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The number of overtakes in an $M/M/2$ queue

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ABSTRACT

The phenomenon of overtaking in queueing systems and queueing networks has been addressed by several authors with various motivations in the last decades. Nevertheless, up to now, for the relatively simple $M/M/2/$ FCFS queue, the distribution of the number of overtakes a stationary customer suffers from was not known. In this paper, we characterize this distribution by its probability generating function. As a consequence, we derive the expectation (which is well-known) and the variance.

1. Introduction

For several decades, the phenomenon of overtaking queueing systems has been investigated. In general, 'overtaking' refers to violations of the FIFO-principle (first in, first out). For multiple-server queueing systems with variabilities of the service time, even the FCFS discipline (first come, first served) allows overtaking: Customers with a relatively long service time may be overtaken by customers with a relatively short service time served by another server.

Early work concerning this topic is due to Whitt [1] and Gordon [2]. In [1], a major goal was relating overtakes in queueing networks to sojourn times. In particular, the possibility of overtakes in a queueing network implies dependencies for the sojourn times at the various nodes of the network. In [2], overtaking is related to social injustice/ unfairness since human customers will judge violations of the principle 'first in, first out' as unfair, see also [3].

At the beginning of the 20th century, the problem of quantifying fairness in queueing systems once again received attention. Following the work of [2,3] and relying on studies [4], principles for defining fairness measures were established [5]. In particular, fairness measure should satisfy a *seniority principle*: For a single-node queueing system (no network), if two jobs with the same service requirement arrive at the same time, the job which arrived first should be completed first. A fairness measure should reflect this judgement in the sense that interchanging the order of service of these two jobs should increase the unfairness / decrease the fairness. Amongst other suggestions (slowdown-based measures [6], the resource allocation queueing fairness measure [7]), the discrimination frequency was proposed [8]. For the evaluation of the discrimination frequency, the number of overtakes a customer suffers from has to be determined.

Another consequence of overtaking in queueing systems was brought up by Kim and Lim [9]: Consider a production line with a production station with c parallel servers. Then due to probabilistic service times, even for first-come, first-served scheduling, overtakes might occur at this station. Depending on the type of production, there might be need for re-ordering the products after service at this station, and hence, additional buffer for re-ordering is required. An additional example mentioned in [9] concerns packet-switched communication networks: In order to decrease the sojourn time, a job might be assigned to several servers. Since overtakes can occur, after service completion, a rearranging might be necessary.

In the literature [1,2,9], two kind of overtakes are distinguished: Let us fix a customer whom we will refer to as *tagged customer*. Then *skips* are overtakes he performs on other customers and *slips* are overtakes he suffers from. With respect to fairness and social injustice, slips are more interesting whereas in other applications (e.g. reordering in production lines), the distribution of the number of skips is relevant, see [9].

In some way, the simplest queue in which overtakes might occur is the stationary $M/M/2$/FCFS queue. Whereas the distribution of skips is well-known for this queueing system (see literature review in the next section), the distribution of slips is much more difficult to obtain, and to the knowledge of the authors, no result characterizing the distribution has been published before. In this paper, we find an explicit representation for the generating function and, as a consequence, for the

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variance of slips. In what follows, we will identify the terms 'overtakes' and 'slips'.

2. Problem setting and literature review

For stationary $M/M/c$/FCFS queues with arrival rate λ and service rate μ at each server (with $\rho := \frac{\lambda}{c\mu} < 1$), skips are well-understood (see e.g. [2]): Due to the PASTA property, the tagged customer finds n customers in the system with probability π_n where $\pi = (\pi_n)_{n=0}^{\infty}$ is the invariant distribution of the continuous-time Markov chain $(N_t)_{t\geq 0}$ of the number of customers in the system. For $n < c-1$, the tagged customer's service starts immediately, and we have n candidates for being overtaken by the tagged customer. For $n \geq c-1$, all other servers are busy when our tagged customer's service starts. Hence, we have $c-1$ candidates for being overtaken. Due to the service times being memoryless, the residual service times of each customer being in service is still exponentially distributed with parameter μ when our tagged customer's service starts. His/her service time is exponentially distributed with parameter μ as well, and due to all service times being independent, the number of skips given m candidates for being overtaken is uniformly distributed on $\{0, 1, ..., m\}$. Using total probability, the distribution of the number of skips a stationary customer performs can be calculated as done in [2,9]. In particular, the expected number of skips computes as

$$\frac{1}{2}\left(c\rho - \frac{\frac{(c\rho)^c}{c!}}{(1-\rho)\sum_{k=0}^{c-1}\frac{(c\rho)^k}{k!} + \frac{(c\rho)^c}{c!}}\right),$$

and for the special case of $c = 2$, we obtain $\frac{\lambda}{\lambda+2\mu} = \frac{\rho}{1+\rho}$.

Using ergodic theorems for Markov chains, the stationary number of skips can also be interpreted as the long-run average number of skips performed by successive customers. Since each skip causes a slip for another customer and vice versa, the long-run average of number of skips and slips coincide. Hence, the above terms also give the expected number of slips for a stationary customer in an $M/M/c$ queue or an $M/M/2$ queue, respectively. In the context of the above-mentioned fairness measures, this fact was used in [10] for determining the expected discrimination frequency. While this equality is true for the expectation, it cannot be true for the distribution or higher moments. As a simple argument, note that the number of skips is bounded by $c-1$ whereas the number of slips is unbounded.

To the knowledge of the authors, in no previous work, explicit formulas for the distribution of the stationary number of slips or even for its variance have been derived, not even for $c = 2$. In [9], a recursion scheme for a generating function related to slips has been established, but no explicit solution was obtained. The only explicit result there was the probability that the tagged customer is overtaken by the next customer, which is given by

$$\frac{c\rho}{2(c\rho+1)}\left(1 - \frac{1}{c^2\rho}\frac{\frac{(c\rho)^c}{c!}}{(1-\rho)\sum_{k=0}^{c-1}\frac{(c\rho)^k}{k!} + \frac{(c\rho)^c}{c!}}\right).$$

For $c = 2$, the next customer is the first candidate to overtake the tagged customer, and hence, the probability of being overtaken by the next customer coincides with the probability of suffering from at least one slip. Simplifying the above term for $c = 2$ results in

$$\mathbb{P}(OV > 0) = \frac{\rho(2+\rho)}{2(1+\rho)(1+2\rho)}. \tag{1}$$

In [2, section 5.2], the distribution of the number of slips in $M/M/c$ queues was derived under the additional assumption of full saturation (all other servers are busy) during the tagged customer's service. As we will point out below, this assumption simplifies the analysis of the number of slips significantly. In this paper, we will present a new approach which allows us to derive an explicit representation of the probability generating function for the stationary number OV of slips in an $M/M/2$/FCFS queue without any additional assumptions. By standard arguments (derivations in 1), we are able to obtain the variance.

