tag{3.1.7}$$

The operator $\phi_i^\dagger$ acts on the i^{th} state vector in a basis element and raises the i^{th} occupation number by one. This is described by the equation

$$\phi_i^\dagger|n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = |n_0\rangle_0 \otimes \cdots \otimes |n_i + 1\rangle_i \otimes \cdots \otimes |n_M\rangle_M \tag{3.1.8}$$

Every basis element of $\mathcal{V}$ is an eigenvector of the operator $\mathcal{N}_i$ with eigenvalue equal to the i^{th} occupation number. This is described by the equation

$$\mathcal{N}_i|n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = n_i|n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M \tag{3.1.9}$$

Lastly, the operator π_i acts on the i^{th} state vector in a basis element. If the i^{th} occupation number is zero π_i acts identically, otherwise it annihilates the basis element. This is described by the equation

$$\pi_i|n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \begin{cases} |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M, & n_i = 0 \\ \\ 0, & n_i \geq 1 \end{cases} \tag{3.1.10}$$

The set of definitions (3.1.7)–(3.1.10) provide a faithful representation of the phase algebras $\mathcal{A}_0, \ldots, \mathcal{A}_M$. Assuming that all operators act linearly, equations (3.1.7)–(3.1.10) completely determine the action of $\{\phi_i, \phi_i^\dagger, \mathcal{N}_i, \pi_i\}$ on the vector space $\mathcal{V}$. Also, from the definition of the inner product (3.1.2) we find that

$$\mathcal{I}\Big(\phi_i|m\rangle, |n\rangle\Big) = \mathcal{I}\Big(|m\rangle, \phi_i^\dagger|n\rangle\Big) \tag{3.1.11}$$

which shows that $\phi_i, \phi_i^\dagger$ are adjoint operators, while $\mathcal{N}_i, \pi_i$ are clearly self-adjoint.

Following subsection 2.1.4 in the previous chapter, we fix appropriate actions for $\mathcal{A}_0, \ldots, \mathcal{A}_M$ on the basis elements of the dual space $\mathcal{V}^*$. In short, the roles of $\phi_i, \phi_i^\dagger$ get interchanged, while $\mathcal{N}_i, \pi_i$ behave as before. More explicitly, the operator $\phi_i^\dagger$ acts on the i^{th} state vector in a dual basis element. If the i^{th} occupation number is zero $\phi_i^\dagger$ annihilates the dual basis element, otherwise it lowers the i^{th} occupation number by one. This is described by the equation

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \phi_i^\dagger = \begin{cases} 0, & n_i = 0 \\ \\ \langle n_0|_0 \otimes \cdots \otimes \langle n_i - 1|_i \otimes \cdots \otimes \langle n_M|_M, & n_i \geq 1 \end{cases} \tag{3.1.12}$$

The operator ϕ_i acts on the i^{th} state vector in a dual basis element and raises the i^{th} occupation number by one. This is described by the equation

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \phi_i = \langle n_0|_0 \otimes \cdots \otimes \langle n_i + 1|_i \otimes \cdots \otimes \langle n_M|_M \tag{3.1.13}$$

Every basis element of $\mathcal{V}^*$ is an eigenvector of the operator $\mathcal{N}_i$ with eigenvalue equal to the i^{th} occupation number. This is described by the equation

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \mathcal{N}_i = n_i \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \tag{3.1.14}$$

Finally, the operator π_i acts on the i^{th} state vector in a dual basis element. If the i^{th} occupation number is zero π_i acts identically, otherwise it annihilates the dual basis element. This is described by the equation

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \pi_i = \begin{cases} \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M, & n_i = 0 \\ \\ 0, & n_i \geq 1 \end{cases} \tag{3.1.15}$$

The set of definitions (3.1.12)–(3.1.15) provide the dual representation of the algebras $\mathcal{A}_0, \ldots, \mathcal{A}_M$. Assuming that all operators act linearly, the equations (3.1.12)–(3.1.15) completely determine the action of $\{\phi_i, \phi_i^\dagger, \mathcal{N}_i, \pi_i\}$ on the dual vector space $\mathcal{V}^*$.

3.1.4 Calculation of $\langle m|n\rangle$

In the interest of self-consistency, in this subsection we check that (3.1.4) is actually obeyed. By virtue of the equation (3.1.8), it is possible to write any basis element of $\mathcal{V}$ in terms of the operators $\phi_i^\dagger$ acting on the vacuum state. Explicitly speaking, we have

$$\begin{aligned} |n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M &= (\phi_0^\dagger)^{n_0} \ldots (\phi_M^\dagger)^{n_M} |0\rangle_0 \otimes \cdots \otimes |0\rangle_M \\ &= (\phi_0^\dagger)^{n_0} \ldots (\phi_M^\dagger)^{n_M} |0\rangle \end{aligned} \tag{3.1.16}$$

Similarly, the equation (3.1.13) makes it possible to write any basis element of $\mathcal{V}^*$ in terms of the operators ϕ_i acting on the dual vacuum state. We find that

$$\begin{aligned} \langle m| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M &= \langle 0|_0 \otimes \cdots \otimes \langle 0|_M (\phi_0)^{m_0} \ldots (\phi_M)^{m_M} \\ &= \langle 0|(\phi_0)^{m_0} \ldots (\phi_M)^{m_M} \end{aligned} \tag{3.1.17}$$

Using equations (3.1.16) and (3.1.17) and the fact that the vacuum expectation value of any element in $\mathcal{A} = \mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_M$ is unambiguously defined, we have

$$\langle m|n\rangle = \langle 0|(\phi_0)^{m_0}\dots(\phi_M)^{m_M}(\phi_0^\dagger)^{n_0}\dots(\phi_M^\dagger)^{n_M}|0\rangle \tag{3.1.18}$$

From the commutation relations (3.1.6) and the fact that $\phi_i|0\rangle = \langle 0|\phi_i^\dagger = 0$ for all $0 \le i \le M$, we calculate (3.1.18) explicitly to obtain

$$\langle m|n\rangle = \prod_{i=0}^{M} \delta_{m_i,n_i} \tag{3.1.19}$$

in agreement with equation (3.1.4).

3.1.5 Hamiltonian $\mathcal{H}$

The Hamiltonian of the phase model is given by

$$\mathcal{H} = -\frac{1}{2}\sum_{i=0}^{M}\left(\phi_i^\dagger\phi_{i+1} + \phi_i\phi_{i+1}^\dagger\right) + \bar{\mathcal{N}} \tag{3.1.20}$$

with $\bar{\mathcal{N}} = \sum_{i=0}^{M}\mathcal{N}_i$ and where the periodicity $\phi_{M+1} = \phi_0$ and $\phi_{M+1}^\dagger = \phi_0^\dagger$ is imposed. The problem of finding eigenvectors $|\Psi\rangle$ of this Hamiltonian can be solved using the quantum inverse scattering method/algebraic Bethe Ansatz. In the forthcoming subsections we recover $\mathcal{H}$ from the transfer matrix of the model and construct the Bethe eigenvectors, using the terminologies which were described in chapter 2.

3.1.6 L-matrix and local intertwining equation

The R-matrix for the phase model depends on two indeterminates x, y and acts in the tensor product $\mathcal{V}_a \otimes \mathcal{V}_b$, where $\mathcal{V}_a, \mathcal{V}_b$ are copies of $\mathbb{C}^2$. It is given by

$$R_{ab}(x,y) = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 0 & x^{\frac{1}{2}}y^{\frac{1}{2}} & 0 \\ 0 & x^{\frac{1}{2}}y^{\frac{1}{2}} & x-y & 0 \\ 0 & 0 & 0 & x \end{pmatrix}_{ab} \tag{3.1.21}$$

and corresponds to the $a_\pm(x,y) = x,\ b_+(x,y) = 0,\ b_-(x,y) = x-y,\ c_\pm(x,y) = x^{\frac{1}{2}}y^{\frac{1}{2}}$ case of (2.2.1). The L-matrix for the phase model depends on a single indeterminate x, and acts in the space $\mathcal{V}_a$. Its entries are operators acting at the m^{th} lattice site, and identically everywhere else. It has the form

$$L_{am}(x) = \begin{pmatrix} x^{-\frac{1}{2}} & \phi_m^\dagger \\ \phi_m & x^{\frac{1}{2}} \end{pmatrix}_a \tag{3.1.22}$$

Using these definitions, the local intertwining equation is given by

$$R_{ab}(x,y)L_{am}(x)L_{bm}(y) = L_{bm}(y)L_{am}(x)R_{ab}(x,y) \tag{3.1.23}$$

This is a 4×4 matrix equation, which gives rise to sixteen scalar identities. Each of these identities may be verified by direct calculation. The L-matrix and R-matrix of the phase model may be found in [7], [10], which use slightly different parametrizations from those that we have adopted.

3.1.7 Monodromy matrix and global intertwining equation

The monodromy matrix is an $(M+1)$-fold product of the L-matrices (3.1.22), taken in the auxiliary space $\mathrm{End}(\mathcal{V}_a)$. It has the form

$$T_a(x) = L_{aM}(x)\dots L_{a0}(x) = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}_a \tag{3.1.24}$$

where $A(x), B(x), C(x), D(x)$ are elements of $\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_M$. The monodromy matrix satisfies the global intertwining equation

$$R_{ab}(x,y)T_a(x)T_b(y) = T_b(y)T_a(x)R_{ab}(x,y) \tag{3.1.25}$$

which follows immediately from the local intertwining equation (3.1.23).[3] The identity (3.1.25) gives sixteen commutation relations between the $A(x), B(x), C(x), D(x)$ operators, but for our purposes we will only require two. These are the equations

$$[B(x), B(y)] = [C(x), C(y)] = 0 \tag{3.1.26}$$

and they are necessary to show that the Bethe eigenvectors are symmetric in their rapidity variables.

3.1.8 Recovering $\mathcal{H}$ from the transfer matrix

Let $t(x) = \mathrm{tr}_a T_a(x) = A(x) + D(x)$ be the transfer matrix of the phase model. The Hamiltonian (3.1.20) may be recovered via the equation

$$\mathcal{H} = \frac{1}{2}\left[x^2 \frac{d}{dx}\Big(x^{-(M+1)/2} t(x)\Big)\right]_{x\to\infty} - \frac{1}{2}\left[\frac{d}{dx}\Big(x^{(M+1)/2} t(x)\Big)\right]_{x\to 0} + \bar{\mathcal{N}} \tag{3.1.27}$$

and using the fact that $[\bar{\mathcal{N}}, t(x)] = 0$, it follows that $[\mathcal{H}, t(x)] = 0$. Hence the eigenvectors of $\mathcal{H}$ may be found by studying the eigenvectors of $t(x)$.

[3] See lemma 1 in chapter 2.

3.1.9 Bethe Ansatz for the eigenvectors

As was explained in theorem 1 of the previous chapter, the eigenvectors of the transfer matrix $t(x)$ are given by

$$|\Psi\rangle = B(y_1)\dots B(y_N)|0\rangle, \quad \langle\Psi| = \langle 0|C(y_N)\dots C(y_1) \tag{3.1.28}$$

where the variables $\{y_1, \dots, y_N\}$ are assumed to obey the Bethe equations (2.3.6). For the present model, we have $a(y_i, y_j) = y_i,\ \alpha(y_i) = y_i^{-(M+1)/2},\ \delta(y_i) = y_i^{(M+1)/2}$. Substituting these expressions into (2.3.6), the Bethe equations for the phase model read

$$y_i^{M+N} = (-)^{N-1} \prod_{\substack{j\neq i\\ j=1}}^{N} y_j \tag{3.1.29}$$

for all $1 \leq i \leq N$. Although these equations are necessary for the success of the Bethe Ansatz, in our subsequent analysis we will study the vectors (3.1.28) *without* imposing the restrictions (3.1.29). We shall continue to call the objects (3.1.28) Bethe eigenvectors, despite the fact that the Bethe equations are superfluous in our calculations.

3.2 Calculation of phase model Bethe eigenvectors

In this section we derive an explicit expression for the Bethe eigenvectors (3.1.28). Although the result that we obtain first appeared in [7], our derivation has some new features. In particular, we map the Bethe eigenvectors to the charged fermionic Fock space of chapter 1, and show that they lie in the GL_∞ orbit of the Fock vacuum.

3.2.1 The maps $\mathcal{M}_\psi$ and $\mathcal{M}_\psi^*$

Definition 1. Let $|n\rangle = |n_0\rangle_0 \otimes \dots \otimes |n_M\rangle_M$ and $\langle n| = \langle n_0|_0 \otimes \dots \otimes \langle n_M|_M$ be basis elements of $\mathcal{V}$ and $\mathcal{V}^*$, respectively, and define

$$\Sigma_0 = \sum_{j=0}^{M} n_j \tag{3.2.1}$$

From this, let $|\nu) = |\nu_1, \dots, \nu_{\Sigma_0})$ and $(\nu| = (\nu_1, \dots, \nu_{\Sigma_0}|$ be partitions in the Fock spaces $\mathcal{F}_\psi^{(0)}$ and $\mathcal{F}_\psi^{*(0)}$ with n_i parts equal to i for all $0 \leq i \leq M$. That is, we let

$$|\nu) = |M^{n_M}, \dots, 1^{n_1}, 0^{n_0}) = |M^{n_M}, \dots, 1^{n_1}) \tag{3.2.2}$$
$$(\nu| = (M^{n_M}, \dots, 1^{n_1}, 0^{n_0}| = (M^{n_M}, \dots, 1^{n_1}| \tag{3.2.3}$$

We define linear maps $\mathcal{M}_\psi : \mathcal{V} \to \mathcal{F}_\psi^{(0)}$ and $\mathcal{M}_\psi^* : \mathcal{V}^* \to \mathcal{F}_\psi^{*(0)}$ whose actions are given by

$$\mathcal{M}_\psi |n\rangle = |\nu), \quad \langle n| \mathcal{M}_\psi^* = (\nu| \tag{3.2.4}$$

Notice that these mappings are *not* one-to-one since they are insensitive to the value of n_0, which only appears as trivial information in the corresponding partition ν. Furthermore, they are isometric in the sense that $\langle m|n\rangle = \langle\langle m|\mathcal{M}_\psi^*, \mathcal{M}_\psi|n\rangle\rangle$ for all $\langle m| \in \mathcal{V}^*, |n\rangle \in \mathcal{V}$ which satisfy $m_0 = n_0$. From equation (3.2.4), the action of $\mathcal{M}_\psi, \mathcal{M}_\psi^*$ on any element of $\mathcal{V}, \mathcal{V}^*$ can be calculated using linearity.

The maps $\mathcal{M}_\psi$ and $\mathcal{M}_\psi^*$ are motivated by section V of [7], which discusses the same correspondence between basis elements of $\mathcal{V}$ and partitions.

Example 1. Fix $M = 4$ and $|n\rangle = |2\rangle_0 \otimes |3\rangle_1 \otimes |0\rangle_2 \otimes |2\rangle_3 \otimes |1\rangle_4$. We have $\Sigma_0 = 2+3+0+2+1 = 8$, and we let

$$|\nu) = |\nu_1, \ldots, \nu_8) = |4^1, 3^2, 2^0, 1^3, 0^2) = |4,3,3,1,1,1) \tag{3.2.5}$$

Then $\mathcal{M}_\psi|n\rangle = |\nu)$. This correspondence is also shown in the figure below.

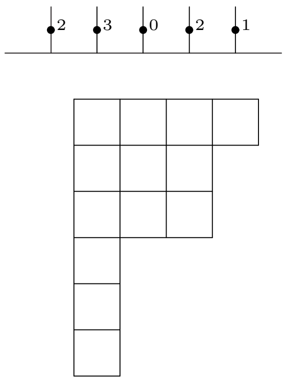

Figure 3.1: Mapping of $|n\rangle = |2\rangle_0 \otimes |3\rangle_1 \otimes |0\rangle_2 \otimes |2\rangle_3 \otimes |1\rangle_4$ to $|\nu) = |4,3,3,1,1,1)$. The top part of the figure represents the state vector $|n\rangle$, while the lower part represents the Young diagram of ν. Each occupation number n_i manifests itself as n_i rows of boxes of length i in the Young diagram.

3.2.2 Admissible basis elements

Definition 2. Let $|m\rangle = |m_0\rangle_0 \otimes \cdots \otimes |m_M\rangle_M$ and $|n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ be basis elements of $\mathcal{V}$. Define the partial sums of occupation numbers

$$\Sigma_i^m = \sum_{j=i}^{M} m_j, \quad \Sigma_i^n = \sum_{j=i}^{M} n_j \tag{3.2.6}$$

for all $0 \leq i \leq M$. We say that $|m\rangle$ is *admissible* to $|n\rangle$, and write $|m\rangle \triangleright |n\rangle$, if and only if

$$\begin{aligned} 0 \leq (\Sigma_i^m - \Sigma_i^n) \leq 1 \quad \text{for all} \quad 1 \leq i \leq M \\ \text{and} \quad (\Sigma_0^m - \Sigma_0^n) = 1 \end{aligned} \tag{3.2.7}$$

This definition extends in an obvious way to the basis elements of $\mathcal{V}^*$. For $\langle m| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M$ and $\langle n| = \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M$, we say that $\langle m|$ is admissible to $\langle n|$, and write $\langle n| \triangleleft \langle m|$, if and only if the above condition on the occupation numbers is satisfied.

Example 2. Fix $M = 4$, and let

$$|m\rangle = |2\rangle_0 \otimes |4\rangle_1 \otimes |0\rangle_2 \otimes |1\rangle_3 \otimes |2\rangle_4 \tag{3.2.8}$$

$$|n\rangle = |2\rangle_0 \otimes |3\rangle_1 \otimes |0\rangle_2 \otimes |2\rangle_3 \otimes |1\rangle_4 \tag{3.2.9}$$

The partial sums are given by

$$\left\{\Sigma_0^m, \Sigma_1^m, \Sigma_2^m, \Sigma_3^m, \Sigma_4^m\right\} = \{9, 7, 3, 3, 2\} \tag{3.2.10}$$

$$\left\{\Sigma_0^n, \Sigma_1^n, \Sigma_2^n, \Sigma_3^n, \Sigma_4^n\right\} = \{8, 6, 3, 3, 1\} \tag{3.2.11}$$

and we find that

$$(\Sigma_i^m - \Sigma_i^n) = \begin{cases} 1, & i = 0 \\ 1, & i = 1 \\ 0, & i = 2 \\ 0, & i = 3 \\ 1, & i = 4 \end{cases} \tag{3.2.12}$$

Therefore, these two basis vectors satisfy the condition $|m\rangle \triangleright |n\rangle$.

3.2.3 Interlacing partitions

Definition 3. Let $|\mu) = |\mu_1, \ldots, \mu_{l+1})$ and $|\nu) = |\nu_1, \ldots, \nu_l)$ be two partitions in $\mathcal{F}_\psi^{(0)}$. We say that $|\mu)$ *interlaces* $|\nu)$, and write $|\mu) \succ |\nu)$, if and only if

$$\mu_i \geq \nu_i \geq \mu_{i+1} \tag{3.2.13}$$

for all $1 \leq i \leq l$. Similarly, for the two partitions $(\mu| = (\mu_1, \ldots, \mu_{l+1}|$ and $(\nu| = (\nu_1, \ldots, \nu_l|$ in $\mathcal{F}_\psi^{*(0)}$ we say that $(\mu|$ interlaces $(\nu|$, and write $(\nu| \prec (\mu|$, if and only if the above condition on the partition parts is satisfied.[4]

Example 3. The partitions

$$|\mu) = |\mu_1, \ldots, \mu_7) = |4, 4, 3, 1, 1, 1, 1) \tag{3.2.14}$$
$$|\nu) = |\nu_1, \ldots, \nu_6) = |4, 3, 3, 1, 1, 1) \tag{3.2.15}$$

obey $\mu_i \geq \nu_i \geq \mu_{i+1}$ for all $1 \leq i \leq 6$. Therefore, we have $|\mu) \succ |\nu)$. This example is further illustrated by the figure below.

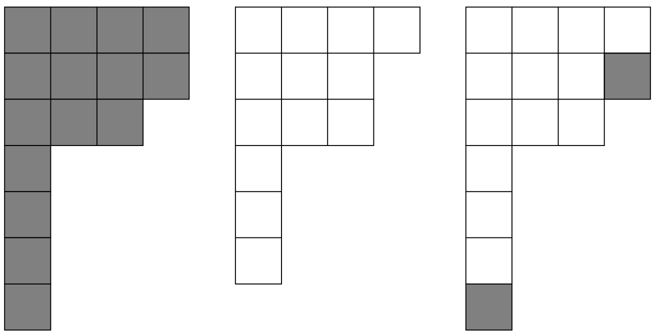

Figure 3.2: Interlacing partitions $|\mu) = |4, 4, 3, 1, 1, 1, 1)$ and $|\nu) = |4, 3, 3, 1, 1, 1)$. The grey Young diagram represents μ, while the white Young diagram represents ν. When the smaller Young diagram is stacked on the larger one, the visible grey boxes comprise the skew diagram μ/ν. Because the partitions interlace, μ/ν is a horizontal strip, [65].

3.2.4 Admissible basis vectors map to interlacing partitions

Lemma 1. Let $|m\rangle = |m_0\rangle_0 \otimes \cdots \otimes |m_M\rangle_M$ and $|n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ be basis elements of $\mathcal{V}$, and let

$$|\mu) = \mathcal{M}_\psi |m\rangle, \quad |\nu) = \mathcal{M}_\psi |n\rangle \tag{3.2.16}$$

be their corresponding partitions in $\mathcal{F}_\psi^{(0)}$. Then if $|m\rangle$ is admissible to $|n\rangle$, the partition $|\mu)$ interlaces $|\nu)$. That is,

[4] The relationship (3.2.13) between two partitions is ubiquitous in the literature, albeit under different nomenclature. For example, the condition $\mu \succ \nu$ is equivalent to saying that the skew diagram μ/ν forms a horizontal strip, which is the terminology preferred in [65] and most other references.

$$|m\rangle \triangleright |n\rangle \implies |\mu) \succ |\nu) \tag{3.2.17}$$

Similarly, at the level of the dual spaces $\mathcal{V}^*$ and $\mathcal{F}_\psi^{*(0)}$ we have

$$\langle n| \triangleleft \langle m| \implies (\nu| \prec (\mu| \tag{3.2.18}$$

Proof. Due to the assumption $|m\rangle \triangleright |n\rangle$ we know that $\Sigma_0^m - \Sigma_0^n = 1$, which implies that the corresponding partitions $|\mu)$ and $|\nu)$ have $\Sigma_0^n + 1$ and Σ_0^n parts, respectively. The proof is achieved by showing that

$$\mu_i \geq \nu_i \geq \mu_{i+1}, \quad \text{for all } 1 \leq i \leq \Sigma_0^n \tag{3.2.19}$$

We will demonstrate this fact by contradiction. Using the definition (3.2.4) of the map $\mathcal{M}_\psi$, we write

$$|\mu) = |M^{m_M}, \ldots, 1^{m_1}, 0^{m_0}), \quad |\nu) = |M^{n_M}, \ldots, 1^{n_1}, 0^{n_0}) \tag{3.2.20}$$

from which we recover the inequalities

$$\Sigma_{\mu_j+1}^m < j \leq \Sigma_{\mu_j}^m \quad \text{for all } 1 \leq j \leq \Sigma_0^m \tag{3.2.21}$$

$$\Sigma_{\nu_j+1}^n < j \leq \Sigma_{\nu_j}^n \quad \text{for all } 1 \leq j \leq \Sigma_0^n \tag{3.2.22}$$

Now suppose that $\nu_i > \mu_i$ for some $1 \leq i \leq \Sigma_0^n$. Then from (3.2.21) we have $\Sigma_{\nu_i}^m \leq \Sigma_{\mu_i+1}^m < i$, while from (3.2.22) we have $i \leq \Sigma_{\nu_i}^n$. Together these imply that $\Sigma_{\nu_i}^m - \Sigma_{\nu_i}^n < 0$, which contradicts the assumption $|m\rangle \triangleright |n\rangle$.

Alternatively, suppose that $\mu_{i+1} > \nu_i$ for some $1 \leq i \leq \Sigma_0^n$. Then from (3.2.21) we have $i + 1 \leq \Sigma_{\mu_{(i+1)}}^m$, while from (3.2.22) we have $\Sigma_{\mu_{(i+1)}}^n \leq \Sigma_{\nu_i+1}^n < i$. Together these imply that $1 < \Sigma_{\mu_{(i+1)}}^m - \Sigma_{\mu_{(i+1)}}^n$, which once again contradicts the assumption $|m\rangle \triangleright |n\rangle$.

Hence we see that to avoid any contradiction the sequence of inequalities (3.2.19) must hold. The following figure provides an example of admissible basis vectors and interlacing partitions.

□

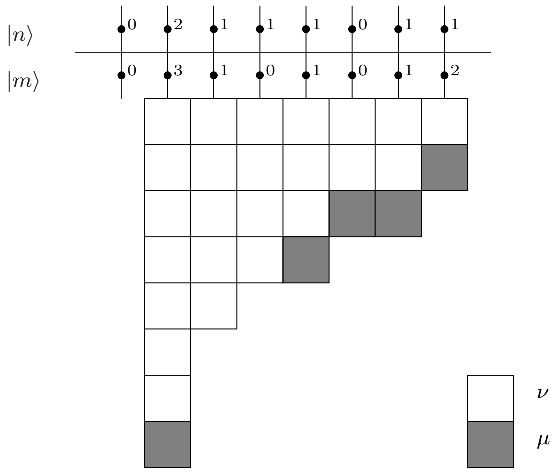

Figure 3.3: Admissible basis vectors and corresponding interlacing partitions. The two rows of numbers represent admissible basis vectors. The top row of occupation numbers gives rise to the white Young diagram, while the bottom row gives rise to the grey Young diagram. When these Young diagrams are stacked, the skew diagram is a horizontal strip.

3.2.5 Calculation of $\mathbb{B}(x)|n\rangle$

Lemma 2. Define $\mathbb{B}(x) = x^{\frac{M}{2}} B(x)$ and let $|n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ be an arbitrary basis vector of $\mathcal{V}$. The action of $\mathbb{B}(x)$ on $|n\rangle$ is given by

$$\mathbb{B}(x)|n\rangle = \sum_{|m\rangle \triangleright |n\rangle} \prod_{i=1}^{M} x^{i(m_i - n_i)} |m\rangle \tag{3.2.23}$$

where the sum is over all basis vectors $|m\rangle = |m_0\rangle_0 \otimes \cdots \otimes |m_M\rangle_M$ which are admissible to $|n\rangle$.

Proof. Let $\langle m| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M$ be an arbitrary basis vector of $\mathcal{V}^*$. We begin by writing the B-operator as a contraction on the auxiliary space $\mathcal{V}_a$, as follows

$$B(x) = \begin{pmatrix} 1 & 0 \end{pmatrix}_a \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}_a \begin{pmatrix} 0 \\ 1 \end{pmatrix}_a = \uparrow_a^* T_a(x) \downarrow_a \tag{3.2.24}$$

which leads to the equation

$$\langle m|B(x)|n\rangle = \uparrow_a^* \otimes \langle m|T_a(x)|n\rangle \otimes \downarrow_a = \uparrow_a^* \otimes \langle m|L_{aM}(x) \ldots L_{a0}(x)|n\rangle \otimes \downarrow_a \tag{3.2.25}$$

By commuting operators and vectors which reside in different spaces we find that

$$\langle m|B(x)|n\rangle = \uparrow^* L^{(M)}(x) \ldots L^{(0)}(x) \downarrow \tag{3.2.26}$$

where we have dropped the redundant subscripts a, and have defined the modified L-matrices

$$L^{(i)}(x) = \begin{pmatrix} \langle m_i|_i x^{-\frac{1}{2}} |n_i\rangle_i & \langle m_i|_i \phi_i^{\dagger} |n_i\rangle_i \\ \langle m_i|_i \phi_i |n_i\rangle_i & \langle m_i|_i x^{\frac{1}{2}} |n_i\rangle_i \end{pmatrix} \tag{3.2.27}$$

for all $0 \le i \le M$. Calculating the entries within these matrices explicitly, we obtain

$$L^{(i)}(x) = \begin{cases} \begin{pmatrix} x^{-\frac{1}{2}} & 0 \\ 0 & x^{\frac{1}{2}} \end{pmatrix} & m_i = n_i \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & m_i = n_i + 1 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & m_i + 1 = n_i \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise} \end{cases} \tag{3.2.28}$$

Using the expression (3.2.28) for $L^{(i)}(x)$, we find that

$$\langle m|B(x)|n\rangle = \uparrow^{*} L^{(M)}(x) \dots L^{(0)}(x) \downarrow = 0 \tag{3.2.29}$$

when $|m\rangle \ntriangleright |n\rangle$. In the case when $|m\rangle \triangleright |n\rangle$, let $\{p_1 < \dots < p_r\}$ be the set of all integers p such that $m_p = n_p + 1$. Similarly, let $\{q_1 < \dots < q_s\}$ be the set of all integers q such that $m_q + 1 = n_q$. The admissibility relation means that necessarily $s = r - 1$ and

$$p_i < q_i < p_{i+1}, \quad \text{for all } 1 \le i \le r-1 \tag{3.2.30}$$

By virtue of the ordering (3.2.30) and the expression (3.2.28) for $L^{(i)}(x)$, we obtain

$$\langle m|B(x)|n\rangle = \uparrow^{*} L^{(M)}(x) \dots L^{(0)}(x) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \uparrow = \tag{3.2.31}$$
$$\uparrow^{*} \overleftarrow{\prod_{i=1}^{r}} \left[\begin{pmatrix} x^{-\frac{1}{2}} & 0 \\ 0 & x^{\frac{1}{2}} \end{pmatrix}^{q_i - p_i - 1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^{-\frac{1}{2}} & 0 \\ 0 & x^{\frac{1}{2}} \end{pmatrix}^{p_i - q_{(i-1)} - 1} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right] \uparrow$$

where we have defined $q_0 = -1$ and $q_r = M + 1$. Calculating this matrix product explicitly, we find

$$\langle m|B(x)|n\rangle = x^{-\frac{M}{2}} \prod_{i=1}^{r-1} x^{p_i - q_i} x^{p_r} = x^{-\frac{M}{2}} \prod_{i=1}^{M} x^{i(m_i - n_i)} \tag{3.2.32}$$

Combining the equations (3.2.29) and (3.2.32) into a single case, we have

$$x^{\frac{M}{2}} \langle m|B(x)|n\rangle = \begin{cases} \displaystyle\prod_{i=1}^{M} x^{i(m_i - n_i)}, & |m\rangle \triangleright |n\rangle \\ 0, & \text{otherwise} \end{cases} \tag{3.2.33}$$

The result (3.2.23) follows from the orthonormality (3.1.19) of the basis vectors of $\mathcal{V}$, and from the definition of $\mathbb{B}(x)$. □

3.2.6 Calculation of $\langle n|\mathbb{C}(x)$

Lemma 3. Define $\mathbb{C}(x) = x^{\frac{M}{2}} C(1/x)$ and let $\langle n| = \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M$ be an arbitrary basis vector of $\mathcal{V}^*$. The action of $\mathbb{C}(x)$ on $\langle n|$ is given by

$$\langle n|\mathbb{C}(x) = \sum_{\langle n| \triangleleft \langle m|} \prod_{i=1}^{M} x^{i(m_i - n_i)} \langle m| \tag{3.2.34}$$

where the sum is over all basis vectors $\langle m| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M$ which are admissible to $\langle n|$.

Proof. A simple modification of the proof of lemma 2. □

3.2.7 Calculation of $\mathcal{M}_\psi \mathbb{B}(x)|n\rangle$ and $\langle n|\mathbb{C}(x)\mathcal{M}_\psi^*$

Let $|n\rangle$ and $\langle n|$ be arbitrary basis vectors of $\mathcal{V}$ and $\mathcal{V}^*$, respectively, and let $|\nu)$ and $(\nu|$ be their corresponding partitions, given by equations (3.2.2) and (3.2.3). Furthermore, define $l = \ell(\nu)$ to be the number of non-zero parts in the partition ν. Using the definition (3.2.4) of the maps $\mathcal{M}_\psi$ and $\mathcal{M}_\psi^*$, the expressions (3.2.23) and (3.2.34) and the result of lemma 1, we obtain

$$\mathcal{M}_\psi \mathbb{B}(x)|n\rangle = \sum_{\nu \prec \mu \subseteq [l+1,M]} x^{|\mu| - |\nu|} |\mu) \tag{3.2.35}$$

$$\langle n|\mathbb{C}(x)\mathcal{M}_\psi^* = \sum_{\nu \prec \mu \subseteq [l+1,M]} x^{|\mu| - |\nu|} (\mu| \tag{3.2.36}$$

Both sums are over all partitions μ which interlace with ν, and whose Young diagrams are contained in the rectangle $[l+1, M]$.

3.2.8 Skew Schur functions

For an arbitrary pair of partitions μ, ν and an indeterminate x, the single variable *skew Schur function* $s_{\mu/\nu}(x)$ is given by

$$s_{\mu/\nu}(x) = \begin{cases} x^{|\mu|-|\nu|}, & \mu \succ \nu \\ 0, & \text{otherwise} \end{cases} \tag{3.2.37}$$

In the case $\nu = \emptyset$ we have $s_{\mu/\nu}(x) = s_\mu(x)$, where $s_\mu(x)$ is the ordinary Schur function in a single variable x. The skew Schur function satisfies the identity

$$s_\mu\{x_1, \ldots, x_n\} = \sum_{\nu \subseteq [n-1,\infty]} s_{\mu/\nu}(x_n) s_\nu\{x_1, \ldots, x_{n-1}\} \tag{3.2.38}$$

where the sum is taken over all partitions ν whose lengths satisfy $\ell(\nu) \leq n-1$, and $s_\mu\{x_1, \ldots, x_n\}$ and $s_\nu\{x_1, \ldots, x_{n-1}\}$ are Schur functions in n and $n-1$ variables, respectively.[5]

3.2.9 Calculation of $\mathcal{M}_\psi \mathbb{B}(x_1) \ldots \mathbb{B}(x_N)|0\rangle$

The purpose of the previous subsection was to provide the equation (3.2.38), which we now use to calculate the phase model Bethe eigenvectors explicitly.

Lemma 4. Let $\{x_1, \ldots, x_N\}$ be a finite set of variables. We claim that

$$\mathcal{M}_\psi \mathbb{B}(x_1) \ldots \mathbb{B}(x_N)|0\rangle = \sum_{\mu \subseteq [N,M]} s_\mu\{x_1, \ldots, x_N\}|\mu) \tag{3.2.39}$$

where $s_\mu\{x_1, \ldots, x_N\}$ is the Schur function in N variables (1.1.67), and the sum is over all partitions μ whose Young diagrams are contained in the rectangle $[N, M]$. This result was originally obtained in [7].

Proof. We begin by specializing equation (3.2.35) to the case $|n\rangle = |0\rangle$, to obtain

$$\mathcal{M}_\psi \mathbb{B}(x)|0\rangle = \sum_{\emptyset \prec \mu \subseteq [1,M]} s_{\mu/\emptyset}(x)|\mu) = \sum_{\mu \subseteq [1,M]} s_\mu(x)|\mu) \tag{3.2.40}$$

where we have used the equation (3.2.37) for the skew Schur function, and the definition $\ell(\emptyset) = 0$. We use equation (3.2.40) as the basis for induction, and assume that

[5] For more information on skew Schur functions, the reader is referred to section 5 of chapter I in [65].

$$\mathcal{M}_\psi \mathbb{B}(x_1)\dots\mathbb{B}(x_{N-1})|0\rangle = \sum_{\nu\subseteq[N-1,M]} s_\nu\{x_1,\dots,x_{N-1}\}|\nu) \tag{3.2.41}$$

for some $N \geq 2$. In terms of the basis vectors of $\mathcal{V}$, this assumption is written as

$$\mathbb{B}(x_1)\dots\mathbb{B}(x_{N-1})|0\rangle = \sum_{|n\rangle|\Sigma_0=N-1} s_\nu\{x_1,\dots,x_{N-1}\}|n\rangle \tag{3.2.42}$$

where the sum is over all basis vectors $|n\rangle$ whose occupation numbers satisfy the condition $\sum_{i=0}^{M} n_i = N-1$, and ν is the partition corresponding to each $|n\rangle$. Acting on (3.2.42) with the composition of operators $\mathcal{M}_\psi \circ \mathbb{B}(x_N)$ and using the fact that the B-operators commute (3.1.26), we obtain

$$\mathcal{M}_\psi \mathbb{B}(x_1)\dots\mathbb{B}(x_N)|0\rangle = \sum_{\nu\subseteq[N-1,M]} s_\nu\{x_1,\dots,x_{N-1}\} \sum_{\nu\prec\mu\subseteq[N,M]} s_{\mu/\nu}(x_N)|\mu) \tag{3.2.43}$$

Since $s_{\mu/\nu}(x_N) = 0$ if $\mu \not\succ \nu$, we may alter the sums appearing in (3.2.43), yielding

$$\begin{aligned}\mathcal{M}_\psi \mathbb{B}(x_1)\dots\mathbb{B}(x_N)|0\rangle &= \sum_{\mu\subseteq[N,M]} \sum_{\nu\subseteq[N-1,M]} s_{\mu/\nu}(x_N) s_\nu\{x_1,\dots,x_{N-1}\}|\mu) \\ &= \sum_{\mu\subseteq[N,M]} \sum_{\nu\subseteq[N-1,\infty]} s_{\mu/\nu}(x_N) s_\nu\{x_1,\dots,x_{N-1}\}|\mu)\end{aligned} \tag{3.2.44}$$

where the final equality holds since every part of μ is less than or equal to M, and therefore $s_{\mu/\nu}(x_N) = 0$ if any part of ν is greater than M. Using the identity (3.2.38) we evaluate the sum over ν explicitly, producing the equation (3.2.39). Therefore by induction the result (3.2.39) must hold for arbitrary $N \geq 1$. □

3.2.10 Calculation of $\langle 0|\mathbb{C}(x_N)\dots\mathbb{C}(x_1)\mathcal{M}_\psi^*$

By following essentially the same steps that were used in the previous subsection, we can also derive the expression

$$\langle 0|\mathbb{C}(x_N)\dots\mathbb{C}(x_1)\mathcal{M}_\psi^* = \sum_{\mu\subseteq[N,M]} s_\mu\{x_1,\dots,x_N\}(\mu| \tag{3.2.45}$$

for the dual Bethe eigenvectors. As before, this sum is taken over all partitions μ whose Young diagrams are contained in the rectangle $[N,M]$.

3.2.11 Charged fermionic expression for Bethe eigenvectors

The first goal of this subsection is to show that $\mathcal{M}_\psi$ maps the phase model Bethe eigenvectors $\mathbb{B}(x_1)\ldots\mathbb{B}(x_N)|0\rangle$ to vectors $g_\psi|0\rangle \in \mathcal{F}_\psi^{(0)}$ which satisfy the charged fermion bilinear identity (1.2.15). In order to do this, we make some definitions. For all integers $i \geq -N$ and $1 \leq j \leq N$ we define

$$c_{i,j}\{x\} = \begin{cases} h_{i+j}\{x\}, & -N \leq i < M \\ 0, & i \geq M \end{cases} \tag{3.2.46}$$

where $h_{i+j}\{x\}$ is a complete symmetric function (1.1.66) in the finite set of variables $\{x\} = \{x_1,\ldots,x_N\}$. Using this expression for the functions $c_{i,j}\{x\}$, for all ordered sets of integers $\{m\} = \{m_1 > \cdots > m_N \geq -N\}$ let us also define the coefficients

$$c_{\{m\}}\{x\} = \det\Big(c_{m_i,j}\{x\}\Big)_{1\leq i,j\leq N} \tag{3.2.47}$$

Since $c_{i,j}\{x\} = 0$ if $i \geq M$, the coefficient $c_{\{m\}}\{x\}$ vanishes if $m_1 \geq M$. Now suppose that $\mu = \{\mu_1,\ldots,\mu_N\}$ is the partition formed by setting $\mu_i = m_i + i$ for all $1 \leq i \leq N$. By the definition of the Schur functions (1.1.67), it follows that

$$c_{\{m\}}\{x\} = \begin{cases} s_\mu\{x\}, & \mu \subseteq [N,M] \\ 0, & \mu \not\subseteq [N,M] \end{cases} \tag{3.2.48}$$

Returning to the expression (3.2.39) for the Bethe eigenvectors, we can use the coefficients (3.2.48) to write

$$\mathcal{M}_\psi \mathbb{B}(x_1)\ldots\mathbb{B}(x_N)|0\rangle = \sum_{\mathrm{card}\{m\}=N} c_{\{m\}}\{x\}\psi_{m_1}\ldots\psi_{m_N}|-N\rangle \tag{3.2.49}$$

where the sum is over all sets of integers $\{m\} = \{m_1 > \cdots > m_N \geq -N\}$, and we have made the identification $|\mu) = \psi_{m_1}\ldots\psi_{m_N}|-N\rangle$. Thanks to their determinant form (3.2.47) and lemma 12 of chapter 1, we see that the coefficients $c_{\{m\}}\{x\}$ satisfy the KP Plücker relations (1.3.40). This implies that the right hand side of (3.2.49) satisfies the charged fermion bilinear identity (1.2.15), as we intended to show.

The second goal of this subsection is to express $\mathcal{M}_\psi\mathbb{B}(x_1)\ldots\mathbb{B}(x_N)|0\rangle$ in the orbit of the Fock vacuum under GL_∞. We know that this is possible using theorem 2 of chapter 1, and the fact that the right hand side of (3.2.49) satisfies the CFBI. We begin by writing the formula (3.2.39) for $\mathcal{M}_\psi\mathbb{B}(x_1)\ldots\mathbb{B}(x_N)|0\rangle$ in terms of the canonical $\mathcal{F}_\psi^{(0)}$ basis (1.1.9), yielding

$$\mathcal{M}_\psi \mathbb{B}(x_1)\dots\mathbb{B}(x_N)|0\rangle = |0\rangle + \sum_{\substack{1\leq m\leq M\\ 1\leq n\leq N}} (-)^{n-1} s_{\{m,1^{n-1}\}}\{x\}\psi_{m-1}\psi^*_{-n}|0\rangle + g^{(1)}_\psi|0\rangle \tag{3.2.50}$$

where $s_{\{m,1^{n-1}\}}\{x\}$ is the Schur function associated to the partition with one part of size m and $n-1$ parts of size 1, and we assume that all monomials within $g^{(1)}_\psi \in Cl^{(0)}_\psi$ consist of at least two $(+1)$ and two (-1) fermions. Because the right hand side of (3.2.50) obeys the CFBI, we can use the method adopted in the proof of lemma 10 in chapter 1 to obtain

$$\mathcal{M}_\psi \mathbb{B}(x_1)\dots\mathbb{B}(x_N)|0\rangle = \exp\left(\sum_{\substack{1\leq m\leq M\\ 1\leq n\leq N}} (-)^{n-1} s_{\{m,1^{n-1}\}}\{x\}\psi_{m-1}\psi^*_{-n}\right)|0\rangle \tag{3.2.51}$$

This result explicitly places $\mathcal{M}_\psi \mathbb{B}(x_1)\dots\mathbb{B}(x_N)|0\rangle$ in the vacuum orbit of GL_∞.

Finally, let us remark that all of these results can be extended to the dual Bethe eigenvectors $\langle 0|\mathbb{C}(x_N)\dots\mathbb{C}(x_1)\mathcal{M}^*_\psi$. For example, by completely analogous reasoning it is possible to show that

$$\langle 0|\mathbb{C}(x_N)\dots\mathbb{C}(x_1)\mathcal{M}^*_\psi = \langle 0|\exp\left(\sum_{\substack{1\leq m\leq M\\ 1\leq n\leq N}} (-)^{n-1} s_{\{m,1^{n-1}\}}\{x\}\psi_{-n}\psi^*_{m-1}\right) \tag{3.2.52}$$

3.3 Scalar product, boxed plane partitions

3.3.1 Plane partitions

Definition 4. A *plane partition* π is a set of non-negative integers $\pi(i,j)$ which satisfy

$$\pi(i,j) \geq \pi(i+1,j), \quad \pi(i,j)\geq \pi(i,j+1) \tag{3.3.1}$$

for all integers $i,j\geq 1$, as well as the finiteness condition

$$\lim_{i\to\infty}\pi(i,j) = \lim_{j\to\infty}\pi(i,j) = 0 \tag{3.3.2}$$

An *M-boxed plane partition* is a set of non-negative integers $\pi(i,j)$ satisfying the above properties, as well as the supplementary condition

$$0 \leq \pi(i,j) \leq M \tag{3.3.3}$$

for all integers $i, j \geq 1$, where $M \geq 1$ is some fixed positive integer.

Plane partitions are two-dimensional analogues of ordinary partitions. They are pictorially represented in one of two ways. The first way uses the notion of a *tableau*, whereby the non-negative integer $\pi(i,j)$ is placed in the coordinate-labelled box (i,j) for all $i, j \geq 1$.

4	2	1	1	1
3	2	1	1	
2	1	1		
1				

Figure 3.4: Tableau representation of a plane partition. An integer $\pi(i,j)$ is assigned to each box (i,j). The top row of the tableau corresponds to the boxes $(1,n)$, where $1 \leq n \leq 5$, while the left-most column corresponds to the boxes $(m,1)$, where $1 \leq m \leq 4$.

The second way is by stacking a column of cubes of height $\pi(i,j)$ over the coordinate-labelled square (i,j) for all $i, j \geq 1$. When viewed in its three-dimensional representation, an M-boxed plane partition has columns of cubes which are maximally of height M.

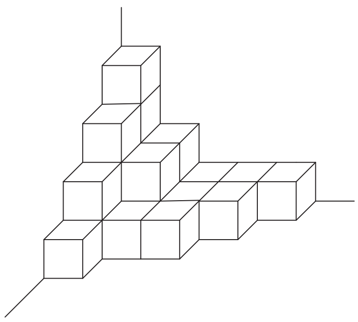

Figure 3.5: Three-dimensional representation of a plane partition. Columns of cubes of height $\pi(i,j)$ are stacked above each square (i,j). This plane partition is 4-boxed.

3.3.2 Diagonal slices of plane partitions

Definition 5. Let π be an arbitrary plane partition. For $i \geq 0$ define the partitions $|\pi_i) \in \mathcal{F}_{\psi}^{(0)}$ whose parts are given by

$$(\pi_i)_j = \pi(j, i+j) \tag{3.3.4}$$

for all $j \geq 1$. Similarly for $i \leq 0$ define the partitions $(\pi_i| \in \mathcal{F}_\psi^{*(0)}$ whose parts are given by

$$(\pi_i)_j = \pi(-i+j, j) \tag{3.3.5}$$

for all $j \geq 1$. The partitions $|\pi_i)$ and $(\pi_i|$ are called the *diagonal slices* of the plane partition π.

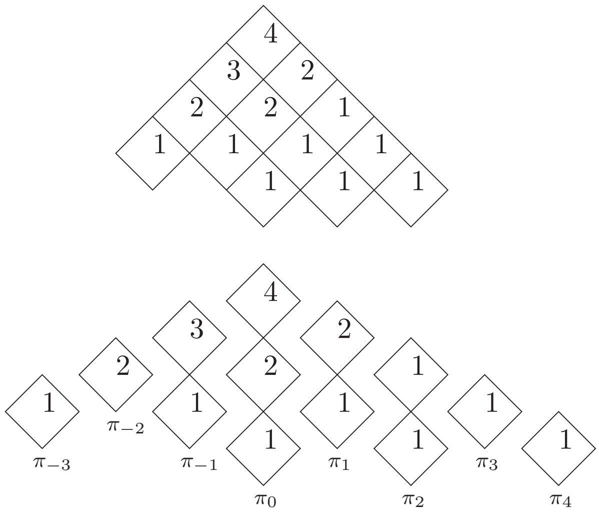

Figure 3.6: Diagonal slices of a plane partition. Each column of boxes represents a regular partition. For example, the column labelled π_0 represents the partition $\{4, 2, 1\}$.

Lemma 5. Let $|\pi_i)$ and $(\pi_i|$ be the diagonal slices of the arbitrary plane partition π. Then we have

$$(\pi_{i-1}| \prec (\pi_i| \text{ for all } i \leq 0, \qquad |\pi_i) \succ |\pi_{i+1}) \text{ for all } i \geq 0 \tag{3.3.6}$$

Proof. This observation is due to Okounkov and Reshetikhin in [74], where it was used to define and study the Schur stochastic process. The proof is immediate from the definition (3.2.13) of interlacing partitions and the defining property (3.3.1) of plane partitions. Since π is a plane partition we have

$$\begin{aligned} \pi(-i+j,j) \geq \pi(-i+1+j,j) \geq \pi(-i+j+1,j+1) \ \text{ for all } i \leq 0,\ j \geq 1 \\ \implies (\pi_i)_j \geq (\pi_{i-1})_j \geq (\pi_i)_{j+1} \ \text{ for all } i \leq 0,\ j \geq 1 \end{aligned} \tag{3.3.7}$$

proving that $(\pi_{i-1}| \prec (\pi_i|$ for all $i \leq 0$. Similarly we find

$$\begin{aligned} \pi(j, i+j) \geq \pi(j, i+1+j) \geq \pi(j+1, i+j+1) \quad \text{for all } i \geq 0,\ j \geq 1 \\ \implies (\pi_i)_j \geq (\pi_{i+1})_j \geq (\pi_i)_{j+1} \quad \text{for all } i \geq 0,\ j \geq 1 \end{aligned} \tag{3.3.8}$$

proving that $|\pi_i) \succ |\pi_{i+1})$ for all $i \geq 0$.

□

3.3.3 Generating M-boxed plane partitions

In this subsection we reproduce the result of [7], where it was shown that the scalar product of the phase model on $M+1$ sites is a generating function for M-boxed plane partitions. This correspondence may be realized by iterating the $|n\rangle = |0\rangle$ case of equation (3.2.35) N times, giving

$$\mathcal{M}_\psi \mathbb{B}(x_1) \dots \mathbb{B}(x_N)|0\rangle = \sum_{[N,M] \supseteq \pi_0 \succ \dots \succ \pi_N = \emptyset} \prod_{i=1}^{N} x_i^{|\pi_{i-1}| - |\pi_i|} |\pi_0) \tag{3.3.9}$$

where the sum is over all interlacing partitions $\{\pi_0 \succ \dots \succ \pi_N\}$ which are subject to $\pi_0 \subseteq [N, M]$ and $\pi_N = \emptyset$. Similarly, one can iterate the $\langle n| = \langle 0|$ case of (3.2.36) N times, giving

$$\langle 0|\mathbb{C}(x_N) \dots \mathbb{C}(x_1) \mathcal{M}_\psi^* = \sum_{\emptyset = \pi_{-N} \prec \dots \prec \pi_0 \subseteq [N,M]} \prod_{i=1}^{N} x_i^{|\pi_{-i+1}| - |\pi_{-i}|} (\pi_0| \tag{3.3.10}$$

where the sum is over all interlacing partitions $\{\pi_{-N} \prec \dots \prec \pi_0\}$ which are subject to $\pi_0 \subseteq [N, M]$ and $\pi_{-N} = \emptyset$. Due to the isometry of the maps (3.2.4) and the orthonormality (1.1.33) of partition states, we thus obtain

$$\langle 0|\mathbb{C}(x_N) \dots \mathbb{C}(x_1) \mathbb{B}(y_1) \dots \mathbb{B}(y_N)|0\rangle = \sum_{\pi \subseteq [N,N,M]} A_\pi\Big(\{x\}, \{y\}\Big) \tag{3.3.11}$$

where the sum is over all plane partitions π which fit inside the box of dimension $N \times N \times M$, and where we have defined the weighting factor

$$A_\pi\Big(\{x\}, \{y\}\Big) = \prod_{i=1}^{N} x_i^{|\pi_{-i+1}| - |\pi_{-i}|} y_i^{|\pi_{i-1}| - |\pi_i|} \tag{3.3.12}$$

which depends on the diagonal slices of π. From equation (3.3.11), we see that the scalar product is a generating function of M-boxed plane partitions. A closed form expression for this generating function can be obtained by using the formulae (3.2.39) and (3.2.45) for the Bethe eigenvectors to show that

$$\langle 0|\mathbb{C}(x_N)\ldots\mathbb{C}(x_1)\mathbb{B}(y_1)\ldots\mathbb{B}(y_N)|0\rangle = \sum_{\mu\subseteq[N,M]} s_\mu\{x\}s_\mu\{y\} \tag{3.3.13}$$

$$= \frac{\displaystyle\sum_{M-1\geq m_1>\cdots>m_N\geq -N} \det\left(x_i^{m_j+N}\right)_{1\leq i,j\leq N} \det\left(y_j^{m_i+N}\right)_{1\leq i,j\leq N}}{\displaystyle\prod_{1\leq i<j\leq N}(x_i-x_j)(y_i-y_j)}$$

where we have used the Jacobi-Trudi identity for Schur functions[6]

$$s_\mu\{x\} = \det\Big(h_{\mu_i-i+j}\{x\}\Big)_{1\leq i,j\leq N} = \frac{\det\left(x_i^{\mu_j-j+N}\right)_{1\leq i,j\leq N}}{\displaystyle\prod_{1\leq i<j\leq N}(x_i-x_j)} \tag{3.3.14}$$

in conjunction with the definition $m_i = \mu_i - i$ for all $1 \leq i \leq N$. Using the Cauchy-Binet identity[7] to convert the sum in the numerator of (3.3.13) into a single determinant, we obtain

$$\langle 0|\mathbb{C}(x_N)\ldots\mathbb{C}(x_1)\mathbb{B}(y_1)\ldots\mathbb{B}(y_N)|0\rangle = \frac{\det\left(\displaystyle\sum_{m=0}^{M+N-1}(x_iy_j)^m\right)_{1\leq i,j\leq N}}{\displaystyle\prod_{1\leq i<j\leq N}(x_i-x_j)(y_i-y_j)} \tag{3.3.15}$$

$$= \frac{\det\Big(\big(1-(x_iy_j)^{M+N}\big)\Big/(1-x_iy_j)\Big)_{1\leq i,j\leq N}}{\displaystyle\prod_{1\leq i<j\leq N}(x_i-x_j)(y_i-y_j)}$$

Equating the right hand sides of (3.3.11) and (3.3.15), we have proved that

$$\sum_{\pi\subseteq[N,N,M]} A_\pi\Big(\{x\},\{y\}\Big) = \frac{\det\Big(\big(1-(x_iy_j)^{M+N}\big)\Big/(1-x_iy_j)\Big)_{1\leq i,j\leq N}}{\displaystyle\prod_{1\leq i<j\leq N}(x_i-x_j)(y_i-y_j)} \tag{3.3.16}$$

which matches the evaluation of this generating function in [7].

[6] See section 3 of chapter I in [65].

[7] See subsection 5.2.5 of the thesis.

3.3.4 Scalar product as a power-sum specialized KP τ-function

We now demonstrate that the phase model scalar product is a specialization of a KP τ-function. The specialization is achieved by setting the KP time variables to power sums in the phase model rapidities. Our starting point is the equation

$$\langle 0| \exp\left(\sum_{m=1}^{\infty} t_m H_m\right) = \sum_{\mu} \chi_\mu\{t\}(\mu| \tag{3.3.17}$$

whose sum is over all partitions μ, which follows from lemma 4 in chapter 1 and the orthonormality (1.1.33) of partitions. Defining $t_m = \frac{1}{m}\sum_{n=1}^{N} x_n^m$ for all $m \geq 1$, equation (3.3.17) becomes

$$\langle 0| \exp\left(\sum_{m=1}^{\infty}\sum_{n=1}^{N} \frac{1}{m} x_n^m H_m\right) = \sum_{\mu\subseteq[N,\infty]} s_\mu\{x\}(\mu| \tag{3.3.18}$$

where the sum is over all partitions μ with maximal length N. Equating the right hand sides of (3.2.39) and (3.2.51) and using the identity (3.3.18), we find

$$\langle 0| \exp\left(\sum_{m=1}^{\infty}\sum_{n=1}^{N} \frac{1}{m} x_n^m H_m\right) \exp X\{y\}|0\rangle = \sum_{\mu\subseteq[N,M]} s_\mu\{x\} s_\mu\{y\} \tag{3.3.19}$$

where $X\{y\} \in A_\infty$ is defined as

$$X\{y\} = \sum_{\substack{1\leq m\leq M\\ 1\leq n\leq N}} (-)^{n-1} s_{\{m,1^{n-1}\}}\{y\} \psi_{m-1}\psi^*_{-n} \tag{3.3.20}$$

Now consider the polynomial KP τ-function $\tau\{t\} = \langle e^{H\{t\}} e^{X\{y\}}\rangle$. Comparing the first line of (3.3.13) with equation (3.3.19), we conclude that

$$\tau\{t\} = \langle 0| \exp\left(\sum_{m=1}^{\infty} t_m H_m\right) \exp X\{y\}|0\rangle = \langle 0|\mathbb{C}(x_N)\dots\mathbb{C}(x_1)\mathbb{B}(y_1)\dots\mathbb{B}(y_N)|0\rangle \tag{3.3.21}$$

under the power-sum specialization $t_m = \frac{1}{m}\sum_{n=1}^{N} x_n^m$ for all $m \geq 1$. This connection between plane partition generating functions and KP τ-functions was suggested in [39], albeit in the context of plane partitions whose column heights are unrestricted. The result of this subsection is at the level of M-boxed plane partitions, and it specializes to the result of [39] in the limit $M \to \infty$.

3.4 Phase model on an infinite lattice

In this section we study the action of the monodromy matrix operators $\mathbb{B}(x)$ and $\mathbb{C}(x)$ when the number of lattice sites becomes infinite. Our main result is lemma 6, showing that in the limit $M \to \infty$ the operators $\mathbb{B}(x)$ and $\mathbb{C}(x)$ acquire equivalent actions to the half-vertex operators $\Gamma_-(x)$ and $\Gamma_+(x)$ from KP theory. This result rests basically on the works [39], [67], [74], [75] which studied the actions of these half-vertex operators.

3.4.1 Calculation of $\mathcal{M}_\psi \mathbb{B}(x)|n\rangle$ and $\langle n|\mathbb{C}(x)\mathcal{M}_\psi^*$ as $M \to \infty$

Lemma 6. Consider the infinite lattice limit of the phase model, which is obtained by taking $M \to \infty$. Let $|n\rangle = \otimes_{j=0}^{\infty} |n_j\rangle_j$ and $\langle n| = \otimes_{j=0}^{\infty} \langle n_j|_j$ be basis vectors of $\mathcal{V}$ and $\mathcal{V}^*$, respectively, in this limit.[8] In addition, let $|\nu)$ and $(\nu|$ be the image states of these basis vectors under the mappings (3.2.4). We claim that

$$\mathcal{M}_\psi \Big[\lim_{M\to\infty} \mathbb{B}(x)|n\rangle \Big] = \Gamma_-(x)|\nu), \quad \Big[\lim_{M\to\infty} \langle n|\mathbb{C}(x) \Big] \mathcal{M}_\psi^* = (\nu|\Gamma_+(x) \tag{3.4.1}$$

where we have defined the KP half-vertex operators[9]

$$\Gamma_-(x) = \exp\left(\sum_{n=1}^{\infty} \frac{x^n}{n} H_{-n} \right), \quad \Gamma_+(x) = \exp\left(\sum_{n=1}^{\infty} \frac{x^n}{n} H_n \right) \tag{3.4.2}$$

and H_{-n}, H_n denote the Heisenberg generators (1.1.39).

Proof. We split the proof into two steps. In the first step, we show that (3.4.1) is equivalent to the statement (3.4.6). In the second step we prove (3.4.6) using the calculus of charged free fermions.

Step 1. Taking the $M \to \infty$ limit of equations (3.2.35) and (3.2.36), we obtain

$$\mathcal{M}_\psi \Big[\lim_{M\to\infty} \mathbb{B}(x)|n\rangle \Big] = \sum_{\mu \succ \nu} x^{|\mu|-|\nu|} |\mu) \tag{3.4.3}$$

$$\Big[\lim_{M\to\infty} \langle n|\mathbb{C}(x) \Big] \mathcal{M}_\psi^* = \sum_{\mu \succ \nu} x^{|\mu|-|\nu|} (\mu| \tag{3.4.4}$$

where the sums are over all partitions μ which interlace with ν, whose parts now have no size restriction. The equations (3.4.1) are therefore equivalent to the statements

[8] When considering such basis vectors, we always assume there exists some integer I such that $n_i = 0$ for all $i > I$. This is necessary to ensure that the vectors contain a finite amount of non-trivial information.

[9] We use the terminology *half-vertex operator* in reference to the fact that $\Gamma_-(x), \Gamma_+(x)$ each constitute one half of a charged fermion vertex operator, [50].

$$\Gamma_-(x)|\nu) = \sum_{\mu\succ\nu} x^{|\mu|-|\nu|}|\mu), \quad (\nu|\Gamma_+(x) = \sum_{\mu\succ\nu} x^{|\mu|-|\nu|}(\mu| \tag{3.4.5}$$

which are entirely at the level of charged free fermions. Due to the orthonormality (1.1.33) of partition states, equations (3.4.5) may be presented in the alternative form

$$(\mu|\Gamma_-(x) = \sum_{\nu\prec\mu} x^{|\mu|-|\nu|}(\nu|, \quad \Gamma_+(x)|\mu) = \sum_{\nu\prec\mu} x^{|\mu|-|\nu|}|\nu) \tag{3.4.6}$$

where the sums are now over partitions ν such that $\nu \prec \mu$. We will find it convenient to prove (3.4.6), as opposed to (3.4.5). The point is that the sums in the former are finite, whereas the sums in the latter are infinite and inherently more difficult to handle. When we succeed in showing (3.4.6), we will have achieved the proof of (3.4.1).

