On a discrete time risk model with delayed claims and a constant dividend barrier

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Abstract

In this paper a compound binomial risk model with a constant dividend barrier is considered. Two types of individual claims, main claims and by-claims, are defined, where by-claims are produced by the main claims and may be delayed for one time period with a certain probability. Some prior work on these time-correlated claims has been done by Yuen and Guo (2001) and the references therein. Formulae for the expected present value of dividend payments up to the time of ruin are obtained for discrete-type individual claims, together with some other results of interest. Explicit expressions for the corresponding results are derived in a special case, for which a comparison is also made to the original discrete model of De Finetti (1957).

Keywords: Compound binomial model; Main claim; By-claim; Dividend.

1 Introduction

In recent years, risk models with correlated claims and models with dividend payments have been two of the major interests in the risk theory literature. The risk model considered in this paper is a compound binomial model with time-correlated individual claims and dividend payments that are ruled by a constant dividend barrier.

A framework of time-correlated claims is built by introducing two kinds of individual claims, namely main claims and by-claims, and allowing possible delays of the occurrence of by-claims. Considerations of delay in claim settlement can be found in Waters and Papatriandafylou (1985), Yuen and Guo (2001) and Wu and Yuen (2004). Other dependence structures in terms of main claims and by-claims are studied in Yuen and Wang (2002) and Wu and Yuen (2003).

Because of the certainty of ruin for a risk model with a constant dividend barrier, the calculation of the expected discounted dividend payments is a major problem of interest in the context, instead of the ruin probability of the business. The very first risk model with dividends in the literature was proposed by De Finetti (1957), in which a discrete time model with very simple periodic gains was studied. References for the results of De Finetti's model can be found in Bühlmann (1970, Section 6.4.5) and Gerber and Shiu (2004, Appendix). Other discrete time risk models involving dividends include the discrete time model with a constant barrier of Claramunt et al. (2002), in which the expected present value of dividends is calculated based on a system of linear equations, and the model considered in Dickson and Waters (2004)

that is used to tackle certain problems in the classical continuous time model. Also, problems relating to dividends have been considered more extensively in the continuous time setting. Related works can be found in Bühlmann (1970), Dickson and Waters (2004), Gerber (1979), Gerber and Shiu (1998, 2004), Højgaard (2002), Li and Garrido (2004), Lin et al. (2003), Paulsen and Gjessing (1997), Zhou (2005) and references therein.

In this paper, Section 2 defines the model of interest, describes various payments, including the premiums, claims and dividends, and lists the notation. In Section 3, difference equations are developed for the expected present value of dividend payments. Then an explicit expression is derived, using the technique of generating functions. Moreover, closed-form solutions for the expected present value of dividends are obtained for two classes of claim size distributions in Section 4. Numerical examples are also provided to illustrate the impact of the delay of by-claims on the expected present value of dividends. Finally, in Section 5, a slightly modified model is discussed, also aiming to evaluate the impact of the delay of by-claims on the expected present value of dividends through a comparison between our results and those given in Gerber and Shiu (2004, Appendix) where there is no delay.

2 The model

We consider a discrete time compound binomial risk model with two types of individual claims: main claims and by-claims. For a detailed description and the intuition for this model, see Yuen and Guo (2001). Let U_k be the total amount of claims up

to the end of the kth time period, $k \in \mathbb{N}^+$ and $U_0 = 0$. We define

$$U_k = U_k^X + U_k^Y, \tag{2.1}$$

where U_k^X and U_k^Y are the total main claims and by-claims, respectively, in the first k time periods.

Random variables $X_1, X_2, \ldots$ denote the sizes of the main claims and are independent and identically distributed (i.i.d.) having a probability function (p.f.) $f_m, m = 1, 2, \ldots$. The probability of having a main claim in each time period is $p, 0 < p < 1$, and the probability of no claim is $q = 1 - p$.

The amounts of by-claims, denoted by $Y_1, Y_2, \ldots$, are also i.i.d. and have another p.f. $g_n, n = 1, 2, \ldots$. X_i and Y_j are independent of each other for all i and j, and their means are denoted by μ_X and μ_Y, respectively. One main claim induces one by-claim, which occurs simultaneously with probability $0 \leq \theta \leq 1$, that is to say, the by-claim may be delayed with probability $1 - \theta$. We only consider a delay of one time period in this paper.