Naturally, in an $M/M/2$ queue, we only have to consider the departure pattern at the server which our tagged customer is not served at, which we will refer to server S. Things would be quite easy if server S never runs empty since then the interdeparture times at server S coincide with the service times. Due to the tagged customer's service time being memoryless, every customer served at server S will leave the system before our tagged customer with probability $\frac{1}{2}$. If server S is busy when our tagged customer's service starts, the first departure of a customer at server S is no overtake. Conversely, with probability $\frac{1}{2}$, the tagged customer leaves the system before the first departure at server S occurs and therefore, the tagged customer performs an overtake himself. If the tagged customer leaves the system between the first and the second departure at server S, the tagged customer neither performs a skip nor suffers from a slip. If the tagged customer leaves the system between the $\ell+1$st and $\ell+2$nd departure at server S, he is overtaken by exactly ℓ customers. Under the condition of server S never running empty, the latter happens with probability $\left(\frac{1}{2}\right)^{\ell+1}\cdot\frac{1}{2} = \frac{1}{2^{\ell+2}}$. Hence, in this case we should expect

$$\mathbb{P}(OV = 0) = \frac{3}{4} \quad \text{and} \quad \mathbb{P}(OV = \ell) = \frac{1}{2^{\ell+2}}, \quad \ell \geq 1$$

with probability generating function

$$\begin{aligned} G(z) &= \sum_{\ell=0}^{\infty}\mathbb{P}(OV = \ell) = \frac{3}{4} + \sum_{n=1}^{\infty}\frac{z^n}{2^{n+2}} = \frac{3}{4} + \frac{1}{4}\cdot\frac{\frac{z}{2}}{1-\frac{z}{2}} \\ &= \frac{3}{4} + \frac{1}{4}\cdot\frac{z}{2-z} \\ &= \frac{3(2-z)+z}{4(2-z)} = \frac{3-z}{2(2-z)}. \end{aligned} \tag{2}$$

Unfortunately, for stable $M/M/2$ queues, that is, $\rho = \frac{\lambda}{2\mu} < 1$, server S will run empty during the tagged customer's service with a positive probability. Then the departure pattern at server S also depends on the arrival times, and things get complicated. Finding a way for dealing with this problem is the main issue of this paper. Nevertheless, the considerations leading to (2) provide a plausibility check for our main result since (2) should be the limit for $\rho \to 1-$.

We conclude this introductory section with summarizing our problem setting and the variables which we use throughout this paper.

- We consider an $M/M/2$/FCFS queue with arrival rate λ, service rate μ at both stations, and we assume $\rho = \frac{\lambda}{2\mu} < 1$ in order to guarantee stability.
- N is the stationary number of customers in the system. Due to the PASTA property [11, Theorem VII.6.7] this distribution coincides with the number of customers found by the arriving tagged customer. As is well-known [12], $\pi_n := \mathbb{P}(N = n)$ is given by $\pi_n = 2\rho^n\pi_0$ for $n \geq 1$ and $\pi_0 = \frac{1-\rho}{1+\rho}$.
- OV is the number of overtakes (slips) the tagged customer suffers from. By G, we denote the corresponding probability generating function

$$G(z) = \sum_{\ell=0}^{\infty}\mathbb{P}(OV = \ell)z^{\ell}.$$

- By L, we denote the number of customers which depart during the tagged customer's service time.
- By B, we denote the number of customers arriving during the tagged customer's waiting time.
- By M, we denote the number of customers in the system when the tagged customer's service begins, excluding the tagged customer.

3. Main results

The distribution of a random variable X taking values in {0, 1, 2, ...} is uniquely determined by its probability generating function G_X where $G_X(z) = \mathbb{E}[z^X] = \sum_{n=0}^{\infty} \mathbb{P}(X = n)z^n$; convergence is guaranteed for $|z| \le 1$. Given $G_X(z)$, we find the probabilities $\mathbb{P}(X = n)$ by expanding $G(z)$ into its Taylor series in 0 (or equivalently, by determining the derivatives in 0 and scaling with $n!$). However, a major benefit of the probability generating function is the easy determination of moments: In case of existence, the kth factorial moment computes as

$$\mathbb{E}[X(X-1)...(X-k+1)] = G_X^{(k)}(1).$$

In particular, for $\mathbb{E}[X^2] < \infty$, the variance can be obtained via

$$\mathbf{VAR}[X] = G_X''(1) + G_X'(1) - (G_X'(1))^2.$$

In total, characterizing the distribution by means of the generating function is sufficient for most practical purposes. In our main result, we provide an explicit representation for the generating function of the number of overtakes (slips) in an $M/M/2$ queue.

Theorem 3.1. *Let G be the generating function for the stationary number OV of overtakes in an $M/M/2$ queue. Then*

$$G(z) = 1 + \frac{1-z}{(1+\rho)(2-z)}((1-\rho)G_0(z) - 1).$$

with

$$G_0(z) = \frac{1}{2-z} + \frac{1-z}{2-z}\cdot\frac{\sqrt{(1+\rho)^2 - 2\rho z} - \rho}{1 + 2\rho(1-z)} = \frac{1 + (1-z)(\rho + \sqrt{(1+\rho)^2 - 2\rho z})}{(2-z)(1+2\rho(1-z))}.$$

3.1. Plausibility checks

We have $G_0(0) = \frac{1+\rho}{1+2\rho}$, yielding $G(0) = 1 + \frac{1-\rho^2-(1+2\rho)}{2(1+\rho)(1+2\rho)} = \frac{2+4\rho+3\rho^2}{2(1+\rho)(1+2\rho)}$, that is,

$$\mathbb{P}(OV > 0) = 1 - \frac{2+4\rho+3\rho^2}{2(1+\rho)(1+2\rho)} = \frac{\rho(2+\rho)}{2(1+\rho)(1+2\rho)}.$$

This result coincides with (1) obtained from [9].

