

The Diffusion Perturbed Compound Poisson Risk Model with a Dividend Barrier

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Abstract

We consider a diffusion perturbed classical compound Poisson risk model in the presence of a constant dividend barrier. Integro-differential equations with certain boundary conditions for the expected discounted penalty (Gerber-Shiu) functions (caused by oscillations or by a claim) are derived and solved. Their solutions can be expressed in terms of the Gerber-Shiu functions in the corresponding perturbed risk model without a barrier. Finally, explicit results are given when the claim sizes are rationally distributed.

Keywords: Compound Poisson process; Diffusion process; Gerber-Shiu function; Integro-differential equation; Time of ruin; Surplus before ruin; Deficit at ruin

1 Introduction

Consider the following classical continuous time surplus process perturbed by a diffusion

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma B(t), \quad t \geq 0, \quad (1)$$

where $\{N(t); t \geq 0\}$ is a Poisson process with parameter λ , denoting the total number of claims from an insurance portfolio. X_1, X_2, \dots , independent of

$\{N(t); t \geq 0\}$, are positive i.i.d. random variables with common distribution function (df) $P(x) = 1 - \bar{P}(x) = P(X \leq x)$, density function $p(x)$, moments $\mu_j = \int_0^\infty x^j p(x) dx$, for $j = 0, 1, 2, \dots$, and the Laplace transform $\hat{p}(s) = \int_0^\infty e^{-sx} p(x) dx$. $\{B(t); t \geq 0\}$ is a standard Wiener process that is independent of the aggregate claims process $S(t) := \sum_{i=1}^{N(t)} X_i$ and $\sigma > 0$ is the dispersion parameter. In the above model, $u = U(0) \geq 0$ is the initial surplus, $c = \lambda \mu_1(1 + \theta)$ is the premium rate per unit time, and $\theta > 0$ is the relative security loading factor.

The classical risk model perturbed by a diffusion was first introduced by Gerber (1970) and has been further studied by many authors during the last few years; e.g., Dufresne and Gerber (1991), Furrer and Schmidli (1994), Schmidli (1995), Gerber and Landry (1998), Wang and Wu (2000), Wang (2001), Tsai (2001, 2003), Tsai and Willmot (2002a,b), Chiu and Yin (2003), and the references therein.

In this paper, a barrier strategy is considered by assuming that there is a horizontal barrier of level $b \geq u$ such that when the surplus reaches level b , the “overflow” will be paid as dividend. Let $U_b(t)$ be the modified surplus process with initial surplus $U_b(0) = u$ under the above barrier strategy.

Define now $T_b = \inf\{t : U_b(t) \leq 0\}$ to be the time of ruin and

$$\Psi_b(u) = P(T_b < \infty | U_b(0) = u), \quad 0 \leq u \leq b,$$

to be the ultimate ruin probability. Further, define

$$\Psi_{b,d}(u) = P(T_b < \infty, U_b(T_b) = 0 | U_b(0) = u), \quad 0 \leq u \leq b,$$

to be the probability of ruin caused by the oscillations in $U_b(t)$ due to the Wiener process $B(t)$ and

$$\Psi_{b,s}(u) = P(T_b < \infty, U_b(T_b) < 0 | U_b(0) = u), \quad 0 \leq u \leq b,$$

to be the probability of ruin caused by a claim. We have that $\Psi_b(u) = \Psi_{b,d}(u) + \Psi_{b,s}(u)$, with $\Psi_{b,d}(0) = 1$ and $\Psi_{b,s}(0) = 0$.

Next, for $\delta > 0$, define

$$\phi_{b,d}(u) = E[e^{-\delta T_b} I(T_b < \infty, U_b(T_b) = 0) | U_b(0) = u], \quad 0 \leq u \leq b,$$

with $\phi_{b,d}(0) = 1$, to be the Laplace transform of the ruin time T_b with respect to δ if the ruin is due to the oscillations. Let $w(x, y)$, for $x, y \geq 0$, be the non-negative values of a penalty function and define

$$\phi_{b,s}(u) = E[e^{-\delta T_b} w(U_b(T_b^-), |U_b(T_b)|) I(T_b < \infty, U_b(T_b) < 0) | U_b(0) = u],$$

with $\phi_{b,s}(0) = 0$, to be the expected discounted penalty (Gerber-Shiu) function if the ruin is caused by a claim. Then

$$\phi_b(u) = \phi_{b,d}(u) + \phi_{b,s}(u)$$

is the expected discounted penalty function.