Step 2. Consider the length l partitions

$$(\mu| = \langle -l|\psi^*_{m_l}\dots\psi^*_{m_1}, \quad |\mu) = \psi_{m_1}\dots\psi_{m_l}|-l\rangle \tag{3.4.7}$$

where $\{m_1 > \cdots > m_l > -l\}$, and the elements of the partitions are given by $\mu_i = m_i + i$ for all $1 \le i \le l$. In order to prove (3.4.6), we must calculate $(\mu|\Gamma_-(x)$ and $\Gamma_+(x)|\mu)$. To progress in this direction, we require the commutation relations

$$\Gamma_-(x)\left(\sum_{n=0}^{\infty}\psi^*_{(i-n)}x^n\right) = \psi^*_i\Gamma_-(x), \quad \Gamma_+(x)\psi_i = \left(\sum_{n=0}^{\infty}\psi_{(i-n)}x^n\right)\Gamma_+(x) \tag{3.4.8}$$

which are derived following the arguments presented in subsection 1.1.8.[10] Applying the relations (3.4.8) repeatedly to the partitions (3.4.7), we find

$$(\mu|\Gamma_-(x) = \langle -l|\left(\sum_{i_l=0}^{\infty}\psi^*_{(m_l-i_l)}x^{i_l}\right)\dots\left(\sum_{i_1=0}^{\infty}\psi^*_{(m_1-i_1)}x^{i_1}\right) \tag{3.4.9}$$

$$\Gamma_+(x)|\mu) = \left(\sum_{i_1=0}^{\infty}\psi_{(m_1-i_1)}x^{i_1}\right)\dots\left(\sum_{i_l=0}^{\infty}\psi_{(m_l-i_l)}x^{i_l}\right)|-l\rangle \tag{3.4.10}$$

where we have used the fact that $\langle -l|\Gamma_-(x) = \langle -l|$ and $\Gamma_+(x)|-l\rangle = |-l\rangle$. Now for arbitrary integers $m > n$ we have the identities

[10] Setting $t_n = x^n/n$ for all $n \ge 1$ in (1.1.46), we obtain $\Gamma_+(x)\Psi(k) = \frac{1}{1-xk}\Psi(k)\Gamma_+(x)$. Extracting the coefficients of k^i from this equation, we prove the second commutation relation in (3.4.8). The first commutation relation may be proved similarly.

$$\left(\sum_{i=0}^{\infty}\psi^*_{(n-i)}x^i\right)\left(\sum_{j=0}^{\infty}\psi^*_{(m-j)}x^j\right)=\left(\sum_{i=0}^{\infty}\psi^*_{(n-i)}x^i\right)\left(\sum_{j=0}^{m-n-1}\psi^*_{(m-j)}x^j\right) \quad (3.4.11)$$

$$\left(\sum_{i=0}^{\infty}\psi_{(m-i)}x^i\right)\left(\sum_{j=0}^{\infty}\psi_{(n-j)}x^j\right)=\left(\sum_{i=0}^{m-n-1}\psi_{(m-i)}x^i\right)\left(\sum_{j=0}^{\infty}\psi_{(n-j)}x^j\right) \quad (3.4.12)$$

which, when substituted into (3.4.9) and (3.4.10), lead to the truncated equations

$$(\mu|\Gamma_-(x)=\langle -l|\left(\sum_{i_l=0}^{m_l-m_{\bar{l}}-1}\psi^*_{(m_l-i_l)}x^{i_l}\right)\dots\left(\sum_{i_1=0}^{m_1-m_2-1}\psi^*_{(m_1-i_1)}x^{i_1}\right) \quad (3.4.13)$$

$$\Gamma_+(x)|\mu)=\left(\sum_{i_1=0}^{m_1-m_2-1}\psi_{(m_1-i_1)}x^{i_1}\right)\dots\left(\sum_{i_l=0}^{m_l-m_{\bar{l}}-1}\psi_{(m_l-i_l)}x^{i_l}\right)|-l\rangle \quad (3.4.14)$$

where we have defined $\bar{l}=l+1$ and $m_{\bar{l}}=-\bar{l}$. The indices in the sums (3.4.13) and (3.4.14) can then be modified to produce the equations

$$(\mu|\Gamma_-(x)=\langle -l|\left(\sum_{m_l\ge n_l>m_{\bar{l}}}\psi^*_{n_l}x^{m_l-n_l}\right)\dots\left(\sum_{m_1\ge n_1>m_2}\psi^*_{n_1}x^{m_1-n_1}\right)=\sum_{\nu\prec\mu}x^{|\mu|-|\nu|}(\nu| \quad (3.4.15)$$

$$\Gamma_+(x)|\mu)=\left(\sum_{m_1\ge n_1>m_2}\psi_{n_1}x^{m_1-n_1}\right)\dots\left(\sum_{m_l\ge n_l>m_{\bar{l}}}\psi_{n_l}x^{m_l-n_l}\right)|-l\rangle=\sum_{\nu\prec\mu}x^{|\mu|-|\nu|}|\nu) \quad (3.4.16)$$

where we have defined partitions $(\nu|=\langle -l|\psi^*_{n_l}\dots\psi^*_{n_1}$ and $|\nu)=\psi_{n_1}\dots\psi_{n_l}|-l\rangle$, which correspond to the ordered set $\{n_1>\dots>n_l\ge -l\}$ via $\nu_i=n_i+i$ for all $1\le i\le l$. Notice that the final equalities of (3.4.15) and (3.4.16) follow from the relationship

$$m_i\ge n_i>m_{i+1}\ \text{ for all } 1\le i\le l \implies \mu_i\ge\nu_i\ge\mu_{i+1}\ \text{ for all } 1\le i\le l \quad (3.4.17)$$

as well as the fact $|\mu|-|\nu|=\sum_{i=1}^{l}(m_i-n_i)$. The equations (3.4.15) and (3.4.16) complete the proof of (3.4.6).

□

3.4.2 Generating plane partitions of arbitrary size

In the previous section we demonstrated that the phase model scalar product on a lattice of size $M+1$ generates M-boxed plane partitions. Accordingly, we expect that in the limit $M \rightarrow \infty$ the scalar product will generate plane partitions whose column heights are arbitrarily large, giving rise to the equation

$$\lim_{M\rightarrow\infty} \langle 0|\mathbb{C}(x_N)\dots\mathbb{C}(x_1)\mathbb{B}(y_1)\dots\mathbb{B}(y_N)|0\rangle = \sum_{\pi\subseteq[N,N,\infty]} A_{\pi}\Big(\{x\},\{y\}\Big) \qquad (3.4.18)$$

where the sum is over all plane partitions π which fit inside the box of dimension $N \times N \times \infty$, with $A_{\pi}(\{x\},\{y\})$ given by (3.3.12). On the other hand, using the result of lemma 6 we are able to write

$$\lim_{M\rightarrow\infty} \langle 0|\mathbb{C}(x_N)\dots\mathbb{C}(x_1)\mathbb{B}(y_1)\dots\mathbb{B}(y_N)|0\rangle = (\emptyset|\Gamma_+(x_N)\dots\Gamma_+(x_1)\Gamma_-(y_1)\dots\Gamma_-(y_N)|\emptyset) \qquad (3.4.19)$$

which lends itself to immediate evaluation, owing to simple commutation relations between the KP half-vertex operators. Explicitly speaking, using the definition (3.4.2) of $\Gamma_-(y)$ and $\Gamma_+(x)$ we find that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{x^m y^n}{mn}[H_m, H_{-n}] = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{x^m y^n}{mn} m\delta_{m,n} = \sum_{m=1}^{\infty} \frac{(xy)^m}{m} \qquad (3.4.20)$$

$$\Longrightarrow \Gamma_+(x)\Gamma_-(y) = \exp\left(\sum_{m=1}^{\infty}\frac{(xy)^m}{m}\right)\Gamma_-(y)\Gamma_+(x) = \frac{1}{1-xy}\Gamma_-(y)\Gamma_+(x)$$

Employing the commutation relation (3.4.20) repeatedly in (3.4.19) and using the fact that $\Gamma_+(x)|\emptyset) = |\emptyset)$, $(\emptyset|\Gamma_-(y) = (\emptyset|$, we obtain

$$\lim_{M\rightarrow\infty} \langle 0|\mathbb{C}(x_N)\dots\mathbb{C}(x_1)\mathbb{B}(y_1)\dots\mathbb{B}(y_N)|0\rangle = \prod_{i,j=1}^{N} \frac{1}{1-x_i y_j} \qquad (3.4.21)$$

Comparing the equations (3.4.18) and (3.4.21), we have proved that

$$\sum_{\pi\subseteq[N,N,\infty]} A_{\pi}\Big(\{x\},\{y\}\Big) = \prod_{i,j=1}^{N} \frac{1}{1-x_i y_j} \qquad (3.4.22)$$

which is a much simpler evaluation of this generating function than in the finite case (3.3.16). Let us remark that this calculation could also have been performed using the first line of (3.3.13) and the identity

$$\sum_{\mu\subseteq[N,\infty]} s_{\mu}\{x\} s_{\mu}\{y\} = \prod_{i,j=1}^{N} \frac{1}{1-x_i y_j} \qquad (3.4.23)$$

from section 4, chapter I of [65]. The proof which we have given is independent of the properties of symmetric functions. As a final observation, let us specialize the variables $\{x_1, \ldots, x_N\}$ and $\{y_1, \ldots, y_N\}$ to

$$x_i = y_i = z^{i-\frac{1}{2}} \quad \text{for all } 1 \leq i \leq N \tag{3.4.24}$$

giving rise to the equation

$$\sum_{\pi \subseteq [N,N,\infty]} z^{|\pi|} = \prod_{i,j=1}^{N} \frac{1}{1 - z^{i+j-1}} \tag{3.4.25}$$

where $|\pi|$ is the *weight* of the plane partition π, defined equal to the sum of all its entries $\pi(i,j)$. Taking the limit $N \to \infty$ we obtain

$$\sum_{\pi} z^{|\pi|} = \prod_{i=1}^{\infty} \frac{1}{(1 - z^i)^i} \tag{3.4.26}$$

where the sum is now over plane partitions π of completely arbitrary dimension. Equation (3.4.26) is a famous generating function for plane partitions, originally found by MacMahon [66]. The idea of deriving this generating function using charged fermions is due to [74], [75] and was explained in detail in [39].

3.5 i-boson model

3.5.1 Space of states $\tilde{\mathcal{V}}$ and inner product $\tilde{\mathcal{I}}$

It is necessary to introduce a space of states $\tilde{\mathcal{V}}$ which is a subspace of $\mathcal{V}$, defined in section 3.1. In brief, a lattice configuration in $\tilde{\mathcal{V}}$ can have an unlimited number of particles occupying the 0^{th} site, but the remaining M sites are occupied by at most one particle. We represent this mathematically by writing

$$\text{Basis}(\tilde{\mathcal{V}}) = \Big\{ |\tilde{n}\rangle = |n_0\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M \ \Big|\ 0 \leq n_1, \ldots, n_M \leq 1 \Big\} \tag{3.5.1}$$

where n_0 ranges over all non-negative integers, while the remaining occupation numbers are constrained by $0 \leq n_1, \ldots, n_M \leq 1$. The inner product between two basis vectors $|\tilde{m}\rangle = |m_0\rangle_0 \otimes \cdots \otimes |m_M\rangle_M$ and $|\tilde{n}\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ is defined to be

$$\tilde{\mathcal{I}}\Big(|\tilde{m}\rangle, |\tilde{n}\rangle\Big) = 2^{m_0} \theta\Big(0 \leq m_0 \leq 1\Big) \delta_{m_0,n_0} \prod_{j=1}^{M} 2^{-m_j} \delta_{m_j,n_j} \tag{3.5.2}$$

where $\theta(x)$ is the Boolean function, with $\theta(x) = 1$ if x is true and $\theta(x) = 0$ if x is false. This corresponds to setting

$$c_0(m_0) = 2^{m_0}\theta\Big(0 \le m_0 \le 1\Big), \quad c_j(m_j) = 2^{-m_j} \text{ for all } 1 \le j \le M \tag{3.5.3}$$

in equation (2.1.8). The dual space of states $\tilde{\mathcal{V}}^*$ has the basis

$$\text{Basis}(\tilde{\mathcal{V}}^*) = \Big\{ \langle \tilde{m}| = \langle m_0|_0 \otimes \langle m_1|_1 \otimes \cdots \otimes \langle m_M|_M \ \Big| \ 0 \le m_1, \ldots, m_M \le 1 \Big\} \tag{3.5.4}$$

where m_0 again ranges over all non-negative integers, while the remaining occupation numbers are constrained by $0 \le m_1, \ldots, m_M \le 1$. The dual space of states $\tilde{\mathcal{V}}^*$ is prescribed the action

$$\langle \tilde{m}|\tilde{n}\rangle = \tilde{\mathcal{I}}\Big(|\tilde{m}\rangle, |\tilde{n}\rangle\Big) \tag{3.5.5}$$

for all $\langle \tilde{m}| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M \in \tilde{\mathcal{V}}^*$ and $|\tilde{n}\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M \in \tilde{\mathcal{V}}$.

3.5.2 i-boson algebra

Let us define the *i-boson algebra*. It is generated by $\{\varphi, \varphi^\dagger, \mathcal{N}\}$ which satisfy the commutation relations

$$[\varphi, \varphi^\dagger] = (-)^{\mathcal{N}}, \quad [\mathcal{N}, \varphi] = -\varphi, \quad [\mathcal{N}, \varphi^\dagger] = \varphi^\dagger \tag{3.5.6}$$

This algebra is the $q = i = \sqrt{-1}$ case of the q-boson algebra (4.1.4), discussed in the next chapter. As we did for the phase model, we consider $M + 1$ copies of the i-boson algebra, generated by $\{\varphi_0, \varphi_0^\dagger, \mathcal{N}_0\}$ through to $\{\varphi_M, \varphi_M^\dagger, \mathcal{N}_M\}$. Recalling the conventions of chapter 2, we denote these algebras by $\tilde{\mathcal{A}}_0, \ldots, \tilde{\mathcal{A}}_M$ with generators $\mathfrak{a}_i^+ = \varphi_i^\dagger, \mathfrak{a}_i^- = \varphi_i, \mathfrak{a}_i^0 = \mathcal{N}_i$. Different copies of the i-boson algebra are assumed to commute, giving rise to the equations

$$[\varphi_i, \varphi_j^\dagger] = \delta_{i,j}(-)^{\mathcal{N}_i}, \quad [\mathcal{N}_i, \varphi_j] = -\delta_{i,j}\varphi_i, \quad [\mathcal{N}_i, \varphi_j^\dagger] = \delta_{i,j}\varphi_i^\dagger \tag{3.5.7}$$

for all $0 \le i, j \le M$.

3.5.3 Representations of i-boson algebras

In direct analogy with the first part of the chapter, we fix representations of the i-boson algebras (3.5.7) on the vector space $\mathcal{V}$ (3.1.1). These representations have much in common with those of the phase algebras (3.1.6). The operator φ_j acts as an annihilator, removing particles from the j^{th} lattice site, and $\varphi_j^\dagger$ acts as a creation operator, adding particles to the j^{th} lattice site. What is different is that these operators also produce certain factors, which are necessary to correctly represent the i-boson algebras. These factors also conspire to ensure that $\tilde{\mathcal{V}} \subset \mathcal{V}$ is closed under the action of $\tilde{\mathcal{A}}_0, \ldots, \tilde{\mathcal{A}}_M$, as we demonstrate below.

We begin by constructing representations for the algebras $\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_M$. For all $1 \leq j \leq M$, the operator φ_j has the action

$$\varphi_j |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \begin{cases} 0, & n_j = 0 \\ \frac{1}{\sqrt{2}} |n_0\rangle_0 \otimes \cdots \otimes |n_j - 1\rangle_j \otimes \cdots \otimes |n_M\rangle_M, & n_j \geq 1 \end{cases} \tag{3.5.8}$$

while for all $1 \leq j \leq M$ the operator $\varphi_j^\dagger$ has the action

$$\varphi_j^\dagger |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1 + (-)^{n_j}}{\sqrt{2}} |n_0\rangle_0 \otimes \cdots \otimes |n_j + 1\rangle_j \otimes \ldots \otimes |n_M\rangle_M \tag{3.5.9}$$

We construct a slightly different representation for the algebra $\tilde{\mathcal{A}}_0$. The operator φ_0 has the action

$$\varphi_0 |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1 - (-)^{n_0}}{\sqrt{2}} |n_0 - 1\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M \tag{3.5.10}$$

while the operator $\varphi_0^\dagger$ has the action

$$\varphi_0^\dagger |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1}{\sqrt{2}} |n_0 + 1\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M \tag{3.5.11}$$

As before, for all $0 \leq j \leq M$ the action of $\mathcal{N}_j$ is given by

$$\mathcal{N}_j |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = n_j |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M \tag{3.5.12}$$

It is straightforward to check that (3.5.8)–(3.5.12) faithfully represent the i-boson algebras $\tilde{\mathcal{A}}_0, \ldots, \tilde{\mathcal{A}}_M$. Assuming all operators act linearly, the equations (3.5.8)–(3.5.12) completely determine the action of $\{\varphi_j, \varphi_j^\dagger, \mathcal{N}_j\}$ on $\mathcal{V}$. Studying (3.5.9), we see that for all $1 \leq j \leq M$ the operator $\varphi_j^\dagger$ annihilates any state with $n_j = 1$. It follows that $\tilde{\mathcal{V}}$ is closed under the action of the i-boson algebras. In addition, from the definition of the inner product (3.5.2) we see that

$$\tilde{\mathcal{I}}\Big(\varphi_j |\tilde{m}\rangle, |\tilde{n}\rangle\Big) = \tilde{\mathcal{I}}\Big(|\tilde{m}\rangle, \varphi_j^\dagger |\tilde{n}\rangle\Big) \tag{3.5.13}$$

for all $|\tilde{m}\rangle, |\tilde{n}\rangle \in \tilde{\mathcal{V}}$. Hence $\varphi_j, \varphi_j^\dagger$ are adjoint operators on $\tilde{\mathcal{V}}$ for all $0 \leq j \leq M$, while $\mathcal{N}_j$ continues to be self-adjoint.

We again follow subsection 2.1.4 to deduce appropriate actions for $\tilde{\mathcal{A}}_0, \ldots, \tilde{\mathcal{A}}_M$ on the dual space $\mathcal{V}^*$. We firstly consider the algebras $\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_M$. For all $1 \leq j \leq M$, the operator $\varphi_j^\dagger$ has the action

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \varphi_j^\dagger = \begin{cases} 0, & n_j = 0 \\ \frac{1}{\sqrt{2}} \langle n_0|_0 \otimes \cdots \otimes \langle n_j - 1|_j \otimes \cdots \otimes \langle n_M|_M, & n_j \geq 1 \end{cases} \tag{3.5.14}$$

while for all $1 \leq j \leq M$ the operator φ_j has the action

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \varphi_j = \frac{1 + (-)^{n_j}}{\sqrt{2}} \langle n_0|_0 \otimes \cdots \otimes \langle n_j + 1|_j \otimes \cdots \otimes \langle n_M|_M \tag{3.5.15}$$

As before, $\tilde{\mathcal{A}}_0$ is assigned its own representation. The operator $\varphi_0^\dagger$ has the action

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \varphi_0^\dagger = \frac{1 - (-)^{n_0}}{\sqrt{2}} \langle n_0 - 1|_0 \otimes \langle n_1|_1 \otimes \cdots \otimes \langle n_M|_M \tag{3.5.16}$$

while the operator φ_0 has the action

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \varphi_0 = \frac{1}{\sqrt{2}} \langle n_0 + 1|_0 \otimes \langle n_1|_1 \otimes \cdots \otimes \langle n_M|_M \tag{3.5.17}$$

Finally, for all $0 \leq j \leq M$ the action of $\mathcal{N}_j$ is given by

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \mathcal{N}_j = n_j \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \tag{3.5.18}$$

The set of definitions (3.5.14)–(3.5.18) provide the dual representation of the algebras $\tilde{\mathcal{A}}_0, \ldots, \tilde{\mathcal{A}}_M$. Assuming that all operators act linearly, the equations (3.5.14)–(3.5.18) completely determine the action of $\{\varphi_j, \varphi_j^\dagger, \mathcal{N}_j\}$ on the dual vector space $\mathcal{V}^*$.

3.5.4 Hamiltonian $\tilde{\mathcal{H}}$

The Hamiltonian of the i-boson model is given by

$$\tilde{\mathcal{H}} = -\frac{1}{2} \sum_{j=0}^{M} \left(\varphi_j^\dagger \varphi_{j+1} + \varphi_j \varphi_{j+1}^\dagger \right) + \bar{\mathcal{N}} \tag{3.5.19}$$

where the periodicity $\varphi_{M+1} = \varphi_0$ and $\varphi_{M+1}^\dagger = \varphi_0^\dagger$ is imposed. This Hamiltonian is deduced simply by taking the $q \to i$ limit of the q-boson Hamiltonian (4.1.17). We now describe the quantum inverse scattering/algebraic Bethe Ansatz scheme for finding eigenvectors $|\Psi\rangle \in \tilde{\mathcal{V}}$ of (3.5.19).

3.5.5 L-matrix and local intertwining equation

The R-matrix for the i-boson model depends on two indeterminates x, y and acts in the tensor product $\mathcal{V}_a \otimes \mathcal{V}_b$, where $\mathcal{V}_a, \mathcal{V}_b$ are copies of $\mathbb{C}^2$. It is given by

$$\tilde{R}_{ab}(x,y) = \begin{pmatrix} x+y & 0 & 0 & 0 \\ 0 & y-x & 2x^{\frac{1}{2}}y^{\frac{1}{2}} & 0 \\ 0 & 2x^{\frac{1}{2}}y^{\frac{1}{2}} & x-y & 0 \\ 0 & 0 & 0 & x+y \end{pmatrix}_{ab} \tag{3.5.20}$$

and corresponds to the $a_{\pm}(x,y) = x+y,\ b_{\pm}(x,y) = \pm(y-x),\ c_{\pm}(x,y) = 2x^{\frac{1}{2}}y^{\frac{1}{2}}$ case of (2.2.1). The L-matrix for the i-boson model depends on a single indeterminate x, and acts in the space $\mathcal{V}_a$. Its entries are operators acting at the m^{th} lattice site, and identically everywhere else. It has the form

$$\tilde{L}_{am}(x) = \begin{pmatrix} x^{-\frac{1}{2}} & 2^{\frac{1}{2}}\varphi_m^{\dagger} \\ 2^{\frac{1}{2}}\varphi_m & x^{\frac{1}{2}} \end{pmatrix}_a \tag{3.5.21}$$

Using these definitions, the local intertwining equation is given by

$$\tilde{R}_{ab}(x,y)\tilde{L}_{am}(x)\tilde{L}_{bm}(y) = \tilde{L}_{bm}(y)\tilde{L}_{am}(x)\tilde{R}_{ab}(x,y) \tag{3.5.22}$$

This is a 4×4 matrix equation, which gives rise to sixteen scalar identities. Each of these identities may be verified by direct calculation, and by using the commutation relations (3.5.7) where appropriate.

3.5.6 Monodromy matrix and global intertwining equation

The monodromy matrix is an $(M+1)$-fold product of the L-matrices (3.5.21), taken in the auxiliary space $\mathrm{End}(\mathcal{V}_a)$. It has the form

$$\tilde{T}_a(x) = \tilde{L}_{aM}(x)\dots\tilde{L}_{a0}(x) = \begin{pmatrix} \tilde{A}(x) & \tilde{B}(x) \\ \tilde{C}(x) & \tilde{D}(x) \end{pmatrix}_a \tag{3.5.23}$$

where $\tilde{A}(x), \tilde{B}(x), \tilde{C}(x), \tilde{D}(x)$ are elements of $\tilde{\mathcal{A}}_0 \otimes \cdots \otimes \tilde{\mathcal{A}}_M$. The monodromy matrix satisfies the global intertwining equation

$$\tilde{R}_{ab}(x,y)\tilde{T}_a(x)\tilde{T}_b(y) = \tilde{T}_b(y)\tilde{T}_a(x)\tilde{R}_{ab}(x,y) \tag{3.5.24}$$

the proof of which is immediate from the local intertwining equation (3.5.22). The identity (3.5.24) gives sixteen commutation relations between the monodromy matrix operators $\tilde{A}(x), \tilde{B}(x), \tilde{C}(x), \tilde{D}(x)$, but for our purposes we will only require two. These are the equations

$$[\tilde{B}(x), \tilde{B}(y)] = [\tilde{C}(x), \tilde{C}(y)] = 0 \tag{3.5.25}$$

and they are necessary to show that the Bethe eigenvectors are symmetric in their rapidity variables.

3.5.7 Recovering $\tilde{\mathcal{H}}$ from the transfer matrix

Let $\tilde{t}(x) = \mathrm{tr}_a \tilde{T}_a(x) = \tilde{A}(x) + \tilde{D}(x)$ be the transfer matrix of the i-boson model. The Hamiltonian (3.5.19) may be recovered via the equation

$$\tilde{\mathcal{H}} = \frac{1}{4}\left[x^2 \frac{d}{dx}\left(x^{-(M+1)/2}\tilde{t}(x)\right)\right]_{x\rightarrow\infty} - \frac{1}{4}\left[\frac{d}{dx}\left(x^{(M+1)/2}\tilde{t}(x)\right)\right]_{x\rightarrow 0} + \tilde{\mathcal{N}} \tag{3.5.26}$$

from which it follows that $[\tilde{\mathcal{H}}, \tilde{t}(x)] = 0$. Hence the eigenvectors of $\tilde{\mathcal{H}}$ may be found by studying the eigenvectors of $\tilde{t}(x)$.

3.5.8 Bethe Ansatz for the eigenvectors

As was explained in theorem 1 of the previous chapter, the eigenvectors of the transfer matrix $\tilde{t}(x)$ are given by

$$|\Psi\rangle = \tilde{B}(y_1)\ldots\tilde{B}(y_N)|0\rangle, \quad \langle\Psi| = \langle 0|\tilde{C}(y_N)\ldots\tilde{C}(y_1) \tag{3.5.27}$$

where the variables $\{y_1, \ldots, y_N\}$ are assumed to obey the Bethe equations (2.3.6). For the present model, $a(y_i, y_j) = y_i + y_j$, $\alpha(y_i) = y_i^{-(M+1)/2}$, $\delta(y_i) = y_i^{(M+1)/2}$. Substituting these expressions into (2.3.6), the Bethe equations for the i-boson model have the decoupled form

$$y_i^{M+1} = (-)^{N+1} \tag{3.5.28}$$

for all $1 \leq i \leq N$. The equations (3.5.28) may be trivially solved, and reflect the inherent simplicity of the model under consideration. In the next section we turn to a more rigorous examination of the eigenvectors (3.5.27), in which the trivial Bethe equations (3.5.28) are not required. We remark that the vectors (3.5.27) are genuinely elements of $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}^*$, owing to the closure of these spaces under the action of the algebras (3.5.7) and the fact that $|0\rangle \in \tilde{\mathcal{V}}$, $\langle 0| \in \tilde{\mathcal{V}}^*$.

3.6 Calculation of i-boson model Bethe eigenvectors

In this section we essentially repeat the calculations of section 3.2, but now in the context of the i-boson model. Our main result is that the i-boson model Bethe eigenvectors can be mapped to the neutral fermionic Fock space of chapter 1, and under this mapping they lie in the O_∞ orbit of the Fock vacuum.

3.6.1 The maps $\mathcal{M}_\phi$ and $\mathcal{M}_\phi^*$

Let us begin by introducing analogues of the maps presented in subsection 3.2.1. Observing that the basis elements of $\tilde{\mathcal{V}} \subset \mathcal{V}$ and $\tilde{\mathcal{V}}^* \subset \mathcal{V}^*$ correspond with *strict* partitions under the maps $\mathcal{M}_\psi$ and $\mathcal{M}_\psi^*$, we are motivated to make the following definition.

Definition 6. Let $|\tilde{n}\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ and $\langle\tilde{n}| = \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M$ be basis elements of $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}^*$, respectively, and define

$$\Sigma_1 = \sum_{j=1}^{M} n_j \tag{3.6.1}$$

From this, let $|\tilde{\nu})$ and $(\tilde{\nu}|$ be the strict partitions in $\mathcal{F}_\phi^{(0)}$ and $\mathcal{F}_\phi^{*(0)}$ with one part equal to j if $n_j = 1$, for all $1 \leq j \leq M$. That is, we let

$$|\tilde{\nu}) = |M^{n_M}, \ldots, 1^{n_1}), \quad \Sigma_1 \text{ even} \qquad |\tilde{\nu}) = |M^{n_M}, \ldots, 1^{n_1}, 0), \quad \Sigma_1 \text{ odd} \tag{3.6.2}$$
$$(\tilde{\nu}| = (M^{n_M}, \ldots, 1^{n_1}|, \quad \Sigma_1 \text{ even} \qquad (\tilde{\nu}| = (M^{n_M}, \ldots, 1^{n_1}, 0|, \quad \Sigma_1 \text{ odd} \tag{3.6.3}$$

We define linear maps $\mathcal{M}_\phi : \tilde{\mathcal{V}} \to \mathcal{F}_\phi^{(0)}$ and $\mathcal{M}_\phi^* : \tilde{\mathcal{V}}^* \to \mathcal{F}_\phi^{*(0)}$ whose actions are given by

$$\mathcal{M}_\phi|\tilde{n}\rangle = 2^{-\ell(\tilde{\nu})}|\tilde{\nu}), \quad \langle\tilde{n}|\mathcal{M}_\phi^* = 2^{-\ell(\tilde{\nu})}(\tilde{\nu}| \tag{3.6.4}$$

Since these mappings do not depend on the value of n_0, they are not one-to-one. We also remark that the maps (3.6.4) are non-isometric, in the sense that

$$\langle\tilde{m}|\tilde{n}\rangle \neq \Big\langle \langle\tilde{m}|\mathcal{M}_\phi^*, \mathcal{M}_\phi|\tilde{n}\rangle \Big\rangle \tag{3.6.5}$$

To show this, let $(\tilde{\mu}|$ and $|\tilde{\nu})$ be the strict partitions corresponding with the respective basis vectors $\langle\tilde{m}|$ and $|\tilde{n}\rangle$ under the maps (3.6.4). Using the orthogonality relation (1.4.22) we find that

$$\Big\langle \langle\tilde{m}|\mathcal{M}_\phi^*, \mathcal{M}_\phi|\tilde{n}\rangle \Big\rangle = 2^{-\ell(\tilde{\mu})}2^{-\ell(\tilde{\nu})}\Big\langle (\tilde{\mu}|, |\tilde{\nu}) \Big\rangle = 2^{-\ell(\tilde{\mu})}\delta_{\tilde{\mu},\tilde{\nu}} = \prod_{j=1}^{M} 2^{-m_j}\delta_{m_j,n_j} \tag{3.6.6}$$

Comparing (3.5.5) with (3.6.6), we see that in general $\langle \tilde{m}|\tilde{n}\rangle \neq \langle\langle \tilde{m}|\mathcal{M}_\phi^*, \mathcal{M}_\phi|\tilde{n}\rangle\rangle$. The only difference between these two quantities is that (3.5.5) is sensitive to the value of m_0, n_0, whereas (3.6.6) is not. Hence the mappings (3.6.4) essentially filter out all information from the 0^{th} site of the lattice.

3.6.2 Calculation of $\tilde{\mathbb{B}}(x)|\tilde{n}\rangle$

Lemma 7. Define $\tilde{\mathbb{B}}(x) = x^{\frac{M}{2}}\tilde{B}(x)$ and let $|\tilde{n}\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ be an arbitrary basis vector of $\tilde{\mathcal{V}}$. The action of $\tilde{\mathbb{B}}(x)$ on $|\tilde{n}\rangle$ is given by

$$\tilde{\mathbb{B}}(x)|\tilde{n}\rangle = \sum_{|\tilde{m}\rangle \triangleright |\tilde{n}\rangle} \prod_{i=1}^{M} 2^{\delta_{(m_i-n_i),1}} x^{i(m_i-n_i)} |\tilde{m}\rangle \tag{3.6.7}$$

where the sum is over all basis vectors $|\tilde{m}\rangle = |m_0\rangle_0 \otimes \cdots \otimes |m_M\rangle_M \in \tilde{\mathcal{V}}$ which are admissible to $|\tilde{n}\rangle$.

Proof. We defer the proof of this equation to the next chapter, where we obtain a similar result (4.3.6) in the context of the q-boson model. Specializing (4.3.6) to the value $q = i$, we obtain (3.6.7) as a corollary. □

3.6.3 Calculation of $\langle \tilde{n}|\tilde{\mathbb{C}}(x)$

Lemma 8. Define $\tilde{\mathbb{C}}(x) = x^{\frac{M}{2}}\tilde{C}(1/x)$ and let $\langle \tilde{n}| = \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M$ be an arbitrary basis vector of $\tilde{\mathcal{V}}^*$. The action of $\tilde{\mathbb{C}}(x)$ on $\langle \tilde{n}|$ is given by

$$\langle \tilde{n}|\tilde{\mathbb{C}}(x) = \sum_{\langle \tilde{n}| \triangleleft \langle \tilde{m}|} \prod_{i=1}^{M} 2^{\delta_{(m_i-n_i),1}} x^{i(m_i-n_i)} \langle \tilde{m}| \tag{3.6.8}$$

where the sum is over all basis vectors $\langle \tilde{m}| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M \in \tilde{\mathcal{V}}^*$ which are admissible to $\langle \tilde{n}|$.

Proof. Again, we defer the proof to the next chapter. Specializing the later result (4.3.16) to the value $q = i$, we obtain (3.6.8) as a corollary. □

3.6.4 Calculation of $\mathcal{M}_\phi \tilde{\mathbb{B}}(x)|\tilde{n}\rangle$ and $\langle \tilde{n}|\tilde{\mathbb{C}}(x)\mathcal{M}_\phi^*$

Let $|\tilde{n}\rangle$ and $\langle \tilde{n}|$ be arbitrary basis vectors of $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}^*$, respectively, and let $|\tilde{\nu})$ and $(\tilde{\nu}|$ be their corresponding strict partitions, given by equations (3.6.2) and (3.6.3). We fix $l = \ell(\tilde{\nu})$ to be the number of non-zero parts of the strict partition $\tilde{\nu}$. For any two strict partitions $\tilde{\mu}, \tilde{\nu}$ let us also define $\#(\tilde{\mu}|\tilde{\nu})$ to be the number of parts in $\tilde{\mu}$ which are not in $\tilde{\nu}$. Using the definition (3.6.4) of the maps $\mathcal{M}_\phi$ and $\mathcal{M}_\phi^*$, the expressions (3.6.7) and (3.6.8) and the result of lemma 1, we obtain

$$\mathcal{M}_{\phi}\tilde{\mathbb{B}}(x)|\tilde{n}\rangle = \sum_{\tilde{\nu}\prec\tilde{\mu}\subseteq[l+1,M]} 2^{\#(\tilde{\mu}|\tilde{\nu})-\ell(\tilde{\mu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}|\tilde{\mu}\rangle \tag{3.6.9}$$

$$\langle\tilde{n}|\tilde{\mathbb{C}}(x)\mathcal{M}_{\phi}^{*} = \sum_{\tilde{\nu}\prec\tilde{\mu}\subseteq[l+1,M]} 2^{\#(\tilde{\mu}|\tilde{\nu})-\ell(\tilde{\mu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}(\tilde{\mu}| \tag{3.6.10}$$

Both sums are over all strict partitions $\tilde{\mu}$ which interlace with $\tilde{\nu}$, and whose Young diagrams are contained within the rectangle $[l+1, M]$.

3.6.5 Skew Schur Q-functions

Before we progress to the calculation of the i-boson model Bethe eigenvectors, we present some formulae from the theory of symmetric functions.[11] For an arbitrary pair of strict partitions $\tilde{\mu}, \tilde{\nu}$ and an indeterminate x, the single variable *skew Schur Q-function* $Q_{\tilde{\mu}/\tilde{\nu}}(x)$ is given by

$$Q_{\tilde{\mu}/\tilde{\nu}}(x) = \begin{cases} 2^{\#(\tilde{\mu}|\tilde{\nu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}, & \tilde{\mu} \succ \tilde{\nu} \\ 0, & \text{otherwise} \end{cases} \tag{3.6.11}$$

In the case $\tilde{\nu} = \emptyset$ we have $Q_{\tilde{\mu}/\tilde{\nu}}(x) = Q_{\tilde{\mu}}(x)$, where $Q_{\tilde{\mu}}(x)$ is the ordinary Schur Q-function in a single variable x. The skew Schur Q-function satisfies the identity

$$Q_{\tilde{\mu}}\{x_1, \ldots, x_n\} = \sum_{\tilde{\nu}\subseteq[n-1,\infty]} Q_{\tilde{\mu}/\tilde{\nu}}(x_n) Q_{\tilde{\nu}}\{x_1, \ldots, x_{n-1}\} \tag{3.6.12}$$

where the sum is taken over all strict partitions $\tilde{\nu}$ whose lengths satisfy $\ell(\tilde{\nu}) \leq n-1$, and $Q_{\tilde{\mu}}\{x_1, \ldots, x_n\}$ and $Q_{\tilde{\nu}}\{x_1, \ldots, x_{n-1}\}$ are Schur Q-functions in n and $n-1$ variables, respectively.

3.6.6 Calculation of $\mathcal{M}_{\phi}\tilde{\mathbb{B}}(x_1)\ldots\tilde{\mathbb{B}}(x_N)|0\rangle$

Lemma 9. Let $\{x_1, \ldots, x_N\}$ be a finite set of variables. We claim that

$$\mathcal{M}_{\phi}\tilde{\mathbb{B}}(x_1)\ldots\tilde{\mathbb{B}}(x_N)|0\rangle = \sum_{\tilde{\mu}\subseteq[N,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x_1, \ldots, x_N\}|\tilde{\mu}\rangle \tag{3.6.13}$$

where $Q_{\tilde{\mu}}\{x_1, \ldots, x_N\}$ is the Schur Q-function in N variables (1.4.51), and the sum is over all strict partitions $\tilde{\mu}$ whose Young diagrams are contained in the rectangle $[N, M]$.

[11] For more information on this material, see section 8 in chapter III of [65].

Proof. Taking the special case $|\tilde{n}\rangle = |0\rangle$ of equation (3.6.9) we obtain

$$\mathcal{M}_{\phi}\tilde{\mathbb{B}}(x)|0\rangle = \sum_{\emptyset \prec \tilde{\mu} \subseteq [1,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}/\emptyset}(x)|\tilde{\mu}\rangle = \sum_{\tilde{\mu} \subseteq [1,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}(x)|\tilde{\mu}\rangle \tag{3.6.14}$$

where we have used the equation (3.6.11) for the skew Schur Q-function, and the definition $\ell(\emptyset) = 0$. We use equation (3.6.14) as the basis for induction, and assume that

$$\mathcal{M}_{\phi}\tilde{\mathbb{B}}(x_1)\dots\tilde{\mathbb{B}}(x_{N-1})|0\rangle = \sum_{\tilde{\nu} \subseteq [N-1,M]} 2^{-\ell(\tilde{\nu})} Q_{\tilde{\nu}}\{x_1,\dots,x_{N-1}\}|\tilde{\nu}\rangle \tag{3.6.15}$$

for some $N \geq 2$. In terms of the basis vectors of $\tilde{\mathcal{V}}$, this assumption is written as

$$\tilde{\mathbb{B}}(x_1)\dots\tilde{\mathbb{B}}(x_{N-1})|0\rangle = \sum_{|\tilde{n}\rangle|\Sigma_0 = N-1} Q_{\tilde{\nu}}\{x_1,\dots,x_{N-1}\}|\tilde{n}\rangle \tag{3.6.16}$$

where the sum is over all basis vectors $|\tilde{n}\rangle$ such that $\sum_{j=0}^{M} n_j = N-1$, and $\tilde{\nu}$ is the strict partition corresponding to each $|\tilde{n}\rangle$. Acting on (3.6.16) with the composition of operators $\mathcal{M}_{\phi} \circ \tilde{\mathbb{B}}(x_N)$ and using the fact that the B-operators commute (3.5.25), we obtain

$$\mathcal{M}_{\phi}\tilde{\mathbb{B}}(x_1)\dots\tilde{\mathbb{B}}(x_N)|0\rangle = \sum_{\tilde{\nu} \subseteq [N-1,M]} Q_{\tilde{\nu}}\{x_1,\dots,x_{N-1}\} \sum_{\tilde{\nu} \prec \tilde{\mu} \subseteq [N,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}/\tilde{\nu}}(x_N)|\tilde{\mu}\rangle \tag{3.6.17}$$

Since $Q_{\tilde{\mu}/\tilde{\nu}}(x_N) = 0$ if $\tilde{\mu} \not\succ \tilde{\nu}$, we may alter the sums appearing in (3.6.17), yielding

$$\begin{aligned} \mathcal{M}_{\phi}\tilde{\mathbb{B}}(x_1)\dots\tilde{\mathbb{B}}(x_N)|0\rangle &= \sum_{\tilde{\mu} \subseteq [N,M]} 2^{-\ell(\tilde{\mu})} \sum_{\tilde{\nu} \subseteq [N-1,M]} Q_{\tilde{\mu}/\tilde{\nu}}(x_N) Q_{\tilde{\nu}}\{x_1,\dots,x_{N-1}\}|\tilde{\mu}\rangle \\ &= \sum_{\tilde{\mu} \subseteq [N,M]} 2^{-\ell(\tilde{\mu})} \sum_{\tilde{\nu} \subseteq [N-1,\infty]} Q_{\tilde{\mu}/\tilde{\nu}}(x_N) Q_{\tilde{\nu}}\{x_1,\dots,x_{N-1}\}|\tilde{\mu}\rangle \end{aligned} \tag{3.6.18}$$

where the final equality holds since the leading part of $\tilde{\mu}$ is less than or equal to M, and therefore $Q_{\tilde{\mu}/\tilde{\nu}}(x_N) = 0$ if the leading part of $\tilde{\nu}$ exceeds M. Using the identity (3.6.12) we evaluate the sum over $\tilde{\nu}$ explicitly, producing the equation (3.6.13). Therefore by induction the result (3.6.13) must hold for arbitrary $N \geq 1$.

□

3.6.7 Calculation of $\langle 0|\tilde{\mathbb{C}}(x_N)\dots\tilde{\mathbb{C}}(x_1)\mathcal{M}_\phi^*$

By following essentially the same steps that were used in the previous subsection, we can also derive the expression

$$\langle 0|\tilde{\mathbb{C}}(x_N)\dots\tilde{\mathbb{C}}(x_1)\mathcal{M}_\phi^* = \sum_{\tilde{\mu}\subseteq[N,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x_1,\dots,x_N\}(\tilde{\mu}| \tag{3.6.19}$$

for the dual Bethe eigenvectors. As before, this sum is taken over all strict partitions $\tilde{\mu}$ whose Young diagrams are contained in the rectangle $[N,M]$.

3.6.8 Neutral fermionic expression for Bethe eigenvectors

In analogy with subsection 3.2.11, we proceed to show that $\mathcal{M}_\phi$ maps the i-boson model Bethe eigenvectors $\tilde{\mathbb{B}}(x_1)\dots\tilde{\mathbb{B}}(x_N)|0\rangle$ to vectors $g_\phi|0\rangle \in \mathcal{F}_\phi^{(0)}$ which satisfy the neutral fermion bilinear identity (1.5.14). In order to do this, we prepare some definitions. For all integers $m > n \geq 0$ we define

$$c_{m,n}\{x\} = \begin{cases} 2^{\delta_{n,0}}\Big(\frac{1}{4}q_m\{x\}q_n\{x\} + \frac{1}{2}\sum_{k=1}^{n}(-)^k q_{m+k}\{x\}q_{n-k}\{x\}\Big), & m \leq M \\ 0, & m > M \end{cases} \tag{3.6.20}$$

where $q_m\{x\}$ is the function (1.4.50) in the variables $\{x\} = \{x_1,\dots,x_N\}$, as defined in chapter 1. Using this expression for the functions $c_{m,n}\{x\}$, for all strict partitions $\tilde{\mu} = \{\mu_1 > \dots > \mu_{2r} \geq 0\}$ let us also define the coefficients

$$c_{\tilde{\mu}}\{x\} = \mathrm{Pf}\Big(c_{\mu_i,\mu_j}\{x\}\Big)_{1\leq i<j\leq 2r} \tag{3.6.21}$$

Since $c_{m,n}\{x\} = 0$ if $m > M$, the coefficient $c_{\tilde{\mu}}\{x\}$ vanishes when $\mu_1 > M$. By the definition of the Schur Q-functions (1.4.51), it follows that

$$c_{\tilde{\mu}}\{x\} = \begin{cases} 2^{-\ell(\tilde{\mu})}Q_{\tilde{\mu}}\{x\}, & \tilde{\mu} \subseteq [N,M] \\ 0, & \tilde{\mu} \not\subseteq [N,M] \end{cases} \tag{3.6.22}$$

Returning to the expression (3.6.13) for the Bethe eigenvectors, we can use the coefficients (3.6.22) to write

$$\mathcal{M}_\phi \tilde{\mathbb{B}}(x_1)\dots\tilde{\mathbb{B}}(x_N)|0\rangle = \sum_{\tilde{\mu}} c_{\tilde{\mu}}\{x\}\phi_{\mu_1}\dots\phi_{\mu_{2r}}|0\rangle \tag{3.6.23}$$

where the sum is over strict partitions $\tilde{\mu} = \{\mu_1 > \cdots > \mu_{2r} \geq 0\}$ with r taking all non-negative values, and where we have made the identification $|\tilde{\mu}) = \phi_{\mu_1} \ldots \phi_{\mu_{2r}}|0\rangle$. Owing to their Pfaffian form (3.6.21) and lemma 22 of chapter 1, we see that the coefficients $c_{\tilde{\mu}}\{x\}$ satisfy the BKP Plücker relations (1.6.37). This implies that the right hand side of (3.6.23) satisfies the neutral fermion bilinear identity (1.5.14), as we intended to show.

Having established the preceding result, we now express $\mathcal{M}_\phi \tilde{\mathbb{B}}(x_1) \ldots \tilde{\mathbb{B}}(x_N)|0\rangle$ in the orbit of the Fock vacuum under O_∞. We know that this is possible using theorem 4 of chapter 1, and the fact that the right hand side of (3.6.23) satisfies the NFBI. We begin by expanding $\mathcal{M}_\phi \tilde{\mathbb{B}}(x_1) \ldots \tilde{\mathbb{B}}(x_N)|0\rangle$ in the $\mathcal{F}_\phi^{(0)}$ basis (1.4.11), yielding

$$\mathcal{M}_\phi \tilde{\mathbb{B}}(x_1) \ldots \tilde{\mathbb{B}}(x_N)|0\rangle = |0\rangle + \sum_{0 \leq n < m \leq M} 2^{\delta_{n,0}-2} Q_{\{m,n\}}\{x\} \phi_m \phi_n |0\rangle + g_\phi^{(1)} |0\rangle \tag{3.6.24}$$

where $Q_{\{m,n\}}\{x\}$ is the Schur Q-function associated to the strict partition with one part of size m, and another part of size n. As usual we assume that all monomials within $g_\phi^{(1)} \in Cl_\phi^{(0)}$ consist of at least four neutral fermions. Because the right hand side of (3.6.24) obeys the NFBI, we can use the method adopted in the proof of lemma 20 in chapter 1 to obtain

$$\mathcal{M}_\phi \tilde{\mathbb{B}}(x_1) \ldots \tilde{\mathbb{B}}(x_N)|0\rangle = \exp\left(\sum_{0 \leq n < m \leq M} 2^{\delta_{n,0}-2} Q_{\{m,n\}}\{x\} \phi_m \phi_n \right) |0\rangle \tag{3.6.25}$$

This result explicitly places $\mathcal{M}_\phi \tilde{\mathbb{B}}(x_1) \ldots \tilde{\mathbb{B}}(x_N)|0\rangle$ in the vacuum orbit of O_∞.

Finally, let us remark that all of these results can be extended to the dual Bethe eigenvectors $\langle 0|\tilde{\mathbb{C}}(x_N) \ldots \tilde{\mathbb{C}}(x_1)\mathcal{M}_\phi^*$. For example, by completely analogous reasoning it is possible to show that

$$\langle 0|\tilde{\mathbb{C}}(x_N) \ldots \tilde{\mathbb{C}}(x_1)\mathcal{M}_\phi^* = \langle 0| \exp\left(\sum_{0 \leq n < m \leq M} 2^{\delta_{n,0}-2} Q_{\{m,n\}}\{x\} \phi_n^* \phi_m^* \right) \tag{3.6.26}$$

3.7 Scalar product, boxed strict plane partitions

3.7.1 Strict plane partitions

Definition 7. A *strict plane partition* $\tilde{\pi}$ is a set of non-negative integers $\pi(i,j)$ which satisfy

$$\pi(i,j) \geq \pi(i+1,j), \quad \pi(i,j) \geq \pi(i,j+1) \tag{3.7.1}$$
$$\pi(i,j) > \pi(i+1,j+1) \tag{3.7.2}$$

for all integers $i, j \geq 1$, as well as the finiteness condition

$$\lim_{i\to\infty} \pi(i,j) = \lim_{j\to\infty} \pi(i,j) = 0 \tag{3.7.3}$$

That is, strict plane partitions obey all of the axioms of ordinary plane partitions, plus the additional constraint (3.7.2) which imposes strictness on every diagonal. Similarly, an *M-boxed strict plane partition* is a set of non-negative integers $\pi(i,j)$ satisfying the above properties, as well as the supplementary condition

$$0 \leq \pi(i,j) \leq M \tag{3.7.4}$$

for all integers $i, j \geq 1$, and where $M \geq 1$ is some fixed positive integer.

To illustrate this definition, we now give the two and three-dimensional diagrams of an exemplary strict plane partition.

4	2	1	1	1
3	2	1		
2	1	1		
1				

Figure 3.7: Tableau representation of a strict plane partition. The integers on the diagonals of this tableau are strictly decreasing. For example, the central diagonal $\{\pi(1,1) = 4, \pi(2,2) = 2, \pi(3,3) = 1\}$ is a strictly decreasing sequence.

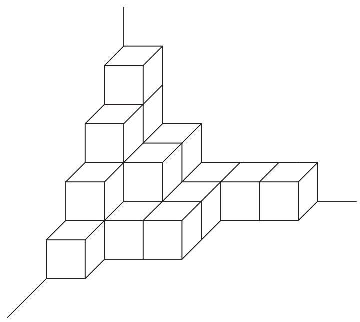

Figure 3.8: Three-dimensional representation of a strict plane partition. This plane partition has six connected horizontal plateaux. The strictness imposed on the diagonals means that all connected horizontal plateaux are one square wide.

3.7.2 Diagonal slices of strict plane partitions

Definition 8. Let $\tilde{\pi}$ be an arbitrary strict plane partition. For $i \geq 0$ define the strict partitions $|\tilde{\pi}_i) \in \mathcal{F}_\phi^{(0)}$ whose elements are given by

$$(\tilde{\pi}_i)_j = \pi(j, i+j) \tag{3.7.5}$$

for all $j \geq 1$. Similarly for $i \leq 0$ define the strict partitions $(\tilde{\pi}_i| \in \mathcal{F}_\phi^{*(0)}$ whose elements are given by

$$(\tilde{\pi}_i)_j = \pi(-i+j, j) \tag{3.7.6}$$

for all $j \geq 1$. The strict partitions $|\tilde{\pi}_i)$ and $(\tilde{\pi}_i|$ comprise the diagonal slices of the strict plane partition $\tilde{\pi}$.

Lemma 10. Let $|\tilde{\pi}_i)$ and $(\tilde{\pi}_i|$ be the diagonal slices of the strict plane partition $\tilde{\pi}$. Then we have

$$(\tilde{\pi}_{i-1}| \prec (\tilde{\pi}_i| \ \text{ for all } i \leq 0, \qquad |\tilde{\pi}_i) \succ |\tilde{\pi}_{i+1}) \ \text{ for all } i \geq 0 \tag{3.7.7}$$

Proof. This is a trivial corollary of the result in subsection 3.3.2. The diagonal slices of $\tilde{\pi}$ are (strict) partitions, and must therefore interlace by lemma 5. □

3.7.3 Connected elements, paths in strict plane partitions

Definition 9. The element $\pi(i,j)$ of π is considered to be *connected* with both the elements $\pi(i+1,j), \pi(i,j+1)$ and, assuming they exist, with $\pi(i-1,j), \pi(i,j-1)$.[12] We indicate that two elements are connected by writing, for example, $\pi(i,j) \sim \pi(i,j+1)$. A set $\mathcal{S}$ of more than two elements in π is connected if, for any two $\pi(i_0,j_0), \pi(i_n,j_n) \in \mathcal{S}$, there exists a subset $\{\pi(i_1,j_1),\ldots,\pi(i_{n-1},j_{n-1})\} \subset \mathcal{S}$ such that $\pi(i_k,j_k) \sim \pi(i_{k+1},j_{k+1})$ for all $0 \leq k \leq n-1$.

Definition 10. Let $\tilde{\pi}$ be an arbitrary strict plane partition. A *path* in $\tilde{\pi}$ is a set of connected elements of $\tilde{\pi}$ which all have the same numerical value. When viewed in the standard three-dimensional representation, paths in a strict plane partition are connected horizontal plateaux which are maximally one square wide. We let $p(\tilde{\pi})$ denote the number of paths possessed by $\tilde{\pi}$. This definition of paths within a strict plane partition was originally given in [35].

Definition 11. Let $\tilde{\pi}$ be a strict plane partition living inside the box $[N,N,M]$, with diagonal slices $\{\emptyset = \tilde{\pi}_{-N} \prec \cdots \prec \tilde{\pi}_{-1} \prec \tilde{\pi}_0 \succ \tilde{\pi}_1 \succ \cdots \succ \tilde{\pi}_N = \emptyset\}$. We associate to this strict plane partition the weighting $B_{\tilde{\pi}}(\{x\},\{y\})$ given by

[12] If $\pi(i,j)$ is on the edge of the plane partition, one or both of $\pi(i-1,j), \pi(i,j-1)$ may not exist.

$$B_{\tilde{\pi}}\Big(\{x\},\{y\}\Big) = 2^{p(\tilde{\pi})} \prod_{i=1}^{N} x_i^{|\tilde{\pi}_{-i+1}|-|\tilde{\pi}_{-i}|} y_i^{|\tilde{\pi}_{i-1}|-|\tilde{\pi}_i|} \tag{3.7.8}$$

where $p(\tilde{\pi})$ is the number of paths in $\tilde{\pi}$.

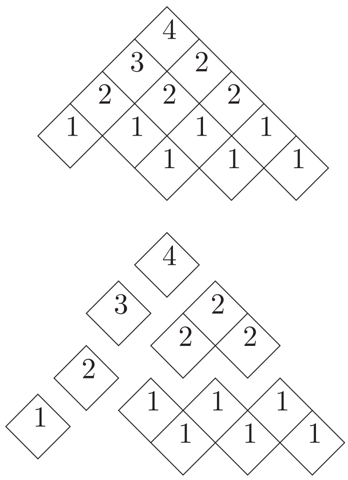

Figure 3.9: Splitting a strict plane partition into its constituent paths.

Lemma 11. Let $\tilde{\pi}$ be a strict plane partition as described in definition 11. Then

$$2^{-\ell(\tilde{\pi}_0)} \prod_{i=1}^{N} 2^{\#(\tilde{\pi}_{-i+1}|\tilde{\pi}_{-i})} \prod_{j=1}^{N} 2^{\#(\tilde{\pi}_{j-1}|\tilde{\pi}_j)} = 2^{p(\tilde{\pi})} \tag{3.7.9}$$

Proof. The proof is well illustrated by an example, so we consider the strict plane partition $\tilde{\pi}$ drawn in figure 3.9. We see that $\tilde{\pi}$ lives inside the box $[5,5,4]$, and its diagonal slices are given by

$$\begin{array}{lll}
\tilde{\pi}_{-5} = \emptyset & & \tilde{\pi}_1 = \{2,1\} \\
\tilde{\pi}_{-4} = \emptyset & & \tilde{\pi}_2 = \{2,1\} \\
\tilde{\pi}_{-3} = \{1\} & \tilde{\pi}_0 = \{4,2,1\} & \tilde{\pi}_3 = \{1\} \\
\tilde{\pi}_{-2} = \{2\} & & \tilde{\pi}_4 = \{1\} \\
\tilde{\pi}_{-1} = \{3,1\} & & \tilde{\pi}_5 = \emptyset
\end{array} \tag{3.7.10}$$

Using these strict partitions, we evaluate

$$\begin{aligned}
&2^{\#(\tilde{\pi}_{-4}|\tilde{\pi}_{-5})} = 2^0 && 2^{\#(\tilde{\pi}_0|\tilde{\pi}_1)} = 2^1 \\
&2^{\#(\tilde{\pi}_{-3}|\tilde{\pi}_{-4})} = 2^1 && 2^{\#(\tilde{\pi}_1|\tilde{\pi}_2)} = 2^0 \\
&2^{\#(\tilde{\pi}_{-2}|\tilde{\pi}_{-3})} = 2^1 && 2^{\#(\tilde{\pi}_2|\tilde{\pi}_3)} = 2^1 \\
&2^{\#(\tilde{\pi}_{-1}|\tilde{\pi}_{-2})} = 2^2 && 2^{\#(\tilde{\pi}_3|\tilde{\pi}_4)} = 2^0 \\
&2^{\#(\tilde{\pi}_0|\tilde{\pi}_{-1})} = 2^2 && 2^{\#(\tilde{\pi}_4|\tilde{\pi}_5)} = 2^1
\end{aligned} \tag{3.7.11}$$

Let us consider these factors, progressing from the extremal diagonal slices of $\tilde{\pi}$ towards its central slice $\tilde{\pi}_0$. We obtain a factor of 2 for every path which begins in $\tilde{\pi}$ and does *not* intersect the central diagonal. Paths that *do* intersect the central diagonal are assigned a factor of 2^2, which is a double counting. We cure this double counting by dividing by 2 for every element in $\tilde{\pi}_0$, that is, by dividing by $2^{\ell(\tilde{\pi}_0)} = 2^3$. The result is

$$2^{-\ell(\tilde{\pi}_0)} \prod_{i=1}^{5} 2^{\#(\tilde{\pi}_{-i+1}|\tilde{\pi}_{-i})} \prod_{j=1}^{5} 2^{\#(\tilde{\pi}_{j-1}|\tilde{\pi}_j)} = 2^6 = 2^{p(\tilde{\pi})} \tag{3.7.12}$$

as required. It is clear that this method extends to arbitrary strict plane partitions. □

3.7.4 Generating M-boxed strict plane partitions

We now derive an analogue of the result given in subsection 3.3.3, this time relating to the scalar product of the i-boson model on $M+1$ sites. The result is that the scalar product between the image Bethe eigenstates (3.6.13) and (3.6.19) is a generating function for M-boxed strict plane partitions. This correspondence may be realized by iterating the $|\tilde{n}\rangle = |0\rangle$ case of equation (3.6.9) N times, giving

$$\mathcal{M}_\phi \tilde{\mathbb{B}}(x_1)\dots\tilde{\mathbb{B}}(x_N)|0\rangle = \sum_{[N,M]\supseteq\tilde{\pi}_0\succ\cdots\succ\tilde{\pi}_N=\emptyset} 2^{-\ell(\tilde{\pi}_0)} \prod_{i=1}^{N} 2^{\#(\tilde{\pi}_{i-1}|\tilde{\pi}_i)} x_i^{|\tilde{\pi}_{i-1}|-|\tilde{\pi}_i|} |\tilde{\pi}_0) \tag{3.7.13}$$

where the sum is over all interlacing strict partitions $\{\tilde{\pi}_0 \succ \cdots \succ \tilde{\pi}_N\}$ subject to $\tilde{\pi}_0 \subseteq [N,M]$ and $\tilde{\pi}_N = \emptyset$. Similarly, one can iterate the $\langle\tilde{n}| = \langle 0|$ case of (3.6.10) N times, giving

$$\langle 0|\tilde{\mathbb{C}}(x_N)\dots\tilde{\mathbb{C}}(x_1)\mathcal{M}_\phi^* = \sum_{\emptyset=\tilde{\pi}_{-N}\prec\cdots\prec\tilde{\pi}_0\subseteq[N,M]} 2^{-\ell(\tilde{\pi}_0)} \prod_{i=1}^{N} 2^{\#(\tilde{\pi}_{-i+1}|\tilde{\pi}_{-i})} x_i^{|\tilde{\pi}_{-i+1}|-|\tilde{\pi}_{-i}|} (\tilde{\pi}_0| \tag{3.7.14}$$

where the sum is over all interlacing strict partitions $\{\tilde{\pi}_{-N} \prec \cdots \prec \tilde{\pi}_0\}$ subject to $\tilde{\pi}_0 \subseteq [N, M]$ and $\tilde{\pi}_{-N} = \emptyset$. By the definition (3.7.8) of $B_{\tilde{\pi}}(\{x\}, \{y\})$, the result (3.7.9) of lemma 11 and the orthogonality (1.4.22) of strict partition states, we thus obtain

$$\Big\langle \langle 0|\tilde{\mathbb{C}}(x_N) \ldots \tilde{\mathbb{C}}(x_1)\mathcal{M}^*_\phi, \mathcal{M}_\phi \tilde{\mathbb{B}}(y_1) \ldots \tilde{\mathbb{B}}(y_N)|0\rangle \Big\rangle = \sum_{\tilde{\pi} \subseteq [N,N,M]} B_{\tilde{\pi}}\Big(\{x\}, \{y\}\Big) \tag{3.7.15}$$

where the sum is taken over all strict plane partitions $\tilde{\pi}$ which fit inside the box of dimension $N \times N \times M$. Hence the scalar product between the image Bethe eigenstates (3.6.13) and (3.6.19) is a generating function for M-boxed strict plane partitions. This generating function is evaluated explicitly by using the equations (3.6.13), (3.6.19) and the orthogonality relation (1.4.22) to give

$$\Big\langle \langle 0|\tilde{\mathbb{C}}(x_N) \ldots \tilde{\mathbb{C}}(x_1)\mathcal{M}^*_\phi, \mathcal{M}_\phi \tilde{\mathbb{B}}(y_1) \ldots \tilde{\mathbb{B}}(y_N)|0\rangle \Big\rangle = \sum_{\tilde{\mu} \subseteq [N,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x\} Q_{\tilde{\mu}}\{y\} \tag{3.7.16}$$

where $Q_{\tilde{\mu}}\{x\}, Q_{\tilde{\mu}}\{y\}$ denote Schur Q-functions. Comparing equations (3.7.15) and (3.7.16), we have proved the result

$$\sum_{\tilde{\pi} \subseteq [N,N,M]} B_{\tilde{\pi}}\Big(\{x\}, \{y\}\Big) = \sum_{\tilde{\mu} \subseteq [N,M]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x\} Q_{\tilde{\mu}}\{y\} \tag{3.7.17}$$

3.7.5 Scalar product as a power-sum specialized BKP τ-function

In this subsection we demonstrate that the scalar product (3.7.16) is a specialization of a BKP τ-function. The specialization is achieved by setting the BKP time variables to power sums in the i-boson model rapidities. This result parallels the one obtained in subsection 3.3.4, in the context of KP τ-functions. We begin with the equation

$$\langle 0| \exp\left(\sum_{m \in \tilde{\mathbb{N}}} t_m \lambda_m \right) = \sum_{\tilde{\mu}} 2^{-\ell(\tilde{\mu})} \mathcal{Q}_{\tilde{\mu}}\{\tilde{t}\} (\tilde{\mu}| \tag{3.7.18}$$

whose sum is over all strict partitions $\tilde{\mu}$, which follows from lemma 15 in chapter 1 and the orthogonality (1.4.22) of strict partitions. Defining $t_m = \frac{2}{m} \sum_{n=1}^{N} x_n^m$ for all $m \in \tilde{\mathbb{N}}$, equation (3.7.18) becomes

$$\langle 0| \exp\left(\sum_{m\in\tilde{\mathbb{N}}}\sum_{n=1}^{N}\frac{2}{m}x_n^m\lambda_m\right) = \sum_{\tilde{\mu}\subseteq[N,\infty]} 2^{-\ell(\tilde{\mu})}Q_{\tilde{\mu}}\{x\}(\tilde{\mu}| \tag{3.7.19}$$

where the sum is over all strict partitions $\tilde{\mu}$ with maximal length N. Equating the right hand sides of (3.6.13) and (3.6.25) and using the identity (3.7.19), we find

$$\langle 0| \exp\left(\sum_{m\in\tilde{\mathbb{N}}}\sum_{n=1}^{N}\frac{2}{m}x_n^m\lambda_m\right) \exp Y\{y\}|0\rangle = \sum_{\tilde{\mu}\subseteq[N,M]} 2^{-\ell(\tilde{\mu})}Q_{\tilde{\mu}}\{x\}Q_{\tilde{\mu}}\{y\} \tag{3.7.20}$$

where $Y\{y\}\in B_\infty$ is defined as

$$Y\{y\} = \sum_{0\le n<m\le M} 2^{\delta_{n,0}-2}Q_{\{m,n\}}\{y\}\phi_m\phi_n \tag{3.7.21}$$

Now consider the polynomial BKP τ-function $\tau\{\tilde{t}\} = \langle e^{\lambda\{\tilde{t}\}}e^{Y\{y\}}\rangle$. Comparing the equations (3.7.16) and (3.7.20) we find that

$$\begin{aligned}\tau\{\tilde{t}\} &= \langle 0|\exp\left(\sum_{m\in\tilde{\mathbb{N}}} t_m\lambda_m\right)\exp Y\{y\}|0\rangle \\ &= \Big\langle \langle 0|\tilde{\mathbb{C}}(x_N)\dots\tilde{\mathbb{C}}(x_1)\mathcal{M}_\phi^*, \mathcal{M}_\phi\tilde{\mathbb{B}}(y_1)\dots\tilde{\mathbb{B}}(y_N)|0\rangle\Big\rangle\end{aligned} \tag{3.7.22}$$

under the power-sum specialization $t_m = \frac{2}{m}\sum_{n=1}^{N}x_n^m$ for all $m\in\tilde{\mathbb{N}}$. This connection between the generating function of strict plane partitions and BKP τ-functions first appeared in [39], but in the context of strict plane partitions whose column heights are unrestricted. The result of this subsection is at the level of M-boxed strict plane partitions, and it specializes to the result of [39] in the limit $M\to\infty$.

3.8 i-boson model on an infinite lattice

This section is the i-boson model analogue of the earlier section 3.4 on the phase model. We study the action of the monodromy matrix operators $\tilde{\mathbb{B}}(x)$ and $\tilde{\mathbb{C}}(x)$ when the number of lattice sites becomes infinite. Our main result is lemma 12, showing that in the limit $M\to\infty$ the operators $\tilde{\mathbb{B}}(x)$ and $\tilde{\mathbb{C}}(x)$ acquire equivalent actions to the half-vertex operators $\tilde{\Gamma}_-(x)$ and $\tilde{\Gamma}_+(x)$ from BKP theory. The actions of these half-vertex operators were studied briefly in [35], and in more detail in [39].

