Misconceptions about density of decimals: Insights from pre-service teachers' work

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Abstract

Extensive studies have documented various difficulties with, and misconceptions about, decimal numeration across different levels of education. This paper reports on pre-service teachers' misconceptions about the density of decimals. Written test data from 140 pre-service teachers, observation of group and classroom discussions provided evidence of pre-service teachers' difficulties in grasping the density notion of decimals. Incorrect analogies resulting from over generalization of knowledge about whole numbers and fractions were identified. Teaching ideas to resolve these difficulties are discussed. Evidence from this research indicates that it is possible to remove misconceptions about density of decimals.

KEYWORDS: decimals, pre-service teachers, misconceptions, density.

A. INTRODUCTION

Difficulties about the teaching and learning of decimal numeration, including various misconceptions about decimals across different levels of education, have been well established (Brousseau, 1997; Brousseau, Brousseau, & Warfield, 2004, 2007; 1995; Stacey, 2005; Steinle, 2004). Misconceptions and difficulties with decimal numeration have also been observed in samples of pre-service teachers (Putt, 1995; Stacey et al., 2001; Thipkong & Davis, 1991). These studies indicated that some pre-service teachers shared misconceptions apparent in younger students, while Steinle & Stacey (2004) found that certain ways of thinking that are commonly observed in younger students, are infrequent in older students. They concluded that the cumulative effect of instruction of many years is that some misconceptions are *covered over*, instead of *overcome*. The fact that

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pre-service teachers' misconceptions might be passed on to their future students provides an impetus for understanding and resolving these difficulties.

Extensive studies reported that existing knowledge about whole numbers was often utilized to interpret decimals (Hiebert, 1992; Moskal & Magone, 2000; Vamvakoussi & Vosniadou, 2004). The overgeneralization of the discrete nature of whole numbers has been identified as one source of difficulty in grasping the continuity properties of decimals. The continuity properties of real numbers (and hence of decimals) and the completeness of the real number line are manifested in several different ways. For example given any decimal, there is another arbitrarily close to it and all monotonically increasing sequences of decimals that are bounded above have a limit. In this paper, we are concerned only with one relatively simple version of the continuity properties, which we call the density property: between any two decimals, there are infinitely many other decimals. This paper will report on a study examining Indonesian pre-service teachers' knowledge about density of decimals. The data reported in this paper is a small part of the larger study to improve Indonesian pre-service teachers' content and pedagogical content knowledge on decimals.

B. LITERATURE REVIEW

One of the features distinguishing decimals from whole numbers is the density of decimals. Empirical studies examining understanding of density involving children and adults documented extensive difficulties in grasping this feature of decimals. Hiebert *et al*., (1991) found improving the continuity aspects of decimals' density was particularly difficult. Working with problems involving continuous models in the written tests and the interviews, such as marking a representation of a decimal number on a number line, or finding a number in between two given decimals such as 0.3 and 0.4 were found to be more challenging than working with discrete-representation task utilizing MAB models. Analysis for this finding suggested that an extra step in finding the unit of the continuous models explained the lower performance on continuous-representation tasks.

Likewise, Merenlouto (2003) found that only a small portion of Finnish students aged 16-17 years old in her study changed their concept of density. She attributed difficulties with grasping density to students' reference to natural numbers and difficulties in extending their frame of reference to rational or real numbers. Furthermore, she contended that this kind of explanation was based on an abstraction from natural numbers properties rather than a radical conceptual change from natural to real numbers. Recent works of Merenlouto & Lehtinen (2004) and Vamvakoussi & Vosniadou (2004) used conceptual change perspective as an instructional strategy in overcoming students' difficulties to grasp the density property.

Difficulties with density were also evident in studies involving pre-service teachers. Menon (2004) found only 59% of 142 pre-service teachers recognized the density of decimals. A similar trend was noted by Tsao (2005) who found that of 12 pre-service teachers involved in her study, only the six high ability students demonstrated an understanding of density.

The nature of incorrect responses with regard to the density is reflected in common misconceptions drawing on analogies between decimals and whole numbers. In general, incorrect answers could be classified in two categories. Firstly, some students believe that there are no decimals between given pairs of decimals. Fuglestad (1996) found that most students in her study of Norwegian students claimed there were no decimals in between two given decimals such as between 3.9 and 4 or between 0.63 and 0.64. Similarly, Bana, Farrell, and McIntosh (1997) reported that the majority of 12 year olds and 14 year olds from Australia, US, Taiwan and Sweden displayed the same problem. Only 62% of 14 year olds from Australia and 78% of 14 year olds from Taiwan showed understanding of decimal density. This evidence reflected incorrect extension of whole number knowledge that there is no whole number in between two consecutive whole numbers such as 63 and 64.