The barrier strategy was initially proposed by De Finetti (1957) for a binomial model. More general barrier strategies for a compound Poisson risk process have been studied in a number of papers and books. These references include Bühlmann (1970), Segerdahl (1970), Gerber (1972, 1979, 1981), Paulsen and Gjessing (1997), Albrecher and Kainhofer (2002), Højgaard (2002), Dickson and Waters (2004), and Gerber and Shiu (2004). The main focus is on optimal dividend payouts and the time of ruin, under various barrier strategies and other economic conditions. More recently, there are some research papers studying the ruin related quantities such as the surplus before ruin and the deficit at ruin by using the Gerber-Shiu function under a barrier strategy, in the classical risk model or Sparre Andersen risk models, e.g., Lin et al. (2003), Li and Garrido (2004).

The main goal of this paper is to evaluate the Gerber-Shiu function $\phi_b(u)$ and its decompositions of $\phi_{b,s}(u)$ and $\phi_{b,d}(u)$ in the above defined diffusion perturbed classical risk model in the presence of a constant dividend barrier b and analyze several of its special cases.

2 Integro-differential Equations and Their Solutions

In this section, we will show that $\phi_{b,d}(u)$ and $\phi_{b,s}(u)$ both satisfy an integro-differential equation with certain boundary conditions. First, the conditions satisfied by $\phi_{b,d}(b)$ and $\phi_{b,s}(b)$ are given in the following two theorems.

Theorem 1 If the initial surplus is b , then we have the following equations:

$$\phi'_{b,d}(b) + \varpi \phi_{b,d}(b) = \left(\frac{\lambda \varpi}{\lambda + \delta} \right) \int_0^b \phi_{b,d}(b-z)p(z)dz, \quad (2)$$

where

$$\varpi = \frac{1}{\sigma} \left[\frac{c}{\sigma} + \sqrt{\left(\frac{c}{\sigma}\right)^2 + 2(\lambda + \delta)} \right].$$

Proof: Let $\nu = c/\sigma$. For $0 < \epsilon < b$, define $T = \inf\{s > 0 : \nu s + B(s) = -\epsilon\}$. It follows from Karatzas and Shreve (1991, p. 197) that

$$P(T \in dt) = \frac{\epsilon}{\sqrt{2\pi t^3}} \exp\left[-\frac{(\epsilon + \nu t)^2}{2t}\right] dt, \quad t > 0, \quad (3)$$

$$P(T < \infty) = \exp(-2\nu\epsilon), \quad (4)$$

$$E[\exp(-\alpha T)] = \exp(-\nu\epsilon - \epsilon\sqrt{\nu^2 + 2\alpha}). \quad (5)$$

Let $\tau = T \wedge W_1$, where W_1 is the occurrence time of the first claim which is exponentially distributed with parameter λ . Then by the strong Markovian property of $U_b(t)$, we have

$$\begin{aligned} \phi_{b,d}(b) &= E[\exp(-\delta\tau)\phi_{b,d}(U_b(\tau))] \\ &= E[\exp(-\delta T)\phi_{b,d}(U_b(T))I(T < W_1)] \\ &\quad + E[\exp(-\delta W_1)\phi_{b,d}(U_b(W_1))I(T \geq W_1)] \\ &= I_1 + I_2. \end{aligned} \quad (6)$$

Note that

$$\begin{aligned} I_1 &= \int_0^\infty \exp(-\delta s)\phi_{b,d}(b - \sigma\epsilon)E[I(s < W_1)]P(T \in ds) \\ &= \phi_{b,d}(b - \sigma\epsilon) \int_0^\infty \exp[-(\lambda + \delta)s]P(T \in ds) \\ &= \phi_{b,d}(b - \sigma\epsilon)E[\exp[-(\lambda + \delta)T]] \\ &= \phi_{b,d}(b - \sigma\epsilon) \exp[-\nu\epsilon - \epsilon\sqrt{\nu^2 + 2(\lambda + \delta)}], \end{aligned}$$

and

$$\begin{aligned} I_2 &= \int_0^\infty \lambda \exp[-(\lambda + \delta)s]E[I(T \geq s)] \int_0^b \phi_{b,d}(b - z)p(z)dz ds \\ &= \lambda \int_0^\infty \exp[-(\lambda + \delta)s]P(T \geq s)ds \int_0^b \phi_{b,d}(b - z)p(z)dz, \end{aligned}$$