Assume that premiums are received at the beginning of each time period with a constant premium rate of 1 per period, and all claim payments are made only at the end of each time period. We introduce a dividend policy to the company that certain amount of dividends will be paid to the policyholder instantly, as long as the surplus of the company at time k is higher than a constant dividend barrier b ($b > 0$). It implies that the dividend payments will only possibly occur at the beginning of each period, right after receiving the premium payment. The surplus at the end of the kth

time period, S_k, is then defined to be, for $k = 1, 2, \ldots,$

$$S_k = u + k - U_k - UD_k, \qquad (S_0 = u). \tag{2.2}$$

Here the initial surplus is $u, u = 1, 2, \ldots, b$. The positive safety loading condition holds if $p(\mu_X + \mu_Y) < 1$. We define UD_k as the sum of dividend payments in the first k periods, for $k = 1, 2, \ldots,$

$$UD_k = D_1 + D_2 + \cdots + D_k, \qquad (UD_0 = 0).$$

Denote by D_n the amount of dividend paid out in period n, for $n = 1, 2, \ldots,$ with definition

$$D_n = \max\{S_{n-1} + 1 - b, 0\}. \tag{2.3}$$

Define $T = \min\{k | S_k \leq 0\}$ to be the time of ruin, $\psi(u; b) = P[T < \infty | S_0 = u]$ to be the ruin probability, and $\phi(u; b) = 1 - \psi(u; b)$, for $u = 1, 2, \ldots, b$, to be the non-ruin probability. Let v be a constant annual discount rate for each period. Then the expected present value of the dividend payments due until ruin is

$$V(u; b) := E\left[\sum_{k=1}^{T} D_k v^{k-1} \middle| S_0 = u\right].$$

3 The expected present value of dividends

To study the expected present value of the dividend payments, $V(u; b)$, we consider the claim occurrences in two scenarios (see Yuen and Guo (2001)). In the first scenario, if a main claim occurs in a certain time period, its associated by-claim also occurs in the same period. Thus the surplus process is renewed at the beginning of the next

time period. The second scenario is simply the complement of the first one, i.e., given a main claim, its associated by-claim will occur one period later. Conditional on the second scenario, we define a complementary surplus process as follows:

$$S_k^* = u + k - U_k - UD_k^* - Y, \qquad k = 1, 2, \ldots, \tag{3.1}$$

with $S_0^* = u$, where UD_k^* is the sum of dividend payments in the first k time periods, and Y is a random variable following the probability function $g_n, n = 1, 2, \ldots$, and is independent of all other claim amounts random variables X_i and Y_j for all i and j. The corresponding ruin probability is denoted by $\psi^*(u; b)$ with $\psi^*(0; b) = \psi^*(1; b) = 1$, the non-ruin probability is denoted by $\phi^*(u; b) = 1 - \psi^*(u; b)$, and the expected present value of the dividend payments is denoted by $V^*(u; b)$. Then conditioning on the occurrences of claims at the end of the first time period, we can set up the following difference equations for $V(u; b)$ and $V^*(u; b)$:

$$
\begin{aligned}
V(u; b) &= v \Bigg\{ qV(u+1; b) + p\theta \sum_{m+n \leq u+1} V(u+1-m-n; b) f_m g_n \\
&\quad + p(1-\theta) \sum_{m=1}^{u+1} V^*(u+1-m; b) f_m \Bigg\}, \quad u = 1, 2, \ldots, b-1, \tag{3.2}
\end{aligned}
$$

$$V^*(1; b) = vq \sum_{n=1}^{2} V(2-n; b) g_n, \tag{3.3}$$

and for $u = 2, 3, \ldots, b-1$,

$$
\begin{aligned}
V^*(u; b) &= v \Bigg\{ q \sum_{n=1}^{u+1} V(u+1-n; b) g_n \\
&\quad + p\theta \sum_{m+n+l \leq u+1} V(u+1-m-n-l; b) f_m g_n g_l \\
&\quad + p(1-\theta) \sum_{m+l \leq u+1} V^*(u+1-m-l; b) f_m g_l \Bigg\}, \tag{3.4}
\end{aligned}
$$

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with boundary conditions:

$$V(0; b) = 0,$$

$$V(b; b) = 1 + V(b - 1; b),$$

$$V^*(0; b) = 0,$$

and

$$V^*(b; b) = 1 + V^*(b - 1; b).$$

The second boundary condition holds because when the initial surplus is b, the premium received at the beginning of the first period will be paid out as a dividend immediately. Except the first dividend payment, the rest of the model is the same as that starting from an initial surplus $b - 1$. The last condition can be explained similarly.