Next, consider the limit behaviour for $\rho \to 1-$ (heavy traffic). As in the introductory section, let server S be the server which does not serve the tagged customer. For $\rho \to 1-$, the probability of server S being busy when the tagged customer's service starts converges to 1. Furthermore the probability that server S runs empty during the tagged customer's service converges to 0. In total, by means of the considerations in the Section 2, the limit behaviour should correspond to (2), that is

$$G(z) \to \frac{3-z}{2(2-z)}$$

as $\rho \to 1-$. Indeed, our main result provides

$$\lim_{\rho\to 1-} G(z) = 1 + \frac{1-z}{2(2-z)}\cdot(-1) = \frac{3-z}{2(2-z)}.$$

Finally, consider $\rho \to 0+$. Then the probability that any other customer is served during the tagged customer's service time converges to 0. Hence, we should have $\mathbb{P}(OV = 0) \to 1$, that is, $G(z) \to 1$ as $\rho \to 0+$. Indeed, Theorem 3.1 guarantees $\lim_{\rho\to 0+} G_0(z) = \frac{1+(1-z)\cdot 1}{(2-z)\cdot 1} = 1$ and hence,

$$\lim_{\rho\to 0+} G(z) = 1 + \frac{1-z}{2-z}\lim_{\rho\to 0+}(G_0(z) - 1) = 1.$$

3.2. Expectation and variance

Using Theorem 3.1, moments of OV can be determined. Here, we focus on $\mathbb{E}[OV]$ and $\mathbf{VAR}[OV]$. For this purpose, we first differentiate G_0'. Note that $\frac{d}{dz}\frac{1}{2-z} = \frac{1}{(2-z)^2}$ and $\frac{d}{dz}\frac{1-z}{2-z} = \frac{d}{dz}\left(1 - \frac{1}{2-z}\right) = -\frac{1}{(1-z)^2}$. Therefore,

$$G_0'(z) = \frac{1}{(2-z)^2} - \frac{1}{(2-z)^2}\cdot\frac{\sqrt{(1+\rho)^2 - 2\rho z} - \rho}{1+2\rho(1-z)} + \frac{1-z}{2-z}\cdot\frac{d}{dz}\frac{\sqrt{(1+\rho)^2 - 2\rho z} - \rho}{1+2\rho(1-z)}.$$

Hence, we obtain $G_0(1) = 1$ and

$$G_0'(1) = 1 - 1\cdot\frac{\sqrt{1+\rho^2} - \rho}{1} + 0 = 1 + \rho - \sqrt{1+\rho^2}.$$

Now, we consider the function G itself. We have

$$G'(z) = \frac{1}{(2-z)^2}\cdot\frac{1-(1-\rho)G_0(z)}{1+\rho} + \frac{1-z}{2-z}\cdot\frac{1-\rho}{1+\rho}G_0'(z) \quad \text{and}$$

$$G''(z) = \frac{2}{(2-z)^3}\cdot\frac{1-(1-\rho)G_0(z)}{1+\rho} - \frac{2}{(2-z)^2}\cdot\frac{1-\rho}{1+\rho}G_0'(z) + \frac{1-z}{2-z}\cdot\frac{1-\rho}{1+\rho}G_0''(z).$$

In particular, we have

$$\mathbb{E}[OV] = G'(1) = \frac{1-(1-\rho)G_0(1)}{1+\rho} = \frac{\rho}{1+\rho},$$

coinciding with the results for skips from [2,9]. Furthermore, we have $G''(1) = \frac{2\rho}{1+\rho} - \frac{2(1-\rho)(1+\rho-\sqrt{1+\rho^2})}{1+\rho}$, yielding $\mathbb{E}[OV^2] = \frac{3\rho - 2 + 2\rho^2 + 2(1-\rho)\sqrt{1+\rho^2}}{1+\rho}$, and finally

$$\begin{aligned}\mathbf{Var}[OV] &= \frac{(1+\rho)(2\rho^2+3\rho-2) - \rho^2 + 2(1-\rho)(1+\rho)\sqrt{1+\rho^2}}{(1+\rho)^2} \\ &= \frac{2\rho^3 + 4\rho^2 + \rho - 2 + 2(1-\rho^2)\sqrt{1+\rho^2}}{(1+\rho)^2} \\ &= \frac{\rho}{(1+\rho)^2} + 2\cdot\frac{\rho^2 + (1-\rho)(\sqrt{1+\rho^2} - 1)}{1+\rho}.\end{aligned}$$

To the knowledge of the authors, this result is completely new.

3.3. Comparison to the number of skips

As pointed out in the introduction, it is quite easy to determine the distribution of the number of skips a stationary customer performs. For the $M/M/2$/FCFS model, this number is either 0 or 1 with expectation $\frac{\rho}{1+\rho}$. Hence, $\frac{\rho}{1+\rho}$ is the probability for one skip and the variance computes as

$$\frac{\rho}{1+\rho} - \frac{\rho^2}{(1+\rho)^2} = \frac{\rho}{(1+\rho)^2}.$$

Since $2\cdot\frac{\rho^2 + (1-\rho)(\sqrt{1+\rho^2}-1)}{1+\rho} > 0$, the variance of overtakes (in the sense of slips) is larger than that of skips. As Fig. 1 demonstrates, this difference is significant for large ρ.

4. Proof of Theorem 3.1

Due to only having two servers, we have a deterministic relationship between the number L of customers departing during the tagged customer's service time and the number OV of overtakes. It is given by

$$OV = \begin{cases} L, & N = 0, \\ \max\{L-1, 0\}, & N \ge 1, \end{cases}$$

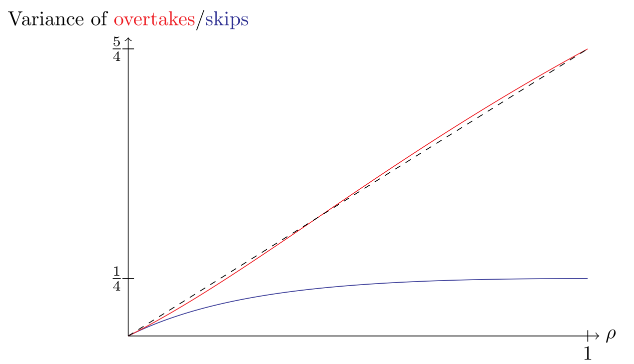

Fig. 1. Comparison of the variances of the number of overtakes and the number of skips. In order to demonstrate that $\mathbf{VAR}[OV]$ is a non-linear function in ρ, we have additionally depicted the graph of $\rho \mapsto \frac{5}{4}\rho$ as a dashed line.

since for $N \geq 1$, the service of the first departing customer began before the tagged customer's service.