3.8.1 Calculation of $\mathcal{M}_\phi \tilde{\mathbb{B}}(x)|\tilde{n}\rangle$ and $\langle\tilde{n}|\tilde{\mathbb{C}}(x)\mathcal{M}_\phi^*$ as $M \to \infty$

Lemma 12. Consider the infinite lattice limit of the i-boson model, obtained by taking $M \to \infty$. Let $|\tilde{n}\rangle = \otimes_{j=0}^{\infty} |n_j\rangle_j$ and $\langle\tilde{n}| = \otimes_{j=0}^{\infty} \langle n_j|_j$ be basis vectors of $\tilde{\mathcal{V}}$ and $\tilde{\mathcal{V}}^*$, respectively, in this limit. In addition, let $2^{-\ell(\tilde{\nu})}|\tilde{\nu})$ and $2^{-\ell(\tilde{\nu})}(\tilde{\nu}|$ be the image states of these basis vectors under the mappings (3.6.4). We claim that

$$\mathcal{M}_\phi \Big[\lim_{M\to\infty} \tilde{\mathbb{B}}(x)|\tilde{n}\rangle\Big] = 2^{-\ell(\tilde{\nu})}\tilde{\Gamma}_-(x)|\tilde{\nu}), \quad \Big[\lim_{M\to\infty} \langle\tilde{n}|\tilde{\mathbb{C}}(x)\Big]\mathcal{M}_\phi^* = 2^{-\ell(\tilde{\nu})}(\tilde{\nu}|\tilde{\Gamma}_+(x) \tag{3.8.1}$$

where we have defined the BKP half-vertex operators[13]

$$\tilde{\Gamma}_-(x) = \exp\left(\sum_{n\in\tilde{\mathbb{N}}} \frac{2}{n} x^n \lambda_{-n}\right), \quad \tilde{\Gamma}_+(x) = \exp\left(\sum_{n\in\tilde{\mathbb{N}}} \frac{2}{n} x^n \lambda_{n}\right) \tag{3.8.2}$$

and λ_{-n}, λ_n denote the Heisenberg generators (1.4.29).

Proof. We split the proof into two steps. In the first step, we show that (3.8.1) is equivalent to the statement (3.8.7). In the second step we prove (3.8.7) using the calculus of neutral free fermions.

Step 1. Taking the $M \to \infty$ limit of equations (3.6.9) and (3.6.10), we obtain

$$\mathcal{M}_\phi \Big[\lim_{M\to\infty} \tilde{\mathbb{B}}(x)|\tilde{n}\rangle\Big] = \sum_{\tilde{\mu}\succ\tilde{\nu}} 2^{\#(\tilde{\mu}|\tilde{\nu})-\ell(\tilde{\mu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}|\tilde{\mu}) \tag{3.8.3}$$

$$\Big[\lim_{M\to\infty} \langle\tilde{n}|\tilde{\mathbb{C}}(x)\Big]\mathcal{M}_\phi^* = \sum_{\tilde{\mu}\succ\tilde{\nu}} 2^{\#(\tilde{\mu}|\tilde{\nu})-\ell(\tilde{\mu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}(\tilde{\mu}| \tag{3.8.4}$$

where the sums are over all strict partitions $\tilde{\mu}$ which interlace with $\tilde{\nu}$, whose parts now have no size restriction. The equations (3.8.1) are therefore equivalent to the statements

$$\tilde{\Gamma}_-(x)|\tilde{\nu}) = \sum_{\tilde{\mu}\succ\tilde{\nu}} 2^{\#(\tilde{\mu}|\tilde{\nu})-\ell(\tilde{\mu})+\ell(\tilde{\nu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}|\tilde{\mu}) \tag{3.8.5}$$

$$(\tilde{\nu}|\tilde{\Gamma}_+(x) = \sum_{\tilde{\mu}\succ\tilde{\nu}} 2^{\#(\tilde{\mu}|\tilde{\nu})-\ell(\tilde{\mu})+\ell(\tilde{\nu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}(\tilde{\mu}| \tag{3.8.6}$$

which are entirely at the level of neutral free fermions. Due to the orthogonality (1.4.22) of strict partition states, equations (3.8.5) and (3.8.6) may be presented in the alternative form

[13] Once again we use the half-vertex nomenclature, since $\tilde{\Gamma}_-(x), \tilde{\Gamma}_+(x)$ each comprise one half of a neutral fermion vertex operator, [50].

$$(\tilde{\mu}|\tilde{\Gamma}_-(x) = \sum_{\tilde{\nu}\prec\tilde{\mu}} 2^{\#(\tilde{\mu}|\tilde{\nu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}(\tilde{\nu}|, \quad \tilde{\Gamma}_+(x)|\tilde{\mu}) = \sum_{\tilde{\nu}\prec\tilde{\mu}} 2^{\#(\tilde{\mu}|\tilde{\nu})} x^{|\tilde{\mu}|-|\tilde{\nu}|}|\tilde{\nu}) \tag{3.8.7}$$

where the sums are now over strict partitions $\tilde{\nu}$ such that $\tilde{\nu} \prec \tilde{\mu}$. The sums in (3.8.7) are finite, whereas those in (3.8.5) and (3.8.6) are infinite. As we mentioned earlier, it is thus easiest to prove (3.8.7), and this will in turn establish the equations (3.8.1).

Step 2. Consider the even-length strict partitions

$$(\tilde{\mu}| = \langle 0|\phi^*_{\mu_{2r}} \dots \phi^*_{\mu_1}, \quad |\tilde{\mu}) = \phi_{\mu_1} \dots \phi_{\mu_{2r}}|0\rangle \tag{3.8.8}$$

where we assume that $\{\mu_1 > \cdots > \mu_{2r} > 0\}$. In order to prove (3.8.7), we must calculate $(\tilde{\mu}|\tilde{\Gamma}_-(x)$ and $\tilde{\Gamma}_+(x)|\tilde{\mu})$. To achieve this we require the commutation relations

$$\tilde{\Gamma}_-(x)\left(\phi^*_i + 2\sum_{n=1}^{\infty}\phi^*_{(i-n)}x^n\right) = \phi^*_i\tilde{\Gamma}_-(x), \quad \tilde{\Gamma}_+(x)\phi_i = \left(\phi_i + 2\sum_{n=1}^{\infty}\phi_{(i-n)}x^n\right)\tilde{\Gamma}_+(x) \tag{3.8.9}$$

which are derived following the arguments presented in subsection 1.4.8.[14] Applying the relations (3.8.9) repeatedly to the strict partitions (3.8.8), we obtain

$$\begin{aligned}(\tilde{\mu}|\tilde{\Gamma}_-(x) &= \langle 0|\left(\phi^*_{\mu_{2r}} + 2\sum_{i_{2r}=1}^{\infty}\phi^*_{(\mu_{2r}-i_{2r})}x^{i_{2r}}\right)\dots\left(\phi^*_{\mu_1} + 2\sum_{i_1=1}^{\infty}\phi^*_{(\mu_1-i_1)}x^{i_1}\right)\\ &= \langle 0|\left(\sum_{i_{2r}=0}^{\infty}(2-\delta_{i_{2r},0})\phi^*_{(\mu_{2r}-i_{2r})}x^{i_{2r}}\right)\dots\left(\sum_{i_1=0}^{\infty}(2-\delta_{i_1,0})\phi^*_{(\mu_1-i_1)}x^{i_1}\right)\end{aligned} \tag{3.8.10}$$

where we have used the fact that $\langle 0|\tilde{\Gamma}_-(x) = \langle 0|$, and

$$\begin{aligned}\tilde{\Gamma}_+(x)|\tilde{\mu}) &= \left(\phi_{\mu_1} + 2\sum_{i_1=1}^{\infty}\phi_{(\mu_1-i_1)}x^{i_1}\right)\dots\left(\phi_{\mu_{2r}} + 2\sum_{i_{2r}=1}^{\infty}\phi_{(\mu_{2r}-i_{2r})}x^{i_{2r}}\right)|0\rangle\\ &= \left(\sum_{i_1=0}^{\infty}(2-\delta_{i_1,0})\phi_{(\mu_1-i_1)}x^{i_1}\right)\dots\left(\sum_{i_{2r}=0}^{\infty}(2-\delta_{i_{2r},0})\phi_{(\mu_{2r}-i_{2r})}x^{i_{2r}}\right)|0\rangle\end{aligned} \tag{3.8.11}$$

[14] Setting $t_n = \frac{2}{n}x^n$ for all $n \in \tilde{\mathbb{N}}$ in (1.4.35), we obtain $\tilde{\Gamma}_+(x)\Phi(k) = \frac{1+xk}{1-xk}\Phi(k)\tilde{\Gamma}_+(x)$. Extracting the coefficients of k^i from this equation, we prove the second commutation relation in (3.8.9). The first commutation relation may be proved similarly.

where we have used the fact that $\tilde{\Gamma}_+(x)|0\rangle = |0\rangle$. The equation (3.8.10) contains infinite sums which can be truncated by means of the identity

$$\left(\sum_{i=0}^{\infty}(2-\delta_{i,0})\phi^*_{(n-i)}x^i\right)\left(\sum_{j=0}^{\infty}(2-\delta_{j,0})\phi^*_{(m-j)}x^j\right) \tag{3.8.12}$$
$$=\left(\sum_{i=0}^{\infty}(2-\delta_{i,0})\phi^*_{(n-i)}x^i\right)\left(\sum_{j=0}^{m-n}(2-\delta_{j,0}-\delta_{j,m-n})\phi^*_{(m-j)}x^j\right)$$

which holds for all integers $m > n > 0$, while the sums in (3.8.11) may be truncated by means of the identity

$$\left(\sum_{i=0}^{\infty}(2-\delta_{i,0})\phi_{(m-i)}x^i\right)\left(\sum_{j=0}^{\infty}(2-\delta_{j,0})\phi_{(n-j)}x^j\right) \tag{3.8.13}$$
$$=\left(\sum_{i=0}^{m-n}(2-\delta_{i,0}-\delta_{i,m-n})\phi_{(m-i)}x^i\right)\left(\sum_{j=0}^{\infty}(2-\delta_{j,0})\phi_{(n-j)}x^j\right)$$

which also holds for all integers $m > n > 0$. Substituting the identity (3.8.12) into (3.8.10) we obtain

$$(\tilde{\mu}|\tilde{\Gamma}_-(x) = \langle 0|\left(\sum_{i_{2r}=0}^{\mu_{2r}}(2-\delta_{i_{2r},0})\phi^*_{(\mu_{2r}-i_{2r})}x^{i_{2r}}\right) \tag{3.8.14}$$
$$\times\left(\sum_{i_{\widetilde{2r}}=0}^{\mu_{\widetilde{2r}}-\mu_{2r}}(2-\delta_{i_{\widetilde{2r}},0}-\delta_{i_{\widetilde{2r}},\mu_{\widetilde{2r}}-\mu_{2r}})\phi^*_{(\mu_{\widetilde{2r}}-i_{\widetilde{2r}})}x^{i_{\widetilde{2r}}}\right)\dots$$
$$\times\left(\sum_{i_1=0}^{\mu_1-\mu_2}(2-\delta_{i_1,0}-\delta_{i_1,\mu_1-\mu_2})\phi^*_{(\mu_1-i_1)}x^{i_1}\right)$$

where we have defined $\widetilde{2r} = 2r-1$, and used the annihilation properties (1.4.8) to truncate the left-most sum. Furthermore, substituting (3.8.13) into (3.8.11) gives

$$\tilde{\Gamma}_+(x)|\tilde{\mu}) = \left(\sum_{i_1=0}^{\mu_1-\mu_2}(2-\delta_{i_1,0}-\delta_{i_1,\mu_1-\mu_2})\phi_{(\mu_1-i_1)}x^{i_1}\right)\dots \tag{3.8.15}$$
$$\times\left(\sum_{i_{\widetilde{2r}}=0}^{\mu_{\widetilde{2r}}-\mu_{2r}}(2-\delta_{i_{\widetilde{2r}},0}-\delta_{i_{\widetilde{2r}},\mu_{\widetilde{2r}}-\mu_{2r}})\phi_{(\mu_{\widetilde{2r}}-i_{\widetilde{2r}})}x^{i_{\widetilde{2r}}}\right)$$
$$\times\left(\sum_{i_{2r}=0}^{\mu_{2r}}(2-\delta_{i_{2r},0})\phi_{(\mu_{2r}-i_{2r})}x^{i_{2r}}\right)|0\rangle$$

where we have again set $\widetilde{2r} = 2r - 1$, and used the annihilation properties (1.4.8) to truncate the right-most sum. The indices in (3.8.14) can then be modified to produce the equation

$$\begin{aligned}(\tilde{\mu}|\tilde{\Gamma}_-(x) = \langle 0| &\left(\sum_{\mu_{2r} \geq \nu_{2r} \geq 0} (2 - \delta_{\mu_{2r},\nu_{2r}}) \phi^*_{\nu_{2r}} x^{\mu_{2r}-\nu_{2r}} \right) \\ &\times \left(\sum_{\mu_{\widetilde{2r}} \geq \nu_{\widetilde{2r}} \geq \mu_{2r}} (2 - \delta_{\mu_{\widetilde{2r}},\nu_{\widetilde{2r}}} - \delta_{\mu_{2r},\nu_{\widetilde{2r}}}) \phi^*_{\nu_{\widetilde{2r}}} x^{\mu_{\widetilde{2r}}-\nu_{\widetilde{2r}}} \right) \dots \\ &\times \left(\sum_{\mu_1 \geq \nu_1 \geq \mu_2} (2 - \delta_{\mu_1,\nu_1} - \delta_{\mu_2,\nu_1}) \phi^*_{\nu_1} x^{\mu_1-\nu_1} \right) = \sum_{\tilde{\nu} \prec \tilde{\mu}} 2^{\#(\tilde{\mu}|\tilde{\nu})} x^{|\tilde{\mu}|-|\tilde{\nu}|} (\tilde{\nu}|\end{aligned} \tag{3.8.16}$$

where we have defined the strict partitions $(\tilde{\nu}| = \langle 0|\phi^*_{\nu_{2r}} \dots \phi^*_{\nu_1}$, while the indices in (3.8.15) can be modified to produce the equation

$$\begin{aligned}\tilde{\Gamma}_+(x)|\tilde{\mu}) = &\left(\sum_{\mu_1 \geq \nu_1 \geq \mu_2} (2 - \delta_{\mu_1,\nu_1} - \delta_{\mu_2,\nu_1}) \phi_{\nu_1} x^{\mu_1-\nu_1} \right) \dots \\ &\times \left(\sum_{\mu_{\widetilde{2r}} \geq \nu_{\widetilde{2r}} \geq \mu_{2r}} (2 - \delta_{\mu_{\widetilde{2r}},\nu_{\widetilde{2r}}} - \delta_{\mu_{2r},\nu_{\widetilde{2r}}}) \phi_{\nu_{\widetilde{2r}}} x^{\mu_{\widetilde{2r}}-\nu_{\widetilde{2r}}} \right) \\ &\times \left(\sum_{\mu_{2r} \geq \nu_{2r} \geq 0} (2 - \delta_{\mu_{2r},\nu_{2r}}) \phi_{\nu_{2r}} x^{\mu_{2r}-\nu_{2r}} \right) |0\rangle = \sum_{\tilde{\nu} \prec \tilde{\mu}} 2^{\#(\tilde{\mu}|\tilde{\nu})} x^{|\tilde{\mu}|-|\tilde{\nu}|} |\tilde{\nu})\end{aligned} \tag{3.8.17}$$

where we have defined the strict partitions $|\tilde{\nu}) = \phi_{\nu_1} \dots \phi_{\nu_{2r}}|0\rangle$. Notice that the final equality in (3.8.16) and (3.8.17) follows from the fact that the summation variables satisfy

$$\mu_i \geq \nu_i \geq \mu_{i+1} \quad \text{for all } 1 \leq i \leq \widetilde{2r}, \qquad \mu_{2r} \geq \nu_{2r} \geq 0 \tag{3.8.18}$$

as well as the fact that $|\tilde{\mu}| - |\tilde{\nu}| = \sum_{i=1}^{2r} (\mu_i - \nu_i)$. It is also straightforward to check that the correct factors $2^{\#(\tilde{\mu}|\tilde{\nu})}$ are recovered from the sums in (3.8.16) and (3.8.17). This completes the proof of (3.8.7) for even-length strict partitions $\tilde{\mu}$. The proof for odd-length strict partitions starts by acting on the states

$$(\tilde{\mu}| = \langle 1|\phi^*_{\mu_{2r-1}} \dots \phi^*_{\mu_1}, \qquad |\tilde{\mu}) = \phi_{\mu_1} \dots \phi_{\mu_{2r-1}}|1\rangle \tag{3.8.19}$$

with $\{\mu_1 > \dots > \mu_{2r-1} > 0\}$, and is achieved following precisely the same procedure as above.

□

3.8.2 Generating strict plane partitions of arbitrary size

In the previous section we demonstrated that the scalar product (3.7.15) on a lattice of size $M+1$ generates M-boxed strict plane partitions. Accordingly, we expect that in the limit $M \to \infty$ it will generate strict plane partitions whose column heights are arbitrarily large, giving rise to the equation

$$\lim_{M\to\infty} \Big\langle \langle 0|\tilde{\mathbb{C}}(x_N)\dots\tilde{\mathbb{C}}(x_1)\mathcal{M}_{\phi}^{*}, \mathcal{M}_{\phi}\tilde{\mathbb{B}}(y_1)\dots\tilde{\mathbb{B}}(y_N)|0\rangle \Big\rangle = \sum_{\tilde{\pi}\subseteq[N,N,\infty]} B_{\tilde{\pi}}\Big(\{x\},\{y\}\Big) \tag{3.8.20}$$

where the sum is over all strict plane partitions $\tilde{\pi}$ which fit inside the box of dimension $N \times N \times \infty$, with $B_{\tilde{\pi}}(\{x\},\{y\})$ given by (3.7.8). On the other hand, using the result of lemma 12 we are able to write

$$\lim_{M\to\infty} \Big\langle \langle 0|\tilde{\mathbb{C}}(x_N)\dots\tilde{\mathbb{C}}(x_1)\mathcal{M}_{\phi}^{*}, \mathcal{M}_{\phi}\tilde{\mathbb{B}}(y_1)\dots\tilde{\mathbb{B}}(y_N)|0\rangle \Big\rangle = (\emptyset|\tilde{\Gamma}_+(x_N)\dots\tilde{\Gamma}_+(x_1)\tilde{\Gamma}_-(y_1)\dots\tilde{\Gamma}_-(y_N)|\emptyset) \tag{3.8.21}$$

which lends itself to immediate evaluation, owing to simple commutation relations between the BKP half-vertex operators. Explicitly speaking, using the definition (3.8.2) of $\tilde{\Gamma}_-(y)$ and $\tilde{\Gamma}_+(x)$ we find that

$$4\sum_{m\in\tilde{\mathbb{N}}}\sum_{n\in\tilde{\mathbb{N}}}\frac{x^m y^n}{mn}[\lambda_m,\lambda_{-n}] = 2\sum_{m\in\tilde{\mathbb{N}}}\sum_{n\in\tilde{\mathbb{N}}}\frac{x^m y^n}{mn} m\delta_{m,n} = 2\sum_{m\in\tilde{\mathbb{N}}}\frac{(xy)^m}{m} \tag{3.8.22}$$

$$\Longrightarrow \tilde{\Gamma}_+(x)\tilde{\Gamma}_-(y) = \exp\left(2\sum_{m\in\tilde{\mathbb{N}}}\frac{(xy)^m}{m}\right)\tilde{\Gamma}_-(y)\tilde{\Gamma}_+(x) = \frac{1+xy}{1-xy}\tilde{\Gamma}_-(y)\tilde{\Gamma}_+(x)$$

Employing the commutation relation (3.8.22) repeatedly in (3.8.21), we obtain

$$\lim_{M\to\infty} \Big\langle \langle 0|\tilde{\mathbb{C}}(x_N)\dots\tilde{\mathbb{C}}(x_1)\mathcal{M}_{\phi}^{*}, \mathcal{M}_{\phi}\tilde{\mathbb{B}}(y_1)\dots\tilde{\mathbb{B}}(y_N)|0\rangle \Big\rangle = \prod_{i,j=1}^{N}\frac{1+x_i y_j}{1-x_i y_j} \tag{3.8.23}$$

Comparing equations (3.8.20) and (3.8.23), we have proved that

$$\sum_{\tilde{\pi}\subseteq[N,N,\infty]} B_{\tilde{\pi}}\Big(\{x\},\{y\}\Big) = \prod_{i,j=1}^{N}\frac{1+x_i y_j}{1-x_i y_j} \tag{3.8.24}$$

which is a simpler evaluation of this generating function than in the finite case (3.7.17). This calculation could also have been performed using (3.7.16) and the identity

$$\sum_{\tilde{\mu}\subseteq[N,\infty]} 2^{-\ell(\tilde{\mu})} Q_{\tilde{\mu}}\{x\} Q_{\tilde{\mu}}\{y\} = \prod_{i,j=1}^{N} \frac{1+x_i y_j}{1-x_i y_j} \tag{3.8.25}$$

from section 8 in chapter III of [65]. We believe our proof is more fundamental, since it is independent of the properties of symmetric functions. As a final observation, let us specialize the variables $\{x_1,\ldots,x_N\}$ and $\{y_1,\ldots,y_N\}$ to

$$x_i = y_i = z^{i-\frac{1}{2}} \quad \text{for all } 1 \leq i \leq N \tag{3.8.26}$$

giving rise to the equation

$$\sum_{\tilde{\pi}\subseteq[N,N,\infty]} 2^{p(\tilde{\pi})} z^{|\tilde{\pi}|} = \prod_{i,j=1}^{N} \frac{1+z^{i+j-1}}{1-z^{i+j-1}} \tag{3.8.27}$$

where $p(\tilde{\pi})$ and $|\tilde{\pi}|$ are the number of paths and weight of the strict plane partition $\tilde{\pi}$, respectively. Taking the limit $N \rightarrow \infty$ we obtain

$$\sum_{\tilde{\pi}} 2^{p(\tilde{\pi})} z^{|\tilde{\pi}|} = \prod_{i=1}^{\infty} \frac{(1+z^i)^i}{(1-z^i)^i} \tag{3.8.28}$$

where the sum is over strict plane partitions $\tilde{\pi}$ of completely arbitrary dimension. The generating function (3.8.28) first appeared in [35]. It was used independently in [87] to study the shifted Schur process, extending the earlier work of [74].

3.9 Conclusion

In this chapter we studied the Bethe eigenvectors of the phase and i-boson models. Both models admit a representation on the vector space $\mathcal{V}$, and we showed that the basis elements of $\mathcal{V}$ can be mapped quite naturally to partitions in $\mathcal{F}_{\psi}^{(0)}, \mathcal{F}_{\phi}^{(0)}$. Our main observation was that under these maps, the Bethe eigenvectors of these models lie in the orbit of the vacuum under GL_{∞} and O_{∞}, respectively. This proved that the corresponding scalar products are power-sum specializations of KP and BKP τ-functions, respectively.

The other key results in this chapter are the lemmas 6 and 12, which pertain to the infinite lattice limit of the phase and i-boson models. We found that when $M \rightarrow \infty$, the action of a B-operator on a general state maps to the action of a charged/neutral half-vertex operator on the image state. Owing to the elementary commutations between the half-vertex operators, we could easily evaluate the scalar products of these models in the $M \rightarrow \infty$ limit. Hence we obtained new proofs of the generating functions for ordinary and strict plane partitions.

This chapter raises several questions which might lead to interesting research in the future. We list two such questions below.

1. *Why do the Bethe eigenvectors of the phase and i-boson models lead to solutions of the KP and BKP hierarchies?* At a superficial level we would expect these two areas of integrability to be unrelated, and yet we have considerable evidence to indicate that this is not the case. Ultimately, we desire a more fundamental explanation for this link between classical and quantum integrable models. Let us also remark that our work complements that of [93], where it was shown that one-point boundary correlation functions of the phase model are τ-functions of the 2-Toda hierarchy [83]. It would be worthwhile to repeat these calculations in the context of the i-boson model.

2. *Is it possible to obtain fermionic proofs of the generating functions for various symmetry classes of plane partitions?* There are many different symmetry classes of plane partitions, whose enumerations are in factorized form.[15] It is possible that the techniques of this chapter could be extended to constructing these restricted plane partitions, and to calculating their generating functions.

[15] See, for example, chapter 6 of [12].

Chapter 4

q-boson model and Hall-Littlewood plane partitions

4.0 Introduction

In chapter 3 we studied the $q \to \infty$ and $q \to i$ limits of the q-boson model. In the $q \to \infty$ case we reviewed the work of [7], before showing that the Bethe eigenvectors map to elements of $\mathcal{F}_\psi$ which satisfy the charged fermionic bilinear identity. We obtained an analogous result in the context of the $q \to i$ limit, where the Bethe eigenvectors map to elements of $\mathcal{F}_\phi$ which satisfy the neutral fermionic bilinear identity. The aim of this chapter is to provide a fermionic description of the q-boson model itself.

The q-boson model was introduced in [8] by applying the Primakov-Holstein transformation to the L-matrix of the spin-$\frac{1}{2}$ XXZ model. The bosons which appear in this model generate a q-deformed Heisenberg algebra, which has several different representations on the vector space discussed in section 3.1. In section 4.1 we discuss one such representation on $\mathcal{V}$ [86], give the q-boson model Hamiltonian $\mathcal{H}$, and construct its eigenvectors using the algebraic Bethe Ansatz.

After this we will find it necessary to introduce a set of t-fermions, where t is a deformation parameter,[1] which generalize the charged fermions of chapter 1. These fermions originally appeared in the papers [51] and [52] by N Jing, where they were used in the context of Hall-Littlewood functions, and their connection with the q-boson model was proposed by P Sulkowski in [82]. In section 4.2 we study the algebra generated by the t-fermions, deriving several useful identities. We define a representation of this algebra on the t-deformed Fock space $\mathcal{F}_\psi(t)$, and calculate inner products between the partition elements of $\mathcal{F}_\psi(t)$. We also state a t-deformed version of the KP half-vertex operator, which is used later in the chapter.

An explicit expression for the q-boson model Bethe eigenvectors was found by

[1] Throughout this chapter the parameters q and t play the same role, and they are related via the equation $t = q^{-2}$. Generally we will work in terms of t, which is the parameter used in the study of Hall-Littlewood functions.

N Tsilevich in [86]. Tsilevich extended the earlier work of Bogoliubov, writing the Bethe eigenvectors as sums over partitions which are weighted by Hall-Littlewood functions. In section 4.3 we reproduce this result and transfer it to the language of the t-deformed fermions, as was suggested in [82]. We map the basis vectors of $\mathcal{V}$ to partitions in the Fock space $\mathcal{F}_\psi(t)$ and calculate the image of the Bethe eigenvectors under this map.

The remainder of the chapter consists of original work. In section 4.4 we extend the results of chapter 3, by showing that the q-boson model scalar product is a generating function for plane partitions inside a box of size $N \times N \times M$. Within this generating function, each plane partition is assigned a weight which depends on the deformation parameter t. All the weights collapse to 1 in the $t \to 0$ $(q \to \infty)$ limit, giving rise to Bogoliubov's generating function as discussed in section 3.3. In the $t \to -1$ $(q \to i)$ limit, all weights assigned to non-strict plane partitions collapse to 0, giving rise to the generating function discussed in section 3.7. The t-weighted generating function for plane partitions was originally found by M Vuletić in [88], using purely combinatorial arguments.

The most interesting result of the chapter is given in section 4.5, which considers the $M \to \infty$ limit of the q-boson model. In this limit, we prove that the action of a B-operator on an arbitrary vector in $\mathcal{V}$ is equivalent to the action of a t-deformed half-vertex operator on the image state in $\mathcal{F}_\psi(t)$.[2] This is an extension of the results obtained in chapter 3, and allows the infinite lattice scalar product to be easily evaluated. The t-deformed half-vertex operators have simple commutation relations, meaning that the scalar product once again factorizes into product form. We thus obtain a fermionic proof of Vuletić's generating function, first proposed in [36].

4.1 q-boson model

In this section we gather together a number of preliminary results pertaining to the q-boson model. We mainly follow [10] and [86]. The sections 3.1 and 3.5 from the previous chapter can be viewed as specializations of the material presented here.

4.1.1 Space of states $\mathcal{V}$ and inner product $\mathcal{I}_t$

Like the phase model in the previous chapter, the q-boson model consists of a lattice of $M+1$ sites which can each be occupied by an unlimited number of particles. For this reason, its space of states $\mathcal{V}$ is the same as that defined in subsection 3.1.1. For any two basis vectors $|m\rangle = |m_0\rangle_0 \otimes \cdots \otimes |m_M\rangle_M, |n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ we define a bilinear inner product $\mathcal{I}_t$ given by

[2] This connection was also noticed in [82], by identifying the basis vectors of $\mathcal{V}$ and partitions in $\mathcal{F}_\psi(t)$ with Hall-Littlewood functions. Our derivation is distinct from that of [82], in that it relies solely on the calculus of the t-deformed fermions.

$$\mathcal{I}_t\Big(|m\rangle, |n\rangle\Big) = \frac{[m_0]_t!}{\prod_{i=1}^{M}[m_i]_t!} \prod_{i=0}^{M} \delta_{m_i,n_i} \tag{4.1.1}$$

where we have adopted the notations

$$[n]_t! = \prod_{i=1}^{n}(1-t^i) \quad \text{for all } n \geq 1, \quad \text{and} \quad [0]_t! = 1 \tag{4.1.2}$$

The inner product between more general states of $\mathcal{V}$ is deduced using bilinearity. Here $t = q^{-2}$ is the deformation parameter characteristic of the model. By setting $t = 0$ ($q \rightarrow \infty$) we recover the inner product (3.1.2) used for the phase model. Also, letting the basis vectors in (4.1.1) be elements of $\tilde{\mathcal{V}}$ and setting $t = -1$ ($q = i$), we recover the inner product (3.5.2) used for the i-boson model.

The dual space of states $\mathcal{V}^*$ acts according to

$$\langle m|n\rangle = \mathcal{I}_t\Big(|m\rangle, |n\rangle\Big) \tag{4.1.3}$$

for all $\langle m| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M \in \mathcal{V}^*$ and $|n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M \in \mathcal{V}$.

4.1.2 q-boson algebra

The q-boson algebra is generated by $\{b, b^\dagger, \mathcal{N}\}$ which satisfy the commutation relations

$$[b, b^\dagger] = q^{-2\mathcal{N}} = t^{\mathcal{N}}, \quad [\mathcal{N}, b] = -b, \quad [\mathcal{N}, b^\dagger] = b^\dagger \tag{4.1.4}$$

where we have again identified $q^{-2} = t$, and now retain this definition throughout the entire chapter without further comment. This algebra collapses to the phase algebra (3.1.5) in the limit $q \rightarrow \infty$, and to the i-boson algebra (3.5.6) in the limit $q \rightarrow i$. As in the previous chapter, we consider $M+1$ copies of the q-boson algebra, generated by $\{b_0, b_0^\dagger, \mathcal{N}_0\}$ through to $\{b_M, b_M^\dagger, \mathcal{N}_M\}$. Adopting the labelling system of chapter 2, we denote these algebras by $\mathcal{A}_0, \ldots, \mathcal{A}_M$ with $\mathfrak{a}_i^+ = b_i^\dagger$, $\mathfrak{a}_i^- = b_i$, $\mathfrak{a}_i^0 = \mathcal{N}_i$. As usual different copies are commuting, giving rise to the equations

$$[b_i, b_j^\dagger] = \delta_{i,j} t^{\mathcal{N}_i}, \quad [\mathcal{N}_i, b_j] = -\delta_{i,j} b_i, \quad [\mathcal{N}_i, b_j^\dagger] = \delta_{i,j} b_i^\dagger \tag{4.1.5}$$

for all $0 \leq i, j \leq M$.

4.1.3 Representations of q-boson algebras

In this subsection, following [86], we fix representations of the q-boson algebras on the vector space $\mathcal{V}$. Quite generally speaking, b_i plays the role of an annihilation operator, removing particles from the i^{th} lattice site. Conversely, $b_i^\dagger$ is a creation operator, adding particles to the i^{th} lattice site. In order to represent the q-boson algebras correctly, these operators must also produce accompanying factors that depend on t. It transpires that there is a certain amount of freedom in choosing these factors. We will make one choice for the representation of $\mathcal{A}_1, \ldots, \mathcal{A}_M$ and a different choice for the representation of $\mathcal{A}_0$, as we describe below.

Firstly we consider the algebras $\mathcal{A}_1, \ldots, \mathcal{A}_M$. For all $1 \leq i \leq M$, the operator b_i has the action

$$b_i |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \begin{cases} 0, & n_i = 0 \\ (1-t)^{-\frac{1}{2}} |n_0\rangle_0 \otimes \cdots \otimes |n_i - 1\rangle_i \otimes \cdots \otimes |n_M\rangle_M, & n_i \geq 1 \end{cases} \tag{4.1.6}$$

while for all $1 \leq i \leq M$ the operator $b_i^\dagger$ has the action

$$b_i^\dagger |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1 - t^{n_i+1}}{(1-t)^{\frac{1}{2}}} |n_0\rangle_0 \otimes \cdots \otimes |n_i + 1\rangle_i \otimes \cdots \otimes |n_M\rangle_M \tag{4.1.7}$$

Secondly we consider the algebra $\mathcal{A}_0$, for which the representation is slightly different. The operator b_0 has the action

$$b_0 |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = \frac{1 - t^{n_0}}{(1-t)^{\frac{1}{2}}} |n_0 - 1\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M \tag{4.1.8}$$

while the operator $b_0^\dagger$ has the action

$$b_0^\dagger |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = (1-t)^{-\frac{1}{2}} |n_0 + 1\rangle_0 \otimes |n_1\rangle_1 \otimes \cdots \otimes |n_M\rangle_M \tag{4.1.9}$$

For all $0 \leq i \leq M$ the action of $\mathcal{N}_i$ is given by

$$\mathcal{N}_i |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M = n_i |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M \tag{4.1.10}$$

The set of definitions (4.1.6)–(4.1.10) faithfully represent the algebras $\mathcal{A}_0, \ldots, \mathcal{A}_M$. The representations of the phase algebras in subsection 3.1.3 are obtained by setting $t = 0$. Similarly, the representations of the i-boson algebras in subsection 3.5.3 are obtained by setting $t = -1$.

By virtue of the definitions (4.1.1) and (4.1.6)–(4.1.9), we notice that for all $|m\rangle, |n\rangle \in \mathcal{V}$ and $0 \leq i \leq M$ the operators $b_i, b_i^\dagger$ satisfy the equation

$$\mathcal{I}_t\Big(b_i|m\rangle, |n\rangle\Big) = \mathcal{I}_t\Big(|m\rangle, b_i^\dagger|n\rangle\Big) \tag{4.1.11}$$

meaning that they are adjoint. Once again, the operator $\mathcal{N}_i$ is self-adjoint. Hence we can immediately deduce actions for $\{b_i, b_i^\dagger, \mathcal{N}_i\}$ on the dual space $\mathcal{V}^*$, as follows. For all $1 \leq i \leq M$, the operator $b_i^\dagger$ has the action

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M b_i^\dagger = \left\{ \begin{array}{ll} 0, & n_i = 0 \\ \\ (1-t)^{-\frac{1}{2}} \langle n_0|_0 \otimes \cdots \otimes \langle n_i - 1|_i \otimes \cdots \otimes \langle n_M|_M, & n_i \geq 1 \end{array} \right. \tag{4.1.12}$$

while for all $1 \leq i \leq M$ the operator b_i has the action

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M b_i = \frac{1-t^{n_i+1}}{(1-t)^{\frac{1}{2}}} \langle n_0|_0 \otimes \cdots \otimes \langle n_i + 1|_i \otimes \cdots \otimes \langle n_M|_M \tag{4.1.13}$$

As before, the algebra $\mathcal{A}_0$ is prescribed its own representation. The operator $b_0^\dagger$ has the action

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M b_0^\dagger = \frac{1-t^{n_0}}{(1-t)^{\frac{1}{2}}} \langle n_0 - 1|_0 \otimes \langle n_1|_1 \otimes \cdots \otimes \langle n_M|_M \tag{4.1.14}$$

while the operator b_0 has the action

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M b_0 = (1-t)^{-\frac{1}{2}} \langle n_0 + 1|_0 \otimes \langle n_1|_1 \otimes \cdots \otimes \langle n_M|_M \tag{4.1.15}$$

For all $0 \leq i \leq M$ the action of $\mathcal{N}_i$ is given by

$$\langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \mathcal{N}_i = n_i \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M \tag{4.1.16}$$

The set of definitions (4.1.12)–(4.1.16) provide the dual representation of the algebras $\mathcal{A}_0, \ldots, \mathcal{A}_M$.

4.1.4 Hamiltonian $\mathcal{H}$

The Hamiltonian of the q-boson model is given by

$$\mathcal{H} = -\frac{1}{2} \sum_{i=0}^{M} \Big(b_i^\dagger b_{i+1} + b_i b_{i+1}^\dagger \Big) + \bar{\mathcal{N}} \tag{4.1.17}$$

where the periodicity $b_{M+1} = b_0$ and $b^{\dagger}_{M+1} = b^{\dagger}_0$ is imposed. The Hamiltonians (3.1.20) and (3.5.19) are trivially recovered by taking the limits $t \to 0$ and $t \to -1$. It follows that the eigenvectors of these earlier Hamiltonians can be recovered by constructing the eigenvectors of (4.1.17) directly, as we do in the remainder of this section.

4.1.5 L-matrix and local intertwining equation

The R-matrix for the q-boson model depends on two indeterminates x, y and acts in the tensor product $\mathcal{V}_a \otimes \mathcal{V}_b$, where $\mathcal{V}_a, \mathcal{V}_b$ are copies of $\mathbb{C}^2$. It is given by

$$R_{ab}(x,y,t) = \left(\begin{array}{cccc} x - ty & 0 & 0 & 0 \\ 0 & t(x-y) & (1-t)x^{\frac{1}{2}}y^{\frac{1}{2}} & 0 \\ 0 & (1-t)x^{\frac{1}{2}}y^{\frac{1}{2}} & x-y & 0 \\ 0 & 0 & 0 & x-ty \end{array} \right)_{ab} \tag{4.1.18}$$

and corresponds to choosing $a_{\pm}(x,y) = x - ty$, $b_{+}(x,y) = t(x-y)$, $b_{-}(x,y) = x-y$, $c_{\pm}(x,y) = (1-t)x^{\frac{1}{2}}y^{\frac{1}{2}}$ in (2.2.1). The L-matrix for the q-boson model depends on a single indeterminate x, and acts in the space $\mathcal{V}_a$. Its entries are operators acting at the m^{th} lattice site, and identically everywhere else. It has the form

$$L_{am}(x,t) = \left(\begin{array}{cc} x^{-\frac{1}{2}} & (1-t)^{\frac{1}{2}} b^{\dagger}_m \\ (1-t)^{\frac{1}{2}} b_m & x^{\frac{1}{2}} \end{array} \right)_a \tag{4.1.19}$$

Notice that both the R-matrix (4.1.18) and L-matrix (4.1.19) collapse to their phase model counterpart in the limit $t \to 0$, and to their i-boson model counterpart in the limit $t \to -1$. Returning to the parent model, the local intertwining equation has the usual form

$$R_{ab}(x,y,t) L_{am}(x,t) L_{bm}(y,t) = L_{bm}(y,t) L_{am}(x,t) R_{ab}(x,y,t) \tag{4.1.20}$$

This is a 4×4 matrix equation, which gives rise to sixteen scalar identities. Each of these identities may be verified by direct calculation, and by using the commutation relations (4.1.5) where appropriate.

4.1.6 Monodromy matrix and global intertwining equation

The monodromy matrix is an $(M+1)$-fold product of the L-matrices (4.1.19), taken in the auxiliary space $\mathrm{End}(\mathcal{V}_a)$. It has the form

$$T_a(x,t) = L_{aM}(x,t) \dots L_{a0}(x,t) = \left(\begin{array}{cc} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{array} \right)_a \tag{4.1.21}$$

where $A(x,t), B(x,t), C(x,t), D(x,t)$ are elements of $\mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_M$. The monodromy matrix satisfies the global intertwining equation

$$R_{ab}(x,y,t)T_a(x,t)T_b(y,t) = T_b(y,t)T_a(x,t)R_{ab}(x,y,t) \tag{4.1.22}$$

the proof of which is immediate from the local intertwining relation (4.1.20). The equation (4.1.22) contains sixteen commutation relations between the monodromy matrix operators $A(x,t), B(x,t), C(x,t), D(x,t)$, but for our purposes we will only require two. These are the equations

$$[B(x,t), B(y,t)] = [C(x,t), C(y,t)] = 0 \tag{4.1.23}$$

and they are necessary to show that the Bethe eigenvectors are symmetric in their rapidity variables.

4.1.7 Recovering $\mathcal{H}$ from the transfer matrix

Let $\mathrm{tr}_a T_a = A(x,t) + D(x,t)$ be the transfer matrix of the q-boson model. The Hamiltonian (4.1.17) may be recovered via the equation

$$\mathcal{H} = \frac{1}{2(1-t)}\left[x^2 \frac{d}{dx}\left(x^{-(M+1)/2}\mathrm{tr}_a T_a\right)\right]_{x\rightarrow\infty} - \frac{1}{2(1-t)}\left[\frac{d}{dx}\left(x^{(M+1)/2}\mathrm{tr}_a T_a\right)\right]_{x\rightarrow 0} + \bar{\mathcal{N}} \tag{4.1.24}$$

from which it follows that $[\mathcal{H}, \mathrm{tr}_a T_a] = 0$. Hence the eigenvectors of $\mathcal{H}$ may be found by studying the eigenvectors of the transfer matrix.

4.1.8 Bethe Ansatz for the eigenvectors

The eigenvectors of the transfer matrix $\mathrm{tr}_a T_a$ are obtained via the Ansatz

$$|\Psi\rangle = B(y_1,t)\ldots B(y_N,t)|0\rangle, \quad \langle\Psi| = \langle 0|C(y_N,t)\ldots C(y_1,t) \tag{4.1.25}$$

where the variables $\{y_1,\ldots,y_N\}$ are assumed to obey the Bethe equations (2.3.6). For the present model, $a(y_i,y_j) = y_i - ty_j$, $\alpha(y_i) = y_i^{-(M+1)/2}$, $\delta(y_i) = y_i^{(M+1)/2}$. Substituting these expressions into (2.3.6), the Bethe equations for the q-boson model read

$$y_i^{M+1} = \prod_{\substack{j\neq i\\ j=1}}^{N} \frac{ty_i - y_j}{y_i - ty_j} \tag{4.1.26}$$

for all $1 \leq i \leq N$. As was the case in the previous chapter, we now proceed towards an explicit evaluation of the eigenvectors (4.1.25) *without* assuming the Bethe equations (4.1.26).

4.2 Charged t-fermions and related definitions

Before we begin our calculation of the Bethe eigenvectors, we digress briefly to discuss a t-deformed species of fermions and its corresponding Fock space $\mathcal{F}_\psi(t)$. These fermions were introduced in [51], and appeared in a slightly modified form in [52]. The material of this section is mostly taken from [36], and will be necessary when we map the Bethe eigenvectors to elements of $\mathcal{F}_\psi(t)$.

4.2.1 Charged t-fermions

Following [52], consider two infinite sets $\{\psi_m(t)\}_{m\in\mathbb{Z}}$ and $\{\psi_m^*(t)\}_{m\in\mathbb{Z}}$, where m runs over all integers. The elements in these sets are called *charged t-fermions* and they satisfy the anticommutation relations

$$\psi_m\psi_n + \psi_n\psi_m = t\psi_{(m+1)}\psi_{(n-1)} + t\psi_{(n+1)}\psi_{(m-1)} \tag{4.2.1}$$
$$\psi_m^*\psi_n^* + \psi_n^*\psi_m^* = t\psi_{(m-1)}^*\psi_{(n+1)}^* + t\psi_{(n-1)}^*\psi_{(m+1)}^* \tag{4.2.2}$$
$$\psi_m\psi_n^* + \psi_n^*\psi_m = t\psi_{(m-1)}\psi_{(n-1)}^* + t\psi_{(n+1)}^*\psi_{(m+1)} + (1-t)^2\delta_{m,n} \tag{4.2.3}$$

for all $m, n \in \mathbb{Z}$.[3] In these equations $t \in \mathbb{C}$ plays the role of a deformation parameter, and for simplicity we will always assume that $|t| < 1$. The charged free fermions of section 1.1 are recovered as the $t = 0$ specialization of equations (4.2.1)–(4.2.3).

4.2.2 Clifford algebra $Cl_\psi(t)$ and identities

The Clifford algebra $Cl_\psi(t)$ is the associative algebra generated by 1 and the charged t-fermions $\{\psi_m(t)\}_{m\in\mathbb{Z}}$ and $\{\psi_m^*(t)\}_{m\in\mathbb{Z}}$, modulo the equations (4.2.1)–(4.2.3). For the purposes of calculation, the t-deformed anticommutation relations can be rather cumbersome. For this reason, we will prove several identities which make the algebra $Cl_\psi(t)$ easier to handle.[4]

Lemma 1. For all $m, n \in \mathbb{Z}$, we have

$$\psi_m^*\psi_n = t\psi_{(n-1)}\psi_{(m-1)}^* + (t^2-1)\sum_{i=0}^{\infty}\psi_{(n+i)}\psi_{(m+i)}^* t^i + (1-t)\delta_{m,n} \tag{4.2.4}$$

Proof. Rearranging the anticommutation relation (4.2.3), we have

$$\psi_m^*\psi_n = t\psi_{(n-1)}\psi_{(m-1)}^* - \psi_n\psi_m^* + t\psi_{(m+1)}^*\psi_{(n+1)} + (1-t)^2\delta_{m,n} \tag{4.2.5}$$

[3] To reduce notational complexity, we will often abbreviate $\psi_m(t) = \psi_m, \psi_m^*(t) = \psi_m^*$ throughout this chapter. This notation is not to be confused with the charged fermions of chapter 1, and the reader should assume that all fermions appearing in this chapter obey the deformed anticommutation relations (4.2.1)–(4.2.3).

[4] These identities were originally proved in [36].

and repeating this rearrangement to replace the term $t\psi^*_{(m+1)}\psi_{(n+1)}$ in (4.2.5), we recover

$$\begin{aligned}\psi^*_m\psi_n = t\psi_{(n-1)}\psi^*_{(m-1)} + (t^2-1)\psi_n\psi^*_m - t\psi_{(n+1)}\psi^*_{(m+1)} + t^2\psi^*_{(m+2)}\psi_{(n+2)}\\ + (1-t)^2(1+t)\delta_{m,n} \qquad (4.2.6)\end{aligned}$$

Iterating this substitution procedure infinitely, we arrive at the equation

$$\psi^*_m\psi_n = t\psi_{(n-1)}\psi^*_{(m-1)} + (t^2-1)\sum_{i=0}^{\infty}\psi_{(n+i)}\psi^*_{(m+i)}t^i + (1-t)^2\sum_{i=0}^{\infty}t^i\delta_{m,n} \qquad (4.2.7)$$

and the proof is achieved by the geometric series identity $\sum_{i=0}^{\infty} t^i = \frac{1}{1-t}$. □

Lemma 2. For arbitrary $m, n \in \mathbb{Z}$, we have

$$\psi_m\psi^*_n = t\psi^*_{(n+1)}\psi_{(m+1)} + (t^2-1)\sum_{i=0}^{\infty}\psi^*_{(n-i)}\psi_{(m-i)}t^i + (1-t)\delta_{m,n} \qquad (4.2.8)$$

Proof. Analogous to the proof of lemma 1. □

Lemma 3. For arbitrary $m \in \mathbb{Z}$ and $n \geq 0$, we propose the identity

$$\psi_{(m-n)}\psi_m + (1-t)\sum_{i=1}^{n}\psi_{(m-n+i)}\psi_{(m-i)} = t\psi_{(m+1)}\psi_{(m-n-1)} \qquad (4.2.9)$$

Proof. Let $\mathcal{P}_n$ denote the proposition (4.2.9). Using the anticommutation relation (4.2.1) we obtain the equations

$$\psi_m\psi_m = t\psi_{(m+1)}\psi_{(m-1)} \qquad (4.2.10)$$

$$\psi_{(m-1)}\psi_m + (1-t)\psi_m\psi_{(m-1)} = t\psi_{(m+1)}\psi_{(m-2)} \qquad (4.2.11)$$

which prove that $\mathcal{P}_0$ and $\mathcal{P}_1$ are true. For $n \geq 2$, we rearrange the left hand side of the proposition $\mathcal{P}_n$ to give

$$\begin{aligned}&\psi_{(m-n)}\psi_m + (1-t)\sum_{i=1}^{n}\psi_{(m-n+i)}\psi_{(m-i)} \qquad (4.2.12)\\ =&\psi_{(m-n)}\psi_m - t\psi_{(m-n+1)}\psi_{(m-1)} + (1-t)\psi_m\psi_{(m-n)}\\ &+\left(\psi_{(m-n+1)}\psi_{(m-1)} + (1-t)\sum_{i=1}^{n-2}\psi_{(m-n+1+i)}\psi_{(m-1-i)}\right)\end{aligned}$$

Assuming that the proposition $\mathcal{P}_{n-2}$ is true, the parenthesized term in (4.2.12) is equal to $t\psi_m\psi_{(m-n)}$, and we recover

$$\begin{aligned}&\psi_{(m-n)}\psi_m + (1-t)\sum_{i=1}^{n}\psi_{(m-n+i)}\psi_{(m-i)} \\ =&\psi_{(m-n)}\psi_m - t\psi_{(m-n+1)}\psi_{(m-1)} + \psi_m\psi_{(m-n)}\end{aligned} \tag{4.2.13}$$

Finally, applying the anticommutation relation (4.2.1) to the right hand side of (4.2.13), we prove that $\mathcal{P}_n$ is true. Therefore $\mathcal{P}_{n-2}$ true $\implies$ $\mathcal{P}_n$ true, and by induction the proposition (4.2.9) holds for all $n \geq 0$. □

4.2.3 Fock representations of $Cl_\psi(t)$

As we did in section 1.1, we introduce a vacuum vector $|0\rangle$ and dual vacuum vector $\langle 0|$. We define actions of $Cl_\psi(t)$ on these vacuum states by setting

$$\psi_m(t)|0\rangle = \psi_n^*(t)|0\rangle = 0, \quad \langle 0|\psi_m^*(t) = \langle 0|\psi_n(t) = 0 \tag{4.2.14}$$

for all integers $m < 0, n \geq 0$. The t-deformed Fock space $\mathcal{F}_\psi(t)$ and its dual $\mathcal{F}_\psi^*(t)$ are the vector spaces generated linearly by the action of $Cl_\psi(t)$ on $|0\rangle$ and $\langle 0|$, respectively.

Lemma 4. For all $l \geq 1$ we define the charged vacuum states

$$|-l\rangle = \psi_{-l}^*(t)\dots\psi_{-1}^*(t)|0\rangle, \quad \langle -l| = \langle 0|\psi_{-1}(t)\dots\psi_{-l}(t) \tag{4.2.15}$$

and propose the identities

$$\psi_m(t)|-l\rangle = \begin{cases} 0, & m < -l \\ |-l+1\rangle, & m = -l \end{cases} \quad \text{and} \quad \langle -l|\psi_m^*(t) = \begin{cases} 0, & m < -l \\ \langle -l+1|, & m = -l \end{cases} \tag{4.2.16}$$

Proof. We prove only the first of the propositions in (4.2.16), as the proof of the second is completely analogous. Let us denote this first proposition by $\mathcal{P}_l$. In the case $m < -1$ we can use the identity (4.2.8) and the annihilation properties (4.2.14) to show that $\psi_m|-1\rangle = \psi_m\psi_{-1}^*|0\rangle = 0$. Furthermore, when $m = -1$ we use the anticommutation relation (4.2.3) to obtain

$$\begin{aligned}\psi_{-1}\psi_{-1}^*|0\rangle &= \Big\{(1-t)^2 + t\psi_{-2}\psi_{-2}^* + t\psi_0^*\psi_0\Big\}|0\rangle \\ &= \Big\{(1+t)(1-t)^2 + t\psi_{-2}\psi_{-2}^* + t^2\psi_{-1}\psi_{-1}^* + t^2\psi_1^*\psi_1\Big\}|0\rangle\end{aligned} \tag{4.2.17}$$

and by application of (4.2.8) we have $\psi_{-2}\psi^*_{-2}|0\rangle = (1-t)|0\rangle$, while by (4.2.4) we see that $\psi^*_1\psi_1|0\rangle = (1-t)|0\rangle$. Substituting these results into (4.2.17), we obtain

$$\psi_{-1}\psi^*_{-1}|0\rangle = t^2\psi_{-1}\psi^*_{-1}|0\rangle + (1-t^2)|0\rangle \tag{4.2.18}$$

and therefore $\psi_{-1}|-1\rangle = |0\rangle$. Hence we have shown that $\mathcal{P}_1$ is true. Now assume that $\mathcal{P}_l$ is true for some $l \geq 1$. In the case $m < -(l+1)$ we use the identity (4.2.8) to write

$$\begin{aligned}\psi_m|-l-1\rangle &= \psi_m\psi^*_{-(l+1)}|-l\rangle \qquad (4.2.19)\\ &= \Big\{t\psi^*_{-l}\psi_{(m+1)} + (t^2-1)\sum_{i=0}^{\infty}\psi^*_{-(i+l+1)}\psi_{(m-i)}t^i\Big\}|-l\rangle = 0\end{aligned}$$

where every term on the right hand side vanishes, because $\mathcal{P}_l$ holds. In the case $m = -(l+1)$ we again use identity (4.2.8) to write

$$\begin{aligned}\psi_{-(l+1)}\psi^*_{-(l+1)}|-l\rangle &= \Big\{t\psi^*_{-l}\psi_{-l} + (t^2-1)\sum_{i=0}^{\infty}\psi^*_{-(i+l+1)}\psi_{-(i+l+1)}t^i + (1-t)\Big\}|-l\rangle\\ &= |-l\rangle \qquad (4.2.20)\end{aligned}$$

where, again, the final equality holds because $\mathcal{P}_l$ is true. This establishes the identity $\psi_{-(l+1)}|-l-1\rangle = |-l\rangle$. Hence $\mathcal{P}_l$ true $\implies$ $\mathcal{P}_{l+1}$ true, and the proof of (4.2.16) is complete by induction. □

4.2.4 Partitions

In direct analogy with section 1.1, we now identify elements of the deformed Fock spaces $\mathcal{F}_\psi(t)$ and $\mathcal{F}^*_\psi(t)$ with partitions. The correspondence is essentially the same as equation (1.1.23), except that all charged fermions are replaced with their t-deformed counterparts. Explicitly, we write

$$\psi_{m_1}(t)\dots\psi_{m_l}(t)|-l\rangle = |\mu_1,\dots,\mu_l), \quad \langle -l|\psi^*_{m_l}(t)\dots\psi^*_{m_1}(t) = (\mu_1,\dots,\mu_l| \tag{4.2.21}$$

where $\mu_i = m_i + i$ for all $1 \leq i \leq l$. The following result is an extension of (1.1.33), and evaluates the form $\langle,\rangle$ between t-deformed partitions.

Lemma 5. Let μ be an arbitrary partition having $p_i(\mu) \geq 0$ parts of size i, for all $i \geq 1$. We associate to this partition a function $b_\mu(t)$, defined as

$$b_\mu(t) = \prod_{i=1}^{\infty} \left(\prod_{j=1}^{p_i(\mu)} (1-t^j) \right) = \prod_{i=1}^{\infty} [p_i(\mu)]_t! \tag{4.2.22}$$

This product is actually finite, since there must exist an $I \geq 1$ such that $p_i(\mu) = 0$ for all $i > I$. Let $(\mu| \in \mathcal{F}^*_\psi(t)$ and $|\nu) \in \mathcal{F}_\psi(t)$ be two arbitrary partitions, given by

$$(\mu| = \langle -l|\psi^*_{m_l}(t)\dots\psi^*_{m_1}(t), \quad |\nu) = \psi_{n_1}(t)\dots\psi_{n_k}(t)|-k\rangle \tag{4.2.23}$$

We claim that

$$\Big\langle (\mu|, |\nu) \Big\rangle = \langle -l|\psi^*_{m_l}(t)\dots\psi^*_{m_1}(t)\psi_{n_1}(t)\dots\psi_{n_k}(t)|-k\rangle = b_\mu(t)\delta_{\mu,\nu} \tag{4.2.24}$$

where $\delta_{\mu,\nu} = 1$ if $\mu = \nu$, and $\delta_{\mu,\nu} = 0$ if μ and ν are different.

Proof. We take the proof from [36]. Using the identity (4.2.4) and the annihilation properties (4.2.14) of the charged t-fermions, it follows that $\langle (\mu|, |\nu)\rangle = 0$ if $m_1 \neq n_1$. In the case when $m_1 = n_1$, we assume that $\{m_1, \dots, m_s\}$ are nearest neighbours for some $1 \leq s \leq l$.[5] That is, we fix

$$m_i = m_{(i+1)} + 1, \text{ for all } 1 \leq i \leq s-1 \tag{4.2.25}$$

but take $m_s > m_{(s+1)} + 1$. Commuting the central pair of t-fermions $\psi^*_{m_1}\psi_{m_1}$ using (4.2.4) and annihilating terms with (4.2.14), we obtain

$$\begin{aligned} \Big\langle (\mu|, |\nu) \Big\rangle &= (1-t)\langle -l|\psi^*_{m_l}\dots\psi^*_{m_2}\psi_{n_2}\dots\psi_{n_k}|-k\rangle \\ &\quad + t\langle -l|\psi^*_{m_l}\dots\psi^*_{m_2}\psi_{m_2}\psi^*_{m_2}\psi_{n_2}\dots\psi_{n_k}|-k\rangle \end{aligned} \tag{4.2.26}$$

where we have recalled that $m_2 = m_1 - 1$. Iterating this calculation on the second term on the right hand side of (4.2.26), we ultimately find

$$\Big\langle (\mu|, |\nu) \Big\rangle = (1-t^s)\langle -l|\psi^*_{m_l}\dots\psi^*_{m_2}\psi_{n_2}\dots\psi_{n_k}|-k\rangle \tag{4.2.27}$$

and we have reduced, by two, the number of fermions appearing in the expectation value. Repeating this overall procedure, we find that $\langle (\mu|, |\nu)\rangle = 0$ if $m_i \neq n_i$ for any $1 \leq i \leq s$. In the case when $m_i = n_i$ for all $1 \leq i \leq s$, we obtain

[5] This is equivalent to assuming that the first s parts of μ have the same size.

$$\Big\langle (\mu|, |\nu) \Big\rangle = \prod_{i=1}^{s} (1-t^i) \langle -l | \psi^*_{m_l} \dots \psi^*_{m_{(s+1)}} \psi_{n_{(s+1)}} \dots \psi_{n_k} | -k \rangle \tag{4.2.28}$$

and we have acquired a factor of $\prod_{i=1}^{s}(1-t^i)$ when the first s parts of μ have the same size, as desired. Finally we can see that $\langle (\mu|, |\nu) \rangle = 0$ unless $l = k$ and $m_i = n_i$ for all $1 \le i \le l$, or equivalently, $\mu = \nu$. If this condition holds, we clearly have $\langle (\mu|, |\nu) \rangle = b_\mu(t)$.

□

4.2.5 t-deformed Heisenberg algebra

In this subsection, once again following [52], we give t-deformed analogues of the operators (1.1.39). We define

$$H_m(t) = \frac{1}{1-t} \sum_{i \in \mathbb{Z}} \psi_i(t) \psi^*_{(i+m)}(t) \tag{4.2.29}$$

for all integers $m > 0$, and

$$H_m(t) = \frac{1}{(1-t)(1-t^{|m|})} \sum_{i \in \mathbb{Z}} \psi_i(t) \psi^*_{(i+m)}(t) \tag{4.2.30}$$

for all integers $m < 0$. Using the t-anticommutation relations (4.2.1)–(4.2.3), it is possible to show that these operators obey the commutation relation[6]

$$[H_m(t), H_n(t)] = \frac{m}{1-t^{|m|}} \delta_{m+n,0} \tag{4.2.31}$$

for all $m, n \in \mathbb{Z}^\times$. Extending the identities (1.1.41), we also have the commutation relations

$$[H_m(t), \psi_n(t)] = \psi_{n-m}(t), \quad [H_m(t), \psi^*_n(t)] = -\psi^*_{m+n}(t) \tag{4.2.32}$$

Lastly, we state the annihilation identities

$$H_m(t) | -l \rangle = 0, \quad \langle -l | H_{-m}(t) = 0 \tag{4.2.33}$$

which are true for all $l \ge 0$ and $m \ge 1$. The relations (4.2.33) are proved by using the definitions (4.2.29) and (4.2.30), the annihilation properties (4.2.14), as well as the algebraic relations (4.2.1)–(4.2.3). For brevity, we omit this proof.

[6]See section III of [52] for a detailed proof of (4.2.31).

4.2.6 t-deformed half-vertex operators

We introduce the Hamiltonians

$$H_{\pm}(x,t) = \sum_{n=1}^{\infty} \frac{1-t^n}{n} x^n H_{\pm n}(t) \tag{4.2.34}$$

where x is an indeterminate. We also define t-analogues of the generating functions (1.1.43), given by

$$\Psi(k) = \sum_{i\in\mathbb{Z}} \psi_i(t) k^i, \quad \Psi^*(k) = \sum_{i\in\mathbb{Z}} \psi_i^*(t) k^{-i} \tag{4.2.35}$$

Using these definitions and the commutation relations (4.2.32), we obtain

$$[H_+(x,t), \Psi(k)] = \sum_{n=1}^{\infty}\sum_{i\in\mathbb{Z}} [H_n(t), \psi_i(t)] \frac{1-t^n}{n} x^n k^i = \Psi(k) \sum_{n=1}^{\infty} \frac{1-t^n}{n} (xk)^n \tag{4.2.36}$$

$$[\Psi^*(k), H_-(x,t)] = \sum_{n=1}^{\infty}\sum_{i\in\mathbb{Z}} [\psi_i^*(t), H_{-n}(t)] \frac{1-t^n}{n} x^n k^{-i} = \Psi^*(k) \sum_{n=1}^{\infty} \frac{1-t^n}{n} \left(\frac{x}{k}\right)^n \tag{4.2.37}$$

which, in turn, imply that

$$e^{H_+(x,t)} \Psi(k) = \frac{1-txk}{1-xk} \Psi(k) e^{H_+(x,t)} \tag{4.2.38}$$

$$\Psi^*(k) e^{H_-(x,t)} = \frac{k-tx}{k-x} e^{H_-(x,t)} \Psi^*(k) \tag{4.2.39}$$

where we have used the formal power series identity

$$\sum_{n=1}^{\infty} \frac{1-t^n}{n} z^n = \log\left(\frac{1-tz}{1-z}\right) \tag{4.2.40}$$

Defining $\Gamma_{\pm}(x,t) = e^{H_{\pm}(x,t)}$ and eliminating the generating parameter k from the equations (4.2.38) and (4.2.39), we obtain

$$\Gamma_+(x,t)\psi_i = \left(\psi_i + (1-t)\sum_{n=1}^{\infty} \psi_{(i-n)} x^n\right) \Gamma_+(x,t) \tag{4.2.41}$$

$$\psi_i^* \Gamma_-(x,t) = \Gamma_-(x,t) \left(\psi_i^* + (1-t)\sum_{n=1}^{\infty} \psi_{(i-n)}^* x^n\right) \tag{4.2.42}$$

for all $i \in \mathbb{Z}$. The operators $\Gamma_\pm(x,t)$ are t-generalizations of $\Gamma_\pm(x)$ as defined in subsection 3.4.1. They each constitute one half of the t-fermion vertex operators in [52], and for this reason we call them *t-deformed half-vertex operators.*

Equations (4.2.41) and (4.2.42) play an important role when we consider the q-boson model on an infinite lattice, in section 4.5. Essentially they extend the equations (3.4.8) from the last chapter to arbitrary t values.