where

$$\begin{aligned} &\lambda \int_0^\infty \exp[-(\lambda + \delta)s]P(T \geq s)ds \\ &= \int_0^\infty \lambda \exp[-(\lambda + \delta)s] [P(T = \infty) + P(s \leq T < \infty)] ds \\ &= \frac{\lambda}{\lambda + \delta} [1 - \exp(-2\nu\epsilon)] + \int_0^\infty \lambda \exp[-(\lambda + \delta)s] \int_s^\infty P(T \in dt) ds \end{aligned}$$

$$\begin{aligned}
&= \frac{\lambda}{\lambda + \delta} [1 - \exp(-2\nu\epsilon)] + \frac{\lambda}{\lambda + \delta} \int_0^\infty \{1 - \exp[-(\lambda + \delta)t]\} P(T \in dt) \\
&= \frac{\lambda}{\lambda + \delta} [1 - \exp(-2\nu\epsilon)] + \frac{\lambda}{\lambda + \delta} \left\{ P(T < \infty) - E[\exp[-(\lambda + \delta)T]] \right\} \\
&= \frac{\lambda}{\lambda + \delta} \left[1 - \exp(-\nu\epsilon - \epsilon\sqrt{\nu^2 + 2(\lambda + \delta)}) \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\phi_{b,d}(b) &= \phi_{b,d}(b - \sigma\epsilon) \exp[-\nu\epsilon - \epsilon\sqrt{\nu^2 + 2(\lambda + \delta)}] \\
&\quad + \frac{\lambda}{\lambda + \delta} \left[1 - \exp(-\nu\epsilon - \epsilon\sqrt{\nu^2 + 2(\lambda + \delta)}) \right] \int_0^b \phi_{b,d}(b - z)p(z)dz.
\end{aligned}$$

Subtracting both sides by $\phi_{b,d}(b - \sigma\epsilon)$, dividing both sides by $\sigma\epsilon$, letting ϵ go to 0, and substituting back $\nu = c/\sigma$, we finally prove that (2) holds. \square

Theorem 2 If the initial surplus is b , then we have the following equations:

$$\phi'_{b,s}(b) + \varpi \phi_{b,s}(b) = \left(\frac{\lambda \varpi}{\lambda + \delta} \right) \left[\int_0^b \phi_{b,d}(b - z)p(z)dz + \omega(b) \right]. \quad (7)$$

Proof: The proof is exactly similar to that of Theorem 1. \square .

Theorem 3 Suppose $p(x)$ is continuously differentiable on $(0, \infty)$, then $\phi_{b,d}(u)$ satisfies the following homogenous integro-differential equation for $0 < u < b$:

$$\frac{\sigma^2}{2} \phi''_{b,d}(u) + c \phi'_{b,d}(u) = (\lambda + \delta) \phi_{b,d}(u) - \lambda \int_0^u \phi_{b,d}(u - x) p(x) dx, \quad (8)$$

with the boundary conditions

$$\phi_{b,d}(0) = 1, \quad (9)$$

$$\left[c + \frac{\lambda + \delta}{\varpi} \right] \phi'_{b,d}(b) + \frac{\sigma^2}{2} \phi''_{b,d}(b) = 0. \quad (10)$$

Proof: The proof of the integro-differential equation (8) is exactly the same as that of the integro-differential equation satisfied by $\phi_{\infty,d}(u)$, see Gerber and Landry (1998, pp. 265-266). The boundary condition (9) is from the definition of $\phi_{b,d}$. Letting u go to b from the left in (8) and noting that (2) holds in Theorem 1, we can prove that the boundary condition (10) holds. \square

Theorem 4 Suppose $p(x)$ is continuously differentiable on $(0, \infty)$ and $\omega(u)$ is twice continuously differentiable on $(0, \infty)$, then $\phi_{b,s}(u)$ satisfies the following non-homogenous integro-differential equation for $0 < u < b$:

$$\frac{\sigma^2}{2}\phi''_{b,s}(u) + c\phi'_{b,s}(u) = (\lambda + \delta)\phi_{b,s}(u) - \lambda \int_0^u \phi_{b,s}(u-x)p(x)dx - \lambda\omega(u), \quad (11)$$

with boundary conditions

$$\phi_{b,s}(0) = 0, \quad (12)$$

$$\left[c + \frac{\lambda + \delta}{\varpi} \right] \phi'_{b,s}(b) + \frac{\sigma^2}{2}\phi''_{b,s}(b) = 0. \quad (13)$$

