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Optimal Stochastic control theory and its' application to Landmark University Development ventures and investment model

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Abstract: *This paper deal with optimal stochastic control theory and its' application to Landmark University Development ventures and investments model. Here Stochastic Differential Equations (SDE) is considered as an ordinary differential equations (ODE) driven by white noise and we justified the connection between the Ito's integral and white noise in the case of non-random integrands interpreted as cost functions.*
Keywords*: Investment, Optimal, Stochastic, Venture, White Noise*

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I. Introduction

Investment consist of making decisions for a manufacturing system, in which different types of events such as operations, failures, preventive, maintenance and raw material supply as well as customer demand fluctuation occur at the same or various time. To better understand and more effectively deal with randomness from various sources, Mathematical models that can characterize the unique feature of every major event are needed of which Landmark University management can be of an immense beneficiary.

Many active researchers and historical figures have made significant contribution to control theory, notable among them includes [1] who considered the problem of optimal investment for the Nerlove-Arrow model under a replenish-able budget by considering an optimal control problem with two state variables for the dynamics of this model and opined that the optimal control is the rate of investment expenditure that is required to maximize the present value of net streams over an infinite time horizon subject to a replenish-able budget.

[2a] examined the continuous-time consumption-portfolio problem for an individual whose income is generated by capital gains and investments in assets with prices assumed to satisfy the geometric Brownion motion. [2b] extend these results for more general utility function, price behavior assumptions, and for income generated also from non-capital gains sources.

[3] Studied the problem of optimal exploration and consumption of a natural resource in the stochastic case while [4] also studied the cyclical consumption patterns and rational addition. [5] Considered the optimal consumption training, working time and leisure over the life cycle.

[6] Investigated the optimal consumption and investment with bankruptcy. In [7], the stochastic optimal control problem of consumption model which is subject to It'o differential equation was discussed. They assumed that the utility function is an increasing function of the consumption rates while [8] Introduced the maximum principle and the bang-bang principle.

An interesting example in one dimension arises in Merton's model of optimal portfolio by [9] and the stochastic approach is used to study the stability and control of tumor and cancer modelled by [10] where he used the optimal control approach to stabilize the tumor and cancer unstable equalized states.[11] Applied the stochastic optimal control to study the problem of optimal control of a stochastic production planning and investment model.

[12] Provides an algorithm for the optimal control of nonlinear stochastic models where the new version of the algorithm OPTCON2, named OPTCON2, was developed to obtain approximate solution of control optimum problems where the objective function is quadratic and the dynamic multivariable system is nonlinear

II. Stochastic Optimal Control Principle

We consider differential equation of the form:

$$dx = f(t, x_t, u_t, z_t) = f(x_t, u_t, t) + Gdz(t) \tag{1}$$

An optimal control problem is specified by giving a performance criterion that grades the possible control function u in order of preference by attaching a number $J(u)$ which represent the cost of u, so that we can choose the control that minimizes it

We let the cost function be of the form:

$$J(u) = E\left\{\int_0^T H(t, x_t, u_t)dt + G(x_t)\right\} \tag{2}$$

Where T is infinite termination time and is not fixed, x_t is the value of one's asset.

H represent the cost deviation from some desired trajectory of x_t or the use of too much control forces or energy. $G(x_t)$ is the cost failure to reach some special target set at terminal.

While the deterministic optimal control problem be of the form

$$\int_0^T H(t, x_t, u_t, z_t)dt + G(x_t) \tag{3}$$

Solving the stochastic optimal control problems defined in the equations (2) and (3).