4.3 Calculation of Bethe eigenvectors

4.3.1 The maps $\mathcal{M}_\psi(t)$ and $\mathcal{M}^*_\psi(t)$

We begin by defining the maps $\mathcal{M}_\psi(t)$ and $\mathcal{M}^*_\psi(t)$, which take the basis vectors of $\mathcal{V}$ and $\mathcal{V}^*$ to partitions in the deformed Fock spaces $\mathcal{F}_\psi(t)$ and $\mathcal{F}^*_\psi(t)$, respectively. These maps are the natural t-extension of those defined in section 3.2.

Definition 1. Let $|n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ and $\langle n| = \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M$ be basis elements of $\mathcal{V}$ and $\mathcal{V}^*$, respectively, and define

$$\Sigma_0 = \sum_{j=0}^{M} n_j \tag{4.3.1}$$

From this, let $|\nu) = |\nu_1, \ldots, \nu_{\Sigma_0})$ and $(\nu| = (\nu_1, \ldots, \nu_{\Sigma_0}|$ be the partitions in $\mathcal{F}_\psi(t)$ and $\mathcal{F}^*_\psi(t)$ with n_i parts equal to i for all $0 \le i \le M$. That is, we let

$$|\nu) = |M^{n_M}, \ldots, 1^{n_1}, 0^{n_0}) = |M^{n_M}, \ldots, 1^{n_1}) \tag{4.3.2}$$
$$(\nu| = (M^{n_M}, \ldots, 1^{n_1}, 0^{n_0}| = (M^{n_M}, \ldots, 1^{n_1}| \tag{4.3.3}$$

We define linear maps $\mathcal{M}_\psi(t) : \mathcal{V} \to \mathcal{F}_\psi(t)$ and $\mathcal{M}^*_\psi(t) : \mathcal{V}^* \to \mathcal{F}^*_\psi(t)$ whose actions are given by

$$\mathcal{M}_\psi(t)|n\rangle = \frac{1}{b_\nu(t)}|\nu), \quad \langle n|\mathcal{M}^*_\psi(t) = \frac{1}{b_\nu(t)}(\nu| \tag{4.3.4}$$

where $b_\nu(t)$ denotes the factor (4.2.22) assigned to the partition ν. These maps are motivated by the orthogonality relation (4.2.24), from which we see that

$$\Big\langle \langle m|\mathcal{M}^*_\psi(t), \mathcal{M}_\psi(t)|n\rangle \Big\rangle = \frac{\langle (\mu|, |\nu)\rangle}{b_\mu(t) b_\nu(t)} = \frac{\delta_{\mu,\nu}}{b_\mu(t)} = \prod_{i=1}^{M} \frac{\delta_{m_i,n_i}}{[m_i]_t!} \tag{4.3.5}$$

In other words, the maps (4.3.4) preserve the inner product (4.1.1) on all sites $\{1, \ldots, M\}$, but project out all information from the 0^{th} site.

4.3.2 Calculation of $\mathbb{B}(x,t)|n\rangle$

Lemma 6. Define $\mathbb{B}(x,t) = x^{\frac{M}{2}} B(x,t)$ and let $|n\rangle = |n_0\rangle_0 \otimes \cdots \otimes |n_M\rangle_M$ be an arbitrary basis vector of $\mathcal{V}$. The action of $\mathbb{B}(x,t)$ on $|n\rangle$ is given by

$$\mathbb{B}(x,t)|n\rangle = \sum_{|m\rangle \rhd |n\rangle} \prod_{i=1}^{M} x^{i(m_i-n_i)} \Big(1 - \delta_{m_i,n_i+1} t^{m_i}\Big) |m\rangle \tag{4.3.6}$$

where the sum is over all basis vectors $|m\rangle = |m_0\rangle_0 \otimes \cdots \otimes |m_M\rangle_M$ which are admissible to $|n\rangle$. In the case $t = 0$ ($q \to \infty$), this equation clearly reduces to lemma 2 from chapter 3. Furthermore, when acting on basis vectors $|\tilde{n}\rangle \in \tilde{\mathcal{V}}$ and with $t = -1$ ($q = i$), the weighting factor in (4.3.6) vanishes if $m_i = 2, n_i = 1$ for any $1 \leq i \leq M$. In this case, the right hand side of (4.3.6) becomes a sum over admissible basis vectors $|\tilde{m}\rangle \in \tilde{\mathcal{V}}$, and we recover lemma 7 from chapter 3.

Proof. We proceed along similar lines to the proof of lemma 2 in chapter 3. Let $\langle m| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M$ be an arbitrary basis vector of $\mathcal{V}^*$. We write the B-operator as a contraction on the auxiliary space $\mathcal{V}_a$, as follows

$$B(x,t) = \begin{pmatrix} 1 & 0 \end{pmatrix}_a \begin{pmatrix} A(x,t) & B(x,t) \\ C(x,t) & D(x,t) \end{pmatrix}_a \begin{pmatrix} 0 \\ 1 \end{pmatrix}_a = \uparrow_a^* T_a(x,t) \downarrow_a \tag{4.3.7}$$

which leads to the equation

$$\langle m|B(x,t)|n\rangle = \uparrow_a^* \otimes \langle m|T_a(x,t)|n\rangle \otimes \downarrow_a = \uparrow_a^* \otimes \langle m|L_{aM}(x,t)\dots L_{a0}(x,t)|n\rangle \otimes \downarrow_a \tag{4.3.8}$$

By commuting operators and vectors which reside in different spaces we obtain

$$\langle m|B(x,t)|n\rangle = \uparrow^* L^{(M)}(x,t)\dots L^{(0)}(x,t) \downarrow \tag{4.3.9}$$

where we have dropped the redundant subscripts a, and have defined the modified L-matrices

$$L^{(i)}(x,t) = \begin{pmatrix} \langle m_i|_i x^{-\frac{1}{2}} |n_i\rangle_i & \langle m_i|_i (1-t)^{\frac{1}{2}} b_i^\dagger |n_i\rangle_i \\ \langle m_i|_i (1-t)^{\frac{1}{2}} b_i |n_i\rangle_i & \langle m_i|_i x^{\frac{1}{2}} |n_i\rangle_i \end{pmatrix} \tag{4.3.10}$$

for all $0 \leq i \leq M$. Calculating the entries within these matrices explicitly yields

$$[m_i]_t! L^{(i)}(x,t) = \begin{cases} \begin{pmatrix} x^{-\frac{1}{2}} & 0 \\ 0 & x^{\frac{1}{2}} \end{pmatrix} & m_i = n_i \\ (1-t^{m_i}) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & m_i = n_i + 1 \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & m_i + 1 = n_i \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise} \end{cases} \tag{4.3.11}$$

for all $1 \leq i \leq M$, as well as

$$\frac{L^{(0)}(x,t)}{[m_0]_t!} = \begin{cases} \begin{pmatrix} x^{-\frac{1}{2}} & 0 \\ 0 & x^{\frac{1}{2}} \end{pmatrix} & m_0 = n_0 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & m_0 = n_0 + 1 \\ (1-t^{n_0}) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & m_0 + 1 = n_0 \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \text{otherwise} \end{cases} \tag{4.3.12}$$

Using the expressions (4.3.11) and (4.3.12) for $L^{(i)}(x,t)$ and $L^{(0)}(x,t)$ respectively, we see that

$$\langle m|B(x,t)|n\rangle = \uparrow^* L^{(M)}(x,t) \dots L^{(0)}(x,t) \downarrow = 0 \tag{4.3.13}$$

when $|m\rangle \not\triangleright |n\rangle$. In the case $|m\rangle \triangleright |n\rangle$, we follow essentially the same argument that was used to calculate $\uparrow^* L^{(M)}(x) \dots L^{(0)}(x) \downarrow$ when proving lemma 2 in chapter 3. The calculation of $\uparrow^* L^{(M)}(x,t) \dots L^{(0)}(x,t) \downarrow$ deviates only up to overall factors depending on t, and we easily surmise that

$$\uparrow^* L^{(M)}(x,t) \dots L^{(0)}(x,t) \downarrow = \frac{x^{-\frac{M}{2}} [m_0]_t! \prod_{i=1}^{M} x^{i(m_i - n_i)} \left(1 - \delta_{m_i, n_i+1} t^{m_i}\right)}{\prod_{i=1}^{M} [m_i]_t!} \tag{4.3.14}$$

when $|m\rangle \triangleright |n\rangle$. Combining the equations (4.3.13) and (4.3.14) into a single case, we have

$$x^{\frac{M}{2}} \langle m|B(x,t)|n\rangle = \begin{cases} \dfrac{[m_0]_t! \displaystyle\prod_{i=1}^{M} x^{i(m_i-n_i)} \Big(1-\delta_{m_i,n_i+1} t^{m_i}\Big)}{\displaystyle\prod_{i=1}^{M} [m_i]_t!}, & |m\rangle \triangleright |n\rangle \\ 0, & \text{otherwise} \end{cases} \tag{4.3.15}$$

The result (4.3.6) follows from the orthogonality (4.1.1) of the basis vectors of $\mathcal{V}$, and from the definition of $\mathbb{B}(x,t)$. □

4.3.3 Calculation of $\langle n|\mathbb{C}(x,t)$

Lemma 7. Define $\mathbb{C}(x,t) = x^{\frac{M}{2}} C(1/x,t)$ and let $\langle n| = \langle n_0|_0 \otimes \cdots \otimes \langle n_M|_M$ be an arbitrary basis vector of $\mathcal{V}^*$. The action of $\mathbb{C}(x,t)$ on $\langle n|$ is given by

$$\langle n|\mathbb{C}(x,t) = \sum_{\langle n| \triangleleft \langle m|} \prod_{i=1}^{M} x^{i(m_i-n_i)} \Big(1-\delta_{m_i,n_i+1} t^{m_i}\Big) \langle m| \tag{4.3.16}$$

where the sum is over all basis vectors $\langle m| = \langle m_0|_0 \otimes \cdots \otimes \langle m_M|_M$ which are admissible to $\langle n|$. Once again, we notice that this result reduces to lemma 3 of chapter 3 in the case $t = 0$ ($q \to \infty$), and to lemma 8 of chapter 3 in the case $t = -1$ ($q = i$).

Proof. A simple modification of the proof of lemma 6. □

4.3.4 Calculation of $\mathcal{M}_\psi(t)\mathbb{B}(x,t)|n\rangle$ and $\langle n|\mathbb{C}(x,t)\mathcal{M}^*_\psi(t)$

Throughout the rest of the chapter we will require the function $p_{\mu/\nu}(t)$,[7] which compares the part multiplicities of the partitions μ, ν and returns

$$p_{\mu/\nu}(t) = \prod_{i=1}^{\infty} \Big(1-\delta_{p_i(\mu),p_i(\nu)+1} t^{p_i(\mu)}\Big) \tag{4.3.17}$$

Let $|n\rangle$ and $\langle n|$ be arbitrary basis vectors of $\mathcal{V}$ and $\mathcal{V}^*$ respectively, and let $|\nu)$ and $(\nu|$ be their corresponding partitions, given by equation (4.3.4). We also fix $l = \ell(\nu)$. Using the definition of the maps (4.3.4), the expressions (4.3.6) and (4.3.16) and the

[7] To translate to the notation of chapter III in [65], we remark that $p_{\mu/\nu}(t) = \varphi_{\mu/\nu}(t)$ and $p_{\nu/\mu}(t) = \psi_{\mu/\nu}(t)$.

relationship between admissible basis vectors and interlacing partitions (lemma 1 of chapter 3), we obtain

$$\mathcal{M}_{\psi}(t)\mathbb{B}(x,t)|n\rangle = \sum_{\nu\prec\mu\subseteq[l+1,M]} x^{|\mu|-|\nu|}\frac{p_{\mu/\nu}(t)}{b_{\mu}(t)}|\mu) \tag{4.3.18}$$

$$\langle n|\mathbb{C}(x,t)\mathcal{M}^{*}_{\psi}(t) = \sum_{\nu\prec\mu\subseteq[l+1,M]} x^{|\mu|-|\nu|}\frac{p_{\mu/\nu}(t)}{b_{\mu}(t)}(\mu| \tag{4.3.19}$$

Both sums are over all partitions μ which interlace with ν, and whose Young diagrams are contained in the rectangle $[l+1,M]$. These equations may be written in the equivalent form

$$\mathcal{M}_{\psi}(t)\mathbb{B}(x,t)|n\rangle = \sum_{\nu\prec\mu\subseteq[l+1,M]} x^{|\mu|-|\nu|}\frac{p_{\nu/\mu}(t)}{b_{\nu}(t)}|\mu) \tag{4.3.20}$$

$$\langle n|\mathbb{C}(x,t)\mathcal{M}^{*}_{\psi}(t) = \sum_{\nu\prec\mu\subseteq[l+1,M]} x^{|\mu|-|\nu|}\frac{p_{\nu/\mu}(t)}{b_{\nu}(t)}(\mu| \tag{4.3.21}$$

where the weighting factors in the sums (4.3.18) and (4.3.19) have been adjusted using the identity[8]

$$\frac{p_{\mu/\nu}(t)}{p_{\nu/\mu}(t)} = \frac{b_{\mu}(t)}{b_{\nu}(t)} \tag{4.3.22}$$

4.3.5 Hall-Littlewood functions

Let $\{x_1,\ldots,x_n\}$ be free variables, and t an additional parameter. Following chapter III of [65], the *Hall-Littlewood function* $P_{\mu}(\{x_1,\ldots,x_n\},t)$ associated to the partition μ is defined as

$$P_{\mu}(\{x_1,\ldots,x_n\},t) = \frac{1}{v_{\mu}(t)}\sum_{\sigma\in S_n}\sigma\left(x_1^{\mu_1}\ldots x_n^{\mu_n}\prod_{1\le i<j\le n}\frac{x_i-tx_j}{x_i-x_j}\right) \tag{4.3.23}$$

where the function $v_{\mu}(t)$ is given by

$$v_{\mu}(t) = \prod_{i=1}^{\infty}\prod_{j=1}^{p_i(\mu)}\left(\frac{1-t^j}{1-t}\right) \tag{4.3.24}$$

[8]See equation (5.12) in chapter III of [65].

The Hall-Littlewood function $P_\mu(\{x\},t)$ specializes to the Schur function $s_\mu\{x\}$ in the limit $t \to 0$. Also, for all strict partitions $\tilde{\mu}$ the function $P_{\tilde{\mu}}(\{x\},t)$ specializes to the Schur Q-function $2^{-\ell(\tilde{\mu})}Q_{\tilde{\mu}}\{x\}$ by setting $t=-1$. Hence the material of chapter 3 is recovered by suitable specializations of the results stated below.

For an arbitrary pair of partitions μ,ν and indeterminates x,t the single variable *skew Hall-Littlewood function* $P_{\mu/\nu}(x,t)$ is given by[9]

$$P_{\mu/\nu}(x,t) = \begin{cases} x^{|\mu|-|\nu|} p_{\nu/\mu}(t), & \mu \succ \nu \\ \\ 0, & \text{otherwise} \end{cases} \tag{4.3.25}$$

In the case $\nu = \emptyset$ we have $P_{\mu/\nu}(x,t) = P_\mu(x,t)$, where $P_\mu(x,t)$ is the ordinary Hall-Littlewood function in a single variable x. The skew Hall-Littlewood function satisfies the identity[10]

$$P_\mu(\{x_1,\dots,x_n\},t) = \sum_{\nu\subseteq[n-1,\infty]} P_{\mu/\nu}(x_n,t)P_\nu(\{x_1,\dots,x_{n-1}\},t) \tag{4.3.26}$$

where the sum is over all partitions ν with length $\ell(\nu) \le n-1$, and $P_\mu(\{x_1,\dots,x_n\},t)$ and $P_\nu(\{x_1,\dots,x_{n-1}\},t)$ are Hall-Littlewood functions in n and $n-1$ variables, respectively.

4.3.6 Calculation of $\mathcal{M}_\psi(t)\mathbb{B}(x_1,t)\dots\mathbb{B}(x_N,t)|0\rangle$

Equipped with the necessary symmetric function identities, we are now able to calculate the q-boson model Bethe eigenvectors explicitly.

Lemma 8. Let $\{x_1,\dots,x_N\}$ be a finite set of variables and t an extra parameter. We claim that

$$\mathcal{M}_\psi(t)\mathbb{B}(x_1,t)\dots\mathbb{B}(x_N,t)|0\rangle = \sum_{\mu\subseteq[N,M]} P_\mu(\{x_1,\dots,x_N\},t)|\mu) \tag{4.3.27}$$

where $P_\mu(\{x_1,\dots,x_N\},t)$ is the Hall-Littlewood function in N variables (4.3.23), and the sum is over all partitions μ whose Young diagrams are contained in the rectangle $[N,M]$. This formula was originally proved in [86].

Proof. We begin by specializing equation (4.3.20) to the case $|n\rangle = |0\rangle$, to obtain

$$\mathcal{M}_\psi(t)\mathbb{B}(x,t)|0\rangle = \sum_{\emptyset\prec\mu\subseteq[1,M]} P_{\mu/\emptyset}(x,t)|\mu) = \sum_{\mu\subseteq[1,M]} P_\mu(x,t)|\mu) \tag{4.3.28}$$

[9] This definition is consistent with equation (5.14′) in chapter III of [65], if one replaces $p_{\nu/\mu}(t)$ with $\psi_{\mu/\nu}(t)$.

[10] See equation (5.5′) in chapter III of [65].

where we have used the equation (4.3.25) for the skew Hall-Littlewood function, and the definitions $b_{\emptyset}(t) = 1, \ell(\emptyset) = 0$. We use equation (4.3.28) as the basis for induction, and assume that

$$\mathcal{M}_{\psi}(t)\mathbb{B}(x_1,t)\dots\mathbb{B}(x_{N-1},t)|0\rangle = \sum_{\nu\subseteq[N-1,M]} P_{\nu}(\{x_1,\dots,x_{N-1}\},t)|\nu\rangle \qquad (4.3.29)$$

for some $N \geq 2$. In terms of the basis vectors of $\mathcal{V}$, this assumption is written as

$$\mathbb{B}(x_1,t)\dots\mathbb{B}(x_{N-1},t)|0\rangle = \sum_{|n\rangle|\Sigma_0=N-1} P_{\nu}(\{x_1,\dots,x_{N-1}\},t)b_{\nu}(t)|n\rangle \qquad (4.3.30)$$

where the sum is over all basis vectors $|n\rangle$ whose occupation numbers satisfy the condition $\sum_{i=0}^{M} n_i = N-1$, and ν is the partition corresponding to each $|n\rangle$. Acting on (4.3.30) with the composition of operators $\mathcal{M}_{\psi}(t) \circ \mathbb{B}(x_N,t)$ and using the fact that the B-operators commute (4.1.23), we obtain

$$\mathcal{M}_{\psi}(t)\mathbb{B}(x_1,t)\dots\mathbb{B}(x_N,t)|0\rangle = \sum_{\nu\subseteq[N-1,M]} P_{\nu}(\{x_1,\dots,x_{N-1}\},t) \sum_{\nu\prec\mu\subseteq[N,M]} P_{\mu/\nu}(x_N,t)|\mu\rangle \qquad (4.3.31)$$

Since $P_{\mu/\nu}(x_N,t) = 0$ if $\mu \not\succ \nu$, we may alter the sums appearing in (4.3.31), yielding

$$\begin{aligned}\mathcal{M}_{\psi}(t)\mathbb{B}(x_1,t)\dots\mathbb{B}(x_N,t)|0\rangle &= \sum_{\mu\subseteq[N,M]}\sum_{\nu\subseteq[N-1,M]} P_{\mu/\nu}(x_N,t)P_{\nu}(\{x_1,\dots,x_{N-1}\},t)|\mu\rangle \\ &= \sum_{\mu\subseteq[N,M]}\sum_{\nu\subseteq[N-1,\infty]} P_{\mu/\nu}(x_N,t)P_{\nu}(\{x_1,\dots,x_{N-1}\},t)|\mu\rangle\end{aligned} \qquad (4.3.32)$$

where the final equality holds since every part of μ is less than or equal to M, and therefore $P_{\mu/\nu}(x_N,t) = 0$ if any part of ν is greater than M. Using the identity (4.3.26) we evaluate the sum over ν explicitly, producing the equation (4.3.27). Therefore by induction the result (4.3.27) must hold for arbitrary $N \geq 1$.

□

4.3.7 Calculation of $\langle 0|\mathbb{C}(x_N,t)\dots\mathbb{C}(x_1,t)\mathcal{M}^{*}_{\psi}(t)$

By following essentially the same steps that were used in the previous subsection, we can also derive the expression

$$\langle 0|\mathbb{C}(x_N,t)\dots\mathbb{C}(x_1,t)\mathcal{M}^{*}_{\psi}(t) = \sum_{\mu\subseteq[N,M]} P_{\mu}(\{x_1,\dots,x_N\},t)(\mu| \qquad (4.3.33)$$

for the dual Bethe eigenvectors. As before, this sum is taken over all partitions μ whose Young diagrams are contained in the rectangle $[N,M]$.

4.4 Scalar product, weighted plane partitions

4.4.1 Levels of paths within plane partitions

Definition 2. Let π be a plane partition. The element $\pi(i,j)$ is said to be at level l if

$$\pi(i,j) = \cdots = \pi(i+l-1, j+l-1) > \pi(i+l, j+l) \tag{4.4.1}$$

for some $l \geq 1$. A *path at level* l is a set of connected elements in π which have the same numerical value and the same level l. We let $p_l(\pi)$ denote the number of paths in π at level l. This definition of levels within a plane partition originally appeared in [88].

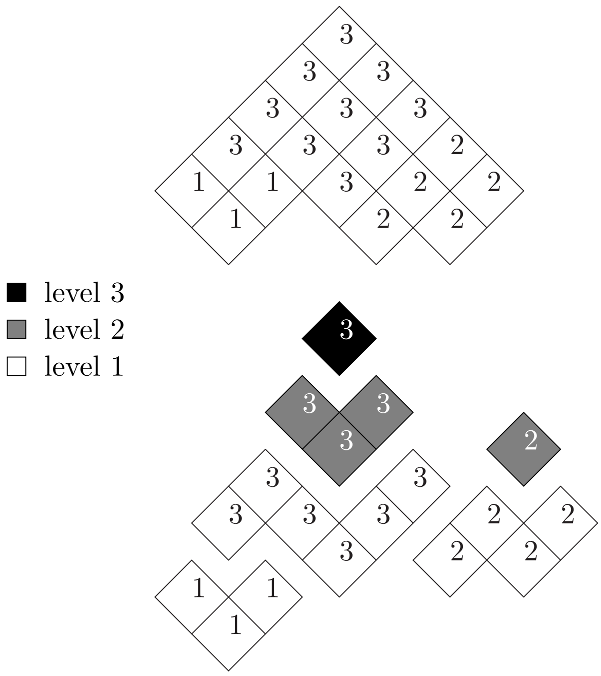

Figure 4.1: Paths at various levels within a plane partition.

4.4.2 Path-weighted plane partitions

In this subsection we assign a t-weighting to plane partitions, which was originally introduced in [88]. In the limit $t = 0$ the weighting becomes 1 for all plane partitions. In the limit $t = -1$ the weighting collapses to zero for all plane partitions which contain paths at level 2, or equivalently, for all plane partitions which are non-strict.

Definition 3. Let π be a plane partition living inside the box $[N, N, M]$, with diagonal slices $\{\emptyset = \pi_{-N} \prec \cdots \prec \pi_{-1} \prec \pi_0 \succ \pi_1 \succ \cdots \succ \pi_N = \emptyset\}$. We associate to this plane partition the weighting $A_\pi(\{x\}, \{y\}, t)$, given by

$$A_\pi\Big(\{x\},\{y\},t\Big) = \prod_{i=1}^{N}\Big(1-t^i\Big)^{p_i(\pi)} x_i^{|\pi_{-i+1}|-|\pi_{-i}|} y_i^{|\pi_{i-1}|-|\pi_i|} \tag{4.4.2}$$

where $p_i(\pi)$ is the number of paths in π at level i. Notice that $A_\pi(\{x\},\{y\},t)$ only takes contributions from paths whose levels are less than or equal to N, but in fact paths at greater levels cannot exist, since $\pi \subseteq [N,N,M]$.

Lemma 9. Let π be a plane partition as described in definition 3. Then

$$\frac{1}{b_{\pi_0}(t)}\prod_{i=1}^{N} p_{\pi_{-i+1}/\pi_{-i}}(t)\prod_{j=1}^{N} p_{\pi_{j-1}/\pi_j}(t) = \prod_{i=1}^{N}\Big(1-t^i\Big)^{p_i(\pi)} \tag{4.4.3}$$

Proof. The proof is best illustrated by an example, so we consider the plane partition drawn in figure 4.1. This plane partition lives inside the box $[5,5,3]$, and its diagonal slices are given by

$$\begin{aligned}
&\pi_{-5} = \emptyset & & & &\pi_1 = \{3,3,2\}\\
&\pi_{-4} = \{1\} & & & &\pi_2 = \{3,2\}\\
&\pi_{-3} = \{3,1\} & &\pi_0 = \{3,3,3\} & &\pi_3 = \{2,2\}\\
&\pi_{-2} = \{3,1\} & & & &\pi_4 = \{2\}\\
&\pi_{-1} = \{3,3\} & & & &\pi_5 = \emptyset
\end{aligned} \tag{4.4.4}$$

Using these partitions and the definition (4.3.17), we evaluate

$$\begin{aligned}
&p_{\pi_{-4}/\pi_{-5}}(t) = 1-t & &p_{\pi_0/\pi_1}(t) = 1-t^3\\
&p_{\pi_{-3}/\pi_{-4}}(t) = 1-t & &p_{\pi_1/\pi_2}(t) = 1-t^2\\
&p_{\pi_{-2}/\pi_{-3}}(t) = 1 & &p_{\pi_2/\pi_3}(t) = 1-t\\
&p_{\pi_{-1}/\pi_{-2}}(t) = 1-t^2 & &p_{\pi_3/\pi_4}(t) = 1-t^2\\
&p_{\pi_0/\pi_{-1}}(t) = 1-t^3 & &p_{\pi_4/\pi_5}(t) = 1-t
\end{aligned} \tag{4.4.5}$$

Multiplying all of these terms together, we obtain a factor of $(1-t^i)$ for every level i path which *does not* intersect the central diagonal. All level i paths which *do* intersect the central diagonal obtain a factor of $(1-t^i)^2$. This double counting is cured by dividing by $b_{\pi_0}(t) = (1-t)(1-t^2)(1-t^3)$. The result is

$$\begin{aligned}
\frac{1}{b_{\pi_0}(t)}\prod_{i=1}^{5} p_{\pi_{-i+1}/\pi_{-i}}(t)\prod_{j=1}^{5} p_{\pi_{j-1}/\pi_j}(t) &= (1-t)^3(1-t^2)^2(1-t^3)\\
&= \prod_{i=1}^{5}\Big(1-t^i\Big)^{p_i(\pi)}
\end{aligned} \tag{4.4.6}$$

as required. This method can be easily extended to an arbitrary plane partition. □

4.4.3 Generating M-boxed path-weighted plane partitions

Aided by the results of the previous subsections, we are now able to calculate the generating function for M-boxed plane partitions with the prescribed weighting (4.4.2). We will see that this generating function is intrinsically related to the scalar product of the q-boson model. We start by iterating the $|n\rangle = |0\rangle$ case of equation (4.3.18) N times successively, which gives

$$\mathcal{M}_\psi(t)\mathbb{B}(x_1,t)\dots\mathbb{B}(x_N,t)|0\rangle = \sum_{[N,M]\supseteq\pi_0\succ\cdots\succ\pi_N=\emptyset} \frac{1}{b_{\pi_0}(t)} \prod_{i=1}^{N} p_{\pi_{i-1}/\pi_i}(t) x_i^{|\pi_{i-1}|-|\pi_i|} |\pi_0\rangle \tag{4.4.7}$$

where the sum is over all interlacing partitions $\{\pi_0 \succ \cdots \succ \pi_N\}$ which are subject to $\pi_0 \subseteq [N,M]$ and $\pi_N = \emptyset$. Similarly, one can iterate the $\langle n| = \langle 0|$ case of (4.3.19) N times successively, obtaining

$$\langle 0|\mathbb{C}(x_N,t)\dots\mathbb{C}(x_1,t)\mathcal{M}^*_\psi(t) = \sum_{\emptyset=\pi_{-N}\prec\cdots\prec\pi_0\subseteq[N,M]} \frac{1}{b_{\pi_0}(t)} \prod_{i=1}^{N} p_{\pi_{-i+1}/\pi_{-i}}(t) x_i^{|\pi_{-i+1}|-|\pi_{-i}|} (\pi_0| \tag{4.4.8}$$

where the sum is over all interlacing partitions $\{\pi_{-N} \prec \cdots \prec \pi_0\}$ which are subject to $\pi_0 \subseteq [N,M]$ and $\pi_{-N} = \emptyset$. By the definition (4.4.2) of $A_\pi(\{x\},\{y\},t)$, the result (4.4.3) of lemma 9 and the orthogonality relation (4.2.24), we therefore obtain

$$\Big\langle \langle 0|\mathbb{C}(x_N,t)\dots\mathbb{C}(x_1,t)\mathcal{M}^*_\psi(t), \mathcal{M}_\psi(t)\mathbb{B}(y_1,t)\dots\mathbb{B}(y_N,t)|0\rangle \Big\rangle = \sum_{\pi\subseteq[N,N,M]} A_\pi\Big(\{x\},\{y\},t\Big) \tag{4.4.9}$$

where the sum is taken over all plane partitions π which fit inside the box $[N,N,M]$. Hence the scalar product between the image Bethe eigenstates (4.3.27) and (4.3.33) is a generating function for M-boxed path-weighted plane partitions. This generating function is evaluated explicitly by using the equations (4.3.27), (4.3.33) and the orthogonality relation (4.2.24) to give

$$\Big\langle \langle 0|\mathbb{C}(x_N,t)\dots\mathbb{C}(x_1,t)\mathcal{M}^*_\psi(t), \mathcal{M}_\psi(t)\mathbb{B}(y_1,t)\dots\mathbb{B}(y_N,t)|0\rangle \Big\rangle = \sum_{\mu\subseteq[N,M]} b_\mu(t) P_\mu(\{x\},t) P_\mu(\{y\},t) \tag{4.4.10}$$

Comparing equations (4.4.9) and (4.4.10), we have proved the result

$$\sum_{\pi \subseteq [N,N,M]} A_\pi\Big(\{x\},\{y\},t\Big) = \sum_{\mu \subseteq [N,M]} b_\mu(t) P_\mu(\{x\},t) P_\mu(\{y\},t) \tag{4.4.11}$$

We remark that the finite generating functions (3.3.16) and (3.7.17) in the previous chapter are obtained from (4.4.11) by setting $t = 0$ and $t = -1$, respectively.

4.5 q-boson model on an infinite lattice

In this section we elaborate upon results which were obtained in [36]. Most of our attention centres on proving theorem 1, which is the t-deformation of lemma 6 in the previous chapter. An independent derivation of theorem 1, using the properties of Hall-Littlewood functions, can be found in [82].

4.5.1 $\mathcal{M}_\psi(t)\mathbb{B}(x,t)|n\rangle$ and $\langle n|\mathbb{C}(x,t)\mathcal{M}^*_\psi(t)$ as $M \to \infty$

Theorem 1. Consider the infinite lattice limit of the q-boson model, obtained by taking $M \to \infty$. Let $|n\rangle = \otimes_{i=0}^{\infty}|n_i\rangle_i$ and $\langle n| = \otimes_{i=0}^{\infty}\langle n_i|_i$ be basis vectors of $\mathcal{V}$ and $\mathcal{V}^*$, respectively, in this limit. Similarly, let $\frac{1}{b_\nu(t)}|\nu)$ and $\frac{1}{b_\nu(t)}(\nu|$ be the image states of these basis vectors under the mappings (4.3.4). We claim that

$$\mathcal{M}_\psi(t)\left[\lim_{M\to\infty} \mathbb{B}(x,t)|n\rangle\right] = \frac{1}{b_\nu(t)}\Gamma_-(x,t)|\nu) \tag{4.5.1}$$

$$\left[\lim_{M\to\infty} \langle n|\mathbb{C}(x,t)\right]\mathcal{M}^*_\psi(t) = \frac{1}{b_\nu(t)}(\nu|\Gamma_+(x,t) \tag{4.5.2}$$

where $\Gamma_\pm(x,t)$ denote the t-deformed half-vertex operators

$$\Gamma_\pm(x,t) = \exp\left(\sum_{n=1}^{\infty}\frac{1-t^n}{n}x^n H_{\pm n}(t)\right) \tag{4.5.3}$$

which were studied in subsection 4.2.6.

Proof. We split the proof into four steps. In the first step, we translate the equations (4.5.1), (4.5.2) to the equivalent statements (4.5.8), (4.5.9) at the level of the t-deformed Fock spaces. Thereafter we focus on proving (4.5.9), since (4.5.8) follows by direct analogy. In the second step we define the function f_0 and use it to express (4.5.9) in the alternative form (4.5.22). The third step contains several identities and a useful change of notation. In the fourth step we prove (4.5.22) using induction.

Step 1. Taking the $M \to \infty$ limit of equations (4.3.20) and (4.3.21), we obtain

$$\mathcal{M}_{\psi}(t)\left[\lim_{M\to\infty}\mathbb{B}(x,t)|n\rangle\right]=\sum_{\mu\succ\nu}x^{|\mu|-|\nu|}\frac{p_{\nu/\mu}(t)}{b_{\nu}(t)}|\mu) \tag{4.5.4}$$

$$\left[\lim_{M\to\infty}\langle n|\mathbb{C}(x,t)\right]\mathcal{M}^{*}_{\psi}(t)=\sum_{\mu\succ\nu}x^{|\mu|-|\nu|}\frac{p_{\nu/\mu}(t)}{b_{\nu}(t)}(\mu| \tag{4.5.5}$$

where the sums are over all partitions μ which interlace ν, but whose parts have no size restriction. The equations (4.5.1) and (4.5.2) are therefore equivalent to the statements

$$\Gamma_{-}(x,t)|\nu)=\sum_{\mu\succ\nu}x^{|\mu|-|\nu|}p_{\nu/\mu}(t)|\mu) \tag{4.5.6}$$

$$(\nu|\Gamma_{+}(x,t)=\sum_{\mu\succ\nu}x^{|\mu|-|\nu|}p_{\nu/\mu}(t)(\mu| \tag{4.5.7}$$

Due to the orthogonality (4.2.24) of partition states, equations (4.5.6) and (4.5.7) may be presented in the alternative form

$$(\mu|\Gamma_{-}(x,t)=\sum_{\nu\prec\mu}x^{|\mu|-|\nu|}p_{\nu/\mu}(t)\frac{b_{\mu}(t)}{b_{\nu}(t)}(\nu|=\sum_{\nu\prec\mu}x^{|\mu|-|\nu|}p_{\mu/\nu}(t)(\nu| \tag{4.5.8}$$

$$\Gamma_{+}(x,t)|\mu)=\sum_{\nu\prec\mu}x^{|\mu|-|\nu|}p_{\nu/\mu}(t)\frac{b_{\mu}(t)}{b_{\nu}(t)}|\nu)=\sum_{\nu\prec\mu}x^{|\mu|-|\nu|}p_{\mu/\nu}(t)|\nu) \tag{4.5.9}$$

where the sums are now over all partitions ν such that $\nu\prec\mu$. In contrast with (4.5.6) and (4.5.7), equations (4.5.8) and (4.5.9) contain only finite sums, which simplifies their analysis. We will give an explicit proof of (4.5.9). The proof of (4.5.8) is omitted, but as it is of such a similar nature we will claim it as a corollary of (4.5.9).

Step 2. (Definition 4.) To every set $\{m\}=\{m_1>\cdots>m_l>-l\}$ we associate a unique collection of integers $\{s\}=\{0=s_0<s_1<\cdots<s_r<s_{r+1}=l\}$ such that the subsets

$$\mathcal{S}_k=\{m_{(s_k+1)}>\cdots>m_{s_{(k+1)}}\} \tag{4.5.10}$$

are comprised of nearest neighbours, for all $0\le k\le r$, with r minimized. We call this the *nearest neighbour partitioning* of $\{m\}$.

Let $\{m\}=\{m_1>\cdots>m_l>-l\}$ and $\{n\}=\{n_1>\cdots>n_l\ge -l\}$ be two arbitrary sets of integers, and fix $n_0=\infty$. For all $0\le j\le r$, we define the functions $f_j(\{m\},\{n\},t)$ by the equation

$$f_j\Big(\{m\},\{n\},t\Big) = \prod_{k=j}^{r}\Big(1-\theta(n_{s_k} > m_{(s_k+1)}+1)\theta(m_{s_{(k+1)}} > n_{s_{(k+1)}})t^{\Delta_k}\Big) \quad (4.5.11)$$

where $\{s\}$ is the set of nearest neighbour points associated to $\{m\}$, $\Delta_k = s_{k+1} - s_k$, and $\theta(z)$ is the Boolean function

$$\theta(z) = \begin{cases} 1, & z \text{ true} \\ 0, & z \text{ false} \end{cases} \quad (4.5.12)$$

(Lemma 10.) Let $\mu = \{\mu_1 \geq \cdots \geq \mu_l > 0\}$ and $\nu = \{\nu_1 \geq \cdots \geq \nu_l \geq 0\}$ be partitions whose $\mathcal{F}_\psi(t)$ equivalents are given by

$$|\mu) = \psi_{m_1}(t)\ldots\psi_{m_l}(t)|-l\rangle, \quad |\nu) = \psi_{n_1}(t)\ldots\psi_{n_l}(t)|-l\rangle \quad (4.5.13)$$

where $m_i = \mu_i - i, n_i = \nu_i - i$ for all $1 \leq i \leq l$. Then if $\mu \succ \nu$, we have

$$p_{\mu/\nu}(t) = f_0\Big(\{m\},\{n\},t\Big) \quad (4.5.14)$$

Proof. (Lemma 10.) Since $\mu \succ \nu$, we know that $m_i \geq n_i > m_{i+1}$ for all $1 \leq i \leq l$. It follows that the inequality $m_i > n_i$ is only allowed if $i \in \{s\}$. Hence all differences between the sets $\{m\},\{n\}$ occur at the points $\{s\}$.

Now we notice that each $\mathcal{S}_k \subset \{m\}$ corresponds to Δ_k parts in μ of the same size. The necessary and sufficient condition for there to be one less part in ν is that $m_{s_{(k+1)}} > n_{s_{(k+1)}}$ and $n_{s_k} > m_{(s_k+1)} + 1$. In such a case $f_0(\{m\},\{n\},t)$ returns a factor of $(1 - t^{\Delta_k})$, which is precisely the same factor returned by $p_{\mu/\nu}(t)$. Iterating this argument across all $0 \leq k \leq r$, we obtain the equality (4.5.14).

To further clarify the proof, we present a short example. Let the partitions μ and ν be given by

$$\mu = \{\mu_1,\mu_2,\mu_3,\mu_4,\mu_5,\mu_6,\mu_7\} = \{7,6,6,6,4,4,2\} \quad (4.5.15)$$

$$\nu = \{\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7\} = \{7,6,6,4,4,3,1\} \quad (4.5.16)$$

These partitions satisfy $\mu \succ \nu$. Furthermore, using the definition (4.3.17) we obtain

$$p_{\mu/\nu}(t) = \prod_{i=1}^{7}\Big(1-\delta_{p_i(\mu),p_i(\nu)+1}t^{p_i(\mu)}\Big) = (1-t^3)(1-t) \quad (4.5.17)$$

Now let $\{m\}$ and $\{n\}$ be the sets formed by fixing, respectively, $m_i = \mu_i - i$ and $n_i = \nu_i - i$ for all $1 \leq i \leq 7$. We find that

$$\{m\} = \{m_1, m_2, m_3, m_4, m_5, m_6, m_7\} = \{6, 4, 3, 2, -1, -2, -5\} \tag{4.5.18}$$
$$\{n\} = \{n_1, n_2, n_3, n_4, n_5, n_6, n_7\} = \{6, 4, 3, 0, -1, -3, -6\} \tag{4.5.19}$$

Each of $\{m_1\}, \{m_2 > m_3 > m_4\}, \{m_5 > m_6\}, \{m_7\}$ are nearest neighbours, and this is the smallest possible decomposition of $\{m\}$ into such subsets. Hence the set $\{s\}$ of nearest neighbour points associated to $\{m\}$ is given by

$$\{s\} = \{s_0, s_1, s_2, s_3, s_4\} = \{0, 1, 4, 6, 7\} \tag{4.5.20}$$

Finally, setting $n_0 = \infty$ and using the definition (4.5.11) we obtain

$$\begin{aligned} f_0\Big(\{m\}, \{n\}, t\Big) &= \Big(1 - \theta(n_0 > m_1 + 1)\theta(m_1 > n_1)t\Big) \\ \times &\Big(1 - \theta(n_1 > m_2 + 1)\theta(m_4 > n_4)t^3\Big)\Big(1 - \theta(n_4 > m_5 + 1)\theta(m_6 > n_6)t^2\Big) \\ \times &\Big(1 - \theta(n_6 > m_7 + 1)\theta(m_7 > n_7)t\Big) = (1 - t^3)(1 - t) \end{aligned} \tag{4.5.21}$$

which is in agreement with (4.5.17).

□

By virtue of lemma 10, we are able to write (4.5.9) in the equivalent form

$$\Gamma_+(x, t)\psi_{m_1} \dots \psi_{m_l} | -l\rangle = \sum_{\nu \prec \mu} x^{|\mu|-|\nu|} f_0\Big(\{m\}, \{n\}, t\Big)\psi_{n_1} \dots \psi_{n_l} | -l\rangle \tag{4.5.22}$$

where we have assumed that $\ell(\mu) = l$ and defined $m_i = \mu_i - i, n_i = \nu_i - i$ for all $1 \leq i \leq l$. We now proceed to calculate the left hand side of (4.5.22), aiming to show that it evaluates to the sum on the right hand side. In order to achieve this we need several identities, which are introduced in the next step.

Step 3. (Identity 1.) Let x be an indeterminate and m an arbitrary integer. We have

$$\begin{aligned} &\left(\sum_{n=0}^{\infty} \psi_{(m-n)} x^{n+1}\right) \left(\psi_m + (1 - t) \sum_{n=1}^{\infty} \psi_{(m-n)} x^n\right) \\ &= \sum_{n=0}^{\infty} \left(\psi_{(m-n)}\psi_m + (1 - t) \sum_{i=1}^{n} \psi_{(m-n+i)}\psi_{(m-i)}\right) x^{n+1} = t\psi_{(m+1)} \sum_{n=1}^{\infty} \psi_{(m-n)} x^n \end{aligned} \tag{4.5.23}$$

where the first equality follows trivially by collecting the coefficient of x^{n+1} for all $n \geq 0$, and the second equality holds due to lemma 3.

(Identity 2.) Let x be an indeterminate, m an arbitrary integer, and fix another integer $s \geq 1$. We obtain

$$\left(\psi_{(m+1)} + (1-t^s)\sum_{i=1}^{\infty}\psi_{(m+1-i)}x^i\right)\left(\psi_m + (1-t)\sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right) \tag{4.5.24}$$
$$= \psi_{(m+1)}\left(\psi_m + (1-t)\sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right) + (1-t^s)\left(\sum_{i=0}^{\infty}\psi_{(m-i)}x^{i+1}\right)\left(\psi_m + (1-t)\sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right)$$
$$= \psi_{(m+1)}\left(\psi_m + (1-t^{s+1})\sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right)$$

where the second line follows trivially by expanding the first, and the final line holds by application of identity 1.

(Identity 3.) Let x be an indeterminate, m an arbitrary integer, and fix two more integers $n \geq 2$, $s \geq 1$. We find

$$\left(\psi_{(m+n)} + (1-t^s)\sum_{i=1}^{\infty}\psi_{(m+n-i)}x^i\right)\left(\psi_m + (1-t)\sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right) \tag{4.5.25}$$
$$= \left(\psi_{(m+n)} + (1-t^s)\sum_{i=1}^{n-1}\psi_{(m+n-i)}x^i\right)\left(\psi_m + (1-t)\sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right)$$
$$+ (1-t^s)x^{n-1}\left(\sum_{i=0}^{\infty}\psi_{(m-i)}x^{i+1}\right)\left(\psi_m + (1-t)\sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right)$$
$$= \left(\psi_{(m+n)} + (1-t^s)\sum_{i=1}^{n-2}\psi_{(m+n-i)}x^i\right)\left(\psi_m + (1-t)\sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right)$$
$$+ (1-t^s)x^{n-1}\psi_{(m+1)}\left(\psi_m + \sum_{i=1}^{\infty}\psi_{(m-i)}x^i\right)$$

where the second line follows trivially by expanding the first, and the final line holds by application of identity 1.

(Change of notation.) We introduce the notation

$$\Psi_m(x,t^s) = \psi_m + (1-t^s)\sum_{i=1}^{\infty}\psi_{(m-i)}x^i \tag{4.5.26}$$

which allows us to abbreviate identity 2 as follows

$$\Psi_{(m+1)}(x,t^s)\Psi_m(x,t) = \psi_{(m+1)}\Psi_m(x,t^{s+1}) \tag{4.5.27}$$

and identity 3 as follows

$$\Psi_{(m+n)}(x,t^s)\Psi_m(x,t) = \left(\psi_{(m+n)} + (1-t^s)\sum_{i=1}^{n-2}\psi_{(m+n-i)}x^i\right)\Psi_m(x,t) \\ + (1-t^s)x^{n-1}\psi_{(m+1)}\Psi_m(x,t^\infty) \tag{4.5.28}$$

where we have defined $t^\infty = 0$, which is perfectly sensible given the assumption $|t| < 1$. The equations (4.5.27) and (4.5.28) are essential to the calculations of the final step.

Step 4. Let us return to the proof of (4.5.22). Our starting point is the commutation relation (4.2.41), which can be expressed in the more succinct form

$$\Gamma_+(x,t)\psi_m = \Psi_m(x,t)\Gamma_+(x,t) \tag{4.5.29}$$

using the notation (4.5.26). Applying the identity (4.5.29) repeatedly to the left hand side of (4.5.22), we find that

$$\Gamma_+(x,t)\psi_{m_1}\dots\psi_{m_l}|-l\rangle = \Psi_{m_1}(x,t)\dots\Psi_{m_l}(x,t)\Gamma_+(x,t)|-l\rangle \\ = \Psi_{m_1}(x,t)\dots\Psi_{m_l}(x,t)|-l\rangle \tag{4.5.30}$$

where the final line follows from the fact that $\Gamma_+(x,t)|-l\rangle = e^{H_+(x,t)}|-l\rangle = |-l\rangle$, which is a simple consequence of the annihilation relation (4.2.33). Hence we see that (4.5.22) is equivalent to the proposition

$$\Psi_{m_1}(x,t)\dots\Psi_{m_l}(x,t)|-l\rangle = \sum_{\nu\prec\mu} x^{|\mu|-|\nu|} f_0\Big(\{m\},\{n\},t\Big)\psi_{n_1}\dots\psi_{n_l}|-l\rangle \tag{4.5.31}$$

In order to prove (4.5.31), fix $\{s\} = \{0 = s_0 < s_1 < \dots < s_r < s_{r+1} = l\}$ as the set arising from the nearest neighbour partitioning of $\{m\}$. We let $\mathcal{P}_j^{(1)}$ denote the proposition

$$\psi_{m_1}\dots\psi_{m_{s_j}} \prod_{k=\bar{s}_j}^{l} \Psi_{m_k}(x,t)|-l\rangle = \sum_{\substack{\nu\prec\mu \\ [\nu]_{s_j}=[\mu]_{s_j}}} x^{|\mu|-|\nu|} f_j\Big(\{m\},\{n\},t\Big)\prod_{k=1}^{l}\psi_{n_k}|-l\rangle \tag{4.5.32}$$

and similarly let $\mathcal{P}_j^{(2)}$ denote the proposition

$$\psi_{m_1}\dots\psi_{m_{s_j}}\Psi_{m_{\bar{s}_j}}(x,0)\prod_{k=\bar{s}_j+1}^{l}\Psi_{m_k}(x,t)|-l\rangle = \sum_{\substack{\nu\prec\mu \\ [\nu]_{s_j}=[\mu]_{s_j}}} x^{|\mu|-|\nu|} f_{j+1}\Big(\{m\},\{n\},t\Big)\prod_{k=1}^{l}\psi_{n_k}|-l\rangle \tag{4.5.33}$$

where we have abbreviated $\bar{s}_j = s_j + 1$ for convenience, and where in both cases the sum is over all partitions ν which satisfy

$$\nu_i = \mu_i, \text{ for all } 1 \le i \le s_j, \qquad \mu_i \ge \nu_i \ge \mu_{i+1}, \text{ for all } \bar{s}_j \le i \le l \tag{4.5.34}$$

Our aim is to prove that $\mathcal{P}_0^{(1)}$ is true, because equation (4.5.32) specializes to (4.5.31) in the case $j = 0$. Firstly we prove the propositions $\mathcal{P}_r^{(1)}$ and $\mathcal{P}_r^{(2)}$. By definition the integers $\{m_{\bar{s}_r} > \cdots > m_l\}$ are nearest neighbours, so we may use the identity (4.5.27) repeatedly to obtain

$$\psi_{m_1} \ldots \psi_{m_{s_r}} \Psi_{m_{\bar{s}_r}}(x, t^{\delta+1}) \overrightarrow{\prod_{k=\bar{s}_r+1}^{l}} \Psi_{m_k}(x,t)|-l\rangle = \overrightarrow{\prod_{k=1}^{l-1}} \psi_{m_k} \Psi_{m_l}(x, t^{\delta+\Delta_r})|-l\rangle \tag{4.5.35}$$

where we have defined $\Delta_r = s_{r+1} - s_r = l - s_r$, and at this stage δ is unspecified. Now due to the annihilation properties (4.2.16), we obtain the truncation

$$\Psi_{m_l}(x, t^{\delta+\Delta_r})|-l\rangle = \left(\psi_{m_l} + (1 - t^{\delta+\Delta_r}) \sum_{m_l > n_l \ge -l} \psi_{n_l} x^{m_l - n_l} \right) |-l\rangle \tag{4.5.36}$$

Substituting this result into (4.5.35), we take the limit $\delta \to 0$ to recover

$$\psi_{m_1} \ldots \psi_{m_{s_r}} \overrightarrow{\prod_{k=\bar{s}_r}^{l}} \Psi_{m_k}(x,t)|-l\rangle = \sum_{\substack{\nu \prec \mu \\ [\nu]_{s_r} = [\mu]_{s_r}}} x^{|\mu|-|\nu|} f_r\Big(\{m\},\{n\},t\Big) \overrightarrow{\prod_{k=1}^{l}} \psi_{n_k}|-l\rangle \tag{4.5.37}$$

and the limit $\delta \to \infty$ to recover

$$\psi_{m_1} \ldots \psi_{m_{s_r}} \Psi_{m_{\bar{s}_r}}(x, 0) \overrightarrow{\prod_{k=\bar{s}_r+1}^{l}} \Psi_{m_k}(x,t)|-l\rangle = \sum_{\substack{\nu \prec \mu \\ [\nu]_{s_r} = [\mu]_{s_r}}} x^{|\mu|-|\nu|} f_{r+1} \overrightarrow{\prod_{k=1}^{l}} \psi_{n_k}|-l\rangle \tag{4.5.38}$$

where we have used the definition (4.5.11) of $f_r(\{m\},\{n\},t)$ to produce (4.5.37), and fixed $f_{r+1} = 1$ to produce (4.5.38). Therefore $\mathcal{P}_r^{(1)}$ and $\mathcal{P}_r^{(2)}$ are true. Now assume that $\mathcal{P}_j^{(1)}$ and $\mathcal{P}_j^{(2)}$ are true for some $1 \le j \le r$, and adopt the notation $\tilde{j} = j - 1$. Since the integers $\{m_{\bar{s}_{\tilde{j}}} > \cdots > m_{s_j}\}$ are nearest neighbours, we may use the identity (4.5.27) repeatedly to show that

$$\psi_{m_1} \dots \psi_{m_{s_{\tilde{j}}}} \Psi_{m_{\bar{s}_{\tilde{j}}}}(x, t^{\delta+1}) \overrightarrow{\prod_{k=\bar{s}_{\tilde{j}}+1}^{l}} \Psi_{m_k}(x,t)|-l\rangle = \overrightarrow{\prod_{k=1}^{s_j-1}} \psi_{m_k} \Psi_{m_{s_j}}(x, t^{\delta+\Delta_{\tilde{j}}}) \overrightarrow{\prod_{k=\bar{s}_j}^{l}} \Psi_{m_k}(x,t)|-l\rangle \tag{4.5.39}$$

By definition the integers m_{s_j} and $m_{\bar{s}_j}$ are *not* nearest neighbours, so we may apply the identity (4.5.28) to the right hand side of (4.5.39), giving

$$\psi_{m_1} \dots \psi_{m_{s_{\tilde{j}}}} \Psi_{m_{\bar{s}_{\tilde{j}}}}(x, t^{\delta+1}) \overrightarrow{\prod_{k=\bar{s}_{\tilde{j}}+1}^{l}} \Psi_{m_k}(x,t)|-l\rangle = \tag{4.5.40}$$

$$\overrightarrow{\prod_{k=1}^{s_j-1}} \psi_{m_k} \Bigg\{ \Big(\psi_{m_{s_j}} + (1-t^{\delta+\Delta_{\tilde{j}}}) \sum_{m_{s_j} > n_{s_j} > m_{\bar{s}_j}+1} \psi_{n_{s_j}} x^{m_{s_j}-n_{s_j}} \Big) \overrightarrow{\prod_{k=\bar{s}_j}^{l}} \Psi_{m_k}(x,t)|-l\rangle$$

$$+(1-t^{\delta+\Delta_{\tilde{j}}}) x^{m_{s_j}-m_{\bar{s}_j}-1} \psi_{(m_{\bar{s}_j}+1)} \Psi_{m_{\bar{s}_j}}(x,0) \overrightarrow{\prod_{k=\bar{s}_j+1}^{l}} \Psi_{m_k}(x,t)|-l\rangle \Bigg\}$$

Applying the assumptions $\mathcal{P}_j^{(1)}$ and $\mathcal{P}_j^{(2)}$ to (4.5.40), we take the limit $\delta \to 0$ to recover

$$\psi_{m_1} \dots \psi_{m_{s_{\tilde{j}}}} \overrightarrow{\prod_{k=\bar{s}_{\tilde{j}}}^{l}} \Psi_{m_k}(x,t)|-l\rangle = \sum_{\substack{\nu \prec \mu \\ [\nu]_{s_{\tilde{j}}} = [\mu]_{s_{\tilde{j}}}}} x^{|\mu|-|\nu|} f_{\tilde{j}}\Big(\{m\},\{n\},t\Big) \overrightarrow{\prod_{k=1}^{l}} \psi_{n_k}|-l\rangle \tag{4.5.41}$$

and the limit $\delta \to \infty$ to recover

$$\psi_{m_1} \dots \psi_{m_{s_{\tilde{j}}}} \Psi_{m_{\bar{s}_{\tilde{j}}}}(x,0) \overrightarrow{\prod_{k=\bar{s}_{\tilde{j}}+1}^{l}} \Psi_{m_k}(x,t)|-l\rangle = \sum_{\substack{\nu \prec \mu \\ [\nu]_{s_{\tilde{j}}} = [\mu]_{s_{\tilde{j}}}}} x^{|\mu|-|\nu|} f_{\tilde{j}}\Big(\{m\},\{n\},t\Big) \overrightarrow{\prod_{k=1}^{l}} \psi_{n_k}|-l\rangle \tag{4.5.42}$$

Hence $\mathcal{P}_j^{(1)}, \mathcal{P}_j^{(2)}$ true $\implies$ $\mathcal{P}_{\tilde{j}}^{(1)}, \mathcal{P}_{\tilde{j}}^{(2)}$ true, proving the propositions (4.5.32) and (4.5.33) in general by induction. Therefore $\mathcal{P}_0^{(1)}$ holds, completing the proof of the theorem.

□

4.5.2 Generating path-weighted plane partitions of arbitrary size

In the previous section we demonstrated that the scalar product (4.4.9) on a lattice of size $M+1$ generates M-boxed path-weighted plane partitions. Accordingly, we expect that in the limit $M \to \infty$ it will generate path-weighted plane partitions whose column heights are arbitrarily large, giving rise to the equation

$$\lim_{M\to\infty} \Big\langle \langle 0|\mathbb{C}(x_N,t)\dots\mathbb{C}(x_1,t)\mathcal{M}^*_\psi(t), \mathcal{M}_\psi(t)\mathbb{B}(y_1,t)\dots\mathbb{B}(y_N,t)|0\rangle \Big\rangle = \sum_{\pi\subseteq[N,N,\infty]} A_\pi\Big(\{x\},\{y\},t\Big) \tag{4.5.43}$$

where the sum is over all plane partitions π which fit inside the box of dimension $N\times N\times\infty$, with $A_\pi(\{x\},\{y\},t)$ given by (4.4.2). On the other hand, using the result of theorem 1 we are able to write

$$\begin{aligned}\lim_{M\to\infty} \Big\langle \langle 0|\mathbb{C}(x_N,t)\dots\mathbb{C}(x_1,t)\mathcal{M}^*_\psi(t),\mathcal{M}_\psi(t)\mathbb{B}(y_1,t)\dots\mathbb{B}(y_N,t)|0\rangle \Big\rangle \\ = (\emptyset|\Gamma_+(x_N,t)\dots\Gamma_+(x_1,t)\Gamma_-(y_1,t)\dots\Gamma_-(y_N,t)|\emptyset)\end{aligned} \tag{4.5.44}$$

which lends itself to immediate evaluation. This is because the t-deformed half-vertex operators, like their counterparts in the previous chapter, obey a simple commutation rule. To derive this rule, we use the commutator (4.2.31) to obtain the equation

$$\begin{aligned}&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1-t^m)(1-t^n)x^m y^n}{mn}[H_m(t),H_{-n}(t)] \\ =&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(1-t^m)(1-t^n)x^m y^n}{mn}\frac{m\delta_{m,n}}{1-t^m} = \sum_{m=1}^{\infty}\frac{(1-t^m)x^m y^m}{m}\end{aligned} \tag{4.5.45}$$

which, in turn, implies that

$$\begin{aligned}\Gamma_+(x,t)\Gamma_-(y,t) &= \exp\left(\sum_{m=1}^{\infty}\frac{(1-t^m)(xy)^m}{m}\right)\Gamma_-(y,t)\Gamma_+(x,t) \\ &= \frac{1-txy}{1-xy}\Gamma_-(y,t)\Gamma_+(x,t)\end{aligned} \tag{4.5.46}$$

Employing the commutation relation (4.5.46) repeatedly in (4.5.44) yields

$$\lim_{M\to\infty} \Big\langle \langle 0|\mathbb{C}(x_N,t)\dots\mathbb{C}(x_1,t)\mathcal{M}^*_\psi(t), \mathcal{M}_\psi(t)\mathbb{B}(y_1,t)\dots\mathbb{B}(y_N,t)|0\rangle \Big\rangle = \prod_{i,j=1}^{N}\frac{1-tx_i y_j}{1-x_i y_j} \tag{4.5.47}$$

Comparing equations (4.5.43) and (4.5.47), we have proved that

$$\sum_{\pi\subseteq[N,N,\infty]} A_\pi\Big(\{x\},\{y\},t\Big) = \prod_{i,j=1}^{N}\frac{1-tx_i y_j}{1-x_i y_j} \tag{4.5.48}$$

which is a simpler evaluation of this generating function than in the finite case (4.4.11). As we observed in the previous chapter, this type of calculation could be performed using a symmetric function identity. This continues to be the case here, and we remark that (4.5.48) could also be obtained using (4.4.11) and the identity

$$\sum_{\mu \subseteq [N,\infty]} b_\mu(t) P_\mu(\{x\},t) P_\mu(\{y\},t) = \prod_{i,j=1}^{N} \frac{1-tx_iy_j}{1-x_iy_j} \tag{4.5.49}$$

from section 4, chapter III of [65]. As we have done in the previous chapter, let us specialize the variables $\{x_1,\ldots,x_N\}$ and $\{y_1,\ldots,y_N\}$ to

$$x_i = y_i = z^{i-\frac{1}{2}} \quad \text{for all } 1 \le i \le N \tag{4.5.50}$$

giving rise to the equation

$$\sum_{\pi \subseteq [N,N,\infty]} \prod_{i=1}^{N} \left(1-t^i\right)^{p_i(\pi)} z^{|\pi|} = \prod_{i,j=1}^{N} \frac{1-tz^{i+j-1}}{1-z^{i+j-1}} \tag{4.5.51}$$

where $|\pi|$ is the weight of the plane partition π, and $p_i(\pi)$ is the number of paths in π at level i. Taking the limit $N \to \infty$ we obtain

$$\sum_{\pi} \prod_{i=1}^{\infty} \left(1-t^i\right)^{p_i(\pi)} z^{|\pi|} = \prod_{i=1}^{\infty} \frac{(1-tz^i)^i}{(1-z^i)^i} \tag{4.5.52}$$

where the sum is now over plane partitions of completely arbitrary dimension. This result specializes to the generating function (3.4.26) by setting $t=0$, and to (3.8.28) by setting $t=-1$. The generating function (4.5.52) first appeared in [88], where it was proved using combinatorial methods. The fermionic proof which we have described was the key result of [36].

4.6 Conclusion

In this chapter we studied the Bethe eigenvectors of the q-boson model. We gave a representation of the q-boson algebra on the vector space $\mathcal{V}$, which collapses to the previously encountered representations of the phase and i-boson algebras in the respective limits $q \to \infty$ and $q \to i$. We defined a map taking basis elements of $\mathcal{V}$ to partitions in the deformed Fock space $\mathcal{F}_\psi(t)$, and calculated the image of the Bethe eigenvectors under this map. However, since the underlying fermions are more complicated than in the previous chapter, we were unable to connect the finite lattice scalar product with a solution of a classical hierarchy.

On the other hand, in the infinite lattice limit of the q-boson model we were able to obtain a result which logically extends the material of chapter 3. We refer to theorem 1, which shows that when $M \to \infty$ the action of a B-operator on a general state maps to the action of a t-deformed half-vertex operator on the image state. Since the t-deformed half-vertex operators obey simple commutation relations, we could evaluate the scalar product in product form in the $M \to \infty$ limit. This provided a new proof of Vuletić's path-weighted generating function for plane partitions.

We now list two questions which arise from this work, which are worthy of further investigation.

1. *Does there exist a t-deformed hierarchy admitting Hall-Littlewood polynomials as τ-functions?* This question does not seem to have been properly addressed in the literature. The closing remarks of [52] discuss this possibility, but in the context of a slightly different t-deformed Clifford algebra than the one presented in this chapter. Assuming the existence of such a hierarchy, we would expect that individual Hall-Littlewood polynomials and the finite lattice q-boson scalar product should be valid solutions.

2. *Can we extend our procedure to calculating the generating functions for other weighted plane partitions?* A more general generating function for plane partitions, related to the Macdonald polynomials, was obtained in [88]. Furthermore, a more general species of deformed fermions appeared in [53], where they were used once again in the context of Macdonald polynomials. It was realized in [36] that the generating function for Macdonald-type plane partitions could be obtained using the fermions of [53], but an explicit proof is yet to be presented.

Chapter 5

XXZ model and the KP hierarchy

5.0 Introduction

The one-dimensional Heisenberg magnet was introduced in 1928 [46] and first solved in 1931 [6], marking the invention of the Bethe Ansatz. It is a model for a one-dimensional lattice of spin-$\frac{1}{2}$ fermions with nearest neighbour interactions, and its Hamiltonian contains three parameters J_x, J_y, J_z which describe the anisotropy of the system.[1] When these three parameters are different, it is called the XYZ model. In this chapter we will consider the case of partial isotropy, with $J_x = J_y \neq J_z$, which is known as the XXZ model or XXZ spin-$\frac{1}{2}$ chain. Despite its extensive history, the XXZ model continues to be of great interest to researchers in the field of quantum integrable models. In this chapter we will describe several original contributions which we have made to the study of this model, outlining in particular its apparent connection with the KP hierarchy.