Proof: The proof of the integro-differential equation (11) is exactly the same as that of the integro-differential equation satisfied by $\phi_{\infty,s}(u)$, see Tsai and Willmot (2002a, pp. 53-54) or Chiu and Yin (2003, pp. 63-64). The boundary condition (12) is from the definition of $\phi_{b,s}$. Letting u go to b from the left in (11) and noting that (7) holds in Theorem 1, we can prove that the boundary condition (13) holds. \square

The solutions of above integro-differential equations with boundary conditions heavily depend on the solutions of the following homogenous integro-differential equation:

$$\frac{\sigma^2}{2}v''(u) + cv'(u) = (\lambda + \delta)v(u) - \lambda \int_0^u v(u-x)p(x)dx, \quad u \geq 0. \quad (14)$$

The general solution of equation (14) is of the form

$$v(u) = \eta_1 v_1(u) + \eta_2 v_2(u), \quad u \geq 0, \quad (15)$$

where $v_1(u)$ and $v_2(u)$ are two linearly independent solutions of (14), which will be discussed in the next section, and η_1, η_2 are any real numbers. Then the solution of the integro-differential equation (8) with boundary conditions (9) and (10) is

$$\phi_{b,d}(u) = \eta_1 v_1(u) + \eta_2 v_2(u), \quad 0 \leq u \leq b, \quad (16)$$

where η_1 and η_2 can be determined by solving the following linear equation system

$$\begin{cases} \eta_1 v_1(0) + \eta_2 v_2(0) = 1, \\ \eta_1 \left\{ \left[c + \frac{\lambda + \delta}{\varpi} \right] v'_1(b) + \frac{\sigma^2}{2}v''_1(b) \right\} + \eta_2 \left\{ \left[c + \frac{\lambda + \delta}{\varpi} \right] v'_2(b) + \frac{\sigma^2}{2}v''_2(b) \right\} = 0. \end{cases}$$

Let $\phi_{\infty,s}(u)$ be the expected discounted penalty function if the ruin is caused by a claim in the perturbed compound Poisson risk model (1) without a barrier.

Tsai and Willmot (2002a) shows that it satisfies the following integro-differential equation for $u \geq 0$:

$$\frac{\sigma^2}{2}\phi''_{\infty,s}(u) + c\phi'_{\infty,s}(u) = (\lambda + \delta)\phi_{\infty,s}(u) - \lambda \int_0^u \phi_{\infty,s}(u-x)p(x)dx - \lambda\omega(u).$$

We note that this equation is the same as equation (11) except $b = \infty$. Then $\phi_{\infty,s}(u)$ can be viewed as a particular solution of (11). It follows from the general theory of differential equations that the solution of the integro-differential equation (11) with boundary conditions (12) and (13) can be expressed as

$$\phi_{b,s}(u) = \phi_{\infty,s}(u) + \vartheta_1 v_1(u) + \vartheta_2 v_2(u), \quad 0 \leq u \leq b, \quad (17)$$

where ϑ_1 and ϑ_2 can be determined by solving the following linear equation system

$$\begin{cases} \vartheta_1 v_1(0) + \vartheta_2 v_2(0) = 0, \\ \vartheta_1 \left\{ \left[c + \frac{\lambda+\delta}{\varpi} \right] v'_1(b) + \frac{\sigma^2}{2} v''_1(b) \right\} + \vartheta_2 \left\{ \left[c + \frac{\lambda+\delta}{\varpi} \right] v'_2(b) + \frac{\sigma^2}{2} v''_2(b) \right\} \\ = - \left[c + \frac{\lambda+\delta}{\varpi} \right] \phi'_{\infty,s}(b) - \frac{\sigma^2}{2} \phi''_{\infty,s}(b). \end{cases}$$

Tsai and Willmot (2002a) have shown that $\phi_{\infty,s}(u)$ satisfies a defective renewal equation, described as follows. Let $\rho = \rho(\delta)$ be the unique non-negative root of the following generalized Lundberg equation:

$$\lambda\hat{p}(s) = \lambda + \delta - cs - \sigma^2 s^2/2, \quad (18)$$

with $\rho(0) = 0$. Let

$$h(y) = \frac{2c}{\sigma^2} e^{-(\rho + \frac{2c}{\sigma^2})y},$$

$$\gamma(y) = \frac{\lambda}{c} \int_y^\infty e^{-\rho(x-y)} p(x) dx,$$

and

$$\gamma_\omega(y) = \frac{\lambda}{c} \int_y^\infty e^{-\rho(x-y)} \omega(x) dx.$$

Then $\phi_{\infty,s}(u)$ satisfies the following defective renewal equation:

$$\phi_{\infty,s}(u) = \int_0^u \phi_{\infty,s}(u-y) g(y) dy + g_\omega(u), \quad u \geq 0, \quad (19)$$

where $g(y) = h * \gamma(y)$ and $g_\omega(u) = h * \gamma_\omega(u)$, with $*$ denoting the convolution operation.