Furthermore, we consider the number B of customers arriving during the tagged customer's waiting time. Obviously, we have $B = 0$ for $N \leq 1$ since there is no need for waiting. For $N = n \geq 2$, the tagged customer has to wait for $n - 1$ service completions. We have $B = b$ if and only if exactly b arrivals occur before the $n - 1$st service completion. Due to the properties of the exponential distribution, the next 'event' is an arrival with probability $\frac{\lambda}{\lambda+2\mu} = \frac{\rho}{1+\rho}$, and the next 'event' is a service completion with probability $\frac{2\mu}{\lambda+2\mu} = \frac{1}{1+\rho}$. Hence, $B|N = n \geq 2$ follows a negative binomial distribution, we have

$$\mathbb{P}(B = b|N = n) = \binom{b+n-2}{b}\left(\frac{\rho}{1+\rho}\right)^b\left(\frac{1}{1+\rho}\right)^{n-1}, \qquad b \in \mathbb{N}_0, n \geq 2.$$

4.1. The number of departures

Using the distributions of N and B, the distribution of M can be determined, where M is the number of customers in the system when the tagged customer's service begins, that is,

$$M = \begin{cases} N & \text{if } N \in \{0, 1\} \\ B & \text{if } N \geq 2 \end{cases}$$

Since the number L of departures during the tagged customer's service time is directly related to the number OV of overtakes, the key problem is determining the probabilities

$$\psi_m(\ell) = \mathbb{P}(L = \ell|M = m).$$

For $M = m > \ell$, we have $L = \ell$ if and only if the server which does not serve the tagged customer completes ℓ services before the tagged customer leaves the system. For each new service beginning at the other server, the probability for completing this service before the tagged customer's service is completed is given by $\frac{1}{2}$. Since ℓ services have to be completed before the tagged customer leaves the system whereas the $\ell + 1$st service has to be completed later, we find

$$\psi_m(\ell) = \frac{1}{2^{\ell+1}}, \qquad m > \ell. \tag{3}$$

For $m \leq \ell$, this result does not hold true since the other server can run empty before it begins the $\ell + 1$st service. Hence, for small m, the probability of observing ℓ departures before the tagged customer leaves the system becomes smaller. To be precise, we note that $\psi_m(\ell)$ monotonically increases in m (in the non-strict sense).

We will not derive explicit formulas for all $\psi_m(\ell)$, but it turns out that recurrence relations can be found which allow to prove our main result. For deriving these recurrence relations, we introduce an

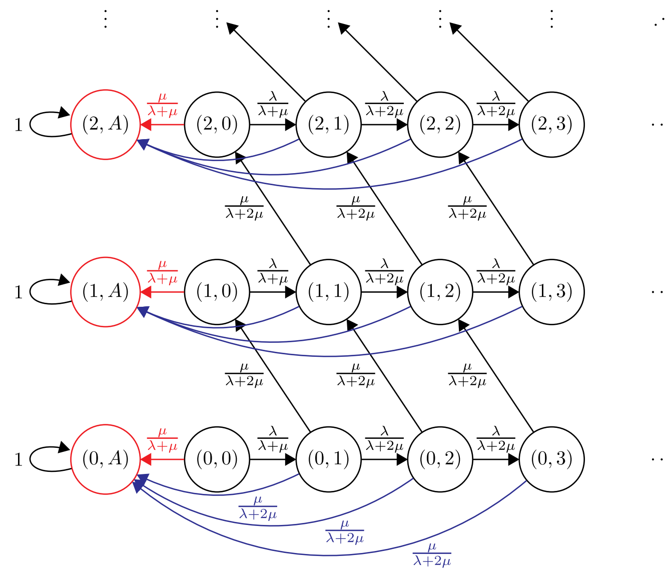

Fig. 2. The Markov chain describing the behaviour of our queue during the time our customer is at the server. The transition probabilities for the transitions symbolized by the blue lines are given by $\frac{\mu}{\lambda+2\mu}$. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

absorbing homogeneous discrete-time Markov chain $(J_n, K_n)_{n=0}^{\infty}$ with state space $\{0, 1, 2, ...\} \times \{A, 0, 1, 2, ...\}$, where $J_0 = 0$, $K_0 = M$ and

- $K_n = k \geq 0$ indicates that the tagged customer is still in the system after the nth 'event' (that is, arrival or departure), and that k other customers are in the system at this time,
- $K_n = A$ indicates that the tagged customer has left the system,
- for $K_n \geq 0$, J_n is the number of departures up to the nth event,
- for $K_n = A$, $J_n = L$ is the number of departures before the tagged customer leaves the system .

Clearly, the memoryless property of the exponential distribution allows this construction, and the state (j, A) with $j \in \{0, 1, 2, ...\}$ become absorbing. For $K_n = 0$, the next event is either an arrival or the tagged customer's departure. By standard arguments for comparison of exponentially distributed random variables, the corresponding probabilities compute as $\frac{\lambda}{\lambda+\mu}$ and $\frac{\mu}{\lambda+\mu}$, respectively. For $K_n > 0$, the next event can be an arrival, the tagged customer's departure or the departure of the customer served at the other server. The corresponding probabilities are $\frac{\lambda}{\lambda+2\mu}$, $\frac{\mu}{\lambda+2\mu}$ and $\frac{\mu}{\lambda+2\mu}$, respectively. In total, the transition probabilities can be illustrated by the Markov graph depicted in Fig. 2.