In section 5.1 we introduce the basics of the XXZ model, including its space of states $\mathcal{V}$, Hamiltonian $\mathcal{H}$, and review the construction of its eigenvectors using the algebraic Bethe Ansatz. Following chapter 2, we also provide graphical representations for the entries of the R-matrix and the monodromy matrix.

One of the most fundamental quantities pertaining to the XXZ model is the domain wall partition function. This object acquires its name because it is equal to the partition function of the six-vertex model [5], under domain wall boundary conditions. In [60], V E Korepin found a set of conditions on the partition function which determine it uniquely. These conditions were subsequently solved by A G Izergin in [48], where a determinant expression for the partition function was obtained. We reproduce these results in section 5.2, working from both an algebraic and a graphical perspective. We conclude the section with the first of our new results, showing that the partition function is a power-sum specialization of a KP τ-function [40].

The scalar product is another object of essential interest in the XXZ model. It

[1] See, for example, chapter II of [61].

is a function of two sets of auxiliary rapidities $\{u\}_N$ and $\{v\}_N$, and its evaluation depends on the restrictions imposed on these rapidities. When $\{u\}_N$ and $\{v\}_N$ are unrestricted, the scalar product can be expressed as a complicated sum over a product of two determinants [61]. On the other hand, when $\{u\}_N$ and $\{v\}_N$ are equal and satisfy the Bethe equations, the scalar product has a compact determinant expression proposed by M Gaudin [45] and proved by Korepin in [60]. In this chapter we are interested in the intermediate case when $\{v\}_N$ satisfies the Bethe equations, but $\{u\}_N$ is unrestricted. Although these conditions are weaker than in Gaudin's case, the scalar product remains expressible as a determinant, as was discovered by N A Slavnov in [81]. This determinant formula has been crucial to the work of N Kitanine et al. on correlation functions, see for example [57], [58], [59].

In section 5.3 we outline a new proof of the Slavnov scalar product formula, which is inspired by the Izergin-Korepin procedure for the partition function. Our proof is based on a sequence of incremental scalar products, S_0 through to S_N, where S_0 is equal to the partition function up to a multiplicative factor, and S_N is the actual scalar product. These scalar products were defined and calculated in [59], but this earlier proof was less elementary and relied on the Drinfel'd twist technique of [68]. We end the section with a new result which extends that of [40], showing that the Slavnov scalar product is a power-sum specialization of a KP τ-function [41].

Motivated by chapters 3 and 4, in section 5.4 we derive an explicit expression for the XXZ model Bethe eigenvectors. That is, we write the Bethe eigenvectors as sums over elementary spin states, and calculate the coefficients within these sums. We find that the coefficients can be expressed as objects which generalize determinants. We name these objects weighted determinants, since they are determinant-like sums over permutations, only each term in the sum is weighted with a factor that depends on pairs of elements in the permutation. It transpires that Hall-Littlewood functions are weighted determinants, which allows us to compare our form for the XXZ eigenvectors with Tsilevich's form for the q-boson eigenvectors, as was given in section 4.3.

5.1 XXZ spin-$\frac{1}{2}$ chain

In this section we introduce the basics of the XXZ spin-$\frac{1}{2}$ chain, describing its solution via the quantum inverse scattering method/algebraic Bethe Ansatz. The notations and conventions that we adopt are basically consistent with those of [59]. For a more general introduction, the reader is referred to chapters VI and VII of the book [61].

5.1.1 Space of states $\mathcal{V}$ and inner product $\mathcal{I}$

The finite length XXZ spin-$\frac{1}{2}$ chain consists of a one-dimensional lattice with M sites. As we mentioned in chapter 2, each site is populated by a single spin-$\frac{1}{2}$ fermion. To each site m we therefore associate a two-dimensional vector space $\mathcal{V}_m$

with the basis

$$\text{Basis}(\mathcal{V}_m) = \Big\{ \uparrow_m, \downarrow_m \Big\} \tag{5.1.1}$$

where for convenience we have adopted the notations

$$\uparrow_m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_m, \quad \downarrow_m = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_m \tag{5.1.2}$$

Physically speaking, $\uparrow_m$ and $\downarrow_m$ represent the spin eigenstates of a spin-$\frac{1}{2}$ fermion at site m. The global vector space $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_M$ has the basis

$$\text{Basis}(\mathcal{V}) = \left\{ |\lambda\rangle = \bigotimes_{m \in \lambda} \uparrow_m \bigotimes_{m \notin \lambda} \downarrow_m \right\} \tag{5.1.3}$$

where $\lambda = \{\lambda_1, \ldots, \lambda_l\}$ ranges over all strict partitions $\{M \geq \lambda_1 > \cdots > \lambda_l > 0\}$ and all lengths $0 \leq l \leq M$ are allowed. The partition $\lambda = \{M, \ldots, 1\}$ corresponds with the state in which all spins are up, while $\lambda = \emptyset$ corresponds with the state in which all spins are down. Since they play an important role in our calculations, we prescribe these states their own special notation, by defining

$$|\Uparrow_M\rangle = \bigotimes_{m=1}^{M} \uparrow_m, \quad |\Downarrow_M\rangle = \bigotimes_{m=1}^{M} \downarrow_m \tag{5.1.4}$$

We will also make use of the notation

$$|\Uparrow_{N/M}\rangle = \bigotimes_{1 \leq m \leq N} \uparrow_m \bigotimes_{N < m \leq M} \downarrow_m, \qquad |\Downarrow_{N/M}\rangle = \bigotimes_{1 \leq m \leq N} \downarrow_m \bigotimes_{N < m \leq M} \uparrow_m \tag{5.1.5}$$

for states whose first N spins are up (down), with all the remaining spins being down (up), respectively. As always, we fix a bilinear inner product $\mathcal{I}$ acting on $\mathcal{V}$. Its action is given by

$$\mathcal{I}\Big(|\lambda\rangle, |\mu\rangle\Big) = \delta_{\lambda,\mu} \tag{5.1.6}$$

for all basis vectors $|\lambda\rangle, |\mu\rangle$. In other words, $\mathcal{I}$ induces orthonormality between the elements of the basis (5.1.3).

As was explained in chapter 2, it is advantageous to define vector spaces dual to those already introduced. To each site m we associate the dual vector space $\mathcal{V}_m^*$ with the basis

$$\text{Basis}(\mathcal{V}_m^*) = \Big\{ \uparrow_m^*, \downarrow_m^* \Big\} \tag{5.1.7}$$

where we have adopted the notations

$$\uparrow_m^* = \begin{pmatrix} 1 & 0 \end{pmatrix}_m, \quad \downarrow_m^* = \begin{pmatrix} 0 & 1 \end{pmatrix}_m \tag{5.1.8}$$

From this, we construct the dual space of states $\mathcal{V}^* = \mathcal{V}_1^* \otimes \cdots \otimes \mathcal{V}_M^*$ whose basis is given by

$$\text{Basis}(\mathcal{V}^*) = \left\{ \langle\lambda| = \bigotimes_{m\in\lambda} \uparrow_m^* \bigotimes_{m\notin\lambda} \downarrow_m^* \right\} \tag{5.1.9}$$

where, as before, $\lambda = \{\lambda_1, \ldots, \lambda_l\}$ ranges over all strict partitions which satisfy $\{M \geq \lambda_1 > \cdots > \lambda_l > 0\}$ and all lengths $0 \leq l \leq M$ are allowed. Similarly to above, we also introduce the notations

$$\langle \Uparrow_M | = \bigotimes_{m=1}^{M} \uparrow_m^*, \quad \langle \Downarrow_M | = \bigotimes_{m=1}^{M} \downarrow_m^* \tag{5.1.10}$$

for the dual total spin up/down states, and

$$\langle \Uparrow_{N/M} | = \bigotimes_{1\leq m\leq N} \uparrow_m^* \bigotimes_{N<m\leq M} \downarrow_m^*, \qquad \langle \Downarrow_{N/M} | = \bigotimes_{1\leq m\leq N} \downarrow_m^* \bigotimes_{N<m\leq M} \uparrow_m^* \tag{5.1.11}$$

for the dual partial spin up/down states. Finally, we fix an action of $\mathcal{V}^*$ on $\mathcal{V}$ by defining

$$\langle\lambda|\mu\rangle = \mathcal{I}\Big(|\lambda\rangle, |\mu\rangle\Big) \tag{5.1.12}$$

for all basis vectors $\langle\lambda| \in \mathcal{V}^*$ and $|\mu\rangle \in \mathcal{V}$. Notice that this definition is consistent with treating $\langle\lambda|\mu\rangle$ as a product of the matrices (5.1.2) and (5.1.8).

5.1.2 $sl_q(2)$ algebra

The algebra which underpins the theory of the XXZ model is a q-deformation of the Lie algebra $sl(2)$. It is denoted $sl_q(2)$ and generated by $\{\sigma^+, \sigma^-, \sigma^z\}$ which satisfy the commutation relations

$$[\sigma^+, \sigma^-] = \frac{q^{\sigma^z/2} - q^{-\sigma^z/2}}{q^{1/2} - q^{-1/2}}, \quad [\sigma^z, \sigma^+] = 2\sigma^+, \quad [\sigma^z, \sigma^-] = -2\sigma^- \tag{5.1.13}$$

By taking the limit $q \to 1$, one recovers the commutation relations of $sl(2)$. As in previous chapters, we will consider M copies of $sl_q(2)$, generated by $\{\sigma_1^+, \sigma_1^-, \sigma_1^z\}$ through to $\{\sigma_M^+, \sigma_M^-, \sigma_M^z\}$. In accordance with our chapter 2 conventions, we denote these algebras by $\mathcal{A}_1, \ldots, \mathcal{A}_M$ with $\mathfrak{a}_m^+ = \sigma_m^+, \mathfrak{a}_m^- = \sigma_m^-, \mathfrak{a}_m^0 = \sigma_m^z$. Different copies of $sl_q(2)$ commute, giving rise to the equations

$$[\sigma_m^+, \sigma_n^-] = \delta_{m,n} \frac{q^{\sigma_m^z/2} - q^{-\sigma_m^z/2}}{q^{1/2} - q^{-1/2}}, \quad [\sigma_m^z, \sigma_n^+] = 2\delta_{m,n}\sigma_m^+, \quad [\sigma_m^z, \sigma_n^-] = -2\delta_{m,n}\sigma_m^- \tag{5.1.14}$$

for all $1 \le m, n \le M$.

5.1.3 Representations of $sl_q(2)$ algebras

It is possible to provide representations of the algebras $\mathcal{A}_1, \ldots, \mathcal{A}_M$ by identifying each of the generators $\{\sigma_m^+, \sigma_m^-, \sigma_m^z\}$ with a 2×2 matrix. In particular, let us define the Pauli matrices

$$\sigma_m^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_m, \quad \sigma_m^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}_m, \quad \sigma_m^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}_m \tag{5.1.15}$$

with $i = \sqrt{-1}$, and the spin-changing matrices

$$\sigma_m^+ = \frac{1}{2}(\sigma_m^x + i\sigma_m^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_m, \quad \sigma_m^- = \frac{1}{2}(\sigma_m^x - i\sigma_m^y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_m \tag{5.1.16}$$

where in all cases the subscript m is used to indicate that the matrices belong to the algebra $\mathcal{A}_m$. It is easy to show that the relations (5.1.14) are faithfully represented with σ_m^z as defined by (5.1.15) and $\sigma_m^\pm$ as defined by (5.1.16).

By virtue of these identifications and the interpretation (5.1.2) of $\uparrow_m, \downarrow_m$ we automatically obtain actions of $\mathcal{A}_1, \ldots, \mathcal{A}_M$ on $\mathcal{V}$. For all $1 \le m \le M$ we have

$$\sigma_m^+ \uparrow_m = 0, \ \sigma_m^+ \downarrow_m = \uparrow_m, \quad \sigma_m^- \uparrow_m = \downarrow_m, \ \sigma_m^- \downarrow_m = 0, \quad \sigma_m^z \uparrow_m = \uparrow_m, \ \sigma_m^z \downarrow_m = -\downarrow_m \tag{5.1.17}$$

Similarly, using the interpretation (5.1.8) of $\uparrow_m^*, \downarrow_m^*$ we deduce actions of $\mathcal{A}_1, \ldots, \mathcal{A}_M$ on $\mathcal{V}^*$. For all $1 \le m \le M$ we obtain

$$\uparrow_m^* \sigma_m^+ = \downarrow_m^*, \; \downarrow_m^* \sigma_m^+ = 0, \quad \uparrow_m^* \sigma_m^- = 0, \; \downarrow_m^* \sigma_m^- = \uparrow_m^*, \quad \uparrow_m^* \sigma_m^z = \uparrow_m^*, \; \downarrow_m^* \sigma_m^z = - \downarrow_m^* \tag{5.1.18}$$

Notice that the actions (5.1.17) and (5.1.18) are consistent with those of the most general context, that is, with equations (2.1.3) and (2.1.11) from chapter 2. The correspondence is realized by setting $\uparrow_m = |\frac{1}{2}\rangle_m$, $\downarrow_m = |-\frac{1}{2}\rangle_m$, $\uparrow_m^* = \langle\frac{1}{2}|_m$, $\downarrow_m^* = \langle-\frac{1}{2}|_m$.

5.1.4 Hamiltonian $\mathcal{H}$

The Hamiltonian of the finite length XXZ spin-$\frac{1}{2}$ chain is given by

$$\mathcal{H} = \sum_{m=1}^{M} \Big(\sigma_m^x \sigma_{m+1}^x + \sigma_m^y \sigma_{m+1}^y + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \Big) \tag{5.1.19}$$

where $\Delta = \frac{1}{2}(q^{1/2} + q^{-1/2})$ is the anisotropy parameter of the model, and the periodicity conditions $\sigma_{M+1}^x = \sigma_1^x, \sigma_{M+1}^y = \sigma_1^y, \sigma_{M+1}^z = \sigma_1^z$ are assumed. Alternatively, the Hamiltonian (5.1.19) may be expressed as

$$\mathcal{H} = \sum_{m=1}^{M} \Big(2\sigma_m^+ \sigma_{m+1}^- + 2\sigma_m^- \sigma_{m+1}^+ + \Delta(\sigma_m^z \sigma_{m+1}^z - 1) \Big) \tag{5.1.20}$$

where we assume $\sigma_{M+1}^{\pm} = \sigma_1^{\pm}$. This is the form which corresponds with the general Hamiltonian (2.1.18) discussed in chapter 2. Once again, we follow the quantum inverse scattering/algebraic Bethe Ansatz procedure for finding the eigenvectors $|\Psi\rangle \in \mathcal{V}$ of $\mathcal{H}$.

5.1.5 R-matrix, crossing symmetry, Yang-Baxter equation

The R-matrix corresponding to the XXZ spin-$\frac{1}{2}$ chain is given by

$$R_{ab}(u,v) = \begin{pmatrix} [u-v+\gamma] & 0 & 0 & 0 \\ 0 & [u-v] & [\gamma] & 0 \\ 0 & [\gamma] & [u-v] & 0 \\ 0 & 0 & 0 & [u-v+\gamma] \end{pmatrix}_{ab} \tag{5.1.21}$$

where we have defined $[u] = 2\sinh u$. The R-matrix is an element of $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b)$. The variables u, v are rapidities associated to the vector spaces $\mathcal{V}_a, \mathcal{V}_b$, and γ is the crossing parameter, related to q via the equation $q = e^{2\gamma}$. Recalling the conventions of chapter 2 we identify the entries of (5.1.21) with vertices, as shown in figure 5.1.

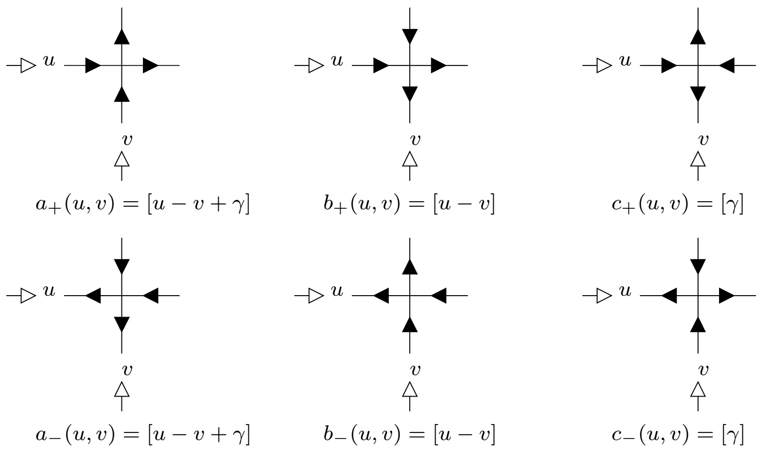

Figure 5.1: Six vertices associated to the XXZ R-matrix. Each entry of the R-matrix (5.1.21) is matched with a vertex.

The entries in the R-matrix (5.1.21) are parametrized in terms of trigonometric functions. An alternative parametrization is obtained by defining

$$x = e^{2u}, \quad y = e^{2v}, \quad q = e^{2\gamma} \tag{5.1.22}$$

Letting $e^{u+v+\gamma} R_{ab}(u,v) = R'_{ab}(x,y)$ under the change of variables (5.1.22), we find

$$R'_{ab}(x,y) = \begin{pmatrix} qx - y & 0 & 0 & 0 \\ 0 & q^{\frac{1}{2}}(x-y) & x^{\frac{1}{2}} y^{\frac{1}{2}}(q-1) & 0 \\ 0 & x^{\frac{1}{2}} y^{\frac{1}{2}}(q-1) & q^{\frac{1}{2}}(x-y) & 0 \\ 0 & 0 & 0 & qx - y \end{pmatrix}_{ab} \tag{5.1.23}$$

Either of the parametrizations (5.1.21) or (5.1.23) may be used in the quantum inverse scattering approach to the XXZ spin-$\frac{1}{2}$ chain. We will generally prefer to use (5.1.21), since its matrix entries are neater. However the parametrization (5.1.23) is necessary to produce polynomial KP τ-functions. We reconcile this problem by performing all calculations in terms of (5.1.21), and then making the simple change of variables (5.1.22) in those places where we discuss KP τ-functions.

Lemma 1. Define $\bar{u} = u + \gamma$ for all rapidities u. The R-matrix has the *crossing symmetry* property

$$R_{ab}(u,v) = -\sigma_b^y R_{ba}(v, \bar{u})^{\mathrm{t}_b} \sigma_b^y \tag{5.1.24}$$

where σ_b^y is the second of the Pauli matrices (5.1.15) acting in $\mathcal{V}_b$, and t_b denotes transposition in the space $\mathrm{End}(\mathcal{V}_b)$.

Proof. We have

$$R_{ba}(v,\bar{u})^{t_b} = \begin{pmatrix} [v-\bar{u}+\gamma] & 0 & 0 & [\gamma] \\ 0 & [v-\bar{u}] & 0 & 0 \\ 0 & 0 & [v-\bar{u}] & 0 \\ [\gamma] & 0 & 0 & [v-\bar{u}+\gamma] \end{pmatrix}_{ba} \tag{5.1.25}$$

which leads to the equation

$$-\sigma_b^y R_{ba}(v,\bar{u})^{t_b} \sigma_b^y = \begin{pmatrix} [\bar{u}-v] & 0 & 0 & 0 \\ 0 & [\bar{u}-v-\gamma] & [\gamma] & 0 \\ 0 & [\gamma] & [\bar{u}-v-\gamma] & 0 \\ 0 & 0 & 0 & [\bar{u}-v] \end{pmatrix}_{ba} \tag{5.1.26}$$

Finally, using the definition $\bar{u} = u+\gamma$ and the fact that $R_{ba}(u,v) = R_{ab}(u,v)$, we prove (5.1.24). The crossing symmetry relation (5.1.24) will be essential in the calculations of later sections.

□

Lemma 2. The R-matrix obeys the *Yang-Baxter equation*

$$R_{ab}(u,v)R_{ac}(u,w)R_{bc}(v,w) = R_{bc}(v,w)R_{ac}(u,w)R_{ab}(u,v) \tag{5.1.27}$$

which holds in $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b \otimes \mathcal{V}_c)$ for all u, v, w.

Proof. This is one of the most essential identities in quantum integrable models, [49]. It can be verified by direct calculation, but for simplicity we omit these details. □

5.1.6 L-matrix and local intertwining equation

The L-matrix for the XXZ model depends on a single indeterminate u, and acts in the space $\mathcal{V}_a$. Its entries are operators acting at the m^{th} lattice site, and identically everywhere else. It has the form

$$L_{am}(u) = \begin{pmatrix} [u+\frac{\gamma}{2}\sigma_m^z] & [\gamma]\sigma_m^- \\ [\gamma]\sigma_m^+ & [u-\frac{\gamma}{2}\sigma_m^z] \end{pmatrix}_a \tag{5.1.28}$$

where we have defined, as before, $[u] = 2\sinh u$. Using the definition of the R-matrix (5.1.21) and the L-matrix (5.1.28), the local intertwining equation is given by

$$R_{ab}(u,v)L_{am}(u)L_{bm}(v) = L_{bm}(v)L_{am}(u)R_{ab}(u,v) \tag{5.1.29}$$

This is 4×4 matrix equation, which gives rise to sixteen scalar identities. Each of these identities may be checked by direct calculation, and by using the commutation relations (5.1.14).

Specializing to the spin-$\frac{1}{2}$ representation of $\{\sigma_m^+, \sigma_m^-, \sigma_m^z\}$ given by equations (5.1.15) and (5.1.16), we find that the L-matrix (5.1.28) takes the form

$$L_{am}(u) = \begin{pmatrix} [u+\frac{\gamma}{2}] & 0 & 0 & 0 \\ 0 & [u-\frac{\gamma}{2}] & [\gamma] & 0 \\ 0 & [\gamma] & [u-\frac{\gamma}{2}] & 0 \\ 0 & 0 & 0 & [u+\frac{\gamma}{2}] \end{pmatrix}_{am} = R_{am}(u, \gamma/2) \qquad (5.1.30)$$

from which we see that it is equal to the R-matrix $R_{am}(u, w_m)$ with $w_m = \frac{\gamma}{2}$. In this representation, the local intertwining equation (5.1.29) becomes

$$R_{ab}(u,v)R_{am}(u,\gamma/2)R_{bm}(v,\gamma/2) = R_{bm}(v,\gamma/2)R_{am}(u,\gamma/2)R_{ab}(u,v) \qquad (5.1.31)$$

which is simply a corollary of the Yang-Baxter equation (5.1.27).

5.1.7 Monodromy matrix and global intertwining equation

In the last subsection we observed that the XXZ L-matrix $L_{am}(u)$ is equal to the R-matrix $R_{am}(u, w_m)$ under the specialization $w_m = \frac{\gamma}{2}$. Using this observation, it is convenient to construct the monodromy matrix as an ordered product of the R-matrices $R_{am}(u, w_m)$, without restricting the variables w_m. That is, we define

$$T_a(u, \{w\}_M) = R_{a1}(u, w_1) \dots R_{aM}(u, w_M) \qquad (5.1.32)$$

The variables $\{w_1, \dots, w_M\}$ are called *inhomogeneities* and the usual monodromy matrix is recovered by setting $w_m = \frac{\gamma}{2}$ for all $1 \le m \le M$. It turns out that the inclusion of the variables $\{w_1, \dots, w_M\}$ simplifies many later calculations. As usual, the contribution from the space $\mathrm{End}(\mathcal{V}_a)$ can be exhibited explicitly by defining

$$T_a(u, \{w\}_M) = \begin{pmatrix} A(u,\{w\}_M) & B(u,\{w\}_M) \\ C(u,\{w\}_M) & D(u,\{w\}_M) \end{pmatrix}_a \qquad (5.1.33)$$

where the matrix entries are all operators acting in $\mathcal{V} = \mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_M$. When writing the monodromy matrix and its entries, it is conventional to display dependence only on the variable associated to $\mathcal{V}_a$, letting $T_a(u, \{w\}_M) = T_a(u)$, $A(u, \{w\}_M) = A(u)$, $B(u, \{w\}_M) = B(u)$, $C(u, \{w\}_M) = C(u)$, $D(u, \{w\}_M) = D(u)$. The diagrammatic version of these operators is given in figure 5.2.

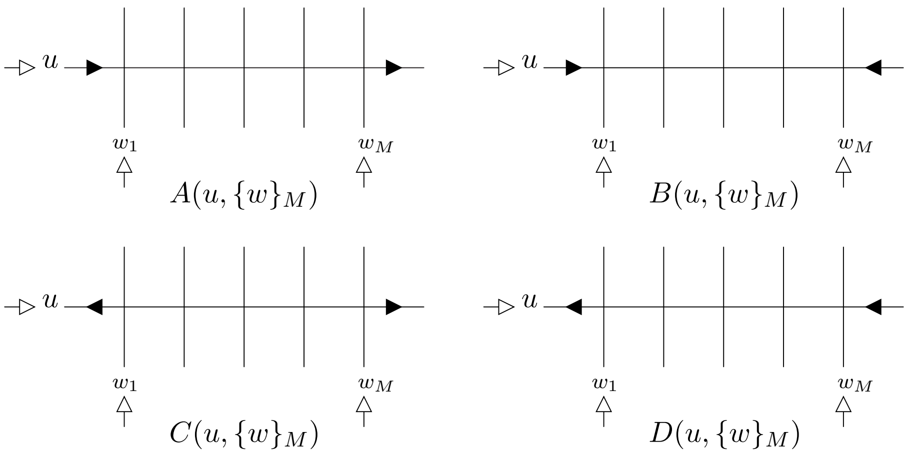

Figure 5.2: Four vertex-strings of the XXZ monodromy matrix. Each entry of (5.1.33) is matched with a string of R-matrix vertices.

Owing to the Yang-Baxter equation (5.1.27) and the definition of the monodromy matrix (5.1.32), we obtain the global intertwining equation

$$R_{ab}(u,v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u,v) \tag{5.1.34}$$

Explicitly writing out the entries of the monodromy matrix, we find that

$$R_{ab}(u,v) \begin{pmatrix} A(u)A(v) & A(u)B(v) & B(u)A(v) & B(u)B(v) \\ A(u)C(v) & A(u)D(v) & B(u)C(v) & B(u)D(v) \\ C(u)A(v) & C(u)B(v) & D(u)A(v) & D(u)B(v) \\ C(u)C(v) & C(u)D(v) & D(u)C(v) & D(u)D(v) \end{pmatrix}_{ab} \tag{5.1.35}$$
$$= \begin{pmatrix} A(v)A(u) & B(v)A(u) & A(v)B(u) & B(v)B(u) \\ C(v)A(u) & D(v)A(u) & C(v)B(u) & D(v)B(u) \\ A(v)C(u) & B(v)C(u) & A(v)D(u) & B(v)D(u) \\ C(v)C(u) & D(v)C(u) & C(v)D(u) & D(v)D(u) \end{pmatrix}_{ab} R_{ab}(u,v)$$

This equation contains sixteen commutation relations between the entries of the monodromy matrix. In the forthcoming calculations we will require

$$B(u)B(v) = B(v)B(u) \tag{5.1.36}$$
$$[\gamma]B(u)D(v) + [u-v]D(u)B(v) = [u-v+\gamma]B(v)D(u) \tag{5.1.37}$$
$$C(u)C(v) = C(v)C(u) \tag{5.1.38}$$
$$[u-v+\gamma]C(u)D(v) = [u-v]D(v)C(u) + [\gamma]C(v)D(u) \tag{5.1.39}$$
$$D(u)D(v) = D(v)D(u) \tag{5.1.40}$$

which are obtained by multiplying the matrices in (5.1.35) and equating components.

Lemma 3. Let $\{u\}_L = \{u_1,\ldots,u_L\}$ and $\{w\}_M = \{w_1,\ldots,w_M\}$ be two sets of variables, with cardinalities $L, M \geq 1$. To each variable u_l we associate an auxiliary space $\mathcal{V}_{a_l}$, while to each variable w_m we associate a quantum space $\mathcal{V}_m$. Define

$$T\Big(\{u\}_L,\{w\}_M\Big) = T_{a_L}(u_L,\{w\}_M)\ldots T_{a_1}(u_1,\{w\}_M) \tag{5.1.41}$$

We claim that

$$T\Big(\{u\}_L,\{w\}_M\Big) = (-)^{LM}\overline{T}_1(w_1,\{\bar{u}\}_L)\ldots\overline{T}_M(w_M,\{\bar{u}\}_L) \tag{5.1.42}$$

where for all $1 \leq m \leq M$ we have defined

$$\overline{T}_m(w_m,\{\bar{u}\}_L) = \left(\begin{array}{cc} D(w_m,\{\bar{u}\}_L) & -B(w_m,\{\bar{u}\}_L) \\ -C(w_m,\{\bar{u}\}_L) & A(w_m,\{\bar{u}\}_L) \end{array}\right)_m \tag{5.1.43}$$

with $\{\bar{u}\}_L = \{u_1+\gamma,\ldots,u_L+\gamma\}$.

Proof. By the definition (5.1.41), we have

$$T\Big(\{u\}_L,\{w\}_M\Big) = \tag{5.1.44}$$
$$\Big(R_{a_L 1}(u_L,w_1)\ldots R_{a_L M}(u_L,w_M)\Big)\ldots\Big(R_{a_1 1}(u_1,w_1)\ldots R_{a_1 M}(u_1,w_M)\Big)$$

Commuting R-matrices which act in different spaces leads to the equation

$$T\Big(\{u\}_L,\{w\}_M\Big) = \tag{5.1.45}$$
$$\Big(R_{a_L 1}(u_L,w_1)\ldots R_{a_1 1}(u_1,w_1)\Big)\ldots\Big(R_{a_L M}(u_L,w_M)\ldots R_{a_1 M}(u_1,w_M)\Big)$$

and using the crossing symmetry relation (5.1.24) on every R-matrix in (5.1.45) we obtain

$$\begin{aligned} T\Big(\{u\}_L,\{w\}_M\Big) &= (-)^{LM}\sigma_1^y\Big(R_{1a_L}(w_1,\bar{u}_L)^{t_1}\ldots R_{1a_1}(w_1,\bar{u}_1)^{t_1}\Big)\sigma_1^y \\ &\times\ldots\times\sigma_M^y\Big(R_{Ma_L}(w_M,\bar{u}_L)^{t_M}\ldots R_{Ma_1}(w_M,\bar{u}_1)^{t_M}\Big)\sigma_M^y \end{aligned} \tag{5.1.46}$$

Using a standard identity of matrix transposition we may reverse the order of the R-matrices in (5.1.46), yielding

$$\begin{aligned} T\Big(\{u\}_L,\{w\}_M\Big) &= (-)^{LM}\sigma_1^y\Big(R_{1a_1}(w_1,\bar{u}_1)\ldots R_{1a_L}(w_1,\bar{u}_L)\Big)^{t_1}\sigma_1^y \\ &\times\ldots\times\sigma_M^y\Big(R_{Ma_1}(w_M,\bar{u}_1)\ldots R_{Ma_L}(w_M,\bar{u}_L)\Big)^{t_M}\sigma_M^y \end{aligned} \tag{5.1.47}$$

Finally we replace each parenthesized term in (5.1.47) with its corresponding monodromy matrix, which gives

$$T\Big(\{u\}_L, \{w\}_M\Big) = (-)^{LM} \sigma_1^y T_1(w_1, \{\bar{u}\}_L)^{\mathrm{t}_1} \sigma_1^y \ldots \sigma_M^y T_M(w_M, \{\bar{u}\}_L)^{\mathrm{t}_M} \sigma_M^y \quad (5.1.48)$$

Letting the monodromy matrices in (5.1.48) be written in the form

$$T_m(w_m, \{\bar{u}\}_L) = \left(\begin{array}{cc} A(w_m, \{\bar{u}\}_L) & B(w_m, \{\bar{u}\}_L) \\ C(w_m, \{\bar{u}\}_L) & D(w_m, \{\bar{u}\}_L) \end{array} \right)_m \quad (5.1.49)$$

for all $1 \leq m \leq M$ and contracting on the quantum spaces $\mathcal{V}_1, \ldots, \mathcal{V}_M$, we recover the result (5.1.42).

□

Lemma 4. The spin-up states $| \Uparrow_M \rangle$ and $\langle \Uparrow_M |$ are eigenvectors of the diagonal elements of the monodromy matrix. Explicitly, we have

$$A(u, \{w\}_M) | \Uparrow_M \rangle = \prod_{j=1}^{M} [u - w_j + \gamma] | \Uparrow_M \rangle, \quad D(u, \{w\}_M) | \Uparrow_M \rangle = \prod_{j=1}^{M} [u - w_j] | \Uparrow_M \rangle \quad (5.1.50)$$

$$\langle \Uparrow_M | A(u, \{w\}_M) = \prod_{j=1}^{M} [u - w_j + \gamma] \langle \Uparrow_M |, \quad \langle \Uparrow_M | D(u, \{w\}_M) = \prod_{j=1}^{M} [u - w_j] \langle \Uparrow_M | \quad (5.1.51)$$

In addition, the spin-down states $| \Downarrow_M \rangle$ and $\langle \Downarrow_M |$ are eigenvectors of the diagonal elements of the monodromy matrix. Explicitly, we have

$$A(u, \{w\}_M) | \Downarrow_M \rangle = \prod_{j=1}^{M} [u - w_j] | \Downarrow_M \rangle, \quad D(u, \{w\}_M) | \Downarrow_M \rangle = \prod_{j=1}^{M} [u - w_j + \gamma] | \Downarrow_M \rangle \quad (5.1.52)$$

$$\langle \Downarrow_M | A(u, \{w\}_M) = \prod_{j=1}^{M} [u - w_j] \langle \Downarrow_M |, \quad \langle \Downarrow_M | D(u, \{w\}_M) = \prod_{j=1}^{M} [u - w_j + \gamma] \langle \Downarrow_M | \quad (5.1.53)$$

Proof. See lemma 2 in chapter 2. □

5.1.8 Recovering $\mathcal{H}$ from the transfer matrix

Let $t(u,\{w\}_M) = A(u,\{w\}_M) + D(u,\{w\}_M)$ denote the transfer matrix of the XXZ model. The Hamiltonian (5.1.19) is recovered via the formula

$$\mathcal{H} = [\gamma] \frac{d}{du} \log \breve{t}(u) \Big|_{u=\frac{\gamma}{2}} \quad \text{where} \quad \breve{t}(u) = \frac{t(u,\{w\}_M)}{\prod_{j=1}^{M}[u - w_j + \gamma]} \Bigg|_{w_1=\cdots=w_M=\frac{\gamma}{2}} \tag{5.1.54}$$

Therefore all eigenvectors of $t(u,\{w\}_M)$ are also eigenvectors of $\mathcal{H}$. Our attention turns, therefore, to finding vectors $|\Psi\rangle \in \mathcal{V}$ satisfying

$$\Big(A(u,\{w\}_M) + D(u,\{w\}_M)\Big)|\Psi\rangle = \tau_\Psi(u,\{w\}_M)|\Psi\rangle \tag{5.1.55}$$

for some suitable constants $\tau_\Psi(u,\{w\}_M)$.

5.1.9 Bethe Ansatz for the eigenvectors

As we explained in a general setting in chapter 2, the eigenvectors of $t(u,\{w\}_M)$ are given by the Ansatz

$$|\Psi\rangle = B(v_1,\{w\}_M)\ldots B(v_N,\{w\}_M)|\Uparrow_M\rangle \tag{5.1.56}$$

where we assume that $N \leq M$, since we annihilate the state $|\Uparrow_M\rangle$ when acting with more B-operators than the number of sites in the spin chain. Similarly, we construct eigenvectors of $t(u,\{w\}_M)$ in the dual space of states via the Ansatz

$$\langle\Psi| = \langle\Uparrow_M|C(v_N,\{w\}_M)\ldots C(v_1,\{w\}_M) \tag{5.1.57}$$

where we again restrict $N \leq M$. To ensure that (5.1.56) and (5.1.57) are genuine eigenvectors, the variables $\{v_1,\ldots,v_N\}$ are required to satisfy the Bethe equations. Using the commutation relations (5.1.35) and the actions (5.1.50), it is possible to show that $|\Psi\rangle, \langle\Psi|$ are eigenvectors of $t(u,\{w\}_M)$ if and only if

$$\prod_{j=1}^{M} \frac{[v_i - w_j + \gamma]}{[v_i - w_j]} = \prod_{j\neq i}^{N} \frac{[v_i - v_j + \gamma]}{[v_i - v_j - \gamma]} \tag{5.1.58}$$

for all $1 \leq i \leq N$.[2] Later in the chapter we will find an explicit expression for the Bethe eigenvectors (5.1.56) and (5.1.57) by expanding them in terms of the bases (5.1.3) and (5.1.9), whereby we have

[2] See theorem 1 in chapter 2. Notice that for the present model

$$a(v_i,v_j) = [v_i - v_j + \gamma], \quad \alpha(v_i,\{w\}_M) = \prod_{j=1}^{M}[v_i - w_j + \gamma], \quad \delta(v_i,\{w\}_M) = \prod_{j=1}^{M}[v_i - w_j] \tag{5.1.59}$$

which, when substituted into (2.3.6), produce the equations (5.1.58).

$$\prod_{i=1}^{N} B(v_i, \{w\}_M)|\Uparrow_M\rangle = \sum_{\lambda|\ell(\lambda)=M-N} b_\lambda\Big(\{v\}_N, \{w\}_M\Big)|\lambda\rangle \tag{5.1.60}$$

$$\langle\Uparrow_M| \prod_{i=1}^{N} C(v_i, \{w\}_M) = \sum_{\lambda|\ell(\lambda)=M-N} c_\lambda\Big(\{v\}_N, \{w\}_M\Big)\langle\lambda| \tag{5.1.61}$$

with both sums taken over all strict partitions λ of $M - N$ integers which satisfy $\{M \geq \lambda_1 > \cdots > \lambda_{M-N} \geq 1\}$. The focus of section 5.4 will be the calculation of the coefficients $b_\lambda(\{v\}_N, \{w\}_M)$ and $c_\lambda(\{v\}_N, \{w\}_M)$.

5.2 Domain wall partition function

In this section we will study Z_N, the domain wall partition function of the XXZ spin-$\frac{1}{2}$ chain. This quantity acquires its name because it is equal to the partition function of the six-vertex model under domain wall boundary conditions. As we will see in sections 5.3 and 5.4, the calculation of Z_N is essential for the explicit evaluation of more complicated objects within the XXZ model, such as its scalar product and Bethe eigenvectors.

5.2.1 Definition of $Z_N(\{v\}_N, \{w\}_N)$

Let $\{v\}_N = \{v_1, \ldots, v_N\}$ and $\{w\}_N = \{w_1, \ldots, w_N\}$ be two sets of variables. The domain wall partition function has the algebraic definition

$$Z_N\Big(\{v\}_N, \{w\}_N\Big) = \langle\Downarrow_N| \prod_{i=1}^{N} B(v_i, \{w\}_N)|\Uparrow_N\rangle \tag{5.2.1}$$

where the ordering of the B-operators is irrelevant, since from equation (5.1.36) we know that they commute.

The relationship of $Z_N(\{v\}_N, \{w\}_N)$ to the coefficients in equation (5.1.60) is easily exposed. Consider the Bethe eigenstate (5.1.60) in the case when $M = N$, which corresponds to acting with the same number of B-operators as the number of sites on the spin chain. In this situation we obtain

$$\prod_{i=1}^{N} B(v_i, \{w\}_N)|\Uparrow_N\rangle = b_\emptyset\Big(\{v\}_N, \{w\}_N\Big)|\Downarrow_N\rangle \tag{5.2.2}$$

from which we see that $b_\emptyset(\{v\}_N, \{w\}_N) = Z_N(\{v\}_N, \{w\}_N)$. Hence the domain wall partition function is, in some sense, the simplest non-trivial example amongst the coefficients $b_\lambda(\{v\}_N, \{w\}_M)$.

5.2.2 Graphical representation of partition function

Using the graphical conventions described in the previous section, the domain wall partition function may be represented as the $N \times N$ lattice shown in figure 5.3.

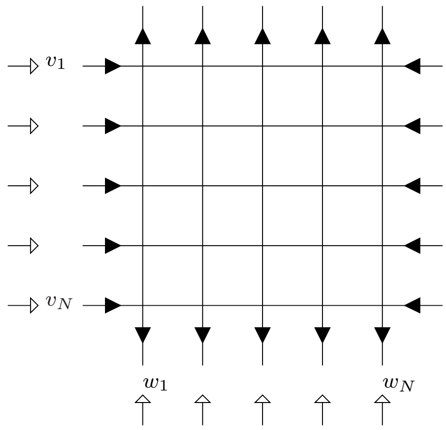

Figure 5.3: Domain wall partition function of the six-vertex model. The top row of upward pointing arrows corresponds with the state vector $| \Uparrow_N \rangle$. The bottom row of downward pointing arrows corresponds with the dual state vector $\langle \Downarrow_N |$. Each horizontal lattice line corresponds to multiplication by a B-operator.

5.2.3 Conditions on $Z_N(\{v\}_N, \{w\}_N)$

In [60] Korepin showed that the function $Z_N(\{v\}_N, \{w\}_N)$ satisfies a set of four conditions which determine it uniquely. We reproduce these facts below, with the following two lemmas.

Lemma 5. Let us adopt the shorthand $Z_N = Z_N(\{v\}_N, \{w\}_N)$. For all $N \geq 2$ we claim that

1. Z_N is symmetric in the $\{w\}_N$ variables.

2. Z_N is a trigonometric polynomial of degree $N-1$ in the rapidity variable v_N.

3. Setting $v_N = w_N - \gamma$, Z_N satisfies the recursion relation

$$Z_N \Big|_{v_N = w_N - \gamma} = [\gamma] \prod_{i=1}^{N-1} [v_i - w_N][w_N - w_i - \gamma] Z_{N-1} \tag{5.2.3}$$

where Z_{N-1} is the domain wall partition function on a square lattice of size $N-1$.

In addition, we have the supplementary condition

4. The partition function on the 1×1 lattice is given by $Z_1 = [\gamma]$.

Proof.

1. We write the domain wall partition function in the form

$$Z_N\Big(\{v\}_N,\{w\}_N\Big) = \langle \Uparrow_N^a | \otimes \langle \Downarrow_N | T\Big(\{v\}_N,\{w\}_N\Big) | \Uparrow_N\rangle \otimes | \Downarrow_N^a\rangle \tag{5.2.4}$$

with $T(\{v\}_N,\{w\}_N)$ given by (5.1.41), and where we have defined the auxiliary states

$$\langle \Uparrow_N^a | = \bigotimes_{i=1}^N \uparrow_{a_i}^*, \quad | \Downarrow_N^a\rangle = \bigotimes_{i=1}^N \downarrow_{a_i} \tag{5.2.5}$$

Using the expressions (5.1.42), (5.1.43) for $T(\{v\}_N,\{w\}_N)$ and contracting on the quantum spaces $\mathcal{V}_1,\ldots,\mathcal{V}_N$ gives

$$Z_N\Big(\{v\}_N,\{w\}_N\Big) = \langle \Uparrow_N^a | C(w_1,\{\bar{v}\}_N)\ldots C(w_N,\{\bar{v}\}_N) | \Downarrow_N^a\rangle \tag{5.2.6}$$

The diagrammatic interpretation of the equivalence (5.2.6) is shown in figure 5.4.

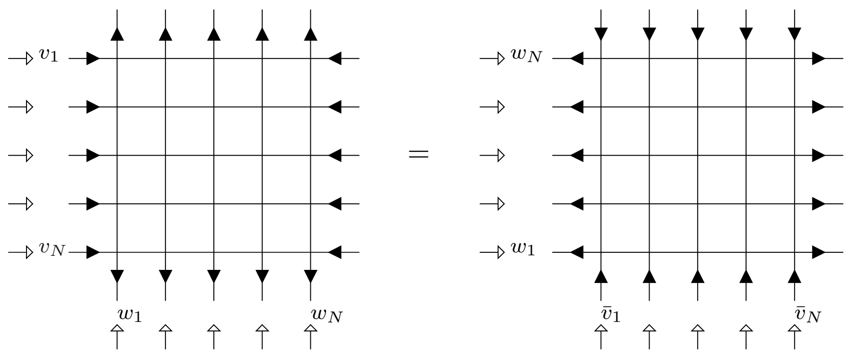

Figure 5.4: Equivalent expressions for the domain wall partition function. The diagram on the left represents a stacking of the operators $B(v_i,\{w\}_N)$. The diagram on the right represents a stacking of the operators $C(w_i,\{\bar{v}\}_N)$.

Thanks to equation (5.1.38) the $C(w_i,\{\bar{v}\}_N)$ operators all commute, proving that $Z_N(\{v\}_N,\{w\}_N)$ is symmetric in $\{w\}_N$.

2. By inserting the set of states $\sum_{n=1}^N \sigma_n^+ | \Downarrow_N\rangle\langle \Downarrow_N | \sigma_n^-$ after the first B-operator appearing in (5.2.1), we obtain the expansion

$$Z_N\Big(\{v\}_N,\{w\}_N\Big) = \sum_{n=1}^N \langle \Downarrow_N | B(v_N,\{w\}_N)\sigma_n^+ | \Downarrow_N\rangle\langle \Downarrow_N | \sigma_n^- \prod_{i=1}^{N-1} B(v_i,\{w\}_N) | \Uparrow_N\rangle \tag{5.2.7}$$

in which all dependence on v_N appears in the first factor within the sum. Hence we proceed to calculate $\langle \Downarrow_N | B(v_N, \{w\}_N) \sigma_n^+ | \Downarrow_N \rangle$ for all $1 \le n \le N$, as shown below.

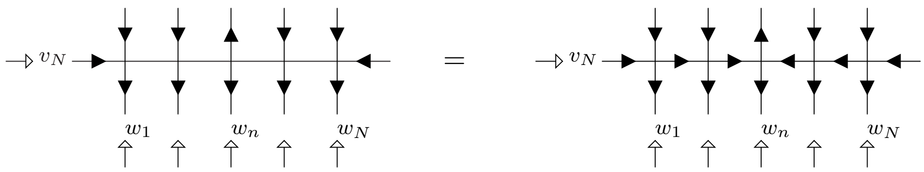

Figure 5.5: Peeling away the bottom row of the partition function. The diagram on the left represents $\langle \Downarrow_N | B(v_N, \{w\}_N) \sigma_n^+ | \Downarrow_N \rangle$, with the internal black arrows being summed over all configurations. The diagram on the right represents the only surviving configuration.

The right hand side of figure 5.5 is simply a product of vertices. Replacing each vertex with its corresponding trigonometric weight, we conclude that

$$\langle \Downarrow_N | B(v_N, \{w\}_N) \sigma_n^+ | \Downarrow_N \rangle = \prod_{1 \le i < n} [v_N - w_i][\gamma] \prod_{n < i \le N} [v_N - w_i + \gamma] \tag{5.2.8}$$

Substituting (5.2.8) into the expansion (5.2.7) gives

$$Z_N\Big(\{v\}_N, \{w\}_N\Big) = \tag{5.2.9}$$
$$[\gamma] \sum_{n=1}^{N} \prod_{1 \le i < n} [v_N - w_i] \prod_{n < i \le N} [v_N - w_i + \gamma] \langle \Downarrow_N | \sigma_n^- \prod_{i=1}^{N-1} B(v_i, \{w\}_N) | \Uparrow_N \rangle$$

From this equation we see that every term in $Z_N(\{v\}_N, \{w\}_N)$ contains a product of exactly $N-1$ trigonometric functions with argument v_N. Therefore $Z_N(\{v\}_N, \{w\}_N)$ is a trigonometric polynomial of degree $N-1$ in the variable v_N.

3. We start from the expansion (5.2.9) of the domain wall partition function, and set $v_N = w_N - \gamma$. This causes all terms in the summation over $1 \le n \le N$ to collapse to zero except the $n = N$ term, giving

$$Z_N\Big(\{v\}_N, \{w\}_N\Big)\Big|_{v_N = w_N - \gamma} = [\gamma] \prod_{i=1}^{N-1} [w_N - w_i - \gamma] \langle \Downarrow_N | \sigma_N^- \prod_{i=1}^{N-1} B(v_i, \{w\}_N) | \Uparrow_N \rangle \tag{5.2.10}$$

We then consider the graphical representation of $\langle \Downarrow_N | \sigma_N^- \prod_{i=1}^{N-1} B(v_i, \{w\}_N) | \Uparrow_N \rangle$, as shown below.

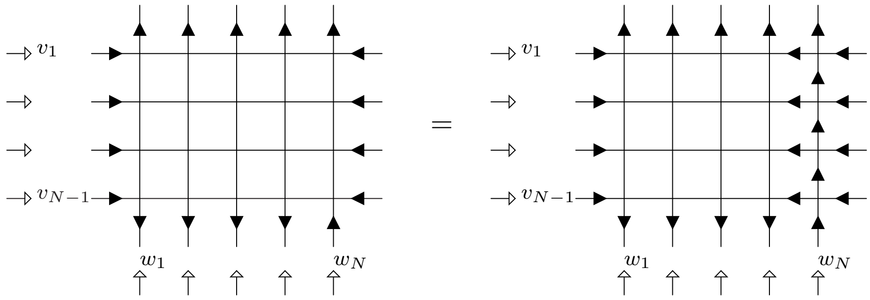

Figure 5.6: Peeling away the right-most column of the partition function. The diagram on the left hand side represents $\langle \Downarrow_N | \sigma_N^- \prod_{i=1}^{N-1} B(v_i, \{w\}_N) | \Uparrow_N \rangle$, with the internal black arrows being summed over all configurations. The diagram on the right contains all surviving configurations.

The right hand side of figure 5.6 represents the $(N-1) \times (N-1)$ domain wall partition function, multiplied by a column of vertices. Replacing these vertices with their trigonometric weights, we find that

$$\langle \Downarrow_N | \sigma_N^- \prod_{i=1}^{N-1} B(v_i, \{w\}_N) | \Uparrow_N \rangle = \prod_{i=1}^{N-1} [v_i - w_N] Z_{N-1}\Big(\{v\}_{N-1}, \{w\}_{N-1}\Big) \qquad (5.2.11)$$

Substituting (5.2.11) into (5.2.10) produces the required recursion relation (5.2.3).

4. Specializing the definition (5.2.1) to the case $N = 1$ gives

$$Z_1(v_1, w_1) = \langle \Downarrow_1 | B(v_1, w_1) | \Uparrow_1 \rangle = \uparrow_{a_1}^* \otimes \downarrow_1^* R_{a_1 1}(v_1, w_1) \uparrow_1 \otimes \downarrow_{a_1} = [\gamma] \qquad (5.2.12)$$

as required. Alternatively, the lattice representation of Z_1 is simply the top right vertex in figure 5.1, whose weight is equal to $[\gamma]$.

□

Lemma 6. Let $\{\breve{Z}_N\}_{N \in \mathbb{N}}$ denote a set of functions $\breve{Z}_N(\{v\}_N, \{w\}_N)$ which satisfy the four conditions of the previous lemma. Then $\breve{Z}_N = Z_N$ for all $N \geq 1$. In other words, the conditions imposed on the domain wall partition function determine it uniquely.

Proof. From condition **4** on $\breve{Z}_1, Z_1$ we know that $\breve{Z}_1 = Z_1$. Hence we may assume that $\breve{Z}_{N-1} = Z_{N-1}$ for some $N \geq 2$. Using this assumption together with condition **3** on $\breve{Z}_N, Z_N$ yields

$$\breve{Z}_N\Big|_{v_N=w_N-\gamma} = [\gamma]\prod_{i=1}^{N-1}[v_i-w_N][w_N-w_i-\gamma]\breve{Z}_{N-1} \tag{5.2.13}$$
$$= [\gamma]\prod_{i=1}^{N-1}[v_i-w_N][w_N-w_i-\gamma]Z_{N-1} = Z_N\Big|_{v_N=w_N-\gamma}$$

Condition **1** on $\breve{Z}_N, Z_N$ states that both are symmetric in the variables $\{w\}_N$. Using this fact in the previous equation, we find that

$$\breve{Z}_N\Big|_{v_N=w_i-\gamma} = Z_N\Big|_{v_N=w_i-\gamma} \quad \text{for all } 1\le i\le N \tag{5.2.14}$$

which proves that $\breve{Z}_N$ and Z_N are equal at N distinct values of v_N. By condition **2**, both functions are trigonometric polynomials of degree $N-1$ in v_N, so their equality at N points implies $\breve{Z}_N = Z_N$ everywhere. This completes the proof by induction.

□

5.2.4 Determinant expression for Z_N

In [48] Izergin found a function which satisfies the four conditions of the previous subsection, and is therefore equal to the domain wall partition function. We present this formula below.

Lemma 7. For all $N \ge 1$ we define

$$\breve{Z}_N\Big(\{v\}_N,\{w\}_N\Big) = \frac{\prod_{i,j=1}^{N}[v_i-w_j+\gamma][v_i-w_j]}{\prod_{1\le i<j\le N}[v_i-v_j][w_j-w_i]}\det\left(\frac{[\gamma]}{[v_i-w_j+\gamma][v_i-w_j]}\right)_{1\le i,j\le N}$$
$$= \frac{\det\left([\gamma]\prod_{k\ne i}^{N}[v_k-w_j+\gamma][v_k-w_j]\right)_{1\le i,j\le N}}{\prod_{1\le i<j\le N}[v_i-v_j][w_j-w_i]} \tag{5.2.15}$$

The functions $\{\breve{Z}_N\}_{N\in\mathbb{N}}$ satisfy the four conditions of lemma 5. Equivalently, the domain wall partition function Z_N is equal to the right hand side of (5.2.15).

Proof.

1. Permuting $w_m \leftrightarrow w_n$ in $\det\Big([\gamma]\prod_{k\neq i}^{N}[v_k - w_j + \gamma][v_k - w_j]\Big)_{1\leq i,j\leq N}$ swaps two columns of the determinant, which introduces a minus sign into the numerator of (5.2.15). Similarly, permuting $w_m \leftrightarrow w_n$ in $\prod_{1\leq i<j\leq N}[w_j - w_i]$ introduces a minus sign into the denominator of (5.2.15). These minus signs cancel, leaving $\breve{Z}_N$ invariant under permutations of the $\{w\}_N$ variables.

2. The numerator of (5.2.15) is a trigonometric polynomial of degree $2N-2$ in v_N, with zeros at the points $v_N = v_n$ for all $1 \leq n \leq N-1$, since such a substitution would render two rows of the determinant equal. The denominator of (5.2.15) is a trigonometric polynomial of degree $N-1$ in v_N, with zeros at the same points. Cancelling these common zeros, $\breve{Z}_N$ is a trigonometric polynomial of degree $N-1$ in v_N.

3. Expanding the determinant in $\breve{Z}_N$ along its N^{th} row, we find

$$\breve{Z}_N\Big(\{v\}_N,\{w\}_N\Big) = \sum_{j=1}^{N} \breve{Z}_{N-1}\Big(\{v\}_{N-1},\{w_1,\dots,\widehat{w_j},\dots,w_N\}\Big) \times \frac{[\gamma]\prod_{k=1}^{N-1}[v_k - w_j+\gamma][v_k - w_j]\prod_{k\neq j}^{N}[v_N - w_k+\gamma][v_N-w_k]}{\prod_{k=1}^{N-1}[v_k - v_N]\prod_{k\neq j}^{N}[w_j - w_k]} \tag{5.2.16}$$

where $\widehat{w_j}$ denotes the omission of that variable. Setting $v_N = w_N - \gamma$ in the above expression, all terms in the summation over $1 \leq j \leq N$ vanish except the $j = N$ term, and we obtain

$$\breve{Z}_N\Big|_{v_N = w_N - \gamma} = [\gamma]\prod_{k=1}^{N-1}[w_N - w_k - \gamma][v_k - w_N]\breve{Z}_{N-1}\Big(\{v\}_{N-1},\{w\}_{N-1}\Big) \tag{5.2.17}$$

which is the desired recursion relation.

4. From the definition (5.2.15) it is clear that $\breve{Z}_1 = [\gamma]$.

□

5.2.5 Partition function as a power-sum specialized KP τ-function

In this subsection we reproduce the main result of [40] where it was shown that, after a normalization and trivial change of variables, the domain wall partition function is a power-sum specialization of a KP τ-function. We start by defining the function Z'_N as follows

$$Z'_N\Big(\{y\}_N,\{z\}_N\Big) = e^{N^2\gamma}\prod_{i=1}^{N} e^{(N-1)(v_i-w_i)} Z_N\Big(\{v\}_N,\{w\}_N\Big) \tag{5.2.18}$$

where we have made the change of variables $e^{2v_i} = y_i, e^{-2w_i} = -z_i, e^{2\gamma} = q$. In contrast to Z_N, which is a trigonometric function, Z'_N is a genuine polynomial in its variables $\{y\}_N$ and $\{z\}_N$. Applying the normalization (5.2.18) to (5.2.15) and performing the prescribed change of variables, we obtain the explicit formula

$$Z'_N\Big(\{y\}_N,\{z\}_N\Big) = \frac{(q-1)^N \det\left(\displaystyle\prod_{k\neq j}^{N}(1+qy_iz_k)(1+y_iz_k)\right)_{1\le i,j\le N}}{\displaystyle\prod_{1\le i<j\le N}(y_i-y_j)(z_j-z_i)} \tag{5.2.19}$$

In order to progress further, we need to define the *elementary symmetric functions* $e_m\{x\}$ in the set of variables $\{x\} = \{x_1,\ldots,x_N\}$. Following section 2 of chapter I in [65], they are given by

$$e_m\{x\} = \mathrm{Coeff}_{z^m}\left[\prod_{i=1}^{N}(1+x_iz)\right] \tag{5.2.20}$$

Using the definition (5.2.20) of the elementary symmetric functions, the entries of the determinant (5.2.19) can be expanded to produce the equation

$$Z'_N\Big(\{y\}_N,\{z\}_N\Big) = \frac{(q-1)^N \det\left(\displaystyle\sum_{k=1}^{2N-1} y_i^{k-1} e_{k-1}\{q\widehat{z}_j\}\cup\{\widehat{z}_j\}\right)_{1\le i,j\le N}}{\displaystyle\prod_{1\le i<j\le N}(y_i-y_j)(z_j-z_i)} \tag{5.2.21}$$

where $\{q\widehat{z}_j\}\cup\{\widehat{z}_j\}$ denotes the set of $2N$ variables $\{qz_1,\ldots,qz_N,z_1,\ldots,z_N\}$ with the omission of the pair $\{qz_j,z_j\}$. The determinant in (5.2.21) can be manipulated further using the Cauchy-Binet identity, which we state below.

Lemma 8. Fix two positive integers $M \ge N \ge 1$. Let A be an arbitrary $N\times M$ matrix with entries $(A)_{i,j} = A_{i,j}$, and B an arbitrary $M\times N$ matrix with entries $(B)_{i,j} = B_{i,j}$. The Cauchy-Binet identity states that

$$\det\left(\sum_{k=1}^{M} A_{i,k}B_{k,j}\right)_{1\le i,j\le N} = \sum_{M\ge k_1>\cdots>k_N\ge 1} \det\Big(A_{i,k_j}\Big)_{1\le i,j\le N}\det\Big(B_{k_i,j}\Big)_{1\le i,j\le N} \tag{5.2.22}$$

where the sum is taken over all strict partitions satisfying $\{M\ge k_1>\cdots>k_N\ge 1\}$.

Proof. See section 2 of chapter I in [43]. □

Applying the Cauchy-Binet identity (5.2.22) to the determinant (5.2.21) gives

$$Z'_N\Big(\{y\}_N,\{z\}_N\Big) = \frac{(q-1)^N}{\prod\limits_{1\le i<j\le N}(y_i-y_j)(z_j-z_i)}\times \tag{5.2.23}$$
$$\sum_{2N-1\ge k_1>\cdots>k_N\ge 1} \det\Big(y_i^{k_j-1}\Big)_{1\le i,j\le N} \det\Big(e_{k_i-1}\{q\widehat{z}_j\}\cup\{\widehat{z}_j\}\Big)_{1\le i,j\le N}$$

We perform a change in summation variables, writing $k_i - 1 = \mu_i - i + N$ for all $1 \le i \le N$. Making this substitution in (5.2.23) yields

$$Z'_N\Big(\{y\}_N,\{z\}_N\Big) = \frac{(q-1)^N}{\prod\limits_{1\le i<j\le N}(y_i-y_j)(z_j-z_i)}\times \tag{5.2.24}$$
$$\sum_{\mu\subseteq[N,N-1]} \det\Big(y_i^{\mu_j-j+N}\Big)_{1\le i,j\le N} \det\Big(e_{\mu_i-i+N}\{q\widehat{z}_j\}\cup\{\widehat{z}_j\}\Big)_{1\le i,j\le N}$$

where the sum is over all partitions μ whose Young diagrams are contained in the rectangle $N\times(N-1)$. By virtue of the Jacobi-Trudi identity (3.3.14), the first determinant in (5.2.24) matches the numerator of a Schur function. Hence we are able to write

$$Z'_N\Big(\{y\}_N,\{z\}_N\Big) = \sum_{\mu\subseteq[N,N-1]} s_\mu\{y\}\zeta_\mu(\{z\},q) \tag{5.2.25}$$

where we have defined the functions[3]

$$\zeta_\mu(\{z\},q) = \frac{(q-1)^N \det\Big(e_{\mu_i-i+N}\{q\widehat{z}_j\}\cup\{\widehat{z}_j\}\Big)_{1\le i,j\le N}}{\prod\limits_{1\le i<j\le N}(z_j-z_i)} \tag{5.2.26}$$

We have thus shown that Z'_N is equal to

$$\tau_{\rm PF}\{t\} = \sum_{\mu\subseteq[N,N-1]} \chi_\mu\{t\}\zeta_\mu(\{z\},q) \tag{5.2.27}$$

under the power-sum specialization $t_n = \frac{1}{n}\sum_{i=1}^N y_i^n$ for all $n \ge 1$. In the next subsection we show that the polynomial $\tau_{\rm PF}\{t\}$ is a KP τ-function, by writing it as an expectation value of charged fermions.

[3] The coefficients $\zeta_\mu(\{z\},q)$ allow an even simpler expression than (5.2.26) by cancelling the Vandermonde factor in the denominator, as was done in [63].

5.2.6 $\tau_{\text{PF}}\{t\}$ as an expectation value of charged fermions

Our aim is to transform the right hand side of (5.2.27) to the fermionic form (1.2.14) of a KP τ-function. To achieve this, we follow the procedure of subsection 3.2.11. For all integers $i \geq -N$ and $1 \leq j \leq N$ we define

$$c_{i,j} = \begin{cases} e_{i+N}\{q\widehat{z}_j\} \cup \{\widehat{z}_j\}, & -N \leq i < N-1 \\ 0, & i \geq N-1 \end{cases} \tag{5.2.28}$$

where $e_{i+N}\{q\widehat{z}_j\} \cup \{\widehat{z}_j\}$ is an elementary symmetric function (5.2.20) in the variables $\{qz_1, \ldots, qz_N, z_1, \ldots, z_N\}$ with $\{qz_j, z_j\}$ omitted. Using this expression, for all ordered sets of integers $\{m\} = \{m_1 > \cdots > m_N \geq -N\}$ we define the coefficients

$$c_{\{m\}}\Big(\{z\}, q\Big) = \frac{(q-1)^N}{\prod\limits_{1 \leq i < j \leq N} (z_j - z_i)} \det\Big(c_{m_i,j}\Big)_{1 \leq i,j \leq N} \tag{5.2.29}$$

Since $c_{i,j} = 0$ if $i \geq N-1$, the coefficient $c_{\{m\}}(\{z\}, q)$ vanishes if $m_1 \geq N-1$. Letting μ be the partition formed by setting $\mu_i = m_i + i$ for all $1 \leq i \leq N$, we conclude that

$$c_{\{m\}}\Big(\{z\}, q\Big) = \begin{cases} \zeta_\mu(\{z\}, q), & \mu \subseteq [N, N-1] \\ 0, & \mu \not\subseteq [N, N-1] \end{cases} \tag{5.2.30}$$

where $\zeta_\mu(\{z\}, q)$ denotes the function (5.2.26). This enables us to write (5.2.27) in the form

$$\tau_{\text{PF}}\{t\} = \langle 0| e^{H\{t\}} \sum_{\text{card}\{m\}=N} c_{\{m\}}\Big(\{z\}, q\Big) \psi_{m_1} \ldots \psi_{m_N} | -N\rangle \tag{5.2.31}$$

where the sum is over all sets of integers $\{m\} = \{m_1 > \cdots > m_N \geq -N\}$, with the identification $|\mu) = \psi_{m_1} \ldots \psi_{m_N} | -N\rangle$. Expanding in the basis (1.1.9) of $\mathcal{F}_\psi^{(0)}$, we have

$$\sum_{\text{card}\{m\}=N} c_{\{m\}}\Big(\{z\}, q\Big) \psi_{m_1} \ldots \psi_{m_N} | -N\rangle = \zeta_\emptyset |0\rangle + \sum_{m=1}^{N-1} \sum_{n=1}^{N} (-)^{n-1} \zeta_{\{m,1^{n-1}\}} \psi_{m-1} \psi^*_{-n} |0\rangle + g_\psi^{(1)} |0\rangle \tag{5.2.32}$$

where we have abbreviated $\zeta_\mu(\{z\}, q) = \zeta_\mu$ for all partitions μ, and where all monomials within $g_\psi^{(1)} \in Cl_\psi^{(0)}$ consist of at least two positive and two negative fermions.