From (3.2) and (3.4) one can rewrite $V^*(u; b)$ as

$$V^*(u; b) = \sum_{n=1}^{u} V(u - n; b) g_n, \qquad u = 2, 3, \ldots, b - 1. \tag{3.5}$$

This result can also be obtained from model (3.1) as

$$V^*(u; b) = E[V(u - Y; b)] = \sum_{n=1}^{u} V(u - n; b) g_n.$$

Substituting (3.5) into (3.2) gives

$$V(1; b) = vqV(2; b) + v^2 pq(1 - \theta) V(1; b) f_1 g_1, \tag{3.6}$$

and for $u = 2, 3, \ldots, b - 1$,

$$V(u; b) = v \left\{ qV(u + 1; b) + p\theta \sum_{m+n \leq u+1} V(u + 1 - m - n; b) f_m g_n \right.$$

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$$+p(1-\theta)\sum_{m+n\leq u+1}V(u+1-m-n;b)f_mg_n\Bigg\}$$

$$=v\Bigg\{qV(u+1;b)+p\sum_{m+n\leq u+1}V(u+1-m-n;b)f_mg_n\Bigg\}, \qquad (3.7)$$

with a new boundary condition:

$$V(b;b)=1+V(b-1;b). \qquad (3.8)$$

To obtain an explicit expression for $V(u;b)$ from (3.6) and (3.7), we define a new function $W(u)$ that satisfies the following difference equation,

$$W(1)=vqW(2)+v^2pq(1-\theta)W(1)f_1g_1, \qquad (3.9)$$

and for $u=2,3,\ldots$,

$$W(u)=v\Bigg\{qW(u+1)+p\sum_{m+n\leq u+1}W(u+1-m-n)f_mg_n\Bigg\}. \qquad (3.10)$$

Apart from a multiplicative constant, the solution of (3.9) and (3.10) is unique. Therefore, we can set $W(1)=1$. It follows from the theory of difference equations that the solution to (3.6) and (3.7) with boundary condition (3.8) is of the form

$$V(u;b)=C(b)W(u), \qquad (3.11)$$

where $C(b)=1/[W(b)-W(b-1)]$.

Remark: Result (3.11) is the discrete counterpart of $V(u;b)$ for the classical model, which is of the form $V(u;b)=h(u)/h'(b)$ for a certain function h. See Gerber (1979).

Let the generating function of $W(u)$ be $\tilde{W}(z):=\sum_{u=1}^{\infty}W(u)z^u$, $-1<\Re(z)<1$. Similarly, $\tilde{f}(z):=\sum_{m=1}^{\infty}f_mz^m$ and $\tilde{g}(z):=\sum_{n=1}^{\infty}g_nz^n$ are probability generating functions (p.g.f.'s) of $\{f_m\}_{m=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$, respectively. Furthermore, we construct

two new generating functions $\tilde{h}(z,1) := q + p\tilde{f}(z)\tilde{g}(z)$ and $\tilde{h}(z,k) := [\tilde{h}(z,1)]^k$. We

denote the probability function of $\tilde{h}(z,k)$ by $h(i,k)$. Yuen and Guo (2001) have

commented that $h(i,k)$ is the probability function of the total claims in the first k

time periods in the compound binomial model with individual claim amount $X_1 + Y_1$.

In the following theorem, we show that $V(u,b)$ can be expressed explicitly in terms

of $h(i,k)$.

Theorem 1 *For the expected present value of dividend payments, $V(u;b)$, of model*

(2.2), we have, for $u = 2, 3, \ldots, b$,

$$V(u;b) \;=\; C(b)\sum_{i=1}^{\infty} v^{i+1}q\Big[vp(1-\theta)f_1g_1h(i+u-1,i) - h(i+u,i)\Big], \qquad (3.12)$$

where

$$C(b) \;=\; \bigg\{\sum_{i=1}^{\infty} v^{i+1}q\Big\{vp(1-\theta)f_1g_1[h(i+b-1,i) - h(i+b-2,i)]$$
$$-[h(i+b,i) - h(i+b-1,i)]\Big\}\bigg\}^{-1},$$

and $V(1;b) = C(b)$.

Proof. The result (3.12) can be derived by using the technique of generating func-

tions. Multiplying both sides of (3.10) by z^u and summing over u from 2 to ∞, we

get

$$\sum_{u=2}^{\infty} W(u)z^u = vq\sum_{u=2}^{\infty} W(u+1)z^u + vp\sum_{u=2}^{\infty}\sum_{m+n\leq u+1} W(u+1-m-n)f_mg_nz^u.$$

Rewriting both sides of the above equation in terms of $\tilde{W}(z)$ yields

$$\tilde{W}(z) - W(1)z = vqz^{-1}\left[\tilde{W}(z) - W(1)z - W(2)z^2\right] + vpz^{-1}\tilde{W}(z)\tilde{f}(z)\tilde{g}(z).$$

From (3.9) and the fact that $W(1) = 1$, the above equation simplifies to

$$\tilde{W}(z)\left\{1 - v\left[q + p\tilde{f}(z)\tilde{g}(z)\right]z^{-1}\right\} = W(1)z - vqW(1) - vqW(2)z$$

$$= vq\left[vp(1-\theta)f_1g_1z - 1\right]. \qquad (3.13)$$

From (3.13) and the definition of $\tilde{h}(z, 1)$ we obtain a final expression for $\tilde{W}(z)$

$$\tilde{W}(z) = \frac{vq\left[vp(1-\theta)f_1g_1z - 1\right]}{1 - v\tilde{h}(z,1)z^{-1}}. \qquad (3.14)$$