The second category of incorrect answer translates knowledge of multiplicative relations between subsequent decimal fractions. For instance, Hart (1981) reported that 22 to 39% students age 12 to 15 year-old thought there were

8, 9, or 10 decimals in between 0.41 and 0.42. Similarly, Tsao (2005) observed the same phenomenon in her study with pre-service teachers. She found that three pre-service teachers from a low ability group believed there were nine decimals in between 1.42 and 1.43 by sequencing only the thousandths: 1.421, 1.422,…, and 1.429.

C. METHODOLOGY

Participants

This paper reported on a small portion of cycle 2 research data which was collected between August to November 2006. The whole study comprised of 2 cycles and adopted design research methodology (Brown, 1992; Gravemeijer, 1994), which cycles through design, teaching experiment, and retrospective analysis phases. Pre-service teachers attending Sanata Dharma University in Yogyakarta from two cohorts participated in cycle 2 study. Out of 140 pre-service teachers, 94 were enrolled in a two-year diploma program run by the elementary teacher training department and the remaining 46 were enrolled in a bachelor of education (secondary) run by science and mathematics education department.

Research instruments

The data reported in this paper came from two pairs of parallel test items on the pre-test and post-test, and from searching episodes of group and classroom discussions pertinent to density of decimals. The pre-test and post-test items are given in Figure 1 below and translated from the original Indonesian version.

Figure 1: Pre-test/post-test items examining knowledge about density of decimals

Pre-test Item 5

How many decimals can you find in between 3.14 and 3.15? Tick one of the options and explain briefly your reasoning.

none, because
1, namely
less than 200, because
more than 200 but finite, because
infinitely many, because

Post-test Item 5

How many decimals can you find in between 2.18 and 2.19? Tick one of the options and explain briefly your reasoning.

none, because
1, namely
less than 200, because
more than 200 but finite, because
infinitely many, because

Pre-test Item 6

How many decimals can you find in between 0.799 and 0.80? Tick one of the options and explain briefly your reasoning.

none, because
1, namely
less than 200, because
more than 200 but finite, because
infinitely many, because

Post-test Item 6

How many decimals can you find in between 0.899 and 0.90? Tick one of the options and explain briefly your reasoning.

none, because
1, namely
less than 200, because
more than 200 but finite, because
infinitely many, because

Insights into the participants' conceptions about density were obtained from observing their strategies to find decimal numbers in between two given decimals. Relevant episodes from video recordings of group and classroom discussions in working with Activity 12 (see Figure 2) will be discussed in this paper to complement the written test data on their knowledge about density of decimals.

Figure 2: Activity 12 in Set 2: Strategies to find decimals in between two given decimals

12. For each pair of decimals in Table A, find decimal numbers in between each pair of decimals if available. Justify your ways to find those decimals and give examples by locating them on the number line.

Table A

1.5	1.51
0.99	0.999
1.7501	1.75011

D. RESULTS AND DISCUSSION

Findings from the written tests

Both cohorts recorded significant improvement on the items involving density of decimals in cycle 2, which indicated the positive impact of addressing the topic in the activities in this cycle. However, it should be noted that despite the fact that both cohorts recorded significant improvement on density, as expected the gap between the mean scores of the two cohorts was quite wide as can be observed in Table 1. The primary pre-service teachers recorded a high proportion of blank responses (about 21%) and showed difficulties with density of decimals.

Table 1: Mean scores pre- and post-test on density items (score ranges from 0-4)

Cohorts	***N***	***Pre-test***		***Post-test***			
		Mean	***SD***	***Mean***	***SD***	***t-value***	***p-value***
Primary	94	1.62	1.9	2.57	1.775	4.359	0.000
Secondary	46	3.17	1.5	3.65	0.971	2.119	0.040

The incorrect responses on density items on the written tests could be classified into three different categories. The first category indicated association of decimal digits with whole numbers which resulted in identifying no decimals in between two given decimals. The second category showed knowledge of the link between decimals and fractions but this knowledge was limited to working with equivalent common fractions with the same denominators. The curriculum sequence reflected in the common mathematics primary school textbooks to introduce decimals of the same lengths might explain this approach. The third approach showed a reliance on a "rounding rule" which was observed in finding the number of decimals in between 0.799 and 0.80. The tendency to use rounding in this case could be also affected by the decimals with repeating decimal digits in 0.799. The following three responses were taken from answers to item 5 and 6 in the pre-test of cycle 2 to illustrate the three categories of incorrect idea on density of decimals.

Riri: There is no decimals in between 3.14, and 3.15 because 3.14 and 3.15 are consecutive numbers.

Agus: There is no decimals in between 3.14, and 3.15 because $3.14 = 3\frac{14}{100}$ and $3.15 = 3\frac{15}{100}$ and there is no number in between $\frac{14}{100}$ and $\frac{15}{100}$.

Igni: There is no decimal in between 0.799 and 0.80 because 0.80 is the result of rounding of 0.799.

Insights from group and classroom discussions about density of decimals

In this section, episodes from a video-recording of one group discussion from the primary cohort and a follow up classroom discussion on Activity 12 will be discussed to give additional insights into the participant's knowledge about density of decimals. In this paper, we report on one group discussion. This particular group was selected because the discussion showed how knowledge about density of decimals evolved from ideas of successive partitioning into ten of an interval between the given pair of decimals.