Up to an overall constant, the coefficients (5.2.29) are determinants. Therefore they satisfy the Plücker relations (1.3.40) automatically, and the right hand side of (5.2.32) must obey the charged fermion bilinear identity (1.2.15). Hence we may follow the procedure used to prove lemma 10 in chapter 1 to obtain

$$\sum_{\mathrm{card}\{m\}=N} c_{\{m\}}\Big(\{z\},q\Big)\psi_{m_1}\dots\psi_{m_N}|-N\rangle = \zeta_\emptyset \exp\left(\sum_{m=1}^{N-1}\sum_{n=1}^{N}(-)^{n-1}\frac{\zeta_{\{m,1^{n-1}\}}}{\zeta_\emptyset}\psi_{m-1}\psi^*_{-n}\right)|0\rangle \tag{5.2.33}$$

Substituting (5.2.33) into (5.2.31), we arrive at the equation

$$\tau_{\mathrm{PF}}\{t\} = \zeta_\emptyset\langle 0|e^{H\{t\}}\exp\left(\sum_{m=1}^{N-1}\sum_{n=1}^{N}(-)^{n-1}\frac{\zeta_{\{m,1^{n-1}\}}}{\zeta_\emptyset}\psi_{m-1}\psi^*_{-n}\right)|0\rangle \tag{5.2.34}$$

which expresses $\tau_{\mathrm{PF}}\{t\}$ in the canonical form of a KP τ-function. The fermionic expression (5.2.34) was originally derived in [40], using a relatively complicated inductive argument. The preceding derivation is simpler in that it only requires lemma 10 of chapter 1.

5.3 Scalar products

In this section we define and calculate a sequence of *intermediate scalar products* S_n, which interpolate between the domain wall partition function and the full scalar product. The domain wall partition function corresponds to the case $n = 0$, whereas the full scalar product is given by the case $n = N$. These functions were originally studied in [59], using Drinfel'd twists in the algebraic Bethe Ansatz setting [68]. Our approach is more elementary, and inspired by the Izergin-Korepin procedure for evaluating the domain wall partition function. The results of this section are to appear subsequently in [89].

5.3.1 Definition of $S_n(\{u\}_n, \{v\}_N, \{w\}_M)$

Let $\{u\}_n = \{u_1, \dots, u_n\}$, $\{v\}_N = \{v_1, \dots, v_N\}$, $\{w\}_M = \{w_1, \dots, w_M\}$ be three sets of variables whose cardinalities satisfy $0 \le n \le N$ and $1 \le N \le M$. We proceed to introduce functions $S_n(\{u\}_n, \{v\}_N, \{w\}_M)$ for all $0 \le n \le N$. In the case $n = 0$ we define

$$S_0\Big(\{v\}_N,\{w\}_M\Big) = \langle \Downarrow_{N/M}|\prod_{j=1}^{N} B(v_j,\{w\}_M)|\Uparrow_M\rangle \tag{5.3.1}$$

where, for conciseness, we have suppressed dependence on the set $\{u\}_0 = \emptyset$. As we will soon show, up to an overall normalization the scalar product S_0 is equal to the domain wall partition function Z_N. Next, for all $1 \le n \le N-1$ we define

$$S_n\Big(\{u\}_n, \{v\}_N, \{w\}_M\Big) = \langle \Downarrow_{\widetilde{N}/M} | \prod_{i=1}^{n} C(u_i, \{w\}_M) \prod_{j=1}^{N} B(v_j, \{w\}_M) | \Uparrow_M \rangle \quad (5.3.2)$$

where we have adopted the notation $\widetilde{N} = N - n$, which is used frequently hereafter. Finally, for $n = N$ we define

$$S_N\Big(\{u\}_N, \{v\}_N, \{w\}_M\Big) = \langle \Uparrow_M | \prod_{i=1}^{N} C(u_i, \{w\}_M) \prod_{j=1}^{N} B(v_j, \{w\}_M) | \Uparrow_M \rangle \quad (5.3.3)$$

which represents the usual scalar product. In all cases (5.3.1)–(5.3.3) we shall assume that the parameters $\{v\}_N$ obey the Bethe equations (5.1.58), while the remaining variables $\{u\}_n$ are considered free. Accordingly, we name these objects *Bethe scalar products*. It turns out that $\{S_n\}_{0 \le n \le N}$ are related by a simple recursion. Hence they provide a convenient way of calculating S_N, starting from Izergin's determinant formula (5.2.15) for Z_N.

5.3.2 Graphical representation of scalar products

We now provide lattice representations of the Bethe scalar products $\{S_n\}_{0 \le n \le N}$. As in the case of the domain wall partition function, these lattices simplify the calculation of the functions themselves.

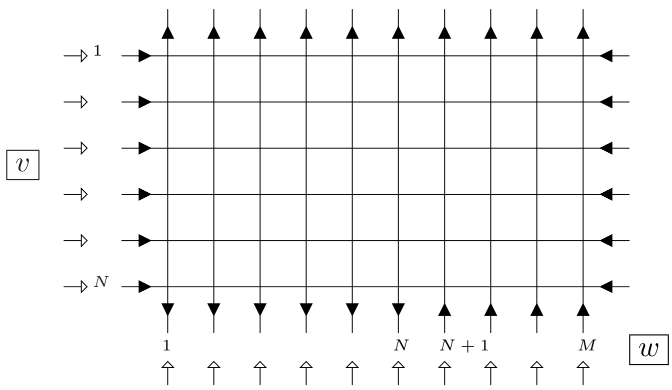

Figure 5.7: Lattice representation of S_0. The top row of arrows corresponds with the state $| \Uparrow_M \rangle$, while the bottom row of arrows corresponds with the dual state $\langle \Downarrow_{N/M} |$. Each horizontal lattice line represents a B-operator $B(v_j, \{w\}_M)$.

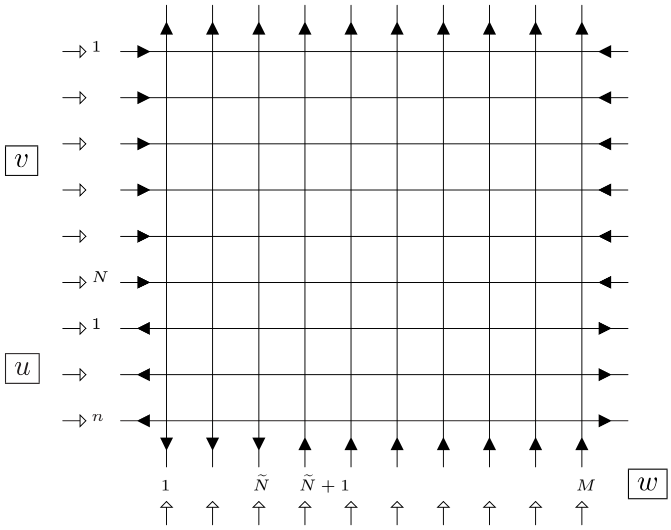

Figure 5.8: Lattice representation of S_n. The top row of arrows corresponds with the state $| \Uparrow_M \rangle$, while the bottom row of arrows corresponds with the dual state $\langle \Downarrow_{\widetilde{N}/M} |$. The highest N horizontal lines represent B-operators $B(v_j, \{w\}_M)$, while the lowest n horizontal lines represent C-operators $C(u_i, \{w\}_M)$.

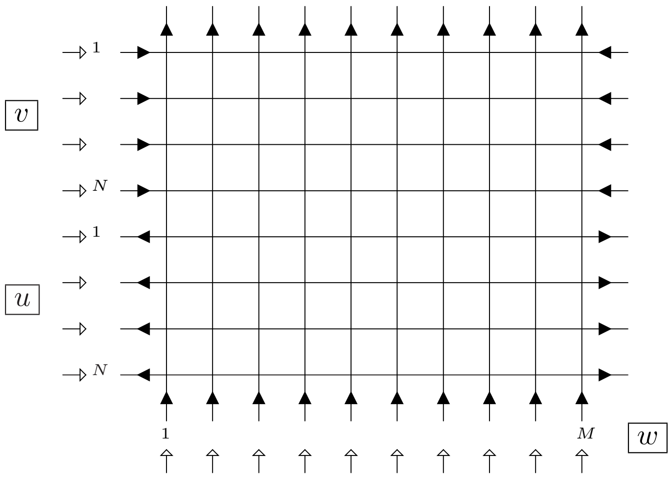

Figure 5.9: Lattice representation of S_N. The top row of arrows corresponds with the state $| \Uparrow_M \rangle$, while the bottom row of arrows corresponds with the dual state $\langle \Uparrow_M |$. The highest N horizontal lines represent B-operators $B(v_j, \{w\}_M)$, while the lowest N horizontal lines represent C-operators $C(u_i, \{w\}_M)$.

5.3.3 Conditions on $S_n(\{u\}_n, \{v\}_N, \{w\}_M)$

Progressing in the same manner as the previous section, we will show that the Bethe scalar products $S_n(\{u\}_n, \{v\}_N, \{w\}_M)$ satisfy a set of properties which determine them uniquely.

Lemma 9. Let us adopt the usual shorthand $S_n = S_n(\{u\}_n, \{v\}_N, \{w\}_M)$. For all $1 \leq n \leq N$ we claim that

1. S_n is symmetric in the variables $\{w_{\widetilde{N}+1}, \ldots, w_M\}$.

2. S_n is a trigonometric polynomial of degree $M-1$ in u_n, with zeros occurring at the points $u_n = w_i - \gamma$, for all $1 \leq i \leq \widetilde{N}$.

3. Setting $u_n = w_{\widetilde{N}+1}$, S_n satisfies the recursion relation

$$S_n\Big|_{u_n = w_{\widetilde{N}+1}} = \prod_{i=1}^{M} [w_{\widetilde{N}+1} - w_i + \gamma] S_{n-1} \tag{5.3.4}$$

where S_{n-1} denotes the Bethe scalar product $S_{n-1}(\{u\}_{n-1}, \{v\}_N, \{w\}_M)$.

In addition, we have the supplementary condition

4. S_0 and Z_N are related via the equation

$$S_0\Big(\{v\}_N, \{w\}_M\Big) = \prod_{i=1}^{N} \prod_{j=N+1}^{M} [v_i - w_j] Z_N\Big(\{v\}_N, \{w\}_N\Big) \tag{5.3.5}$$

where we have defined $\{w\}_N = \{w_1, \ldots, w_N\}$.

Proof.

1. We introduce the auxiliary state vectors

$$\langle \Uparrow_N^a | = \bigotimes_{i=1}^{N} \uparrow_{a_i}^*, \quad \langle \Downarrow_n^b | = \bigotimes_{i=1}^{n} \downarrow_{b_i}^*, \quad | \Downarrow_N^a \rangle = \bigotimes_{i=1}^{N} \downarrow_{a_i}, \quad | \Uparrow_n^b \rangle = \bigotimes_{i=1}^{n} \uparrow_{b_i} \tag{5.3.6}$$

which allow us to write

$$\begin{aligned} &S_n\Big(\{u\}_n, \{v\}_N, \{w\}_M\Big) = \\ &\langle \Downarrow_{\widetilde{N}/M} | \otimes \langle \Uparrow_N^a | \otimes \langle \Downarrow_n^b | T\Big(\{v\}_N \cup \{u\}_n, \{w\}_M\Big) | \Uparrow_n^b \rangle \otimes | \Downarrow_N^a \rangle \otimes | \Uparrow_M \rangle \end{aligned} \tag{5.3.7}$$

where we have defined

$$\begin{aligned} &T\Big(\{v\}_N \cup \{u\}_n, \{w\}_M\Big) = \\ &T_{b_n}(u_n, \{w\}_M) \ldots T_{b_1}(u_1, \{w\}_M) T_{a_N}(v_N, \{w\}_M) \ldots T_{a_1}(v_1, \{w\}_M) \end{aligned} \tag{5.3.8}$$

By application of lemma 3 we thus obtain

$$T\Big(\{v\}_N \cup \{u\}_n, \{w\}_M\Big) = (-)^{M\widetilde{N}} \overline{T}_1(w_1, \{\bar{v}\}_N \cup \{\bar{u}\}_n) \ldots \overline{T}_M(w_M, \{\bar{v}\}_N \cup \{\bar{u}\}_n)$$

where for all $1 \leq i \leq M$ we have set

$$\overline{T}_i(w_i, \{\bar{v}\}_N \cup \{\bar{u}\}_n) = \left(\begin{array}{cc} D(w_i, \{\bar{v}\}_N \cup \{\bar{u}\}_n) & -B(w_i, \{\bar{v}\}_N \cup \{\bar{u}\}_n) \\ -C(w_i, \{\bar{v}\}_N \cup \{\bar{u}\}_n) & A(w_i, \{\bar{v}\}_N \cup \{\bar{u}\}_n) \end{array} \right)_i \tag{5.3.9}$$

with $\{\bar{v}\}_N \cup \{\bar{u}\}_n = \{v_1 + \gamma, \ldots, v_N + \gamma, u_1 + \gamma, \ldots, u_n + \gamma\}$. Finally, contracting on the quantum spaces $\mathcal{V}_1, \ldots, \mathcal{V}_M$ gives

$$S_n\Big(\{u\}_n, \{v\}_N, \{w\}_M\Big) = (-)^{(M+1)\widetilde{N}} \times \tag{5.3.10}$$

$$\langle \Uparrow_N^a | \otimes \langle \Downarrow_n^b | \prod_{i=1}^{\widetilde{N}} C(w_i, \{\bar{v}\}_N \cup \{\bar{u}\}_n) \prod_{j=\widetilde{N}+1}^{M} D(w_j, \{\bar{v}\}_N \cup \{\bar{u}\}_n) | \Uparrow_n^b \rangle \otimes | \Downarrow_N^a \rangle$$

The diagrammatic interpretation of (5.3.10) is shown in figure 5.10.

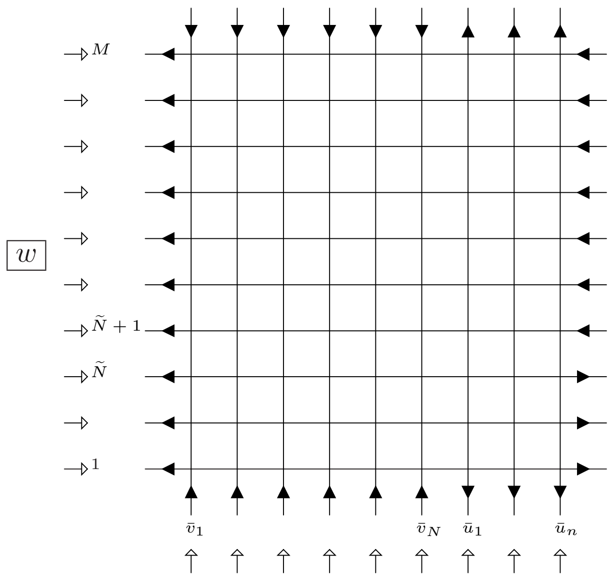

Figure 5.10: Alternative graphical representation of S_n. Neglecting an overall minus sign, S_n is equal to this lattice, which is essentially a rotation of figure 5.8. The top row of arrows represents the state $| \Downarrow_N^a \rangle \otimes | \Uparrow_n^b \rangle$, while the bottom row of arrows represents the dual state $\langle \Uparrow_N^a | \otimes \langle \Downarrow_n^b |$. The lowest $\widetilde{N}$ horizontal lines represent C-operators, with the remaining lines representing D-operators.

By virtue of (5.1.40) the D-operators in (5.3.10) commute, proving that S_n is symmetric in the variables $\{w_{\widetilde{N}+1}, \ldots, w_M\}$.

2. Inserting the set of states $\sum_{m>\widetilde{N}} \sigma_m^-| \Downarrow_{\widetilde{N}/M}\rangle\langle\Downarrow_{\widetilde{N}/M} |\sigma_m^+$ after the first C-operator appearing in (5.3.2), we obtain the expansion

$$S_n\Big(\{u\}_n, \{v\}_N, \{w\}_M\Big) = \sum_{m>\widetilde{N}} \langle\Downarrow_{\widetilde{N}/M} |C(u_n, \{w\}_M)\sigma_m^-| \Downarrow_{\widetilde{N}/M}\rangle \times\langle\Downarrow_{\widetilde{N}/M} |\sigma_m^+ \prod_{i=1}^{n-1} C(u_i, \{w\}_M) \prod_{j=1}^{N} B(v_j, \{w\}_M)| \Uparrow_M\rangle \tag{5.3.11}$$

in which all dependence on u_n appears in the first factor of the sum. We therefore wish to calculate $\langle\Downarrow_{\widetilde{N}/M} |C(u_n, \{w\}_M)\sigma_m^-| \Downarrow_{\widetilde{N}/M}\rangle$ for all $\widetilde{N} < m \leq M$, and do so by identifying it with the string of vertices shown below.

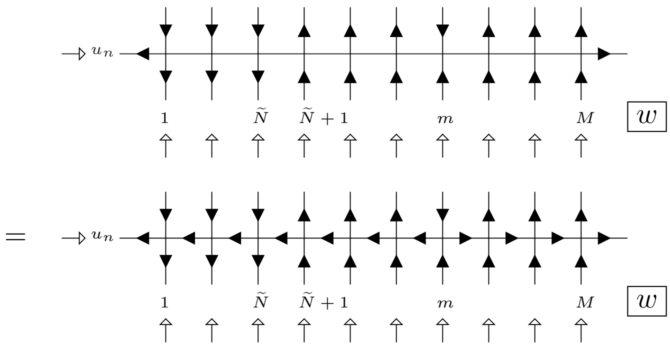

Figure 5.11: Peeling away the bottom row of S_n. The upper diagram represents $\langle\Downarrow_{\widetilde{N}/M} |C(u_n, \{w\}_M)\sigma_m^-| \Downarrow_{\widetilde{N}/M}\rangle$, with the internal black arrows being summed over all configurations. The lower diagram represents the only surviving configuration.

Replacing each vertex in figure 5.11 with its corresponding trigonometric weight, we find that

$$\langle\Downarrow_{\widetilde{N}/M} |C(u_n, \{w\}_M)\sigma_m^-| \Downarrow_{\widetilde{N}/M}\rangle = \prod_{i=1}^{\widetilde{N}}[u_n - w_i + \gamma] \prod_{\widetilde{N}<i<m} [u_n - w_i][\gamma] \prod_{m<i\leq M} [u_n - w_i + \gamma] \tag{5.3.12}$$

Substituting (5.3.12) into the expansion (5.3.11) gives

$$S_n = [\gamma]\prod_{i=1}^{\widetilde{N}}[u_n - w_i + \gamma] \sum_{m>\widetilde{N}} \prod_{\widetilde{N}<i<m} [u_n - w_i] \prod_{m<i\leq M} [u_n - w_i + \gamma] \times\langle\Downarrow_{\widetilde{N}/M} |\sigma_m^+ \prod_{i=1}^{n-1} C(u_i, \{w\}_M) \prod_{j=1}^{N} B(v_j, \{w\}_M)| \Uparrow_M\rangle \tag{5.3.13}$$

From the last expression it is apparent that S_n is a trigonometric polynomial of degree $M-1$ in u_n. Furthermore, $\widetilde{N}$ of the zeros of this polynomial are contained in the factor $\prod_{i=1}^{\widetilde{N}}[u_n - w_i + \gamma]$.

3. Setting $u_n = w_{\widetilde{N}+1}$ in equation (5.3.13), all terms in the sum collapse to zero except the term corresponding to $m = \widetilde{N}+1$, and we obtain

$$S_n\Big|_{u_n=w_{\widetilde{N}+1}} = \prod_{i=1}^{M}[w_{\widetilde{N}+1} - w_i + \gamma]\langle\Downarrow_{\widetilde{N}+1/M} | \prod_{i=1}^{n-1} C(u_i,\{w\}_M) \prod_{j=1}^{N} B(v_j,\{w\}_M)| \Uparrow_M\rangle$$
$$= \prod_{i=1}^{M}[w_{\widetilde{N}+1} - w_i + \gamma] S_{n-1}\Big(\{u\}_{n-1},\{v\}_N,\{w\}_M\Big) \tag{5.3.14}$$

where we have performed the trivial rearrangements $[\gamma] = [w_{\widetilde{N}+1} - w_{\widetilde{N}+1} + \gamma]$ and $\langle\Downarrow_{\widetilde{N}/M} |\sigma^+_{\widetilde{N}+1} = \langle\Downarrow_{\widetilde{N}+1/M} |$ to produce the first line of (5.3.14), while the second line follows directly from the definition of S_{n-1}. Hence we have proved the recursion relation (5.3.4). The diagrammatic interpretation of this identity is given below.

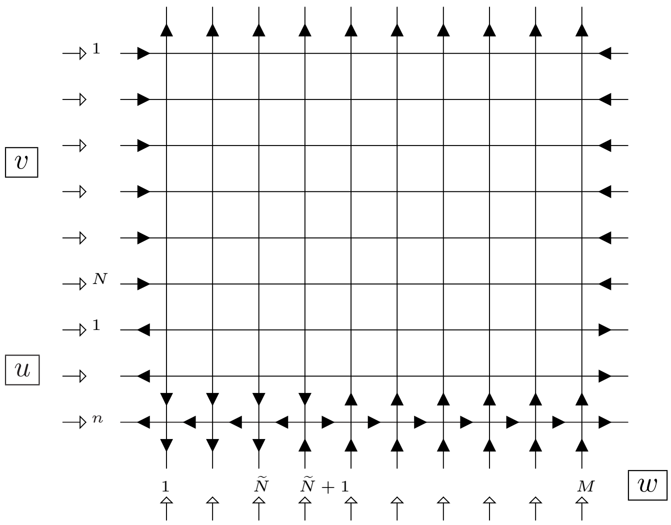

Figure 5.12: Freezing the last row of the S_n lattice. In general, the vertex at the intersection of the u_n and $w_{\widetilde{N}+1}$ lines can be of type $b_-(u_n, w_{\widetilde{N}+1})$ or $c_-(u_n, w_{\widetilde{N}+1})$. Setting $u_n = w_{\widetilde{N}+1}$ causes all configurations with a $b_-(u_n, w_{\widetilde{N}+1})$ vertex to vanish, and we are left with a frozen row of vertices as shown. This row of vertices produces the prefactor in (5.3.4), whilst the remainder of the lattice represents S_{n-1}.

4. From the definition of the state vectors (5.3.6) we obtain

$$S_0\Big(\{v\}_N,\{w\}_M\Big) = \langle\Downarrow_{N/M} | \otimes \langle\Uparrow^a_N |T\Big(\{v\}_N,\{w\}_M\Big)| \Downarrow^a_N\rangle \otimes | \Uparrow_M\rangle \tag{5.3.15}$$

with $T(\{v\}_N, \{w\}_M)$ given by (5.1.41). Using lemma 3 and contracting on the quantum spaces $\mathcal{V}_1, \ldots, \mathcal{V}_M$ gives

$$S_0\Big(\{v\}_N, \{w\}_M\Big) = (-)^{(M+1)N} \langle \Uparrow_N^a | \prod_{i=1}^{N} C(w_i, \{\bar{v}\}_N) \prod_{j=N+1}^{M} D(w_j, \{\bar{v}\}_N) | \Downarrow_N^a \rangle \tag{5.3.16}$$

Now since $| \Downarrow_N^a \rangle$ is an eigenvector of the D-operators, as can be seen from equation (5.1.52), we have

$$S_0\Big(\{v\}_N, \{w\}_M\Big) = (-)^{(M+1)N} \prod_{i=1}^{N} \prod_{j=N+1}^{M} [w_j - \bar{v}_i + \gamma] \langle \Uparrow_N^a | \prod_{i=1}^{N} C(w_i, \{\bar{v}\}_N) | \Downarrow_N^a \rangle \tag{5.3.17}$$

or equivalently, substituting $\bar{v}_i = v_i + \gamma$ for all $1 \le i \le N$ into the previous equation,

$$S_0\Big(\{v\}_N, \{w\}_M\Big) = \prod_{i=1}^{N} \prod_{j=N+1}^{M} [v_i - w_j] \langle \Uparrow_N^a | \prod_{i=1}^{N} C(w_i, \{\bar{v}\}_N) | \Downarrow_N^a \rangle \tag{5.3.18}$$

Comparing with the alternative expression (5.2.6) for the domain wall partition function, equation (5.3.18) completes the proof of (5.3.5). A graphical version of this identity is given below.

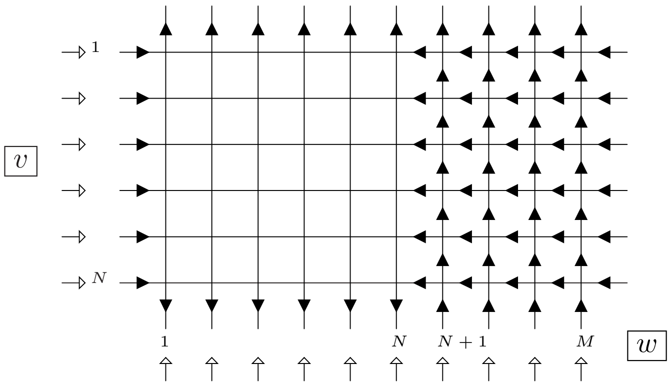

Figure 5.13: Equivalence between S_0 and Z_N. The final $M - N$ columns of the S_0 lattice must assume the configuration shown. All other configurations vanish. The block of vertices thus obtained corresponds with the prefactor in (5.3.5), whilst the remainder of the lattice represents Z_N.

□

5.3.4 Determinant expression for $S_n(\{u\}_n, \{v\}_N, \{w\}_M)$

Lemma 10. Let us define the functions

$$f_i(w) = \frac{[\gamma]}{[v_i - w]} \prod_{k \neq i}^{N} [v_k - w + \gamma] \tag{5.3.19}$$

$$g_i(u) = \frac{[\gamma]}{[v_i - u]} \left(\prod_{k \neq i}^{N} [v_k - u + \gamma] \prod_{k=1}^{M} [u - w_k + \gamma] - \prod_{k \neq i}^{N} [v_k - u - \gamma] \prod_{k=1}^{M} [u - w_k] \right) \tag{5.3.20}$$

Using these definitions, we construct the $N \times N$ matrix

$$\mathcal{M}_n\Big(\{u\}_n, \{v\}_N, \{w\}_M\Big) = \begin{pmatrix} f_1(w_1) & \cdots & f_1(w_{\widetilde{N}}) & g_1(u_n) & \cdots & g_1(u_1) \\ \vdots & & \vdots & \vdots & & \vdots \\ f_N(w_1) & \cdots & f_N(w_{\widetilde{N}}) & g_N(u_n) & \cdots & g_N(u_1) \end{pmatrix} \tag{5.3.21}$$

Assuming that the parameters $\{v\}_N$ satisfy the Bethe equations (5.1.58), we have

$$S_n = \frac{\displaystyle\prod_{i=1}^{N} \prod_{j=1}^{M} [v_i - w_j] \det \mathcal{M}_n\Big(\{u\}_n, \{v\}_N, \{w\}_M\Big)}{\displaystyle\prod_{i=1}^{n} \prod_{j=1}^{\widetilde{N}} [u_i - w_j] \prod_{1 \leq i < j \leq n} [u_i - u_j] \prod_{1 \leq i < j \leq N} [v_i - v_j] \prod_{1 \leq i < j \leq \widetilde{N}} [w_j - w_i]} \tag{5.3.22}$$

The expression (5.3.22) for the intermediate Bethe scalar product S_n originally appeared in appendix C of [59].

Proof. Firstly, one must show that S_n is uniquely determined by the set of conditions in lemma 9. This is accomplished using similar arguments to those presented in lemma 6, and we shall assume this fact *a priori*. Hence it will be sufficient to show that the expression (5.3.22) satisfies the list of properties given in lemma 9.

1. All dependence of (5.3.22) on the variables $\{w_{\widetilde{N}+1}, \ldots, w_M\}$ occurs in the factor $\prod_{i=1}^{N} \prod_{j=1}^{M} [v_i - w_j]$ and in the functions $g_i(u_j)$ in the determinant. Clearly, these terms are invariant under the permutation $w_i \leftrightarrow w_j$ for all $i \neq j$. Hence the expression (5.3.22) is symmetric in $\{w_{\widetilde{N}+1}, \ldots, w_M\}$.

2. Consider the expression (5.3.20) for $g_i(u_n)$. Since the variables $\{v\}_N$ satisfy the Bethe equations (5.1.58), the numerator of $g_i(u_n)$ vanishes in the limit $u_n \to v_i$. It follows that the pole in (5.3.20) is removable, and therefore $g_i(u_n)$ is a trigonometric polynomial of degree $M+N-2$ in u_n. Using this fact, we see that (5.3.22) is a quotient of trigonometric polynomials in u_n. The polynomial in the numerator has degree $M+N-2$, while the polynomial in the denominator has degree $N-1$. We must show that every zero in the denominator is cancelled by a zero in the numerator.

Setting $u_n = u_j$ for $1 \le j \le n-1$ causes two columns of the determinant to become equal, producing $n-1$ zeros in the numerator which cancel $n-1$ of the zeros in the denominator. Furthermore since

$$g_i(w_j) = \frac{[\gamma]}{[v_i - w_j]} \prod_{k\neq i}^{N} [v_k - w_j + \gamma] \prod_{k=1}^{M} [w_j - w_k + \gamma] = \prod_{k=1}^{M} [w_j - w_k + \gamma] f_i(w_j) \tag{5.3.23}$$

it follows that by setting $u_n = w_j$ for $1 \le j \le \widetilde{N}$, two columns of the determinant are equal up to a multiplicative factor, producing $\widetilde{N}$ zeros in the numerator which cancel $\widetilde{N}$ of the zeros in the denominator. This proves that the expression (5.3.22) is a trigonometric polynomial of degree $M-1$ in u_n.

Finally, since

$$\begin{aligned} g_i(w_j - \gamma) &= \frac{-[\gamma]}{[v_i - w_j + \gamma]} \prod_{k\neq i}^{N} [v_k - w_j] \prod_{k=1}^{M} [w_j - w_k - \gamma] \\ &= -\prod_{k=1}^{M} [w_j - w_k - \gamma] \prod_{k=1}^{N} \frac{[v_k - w_j]}{[v_k - w_j + \gamma]} f_i(w_j) \end{aligned} \tag{5.3.24}$$

we see that by setting $u_n = w_j - \gamma$ for all $1 \le j \le \widetilde{N}$, two columns of the determinant are equal up to a multiplicative factor, producing the $\widetilde{N}$ zeros which (5.3.22) requires in order to satisfy property **2**.

3. Using equation (5.3.23) and the definition of the matrix (5.3.21), it is clear that

$$\begin{aligned} \det \mathcal{M}_n\Big(\{u\}_n, \{v\}_N, \{w\}_M\Big)\Big|_{u_n = w_{\widetilde{N}+1}} &= \prod_{k=1}^{M} [w_{\widetilde{N}+1} - w_k + \gamma] \\ &\quad \times \det \mathcal{M}_{n-1}\Big(\{u\}_{n-1}, \{v\}_N, \{w\}_M\Big) \end{aligned} \tag{5.3.25}$$

Furthermore, we notice the trivial product identity

$$\left(\prod_{i=1}^{n} \prod_{j=1}^{\widetilde{N}} [u_i - w_j] \prod_{1 \le i < j \le n} [u_i - u_j] \prod_{1 \le i < j \le \widetilde{N}} [w_j - w_i] \right) \Bigg|_{u_n = w_{\widetilde{N}+1}} = \tag{5.3.26}$$

$$\prod_{i=1}^{n-1} \prod_{j=1}^{\widetilde{N}+1} [u_i - w_j] \prod_{1 \le i < j \le n-1} [u_i - u_j] \prod_{1 \le i < j \le \widetilde{N}+1} [w_j - w_i]$$

Combining the results (5.3.25) and (5.3.26), we find that the expression (5.3.22) satisfies the recursion relation (5.3.4).

4. Taking the $n = 0$ case of (5.3.22) yields

$$S_0\Big(\{v\}_N, \{w\}_M\Big) = \frac{\displaystyle\prod_{i=1}^{N} \prod_{j=1}^{M} [v_i - w_j] \det \mathcal{M}_0}{\displaystyle\prod_{1 \le i < j \le N} [v_i - v_j][w_j - w_i]} \tag{5.3.27}$$

with the matrix $\mathcal{M}_0$ given by

$$\mathcal{M}_0 = \begin{pmatrix} \frac{[\gamma]}{[v_1 - w_1]} \prod_{k \ne 1} [v_k - w_1 + \gamma] & \cdots & \frac{[\gamma]}{[v_1 - w_N]} \prod_{k \ne 1} [v_k - w_N + \gamma] \\ \vdots & & \vdots \\ \frac{[\gamma]}{[v_N - w_1]} \prod_{k \ne N} [v_k - w_1 + \gamma] & \cdots & \frac{[\gamma]}{[v_N - w_N]} \prod_{k \ne N} [v_k - w_N + \gamma] \end{pmatrix} \tag{5.3.28}$$

Comparing (5.3.27) with the determinant expression (5.2.15) for the domain wall partition function, we see that $S_0 = \prod_{i=1}^{N} \prod_{j=N+1}^{M} [v_i - w_j] Z_N$, as required.

□

5.3.5 Evaluation of $S_N(\{u\}_N, \{v\}_N, \{w\}_M)$

Let us now consider the $n = N$ case of equation (5.3.22) in more detail. For purely aesthetic purposes, we simultaneously reverse the order of the columns in the matrix $\mathcal{M}_N$ and the order of the variables in the Vandermonde $\prod_{1 \le i < j \le N} [u_i - u_j]$. We also take the transpose of the matrix $\mathcal{M}_N$. The formula (5.3.22) is invariant under these transformations, and we obtain

$$S_N\Big(\{u\}_N,\{v\}_N,\{w\}_M\Big) = \frac{[\gamma]^N \prod_{i=1}^{N}\prod_{j=1}^{M}[v_i - w_j]}{\prod_{1\leq i<j\leq N}[u_j-u_i][v_i-v_j]} \times \tag{5.3.29}$$
$$\det\left(\frac{\prod_{k\neq j}^{N}[v_k-u_i-\gamma]\prod_{k=1}^{M}[u_i-w_k] - \prod_{k\neq j}^{N}[v_k-u_i+\gamma]\prod_{k=1}^{M}[u_i-w_k+\gamma]}{[u_i-v_j]}\right)_{1\leq i,j\leq N}$$

The expression (5.3.29) for the Bethe scalar product was first proved by Slavnov in [81]. The original proof required a recursion relation between scalar products of dimension N and $N-1$, which can be found in section 3, chapter IX of [61]. Although this earlier proof applies universally to quantum models with the R-matrix (5.1.21), it seems less transparent than the simple Izergin-Korepin proof which we have proposed in the context of the XXZ spin-$\frac{1}{2}$ chain.

5.3.6 Scalar product as a power-sum specialized KP τ-function

In this subsection we consider the Bethe scalar product S'_N, which arises from a normalization and change of variables applied to S_N. We will show that, by virtue of its determinant form, S'_N is a specialization of a polynomial KP τ-function [41]. As in the case of the domain wall partition function Z'_N, this specialization is achieved by setting the τ-function time variables to power sums of rapidities. The procedure we follow is fairly unremarkable, and requires only elementary determinant operations. However the result itself is certainly interesting, since it provides a link between solutions of a quantum model (Bethe eigenvectors of the XXZ spin-$\frac{1}{2}$ chain) and solutions of a classical hierarchy (KP τ-functions).

Letting $S_N(\{u\}_N,\{v\}_N,\{w\}_M)$ denote the full Bethe scalar product, we define

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big) = e^{N^2\gamma}\prod_{i=1}^{N} e^{(M-1)(v_i-u_i)}\prod_{j=1}^{M} e^{2Nw_j} S_N\Big(\{u\}_N,\{v\}_N,\{w\}_M\Big) \tag{5.3.30}$$

where we have set $e^{-2u_i} = x_i, e^{2v_i} = y_i, e^{2w_i} = z_i, e^{2\gamma} = q$. Applying this change of variables to (5.3.29) and extracting a factor of $(-)^{N-1}$ from each row of the determinant leads to the explicit formula

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big)=\frac{(q-1)^N\prod_{i=1}^{N}\prod_{j=1}^{M}(y_i-z_j)}{\prod_{1\le i<j\le N}(x_i-x_j)(y_i-y_j)}\times \tag{5.3.31}$$
$$\det\left(\frac{q^{N-1}\prod_{k\neq j}^{N}\Big(1-x_i\frac{y_k}{q}\Big)\prod_{k=1}^{M}(1-x_iz_k)-q^{\frac{M}{2}}\prod_{k\neq j}^{N}(1-qx_iy_k)\prod_{k=1}^{M}\Big(1-x_i\frac{z_k}{q}\Big)}{1-x_iy_j}\right)_{1\le i,j\le N}$$

Expanding the numerator of the determinant in (5.3.31) in terms of elementary symmetric functions (5.2.20), we obtain the equivalent expression

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big)=\frac{(q-1)^N\prod_{i=1}^{N}\prod_{j=1}^{M}(y_i-z_j)}{\prod_{1\le i<j\le N}(x_i-x_j)(y_i-y_j)}\times \tag{5.3.32}$$
$$\det\left(\frac{\sum_{k=0}^{M+N-1}(-x_i)^k\left[q^{N-1}e_k\Big\{\frac{\widehat{y_j}}{q}\Big\}\cup\{z\}-q^{\frac{M}{2}}e_k\{q\widehat{y_j}\}\cup\Big\{\frac{z}{q}\Big\}\right]}{1-x_iy_j}\right)_{1\le i,j\le N}$$

From here, we wish to remove the pole within the determinant in (5.3.32). This is done using the fact that the rapidities $\{v\}_N$ obey the Bethe equations (5.1.58). Employing the change of variables $e^{2v_i}=y_i, e^{2w_i}=z_i, e^{2\gamma}=q$ in (5.1.58), the Bethe equations take the form

$$\beta_i=q^{N-1}\prod_{j\neq i}^{N}\Big(1-\frac{y_j}{qy_i}\Big)\prod_{j=1}^{M}\Big(1-\frac{z_j}{y_i}\Big)-q^{\frac{M}{2}}\prod_{j\neq i}^{N}\Big(1-\frac{qy_j}{y_i}\Big)\prod_{j=1}^{M}\Big(1-\frac{z_j}{qy_i}\Big)=0 \tag{5.3.33}$$

or in terms of elementary symmetric functions

$$\beta_i=\sum_{k=0}^{M+N-1}(-y_i)^{-k}\left[q^{N-1}e_k\Big\{\frac{\widehat{y_i}}{q}\Big\}\cup\{z\}-q^{\frac{M}{2}}e_k\{q\widehat{y_i}\}\cup\Big\{\frac{z}{q}\Big\}\right]=0 \tag{5.3.34}$$

where we have denoted the left hand side of (5.3.33) and (5.3.34) by β_i, since these equations correspond with the i^{th} Bethe equation. Using (5.3.34), we see that the numerator of the determinant (5.3.32) collapses to zero when $x_i = 1/y_j$. Hence we may remove the pole in this determinant, yielding[4]

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big) = \frac{(q-1)^N \prod_{i=1}^{N}\prod_{j=1}^{M}(y_i - z_j)}{\prod_{1\le i<j\le N}(x_i-x_j)(y_i-y_j)} \times \tag{5.3.35}$$
$$\det\left(\sum_{k=0}^{M+N-2} x_i^k y_j^k \sum_{l=0}^{k}(-y_j)^{-l}\left[q^{N-1}e_l\Big\{\frac{\widehat{y_j}}{q}\Big\}\cup\{z\} - q^{\frac{M}{2}}e_l\{q\widehat{y_j}\}\cup\Big\{\frac{z}{q}\Big\}\right]\right)_{1\le i,j\le N}$$

or more succinctly

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big) = \frac{(q-1)^N \prod_{i=1}^{N}\prod_{j=1}^{M}(y_i - z_j)}{\prod_{1\le i<j\le N}(x_i-x_j)(y_i-y_j)} \det\left(\sum_{k=0}^{M+N-2} x_i^k\Big[y_j^k\beta_j\Big]_+\right)_{1\le i,j\le N} \tag{5.3.36}$$

where $[y_j^k\beta_j]_+$ denotes all terms in $y_j^k\beta_j$ which have non-negative degree in y_j. Now we may follow the procedure of subsection 5.2.5 to bring (5.3.36) towards the form of a KP τ-function. Applying the Cauchy-Binet identity (5.2.22) to the determinant in (5.3.36), we obtain

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big) = \frac{(q-1)^N \prod_{i=1}^{N}\prod_{j=1}^{M}(y_i - z_j)}{\prod_{1\le i<j\le N}(x_i-x_j)(y_i-y_j)} \times \tag{5.3.37}$$
$$\sum_{M+N-1\ge k_1>\cdots>k_N\ge 1}\det\Big(x_i^{k_j-1}\Big)_{1\le i,j\le N}\det\Big(\Big[y_j^{k_i-1}\beta_j\Big]_+\Big)_{1\le i,j\le N}$$

We perform a change in summation variables, writing $k_i - 1 = \mu_i - i + N$ for all $1 \le i \le N$, which gives

[4]If the arbitrary degree-m polynomial $P(x) = \sum_{k=0}^{m} x^k p_k$ has a zero at the point $x = 1/y$, then $\frac{P(x)}{1-xy} = \sum_{k=0}^{m-1} x^k \sum_{l=0}^{k} y^{k-l} p_l$. We use this fact to establish equation (5.3.35).

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big) = \frac{(q-1)^N \prod_{i=1}^{N}\prod_{j=1}^{M}(y_i - z_j)}{\prod_{1\le i<j\le N}(x_i - x_j)(y_i - y_j)} \times \tag{5.3.38}$$
$$\sum_{\mu\subseteq[N,M-1]} \det\left(x_i^{\mu_j - j + N}\right)_{1\le i,j\le N} \det\left(\left[y_j^{\mu_i - i + N}\beta_j\right]_+\right)_{1\le i,j\le N}$$

where the sum is taken over all partitions $\mu = \{\mu_1 \geq \cdots \geq \mu_N \geq 0\}$ whose Young diagrams fit inside the rectangle $[N, M-1]$. Finally, by virtue of the Jacobi-Trudi identity (3.3.14) we have

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big) = \sum_{\mu\subseteq[N,M-1]} s_\mu\{x\}\varsigma_\mu(\{y\},\{z\},q) \tag{5.3.39}$$

with $s_\mu\{x\}$ a Schur function in the variables $\{x\} = \{x_1,\ldots,x_N\}$ and where we have defined the function

$$\varsigma_\mu(\{y\},\{z\},q) = \frac{(q-1)^N \prod_{i=1}^{N}\prod_{j=1}^{M}(y_i - z_j) \det\left(\left[y_j^{\mu_i - i + N}\beta_j\right]_+\right)_{1\le i,j\le N}}{\prod_{1\le i<j\le N}(y_i - y_j)} \tag{5.3.40}$$

This expression for the coefficients $\varsigma_\mu(\{y\},\{z\},q)$ is significantly more compact than the one which appeared in [41]. Written in the form (5.3.40), $\varsigma_\mu(\{y\},\{z\},q)$ is not far removed from a Schur function in the variables $\{y\}$. The only difference is the presence of the Bethe variables β_j in each entry of the determinant.

We have thus shown that $S'_N(\{x\}_N,\{y\}_N,\{z\}_M)$ is equal to

$$\tau_{\mathrm{SP}}\{t\} \sum_{\mu\subseteq[N,M-1]} \chi_\mu\{t\}\varsigma_\mu(\{y\},\{z\},q) \tag{5.3.41}$$

under the power-sum specialization $t_n = \frac{1}{n}\sum_{i=1}^{N} x_i^n$ for all $n \geq 1$. In the next subsection we show that the polynomial $\tau_{\mathrm{SP}}\{t\}$ is a KP τ-function, by writing it as an expectation value of charged fermions.

5.3.7 $\tau_{\mathrm{SP}}\{t\}$ as an expectation value of charged fermions

Proceeding in direct analogy with subsection 5.2.6, it is possible to express the right hand side of (5.3.41) in the canonical form (1.2.14) of a KP τ-function. As usual, this is made possible by the fact that the coefficients $\varsigma_\mu(\{y\},\{z\},q)$ are determinants. To avoid needless repetition we omit the details of this calculation and quote only the result, which reads

$$\tau_{\mathrm{SP}}\{t\} = \varsigma_\emptyset \langle 0| e^{H\{t\}} \exp\left(\sum_{m=1}^{M-1} \sum_{n=1}^{N} (-)^{n-1} \frac{\varsigma_{\{m,1^{n-1}\}}}{\varsigma_\emptyset} \psi_{m-1}\psi^*_{-n} \right) |0\rangle \tag{5.3.42}$$

where we have abbreviated $\varsigma_\mu(\{y\},\{z\},q) = \varsigma_\mu$ for all partitions μ. As was noticed in [41], equation (5.3.42) allows us to identify the XXZ Bethe eigenvectors with elements of $\mathcal{F}_\psi^{(0)}$ which lie in the GL_∞ orbit of the Fock vacuum. By writing the scalar product (5.3.30) in the form

$$S'_N\Big(\{x\}_N,\{y\}_N,\{z\}_M\Big) = \langle 0| \prod_{i=1}^{N} \mathbb{C}(x_i,\{z\}_M) \prod_{j=1}^{N} \mathbb{B}(y_j,\{z\}_M)|0\rangle \tag{5.3.43}$$

for some suitably renormalized monodromy matrix operators, we have

$$\langle 0| \prod_{i=1}^{N} \mathbb{C}(x_i,\{z\}_M) \prod_{j=1}^{N} \mathbb{B}(y_j,\{z\}_M)|0\rangle = \varsigma_\emptyset \left\langle \prod_{i=1}^{N} \left(e^{\sum_{n=1}^{\infty} \frac{1}{n} x_i^n H_n} \right) e^{X(\{y\},\{z\},q)} \right\rangle \tag{5.3.44}$$

where $X(\{y\},\{z\},q) \in A_\infty$ is given by

$$X(\{y\},\{z\},q) = \sum_{m=1}^{M-1} \sum_{n=1}^{N} (-)^{n-1} \frac{\varsigma_{\{m,1^{n-1}\}}}{\varsigma_\emptyset} \psi_{m-1}\psi^*_{-n} \tag{5.3.45}$$

The identity (5.3.44) is suggestive of a map from $\mathcal{V}$ to $\mathcal{F}_\psi^{(0)}$ under which

$$\prod_{j=1}^{N} \mathbb{B}(y_j,\{z\}_M)|0\rangle \longmapsto \varsigma_\emptyset e^{X(\{y\},\{z\},q)}|0\rangle \tag{5.3.46}$$

Such a mapping is reminiscent of those which were discussed in chapter 3, albeit in the context of simpler models.

5.4 Calculation of Bethe eigenvectors

In this section we calculate the Bethe eigenvector coefficients $b_\lambda(\{v\}_N,\{w\}_M)$, as given in equation (5.1.60). We claim an analogous result for the coefficients $c_\lambda(\{v\}_N,\{w\}_M)$ in equation (5.1.61), but for the sake of brevity we do not present an explicit proof. The results of this section are to appear in [90].

5.4.1 Weighted determinants

Definition 1. Let $\mathcal{W}$ and $\mathcal{M}$ be $M \times M$ matrices and fix an integer $l \leq M$. The *weighted determinant* $\overline{\det}_l(\mathcal{W}, \mathcal{M})$ is defined as

$$\overline{\det}_l(\mathcal{W}, \mathcal{M}) = \sum_{\sigma \in S_M} \left(\prod_{1 \leq i < j \leq l} \mathcal{W}_{\sigma_i, \sigma_j} \right) \operatorname{sgn}(\sigma) \prod_{i=1}^{M} \mathcal{M}_{i, \sigma_i} \tag{5.4.1}$$

where the sum is over all permutations σ of the integers $\{1, \ldots, M\}$, and the sign of the permutation $\operatorname{sgn}(\sigma)$ is given by

$$\operatorname{sgn}(\sigma) = \prod_{1 \leq i < j \leq M} \operatorname{sgn}(\sigma_j - \sigma_i) \tag{5.4.2}$$

We will call $\mathcal{W}$ a *weighting matrix,* and the ordinary determinant is recovered by setting its entries to $\mathcal{W}_{i,j} = 1$ for all $1 \leq i, j \leq M$. Hence equation (5.4.1) may be viewed as a generalization of the determinant. In a regular determinant, each term in the sum is weighted with the sign (5.4.2) that depends on pairs of elements $\{\sigma_i, \sigma_j\}$ in the permutation. The weighted determinant is different in that it allows the weighting to be a more general function of pairs of elements in the permutation.

Example 1. Let us consider the Hall-Littlewood function $P_\mu(\{x\}, t)$, as given by the sum (4.3.23) in chapter 4. By explicitly performing the permutation σ of the variables $\{x_1, \ldots, x_N\}$ within this sum, we obtain

$$P_\mu(\{x\}, t) = \frac{\displaystyle\sum_{\sigma \in S_N} \left(\prod_{1 \leq i < j \leq N} (x_{\sigma_i} - t x_{\sigma_j}) \right) \operatorname{sgn}(\sigma) \prod_{i=1}^{N} x_{\sigma_i}^{\mu_i}}{\displaystyle v_\mu(t) \prod_{1 \leq i < j \leq N} (x_i - x_j)} \tag{5.4.3}$$

where the constant term $v_\mu(t)$ is given by (4.3.24). We define $N \times N$ matrices $\mathcal{M}_\mu\{x\}$ and $\mathcal{W}(\{x\}, t)$ whose entries are given by

$$\Big(\mathcal{M}_\mu\{x\} \Big)_{i,j} = x_j^{\mu_i}, \quad \Big(\mathcal{W}(\{x\}, t) \Big)_{i,j} = x_i - t x_j \tag{5.4.4}$$

for all $1 \leq i, j \leq N$. Then the Hall-Littlewood function $P_\mu(\{x\}, t)$ can be expressed in the form

$$P_\mu(\{x\}, t) = \frac{\overline{\det}_N \Big(\mathcal{W}(\{x\}, t), \mathcal{M}_\mu\{x\} \Big)}{\displaystyle v_\mu(t) \prod_{1 \leq i < j \leq N} (x_i - x_j)} \tag{5.4.5}$$

which is a ratio of a weighted determinant and the Vandermonde in $\{x_1, \ldots, x_N\}$. One can view (5.4.5) as the Hall-Littlewood analogue of equation (3.3.14) for Schur functions. We remark that (5.4.5) in conjunction with (4.3.27), (4.3.33) allows us to write the q-boson model Bethe eigenvectors in the form

$$\mathcal{M}_\psi(t)\mathbb{B}(x_1,t)\ldots\mathbb{B}(x_N,t)|0\rangle = \sum_{\mu\subseteq[N,M]} \frac{\overline{\det}_N\Big(\mathcal{W}(\{x\},t),\mathcal{M}_\mu\{x\}\Big)}{v_\mu(t)\prod\limits_{1\le i<j\le N}(x_i-x_j)}|\mu) \tag{5.4.6}$$

$$\langle 0|\mathbb{C}(x_N,t)\ldots\mathbb{C}(x_1,t)\mathcal{M}^*_\psi(t) = \sum_{\mu\subseteq[N,M]} \frac{\overline{\det}_N\Big(\mathcal{W}(\{x\},t),\mathcal{M}_\mu\{x\}\Big)}{v_\mu(t)\prod\limits_{1\le i<j\le N}(x_i-x_j)}(\mu| \tag{5.4.7}$$

The aim of this section is to obtain an analogous result in the context of the XXZ model Bethe eigenvectors. That is, we wish to evaluate the expansion coefficients in (5.1.60), (5.1.61) as weighted determinants divided by a Vandermonde. This proposal is motivated by the preceding equations and the close relationship of the q-boson and XXZ models.

5.4.2 Isolating coefficients

Starting from (5.1.60) and using the orthonormality of the basis vectors (5.1.3), we isolate coefficients as follows

$$b_\lambda\Big(\{v\}_N,\{w\}_M\Big) = \langle\lambda|\prod_{i=1}^{N} B(v_i,\{w\}_M)|\Uparrow_M\rangle \tag{5.4.8}$$

For the purpose of computation, the formula (5.4.8) is not immediately helpful. In order to bring it into a more convenient form, we begin by writing

$$b_\lambda\Big(\{v\}_N,\{w\}_M\Big) = \langle\lambda|\otimes\langle\Uparrow^a_N|T\Big(\{v\}_N,\{w\}_M\Big)|\Downarrow^a_N\rangle\otimes|\Uparrow_M\rangle \tag{5.4.9}$$

where we have defined

$$T\Big(\{v\}_N,\{w\}_M\Big) = T_{a_N}(v_N,\{w\}_M)\ldots T_{a_1}(v_1,\{w\}_M) \tag{5.4.10}$$

Here the spaces $\mathcal{V}_{a_1},\ldots,\mathcal{V}_{a_N}$ assigned to the monodromy matrices are auxiliary, and we have defined the auxiliary state vectors

$$\langle\Uparrow^a_N| = \bigotimes_{i=1}^{N}\uparrow^*_{a_i}, \quad |\Downarrow^a_N\rangle = \bigotimes_{i=1}^{N}\downarrow_{a_i} \tag{5.4.11}$$

Using the result of lemma 3 we may then write

$$T\Big(\{v\}_N, \{w\}_M\Big) = (-)^{MN}\overline{T}_1(w_1, \{\bar{v}\}_N)\dots\overline{T}_M(w_M, \{\bar{v}\}_N) \tag{5.4.12}$$

where for all $1 \le i \le M$ we have defined

$$\overline{T}_i(w_i, \{\bar{v}\}_N) = \begin{pmatrix} D(w_i, \{\bar{v}\}_N) & -B(w_i, \{\bar{v}\}_N) \\ -C(w_i, \{\bar{v}\}_N) & A(w_i, \{\bar{v}\}_N) \end{pmatrix}_i \tag{5.4.13}$$

with $\{\bar{v}\}_N = \{v_1 + \gamma, \dots, v_N + \gamma\}$. Substituting the expression (5.4.12) into (5.4.9) and contracting on the quantum spaces $\mathcal{V}_1, \dots, \mathcal{V}_M$ gives

$$b_\lambda\Big(\{v\}_N, \{w\}_M\Big) = (-)^{(M+1)N}\langle \Uparrow_N^a | \overrightarrow{\prod_{i=1}^{M}} O_i(w_i, \{\bar{v}\}_N)| \Downarrow_N^a\rangle \tag{5.4.14}$$

where we have recalled the fact $\ell(\lambda) = M - N$ and defined the operators

$$O_i(w_i, \{\bar{v}\}_N) = \begin{cases} D(w_i, \{\bar{v}\}_N), & i \in \lambda \\ C(w_i, \{\bar{v}\}_N), & i \notin \lambda \end{cases} \tag{5.4.15}$$

Throughout the rest of this section we will set $l = M - N$ for convenience. In terms of this notation, the expression (5.4.14) contains l D-operators and N C-operators.

5.4.3 Eliminating operators

The expression (5.4.14) is more amenable to calculation than (5.4.8). It allows us to employ the strategy that was used in [11], [33] and [15] to calculate the one and two-point boundary correlators of the six-vertex model, respectively. Our method is purely algebraic and therefore closest to that of [11], while [33] took a more graphical perspective.

The idea is to eliminate the l operators $D(w_{\lambda_i}, \{\bar{v}\}_N)$ from (5.4.14) by employing the commutation relation (5.1.39) repeatedly. This will have the effect of shuffling these l operators to the right until they act on the eigenvector $| \Downarrow_N^a\rangle$. This ultimately leaves us with a domain wall partition function (5.2.6), whose explicit form is given by lemma 7. To begin in this direction, we use (5.1.39) and (5.1.52) repeatedly to establish that

$$D(w_\lambda, \{\bar{v}\}_N) \prod_{j\in J} C(w_j, \{\bar{v}\}_N)| \Downarrow_N^a\rangle = \sum_{i\in J\cup\lambda} \frac{\prod_{j=1}^{N}[w_i - v_j] \prod_{j\in J}[w_i - w_j - \gamma]}{\prod_{\substack{j\in J\cup\lambda \\ j\neq i}}[w_i - w_j]} \prod_{\substack{j\in J\cup\lambda \\ j\neq i}} C(w_j, \{\bar{v}\}_N)| \Downarrow_N^a\rangle \tag{5.4.16}$$

where J is an arbitrary indexing set, and λ an arbitrary extra label. By employing this identity l times, once for each operator $D(w_{\lambda_i}, \{\bar{v}\}_N)$ in (5.4.14), we obtain

$$b_\lambda\Big(\{v\}_N, \{w\}_M\Big) = \sum_{\sigma_1=\lambda_1}^{M} \cdots \sum_{\substack{\sigma_l=\lambda_l \\ \sigma_l \neq \sigma_1,\ldots,\sigma_{l-1}}}^{M} \frac{\prod_{i=1}^{N}[w_{\sigma_1} - v_i] \prod_{i=\lambda_1+1}^{M} [w_{\sigma_1} - w_i - \gamma]}{\prod_{\substack{i=\lambda_1 \\ i \neq \sigma_1}}^{M} [w_{\sigma_1} - w_i]} \cdots$$

$$\times \frac{\prod_{i=1}^{N}[w_{\sigma_l} - v_i] \prod_{\substack{i=\lambda_l+1 \\ i \neq \sigma_1,\ldots,\sigma_{l-1}}}^{M} [w_{\sigma_l} - w_i - \gamma]}{\prod_{\substack{i=\lambda_l \\ i \neq \sigma_1,\ldots,\sigma_l}}^{M} [w_{\sigma_l} - w_i]} \langle \Uparrow_N^a | \prod_{j \neq \sigma_1,\ldots,\sigma_l}^{M} C(w_j, \{\bar{v}\}_N) | \Downarrow_N^a \rangle \quad (5.4.17)$$

The final term in the sum (5.4.17) is the domain wall partition function, given explicitly by

$$\langle \Uparrow_N^a | \prod_{j \neq \sigma_1,\ldots,\sigma_l}^{M} C(w_j, \{\bar{v}\}_N) | \Downarrow_N^a \rangle = \frac{\det\Big([\gamma] \prod_{k \neq i}^{N} [v_k - w_j + \gamma][v_k - w_j]\Big)_{\substack{1 \leq i \leq N \\ j \neq \sigma_1,\ldots,\sigma_l}}}{\prod_{1 \leq i < j \leq N} [v_i - v_j] \prod_{\substack{1 \leq i < j \leq M \\ i,j \neq \sigma_1,\ldots,\sigma_l}} [w_j - w_i]}$$

(5.4.18)

We manipulate the factors which occur in the denominator of the summand of (5.4.17), and in the denominator of the partition function (5.4.18). We find that

$$\prod_{\substack{i=\lambda_1 \\ i \neq \sigma_1}}^{M} \frac{1}{[w_{\sigma_1} - w_i]} \cdots \prod_{\substack{i=\lambda_l \\ i \neq \sigma_1,\ldots,\sigma_l}}^{M} \frac{1}{[w_{\sigma_l} - w_i]} \prod_{\substack{1 \leq i < j \leq M \\ i,j \neq \sigma_1,\ldots,\sigma_l}} \frac{1}{[w_j - w_i]} = \quad (5.4.19)$$

$$\prod_{1 \leq i < j \leq l} \mathrm{sgn}(\sigma_j - \sigma_i) \frac{(-)^{M+\sigma_1} \prod_{i=1}^{\lambda_1 - 1} [w_{\sigma_1} - w_i] \ldots (-)^{M+\sigma_l} \prod_{i=1}^{\lambda_l - 1} [w_{\sigma_l} - w_i]}{\prod_{1 \leq i < j \leq M} [w_j - w_i]}$$

Substituting (5.4.19) into (5.4.17) gives

$$b_\lambda\Big(\{v\}_N,\{w\}_M\Big) = \frac{\displaystyle\sum_{\sigma_1,\ldots,\sigma_l=1}^{M}\ \prod_{1\le i<j\le l} \mathrm{sgn}(\sigma_j-\sigma_i)}{\displaystyle\prod_{1\le i<j\le N}[v_i-v_j]\prod_{1\le i<j\le M}[w_j-w_i]} \tag{5.4.20}$$
$$\times (-)^{M+\sigma_1}\prod_{i=1}^{N}[w_{\sigma_1}-v_i]\prod_{i=1}^{\lambda_1-1}[w_{\sigma_1}-w_i]\prod_{i=\lambda_1+1}^{M}[w_{\sigma_1}-w_i-\gamma]\ldots$$
$$\times (-)^{M+\sigma_l}\prod_{i=1}^{N}[w_{\sigma_l}-v_i]\prod_{i=1}^{\lambda_l-1}[w_{\sigma_l}-w_i]\prod_{\substack{i=\lambda_l+1\\ i\neq\sigma_1,\ldots,\sigma_{l-1}}}^{M}[w_{\sigma_l}-w_i-\gamma]$$
$$\times \det\Big([\gamma]\prod_{k\neq i}^{N}[v_k-w_j+\gamma][v_k-w_j]\Big)_{\substack{1\le i\le N\\ j\neq\sigma_1,\ldots,\sigma_l}}$$

where the summation indices $\sigma_1,\ldots,\sigma_l$ are assumed to be distinct but are now allowed to range from 1 to M, since all terms corresponding to $\sigma_i<\lambda_i$ vanish trivially on account of the products $\prod_{j=1}^{\lambda_i-1}[w_{\sigma_i}-w_j]$. Manipulating the summand of (5.4.20) slightly, we obtain

$$b_\lambda\Big(\{v\}_N,\{w\}_M\Big) = \frac{\displaystyle\sum_{\sigma_1,\ldots,\sigma_l=1}^{M}\ \prod_{1\le i<j\le l}\Big(\mathrm{sgn}(\sigma_j-\sigma_i)\Big/[w_{\sigma_i}-w_{\sigma_j}+\gamma]\Big)}{\displaystyle\prod_{1\le i<j\le N}[v_i-v_j]\prod_{1\le i<j\le M}[w_j-w_i]} \tag{5.4.21}$$
$$\times (-)^{1+\sigma_1}\prod_{i=1}^{N}[v_i-w_{\sigma_1}]\prod_{i=1}^{\lambda_1-1}[w_i-w_{\sigma_1}]\prod_{i=\lambda_1+1}^{M}[w_i-w_{\sigma_1}+\gamma]\ldots$$
$$\times (-)^{l+\sigma_l}\prod_{i=1}^{N}[v_i-w_{\sigma_l}]\prod_{i=1}^{\lambda_l-1}[w_i-w_{\sigma_l}]\prod_{i=\lambda_l+1}^{M}[w_i-w_{\sigma_l}+\gamma]$$
$$\times (-)^{lN}\det\Big([\gamma]\prod_{k\neq i}^{N}[v_k-w_j+\gamma][v_k-w_j]\Big)_{\substack{1\le i\le N\\ j\neq\sigma_1,\ldots,\sigma_l}}$$

The expression (5.4.21) allows us to put $b_\lambda(\{v\}_N,\{w\}_M)$ in the form of a weighted determinant, as we will show in the next subsection.