From state (k, ℓ) with $k \geq 0$, the probability for a transition to (A, ℓ) is at least $\frac{\mu}{\lambda+2\mu}$. Therefore, the probability that no absorption occurs until time n is bounded by $\left(1 - \frac{\mu}{\lambda+2\mu}\right)^n$. Hence, for $n \to \infty$ this probability converges, that is, eventual absorption is almost sure. An absorption in state (ℓ, A) means that ℓ departures have occured before the tagged customer leaves the system, as we start in some state of the form $(0, \cdot)$. Thus we can relate the probability $\psi_m(\ell) = \mathbb{P}(L = \ell|M = m)$, which is the probability that l departures occur before the tagged customer leaves the system given that m customers where in the system before the service started, with the Markov chain. More precisely, this probability is given as the probability that an absorption in state (ℓ, A) happens given that the Markov chain starts in $(0, m)$. Using this interpretation we will now derive a recurrence scheme for $\phi_m(\ell)$ making extensive use of the repetitive structure of the transitions in the CTMC: The probability for an absorption in state (ℓ, A) subject to the initial value $(1, m)$ coincides with the probability for an absorption in state

$(\ell-1, A)$ subject to the initial value $(0, m)$. For $m \geq 1$ and $\ell \geq 1$, there are two transitions which allow absorption in (ℓ, A):

- With probability $\frac{\mu}{\lambda+2\mu}$, a transition to $(1, m-1)$ occurs. According to the remark concerning the repetitive structure, then the absorption will occur in (ℓ, A) with probability $\psi_{m-1}(\ell-1)$.
- With probability $\frac{\lambda}{\lambda+2\mu}$, a transition to $(0, m+1)$ occurs. After this transition, the absorption will occur in (ℓ, A) with probability $\psi_{m+1}(\ell)$.

Therefore, we obtain

$$\psi_m(\ell) = \frac{\mu}{\lambda+2\mu}\psi_{m-1}(\ell-1) + \frac{\lambda}{\lambda+2\mu}\psi_{m+1}(\ell)$$

for $m \geq 1$ and $\ell \geq 1$. An absorption in level $\ell = 0$ can occur in the first step, whereas it cannot occur once the Markov chain has reached level 1. Therefore, we have

$$\psi_m(0) = \frac{\mu}{\lambda+2\mu} + \frac{\lambda}{\lambda+2\mu}\psi_{m+1}(0).$$

For $m = 0$, a direct transition to level 1 is not possible. Note that this fact causes slightly different transition probabilities, we have

$$\psi_0(\ell) = \frac{\lambda}{\lambda+\mu}\psi_1(\ell)$$

for $\ell \geq 1$ and

$$\psi_0(0) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\psi_1(0).$$

Using $\rho = \frac{\lambda}{2\mu}$, we obtain

$$\psi_m(\ell) = \begin{cases} \frac{1}{2}\cdot\frac{1}{1+\rho}\psi_{m-1}(\ell-1) + \frac{\rho}{1+\rho}\psi_{m+1}(\ell), & m \geq 1,\ \ell \geq 1,\\ \frac{1}{2}\cdot\frac{1}{1+\rho} + \frac{\rho}{1+\rho}\psi_{m+1}(0), & m \geq 1,\ \ell = 0,\\ \frac{2\rho}{1+2\rho}\psi_1(\ell), & m = 0,\ \ell \geq 1,\\ \frac{1}{1+2\rho} + \frac{2\rho}{1+2\rho}\psi_1(0), & m = \ell = 0. \end{cases}$$

4.2. Equations for $\mathbb{P}(OV = \ell)$

Next, we want to use the equations for $\psi_m(\ell)$ for deriving equations for $\mathbb{P}(OV = \ell)$. We begin with considering $OV|M = m$. For $M = 0$, we have $OV = L$, whereas for $M \geq 1$, we have $OV = \max\{L-1, 0\}$, and hence, we obtain

$$\begin{aligned} \mathbb{P}(OV = \ell|M = 0) &= \psi_0(\ell),\\ \mathbb{P}(OV = 0|M = m) &= \psi_m(0) + \psi_m(1), \qquad m \geq 1,\\ \mathbb{P}(OV = \ell|M = m) &= \psi_m(\ell+1), \qquad m \geq 1,\ \ell \geq 1. \end{aligned}$$

Now, instead of conditions $M = m$, we want to consider conditions $N = n$. Note that for $N \leq 1$, we have $M = N$ since the tagged customer's service starts immediately on arrival. For $N \geq 2$, we have

$$\begin{aligned} \mathbb{P}(OV = \ell|N = n) &= \sum_{b=0}^{\infty}\mathbb{P}(B = b|N = n)\mathbb{P}(OV = \ell|N = n, B = b)\\ &= \sum_{b=0}^{\infty}\mathbb{P}(B = b|N = n)\mathbb{P}(OV = \ell|M = b+1). \end{aligned}$$

In total, we obtain

$$\begin{aligned} \mathbb{P}(OV = \ell|N = 0) &= \psi_0(\ell), \qquad \ell \in \mathbb{N}_0\\ \mathbb{P}(OV = 0|N = 1) &= \psi_1(0) + \psi_1(1),\\ \mathbb{P}(OV = \ell|N = 1) &= \psi_1(\ell+1), \qquad \ell \geq 1,\\ \mathbb{P}(OV = 0|N = n) &= \sum_{b=0}^{\infty}\binom{b+n-2}{b}\left(\frac{\rho}{1+\rho}\right)^b\left(\frac{1}{1+\rho}\right)^{n-1}\cdot\\ &\qquad (\psi_{b+1}(0) + \psi_{b+1}(1)), \qquad n \geq 2,\\ \mathbb{P}(OV = \ell|N = n) &= \sum_{b=0}^{\infty}\binom{b+n-2}{b}\left(\frac{\rho}{1+\rho}\right)^b\left(\frac{1}{1+\rho}\right)^{n-1}\cdot\\ &\qquad \psi_{b+1}(\ell+1),\\ &\qquad n \geq 2,\ \ell \geq 1. \end{aligned}$$