Rewriting $[1 - v\tilde{h}(z,1)z^{-1}]^{-1}$ in terms of a power series in z, we have

$$\tilde{W}(z) = v^2pq(1-\theta)f_1g_1\sum_{k=0}^{\infty}v^k\tilde{h}(z,k)z^{1-k} - vq\sum_{k=0}^{\infty}v^k\tilde{h}(z,k)z^{-k}.$$

Comparing the coefficients of z^u in both sides gives, for $u = 2, 3, \ldots, b$,

$$W(u) = v^2pq(1-\theta)f_1g_1\sum_{i=u}^{\infty}v^{i+1-u}h(i, i+1-u) - vq\sum_{i=u+1}^{\infty}v^{i-u}h(i, i-u)$$

$$= \sum_{i=1}^{\infty}v^{i+1}q\left[vp(1-\theta)f_1g_1h(i+u-1, i) - h(i+u, i)\right].$$

The above result together with (3.11) gives us the explicit expression for $V(u; b)$ as in (3.12). $\qquad \square$

To end this section, we show that the ruin is certain in the risk model described in (2.2). For $b = 1$, since $\phi(1; 1) = q\phi(1; 1)$ and $0 < q < 1$, then $\phi(1; 1) = 0$. For $b = 2$, $0 \leq \phi(1; 2) \leq \phi(2; 2) = q\phi(2; 2) + p(1-\theta)\phi^*(1; 2)f_1 = q\phi(2; 2)$, then $\phi(1; 2) = \phi(2; 2) = 0$. The following theorem shows that ruin is certain for $b \geq 3$ under certain conditions.

Theorem 2 *The ruin probability in a compound binomial risk model with delayed claims and a constant dividend barrier is one, i.e., $\psi(u; b) = 1$, for $u = 1, 2, \ldots, b$, provided that $\sum_{m+n \leq b-1} f_m g_n < 1$, for $b \geq 3$.*

Proof. Since $\psi(u; b) \geq \psi(b; b)$ for $u = 0, 1, 2, \ldots b$, then it is sufficient to prove that

$\psi(b; b) = 1$ or $\phi(b; b) = 0$ for $b \geq 3$.

Conditioning on the occurrences of claims at the end of the first time period gives

$$
\begin{aligned}
\phi(b; b) &= q\,\phi(b; b) + p\,\theta \sum_{m+n \leq b-1} \phi(b - m - n; b) f_m\, g_n \\
&\quad + p(1 - \theta) \sum_{m=1}^{b-1} \phi^*(b - m; b) f_m,
\end{aligned}
\tag{3.15}
$$

$$
\phi^*(u; b) = \sum_{n=1}^{u-1} \phi(u - n; b) g_n, \quad u = 2, 3, \ldots, b.
\tag{3.16}
$$

Substituting (3.16) into (3.15) yields

$$
\phi(b; b) = q\,\phi(b; b) + p \sum_{m+n \leq b-1} \phi(b - m - n; b) f_m g_n.
$$

It follows from the inequality $\phi(b - m - n; b) \leq \phi(b; b)$ that

$$
\phi(b; b) \leq q\,\phi(b; b) + p\phi(b; b) \sum_{m+n \leq b-1} f_m\, g_n.
\tag{3.17}
$$

Since $\sum_{m+n \leq b-1} f_m g_n < 1$ and $0 \leq \phi(b; b) \leq 1$, then inequality (3.17) gives $\phi(b; b) = 0$,

this implies that $\psi(b; b) = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4 Two Classes of Claim Size Distributions

Equation (3.12) gives an explicit expression for $V(u; b)$ in terms of $h(i, k)$. However,

it can be seen that this formula is not computationally tractable. In this section, we

consider two special cases for the distribution of $X_1 + Y_1$ such that $W(u)$ has a rational

generating function which can be easily inverted. One case is that the probability

function of $X_1 + Y_1$ has finite support such that its p.g.f. is a polynomial, and the

other case is that $X_1 + Y_1$ has a discrete K_n distribution, i.e., the p.g.f. of $X_1 + Y_1$ is

a ratio of two polynomials with certain conditions.