5.4.4 Weighted determinant expression for $b_\lambda(\{v\}_N,\{w\}_M)$

By expanding the determinant in the summand of (5.4.21) we arrive at the expression

$$b_\lambda\Big(\{v\}_N,\{w\}_M\Big) = \frac{\prod_{i=1}^{N}\prod_{j=1}^{M}[v_i - w_j]}{\prod_{1\le i<j\le N}[v_i - v_j]\prod_{1\le i<j\le M}[w_j - w_i]}\times \tag{5.4.22}$$

$$\sum_{\sigma\in S_M}\frac{\prod_{1\le i<j\le M}\operatorname{sgn}(\sigma_j-\sigma_i)}{\prod_{1\le i<j\le l}[w_{\sigma_i} - w_{\sigma_j}+\gamma]}\prod_{i=1}^{l} b_{\lambda_i}\left(w_{\sigma_i},\{w\}_M\right)\prod_{i=1}^{N} f_i\left(w_{\sigma_{l+i}},\{v\}_N\right)$$

where the sum is now taken over all permutations $\sigma \in S_M$, and where we have defined the functions

$$f_i\left(w,\{v\}_N\right) = \frac{[\gamma]}{[v_i - w]}\prod_{k\neq i}^{N}[v_k - w + \gamma] \tag{5.4.23}$$

$$b_{\lambda_i}\left(w,\{w\}_M\right) = \prod_{j=1}^{\lambda_i-1}[w_j - w]\prod_{j=\lambda_i+1}^{M}[w_j - w + \gamma] \tag{5.4.24}$$

Recalling the definition of the weighted determinant (5.4.1), we are able to write

$$b_\lambda\Big(\{v\}_N,\{w\}_M\Big) = \frac{\prod_{i=1}^{N}\prod_{j=1}^{M}[v_i - w_j]\,\overline{\det}_l\Big(\mathcal{W}\{w\},\mathcal{B}_\lambda\{v\}_N\Big)}{\prod_{1\le i<j\le N}[v_i - v_j]\prod_{1\le i<j\le M}[w_j - w_i]} \tag{5.4.25}$$

where the components of the $M\times M$ weighting matrix $\mathcal{W}\{w\}$ are given by

$$\Big(\mathcal{W}\{w\}\Big)_{i,j} = \frac{1}{[w_i - w_j + \gamma]} \tag{5.4.26}$$

and the $M\times M$ matrix $\mathcal{B}_\lambda\{v\}_N$, whose form depends only on the strict partition λ, is defined as

$$\mathcal{B}_\lambda\{v\}_N = \begin{pmatrix} b_{\lambda_1}\left(w_1,\{w\}_M\right) & \cdots & b_{\lambda_1}\left(w_M,\{w\}_M\right) \\ \vdots & & \vdots \\ b_{\lambda_l}\left(w_1,\{w\}_M\right) & \cdots & b_{\lambda_l}\left(w_M,\{w\}_M\right) \\ f_1\left(w_1,\{v\}_N\right) & \cdots & f_1\left(w_M,\{v\}_N\right) \\ \vdots & & \vdots \\ f_N\left(w_1,\{v\}_N\right) & \cdots & f_N\left(w_M,\{v\}_N\right) \end{pmatrix} \tag{5.4.27}$$

As a consistency check, we evaluate some special cases of (5.4.25). When λ is equal to the staircase partition $\{M, \ldots, 1\}$ we have $N = 0$, and accordingly we require that $b_{\{M,\ldots,1\}} = 1$. We observe that the matrix $\mathcal{B}_{\{M,\ldots,1\}}$ is given by

$$\mathcal{B}_{\{M,\ldots,1\}} = \begin{pmatrix} 0 & 0 & b_M(w_M, \{w\}_M) \\ 0 & \cdot^{\cdot^{\cdot}} & 0 \\ b_1(w_1, \{w\}_M) & 0 & 0 \end{pmatrix} \tag{5.4.28}$$

where all entries which are not on the indicated diagonal are zero. The weighted determinant in (5.4.25) is trivially evaluated in this case, and we obtain $b_{\{M,\ldots,1\}} = 1$, as expected. Similarly, when λ is equal to the empty partition $\emptyset$ we have $M = N$, and accordingly we require $b_\emptyset = Z_N$. In this case the weighted determinant (5.4.25) collapses to the Izergin-Korepin expression (5.2.15), as required. In general, the formula (5.4.25) serves to interpolate between these extremal cases.

Substituting the coefficients (5.4.25) into (5.1.60), we can write the Bethe eigenvector $\prod_{i=1}^{N} B(v_i, \{w\}_M)| \Uparrow_M\rangle$ in the form

$$\prod_{i=1}^{N} B(v_i, \{w\}_M)| \Uparrow_M\rangle = \sum_{\lambda|\ell(\lambda)=l} \frac{\displaystyle\prod_{i=1}^{N}\prod_{j=1}^{M}[v_i - w_j]\overline{\det}_l\Big(\mathcal{W}\{w\}, \mathcal{B}_\lambda\{v\}_N\Big)}{\displaystyle\prod_{1\le i<j\le N}[v_i - v_j]\prod_{1\le i<j\le M}[w_j - w_i]}|\lambda\rangle \tag{5.4.29}$$

where the sum is taken over all strict partitions λ of l integers which satisfy the inequalities $\{M \ge \lambda_1 > \cdots > \lambda_l \ge 1\}$.

5.4.5 Weighted determinant expression for $c_\lambda(\{v\}_N, \{w\}_M)$

By proceeding in an analogous fashion, it is possible to show that

$$c_\lambda\Big(\{v\}_N, \{w\}_M\Big) = \frac{\displaystyle\prod_{i=1}^{N}\prod_{j=1}^{M}[v_i - w_j + \gamma]\overline{\det}_l\Big(\mathcal{W}'\{w\}, \mathcal{C}_\lambda\{v\}_N\Big)}{\displaystyle\prod_{1\le i<j\le N}[v_i - v_j]\prod_{1\le i<j\le M}[w_j - w_i]} \tag{5.4.30}$$

where the components of the $M \times M$ weighting matrix $\mathcal{W}'\{w\}$ are given by

$$\Big(\mathcal{W}'\{w\}\Big)_{i,j} = \frac{1}{[w_i - w_j - \gamma]} \tag{5.4.31}$$

and the $M \times M$ matrix $\mathcal{C}_\lambda\{v\}_N$, whose form depends only on the strict partition λ, is defined as

$$\mathcal{C}_\lambda\{v\}_N = \begin{pmatrix} c_{\lambda_1}(w_1,\{w\}_M) & \cdots & c_{\lambda_1}(w_M,\{w\}_M) \\ \vdots & & \vdots \\ c_{\lambda_l}(w_1,\{w\}_M) & \cdots & c_{\lambda_l}(w_M,\{w\}_M) \\ f'_1(w_1,\{v\}_N) & \cdots & f'_1(w_M,\{v\}_N) \\ \vdots & & \vdots \\ f'_N(w_1,\{v\}_N) & \cdots & f'_N(w_M,\{v\}_N) \end{pmatrix} \tag{5.4.32}$$

with the functions

$$f'(w,\{v\}_N) = \frac{[\gamma]}{[v_i - w + \gamma]} \prod_{k\neq i}^{N} [v_k - w] \tag{5.4.33}$$

$$c_{\lambda_i}(w,\{w\}_M) = \prod_{j=1}^{\lambda_i - 1} [w_j - w] \prod_{j=\lambda_i+1}^{M} [w_j - w - \gamma] \tag{5.4.34}$$

Substituting the coefficients (5.4.30) into (5.1.61), we can write the dual Bethe eigenvector $\langle \Uparrow_M | \prod_{i=1}^{N} C(v_i,\{w\}_M)$ in the form

$$\langle \Uparrow_M | \prod_{i=1}^{N} C(v_i,\{w\}_M) = \sum_{\lambda|\ell(\lambda)=l} \frac{\prod_{i=1}^{N}\prod_{j=1}^{M}[v_i - w_j + \gamma]\overline{\det}_l\Big(\mathcal{W}'\{w\},\mathcal{C}_\lambda\{v\}_N\Big)}{\prod_{1\le i<j\le N}[v_i - v_j]\prod_{1\le i<j\le M}[w_j - w_i]} \langle \lambda | \tag{5.4.35}$$

where the sum is taken over all strict partitions λ of l integers which satisfy the inequalities $\{M \geq \lambda_1 > \cdots > \lambda_l \geq 1\}$.

5.5 Conclusion

In this chapter we presented several new results in the context of the XXZ spin-$\frac{1}{2}$ model. Admittedly these results are rather distinct in nature, but we believe that they are each interesting in their own right. We summarize our findings in the following paragraphs.

1. In section 5.3 we gave a new proof of the Slavnov determinant formula for the Bethe scalar product. Our proof relied on evaluating the intermediate Bethe scalar products S_0 through to S_N, which obey a set of Izergin/Korepin type conditions. We believe this method refines Slavnov's original proof insofar as our recursion relations have a simple graphical representation, whereas the recursion relation used in [81] can only be derived by complicated arguments, and we feel it presents a more elementary alternative to the method of [59]. We hope there is potential to extend our graphical proof to the calculation of scalar products in a variety of lattice models. For example, starting from the domain wall partition functions obtained in [13], it should be possible to apply our method to the calculation of scalar products of the higher spin XXZ models [26].

2. By virtue of their determinant form, we were able to show that the partition function and Bethe scalar product are power-sum specializations of KP τ-functions. These provide further examples of quantum mechanical quantities which unexpectedly solve a classical hierarchy. At present we lack a deeper understanding of this classical/quantum relationship, but there is sufficient evidence to suggest that one exists. To achieve this aim, it may be necessary to study the link between these XXZ quantities and the discrete version of the KP hierarchy, as was done in [34]. In particular, it would be worthwhile to relate the results of [34] to those of [62], in which a family of *transfer matrices* were shown to obey a discrete KP equation. The review article [84] also provides some insights on a connection with the 2-component KP hierarchy.

3. In section 5.4 we evaluated the XXZ Bethe eigenvectors by writing them in terms of the elementary spin basis, and evaluating the expansion coefficients. We found that these coefficients can be expressed as a generalization of the determinant, which we called a weighted determinant. Importantly, the Bethe equations were not used in these calculations. If it were possible to incorporate the Bethe equations in some way, it is conceivable that even simpler expressions could be obtained.

Chapter 6

Free fermion condition in lattice models

6.0 Introduction

In the preceding chapters we have noticed the appearance of fermions in the scalar product of the q-boson model and its limiting cases, and the XXZ spin-$\frac{1}{2}$ model. The fermionization in sections 3.4, 3.8 and 4.5 was particularly special, in that it caused the scalar product to factorize into product form. In this chapter we calculate the partition function and scalar product of several different lattice models. We find that these quantities also admit a factorization into product form, a fact which reflects the free fermionic nature of the corresponding models.

The starting point of our studies is the six-vertex model with crossing parameter set to $\gamma = \pi i/2$. Following the literature, we refer to this as the free fermion point of the six-vertex model. At the free fermion point the Izergin-Korepin formula for the partition function becomes a Cauchy determinant, and therefore factorizes into product form. We review this fact in section 6.1, writing the partition function as an expectation value of KP vertex operators. We will also show that the Slavnov formula for the scalar product admits a similar factorization. This leads to a product expression for the Bethe scalar product at the free fermion point.

In [29], [30], [31], B U Felderhof introduced an elliptic eight-vertex model whose weights depend not only on rapidity variables, but on two additional parameters called external fields. In contrast to the eight-vertex model of R J Baxter [5], where vertices are invariant under conjugation of state variables, in the Felderhof model no such symmetry exists between vertices, creating an extra degree of anisotropy in the corresponding lattice model. We consider the trigonometric limit of Felderhof's model in section 6.2. In this limit two of the vertices collapse to zero, leading to the model studied by T Deguchi and A Akutsu in [24] which generalizes the free fermion point of the six-vertex model. We are able to calculate the partition function [14] and scalar product of this model, aided by the asymmetry of its vertex weights. Both the partition function and scalar product factorize into product form, indicating that the trigonometric Felderhof model remains free fermionic despite the presence of external fields.

In the remainder of the chapter we turn our attention to a more general class of lattice models. The most fundamental of these was derived from the eight-vertex model by Baxter [4], and is called the eight-vertex solid-on-solid (SOS) model. The weights of the SOS model are parametrized by elliptic functions, and they are graphically represented by squares with a dynamical or height variable attached at their corners. The six-vertex model is recovered as a special case by taking, simultaneously, the trigonometric and heightless limits of the SOS model. In section 6.3 we review these facts and define the domain wall partition function of the SOS model. We do this algebraically, by following the quantum inverse scattering method for models with a height parameter, see for example [32]. As we did for the six-vertex model in section 6.1, we study the $\gamma = \pi i/2$ specialization of the crossing parameter. In this limit the partition function factorizes into product form. This suggests the existence of a free fermion point even in models which possess a height parameter.

Motivated by the results of section 6.2, in section 6.4 we introduce external fields into the weights of the SOS model at its free fermion point. This was originally achieved by Deguchi and Akutsu [25] at the level of trigonometric functions, and extended to an elliptic parametrization in [38], so we refer to this as the elliptic Deguchi-Akutsu height model. Due to the inherent asymmetry between the weights of this model, we are able to calculate its domain wall partition function using essentially the same techniques that were employed in section 6.2. The partition function is obtained in factorized product form [38], showing that the elliptic Deguchi-Akutsu height model is free fermionic despite the presence of height and external field parameters.

6.1 Free fermion point of six-vertex model

6.1.1 The limit $\gamma = \pi i/2$

The free fermion point of the XXZ/six-vertex model is obtained by setting the crossing parameter to $\gamma = \pi i/2$.[1] In this limit the anisotropy parameter $\Delta = \frac{1}{2}(e^{\gamma} + e^{-\gamma})$ vanishes, and the Hamiltonian (5.1.19) for the model becomes

$$\mathcal{H} = \sum_{j=1}^{M} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y \right) = 2 \sum_{j=1}^{M} \left(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+ \right) \tag{6.1.1}$$

This Hamiltonian incorporates only interactions of the x and y components of the spin, and so it is said to correspond to the XX0 model. In the same limit, the R-matrix (5.1.21) reads

[1] Throughout the entire chapter, we will reserve the letter i to represent $\sqrt{-1}$.

$$R_{ab}(u,v) = \begin{pmatrix} \langle u-v \rangle & 0 & 0 & 0 \\ 0 & [u-v] & \langle 0 \rangle & 0 \\ 0 & \langle 0 \rangle & [u-v] & 0 \\ 0 & 0 & 0 & \langle u-v \rangle \end{pmatrix}_{ab} \tag{6.1.2}$$

where we have employed the notations $[u] = 2\sinh u$, $\langle u \rangle = 2i\cosh u$ and made use of the fact that $[u + \pi i/2] = \langle u \rangle$. We will carry these conventions throughout the next few subsections.

6.1.2 Domain wall partition function in the limit $\gamma = \pi i/2$

Recalling Izergin's determinant expression (5.2.15) for the domain wall partition function, we set $\gamma = \pi i/2$ to obtain

$$Z_N\Big|_{\gamma=\pi i/2} = \frac{\langle 0 \rangle^N \prod_{j,k=1}^{N} \langle v_j - w_k \rangle [v_j - w_k]}{\prod_{1 \le j < k \le N} [v_j - v_k][w_k - w_j]} \det\left(\frac{1}{\langle v_j - w_k \rangle [v_j - w_k]}\right)_{1 \le j,k \le N} \tag{6.1.3}$$

The determinant in (6.1.3) is in Cauchy form, and obeys the factorization[2]

$$\det\left(\frac{1}{\langle v_j - w_k \rangle [v_j - w_k]}\right)_{1 \le j,k \le N} = \frac{\prod_{1 \le j < k \le N} \langle v_j - v_k \rangle [v_j - v_k] \langle w_k - w_j \rangle [w_k - w_j]}{\prod_{j,k=1}^{N} \langle v_j - w_k \rangle [v_j - w_k]} \tag{6.1.5}$$

Substituting (6.1.5) into (6.1.3) and performing various cancellations, the domain wall partition function has the factorized expression

$$Z_N\Big|_{\gamma=\pi i/2} = \langle 0 \rangle^N \prod_{1 \le j < k \le N} \langle v_j - v_k \rangle \langle w_k - w_j \rangle \tag{6.1.6}$$

[2] By using the identity $\langle v \rangle [v] = i[2v]$ we can express (6.1.5) in the more standard form

$$\det\left(\frac{1}{[2v_j - 2w_k]}\right)_{1 \le j,k \le N} = \frac{\prod_{1 \le j < k \le N} [2v_j - 2v_k][2w_k - 2w_j]}{\prod_{j,k=1}^{N} [2v_j - 2w_k]} \tag{6.1.4}$$

Equation (6.1.6) may also be found, for example, in [11]. The aim of this chapter is to find analogues of the formula (6.1.6) across several different free fermionic models.

6.1.3 Partition function and free fermions

In this subsection we study the appearance of free fermions in the partition function at the point $\gamma = \pi i/2$. We achieve this by writing lemma 2 from chapter 1 in terms of charged fermion generating functions, as follows

$$\Big\langle \Psi(y_1)\dots\Psi(y_N)\Psi^*(1/z_N)\dots\Psi^*(1/z_1)\Big\rangle = \det\left(\sum_{k\in\mathbb{Z}}\sum_{l\in\mathbb{Z}}\langle\psi_k\psi_l^*\rangle y_p^k z_q^l\right)_{1\le p,q\le N} \tag{6.1.7}$$

where $\Psi(y_p)$ and $\Psi^*(1/z_q)$ are given by (1.1.43). Evaluating the vacuum expectation value within this determinant, we find

$$\begin{aligned}\Big\langle \Psi(y_1)\dots\Psi(y_N)\Psi^*(1/z_N)\dots\Psi^*(1/z_1)\Big\rangle &= \det\left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\delta_{k,l}y_p^{-k}z_q^{-l}\right)_{1\le p,q\le N}\\ &= \det\left(\frac{1}{y_p z_q - 1}\right)_{1\le p,q\le N}\end{aligned} \tag{6.1.8}$$

for arbitrary $\{y_1,\dots,y_N\}$ and $\{z_1,\dots,z_N\}$. Using this result in the expression (5.2.19) for the rescaled partition function, we obtain

$$\begin{aligned}Z'_N\Big|_{q=-1} &= \frac{2^N\prod\limits_{j,k=1}^{N}(1-y_j^2z_k^2)}{\prod\limits_{1\le j<k\le N}(y_j-y_k)(z_k-z_j)}\det\left(\frac{1}{y_j^2z_k^2-1}\right)_{1\le j,k\le N}\\ &= \frac{2^N\prod\limits_{j,k=1}^{N}(1-y_j^2z_k^2)}{\prod\limits_{1\le j<k\le N}(y_j-y_k)(z_k-z_j)}\Big\langle\Psi(y_1^2)\dots\Psi(y_N^2)\Psi^*(1/z_N^2)\dots\Psi^*(1/z_1^2)\Big\rangle\end{aligned} \tag{6.1.9}$$

Equation (6.1.9) demonstrates the presence of charged fermions in the partition function when $e^{2\gamma} = q = -1$, and provides a justification for the free fermionic nomenclature which is assigned to this point.

6.1.4 Bethe scalar product in the limit $\gamma = \pi i/2$

Consider the determinant expression (5.3.29) for the Bethe scalar product of the XXZ model. Setting $\gamma = \pi i/2$, we find that

$$S_N\Big(\{u\}_N, \{v\}_N, \{w\}_M\Big)\Big|_{\gamma=\pi i/2} = \frac{\langle 0\rangle^N \prod_{j=1}^{N}\prod_{k=1}^{M}[v_j - w_k] \prod_{j,k=1}^{N}\langle u_j - v_k\rangle}{\prod_{1\le j<k\le N}[u_k - u_j][v_j - v_k]} \tag{6.1.10}$$
$$\times \det\left(\frac{\prod_{l=1}^{M}[u_j - w_l] + (-)^N \prod_{l=1}^{M}\langle u_j - w_l\rangle}{\langle u_j - v_k\rangle[u_j - v_k]}\right)_{1\le j,k\le N}$$

where we have obtained the prefactor $\prod_{j,k=1}^{N}\langle u_j - v_k\rangle$ in (6.1.10) using the fact that $\prod_{k=1}^{N}\langle u_j - v_k\rangle = (-)^N \prod_{k=1}^{N}\langle u_j - v_k - \pi i\rangle$. Since the numerator of the determinant in (6.1.10) is common to the entire j^{th} row, we extract it as another prefactor, which gives

$$S_N\Big(\{u\}_N, \{v\}_N, \{w\}_M\Big)\Big|_{\gamma=\pi i/2} = \frac{\langle 0\rangle^N \prod_{j=1}^{N}\prod_{k=1}^{M}[v_j - w_k] \prod_{j,k=1}^{N}\langle u_j - v_k\rangle}{\prod_{1\le j<k\le N}[u_k - u_j][v_j - v_k]} \times \tag{6.1.11}$$
$$\prod_{j=1}^{N}\left(\prod_{k=1}^{M}[u_j - w_k] + (-)^N \prod_{k=1}^{M}\langle u_j - w_k\rangle\right) \frac{\prod_{1\le j<k\le N}\langle u_j - u_k\rangle[u_j - u_k]\langle v_k - v_j\rangle[v_k - v_j]}{\prod_{j,k=1}^{N}\langle u_j - v_k\rangle[u_j - v_k]}$$

where we have used the factorization of the resulting Cauchy determinant to produce the final term in (6.1.11). Cancelling various factors within (6.1.11), we obtain the formula

$$S_N\Big|_{\gamma=\pi i/2} = \langle 0\rangle^N \prod_{1\le j<k\le N}\langle u_k - u_j\rangle\langle v_j - v_k\rangle \prod_{j=1}^{N}\prod_{k=1}^{M}[v_j - w_k] \tag{6.1.12}$$
$$\times \prod_{j,k=1}^{N}\frac{1}{[u_j - v_k]} \prod_{j=1}^{N}\left(\prod_{k=1}^{M}[u_j - w_k] + (-)^N \prod_{k=1}^{M}\langle u_j - w_k\rangle\right)$$

Let us remark that when $\gamma = \pi i/2$ the Bethe equations (5.1.58) decouple into the form

$$\prod_{k=1}^{M}[v_j - w_k] + (-)^N \prod_{k=1}^{M}\langle v_j - w_k\rangle = 0 \tag{6.1.13}$$

for all $1 \leq j \leq N$. Hence all poles present in (6.1.12) are removable, by virtue of the constraints (6.1.13) on $\{v\}_N$. In the next section we will derive an analogue of (6.1.12) in the context of a more general free fermionic model.

6.2 Trigonometric Felderhof model

In this section we devote our attention to the trigonometric Felderhof model. Most of the material that we present originally appeared in [14], [37].

6.2.1 R-matrix and Yang-Baxter equation

The R-matrix (6.1.2) may be generalized to include extra variables, in such a way that the Yang-Baxter equation remains satisfied. This leads to the trigonometric limit of the model introduced by Felderhof in [29], [30], [31], and accordingly we call it the *trigonometric Felderhof model.* This model was also studied in [24], as the first in a hierarchy of vertex models with increasing spin. Explicitly speaking, the R-matrix for the trigonometric Felderhof model is given by

$$R_{ab}(u,p,v,q) = \begin{pmatrix} a_+(u,p,v,q) & 0 & 0 & 0 \\ 0 & b_+(u,p,v,q) & c_+(u,p,v,q) & 0 \\ 0 & c_-(u,p,v,q) & b_-(u,p,v,q) & 0 \\ 0 & 0 & 0 & a_-(u,p,v,q) \end{pmatrix}_{ab} \tag{6.2.1}$$

where we have defined the functions

$$a_\pm(u,p,v,q) = [\pm(u-v) + p + q] \tag{6.2.2}$$

$$b_\pm(u,p,v,q) = [u - v \pm (q-p)] \tag{6.2.3}$$

$$c_\pm(u,p,v,q) = [2p]^{\frac{1}{2}}[2q]^{\frac{1}{2}} \tag{6.2.4}$$

with $[u] = 2\sinh u$ as usual.[3] The R-matrix is an element of $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b)$, and the variables u, v are rapidities associated to the respective vector spaces $\mathcal{V}_a, \mathcal{V}_b$. The new features of this R-matrix are the variables p, q. These are called *external fields*, and are associated to the respective vector spaces $\mathcal{V}_a, \mathcal{V}_b$. We recover the free fermion point of the six-vertex model by setting $p = q = \frac{\pi i}{4}$.

[3] The parametrization of [24] is recovered by multiplying all weights by $e^{u-v+p+q}$ and setting $e^{2p} = \alpha, e^{2q} = \beta$.

The entries of the R-matrix (6.2.1) admit the same graphical representation as those of the R-matrix (5.1.21) in chapter 5. The only difference is that each vertex line now accommodates a rapidity variable and an external field. Hence we identify the functions (6.2.2)–(6.2.4) with the vertices shown below.

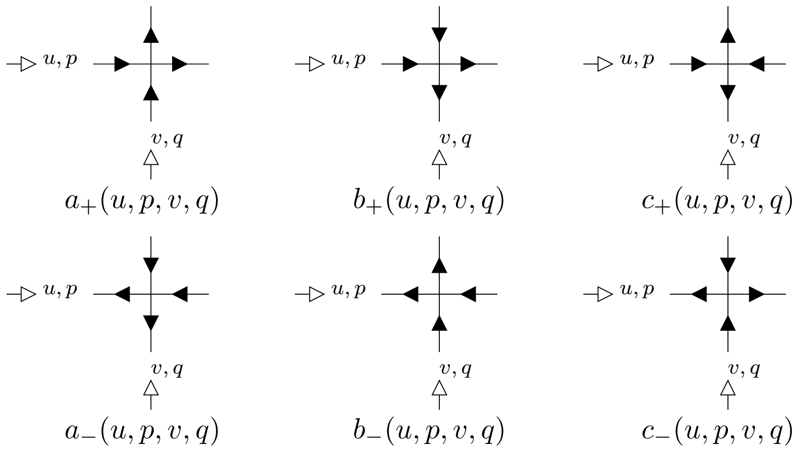

Figure 6.1: Six vertices of the trigonometric Felderhof model.

With the following result we see that the Yang-Baxter equation continues to hold, even in the presence of the external fields.

Lemma 1. The R-matrix (6.2.1) obeys the Yang-Baxter equation

$$R_{ab}(u,p,v,q)R_{ac}(u,p,w,r)R_{bc}(v,q,w,r) = R_{bc}(v,q,w,r)R_{ac}(u,p,w,r)R_{ab}(u,p,v,q) \tag{6.2.5}$$

This is an identity in $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b \otimes \mathcal{V}_c)$, true for all u, v, w and p, q, r.

Proof. By direct computation. □

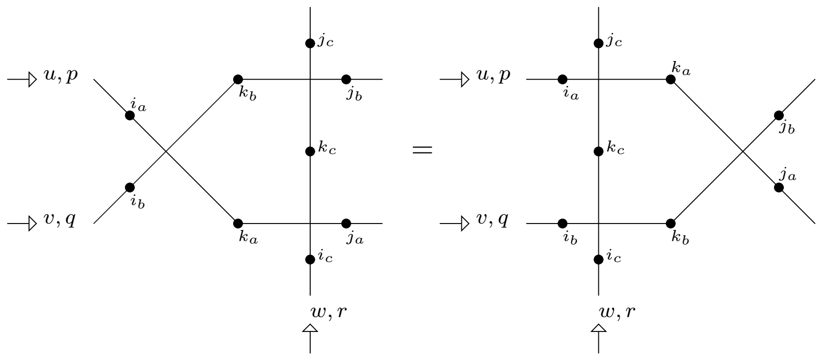

Figure 6.2: Yang-Baxter equation for the trigonometric Felderhof model.

6.2.2 Monodromy matrix and intertwining equation

The monodromy matrix is an ordered product of R-matrices, given by

$$T_a(u,p,\{w,r\}_M) = R_{a1}(u,p,w_1,r_1)\dots R_{aM}(u,p,w_M,r_M) \tag{6.2.6}$$

with the multiplication taken in the space $\mathrm{End}(\mathcal{V}_a)$. We write the contribution from the space $\mathrm{End}(\mathcal{V}_a)$ explicitly, by defining

$$T_a(u,p,\{w,r\}_M) = \begin{pmatrix} A(u,p,\{w,r\}_M) & B(u,p,\{w,r\}_M) \\ C(u,p,\{w,r\}_M) & D(u,p,\{w,r\}_M) \end{pmatrix}_a \tag{6.2.7}$$

where the matrix entries are all operators acting in $\mathcal{V}_1 \otimes \cdots \otimes \mathcal{V}_M$. For a graphical representation of these operators, we refer the reader to figure 5.2 in chapter 5. The correspondence is exactly the same, except that the rapidity u is now accompanied by the external field p, and each w_j by an r_j. Due to the Yang-Baxter equation (6.2.5), we obtain the intertwining equation

$$\begin{aligned} &R_{ab}(u,p,v,q)T_a(u,p,\{w,r\}_M)T_b(v,q,\{w,r\}_M) = \\ &T_b(v,q,\{w,r\}_M)T_a(u,p,\{w,r\}_M)R_{ab}(u,p,v,q) \end{aligned} \tag{6.2.8}$$

As usual, this leads to sixteen commutation relations amongst the entries of the monodromy matrix (6.2.7). Of these commutation relations, two have particular significance in our later calculations. They are given by

$$\begin{aligned} &[u-v+p+q]B(u,p,\{w,r\}_M)B(v,q,\{w,r\}_M) = \\ &[v-u+p+q]B(v,q,\{w,r\}_M)B(u,p,\{w,r\}_M) \end{aligned} \tag{6.2.9}$$

$$\begin{aligned} &[v-u+p+q]C(u,p,\{w,r\}_M)C(v,q,\{w,r\}_M) = \\ &[u-v+p+q]C(v,q,\{w,r\}_M)C(u,p,\{w,r\}_M) \end{aligned} \tag{6.2.10}$$

6.2.3 Domain wall partition function $Z_N(\{v,q\}_N, \{w,r\}_N)$

The domain wall partition function of the trigonometric Felderhof model has the algebraic definition

$$Z_N\Big(\{v,q\}_N,\{w,r\}_N\Big) = \langle \Downarrow_N | \overleftarrow{\prod_{j=1}^{N}} B(v_j,q_j,\{w,r\}_N) | \Uparrow_N \rangle \tag{6.2.11}$$

This naturally extends the definition of the domain wall partition function (5.2.1) to a model containing external fields. Notice that we must define an ordering of the B-operators in (6.2.11), since by (6.2.9) they do not commute.

Similarly to the previous chapter, we represent the domain wall partition function by an $N \times N$ lattice, as shown in figure 6.3.

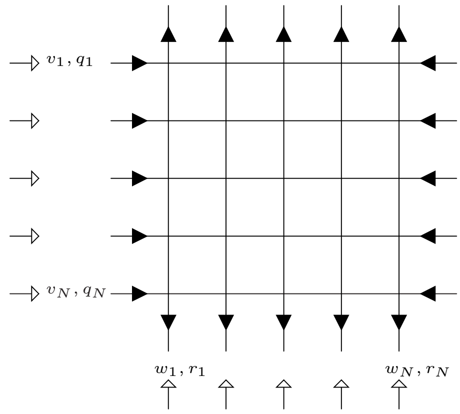

Figure 6.3: Domain wall partition function of the trigonometric Felderhof model. The top row of arrows corresponds with the state vector $| \Uparrow_N \rangle$. The bottom row of arrows corresponds with the dual state vector $\langle \Downarrow_N |$. Each horizontal lattice line corresponds to multiplication by a $B(v_j, q_j, \{w, r\}_N)$ operator. Notice that the ordering of these lattice lines respects the ordering of B-operators defined in (6.2.11).

6.2.4 Conditions on $Z_N(\{v,q\}_N, \{w,r\}_N)$

We now progress towards calculating the domain wall partition function (6.2.11). The procedure begins with the following result from [37], which establishes a set of Korepin-type conditions on $Z_N(\{v,q\}_N, \{w,r\}_N)$.

Lemma 2. We adopt the shorthand $Z_N = Z_N(\{v,q\}_N, \{w,r\}_N)$. For all $N \geq 2$ we claim that

1. Z_N is a trigonometric polynomial of degree $N-1$ in the rapidity variable v_N.

2. The zeros of Z_N occur at the points $v_N = v_j + q_j + q_N$, for all $1 \leq j \leq N-1$.

3. Setting $v_N = w_N + q_N + r_N$, Z_N satisfies the recursion relation

$$Z_N \Big|_{v_N = w_N + q_N + r_N} = [2q_N]^{\frac{1}{2}} [2r_N]^{\frac{1}{2}} \prod_{j=1}^{N-1} [w_N - w_j + r_j + r_N][v_j - w_N + q_j - r_N] Z_{N-1} \tag{6.2.12}$$

where Z_{N-1} is the domain wall partition function on a square lattice of size $N-1$.

In addition, we have the supplementary condition

4. The partition function on the 1×1 lattice is given by $Z_1 = [2q_1]^{\frac{1}{2}} [2r_1]^{\frac{1}{2}}$.

Proof.

1. By inserting the set of states $\sum_{n=1}^{N} \sigma_n^{+} | \Downarrow_N \rangle \langle \Downarrow_N | \sigma_n^{-}$ after the first B-operator appearing in (6.2.11), we obtain the expansion

$$Z_N\Big(\{v,q\}_N, \{w,r\}_N\Big) = \sum_{n=1}^{N} \langle \Downarrow_N | B(v_N, q_N, \{w,r\}_N) \sigma_n^{+} | \Downarrow_N \rangle \qquad (6.2.13)$$
$$\times \langle \Downarrow_N | \sigma_n^{-} \overleftarrow{\prod_{j=1}^{N-1}} B(v_j, q_j, \{w,r\}_N) | \Uparrow_N \rangle$$

in which all dependence on v_N appears in the first factor within the sum. Hence we shall calculate $\langle \Downarrow_N | B(v_N, q_N, \{w,r\}_N) \sigma_n^{+} | \Downarrow_N \rangle$ for all $1 \leq n \leq N$, as shown below.

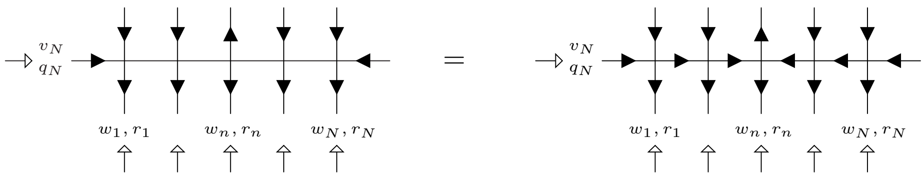

Figure 6.4: Peeling away the bottom row of the trigonometric Felderhof partition function. The diagram on the left represents $\langle \Downarrow_N | B(v_N, q_N, \{w,r\}_N) \sigma_n^{+} | \Downarrow_N \rangle$, with the internal black arrows being summed over all configurations. The diagram on the right represents the only surviving configuration.

The right hand side of figure 6.4 represents a product of vertices. Replacing each vertex with its corresponding trigonometric weight, we have

$$\langle \Downarrow_N | B(v_N, q_N, \{w,r\}_N) \sigma_n^{+} | \Downarrow_N \rangle = \qquad (6.2.14)$$
$$[2q_N]^{\frac{1}{2}} [2r_n]^{\frac{1}{2}} \prod_{1 \leq j < n} [v_N - w_j + r_j - q_N] \prod_{n < j \leq N} [w_j - v_N + q_N + r_j]$$

Substituting (6.2.14) into the expansion (6.2.13) gives

$$Z_N\Big(\{v,q\}_N, \{w,r\}_N\Big) = \sum_{n=1}^{N} [2q_N]^{\frac{1}{2}} [2r_n]^{\frac{1}{2}} \prod_{1 \leq j < n} [v_N - w_j + r_j - q_N] \qquad (6.2.15)$$
$$\times \prod_{n < j \leq N} [w_j - v_N + q_N + r_j] \langle \Downarrow_N | \sigma_n^{-} \overleftarrow{\prod_{j=1}^{N-1}} B(v_j, q_j, \{w,r\}_N) | \Uparrow_N \rangle$$

From (6.2.15) we see that every term in $Z_N(\{v,q\}_N, \{w,r\}_N)$ contains a product of exactly $N-1$ trigonometric functions with argument v_N. Thus $Z_N(\{v,q\}_N, \{w,r\}_N)$ is a trigonometric polynomial of degree $N-1$ in the variable v_N.

2. We multiply the partition function (6.2.11) by $\prod_{j=1}^{N-1}[v_N - v_j + q_j + q_N]$ and repeatedly use the commutation relation

$$[v_N - v_j + q_j + q_N]B(v_N, q_N, \{w, r\}_N)B(v_j, q_j, \{w, r\}_N) = \\ [v_j - v_N + q_j + q_N]B(v_j, q_j, \{w, r\}_N)B(v_N, q_N, \{w, r\}_N) \tag{6.2.16}$$

which is a rewriting of (6.2.9), to change the order of the B-operators. We obtain

$$\prod_{j=1}^{N-1}[v_N - v_j + q_j + q_N]Z_N\Big(\{v, q\}_N, \{w, r\}_N\Big) = \\ \prod_{j=1}^{N-1}[v_j - v_N + q_j + q_N]\langle\Downarrow_N|\prod_{j=1}^{N-1} B(v_j, q_j, \{w, r\}_N)B(v_N, q_N, \{w, r\}_N)|\Uparrow_N\rangle \tag{6.2.17}$$

Graphically, we depict (6.2.17) with the following diagrams.

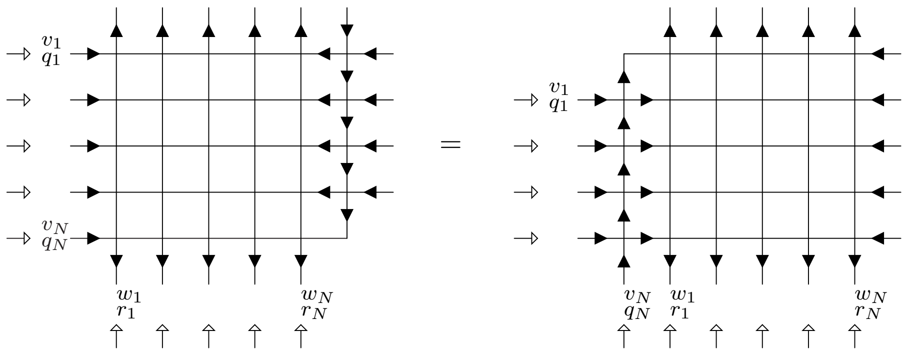

Figure 6.5: Reordering the lattice lines of the trigonometric Felderhof partition function. The diagram on the left is the domain wall partition function multiplied by the string of vertices $\prod_{j=1}^{N-1} a_-(v_j, q_j, v_N, q_N)$, and it corresponds with the left hand side of (6.2.17). Each vertex can be threaded through the lattice using the Yang-Baxter equation, which ultimately produces the diagram on the right. This diagram represents the domain wall partition function with its N^{th} row transferred to the top of the lattice, multiplied by the string of vertices $\prod_{j=1}^{N-1} a_+(v_j, q_j, v_N, q_N)$. Clearly, this corresponds with the right hand side of (6.2.17).

The right hand side of (6.2.17) is a trigonometric polynomial of degree $2N - 2$ in v_N, with zeros at the points $v_N = v_j + q_j + q_N$ for all $1 \le j \le N - 1$. Therefore the partition function $Z_N(\{v, q\}_N, \{w, r\}_N)$ must have zeros at the same points.

3. We start from the expansion (6.2.15) of the domain wall partition function, and set $v_N = w_N + q_N + r_N$. This causes all terms in the summation over $1 \le n \le N$ to collapse to zero except the $n = N$ term, and we obtain

$$Z_N\Big(\{v,q\}_N,\{w,r\}_N\Big)\Big|_{v_N=w_N+q_N+r_N} = \tag{6.2.18}$$
$$[2q_N]^{\frac{1}{2}}[2r_N]^{\frac{1}{2}}\prod_{j=1}^{N-1}[w_N-w_j+r_j+r_N]\langle\Downarrow_N|\sigma_N^- \overleftarrow{\prod_{j=1}^{N-1}} B(v_j,q_j,\{w,r\}_N)|\Uparrow_N\rangle$$

This equation can be further simplified by considering the graphical representation of $\langle\Downarrow_N|\sigma_N^-\overleftarrow{\prod}_{j=1}^{N-1} B(v_j,q_j,\{w,r\}_N)|\Uparrow_N\rangle$, as shown below.

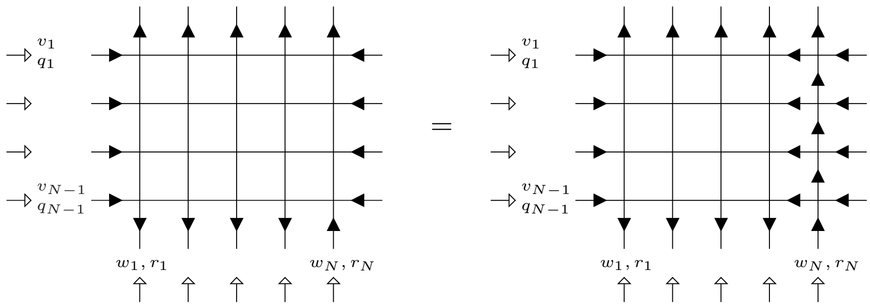

Figure 6.6: Peeling the right-most column of the trigonometric Felderhof partition function. The diagram on the left represents $\langle\Downarrow_N|\sigma_N^-\overleftarrow{\prod}_{j=1}^{N-1} B(v_j,q_j,\{w,r\}_N)|\Uparrow_N\rangle$, with the internal black arrows being summed over all configurations. The diagram on the right contains all surviving configurations.

The right hand side of figure 6.6 represents the $(N-1)\times(N-1)$ partition function, multiplied by a column of vertices. Replacing these vertices with their trigonometric weights, we have

$$\langle\Downarrow_N|\sigma_N^-\overleftarrow{\prod_{j=1}^{N-1}} B(v_j,q_j,\{w,r\}_N)|\Uparrow_N\rangle = \prod_{j=1}^{N-1}[v_j-w_N+q_j-r_N]Z_{N-1} \tag{6.2.19}$$

Substituting (6.2.19) into (6.2.18) we recover the required recursion relation (6.2.12).

4. Specializing the definition (6.2.11) to the case $N=1$ gives

$$Z_1(v_1,q_1,w_1,r_1) = \langle\Downarrow_1|B(v_1,q_1,\{w,r\}_1)|\Uparrow_1\rangle \tag{6.2.20}$$
$$=\uparrow_{a_1}^*\otimes\downarrow_1^* R_{a_1 1}(v_1,q_1,w_1,r_1)\uparrow_1\otimes\downarrow_{a_1}= [2q_1]^{\frac{1}{2}}[2r_1]^{\frac{1}{2}}$$

as required. Alternatively, the 1×1 partition function is the top-right vertex in figure 6.1, whose weight is equal to $[2q_1]^{\frac{1}{2}}[2r_1]^{\frac{1}{2}}$.

□

6.2.5 Factorized expression for $Z_N(\{v,q\}_N,\{w,r\}_N)$

The conditions **1**–**4** are strong constraints on $Z_N(\{v,q\}_N,\{w,r\}_N)$. Not only do they specify $Z_N(\{v,q\}_N,\{w,r\}_N)$ uniquely, they lead to its direct evaluation, as we demonstrate below.

Lemma 3. The domain wall partition function has the factorized expression

$$Z_N\Big(\{v,q\}_N,\{w,r\}_N\Big) = \prod_{j=1}^{N}[2q_j]^{\frac{1}{2}}[2r_j]^{\frac{1}{2}} \prod_{1\le j<k\le N}[v_j-v_k+q_j+q_k][w_k-w_j+r_j+r_k] \tag{6.2.21}$$

Specializing the external fields to $q_j = r_j = \frac{\pi i}{4}$ for all $1 \le j \le N$, we recover the partition function of the six-vertex model at its free fermion point (6.1.6). The result (6.2.21) was first obtained in [14] using a complicated recursion relation. A more straightforward proof, based on solving the conditions **1**–**4**, subsequently appeared in [37]. It is the latter proof which we present below.

Proof. From condition **1** and **2** on $Z_N(\{v,q\}_N,\{w,r\}_N)$, we know that it must have the form

$$Z_N\Big(\{v,q\}_N,\{w,r\}_N\Big) = \mathcal{C}\Big(\{v\}_{N-1},\{q\}_N,\{w,r\}_N\Big)\prod_{j=1}^{N-1}[v_j-v_N+q_j+q_N] \tag{6.2.22}$$

where $\mathcal{C}$ does not depend on v_N, but depends on all other variables. Evaluating (6.2.22) at $v_N = w_N + q_N + r_N$ and comparing with condition **3** on Z_N, we obtain

$$\begin{aligned} Z_N\Big|_{v_N=w_N+q_N+r_N} &= \mathcal{C}\Big(\{v\}_{N-1},\{q\}_N,\{w,r\}_N\Big)\prod_{j=1}^{N-1}[v_j-w_N+q_j-r_N] \qquad (6.2.23)\\ &= [2q_N]^{\frac{1}{2}}[2r_N]^{\frac{1}{2}}\prod_{j=1}^{N-1}[w_N-w_j+r_j+r_N][v_j-w_N+q_j-r_N]Z_{N-1} \end{aligned}$$

from which we extract the equation

$$\mathcal{C} = [2q_N]^{\frac{1}{2}}[2r_N]^{\frac{1}{2}}\prod_{j=1}^{N-1}[w_N-w_j+r_j+r_N]Z_{N-1}\Big(\{v,q\}_{N-1},\{w,r\}_{N-1}\Big) \tag{6.2.24}$$

Substituting this expression for $\mathcal{C}$ into (6.2.22), we obtain the recurrence

$$Z_N\Big(\{v,q\}_N,\{w,r\}_N\Big) = [2q_N]^{\frac{1}{2}}[2r_N]^{\frac{1}{2}}\times \tag{6.2.25}$$
$$\prod_{j=1}^{N-1}[v_j - v_N + q_j + q_N][w_N - w_j + r_j + r_N] Z_{N-1}\Big(\{v,q\}_{N-1},\{w,r\}_{N-1}\Big)$$

whose basis is given by condition **4**. This recurrence is trivially solved to produce the formula (6.2.21). □

6.2.6 Scalar products $S_n(\{u,p\}_n,\{v,q\}_N,\{w,r\}_M)$

Let $\{u\}_n = \{u_1,\ldots,u_n\}, \{v\}_N = \{v_1,\ldots,v_N\}, \{w\}_M = \{w_1,\ldots,w_M\}$ be sets of rapidities, and $\{p\}_n = \{p_1,\ldots,p_n\}, \{q\}_N = \{q_1,\ldots,q_N\}, \{r\}_M = \{r_1,\ldots,r_M\}$ the corresponding sets of external fields. The cardinalities of these sets are assumed to satisfy $0 \le n \le N$ and $1 \le N \le M$. For $n = 0$ we define

$$S_0\Big(\{v,q\}_N,\{w,r\}_M\Big) = \langle \Downarrow_{N/M} | \overleftarrow{\prod_{k=1}^{N}} B(v_k,q_k,\{w,r\}_M)| \Uparrow_M\rangle \tag{6.2.26}$$

Similarly to the last chapter, we will find that S_0 is equal to the trigonometric Felderhof partition function Z_N, up to an overall normalization. Next, for all $1 \le n \le N-1$ we define

$$S_n\Big(\{u,p\}_n,\{v,q\}_N,\{w,r\}_M\Big) = \langle \Downarrow_{\widetilde{N}/M} | \overleftarrow{\prod_{j=1}^{n}} C(u_j,p_j,\{w,r\}_M) \overleftarrow{\prod_{k=1}^{N}} B(v_k,q_k,\{w,r\}_M)| \Uparrow_M\rangle \tag{6.2.27}$$

with $\widetilde{N} = N - n$. Finally, in the case $n = N$ we fix

$$S_N\Big(\{u,p\}_N,\{v,q\}_N,\{w,r\}_M\Big) = \langle \Uparrow_M | \overleftarrow{\prod_{j=1}^{N}} C(u_j,p_j,\{w,r\}_M) \overleftarrow{\prod_{k=1}^{N}} B(v_k,q_k,\{w,r\}_M)| \Uparrow_M\rangle \tag{6.2.28}$$

The scalar products (6.2.26)–(6.2.28) are the trigonometric Felderhof analogues of those defined in subsection 5.3.1. They have identical graphical representations to those described in subsection 5.3.2, except that every rapidity variable is now accompanied by an appropriate external field. In the following subsection we give a set of conditions on these scalar products, using similar techniques to those of the previous chapter.

6.2.7 Conditions on $S_n(\{u,p\}_n,\{v,q\}_N,\{w,r\}_M)$

Lemma 4. For all $1 \le n \le N$ we claim that

1. S_n is invariant under the simultaneous permutation of variables $\{w_j,r_j\} \leftrightarrow \{w_k,r_k\}$ for all $j,k \in \{\widetilde{N}+1,\ldots,M\}$.

2. S_n is a trigonometric polynomial of degree $M-1$ in u_n, with zeros occurring at the points $u_n = p_n + w_j + r_j$, for all $1 \le j \le \widetilde{N}$.

3. Setting $u_n + p_n = w_{\widetilde{N}+1} + r_{\widetilde{N}+1}$, S_n satisfies the recursion relation

$$S_n\Big|_{u_n+p_n=w_{\widetilde{N}+1}+r_{\widetilde{N}+1}} = [2p_n]^{\frac{1}{2}}[2r_{\widetilde{N}+1}]^{\frac{1}{2}} \prod_{1\le j<\widetilde{N}+1} [w_j - w_{\widetilde{N}+1} + r_j - r_{\widetilde{N}+1} + 2p_n] \times \prod_{\widetilde{N}+1<j\le M} [w_{\widetilde{N}+1} - w_j + r_j + r_{\widetilde{N}+1}] S_{n-1} \tag{6.2.29}$$

where we have abbreviated $S_{n-1} = S_{n-1}(\{u,p\}_{n-1},\{v,q\}_N,\{w,r\}_M)$.

In addition, we have the supplementary condition

4. S_0 and Z_N are related via the equation

$$S_0\Big(\{v,q\}_N,\{w,r\}_M\Big) = \prod_{j=1}^{N}\prod_{k=N+1}^{M} [v_j - w_k + q_j - r_k] Z_N\Big(\{v,q\}_N,\{w,r\}_N\Big) \tag{6.2.30}$$

Proof. The proof of properties **1**–**4** is analogous to the proof of lemma 9 in chapter 5. There, we presented an algebraic proof of the properties. Here, we outline a less technical graphical proof.

1. For any $\widetilde{N}+1 < j \le M$, multiplying $S_n(\{u,p\}_n,\{v,q\}_N,\{w,r\}_M)$ by the function $a_+(w_j,r_j,w_{j-1},r_{j-1})$ is equivalent to attaching an a_+ vertex at the base of the lattice, as shown in figure 6.7.

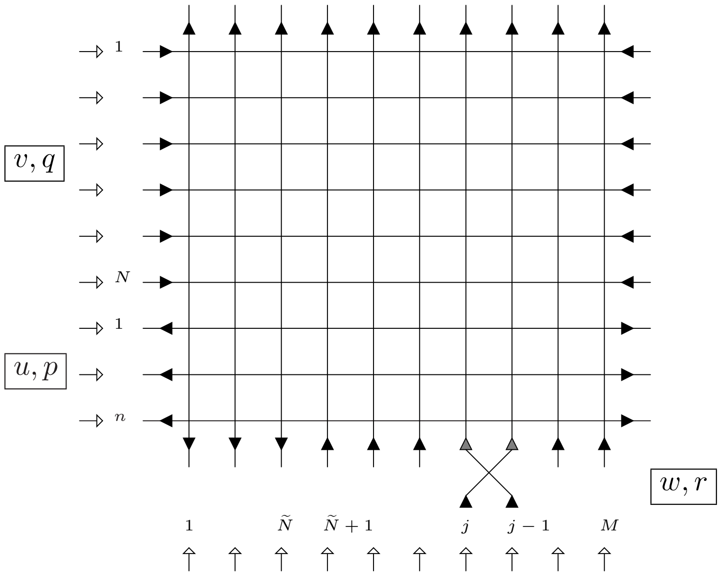

Figure 6.7: Attaching an $a_+(w_j, r_j, w_{j-1}, r_{j-1})$ vertex to the S_n lattice. The points marked with grey arrows are considered to be summed over all arrow configurations, but the only non-zero configuration is the one shown.

The attached vertex can be translated vertically through the lattice using the graphical version of the Yang-Baxter equation, as given by figure 6.2. It ultimately emerges from the top of the lattice, still as an $a_+(w_j, r_j, w_{j-1}, r_{j-1})$ vertex, and the $(j-1)^{\text{th}}$ and j^{th} lattice columns are swapped in the process. The result of this procedure is shown in figure 6.8.

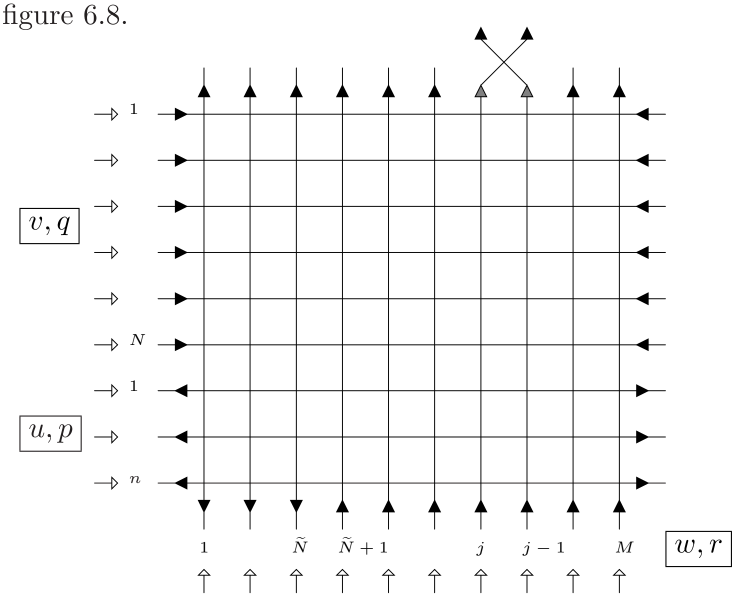

Figure 6.8: Extracting the $a_+(w_j, r_j, w_{j-1}, r_{j-1})$ vertex from the S_n lattice. Once again, the grey arrows indicate the only surviving configuration in the summation at those points.

Cancelling the common factor $a_+(w_j, r_j, w_{j-1}, r_{j-1})$ from figures 6.7 and 6.8, we conclude that S_n is invariant under swapping the $(j-1)^{\text{th}}$ and j^{th} lattice columns, for all $\widetilde{N}+1 < j \le M$. An arbitrary permutation of the lattice columns is just a composition of such swaps. Therefore S_n is invariant under permuting its j^{th} and k^{th} columns, for all $\widetilde{N}+1 \le j, k \le M$.

2. Consider the graphical representation of the scalar product S_n, as given below.

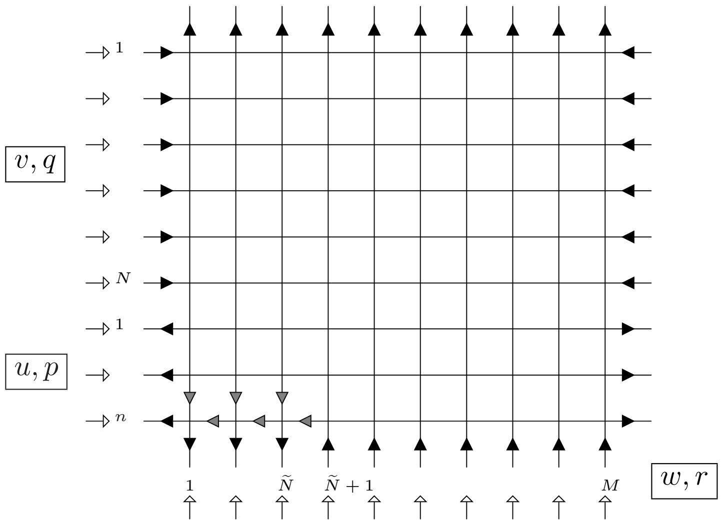

Figure 6.9: Lattice representation of S_n, with frozen vertices included. The grey arrows indicate points which are summed over all configurations, but whose only non-zero configuration is the one shown.

We examine the final row of this lattice, through which the variable u_n flows. Every non-zero configuration of this row contains a $c_-(u_n, p_n, w_j, r_j)$ vertex, which by (6.2.4) does not depend on u_n, and $M-1$ other vertices which are trigonometric polynomials of degree 1 in u_n. It follows that S_n is a trigonometric polynomial of degree $M-1$ in u_n.

Furthermore, all surviving configurations of the final row contain the $\widetilde{N}$ vertices as shown in figure 6.9. Consequentially, S_n contains the factor

$$\prod_{j=1}^{\widetilde{N}} a_-(u_n, p_n, w_j, r_j) = \prod_{j=1}^{\widetilde{N}} [w_j - u_n + p_n + r_j] \tag{6.2.31}$$

which gives rise to zeros at $u_n = p_n + w_j + r_j$ for all $1 \le j \le \widetilde{N}$.

3. Consider the vertex at the intersection of the u_n and $w_{\widetilde{N}+1}$ lines in figure 6.9. In any given lattice configuration, this can be of type $b_-(u_n, p_n, w_{\widetilde{N}+1}, r_{\widetilde{N}+1})$ or $c_-(u_n, p_n, w_{\widetilde{N}+1}, r_{\widetilde{N}+1})$. Setting $u_n + p_n = w_{\widetilde{N}+1} + r_{\widetilde{N}+1}$ results in the cancellation of all terms containing $b_-(u_n, p_n, w_{\widetilde{N}+1}, r_{\widetilde{N}+1})$, and freezes the entire final row of the lattice to the configuration below.

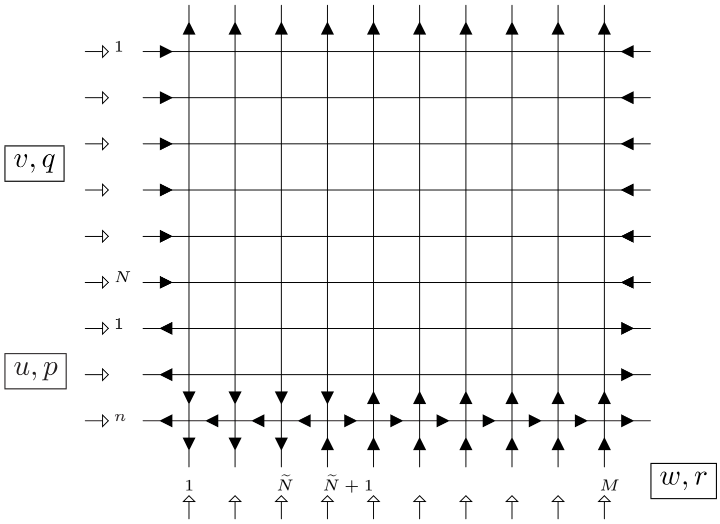

Figure 6.10: Freezing the entire last row of the S_n lattice. The last row of vertices produces the prefactor in (6.2.29), while the remainder of the lattice represents S_{n-1}.

From the diagram we see that setting $u_n + p_n = w_{\widetilde{N}+1} + r_{\widetilde{N}+1}$ reduces S_n to S_{n-1}, up to a multiplicative factor. This factor is evaluated by matching each vertex in the final row with its trigonometric weight, giving

$$S_n \Big|_{u_n+p_n=w_{\widetilde{N}+1}+r_{\widetilde{N}+1}} = \prod_{1\le j<\widetilde{N}+1} a_-\Big(w_{\widetilde{N}+1} + r_{\widetilde{N}+1} - p_n, p_n, w_j, r_j\Big)\times \tag{6.2.32}$$
$$c_-\Big(w_{\widetilde{N}+1} + r_{\widetilde{N}+1} - p_n, p_n, w_{\widetilde{N}+1}, r_{\widetilde{N}+1}\Big) \prod_{\widetilde{N}+1<j\le M} a_+\Big(w_{\widetilde{N}+1} + r_{\widetilde{N}+1} - p_n, p_n, w_j, r_j\Big) S_{n-1}$$

Using the explicit formulae (6.2.2) and (6.2.4) for the functions appearing in (6.2.32), we obtain the required recursion relation (6.2.29).

4. The scalar product S_0 is represented by the lattice below.

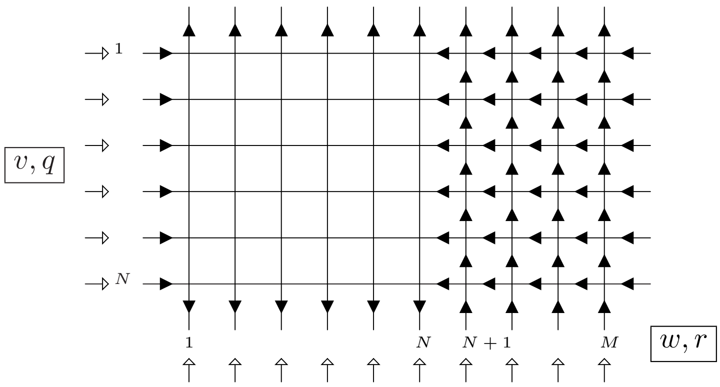

Figure 6.11: Frozen vertices within S_0. The final $M-N$ columns of vertices produce the prefactor in (6.2.30), while the remainder of the lattice represents Z_N.

The vertices in the last $M-N$ columns of a given lattice configuration must be of the form $b_-(v_j,q_j,w_k,r_k)$, or else the configuration vanishes. Peeling away this block of frozen vertices, we find that S_0 is equal to Z_N up to the overall factor $\prod_{j=1}^{N}\prod_{k=N+1}^{M} b_-(v_j,q_j,w_k,r_k)$.

□

6.2.8 Factorized expression for $S_n(\{u,p\}_n,\{v,q\}_N,\{w,r\}_M)$

Lemma 5. We shall assume that (v_j+q_j) satisfies the equation

$$(-)^N \prod_{k=1}^{M}[v_j+q_j-w_k+r_k]+\prod_{k=1}^{M}[v_j+q_j-w_k-r_k]=0 \tag{6.2.33}$$

for all $1\le j\le N$.[4] In the presence of this constraint, the scalar product S_n has the factorized expression

$$S_n\Big(\{u,p\}_n,\{v,q\}_N,\{w,r\}_M\Big)=\prod_{j=1}^{n}[2p_j]^{\frac{1}{2}}\prod_{j=1}^{N}[2q_j]^{\frac{1}{2}}\prod_{j=1}^{\widetilde{N}}[2r_j]^{\frac{1}{2}}\times \tag{6.2.34}$$
$$\prod_{1\le j<k\le n}[u_k-u_j+p_j+p_k]\prod_{1\le j<k\le N}[v_j-v_k+q_j+q_k]\prod_{1\le j<k\le \widetilde{N}}[w_k-w_j+r_j+r_k]\times$$
$$\prod_{j=1}^{n}\prod_{k=1}^{\widetilde{N}}[w_k-u_j+p_j+r_k]\prod_{j=1}^{N}\prod_{k=\widetilde{N}+1}^{M}[v_j-w_k+q_j-r_k]\times$$
$$\prod_{j=1}^{n}\prod_{k=1}^{N}\frac{1}{[u_j-v_k+p_j-q_k]}\prod_{j=1}^{n}\left((-)^N\prod_{k=1}^{M}[u_j-w_k+p_j+r_k]+\prod_{k=1}^{M}[u_j-w_k+p_j-r_k]\right)$$

Proof. We begin by stating that the conditions **1–4** are uniquely determining.[5] Hence we need only verify that (6.2.34) satisfies properties **1–4**.

1. By studying (6.2.34) we see that S_n has dependence on $\{w_{\widetilde{N}+1},\dots,w_M\}$ and $\{r_{\widetilde{N}+1},\dots,r_M\}$ only through the terms

$$\prod_{j=1}^{N}\prod_{k=\widetilde{N}+1}^{M}[v_j-w_k+q_j-r_k],\qquad \prod_{j=1}^{n}\left((-)^N\prod_{k=1}^{M}[u_j-w_k+p_j+r_k]+\prod_{k=1}^{M}[u_j-w_k+p_j-r_k]\right)$$

Both of these terms are invariant under the permutation $\{w_j,r_j\}\leftrightarrow\{w_k,r_k\}$ for all $j,k\in\{\widetilde{N}+1,\dots,M\}$.