We let $V(x,t)$ be the current value function for the expected value of the objective function of the equation (3) from t to T:

$$V(x,t) = \max_u E\left\{\int_t^T H(x,u,z,t)dt + G(x_t)\right\} \tag{4}$$

As the optimal policy is followed from t to T. Given $X_t = x$. then by the optimality principle,

$$V(x,t) = \max_u E[H(x,u,z,t)dt + V(x + dx_t, t + dt)] \tag{5}$$

Applying Taylor's expansion we get:

$$V(x + dx_t, t + dt) = V(x,t) + V_t dt + V_x dx_t + \frac{1}{2}V_{xx}(dx_t)^2 + \frac{1}{2}V_{tt}(dt)^2 + \frac{1}{2}V_{xt}dx_t dt \tag{6}$$

Equation (4) can be formally written thus:

$$\left(\dot{X}(t)\right)^2 = f^2(dt)^2 + G^2(dz_t)^2 + 2fGdz_t dt. \tag{7}$$

$$dx - tdt = f(dt)^2 + Gdz_t dt \tag{8}$$

By the multiplication rules of the stochastic calculus:

$$(dz_t)^2 = dt, dz_t dt = 0, (dt)^2 = 0 \tag{9}$$

We substitute from the equation (2) into the equation (1), apply the equations (3), (4) and equation (5) yields:

$$V(x,t) = \max_u \left[Hdt + V(x,t) + V_t dt + V_x f dX_t + \frac{1}{2}V_{xx}G^2 dt + 0(dt)\right] \tag{10}$$

Conceding the term $V(x,t)$ on both sides of the equation (10) as $dt \to 0$ we derive an Hamilton-Jacob-Bellman equation of the form.

$$0 = \max_{u} \left[H + \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f + \frac{1}{2} G^2 \frac{\partial^2 V}{\partial x^2} \right] \qquad (11)$$

For the current value function $V(x,t)$ with the boundary condition $V(x,T) = S(x,T)$

III. Application Of Stochastic Optimal Control In Landmark University Development Ventures(LMDV) And Investment Model

In this model we considered a (LMD) venture producing a single homogeneous good with a finished goods warehouse and define the parameters and the variables used as follows:

Y_t = The inventory level at time t

V_t = The production rate at time t

R_t = The demand rate at time t

T = The length of the planning period

$\bar{y}$ = The factory inventory goal level

$\bar{v}$ = The factory production goal level

y_0 = The initial inventory level

h = The inventory holding cost coefficient

c = The production cost coefficient

B = The salvage value per unit of the inventory at time T

z_t = The standard Wiener process

σ = The diffusion coefficient

The venture inventory goal level $\bar{y}$ is a safety stock level that the company want to keep on hand. Besides, the venture production goal level $\bar{v}$ is the most efficient level at which it is desired to run the factory.

Applying the above listed parameters and notation we can then described the condition of the model by considering the stock-flow equation of the model as:

$$dx(t) = V(t) - R(t); X(0) = x(0). \qquad (12)$$

Here, the inventory level at time t is increased by the production and decreased by the demand.
Modifying equation (12) to generalize the possible stochastic behaviour in the model, we let the inventory level Y_t to be controlled by the Ito stochastic differential equation.
Thus:

$$\dot{X}(t) = (V_t - R)dt + \sigma dz_t, X(0) = x(0). \qquad (13)$$

The process dz_t can be expressed formally as $w(t)dt,$ where $w(t)$ is refers to as a white noise process. Equation (13) can be interpreted as sales returns or as inventory spoilage that are random in nature. The production rate model does not restricted to be nonnegative i.e it do permit disposable $(V_t < 0)$ and allowed the inventory level to be negative.
We adopt the modified Hamilton-Jacob principle produced by Hamilton-Jacob-Bellman equation satisfied by certain value function to solve the problem.
Here, we wish to minimize the expected total cost determined by the following functional integral:

$$J(u) = E\left\{ \int_0^T \left[c\left(V_t - \bar{v} \right)^2 + h\left(Y_t - \bar{y} \right)^2 \right] dt + BY_T \right\} \qquad (14)$$

We assume that $V(x,t)$ denotes the minimum value of the expected total cost from t to the horizon T with $X_t = x$ using the optimal policy from t to T. The function is then given as:

$$V(x,t)=\min_{V_t} E\left\{\int_t^T\left[c\left(V_t-\bar{v}\right)^2+h\left(Y_t-\bar{y}\right)^2\right]dt+BY_T\right\}. \quad (15)$$