4.1 Claim amount distributions with finite support

Now assume that the distribution of $X_1 + Y_1$ has finite support, e.g., for $N = 2, 3, \ldots$,

$$(f * g)_x = P(X_1 + Y_1 = x) = \pi_x, \qquad x = 2, 3, \ldots, N,$$

where $*$ denotes convolution. Then

$$D_N(z) := \tilde{h}(z, 1) = q + p \sum_{x=2}^{N} z^x \pi_x, \qquad -1 < \Re(z) < 1, \tag{4.1}$$

is a polynomial of degree N. Then $\tilde{W}(z)$ in (3.14) simplifies to

$$\begin{aligned} \tilde{W}(z) &= \frac{v^2 pq(1-\theta)\pi_2 z^2 - vqz}{z - vD_N(z)} = z \frac{vq - v^2 pq(1-\theta)\pi_2 z}{vD_N(z) - z} \\ &= \frac{z}{p\,\pi_N} \frac{q - vpq(1-\theta)\pi_2 z}{(z - R_1)(z - R_2)\cdots(z - R_N)}, \end{aligned}$$

where $R_1, R_2, \ldots, R_N$ are the N roots of the equation of $vD_N(z) - z = 0$ in the whole complex plane. Further, if $R_1, R_2, \ldots, R_N$ are distinct, then by partial fractions, we have

$$\tilde{W}(z) = \frac{1}{p\,\pi_N} \sum_{i=1}^{N} \frac{a_i z}{R_i - z},$$

where

$$a_i = \frac{vpq(1-\theta)R_i - q}{\prod_{j=1, j\neq i}^{N}(R_i - R_j)}, \quad i = 1, 2, \ldots, N.$$

Inverting the p.g.f. $\tilde{W}(z)$ yields

$$W(u) = \sum_{i=1}^{N} \frac{a_i}{p\pi_N} R_i^{-u}, \qquad u = 1, 2, \ldots. \tag{4.2}$$

Now $V(u; b) = C(b)W(u)$, for $u = 1, 2, \ldots, b - 1$, and as $V(b; b) = 1 + V(b-1; b)$, then

$$V(u; b) = \frac{W(u)}{W(b) - W(b-1)} = \frac{\sum_{i=1}^{N} a_i R_i^{-u}}{\sum_{i=1}^{N} a_i(1 - R_i)R_i^{-b}}, \qquad u = 1, 2, \ldots, b. \tag{4.3}$$

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Example 1 In this example, we assume $f_1 = g_1 = 1$. Then $U_k - U_{k-1}$ can only take three possible values: 1, 0, or -1. This generalizes De Finetti's original model where periodic gains are +1 or -1. The p.g.f. of $W(u)$ in (3.14), has a simplified expression

$$\tilde{W}(z) = \frac{zvq\,[1 - vp(1 - \theta)z]}{vpz^2 - z + vq}. \tag{4.4}$$

Let $0 < R_1 < 1 < R_2$ be the solutions of the equation $vpz^2 - z + vq = 0$. Then by partial fractions, (4.4) can be rewritten as

$$\tilde{W}(z) = \frac{1}{p}\left(\frac{a_1 z}{R_1 - z} + \frac{a_2 z}{R_2 - z}\right),$$

where

$$a_1 = \frac{vpq(1 - \theta)R_1 - q}{R_1 - R_2}, \qquad a_2 = \frac{vpq(1 - \theta)R_2 - q}{R_2 - R_1}.$$

Substituting them into (4.3) gives, for $u = 1, 2, \ldots, b$,

$$\begin{aligned}
V(u; b) &= \frac{a_1 R_1^{-u} + a_2 R_2^{-u}}{a_1(1 - R_1)R_1^{-b} + a_2(1 - R_2)R_2^{-b}} \\
&= \frac{(R_1 + \theta R_2)R_2^{-u} - (R_2 + \theta R_1)R_1^{-u}}{R_2^{-b}(1 - R_2)(R_1 + \theta R_2) - R_1^{-b}(1 - R_1)(R_2 + \theta R_1)}.
\end{aligned} \tag{4.5}$$

The last equality holds because of the property of roots R_1 and R_2 that $R_1 + R_2 = (vp)^{-1}$. Two extreme cases of (4.5) are

$$V(u; b) = \frac{R_2^{-u} - R_1^{-u}}{R_2^{-b}(1 - R_2) - R_1^{-b}(1 - R_1)} \qquad \text{for} \quad \theta = 1,$$

and

$$V(u; b) = \frac{R_2^{-(u+1)} - R_1^{-(u+1)}}{R_2^{-(b+1)}(1 - R_2) - R_1^{-(b+1)}(1 - R_1)} \qquad \text{for} \quad \theta = 0.$$

Another value of interest in Example 1 is the optimal dividend barrier b^*, which is the optimal value of b that maximizes $V(u; b)$ for a given u. From (4.5) we know that b^* is the solution of equation

$$\frac{d}{db}\left[R_2^{-b}(1 - R_2)(R_1 + \theta R_2) - R_1^{-b}(1 - R_1)(R_2 + \theta R_1)\right] = 0.$$

We have

$$b^* = \frac{\ln \frac{(R_2 - 1)(R_1 + \theta R_2) \ln(R_2)}{(R_1 - 1)(R_2 + \theta R_1) \ln(R_1)}}{\ln(R_2) - \ln(R_1)},$$

which does not depend on the initial surplus u. Practically, we round b^* to the closest integral value.