Finally, we use total probability. First, we consider $\mathbb{P}(OV = \ell)$ and having

$$\begin{aligned} \sum_{n=2}^{\infty}\binom{b+n-2}{b}\left(\frac{\rho}{1+\rho}\right)^{n-2} &= \sum_{n=0}^{\infty}\binom{b+n}{n}\left(\frac{\rho}{1+\rho}\right)^n\\ &= \sum_{n=0}^{\infty}\binom{-(b+1)}{n}\left(-\frac{\rho}{1+\rho}\right)^n\\ &= \left(1-\frac{\rho}{1+\rho}\right)^{-(b+1)} = (1+\rho)^{b+1} \end{aligned}$$

in mind, we obtain

$$\begin{aligned} \mathbb{P}(OV = 0) &= \sum_{n=0}^{\infty}\pi_n\mathbb{P}(OV = 0|N = n)\\ &= \pi_0\psi_0(0) + \pi_0\cdot 2\rho(\psi_1(0) + \psi_1(1))\\ &\quad + \pi_0\cdot 2\sum_{n=2}^{\infty}\rho^n\sum_{b=0}^{\infty}\binom{b+n-2}{b}\left(\frac{\rho}{1+\rho}\right)^b\left(\frac{1}{1+\rho}\right)^{n-1}\cdot\\ &\quad (\psi_{b+1}(0) + \psi_{b+1}(1))\\ &= \pi_0\psi_0(0) + \pi_0\cdot 2\rho(\psi_1(0) + \psi_1(1))\\ &\quad + \pi_0\cdot 2\rho\sum_{b=0}^{\infty}\left(\frac{\rho}{1+\rho}\right)^{b+1}(\psi_{b+1}(0)\\ &\quad + \psi_{b+1}(1))\sum_{n=2}^{\infty}\binom{b+n-2}{n-2}\left(\frac{\rho}{1+\rho}\right)^{n-2}\\ &= \pi_0\psi_0(0) + \pi_0\cdot 2\rho(\psi_1(0) + \psi_1(1))\\ &\quad + \pi_0\cdot 2\rho\sum_{b=0}^{\infty}\left(\frac{\rho}{1+\rho}\right)^{b+1}(\psi_{b+1}(0) + \psi_{b+1}(1))(1+\rho)^{b+1}\\ &= \pi_0\Bigg(\psi_0(0) + 2\rho(\psi_1(0) + \psi_1(1))\\ &\quad + 2\rho^2\sum_{b=0}^{\infty}\rho^b(\psi_{b+1}(0) + \psi_{b+1}(1))\Bigg) \end{aligned}$$

Analogously, for $n \geq 1$, we derive

$$\begin{aligned} \mathbb{P}(OV = \ell) &= \pi_0\Bigg(\psi_0(\ell) + 2\rho\psi_1(\ell+1)\\ &\quad + 2\rho^2\sum_{n=2}^{\infty}\rho^{n-2}\sum_{b=0}^{\infty}\binom{b+n-2}{b}\left(\frac{\rho}{1+\rho}\right)^b\left(\frac{1}{1+\rho}\right)^{n-1}\\ &\quad \psi_{b+1}(\ell+1)\Bigg)\\ &= \pi_0\left(\psi_0(\ell) + 2\rho\psi_1(\ell+1) + 2\rho^2\sum_{b=0}^{\infty}\rho^b\psi_{b+1}(\ell+1)\right)\\ &= \pi_0\left(\psi_0(\ell) + 2\rho(1+\rho)\psi_1(\ell+1) + 2\rho\sum_{b=2}^{\infty}\rho^b\psi_b(\ell+1)\right). \end{aligned}$$

4.3. Generating functions and a difference equation

Define

$$G_m(z) = \sum_{\ell=0}^{\infty} \psi_m(\ell) z^{\ell}.$$

The equations for $\psi_0(\ell)$ directly lead to

$$\begin{aligned} G_0(z) &= \frac{1}{1+2\rho} + \frac{2\rho}{1+2\rho}\psi_1(0) + \sum_{\ell=1}^{\infty} \frac{2\rho}{1+2\rho}\psi_1(\ell) z^{\ell} \\ &= \frac{1}{1+2\rho} + \frac{2\rho}{1+2\rho} G_1(z). \end{aligned}$$

For $m \geq 1$, we obtain

$$\begin{aligned} G_m(z) &= \frac{1}{2}\cdot\frac{1}{1+\rho} + \frac{\rho}{1+\rho}\psi_{m+1}(0) + \frac{1}{2}\cdot\frac{1}{1+\rho}\sum_{\ell=1}^{\infty}\psi_{m-1}(\ell-1)z^{\ell} \\ &\quad + \frac{\rho}{1+\rho}\sum_{\ell=1}^{\infty}\psi_{m+1}(\ell)z^{\ell} \\ &= \frac{1}{2(1+\rho)} + \frac{z}{2(1+\rho)}G_{m-1}(z) + \frac{\rho}{1+\rho}G_{m+1}(z). \end{aligned} \tag{4}$$

Hence, for any z with $|z| \leq 1$, $G_m(z)$ satisfies a linear (inhomogeneous) difference equation with constant coefficients. It will turn out that the definition of $G_0(z)$ corresponds to the use in the main result Theorem 3.1.

4.4. Solving the difference equation

For $m > \ell$, we have $\psi_m(\ell) = \frac{1}{2^{\ell+1}}$. Furthermore, remember that $\psi_m(\ell)$ monotonically increases in m. Hence, for $z \in [0, 1]$ monotone convergence guarantees that

$$\lim_{m\to\infty} G_m(z) = \sum_{\ell=0}^{\infty}\lim_{m\to\infty}\psi_m(\ell)z^{\ell} = \sum_{\ell=0}^{\infty}\frac{z^{\ell}}{2^{\ell+1}} = \frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}} = \frac{1}{2-z}. \tag{5}$$

By using dominated convergence, this result extends to all $z \in \mathbb{C}$ with $|z| \leq 1$.