Properties of $\phi_{\infty,s}(u)$ and its applications have been studied extensively by Tsai (2001, 2003), Tsai and Willmot (2002a, 2002b), Chiu and Yin (2003), and Li and Garrido (2005) for $n = 1$. Therefore, we may use the properties of $\phi_{\infty,s}(u)$ to analyze $\phi_{b,s}(u)$.

3 Analysis of the Function $v(u)$

The solution of the homogenous equation (14) is uniquely determined by the initial conditions $v(0)$ and $v'(0)$ and can be solved by Laplace transforms. Let $\hat{v}(s) = \int_0^\infty e^{-sx}v(x)dx$ be the Laplace transform of $v(u)$. Taking Laplace transforms on both sides of (14) gives

$$\left[\frac{1}{2}\sigma^2 s^2 + cs - (\lambda + \delta) + \lambda \hat{p}(s) \right] \hat{v}(s) = \frac{\sigma^2}{2}v(0) s + cv(0) + \frac{\sigma^2}{2}v'(0). \quad (20)$$

Since $\sigma^2 \rho^2/2 + c\rho - (\lambda + \delta) + \lambda \hat{p}(\rho) = 0$, then (20) can be rewritten as

$$\begin{aligned} & \left\{ 1 - \left(\frac{2\lambda/\sigma^2}{s + \rho + 2c/\sigma^2} \right) \left[\frac{\hat{p}(\rho) - \hat{p}(s)}{s - \rho} \right] \right\} \hat{v}(s) \\ &= \frac{v(0)}{s + \rho + 2c/\sigma^2} + \frac{v(0)(\rho + 2c/\sigma^2) + v'(0)}{(s - \rho)(s + \rho + 2c/\sigma^2)}. \end{aligned} \quad (21)$$

Inverting it yields

$$\begin{aligned} v(u) &= \int_0^u v(u-y)g(y)dy + \frac{\sigma^2 v(0)}{2c}h(u) \\ &+ \frac{\sigma^2 [v(0)(\rho + 2c/\sigma^2) + v'(0)]}{2c}e^{\rho u} * h(u), \quad u \geq 0. \end{aligned} \quad (22)$$

We remark that equation (22) is defective renewal equation, since $g(y)$ is a defective density function with $\int_0^\infty g(y)dy = (c\rho + \sigma^2 \rho^2/2 - \delta)/(c\rho + \sigma^2 \rho^2/2) < 1$, see Gerber and Landry (1998, eq. (16)).

One can find two linearly independent solutions $v_1(u)$ and $v_2(u)$ by specifying the initial conditions $v_i(0)$ and $v'_i(0)$ for $i = 1, 2$. For example, setting $v_1(0) = 1$ and $v'_1(0) = -(\rho + 2c/\sigma^2)$ yields

$$v_1(u) = \int_0^u v_1(u-y)g(y)dy + \frac{\sigma^2}{2c}h(u), \quad u \geq 0, \quad (23)$$

and setting $v_2(0) = 0$ and $v'_2(0) = 1$ yields

$$v_2(u) = \int_0^u v_2(u-y)g(y)dy + \frac{\sigma^2}{2c}e^{\rho u} * h(u), \quad u \geq 0. \quad (24)$$

To prove that $v_1(u)$ and $v_2(u)$ are linearly independent, we assume that there are constants c_1 and c_2 such that $c_1 v_1(u) + c_2 v_2(u) \equiv 0$, for any $u \geq 0$. Then we have $c_1 v_1(0) + c_2 v_2(0) = 0$ and $c_1 v'_1(0) + c_2 v'_2(0) = 0$. Solving these two equations gives $c_1 = c_2 = 0$. This proves that $v_1(u)$ and $v_2(u)$ are linearly independent.