Furthermore, we can prove the following result.

Theorem 3 *For the risk model considered in Example 1, the expected present value of the dividend payments up to the time of ruin, $V(u; b)$, increases as the probability of a delay of the by-claims is increasing as well.*

Proof. The theorem can be proved by the following fact:

$$\frac{d}{d\theta}\left[\frac{(R_1 + \theta R_2)R_2^{-u} - (R_2 + \theta R_1)R_1^{-u}}{R_2^{-b}(1 - R_2)(R_1 + \theta R_2) - R_1^{-b}(1 - R_1)(R_2 + \theta R_1)}\right]$$
$$= \frac{R_2^{-u}R_1^{-b}(1 - R_1)(R_1^2 - R_2^2) + R_2^{-b}R_1^{-u}(1 - R_2)(R_2^2 - R_1^2)}{\left[R_2^{-b}(1 - R_2)(R_1 + \theta R_2) - R_1^{-b}(1 - R_1)(R_2 + \theta R_1)\right]^2} < 0,$$

since $0 < R_1 < 1 < R_2$. $\qquad\square$

In Example 1, let $p = 0.45, v = 0.95, b = 10$, then we have $R_1 = 0.78786, R_2 = 1.55132$. Table 1 summaries the results for $V(u; b)$ for $\theta = 0, 0.25, 0.5, 0.75, 1$, and $u = 1, \ldots, 10$. The numbers show that the higher the initial surplus of the insurance

Table 1: Values of $V(u; 10)$ when $f_1 = g_1 = 1$

$V(u; 10)$	$\theta = 0$	0.25	0.5	0.75	1
$u = 1$	0.40851	0.36231	0.32549	0.29547	0.27052
2	0.60719	0.57724	0.55338	0.53392	0.51775
3	0.82786	0.80834	0.79279	0.78011	0.76957
4	1.08763	1.07477	1.06453	1.05618	1.04924
5	1.40424	1.39561	1.38874	1.38313	1.37847
6	1.79767	1.79167	1.78689	1.78300	1.77976
7	2.29159	2.28717	2.28365	2.28078	2.27839
8	2.91499	2.91144	2.90862	2.90631	2.90439
9	3.70400	3.70082	3.69829	3.69623	3.69451
10	4.70400	4.70082	4.69829	4.69623	4.69451

Table 2: Values of $V(1; b)$ when $\theta = 0.5$

	$b = 1$	2	3	4	5
$V(1; b)$	1.35364	1.42832	1.35958	1.19780	1.00398
	6	7	8	9	10
	0.81751	0.65524	0.52082	0.41219	0.32549

company, the higher the expected present value of dividend payments prior to the time of ruin. They also confirm Theorem 3 that $V(u; b)$ is increasing as the probability of the delay of by-claims is increasing, i.e., θ is decreasing. Moreover, the impact of the delay of by-claims on $V(u; b)$ is getting smaller as u increases.

With fixed $\theta = 0.5$ and $u = 1$, we get the optimal dividend barrier $b^* = 2$. In Table 2, the expected present values of dividend payments $V(1; b)$ for $b = 1, 2, \ldots, 10$ are provided. The values confirm the fact that when $b = 2$, the policyholders receive the most dividends prior to ruin.

4.2 K_n claim amount distributions

Li (2005a, b) studies a class of discrete Sparre Andersen risk models in which the claims inter-arrival times are K_n distributed. This class of distributions includes geometric, negative binomial, discrete phase-type, as well as linear combinations (including mixtures) of these.

For the two independent claim amount random variables X_1 and Y_1, if they have K_n distributions, so does their sum. Therefore, in this subsection, we assume that $(f * g)_x = P(X_1 + Y_1 = x)$ is K_n distributed for $x = 2, 3, \ldots$, and $n = 1, 2, \ldots$, i.e., the p.g.f. of $f * g$ is given by

$$\tilde{f}(z)\tilde{g}(z) = \frac{z^2 E_{n-1}(z)}{\prod_{i=1}^{n}(1 - z\,q_i)}, \quad \Re(z) < \min\left\{\frac{1}{q_i} : 1 \leq i \leq n\right\},$$

where $0 < q_i < 1$, for $i = 1, 2, \ldots, n$ and $E_{n-1}(z) = \sum_{k=0}^{n-1} z^k e_k$ is a polynomial of degree $n - 1$ or less with $E_{n-1}(1) = \prod_{i=1}^{n}(1 - q_i)$. Then $\tilde{W}(z)$ can be transformed to the following rational function

$$\tilde{W}(z) = \frac{z[v^2 pq(1 - \theta)f_1 g_1 z - vq]\prod_{i=1}^{n}(1 - zq_i)}{z\prod_{i=1}^{n}(1 - zq_i) - vq\prod_{i=1}^{n}(1 - zq_i) - vpz^2 E_{n-1}(z)}.$$