For the function value $V(x,t)$ satisfying the Hamilton-Jacobi-Bellman equation;

$$0=\max_u\left\{-c\left(v-\bar{v}\right)^2-h\left(y-\bar{y}\right)^2+\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}(v-R)+\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2}\right\}. \quad (16)$$

With the boundary condition

$$V(x,t)=Bx \quad (17)$$

Minimizing the expression with respect to y by taking its' derivative with respect to y

Setting the expression to zero, we get:

$$\frac{\partial V}{\partial x}-2c\left(v-\bar{v}\right)=0 \quad (18)$$

Then the optimal production rate that minimizes the total cost can be expressed as a function of the current value function in the following equation:

$$v(x,t)=\frac{1}{2c}\frac{\partial V}{\partial x}+\bar{u} \quad (19)$$

Putting equation (19) into the equation (16) yields:

$$0=-\left[\frac{1}{4c}\left(\frac{\partial V}{\partial x}\right)^2+h\left(y-\bar{y}\right)^2\right]+\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}\left[\frac{1}{2c}\left(\frac{\partial V}{\partial x}\right)+\bar{v}-R(t)\right]+\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial x^2} \quad (20)$$

This is a nonlinear partial differential equation that must be satisfied by the current value function $V(x,t)$ with the boundary condition (17).

For the production rate to be nonnegative, the optimal production rate would be change

$$v(x,t)=\max\left[0,\frac{1}{2c}\frac{\partial V}{\partial x}+\bar{v}\right] \quad (21)$$

We seek the application of the Hamilton-Jacobi-Bellman equation in solving the stochastic production problem having different demand rate.

Solving the nonlinear partial differential equation (11), we assumed that the solution takes the form:

$$V(x,t)=A(t)x^2+B(t)x+C(t) \quad (22)$$

$$\frac{\partial V}{\partial x}=2Ax+B \text{ and } \frac{\partial^2 V}{\partial x^2}=2A$$

$$\frac{\partial V}{\partial t}=\dot{A}x^2+\dot{B}x+\dot{C} \quad (23)$$

where the dot denotes the differentiation with respect to time.

Substituting equation (23) into the equation (11) and collecting the like terms yields:

$$\frac{1}{4c}[2Ax+B]^2-h\left[x^2-2\bar{y}x+\bar{y}^2\right]+\left[\dot{A}x^2+\dot{B}x+\dot{C}\right]+[2Ax+B]\left(\bar{v}-R\right)+\sigma^2A=0 \quad (24)$$

For any value of x, equation (24) hold, and the following nonlinear ordinary differential equations system are derived

$$\left.\begin{aligned}c\dot{A}&=ch-A^2\\ c\dot{B}+AB&=-2ch\bar{y}-2cA(\bar{v}-R)\\ 4c\dot{C}&=4ch\bar{y}-B^2-4c(\bar{y}-R)B-4c\sigma^2A\end{aligned}\right\}\tag{25}$$

Solving the nonlinear equations for different cases of demand rate with the boundary conditions:

$A(T)=0; B(T)=D; M(T)=0$, and assume that the time t changed to $r=e^{2\sqrt{\frac{h}{c}}(t-T)}$

We then have; $\frac{\partial}{\partial t}=2\sqrt{\frac{h}{c}}r\frac{\partial}{\partial r}$;

With the time horizon,

At $t=0\rightarrow r=e^{-2\sqrt{\frac{h}{c}}T}$; as $t=T\rightarrow r=1$

$$\Rightarrow\ r\in\left[e^{-2\sqrt{\frac{h}{c}}T},1\right].$$

We expand $c\dot{A}/(ch-A^2)$ in the equation (25) by partial fraction

Then: $A(r)=\frac{\sqrt{ch}(r-1)}{r+1}$.