Indeed, $m \mapsto \frac{1}{2-z}$ solves the difference equation (4), that is, the constant sequence $\left(\frac{1}{2-z}\right)_{m=0}^{\infty}$ is a solution. The general theory of inhomogeneous linear difference equations with constant coefficients guarantees that each solution has the form

$$G_m(z) = \frac{1}{2-z} + \gamma x^m + \delta y^m,$$

where γ, δ, x, y are constant in m (but not in z), and x, y are the roots of the characteristic polynomial

$$\frac{\rho}{1+\rho}x^2 - x + \frac{z}{2(1+\rho)}.$$

Solving the corresponding quadratic equation, we find $x = \frac{1+\rho-\sqrt{(1+\rho)^2-2\rho z}}{2\rho} \in (0, 1)$ and $y = \frac{1+\rho+\sqrt{(1+\rho)^2-2\rho z}}{2\rho} > 1$. Since $G_m(z)$ converges to $\frac{1}{2-z}$, we obtain $\delta = 0$, and the initial condition yields

$$\frac{1}{2-z} + \gamma = \frac{1}{1+2\rho} + \frac{2\rho}{1+2\rho}\cdot\frac{1}{2-z} + \frac{2\rho}{1+2\rho}\gamma x,$$

that is,

$$\begin{aligned} \gamma &= \frac{\frac{1}{1+2\rho} - \left(1 - \frac{2\rho}{1+2\rho}\right)\frac{1}{2-z}}{1 - \frac{2\rho x}{1+2\rho}} = \frac{1 - \frac{1}{2-z}}{1+2\rho-2\rho x} \\ &= \frac{1-z}{2-z}\cdot\frac{1}{1+2\rho-(1+\rho)+\sqrt{(1+\rho)^2-2\rho z}} \\ &= \frac{1-z}{2-z}\cdot\frac{1}{\rho+\sqrt{(1+\rho)^2-2\rho z}} = \frac{1-z}{2-z}\cdot\frac{\rho-\sqrt{(1+\rho)^2-2\rho z}}{\rho^2-(1+\rho)^2+2\rho z} \\ &= \frac{1-z}{2-z}\cdot\frac{\rho-\sqrt{(1+\rho)^2-2\rho z}}{-1-2\rho+2\rho z} = \frac{1-z}{2-z}\cdot\frac{\sqrt{(1+\rho)^2-2\rho z}-\rho}{1+2\rho(1-z)} \end{aligned}$$

In particular, $G_0(z) = \frac{1}{2-z} + \gamma(z)$ corresponds to the representation of $G_0(z)$ in Theorem 3.1.

4.5. Relating $G(z)$ and $G_0(z)$

Finally, we want to relate $G_0(z)$ and

$$G(z) = \sum_{\ell=0}^{\infty}\mathbb{P}(OV = \ell)z^{\ell}.$$

From above, we use

$$\begin{aligned} \mathbb{P}(OV = 0) &= \pi_0\Bigg(\psi_0(0) + 2\rho(\psi_1(0) + \psi_1(1)) \\ &\quad + 2\rho\sum_{m=1}^{\infty}\rho^m(\psi_m(0) + \psi_m(1))\Bigg), \\ \mathbb{P}(OV = \ell) &= \pi_0\left(\psi_0(\ell) + 2\rho\psi_1(\ell+1) + 2\rho\sum_{m=1}^{\infty}\rho^m\psi_m(\ell+1)\right). \end{aligned}$$

Furthermore, (3) guarantees $\psi_m(0) = \frac{1}{2}$ for $m \geq 1$, implying $\sum_{\ell=0}^{\infty}\psi_m(\ell+1)z^{\ell} = \frac{1}{z}\left(G_m(z) - \frac{1}{2}\right)$. Using these facts, we obtain

$$\begin{aligned} \frac{G(z)}{\pi_0} &= \left(\psi_0(0) + 2\rho(\psi_1(0) + \psi_1(1)) + 2\rho\sum_{m=1}^{\infty}\rho^m(\psi_m(0) + \psi_m(1))\right) \\ &\quad + \sum_{\ell=1}^{\infty}\left(\psi_0(\ell) + 2\rho\psi_1(\ell+1) + 2\rho\sum_{m=1}^{\infty}\rho^m\psi_m(\ell+1)\right)z^{\ell} \\ &= G_0(z) + 2\rho\psi_1(0) + 2\rho\sum_{\ell=0}^{\infty}\psi_1(\ell+1)z^{\ell} + 2\rho\sum_{m=1}^{\infty}\rho^m\psi_m(0) \\ &\quad + 2\rho\sum_{m=1}^{\infty}\rho^m\sum_{\ell=0}^{\infty}\psi_m(\ell+1)z^{\ell} \\ &= G_0(z) + \rho + \frac{2\rho}{z}\left(G_1(z) - \frac{1}{2}\right) + \frac{\rho^2}{1-\rho} \\ &\quad + \frac{2\rho}{z}\sum_{m=1}^{\infty}\rho^m\left(G_m(z) - \frac{1}{2}\right) \\ &= G_0(z) + \frac{2\rho}{z}G_1(z) + \frac{2\rho}{z}\sum_{m=1}^{\infty}\rho^m G_m(z) - \frac{\rho}{1-\rho}\cdot\frac{1-z}{z} \end{aligned} \tag{6}$$

$$= G_0(z) + \frac{2\rho(1+\rho)}{z}G_1(z) + \frac{2\rho}{z}\sum_{m=1}^{\infty}\rho^m\cdot\rho G_{m+1}(z) - \frac{\rho}{1-\rho}\cdot\frac{1-z}{z} \tag{7}$$

Restarting with the representation (6), we also find

$$\begin{aligned}\frac{G(z)}{\pi_0} &= G_0(z) - \frac{2\rho}{z}G_0(z) + \frac{2\rho}{z}G_1(z) + \frac{2\rho}{z}\sum_{m=1}^{\infty}\rho^{m-1}G_{m-1}(z) \\ &\quad - \frac{\rho}{1-\rho}\cdot\frac{1-z}{z} \\ &= \frac{z-2\rho}{z}G_0(z) + \frac{2\rho}{z}G_1(z) + \frac{2\rho}{z}\sum_{m=1}^{\infty}\rho^{m}\cdot\frac{1}{\rho}G_{m-1}(z) \\ &\quad - \frac{\rho}{1-\rho}\cdot\frac{1-z}{z}. \end{aligned} \tag{8}$$