Remarks:

1. Gerber and Landry (1998, eq. (17)) have shown that $\phi_{\infty,d}(u)$ with $\phi_{\infty,d}(0) = 1$ also satisfies the defective renewal equation (23). By the uniqueness of the solution of the defective renewal equation (23), we have $v_1(u) = \phi_{\infty,d}(u)$.
2. By comparing (23) and (24), we can easily prove that

$$v_2(u) = e^{\rho u} * v_1(u) = e^{\rho u} * \phi_{\infty,d}(u) = \int_0^u \phi_{\infty,d}(u-x)e^{\rho x} dx, \quad u \geq 0.$$

4 Main Results

Under the assumptions on $v_1(u)$ and $v_2(u)$ stated in the previous section, eq. (16) gives for $0 \leq u \leq b$:

$$\phi_{b,d}(u) = \phi_{\infty,d}(u) - \frac{[c + (\lambda + \delta)/\varpi] \phi'_{\infty,d}(b) + (\sigma^2/2)\phi''_{\infty,d}(b)}{[c + (\lambda + \delta)/\varpi] v'_2(b) + (\sigma^2/2)v''_2(b)} e^{\rho u} * \phi_{\infty,d}(u), \quad (25)$$

and (17) gives for $0 \leq u \leq b$:

$$\phi_{b,s}(u) = \phi_{\infty,s}(u) - \frac{[c + (\lambda + \delta)\varpi] \phi'_{\infty,s}(b) + (\sigma^2/2)\phi''_{\infty,s}(b)}{[c + (\lambda + \delta)/\varpi] v'_2(b) + (\sigma^2/2)v''_2(b)} e^{\rho u} * \phi_{\infty,d}(u). \quad (26)$$

In particular, if $\delta = 0$ and $w(x, y) = 1$, then $\rho = 0$, and $\phi_{b,d}(u)$ and $\phi_{b,s}(u)$ simplify to the ruin probabilities $\Psi_{b,d}(u)$ and $\Psi_{b,s}(u)$, respectively. We have the following results for $0 \leq u \leq b$:

$$\begin{aligned} \Psi_{b,d}(u) &= \Psi_{\infty,d}(u) - \frac{(c + \lambda/\varpi) \Psi'_{\infty,d}(b) + (\sigma^2/2)\Psi''_{\infty,d}(b)}{(c + \lambda/\varpi) \Psi_{\infty,d}(b) + (\sigma^2/2)\Psi'_{\infty,d}(b)} \int_0^u \Psi_{\infty,d}(x) dx, \\ \Psi_{b,s}(u) &= \Psi_{\infty,s}(u) - \frac{(c + \lambda/\varpi) \Psi'_{\infty,s}(b) + (\sigma^2/2)\Psi''_{\infty,s}(b)}{(c + \lambda/\varpi) \Psi_{\infty,d}(b) + (\sigma^2/2)\Psi'_{\infty,d}(b)} \int_0^u \Psi_{\infty,d}(x) dx. \end{aligned}$$

Dufrense and Gerber (1991, Eq. (4.7)) shows that

$$\Phi'_{\infty}(u) = \frac{2(c - \lambda\mu_1)}{\sigma^2} \Psi_{\infty,d}(u),$$

where $\Phi_{\infty}(u)$ is the non-ruin probability of the risk model (1). Then $\Psi_{b,d}(u)$ and $\Psi_{b,s}(u)$ can be expressed for $0 \leq u \leq b$ as

$$\Psi_{b,d}(u) = \Psi_{\infty,d}(u) - \frac{(c + \lambda/\varpi) \Psi'_{\infty,d}(b) + (\sigma^2/2)\Psi''_{\infty,d}(b)}{(c + \lambda/\varpi) \Phi'_{\infty}(b) + (\sigma^2/2)\Phi''_{\infty}(b)} \Phi_{\infty}(u), \quad (27)$$

$$\Psi_{b,s}(u) = \Psi_{\infty,s}(u) - \frac{(c + \lambda/\varpi) \Psi'_{\infty,s}(b) + (\sigma^2/2)\Psi''_{\infty,s}(b)}{(c + \lambda/\varpi) \Phi'_{\infty}(b) + (\sigma^2/2)\Phi''_{\infty}(b)} \Phi_{\infty}(u). \quad (28)$$

Remark: Since $\Psi_{\infty,s}(u) + \Psi_{\infty,d}(u) = \Psi_{\infty}(u) = 1 - \Phi_{\infty}(u)$, then it follows from (27) and (28) that $\Psi_{b,d}(u) + \Psi_{b,s}(u) = 1$ for $0 \leq u \leq b$. This shows that ruin is certain under the constant dividend barrier strategy.