Since the denominator of the above equation is a polynomial of degree $n + 1$, it can be factored as $[(-1)^n \prod_{i=1}^{n} q_i - vpe_{n-1}]\prod_{i=1}^{n+1}(z - R_i)$, where $R_1, R_2, \ldots, R_{n+1}$ are the $n + 1$ zeros of the denominator. We remark that $(-1)^n \prod_{i=1}^{n} q_i - vpe_{n-1} = (-1)^n vq/\prod_{i=1}^{n+1} R_i$. Then $\tilde{W}(z)$ simplifies to

$$
\begin{aligned}
\tilde{W}(z) &= \left[\prod_{i=1}^{n+1} R_i\right] \frac{[vp(1 - \theta)f_1 g_1 z^2 - z]\prod_{i=1}^{n}(zq_i - 1)}{\prod_{i=1}^{n+1}(z - R_i)} \\
&= \left[\prod_{i=1}^{n+1} R_i\right][z - vp(1 - \theta)f_1 g_1 z^2]\sum_{i=1}^{n+1} \frac{r_i}{(R_i - z)},
\end{aligned}
$$

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where

$$r_i = \frac{\prod_{j=1}^{n}(R_i q_j - 1)}{\prod_{j=1,j\neq i}^{n+1}(R_i - R_j)}, \quad i = 1, 2, \ldots, n+1.$$

Inverting $\tilde{W}(z)$ gives

$$W(1) = \left[\prod_{i=1}^{n+1} R_i\right] \sum_{i=1}^{n+1} r_i R_i^{-1} = \sum_{i=1}^{n+1} r_i \left(\prod_{j=1,j\neq i}^{n+1} R_j\right) = 1,$$

and

$$W(u) = \left[\prod_{i=1}^{n+1} R_i\right] \sum_{i=1}^{n+1} r_i \left[1 - vp(1-\theta)f_1 g_1 R_i\right] R_i^{-u}, \quad u = 2, 3, \ldots.$$

Now that $C(b) = 1/[W(b) - W(b-1)]$, then finally we have

$$V(1;b) = C(b) = \frac{1}{\left[\prod_{i=1}^{n+1} R_i\right] \sum_{i=1}^{n+1} r_i \left[1 - vp(1-\theta)f_1 g_1 R_i\right](1 - R_i)R_i^{-b}}, \qquad (4.6)$$

and

$$
\begin{aligned}
V(u;b) &= C(b)W(u) \\
&= \frac{\sum_{i=1}^{n+1} r_i \left[1 - vp(1-\theta)f_1 g_1 R_i\right] R_i^{-u}}{\sum_{i=1}^{n+1} r_i \left[1 - vp(1-\theta)f_1 g_1 R_i\right](1 - R_i)R_i^{-b}}, \qquad u = 2, \ldots, b. \qquad (4.7)
\end{aligned}
$$

Example 2 In this example, we assume that the main claim X_1 follows a geometric distribution with $f_x = \beta(1-\beta)^{x-1}, 0 < \beta < 1, x = 1, 2, \ldots$, and the by-claim Y_1 follows a geometric distribution with $g_x = \gamma(1-\gamma)^{x-1}, 0 < \gamma < 1, x = 1, 2, \ldots$, so that

$$\tilde{f}(z)\tilde{g}(z) = \frac{(1-\beta)(1-\gamma)z^2}{(1-\beta z)(1-\gamma z)}.$$

Here $n = 2, q_1 = \beta, q_2 = \gamma$, and $E_{n-1}(z) = (1-\beta)(1-\gamma)$. Let R_1, R_2, R_3 be the three roots of the equation

$$z(1-\beta z)(1-\gamma z) - vq(1-\beta z)(1-\gamma z) - vpz^2(1-\beta)(1-\gamma) = 0.$$

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Then we have

$$\tilde{W}(z) = (R_1 R_2 R_3)\Big[z - vp(1-\theta)\beta\gamma z^2\Big]\sum_{i=1}^{3}\frac{r_i}{R_i - z}, \qquad (4.8)$$

where

$$r_1 = \frac{(R_1\beta - 1)(R_1\gamma - 1)}{(R_1 - R_2)(R_1 - R_3)}, r_2 = \frac{(R_2\beta - 1)(R_2\gamma - 1)}{(R_2 - R_1)(R_2 - R_3)}, r_3 = \frac{(R_3\beta - 1)(R_3\gamma - 1)}{(R_3 - R_2)(R_3 - R_1)}.$$