Sari So the number of numbers in between these two numbers is infinite…

Aris: But the way we find them is by first partitioning into ten then we find that there are infinitely many numbers

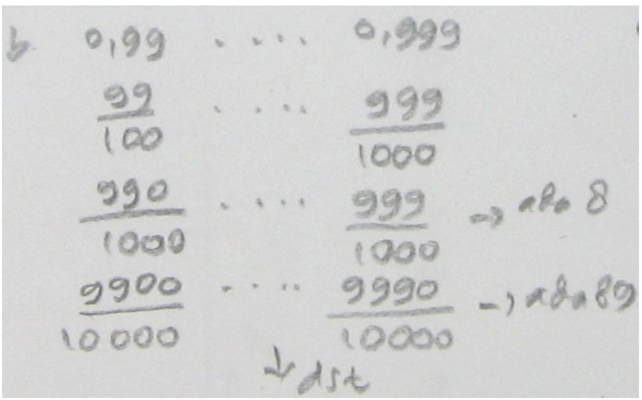
Bayu: So first, we divide the interval into ten equal parts

Aris: Yap, we divide the interval into ten equal parts so the conclusion there are infinitely many

Sari You need to write that down… so first we divide the interval between every pair of numbers into ten then after dividing into ten parts we know that we can continue divide the interval into ten parts. So the conclusion is there are infinitely many numbers.

The following episode of classroom discussion revolved around a strategy proposed by one group to find decimals in between 0.99 and 0.999 (see Figure 3). The lecturer facilitated the whole class discussion by inviting Ratna to respond to a strategy proposed by Rori's group (see Figure 3). The following scripts described parts of the whole class discussion involving Ratna, Lecturer, and Rori. As can be observed in Figure 3, Rori's group first converted the given decimals into corresponding equivalent fractions then after converting them into fractions with common denominators, they found different numbers of fractions in between. Counting only the number of thousandths, Rori's group found there were 8 thousandths in between $\frac{990}{1000}$ and $\frac{999}{1000}$ whilst counting only ten thousandths, there were 89 ten thousandths in between $\frac{9900}{10000}$ and $\frac{9990}{10000}$.

Figure 3: Numbers in between 0.99 and 0.999 taken from Rori's group worksheet



The follow up classroom discussion below highlighted a problematic aspect of a strategy in finding decimals in between two decimals proposed by Rori's group. Ratna's comment indicated a pedagogical concern about the jumps in thinking which was unexplained in Rori's strategy. However, neither Ratna's and the lecturer's comment have addressed the conflicting answers of finding from Roni's strategy and the fact that this strategy showed that Rori's group might lack understanding about density of decimals. Furthermore, this strategy showed lack of pedagogical awareness that the inconsistent answers might cause confusion to children. In respect to understanding of density, this strategy of solving the problem most likely will inhibit children to grasp the density property of decimals as the process of getting into the infinitely many numbers of decimals is not clear as pointed out by Ratna.

Lecturer:	I remember that Ratna has some comments about Rori's work. Perhaps you can share with us your thinking.
Ratna:	I think this strategy is only suitable for the higher class but I think it should not be given to a class that just learn about decimals. I am afraid children will miss the process... This approach might be a quicker way to find numbers in between the given decimals.
Lecturer:	So according to Ratna, the process of finding the numbers in between two decimals using the number line should not be skipped because it helps children to understand. Perhaps Rori would like to respond?
Rori:	I agree with Ratna that the process is important besides the approach we presented is a faster way to find decimals in between two given decimals.
Lecturer:	In my opinion, Rori's approach is an efficient one and it is easy to understand for children who already understand decimals. For children, sometimes they still questioned whether 0.44 and 0.440 has the same value.

A strategy proposed by Rori's group has given insights not only about the participant's lack of understanding about density but also showed a tendency to work only with decimals and fractions from 'the same worlds' which all have the same length of decimal digits or the same denominators. This incorrect strategy is rooted on the discrete nature of whole numbers in solving this problem. A similar trend was observed and reported by Merenlouto & Lehtinen (2002) in the following quote:

> Even at the higher levels of education, students seem to be unaware of their thinking about numbers or the fundamental difference between natural and rational numbers. Because of the operational justification of the extension of number concept, little attention is paid to the underlying general principles of the different number domains in the curriculum. (p.522)

E. CONCLUSION AND IMPLICATIONS

Findings from this study showed that understanding the density property of decimals was not easy. Despite significant improvement observed in the written test results of both cohorts on density items, episodes from group and classroom discussion documented the tendency to incorrectly apply ideas of "discreteness" from whole numbers to decimals. Mathematical textbooks often provide exercises where students need to work only with decimals with the same number of decimal places to avoid complications. This might constrain students from appreciating the continuous property of decimals including density.

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