Next, we will show that if p is rationally distributed then both v_1 and v_2 have a rational Laplace transform, which can be inverted explicitly by partial fractions as follows. Let us assume that claim size X is rationally distributed, i.e.,

$$\hat{p}(s) = \frac{Q_{m-1}(s)}{Q_m(s)}, \quad \Re(s) \in (h_X, \infty), \quad (29)$$

where $m \in \mathbb{N}^+$, $h_X := \inf\{s \in R : E[e^{-sX}] < \infty\}$, Q_m is a polynomial of degree m with leading coefficient 1, Q_{m-1} is a polynomial of degree $m-1$ or less, and Q_m and Q_{m-1} do not have any common zeros. Further, since $\hat{p}(s)$ is finite for all s , with $\Re(s) > 0$, equation $Q_m(s) = 0$ has no roots with positive real parts. This class of distributions is widely used in applied probability applications, which includes, as special cases, Erlangs and part of phase-type, Coxian distributions, as well as mixture of them. Further discussions on rational distributions can be found in Cox (1955) and Neuts (1981, Chapter 2).

Substituting (29) into (20) with $v_1(0) = 1$ and $v_1'(0) = -(\rho + 2c/\sigma^2)$ and multiplying $Q_m(s)$ to both the denominator and numerator yields

$$\hat{v}_1(s) = \frac{(\sigma^2/2)(s - \rho)Q_m(s)}{[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s)}. \quad (30)$$

Since $[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s)$ is a polynomial of degree $m+2$ with leading coefficient $\sigma^2/2$, then it can be factored as

$$[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s) = (\sigma^2/2)(s - \rho) \prod_{i=1}^{m+1} (s + R_i),$$

where $\rho > 0$ and $-R_1, -R_2, \dots, -R_{m+1}$, with $\Re(R_i) > 0, i = 1, 2, \dots, m+1$, are all the roots of the equation $[\sigma^2 s^2/2 + cs - (\lambda + \delta)]Q_m(s) + \lambda Q_{m-1}(s) = 0$ on the whole complex plane. We remark ρ is also the unique positive root of the generalized Lundberg equation (18). If R_1, R_2, \dots, R_{m+1} are distinct, then by partial fractions, (30) can be rewritten as

$$\hat{v}_1(s) = \frac{Q_m(s)}{\prod_{i=1}^{m+1} (s + R_i)} = \sum_{i=1}^{m+1} \frac{\alpha_i}{s + R_i}, \quad (31)$$

where $\alpha_i = Q_m(-R_i)/\prod_{j=1, j \neq i}^{m+1} (R_j - R_i), i = 1, 2, \dots, m+1$. Inverting it gives

$$v_1(u) = \sum_{i=1}^{m+1} \alpha_i e^{-R_i u}, \quad u \geq 0. \quad (32)$$

Then we have for $u \geq 0$

$$v_2(u) = \int_0^u v_1(u-x)e^{\rho x} dx = \frac{Q_m(\rho)}{\prod_{i=1}^{m+1}(\rho + R_i)} e^{\rho u} - \sum_{i=1}^{m+1} \frac{\alpha_i}{(\rho + R_i)} e^{-R_i u}. \quad (33)$$

5 An Example

In this section, we will illustrate some explicit results when the claim sizes are exponentially distributed.

Suppose that the claim sizes are exponentially distributed with density function

$$p(x) = \kappa e^{-\kappa x}, \quad x \geq 0,$$

and Laplace transform $\hat{p}(s) = \kappa/(s + \kappa)$. The equation

$$[\sigma^2 s^2/2 + c s - (\lambda + \delta)](s + \kappa) + \lambda \kappa = 0 \quad (34)$$

has one positive root, say ρ , and two negative roots, say $-R_1, -R_2$. Then (32) gives

$$v_1(u) = \phi_{\infty, d}(u) = \frac{\kappa - R_1}{R_2 - R_1} e^{-R_1 u} + \frac{\kappa - R_2}{R_1 - R_2} e^{-R_2 u}, \quad u \geq 0,$$

and

$$\begin{aligned} v_2(u) &= \int_0^u v_1(u-x)e^{\rho x} dx \\ &= \frac{\rho + \kappa}{(\rho + R_1)(\rho + R_2)} e^{\rho u} + \frac{R_1 - \kappa}{(\rho + R_1)(R_2 - R_1)} e^{-R_1 u} \\ &\quad + \frac{R_2 - \kappa}{(\rho + R_2)(R_1 - R_2)} e^{-R_2 u}, \quad u \geq 0. \end{aligned}$$