Then (4.6) and (4.7) simplify to

$$V(1;b) = \frac{1}{(R_1 R_2 R_3)\sum_{i=1}^{3} r_i\Big[1 - vp(1-\theta)\beta\gamma R_i\Big](1 - R_i)R_i^{-b}},$$

and

$$V(u;b) = \frac{\sum_{i=1}^{3} r_i\Big[1 - vp(1-\theta)\beta\gamma R_i\Big]R_i^{-u}}{\sum_{i=1}^{3} r_i\Big[1 - vp(1-\theta)\beta\gamma R_i\Big](1 - R_i)R_i^{-b}}, \qquad u = 2,\ldots,b.$$

As an example, let $p = 0.35, v = 0.95, b = 10, \beta = \gamma = 0.8$. From the above results

we get $R_1 = 0.64044, R_2 = 1.01812, R_3 = 1.47972, r_1 = 0.75021, r_2 = -0.19739$, and

$r_3 = 0.08718$. The values of $V(u;10)$ for $\theta = 0, 0.25, 0.5, 0.75, 1$, and $u = 1,\ldots,10$

are listed in Table 3. We observe the same features as in Example 1, that $V(u;b)$ is

an increasing function with respect to u, and a decreasing function over θ. Also, the

impact of the delay of by-claims on the expected present value of dividends is reduced

for a higher initial surplus of the company.

5   The effect of timing of dividends

To further explore the impact of the delay of by-claims on $V(u;b)$, in this section

we consider a different timing for dividend payments, assuming $f_1 = g_1 = 1$. The

decision to pay a dividend to policyholders is now made only at the end of each time

Table 3: Values of $V(u; 10)$ for geometric distributed claims

$V(u; 10)$	$\theta = 0$	0.25	0.5	0.75	1
$u = 1$	0.05164	0.04968	0.04786	0.04617	0.04460
2	0.07264	0.07252	0.07242	0.07232	0.07223
3	0.11652	0.11637	0.11624	0.11612	0.11601
4	0.18535	0.18519	0.18504	0.18490	0.18477
5	0.29301	0.29284	0.29268	0.29253	0.29239
6	0.46121	0.46104	0.46087	0.46072	0.46058
7	0.72391	0.72373	0.72356	0.72340	0.72326
8	1.13409	1.13391	1.13374	1.13358	1.13344
9	1.77455	1.77437	1.77420	1.77404	1.77390
10	2.77455	2.77437	2.77420	2.77404	2.77390

period, by measuring the surplus of the company relative to the barrier b. Thus dividends can only be paid at the end of each time period, even though the surplus of the company at the beginning of the period may have exceeded the barrier b. We will use $V_1(u; b)$ to denote the expected present value of dividend payments up to the time of ruin in this case and a closed-form solution for $V_1(u; b)$ is obtained in the following.

When $\theta = 1$, there are no delays for the by-claims, the model turns to be the one proposed in De Finetti (1957) since the surplus increases or decreases by 1 each period. An explicit expression of $V_1(u; b)$ has been provided in Gerber and Shiu (2004). Thus when $0 < \theta < 1$, the explicit expression for $V_1(u; b)$ will show us the impact of the possible delay of the by-claims on the expected present value of dividend payments.

It can be shown that $V_1(u; b)$ satisfies difference equations

$$V_1(1; b) = vqV_1(2; b) + v^2pq(1 - \theta)V_1(1; b), \tag{5.1}$$

and

$$V_1(u; b) = vqV_1(u + 1; b) + vpV_1(u - 1; b), \qquad u = 2, 3, \ldots, b - 1, \tag{5.2}$$

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with boundary condition:

$$V_1(b; b) \quad = \quad vq[1 + V_1(b; b)] + vpV_1(b - 1; b). \tag{5.3}$$

It is straightforward to show that $V_1(u; b)$ has a closed-form solution with the form of $Ar^u + Bs^u$, where r and s are the roots of the quadratic equation $vqx^2 - x + vp = 0$, and $0 < s < 1 < r$. From (5.1) and (5.3) one can determine the coefficients A and B as follows:

$$A = \frac{r + s\theta}{r^b(r - 1)(r + s\theta) - s^b(s - 1)(s + r\theta)},$$

and

$$B = \frac{s + r\theta}{s^b(s - 1)(s + r\theta) - r^b(r - 1)(r + s\theta)}.$$

Therefore

$$V_1(u; b) = \frac{(r + s\theta)r^u - (s + r\theta)s^u}{r^b(r - 1)(r + s\theta) - s^b(s - 1)(s + r\theta)}, \qquad u = 1, 2, \ldots, b. \tag{5.4}$$

The result of Theorem 3 is still true in this case, and it indicates that if $\theta < 1$, i.e., by-claims can be delayed, result (5.4) is always bigger than the result (A12) of Gerber and Shiu (2004),

$$V_1(u; b) = \frac{r^u - s^u}{r^b(r - 1) - s^b(s - 1)}, \qquad u = 1, 2, \ldots, b,$$

which is a special case of the former result with $\theta = 1$.

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