[4] The equations (6.2.33) constitute the Bethe equations for the trigonometric Felderhof model. By this, we mean that the state vector $\overleftarrow{\prod}_{j=1}^{N} B(v_j,q_j,\{w,r\}_M)|\Uparrow_M\rangle$ is an eigenvector of $A(u,p,\{w,r\}_M)+D(u,p,\{w,r\}_M)$ if and only if the equations (6.2.33) are obeyed. We will not prove this fact explicitly, but it can be derived by extending theorem 1 in chapter 2 to models with external fields.

[5] This is proved along very similar lines to lemma 6 in the previous chapter.

2. Since $(v_j + q_j)$ is a root of the equation (6.2.33) for all $1 \le j \le N$, it follows that

$$\prod_{j=1}^{n}\prod_{k=1}^{N} \frac{1}{[u_j - v_k + p_j - q_k]} \prod_{j=1}^{n} \left((-)^N \prod_{k=1}^{M} [u_j - w_k + p_j + r_k] + \prod_{k=1}^{M} [u_j - w_k + p_j - r_k] \right)$$

is a trigonometric polynomial in u_n of degree $M - N$. The remaining terms in (6.2.34) comprise a trigonometric polynomial of degree $N - 1$ in u_n. Therefore the entire expression (6.2.34) is a trigonometric polynomial of degree $M - 1$ in u_n. In addition, the required factor $\prod_{j=1}^{\widetilde{N}} [w_j - u_n + p_n + r_j]$ is present in (6.2.34).

3. The recursion relation (6.2.29) is proved by setting $u_n + p_n = w_{\widetilde{N}+1} + r_{\widetilde{N}+1}$ in (6.2.34) and rearranging the factors in the resulting equation. Since this procedure is elementary in nature, we shall omit the details.

4. Setting $n = 0$ in (6.2.34) gives

$$S_0\Big(\{v,q\}_N, \{w,r\}_M\Big) = \prod_{j=1}^{N} \prod_{k=N+1}^{M} [v_j - w_k + q_j - r_k] \times \tag{6.2.35}$$
$$\prod_{j=1}^{N} [2q_j]^{\frac{1}{2}} [2r_j]^{\frac{1}{2}} \prod_{1 \le j < k \le N} [v_j - v_k + q_j + q_k][w_k - w_j + r_j + r_k]$$

Comparing equation (6.2.35) with the factorized expression (6.2.21) for the domain wall partition function, we verify (6.2.30).

□

6.2.9 Evaluation of $S_N(\{u,p\}_N, \{v,q\}_N, \{w,r\}_M)$

For completeness, we write the $n = N$ case of equation (6.2.34) explicitly. We have

$$S_N\Big(\{u,p\}_N, \{v,q\}_N, \{w,r\}_M\Big) = \prod_{j=1}^{N} [2p_j]^{\frac{1}{2}} [2q_j]^{\frac{1}{2}} \times \tag{6.2.36}$$
$$\prod_{1 \le j < k \le N} [u_k - u_j + p_j + p_k][v_j - v_k + q_j + q_k] \prod_{j=1}^{N} \prod_{k=1}^{M} [v_j - w_k + q_j - r_k] \times$$
$$\prod_{j,k=1}^{N} \frac{1}{[u_j - v_k + p_j - q_k]} \prod_{j=1}^{N} \left((-)^N \prod_{k=1}^{M} [u_j - w_k + p_j + r_k] + \prod_{k=1}^{M} [u_j - w_k + p_j - r_k] \right)$$

Specializing the external fields to $p_j = q_j = \frac{\pi i}{4}$, $r_k = \frac{\pi i}{4}$ for all $1 \le j \le N$, $1 \le k \le M$ gives the XXZ Bethe scalar product at the free fermion point (6.1.12).

6.3 Free fermion point of SOS model

6.3.1 Jacobi theta functions

We begin by presenting some basic theory on elliptic functions, taken from chapter 15 of [5]. Up to overall normalization and scaling of the variable u, the Jacobi theta functions $H(u), H_1(u)$ are given by

$$H(u) = 2 \sinh u \prod_{n=1}^{\infty} \left(1 - 2\mathsf{q}^{2n} \cosh(2u) + \mathsf{q}^{4n}\right) \left(1 - \mathsf{q}^{2n}\right) \tag{6.3.1}$$

$$H_1(u) = 2i \cosh u \prod_{n=1}^{\infty} \left(1 + 2\mathsf{q}^{2n} \cosh(2u) + \mathsf{q}^{4n}\right) \left(1 - \mathsf{q}^{2n}\right) \tag{6.3.2}$$

where $0 < \mathsf{q} < 1$ is the elliptic nome. Taking the limit $\mathsf{q} \to 0$, these functions collapse to their trigonometric versions, $2\sinh u$ and $2i\cosh u$ respectively. Henceforth, we will find it convenient to define

$$[u] = H\left(\frac{\pi i u}{2}\right), \qquad \langle u \rangle = H_1\left(\frac{\pi i u}{2}\right) = [u+1] \tag{6.3.3}$$

which should not be confused with the previous usage of this notation, in the first part of the chapter. The theta function $[u]$ is entire and has the quasi-periodicity properties

$$[u+2] = -[u], \quad [u - 2i\log(\mathsf{q})/\pi] = -\frac{1}{\mathsf{q}} \exp(-\pi i u)[u] \tag{6.3.4}$$

From these properties we obtain the following result, which will be used later in the chapter.

Theorem 1. If $f(u)$ is an entire function that satisfies the quasi-periodicity conditions

$$f(u+2) = (-)^N f(u), \quad f(u - 2i\log(\mathsf{q})/\pi) = \frac{(-)^N}{\mathsf{q}^N} \exp\Big(\pi i(\eta - Nu)\Big) f(u) \tag{6.3.5}$$

for some constant η, then

$$f(u) = \mathcal{C} \left(\prod_{j=1}^{N-1} [u - \zeta_j] \right) [u - \eta + \sum_{j=1}^{N-1} \zeta_j] \tag{6.3.6}$$

where $\mathcal{C}$ and $\zeta_1, \ldots, \zeta_{N-1}$ are suitably chosen constants.

Proof. Choose a period rectangle R such that $f(u)$ has no zeros on the boundary ∂R, and integrate $\frac{f'(u)}{f(u)}$ around the anti-clockwise contour formed by ∂R. From the quasi-periodicity conditions it follows that

$$\oint_{\partial R} \frac{f'(u)}{f(u)} du = 2\pi i N \tag{6.3.7}$$

Hence the sum of residues of $\frac{f'(u)}{f(u)}$ in R is equal to N, proving that $f(u)$ has exactly N zeros in R, if we count a zero of order n with multiplicity n. Write the zeros as $\zeta_1, \ldots, \zeta_N$, and define the function $\phi(u) = \prod_{j=1}^{N} [u - \zeta_j]$. By construction $\frac{f'(u)}{f(u)} - \frac{\phi'(u)}{\phi(u)}$ is doubly periodic and holomorphic, and therefore

$$\frac{f'(u)}{f(u)} - \frac{\phi'(u)}{\phi(u)} = \frac{d}{du} \log \frac{f(u)}{\phi(u)} = \kappa \tag{6.3.8}$$

where κ is a constant. Integrating, we obtain $f(u) = \mathcal{C} e^{\kappa u} \prod_{j=1}^{N} [u - \zeta_j]$. Finally, using the quasi-periodicity properties of $f(u)$ we obtain $\kappa = 0$ and $\eta = \sum_{j=1}^{N} \zeta_j$, which concludes the proof. □

6.3.2 R-matrix and dynamical Yang-Baxter equation

The eight-vertex solid-on-solid (SOS) model was introduced by Baxter in [4], using the vertex/interaction-round-a-face(IRF) correspondence. The main feature distinguishing this model from its vertex counterpart is the *dynamical* or *height parameter* appearing in its weights, as we describe below. In the parametrization of [23], the R-matrix for the SOS model is given by

$$R_{ab}(u, v, h) = \begin{pmatrix} a_+(u, v) & 0 & 0 & 0 \\ 0 & b_+(u, v, h) & c_+(u, v, h) & 0 \\ 0 & c_-(u, v, h) & b_-(u, v, h) & 0 \\ 0 & 0 & 0 & a_-(u, v) \end{pmatrix}_{ab} \tag{6.3.9}$$

where we have defined the weights

$$a_\pm(u, v) = H(\gamma(u - v + 1)) \tag{6.3.10}$$

$$b_\pm(u, v, h) = \pm i \frac{H(\gamma(\xi + h \mp 1))}{H(\gamma(\xi + h))} H(\gamma(u - v)) \tag{6.3.11}$$

$$c_\pm(u, v, h) = \frac{H(\gamma(\xi + h \pm v \mp u))}{H(\gamma(\xi + h))} H(\gamma) \tag{6.3.12}$$

with $H(u)$ given by (6.3.1), γ denoting the crossing parameter, h the height variable, and ξ an arbitrary extra parameter which we shall hereafter fix to $\xi = 1$. The factors of $\pm i$ appearing in the $b_\pm$ weights constitute a trivial gauge transformation which does not affect the Yang-Baxter equation. We have introduced these factors to unify this model with the one discussed in the next section.

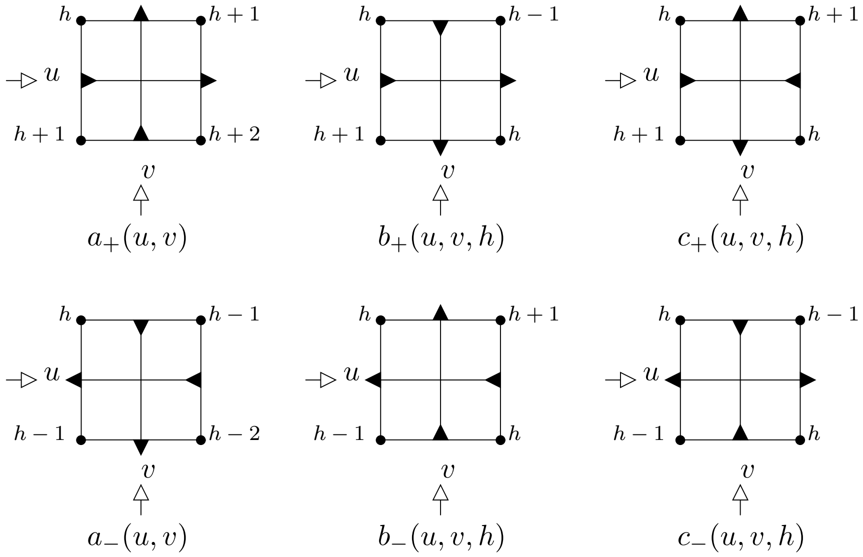

Figure 6.12: Weights of the SOS model. Each entry of the R-matrix (6.3.9) is paired with a *face.* The faces contain all the information normally present in a vertex, plus four surrounding points with prescribed dynamical variables. The dynamical variable in the top-left corner matches that of the R-matrix itself. The values at the remaining corners are obtained by circuiting the face clockwise from the top-left, adding 1 to the previous value each time a black arrow points outward, and subtracting 1 from the previous value each time a black arrow points inward.

Often, when the variables of a particular R-matrix are clear from context, we will abbreviate $R_{ab}(u,v,h) = R_{ab}(h)$. As usual, we have placed the subscript ab on the R-matrix to denote the fact that it is an element of $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b)$. Let

$$e_a^+ = \uparrow_a, \quad e_a^- = \downarrow_a \tag{6.3.13}$$

be the canonical basis vectors of the two dimensional vector space $\mathcal{V}_a$. The *dynamical* R-matrix $R_{ab}(h+\sigma_c^z)$ is an element of $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b \otimes \mathcal{V}_c)$, whose action is given by

$$R_{ab}(h+\sigma_c^z)(e_a^i \otimes e_b^j \otimes e_c^k) = R_{ab}(h+k)(e_a^i \otimes e_b^j) \otimes e_c^k \tag{6.3.14}$$

where σ_c^z denotes the third of the Pauli matrices (5.1.15) acting in $\mathcal{V}_c$, and $i,j,k \in \{+1,-1\}$ are fixed indices.

Lemma 6. Associate the variables u,v,w to the respective vector spaces $\mathcal{V}_a, \mathcal{V}_b, \mathcal{V}_c$ so that, for example, $R_{ab}(h)$ is understood to equal $R_{ab}(u,v,h)$. The *dynamical Yang-Baxter equation* is the identity

$$R_{ab}(h)R_{ac}(h+\sigma_b^z)R_{bc}(h) = R_{bc}(h+\sigma_a^z)R_{ac}(h)R_{ab}(h+\sigma_c^z) \tag{6.3.15}$$

which holds in $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b \otimes \mathcal{V}_c)$.[6]

Proof. We act with the equation (6.3.15) on the vector $e_a^{j_a} \otimes e_b^{j_b} \otimes e_c^{j_c}$, where $j_a, j_b, j_c \in \{+1, -1\}$ are fixed indices. We obtain

$$\begin{aligned} R_{ab}(h)R_{ac}(h+\sigma_b^z)R_{bc}(h)e_a^{j_a} \otimes e_b^{j_b} \otimes e_c^{j_c} &= R_{ab}(h)R_{ac}(h+\sigma_b^z)R_{k_c j_c}^{k_b j_b}(h)e_a^{j_a} \otimes e_b^{k_b} \otimes e_c^{k_c} \\ &= R_{ab}(h)R_{i_c k_c}^{k_a j_a}(h+k_b)R_{k_c j_c}^{k_b j_b}(h)e_a^{k_a} \otimes e_b^{k_b} \otimes e_c^{i_c} \\ &= R_{i_b k_b}^{i_a k_a}(h)R_{i_c k_c}^{k_a j_a}(h+k_b)R_{k_c j_c}^{k_b j_b}(h)e_a^{i_a} \otimes e_b^{i_b} \otimes e_c^{i_c} \end{aligned}$$

for the left hand side of (6.3.15), and

$$\begin{aligned} R_{bc}(h+\sigma_a^z)R_{ac}(h)R_{ab}(h+\sigma_c^z)e_a^{j_a} \otimes e_b^{j_b} \otimes e_c^{j_c} &= R_{bc}(h+\sigma_a^z)R_{ac}(h)R_{k_b j_b}^{k_a j_a}(h+j_c)e_a^{k_a} \otimes e_b^{k_b} \otimes e_c^{j_c} \\ &= R_{bc}(h+\sigma_a^z)R_{k_c j_c}^{i_a k_a}(h)R_{k_b j_b}^{k_a j_a}(h+j_c)e_a^{i_a} \otimes e_b^{k_b} \otimes e_c^{k_c} \\ &= R_{i_c k_c}^{i_b k_b}(h+i_a)R_{k_c j_c}^{i_a k_a}(h)R_{k_b j_b}^{k_a j_a}(h+j_c)e_a^{i_a} \otimes e_b^{i_b} \otimes e_c^{i_c} \end{aligned}$$

for the right hand side of (6.3.15), where in both cases all repeated indices are summed over $\{+1, -1\}$. Equating these two sides again, we have

$$R_{i_b k_b}^{i_a k_a}(h)R_{i_c k_c}^{k_a j_a}(h+k_b)R_{k_c j_c}^{k_b j_b}(h) = R_{i_c k_c}^{i_b k_b}(h+i_a)R_{k_c j_c}^{i_a k_a}(h)R_{k_b j_b}^{k_a j_a}(h+j_c) \tag{6.3.16}$$

for all fixed indices $\{i_a, i_b, i_c, j_a, j_b, j_c\} \in \{+1, -1\}$, where summation is implied over k_a, k_b, k_c. Equation (6.3.16) matches the Yang-Baxter equation listed in [22] and can be verified directly using the entries of the R-matrix (6.3.9). □

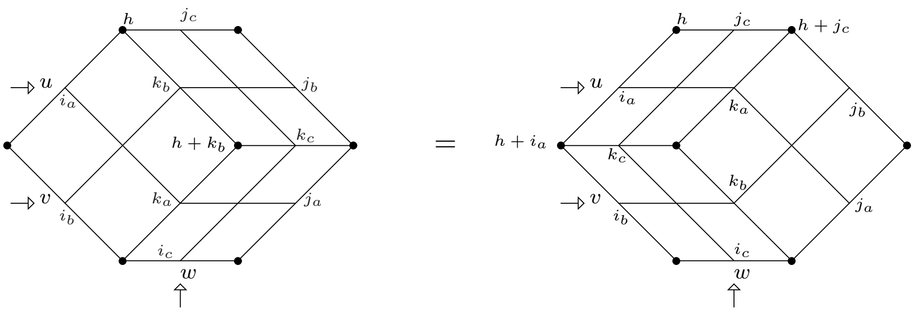

Figure 6.13: Yang-Baxter equation for the SOS model. The external indices $\{i_a, i_b, i_c, j_a, j_b, j_c\}$ are held fixed on both sides of the equation, while $\{k_a, k_b, k_c\}$ are summed over $\{+1, -1\}$. This figure is the graphical equivalent of (6.3.16), as can be seen by matching each face with its corresponding R-matrix entry.

[6] See [32].

Remark 1. Consider the case when the crossing parameter is set to $\gamma = \pi i/2$. From the form of the R-matrix weights (6.3.10)–(6.3.12), the fact that $[u] = H(\pi i u/2)$ and the first of the quasi-periodicity conditions (6.3.4), we see that

$$R_{ab}(u, v, h+1) = R_{ab}(u, v, h-1) \tag{6.3.17}$$

at this special value of γ. This means that the action of the dynamical R-matrix trivializes, because

$$R_{ab}(h + \sigma_c^z)(e_a^i \otimes e_b^j \otimes e_c^k) = R_{ab}(h+1)(e_a^i \otimes e_b^j) \otimes e_c^k \tag{6.3.18}$$

regardless of the value of $k \in \{+1, -1\}$. Therefore when $\gamma = \pi i/2$, the Yang-Baxter equation (6.3.15) becomes

$$R_{ab}(h) R_{ac}(h+1) R_{bc}(h) = R_{bc}(h+1) R_{ac}(h) R_{ab}(h+1) \tag{6.3.19}$$

which is the form more typical of a vertex model. In the next section we will consider a model whose R-matrix satisfies a more general version of (6.3.19).

6.3.3 Important commutation relation

Lemma 7. We have the commutation relation

$$R_{ab}(u_a, v_b, h + \sigma_c^z + \sigma_d^z) R_{cd}(u_c, v_d, h) = R_{cd}(u_c, v_d, h) R_{ab}(u_a, v_b, h + \sigma_c^z + \sigma_d^z) \tag{6.3.20}$$

in $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b \otimes \mathcal{V}_c \otimes \mathcal{V}_d)$.

Proof. We act with (6.3.20) on the vector $e_a^{j_a} \otimes e_b^{j_b} \otimes e_c^{j_c} \otimes e_d^{j_d}$, where $j_a, j_b, j_c, j_d \in \{+1, -1\}$ are fixed indices. We obtain

$$\begin{aligned} R_{ab}(h + \sigma_c^z + \sigma_d^z) R_{cd}(h) e_a^{j_a} \otimes e_b^{j_b} \otimes e_c^{j_c} \otimes e_d^{j_d} &= R_{ab}(h + \sigma_c^z + \sigma_d^z) R_{k_d j_d}^{k_c j_c}(h) e_a^{j_a} \otimes e_b^{j_b} \otimes e_c^{k_c} \otimes e_d^{k_d} \\ &= R_{k_b j_b}^{k_a j_a}(h + k_c + k_d) R_{k_d j_d}^{k_c j_c}(h) e_a^{k_a} \otimes e_b^{k_b} \otimes e_c^{k_c} \otimes e_d^{k_d} \end{aligned}$$

for the left hand side of (6.3.20), and

$$\begin{aligned} R_{cd}(h) R_{ab}(h + \sigma_c^z + \sigma_d^z) e_a^{j_a} \otimes e_b^{j_b} \otimes e_c^{j_c} \otimes e_d^{j_d} &= R_{cd}(h) R_{k_b j_b}^{k_a j_a}(h + j_c + j_d) e_a^{k_a} \otimes e_b^{k_b} \otimes e_c^{j_c} \otimes e_d^{j_d} \\ &= R_{k_d j_d}^{k_c j_c}(h) R_{k_b j_b}^{k_a j_a}(h + j_c + j_d) e_a^{k_a} \otimes e_b^{k_b} \otimes e_c^{k_c} \otimes e_d^{k_d} \end{aligned}$$

for the right hand side of (6.3.20), with all repeated indices summed over $\{+1, -1\}$. These two sides are equal since $R_{k_d j_d}^{k_c j_c}(h)$ is zero unless $k_c + k_d = j_c + j_d$. □

6.3.4 Dynamical monodromy matrix

The dynamical monodromy matrix is defined as

$$T_a(u,\{w\}_M,h) = R_{a1}(u,w_1,h)\dots R_{aM}\Big(u,w_M,h+\sum_{j=1}^{M-1}\sigma_j^z\Big) \tag{6.3.21}$$
$$= \prod_{m=1}^{M}{}^{\rightarrow} R_{am}\Big(u,w_m,h+\sum_{j=1}^{m-1}\sigma_j^z\Big)$$

with the multiplication taken in the space $\mathrm{End}(\mathcal{V}_a)$. We write the contribution from the space $\mathrm{End}(\mathcal{V}_a)$ explicitly, by defining

$$T_a(u,\{w\}_M,h) = \begin{pmatrix} A(u,\{w\}_M,h) & B(u,\{w\}_M,h) \\ C(u,\{w\}_M,h) & D(u,\{w\}_M,h) \end{pmatrix}_a \tag{6.3.22}$$

Lemma 8. By virtue of the dynamical Yang-Baxter equation (6.3.15), we obtain the intertwining equation

$$R_{ab}(u,v,h)T_a(u,\{w\}_M,h+\sigma_b^z)T_b(v,\{w\}_M,h) = \tag{6.3.23}$$
$$T_b(v,\{w\}_M,h+\sigma_a^z)T_a(u,\{w\}_M,h)R_{ab}\Big(u,v,h+\sum_{j=1}^{M}\sigma_j^z\Big)$$

Proof. Starting from the definition (6.3.21) of the dynamical monodromy matrix, we find

$$R_{ab}(u,v,h)T_a(u,\{w\}_M,h+\sigma_b^z)T_b(v,\{w\}_M,h) = \tag{6.3.24}$$
$$R_{ab}(u,v,h)\prod_{m=1}^{M}{}^{\rightarrow} R_{am}\Big(u,w_m,h+\sigma_b^z+\sum_{j=1}^{m-1}\sigma_j^z\Big)\prod_{n=1}^{M}{}^{\rightarrow} R_{bn}\Big(v,w_n,h+\sum_{j=1}^{n-1}\sigma_j^z\Big)$$

Using the commutation relation (6.3.20) we can change the order of the R-matrices appearing on the right hand side of (6.3.24), giving

$$R_{ab}(u,v,h)T_a(u,\{w\}_M,h+\sigma_b^z)T_b(v,\{w\}_M,h) = \tag{6.3.25}$$
$$R_{ab}(u,v,h)\prod_{m=1}^{M}{}^{\rightarrow}\left[R_{am}\Big(u,w_m,h+\sigma_b^z+\sum_{j=1}^{m-1}\sigma_j^z\Big)R_{bm}\Big(v,w_m,h+\sum_{j=1}^{m-1}\sigma_j^z\Big)\right]$$

Successively applying the dynamical Yang-Baxter equation (6.3.15) to the right hand side of (6.3.25), we obtain

$$R_{ab}(u,v,h)T_a(u,\{w\}_M,h+\sigma_b^z)T_b(v,\{w\}_M,h) = \tag{6.3.26}$$
$$\overrightarrow{\prod_{m=1}^{M}}\left[R_{bm}\Big(v,w_m,h+\sigma_a^z+\sum_{j=1}^{m-1}\sigma_j^z\Big)R_{am}\Big(u,w_m,h+\sum_{j=1}^{m-1}\sigma_j^z\Big)\right]R_{ab}\Big(u,v,h+\sum_{j=1}^{M}\sigma_j^z\Big)$$

Using the commutation relation (6.3.20) we then restore the previous ordering of R-matrices on the right hand side of (6.3.26), giving the result (6.3.23). □

Remark 2. By virtue of equation (6.3.18), in the limit $\gamma = \pi i/2$ the dynamical monodromy matrix (6.3.21) becomes

$$T_a(u,\{w\}_M,h) = R_{a1}(u,w_1,h)\dots R_{aM}(u,w_M,h+M-1) = \overrightarrow{\prod_{m=1}^{M}} R_{am}(u,w_m,h+m-1) \tag{6.3.27}$$

and satisfies the intertwining equation

$$\begin{aligned} &R_{ab}(u,v,h)T_a(u,\{w\}_M,h+1)T_b(v,\{w\}_M,h) = \\ &T_b(v,\{w\}_M,h+1)T_a(u,\{w\}_M,h)R_{ab}(u,v,h+M) \end{aligned} \tag{6.3.28}$$

In the next section, we will consider a model whose monodromy matrix is a generalization of (6.3.27), and satisfies a more general version of (6.3.28).

6.3.5 Domain wall partition function

The domain wall partition function of the SOS model is defined as

$$Z_N\Big(\{v\}_N,\{w\}_N,h\Big) = \langle \Downarrow_N | \overleftarrow{\prod_{j=1}^{N}} B(v_j,\{w\}_N,h+j-1) | \Uparrow_N\rangle \tag{6.3.29}$$

An explicit expression for (6.3.29) was obtained by H Rosengren in [77]. It was observed that $Z_N(\{v\}_N,\{w\}_N,h)$ does *not* admit a determinant representation for general values of the crossing parameter γ. However in the limit $\gamma = \pi i/2$, we have the following result.

Lemma 9. Setting $\gamma = \pi i/2$ the domain wall partition function (6.3.29) has the factorized form

$$Z_N\Big|_{\gamma=\pi i/2} = \frac{\Big[h+N+\sum_{j=1}^{N}(w_j-v_j)\Big]}{[h+N]}\langle 0\rangle^N \prod_{1\le j<k\le N}\langle v_j-v_k\rangle\langle w_k-w_j\rangle \tag{6.3.30}$$

This is the height model analogue of equation (6.1.6). The vertex model expression (6.1.6) can be recovered by taking the trigonometric $\mathsf{q} \to 0$ and heightless $h \to i\infty$ limits. We also remark that just as (6.1.6) is related to the factorization of a Cauchy-type determinant, the expression (6.3.30) seems to be related to the factorization of a Frobenius-type determinant [42].

Proof. We will defer the proof to the next section. There, we will calculate the domain wall partition function of a model which generalizes the $\gamma = \pi i/2$ limit of the SOS model. Specializing that result, we will recover (6.3.30) as a corollary. □

6.4 Elliptic Deguchi-Akutsu height model

In this section we study an elliptic extension of the height model discovered by Deguchi and Akutsu in [25]. The results which we present were originally obtained in [38].

6.4.1 R-matrix and Yang-Baxter equation

The R-matrix for the elliptic Deguchi-Akutsu height model is given by

$$R_{ab}(u,p,v,q,h) = \begin{pmatrix} a_+(u,p,v,q) & 0 & 0 & 0 \\ 0 & b_+(u,p,v,q,h) & c_+(u,p,v,q,h) & 0 \\ 0 & c_-(u,p,v,q,h) & b_-(u,p,v,q,h) & 0 \\ 0 & 0 & 0 & a_-(u,p,v,q) \end{pmatrix}_{ab} \tag{6.4.1}$$

where we have defined the functions

$$a_\pm(u,p,v,q) = [\pm(u-v)+p+q] \tag{6.4.2}$$

$$b_\pm(u,p,v,q,h) = \frac{[h]^{\frac{1}{2}}[h+2p+2q]^{\frac{1}{2}}}{[h+2p]^{\frac{1}{2}}[h+2q]^{\frac{1}{2}}}[u-v\pm(q-p)] \tag{6.4.3}$$

$$c_\pm(u,p,v,q,h) = \frac{[2p]^{\frac{1}{2}}[2q]^{\frac{1}{2}}}{[h+2p]^{\frac{1}{2}}[h+2q]^{\frac{1}{2}}}[\pm(v-u)+p+q+h] \tag{6.4.4}$$

with $[u]$ given by (6.3.3). We have placed the subscript ab on the R-matrix to denote the fact that it is an element of $\mathrm{End}(\mathcal{V}_a \otimes \mathcal{V}_b)$. When the variables of a particular R-matrix are clear from context, we will abbreviate $R_{ab}(u,p,v,q,h) = R_{ab}(h)$. The $\gamma = \pi i/2$ limit of the SOS R-matrix (6.3.9) is recovered by setting $p = q = \frac{1}{2}$ in (6.4.1).

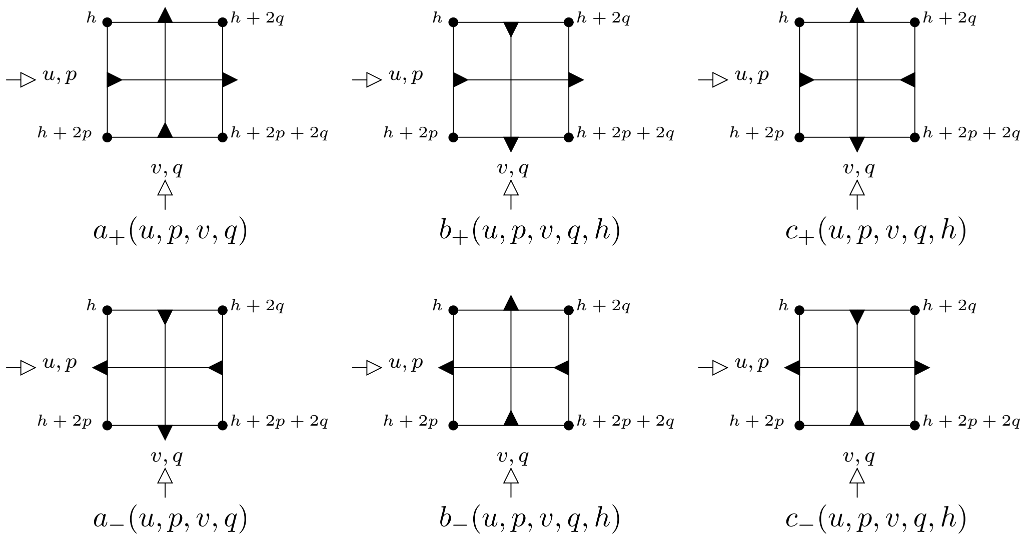

Figure 6.14: Weights of the elliptic Deguchi-Akutsu height model. Each entry of the R-matrix (6.4.1) is paired with a face. Similarly to the SOS model, the dynamical variable at the top-left corner matches that of the R-matrix. In contrast, the dynamical variables at the remaining corners are fixed and do not depend on the black arrows.

Lemma 10. Associate the variables $\{u,p\},\{v,q\},\{w,r\}$ to the respective vector spaces $\mathcal{V}_a,\mathcal{V}_b,\mathcal{V}_c$ so that, for example, $R_{ab}(h)$ is understood to equal $R_{ab}(u,p,v,q,h)$. The R-matrix (6.4.1) satisfies the Yang-Baxter equation

$$R_{ab}(h)R_{ac}(h+2q)R_{bc}(h) = R_{bc}(h+2p)R_{ac}(h)R_{ab}(h+2r) \tag{6.4.5}$$

which holds in $\mathrm{End}(\mathcal{V}_a\otimes\mathcal{V}_b\otimes\mathcal{V}_c)$. Setting $p=q=r=\frac{1}{2}$, we recover (6.3.19).

Proof. By direct computation. To check each component of (6.4.5) one needs various theta function identities, which may be found in chapter 15 of [5].

□

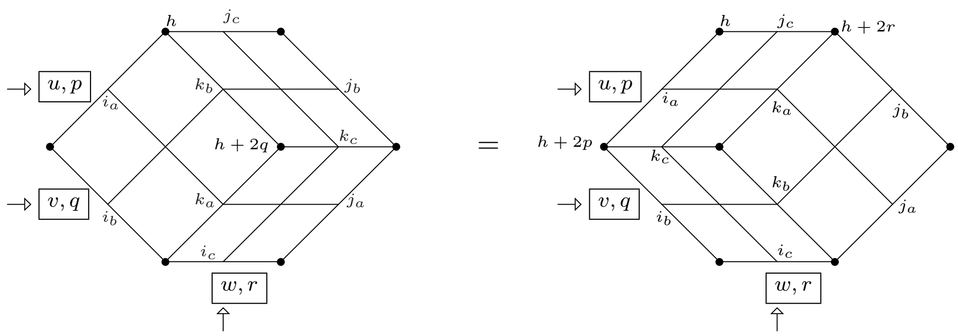

Figure 6.15: Yang-Baxter equation for the elliptic Deguchi-Akutsu height model.

6.4.2 Monodromy matrix and intertwining equation

For all integers $n \geq 1$, we define partial sums of the external field variables $\{p\}, \{q\}, \{r\}$ as follows

$$\overline{p}_n = \sum_{j=1}^{n} p_j, \quad \overline{q}_n = \sum_{j=1}^{n} q_j, \quad \overline{r}_n = \sum_{j=1}^{n} r_j \tag{6.4.6}$$

In addition, we fix $\overline{p}_0 = \overline{q}_0 = \overline{r}_0 = 0$. The monodromy matrix is an ordered product of R-matrices, given by

$$\begin{aligned} T_a(u,p,\{w,r\}_M,h) &= R_{a1}(u,p,w_1,r_1,h)\dots R_{aM}(u,p,w_M,r_M,h+2\overline{r}_{M-1}) \\ &= \overrightarrow{\prod_{m=1}^{M}} R_{am}(u,p,w_m,r_m,h+2\overline{r}_{m-1}) \end{aligned} \tag{6.4.7}$$

with the multiplication taken in the space $\mathrm{End}(\mathcal{V}_a)$. We write the contribution from the space $\mathrm{End}(\mathcal{V}_a)$ explicitly, by defining

$$T_a(u,p,\{w,r\}_M,h) = \left(\begin{array}{cc} A(u,p,\{w,r\}_M,h) & B(u,p,\{w,r\}_M,h) \\ C(u,p,\{w,r\}_M,h) & D(u,p,\{w,r\}_M,h) \end{array} \right)_a \tag{6.4.8}$$

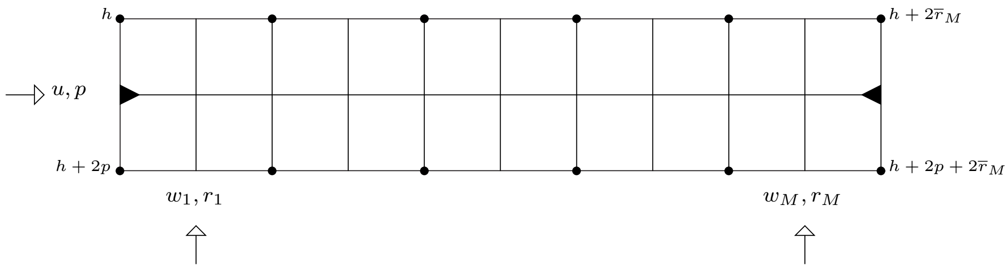

Figure 6.16: Graphical representation of the operator $B(u,p,\{w,r\}_M,h)$. This should be viewed as a string of M conjoined faces, each representing an entry of the R-matrix (6.4.1). The external horizontal black arrows are frozen to values which select the $(+1,-1)$ component of the matrix (6.4.8), while the internal horizontal black arrows are summed over all configurations.

Lemma 11. By virtue of the Yang-Baxter equation (6.4.5), we obtain the intertwining equation

$$\begin{aligned} &R_{ab}(u,p,v,q,h)T_a(u,p,\{w,r\}_M,h+2q)T_b(v,q,\{w,r\}_M,h) = \\ &T_b(v,q,\{w,r\}_M,h+2p)T_a(u,p,\{w,r\}_M,h)R_{ab}(u,p,v,q,h+2\overline{r}_M) \end{aligned} \tag{6.4.9}$$

Setting $p = q = \frac{1}{2}$ and $r_j = \frac{1}{2}$ for all $1 \leq j \leq M$, we recover the equation (6.3.28).

Proof. Starting from the definition (6.4.7) of the monodromy matrix, we find

$$R_{ab}(u,p,v,q,h)T_a(u,p,\{w,r\}_M,h+2q)T_b(v,q,\{w,r\}_M,h) = \qquad (6.4.10)$$
$$R_{ab}(h)\overrightarrow{\prod_{m=1}^{M}} R_{am}(h+2q+2\overline{r}_{m-1})\overrightarrow{\prod_{n=1}^{M}} R_{bn}(h+2\overline{r}_{n-1})$$

We change the order of the R-matrices on the right hand side of (6.4.10), by commuting those which act in different spaces to obtain

$$R_{ab}(u,p,v,q,h)T_a(u,p,\{w,r\}_M,h+2q)T_b(v,q,\{w,r\}_M,h) = \qquad (6.4.11)$$
$$R_{ab}(h)\overrightarrow{\prod_{m=1}^{M}}\Big(R_{am}(h+2q+2\overline{r}_{m-1})R_{bm}(h+2\overline{r}_{m-1})\Big)$$

Successively applying the Yang-Baxter equation (6.4.5) to the right hand side of (6.4.11), we have

$$R_{ab}(u,p,v,q,h)T_a(u,p,\{w,r\}_M,h+2q)T_b(v,q,\{w,r\}_M,h) = \qquad (6.4.12)$$
$$\overrightarrow{\prod_{m=1}^{M}}\Big(R_{bm}(h+2p+2\overline{r}_{m-1})R_{am}(h+2\overline{r}_{m-1})\Big)R_{ab}(h+2\overline{r}_M)$$

Restoring the previous ordering to the R-matrices on the right hand side of (6.4.12), we recover the result (6.4.9).

□

As always, (6.4.9) generates sixteen commutation relations amongst the entries of the monodromy matrix (6.4.8). For our purposes, the most important of these is

$$[u-v+p+q]B(u,p,\{w,r\}_M,h+2q)B(v,q,\{w,r\}_M,h) = \qquad (6.4.13)$$
$$[v-u+p+q]B(v,q,\{w,r\}_M,h+2p)B(u,p,\{w,r\}_M,h)$$

and is graphically depicted by the diagrams below.

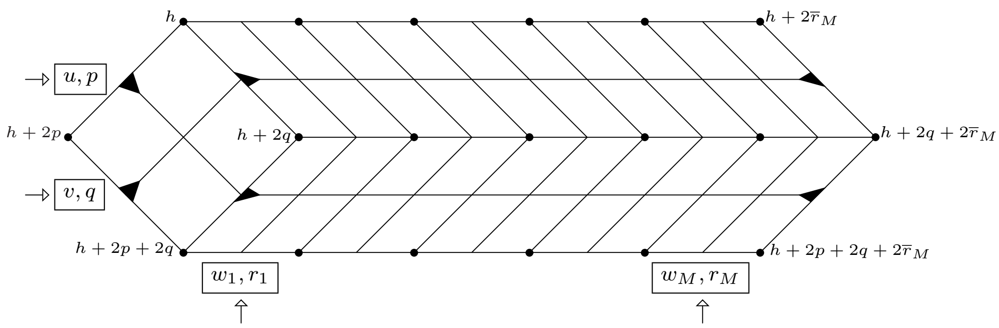

Figure 6.17: Product of two B-operators before commutation.

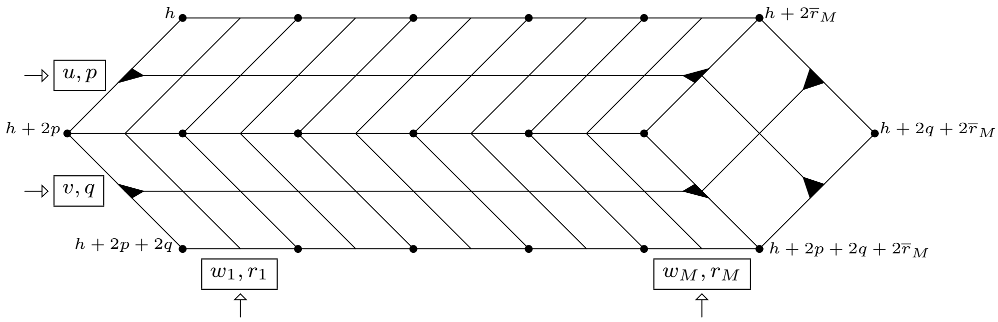

Figure 6.18: Product of two B-operators after commutation.

Similarly to equation (6.2.9) in the context of the trigonometric Felderhof model, (6.4.13) plays a crucial role in the evaluation of the domain wall partition function, which we define in the next subsection.

6.4.3 Domain wall partition function

The domain wall partition function of the elliptic Deguchi-Akutsu height model is defined as

$$Z_N\Big(\{v,q\}_N, \{w,r\}_N, h\Big) = \langle \Downarrow_N | \overleftarrow{\prod_{j=1}^{N}} B\Big(v_j, q_j, \{w,r\}_N, h + 2\overline{q}_{j-1}\Big) | \Uparrow_N \rangle \quad (6.4.14)$$

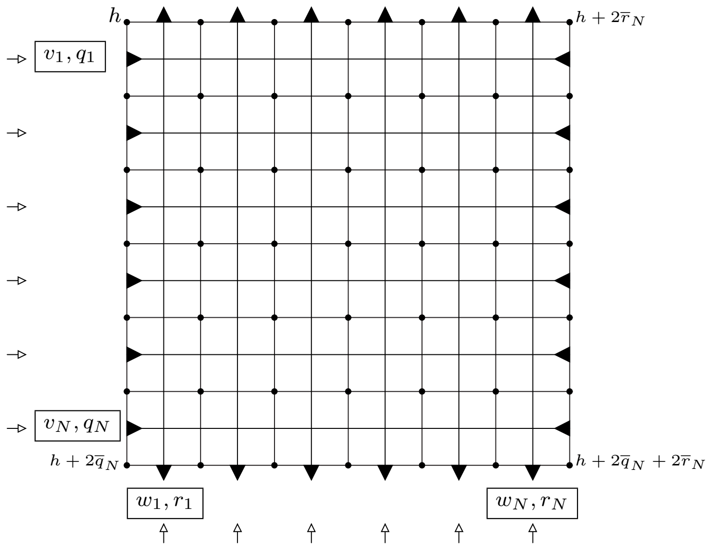

Figure 6.19: Domain wall partition function of the elliptic Deguchi-Akutsu model. The top row of arrows corresponds with the state vector $| \Uparrow_N \rangle$. The bottom row of arrows corresponds with the dual state vector $\langle \Downarrow_N |$. Each horizontal lattice line represents multiplication by a $B(v_j, q_j, \{w,r\}_N, h + 2\overline{q}_{j-1})$ operator. The ordering of lattice lines agrees with the ordering of B-operators in (6.4.14).

6.4.4 Conditions on $Z_N(\{v,q\}_N,\{w,r\}_N,h)$

We provide a set of Korepin-type conditions on $Z_N(\{v,q\}_N,\{w,r\}_N,h)$, which were originally obtained in [38].

Lemma 12. As always, we make the abbreviation $Z_N = Z_N(\{v,q\}_N,\{w,r\}_N,h)$. For all $N \geq 2$ we claim that

1. Z_N is an entire function in v_N that satisfies the quasi-periodicity conditions

$$Z_N\Big|_{v_N \rightarrow v_N+2} = (-)^N Z_N \tag{6.4.15}$$

$$Z_N\Big|_{v_N \rightarrow v_N - 2i\log(\mathsf{q})/\pi} = \frac{(-)^N}{\mathsf{q}^N} \exp\Big(\pi i(\eta - N v_N)\Big) Z_N \tag{6.4.16}$$

where we have defined

$$\eta = h + 2\sum_{j=1}^{N-1} q_j + N q_N + \sum_{j=1}^{N} (w_j + r_j) \tag{6.4.17}$$

2. Z_N has simple zeros at the points $v_N = v_j + q_j + q_N$, for all $1 \leq j \leq N-1$.

3. Setting $v_N = w_N + q_N + r_N$, Z_N satisfies the recursion relation

$$\begin{aligned} Z_N\Big|_{v_N = w_N+q_N+r_N} &= \frac{[h+2\overline{q}_{N-1}]^{\frac{1}{2}}[h+2\overline{r}_{N-1}]^{\frac{1}{2}}}{[h+2\overline{q}_N]^{\frac{1}{2}}[h+2\overline{r}_N]^{\frac{1}{2}}}[2q_N]^{\frac{1}{2}}[2r_N]^{\frac{1}{2}} \\ &\times \prod_{j=1}^{N-1}[w_N - w_j + r_j + r_N][v_j - w_N + q_j - r_N] Z_{N-1} \end{aligned} \tag{6.4.18}$$

where Z_{N-1} is the domain wall partition function on a square lattice of size $N-1$.

In addition, we have the supplementary condition

4. The partition function on the 1×1 lattice is given by

$$Z_1(v_1,q_1,w_1,r_1,h) = \frac{[2q_1]^{\frac{1}{2}}[2r_1]^{\frac{1}{2}}}{[h+2q_1]^{\frac{1}{2}}[h+2r_1]^{\frac{1}{2}}}[w_1 - v_1 + q_1 + r_1 + h] \tag{6.4.19}$$

Proof.

1. Inserting the set of states $\sum_{n=1}^{N} \sigma_n^{+}|\Downarrow_N\rangle\langle\Downarrow_N|\sigma_n^{-}$ after the first B-operator appearing in (6.4.14), the domain wall partition function may be written in the form

$$Z_N\Big(\{v,q\}_N,\{w,r\}_N,h\Big) = \sum_{n=1}^{N} \langle \Downarrow_N |B\Big(v_N,q_N,\{w,r\}_N,h+2\overline{q}_{N-1}\Big)\sigma_n^+| \Downarrow_N\rangle$$
$$\times \langle \Downarrow_N |\sigma_n^- \prod_{j=1}^{N-1} B\Big(v_j,q_j,\{w,r\}_N,h+2\overline{q}_{j-1}\Big)| \Uparrow_N\rangle \quad (6.4.20)$$

in which all dependence on v_N appears in the first factor within the sum. We therefore proceed to calculate $\langle \Downarrow_N |B(v_N,q_N,\{w,r\}_N,h+2\overline{q}_{N-1})\sigma_n^+| \Downarrow_N\rangle$ for all $1 \le n \le N$, as shown below.

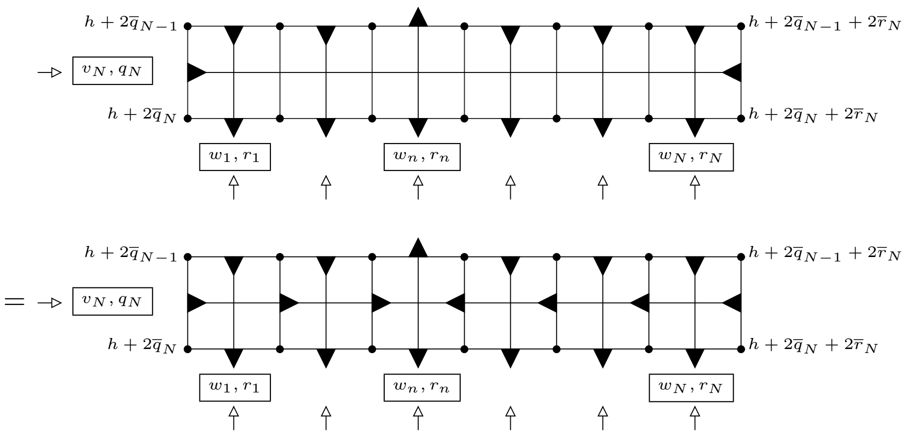

Figure 6.20: Peeling away the bottom row of the elliptic Deguchi-Akutsu partition function. The top diagram represents $\langle \Downarrow_N |B(v_N,q_N,\{w,r\}_N,h+2\overline{q}_{N-1})\sigma_n^+| \Downarrow_N\rangle$, with the internal black arrows being summed over all configurations. The lower diagram represents the only surviving configuration.

Replacing each face in figure 6.20 with its corresponding elliptic weight, we thus obtain

$$\langle \Downarrow_N |B\Big(v_N,q_N,\{w,r\}_N,h+2\overline{q}_{N-1}\Big)\sigma_n^+| \Downarrow_N\rangle = c_+(v_N,q_N,w_n,r_n,h+2\overline{q}_{N-1}+2\overline{r}_{n-1})\times$$
$$\prod_{1\le j<n} b_+(v_N,q_N,w_j,r_j,h+2\overline{q}_{N-1}+2\overline{r}_{j-1}) \prod_{n<j\le N} a_-(v_N,q_N,w_j,r_j) \quad (6.4.21)$$

Recalling the explicit form of these functions, as given by equations (6.4.2)–(6.4.4), it is clear that $\langle \Downarrow_N |B(v_N,q_N,\{w,r\}_N,h+2\overline{q}_{N-1})\sigma_n^+| \Downarrow_N\rangle$ is entire in v_N. Furthermore, we immediately see that

$$\langle \Downarrow_N |B\Big(v_N+2, q_N, \{w,r\}_N, h+2\overline{q}_{N-1}\Big)\sigma_n^+| \Downarrow_N\rangle = \\ (-)^N \langle \Downarrow_N |B\Big(v_N, q_N, \{w,r\}_N, h+2\overline{q}_{N-1}\Big)\sigma_n^+| \Downarrow_N\rangle \tag{6.4.22}$$

which proves that the first quasi-periodicity condition (6.4.15) holds. The proof of the second quasi-periodicity condition (6.4.16) is established using the identities

$$b_+(v_N - 2i\log(\mathsf{q})/\pi, q_N, w_j, r_j, h+2\overline{q}_{N-1}+2\overline{r}_{j-1}) = \\ -\frac{1}{\mathsf{q}}\exp\Big(\pi i(w_j - r_j + q_N - v_N)\Big) b_+(v_N, q_N, w_j, r_j, h+2\overline{q}_{N-1}+2\overline{r}_{j-1}) \tag{6.4.23}$$

$$c_+(v_N - 2i\log(\mathsf{q})/\pi, q_N, w_n, r_n, h+2\overline{q}_{N-1}+2\overline{r}_{n-1}) = \\ -\frac{1}{\mathsf{q}}\exp\Big(\pi i(w_n + q_N + r_n + h + 2\overline{q}_{N-1}+2\overline{r}_{n-1} - v_N)\Big) c_+(v_N, q_N, w_n, r_n, h+2\overline{q}_{N-1}+2\overline{r}_{n-1}) \tag{6.4.24}$$

$$a_-(v_N - 2i\log(\mathsf{q})/\pi, q_N, w_j, r_j) = -\frac{1}{\mathsf{q}}\exp\Big(\pi i(w_j + r_j + q_N - v_N)\Big) a_-(v_N, q_N, w_j, r_j) \tag{6.4.25}$$

which, when substituted into (6.4.21), produce the equation

$$\langle \Downarrow_N |B\Big(v_N - 2i\log(q)/\pi, q_N, \{w,r\}_N, h+2\overline{q}_{N-1}\Big)\sigma_n^+| \Downarrow_N\rangle = \\ \frac{(-)^N}{\mathsf{q}^N}\exp\Big(\pi i(\eta - N v_N)\Big)\langle \Downarrow_N |B\Big(v_N, q_N, \{w,r\}_N, h+2\overline{q}_{N-1}\Big)\sigma_n^+| \Downarrow_N\rangle \tag{6.4.26}$$

where η is given by (6.4.17). Hence the second quasi-periodicity condition (6.4.16) also holds.

2. We rearrange the expression (6.4.14) for the partition function by using the commutation relation

$$[v_N - v_j + q_j + q_N] B\Big(v_N, q_N, \{w,r\}_N, h+2\overline{q}_j\Big) B\Big(v_j, q_j, \{w,r\}_N, h+2\overline{q}_{j-1}\Big) = \\ [v_j - v_N + q_j + q_N] B\Big(v_j, q_j, \{w,r\}_N, h+2\overline{q}_{j-1}+2q_N\Big) B\Big(v_N, q_N, \{w,r\}_N, h+2\overline{q}_{j-1}\Big) \tag{6.4.27}$$

repeatedly to change the order of the B-operators. This effectively amounts to using figures 6.17 and 6.18 to reorder the horizontal lattice lines in figure 6.19. We obtain

$$\prod_{j=1}^{N-1} [v_N - v_j + q_j + q_N] Z_N\Big(\{v,q\}_N, \{w,r\}_N, h\Big) = \tag{6.4.28}$$
$$\prod_{j=1}^{N-1} [v_j - v_N + q_j + q_N] \langle \Downarrow_N | \overleftarrow{\prod_{j=1}^{N-1}} B\Big(v_j, q_j, \{w,r\}_N, h + 2\overline{q}_{j-1} + 2q_N\Big) B\Big(v_N, q_N, \{w,r\}_N, h\Big) | \Uparrow_N \rangle$$

The right hand side of (6.4.28) is an entire function in v_N, with simple zeros at the points $v_N = v_j + q_j + q_N$ for all $1 \leq j \leq N-1$. Therefore the partition function $Z_N(\{v,q\}_N, \{w,r\}_N, h)$ must have simple zeros at the same points.

3. We start from the expansion (6.4.20) of the domain wall partition function and set $v_N = w_N + q_N + r_N$. This causes all terms in the summation over $1 \leq n \leq N$ to collapse to zero except the $n = N$ term, giving

$$\begin{aligned} Z_N\Big|_{v_N = w_N + q_N + r_N} &= c_+(w_N + q_N + r_N, q_N, w_N, r_N, h + 2\overline{q}_{N-1} + 2\overline{r}_{N-1}) \\ &\times \prod_{j=1}^{N-1} b_+(w_N + q_N + r_N, q_N, w_j, r_j, h + 2\overline{q}_{N-1} + 2\overline{r}_{j-1}) \\ &\times \langle \Downarrow_N | \sigma_N^- \overleftarrow{\prod_{j=1}^{N-1}} B\Big(v_j, q_j, \{w,r\}_N, h + 2\overline{q}_{j-1}\Big) | \Uparrow_N \rangle \end{aligned} \tag{6.4.29}$$

We then calculate $\langle \Downarrow_N | \sigma_N^- \overleftarrow{\prod}_{j=1}^{N-1} B(v_j, q_j, \{w,r\}_N, h + 2\overline{q}_{j-1}) | \Uparrow_N \rangle$ using the diagram shown below.

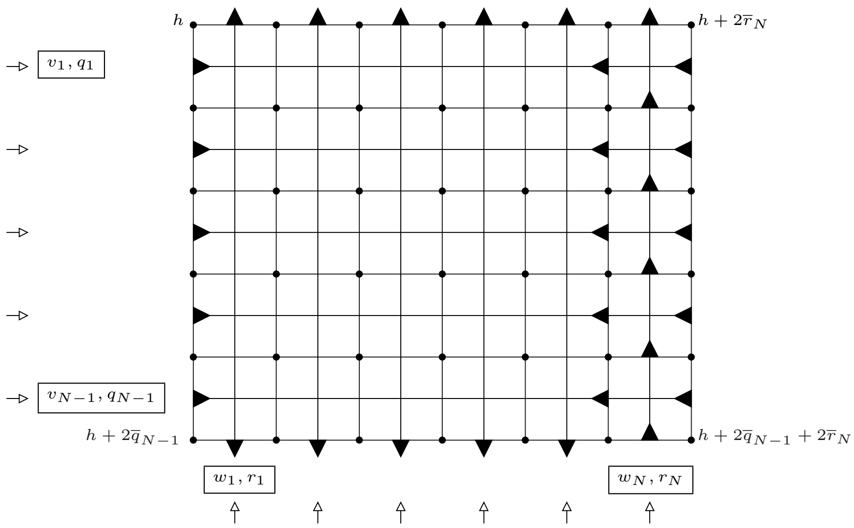

Figure 6.21: Peeling the right-most column of the elliptic Deguchi-Akutsu partition function. This diagram represents $\langle \Downarrow_N | \sigma_N^- \overleftarrow{\prod}_{j=1}^{N-1} B(v_j, q_j, \{w,r\}_N, h+2\overline{q}_{j-1}) | \Uparrow_N \rangle$, with the internal black arrows being summed over all configurations. Each surviving configuration must contain the column of faces shown on the right of the lattice.

Figure 6.21 represents the $(N-1)\times(N-1)$ domain wall partition function, multiplied by a column of faces. Replacing these faces with their elliptic weights, we obtain

$$\langle \Downarrow_N | \sigma_N^- \overleftarrow{\prod_{j=1}^{N-1}} B\Big(v_j, q_j, \{w,r\}_N, h+2\overline{q}_{j-1}\Big) | \Uparrow_N \rangle = \tag{6.4.30}$$
$$\prod_{j=1}^{N-1} b_-(v_j, q_j, w_N, r_N, h+2\overline{q}_{j-1}+2\overline{r}_{N-1}) Z_{N-1}\Big(\{v,q\}_{N-1}, \{w,r\}_{N-1}, h\Big)$$

Finally, substituting (6.4.30) into (6.4.29) gives

$$\begin{aligned} Z_N\Big|_{v_N=w_N+q_N+r_N} &= c_+(w_N+q_N+r_N, q_N, w_N, r_N, h+2\overline{q}_{N-1}+2\overline{r}_{N-1}) \\ &\times \prod_{j=1}^{N-1} b_+(w_N+q_N+r_N, q_N, w_j, r_j, h+2\overline{q}_{N-1}+2\overline{r}_{j-1}) \\ &\times \prod_{j=1}^{N-1} b_-(v_j, q_j, w_N, r_N, h+2\overline{q}_{j-1}+2\overline{r}_{N-1}) Z_{N-1} \end{aligned} \tag{6.4.31}$$

Using the explicit formulae (6.4.3) and (6.4.4) for the functions in (6.4.31), it is straightforward to recover the recursion relation (6.4.18).

4. Specializing the definition (6.4.14) to the case $N=1$ gives

$$\begin{aligned} Z_1(v_1, q_1, w_1, r_1, h) &= \langle \Downarrow_1 | B(v_1, q_1, \{w,r\}_1, h) | \Uparrow_1 \rangle \\ &= \uparrow_{a_1}^* \otimes \downarrow_1^* R_{a_1 1}(v_1, q_1, w_1, r_1, h) \uparrow_1 \otimes \downarrow_{a_1} = c_+(v_1, q_1, w_1, r_1, h) \end{aligned} \tag{6.4.32}$$

as required. Alternatively, the 1×1 partition function is the top-right face in figure 6.14, whose weight is equal to $c_+(v_1, q_1, w_1, r_1, h)$.

□

6.4.5 Evaluation of $Z_N(\{v,q\}_N, \{w,r\}_N, h)$

The conditions **1**–**4** determine $Z_N(\{v,q\}_N, \{w,r\}_N, h)$ uniquely and give a direct algorithm for its evaluation.

Lemma 13. The domain wall partition function has the factorized expression

$$Z_N\Big(\{v,q\}_N, \{w,r\}_N, h\Big) = \frac{[h+\overline{w}_N-\overline{v}_N+\overline{q}_N+\overline{r}_N]}{[h+2\overline{q}_N]^{\frac{1}{2}}[h+2\overline{r}_N]^{\frac{1}{2}}} \times \tag{6.4.33}$$
$$\prod_{j=1}^{N} [2q_j]^{\frac{1}{2}} [2r_j]^{\frac{1}{2}} \prod_{1\le j<k\le N} [v_j-v_k+q_j+q_k][w_k-w_j+r_j+r_k]$$

This expression was originally proved in [38]. Setting $q_j = r_j = \frac{1}{2}$ for all $1 \le j \le N$, we recover the formula (6.3.30) for the partition function of the SOS model at its free fermion point. In addition, taking the trigonometric $\mathtt{q} \to 0$ and heightless $h \to i\infty$ limits of (6.4.33), we recover the partition function of the trigonometric Felderhof model (6.2.21).

Proof. From theorem 1 and conditions **1** and **2** on $Z_N(\{v,q\}_N, \{w,r\}_N, h)$, we know that it must have the form

$$Z_N\Big(\{v,q\}_N, \{w,r\}_N, h\Big) = \mathcal{C}\Big(\{v\}_{N-1}, \{q\}_N, \{w,r\}_N, h\Big) \times \prod_{j=1}^{N-1} [v_j - v_N + q_j + q_N][h + \overline{w}_N - \overline{v}_N + \overline{q}_N + \overline{r}_N] \tag{6.4.34}$$

where $\mathcal{C}$ does not depend on v_N, but depends on all other variables. Evaluating (6.4.34) at $v_N = w_N + q_N + r_N$, we obtain

$$Z_N\Big|_{v_N = w_N + q_N + r_N} = \mathcal{C}\Big(\{v\}_{N-1}, \{q\}_N, \{w,r\}_N, h\Big) \times \prod_{j=1}^{N-1} [v_j - w_N + q_j - r_N][h + \overline{w}_{N-1} - \overline{v}_{N-1} + \overline{q}_{N-1} + \overline{r}_{N-1}] \tag{6.4.35}$$

Comparing (6.4.35) with condition **3** on $Z_N(\{v,q\}_N, \{w,r\}_N, h)$, we arrive at the expression

$$\mathcal{C}\Big(\{v\}_{N-1}, \{q\}_N, \{w,r\}_N, h\Big) = \frac{[h + 2\overline{q}_{N-1}]^{\frac{1}{2}} [h + 2\overline{r}_{N-1}]^{\frac{1}{2}}}{[h + 2\overline{q}_N]^{\frac{1}{2}} [h + 2\overline{r}_N]^{\frac{1}{2}}} \times \frac{[2q_N]^{\frac{1}{2}} [2r_N]^{\frac{1}{2}} \prod_{j=1}^{N-1} [w_N - w_j + r_j + r_N]}{[h + \overline{w}_{N-1} - \overline{v}_{N-1} + \overline{q}_{N-1} + \overline{r}_{N-1}]} Z_{N-1}\Big(\{v,q\}_{N-1}, \{w,r\}_{N-1}, h\Big) \tag{6.4.36}$$

Finally, substituting this expression for $\mathcal{C}$ into (6.4.34), we obtain the recurrence

$$Z_N\Big(\{v,q\}_N, \{w,r\}_N, h\Big) = \frac{[h + 2\overline{q}_{N-1}]^{\frac{1}{2}} [h + 2\overline{r}_{N-1}]^{\frac{1}{2}}}{[h + 2\overline{q}_N]^{\frac{1}{2}} [h + 2\overline{r}_N]^{\frac{1}{2}}} \frac{[h + \overline{w}_N - \overline{v}_N + \overline{q}_N + \overline{r}_N]}{[h + \overline{w}_{N-1} - \overline{v}_{N-1} + \overline{q}_{N-1} + \overline{r}_{N-1}]} [2q_N]^{\frac{1}{2}} [2r_N]^{\frac{1}{2}} \times \prod_{j=1}^{N-1} [v_j - v_N + q_j + q_N][w_N - w_j + r_j + r_N] Z_{N-1}\Big(\{v,q\}_{N-1}, \{w,r\}_{N-1}, h\Big) \tag{6.4.37}$$

whose basis is given by condition **4**. This recurrence is trivially solved to produce the formula (6.4.33).

□

6.5 Conclusion

In this chapter we have investigated the free fermion condition in lattice models. The starting point in our studies was the free fermion point of the six-vertex model, whose partition function and Bethe scalar product both factorize into product form. The main result of the chapter is that this factorization persists when non-trivial external fields are introduced into the model. Indeed, we found that the partition function and Bethe scalar product both factorize when considering the trigonometric Felderhof model, which is an external field deformation of the free fermionic six-vertex model.

Another key result was the observation that these ideas can be extended to models with a height parameter. We claimed that at its free fermion point, the SOS model has a factorized domain wall partition function. This result indicates that the free fermion point is a powerful restriction, since for general values of the crossing parameter the partition function has a relatively complicated non-determinant expression [77]. Furthermore, we showed that this factorization persists at the level of the elliptic Deguchi-Akutsu height model, which contains external fields.

There is potential for further work in the context of these models. We list some of these problems below.

1. It should be possible to extend the results obtained for the trigonometric Felderhof model to the calculation of its one and two-point correlation functions. In particular, drawing on the work of [59], one could express the local spin operators $\sigma^{\pm}_m, \sigma^z_m$ in terms of the monodromy matrix operators. At the very least, the calculation of the one-point functions is then facilitated by our expression for the Bethe scalar product.

2. The situation is more complicated for the elliptic Deguchi-Akutsu height model, and the evaluation of its scalar product remains unsolved. The difficulty essentially arises from the fact that the $c_{\pm}$ faces depend on the rapidities, unlike the $c_{\pm}$ vertices in the trigonometric Felderhof model. This leads to an extra zero in the height model scalar product, compared with its vertex model counterpart.

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