Then (25) gives

$$\begin{aligned} \phi_{b, d}(u) &= \frac{\kappa - R_1}{R_2 - R_1} \left(1 + \frac{\xi}{\rho + R_1}\right) e^{-R_1 u} + \frac{\kappa - R_2}{R_1 - R_2} \left(1 + \frac{\xi}{\rho + R_2}\right) e^{-R_2 u} \\ &\quad - \frac{\xi(\kappa + \rho)}{(\rho + R_1)(\rho + R_2)} e^{\rho u}, \quad 0 \leq u \leq b, \end{aligned}$$

where

$$\xi = \frac{(\sigma^2/2)v_1''(b) + [c + (\lambda + \delta)/\varpi] v_1'(b)}{(\sigma^2/2)v_2''(b) + [c + (\lambda + \delta)/\varpi] v_2'(b)}.$$

The evaluation of $\phi_{b,s}(u)$ depends on the choice of the penalty function $w(x, y)$, e.g., if $w(x, y) = 1$, then $\omega(u) = \bar{P}(u) = e^{-\kappa u}$, and Tsai and Willmot (2002a) shows that the Laplace transform of $\phi_{\infty,s}(u)$ can be expressed as

$$\hat{\phi}_{\infty,s}(s) = \frac{\lambda(s + \kappa)[\hat{\omega}(\rho) - \hat{\omega}(s)]}{[\sigma^2 s^2/2 + c s - (\lambda + \delta)](s + \kappa) + \lambda \kappa} = \frac{2\lambda/[\sigma^2(\rho + \kappa)]}{(s + R_1)(s + R_2)}, \quad (35)$$

inverting it yields

$$\phi_{\infty,s}(u) = \frac{2\lambda}{\sigma^2(\rho + \kappa)(R_2 - R_1)} [e^{-R_1 u} - e^{-R_2 u}], \quad u \geq 0. \quad (36)$$

Then (26) gives

$$\begin{aligned} \phi_{b,s}(u) &= \frac{1}{(R_2 - R_1)} \left[\frac{2\lambda}{\sigma^2(\rho + \kappa)} - \frac{\zeta(R_1 - \kappa)}{\rho + R_1} \right] e^{-R_1 u} \\ &+ \frac{1}{(R_1 - R_2)} \left[\frac{2\lambda}{\sigma^2(\rho + \kappa)} - \frac{\zeta(R_2 - \kappa)}{\rho + R_2} \right] e^{-R_2 u} - \frac{\zeta(\rho + \kappa)}{(\rho + R_1)(\rho + R_2)} e^{\rho u}, \end{aligned}$$

where

$$\zeta = \frac{(\sigma^2/2)\phi''_{\infty,s}(b) + [c + (\lambda + \delta)/\varpi] \phi'_{\infty,s}(b)}{(\sigma^2/2)v''_2(b) + [c + (\lambda + \delta)/\varpi] v'_2(b)}.$$

Now, let $c = 1.1$, $\lambda = 1$, $\kappa = 1$, $\sigma = 0.5$, $\delta = 0.05$, $b = 10$. The roots of equation (34) are: $\rho = 0.1812$, $-R_1 = -0.2264$, $-R_2 = -9.7548$. Then

$$\begin{aligned} \phi_{10,d}(u) &= 0.08 e^{-0.2264 u} + 0.9183 e^{-9.7548 u} + 0.0016 e^{0.1812 u}, \\ \phi_{10,s}(u) &= 0.7011 e^{-0.2264 u} - 0.7153 e^{-9.7548 u} + 0.0142 e^{0.1812 u}, \quad 0 \leq u \leq 10. \end{aligned}$$

If the discount factor δ is set to be $\delta = 0$, then $\rho = 0$, $R_1 = 0.0823$, $R_2 = 9.7177$. In this case, $\phi_{10,d}(u)$ simplifies to the ruin probability due to oscillations $\Psi_{10,d}(u)$ and $\phi_{10,s}(u)$ simplifies to the ruin probability caused by a claim $\Psi_{10,s}(u)$, which are given as follows.

$$\begin{aligned} \Psi_{10,d}(u) &= 0.1029 - 0.8971 e^{-9.7177 u}, \\ \Psi_{10,s}(u) &= 0.8971 - 0.8971 e^{-9.7177 u}, \quad 0 \leq u \leq 10. \end{aligned}$$

The total probability of ruin $\Psi_{10}(u) = \Psi_{10,d}(u) + \Psi_{10,s}(u) = 1$, for all $0 \leq u \leq 10$, this is because the ruin is certain under the constant barrier strategy.

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