Multiplying (6) by $2 + 2\rho$, (7) by -2 and (8) by $-z\rho$ and summing up the resulting equations leads to

$$\begin{aligned}\rho\cdot\frac{(2-z)G(z)}{\pi_0} &= \frac{(2+2\rho)G(z) - 2G(z) - z\rho G(z)}{\pi_0} \\ &= (2 + 2\rho - 2 - \rho(z-2\rho))G_0(z) \\ &\quad + \left(\frac{4(1+\rho)\rho}{z} - \frac{4\rho(1+\rho)}{z} - 2\rho^2\right)G_1(z) \\ &\quad + \frac{2\rho}{z}\sum_{m=1}^{\infty}\rho^m((2+2\rho)G_m(z) - 2\rho G_{m+1}(z) \\ &\quad - zG_{m-1}(z)) \\ &\quad - \frac{\rho}{1-\rho}\cdot\frac{1-z}{z}\cdot(2-z)\rho.\end{aligned}$$

Since $G_m(z)$ satisfies the difference equation (4) and the factor ρ occurs on both sides of the equation, we can simplify this term to

$$\begin{aligned}\frac{(2-z)G(z)}{\pi_0} &= (2 + 2\rho - z)G_0(z) - 2\rho G_1(z) + \frac{2\rho}{z(1-\rho)} \\ &\quad - \frac{\rho}{1-\rho}\cdot\frac{2-3z+z^2}{z} \\ &= (1-z)G_0(z) + 1 + \frac{\rho(3-z)}{1-\rho} = (1-z)G_0(z) \\ &\quad + \frac{1+\rho(2-z)}{1-\rho},\end{aligned}$$

and with $\pi_0 = \frac{1-\rho}{1+\rho}$, we finally derive

$$\begin{aligned}G(z) &= \frac{1-\rho}{1+\rho}\cdot\frac{1-z}{2-z}\cdot G_0(z) + \frac{1+\rho(2-z)}{(1+\rho)(2-z)} \\ &= \frac{1-\rho}{1+\rho}\cdot\frac{1-z}{2-z}\cdot G_0(z) + 1 - \frac{1-z}{(1+\rho)(2-z)} \\ &= 1 + \frac{1-z}{(1+\rho)(2-z)}((1-\rho)G_0(z) - 1).\end{aligned}$$

This concludes the proof of Theorem 3.1.

5. Conclusion and further research

We have found an explicit representation of the probability generating function of the stationary number of overtakes (slips) in an $M/M/2$/FCFS queue. As a consequence, we have found an explicit term for the variance which can be compared to the variance of the number of skips. To the knowledge authors, up to now, only the expectation of the number of overtakes (coinciding with the stationary number of skips) and the probability of at least one overtake were known. Many calculations were – more or less – based on standard methods for solving difference equations, computing probabilities by total probability, ... We want to emphasize on the key ideas:

- Due to the possibility of server S (which does not serve the tagged customer) running empty, the departure process does not only depend on the service pattern but also on the arrival process. In order to count the number of departures from server S during the tagged customer's service time, we introduced an absorbing Markov chain for which the absorbing state corresponds to the number of departures, resulting in the recurrence scheme for the probabilities $\psi_m(\ell)$.
- After using this scheme for obtaining a difference equation for $G_m(z) = \sum \psi_m(\ell)z^\ell$, we have used $\psi_m(\ell) = \frac{1}{2^{\ell+1}}$ for $m > \ell$ to prove that $G_m(z) \overset{m\to\infty}{\to} \frac{1}{2-z}$. This observation enabled us to solve the difference equation for $G_m(z)$, since otherwise we would have had only one side condition for finding two parameters (γ and δ in the proof).

Obviously, a straight-forward generalization would be the consideration of $M/M/c$/FCFS queues for $c > 2$. Most considerations still hold true, in particular, the Markov chain can be constructed similarly (with appropriate slight changes of the transition probabilities), recurrence relations for $\psi_m(\ell)$ can be found again, and a difference equation for $G_m(z)$ arises. Hence, information on the number of departures during the tagged customer's service can be obtained. Unfortunately, for $c > 2$, this is not sufficient for deriving information on the number of overtakes: For $c = 2$ either all departures (for $N = 0$) or all departures but the first one (for $N \geq 1$) are overtakes. For $c > 2$, there is no such deterministic relationship. Hence, this case requires further research.

Another interesting case is the $M/M/\infty$/FCFS model. In constrast to $M/M/c$ for $c > 2$, we have a quite easy deterministic relationship between the deparature process and the amount of slips: We have to determine the number of customers arriving **and** leaving during the tagged customer's service. Due to the infinite number of servers, for this determination, we can ignore all customers present in the system before the tagged customer's service starts. Hence, we could consider a two-dimensional Markov chain (J_n, K_n) where

- $K_n = k \geq 0$ indicates that the tagged customer is still in the system after the nth event, and that there are k customers in the system which have arrived after the tagged customer,
- $K_n = A$ indicates that the tagged customer has left the system,
- and J_n is the number of departures of customers arriving during the tagged customer's service up to the nth event or before the tagged customer leaves the system, respectively.

With $\psi_m(\ell)$ denoting the probability of absorption in (ℓ, A) given the initial state $(0, m)$, we directly find that $\psi_0(\ell)$ is the probability of ℓ slips, that is, with $G_m(z)$ defined as above, $G_0(z)$ is the generating function of the stationary number of slips. On the one hand, this seems to imply an even more direct relationship between the absorption probabilities of the Markov chain and the distribution of the number of slips. Unfortunately, on the other hand, the transition probabilities change in comparison to those depicted in Fig. 2: Transitions from (m, ℓ) to $(m+1, \ell)$ occur with probability $\frac{\lambda}{\lambda+(m+1)\mu}$, transitions to $(m-1, \ell+1)$ occur with probability $\frac{m\mu}{\lambda+(m+1)\mu}$, and transitions to the absorbing state (m, A) occur with probability $\frac{\mu}{\lambda+(m+1)\mu}$. Due to this dependency on m, $G_m(z)$ is no longer a solution of a difference equation with constant coefficients anymore, and thus things get more complicated. Hence, the analysis of slips in the $M/M/\infty$/FCFS model requires further research, too.

We have mentioned that the number of overtakes is important for the analysis of fairness measures. A major goal for introducing fairness measures was the comparison of fairness induced by various scheduling disciplines. Hence, an analysis of the distribution of overtakes for other scheduling disciplines than FCFS would be desirable. Future research could deal with the question whether or not similar approaches can also be applied to other service disciplines in $M/M/2$ or $M/M/c$ queues.

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