IV. General Cases For The Demand Rates In The LMD Venture

We shall consider the first case of the demand rate as constant $(R(t)=R_0=const)$ and the optimal production rate is given as:

$$v(x,t)=\frac{1}{c}\left[\left(\frac{\sqrt{ch}(r-1)}{r+1}\right)x-c(\bar{v}-R_0)+\left[B+2c(\bar{v}-R_0)\right]\frac{\sqrt{r}}{r+1}-\bar{y}\sqrt{ch\left(\frac{r-1}{r+1}\right)}\right]+\bar{v}\tag{26}$$

While the functions $B(r), C(r)$ are as followed

$$B(r)=-2c(\bar{v}-R_0)+2\left[B+2c(\bar{v}-R_0)\right]\frac{\sqrt{r}}{r+1}-2\bar{y}\sqrt{ch}\left(\frac{r-1}{r+1}\right).\tag{27}$$

$$C(r)=\int\left\{\frac{1}{2r}\left[\sqrt{ch}\,\bar{y}^2-\frac{B^2}{4\sqrt{ch}}-\sqrt{\frac{c}{h}}(\bar{v}-R_0)B-\sqrt{\frac{c}{h}}\sigma^2A\right]\right\}dr+C_0\tag{28}$$

Equation (5) was solved after taking the average with respect to the state variable x by substituting for the optimal production rate from the equation (26) to find the expected inventory level.

$$\dot{E}(x)-\frac{A}{c}E(x)=\frac{1}{c}\left\{\left[B+2c(\bar{v}-R_0)\right]\frac{\sqrt{r}}{r+1}-\bar{y}\sqrt{ch}\left(\frac{r-1}{r+1}\right)\right\}$$

$$E(x)=\bar{y}-\frac{B+2(v-R_0)}{2\sqrt{ch}\sqrt{r}}+\left[y_0-\bar{y}+\frac{B+2(v-R_0)}{2\sqrt{ch}\sqrt{r_0}}\right]\frac{\sqrt{r_0}}{1+r_0}\left(\frac{r+1}{\sqrt{r}}\right).\tag{29}$$

It was noted that when the demand rate is constant and is equal to the production goal rate $\bar{v}$, then the expected inventory level will be:

$$E(x) = \bar{y} + (y_0 - \bar{y})\left[\frac{\sqrt{r_0}(1+r)}{(1+r_0)\sqrt{r}}\right] - \frac{B}{2\sqrt{ch}}\left[1 - \frac{(1+r)}{(1+r_0)}\right]\frac{1}{\sqrt{r}} \quad (30)$$

And the expected total cost is:

$$E(V(x(r),T)) = B\left\{\bar{y} + (y_0 - \bar{y})\frac{\sqrt{r_0}(1+r)}{(1+r_0)\sqrt{r}} - \frac{B}{2\sqrt{ch}}\left[\frac{1}{\sqrt{r}} - \frac{(1+r)}{(1+r_0)\sqrt{r}}\right]\right\}. \quad (31)$$

The case when demand rate is a time varying rate. i.e. $R(t) = \bar{v} - \frac{2r\sqrt{r}}{r-1}$

The optimal production rate is then given as:

$$v(x,t) = \frac{1}{c}\left[\left(\frac{\sqrt{ch}(r-1)}{r+1}\right)y + \frac{(c+b)\sqrt{r}}{1+r} - \sqrt{\frac{c}{h}}\left(\sqrt{ch}\frac{r\sqrt{r}}{r+1} + h\bar{y}\frac{r-1}{r+1}\right)\right] + \bar{v} \quad (32)$$

Then, the function $B(r)$ is

$$B(r) = \frac{2(c+b\sqrt{r}}{1+r} - 2\sqrt{\frac{c}{h}}\left(\sqrt{ch}\frac{r\sqrt{r}}{r+1} + h\bar{y}\frac{r-1}{r+1}.\right) \quad (33)$$

The expected value is then calculated as:

$$E(x) = \frac{1}{2}\sqrt{\frac{c}{h}}\left[\left(\frac{c+B}{c}\right)\frac{\sqrt{r}}{r+1}Inr - 2\sqrt{\frac{h}{c}}\bar{y} + \frac{\sqrt{r}}{1+r}In(r-1)^2\right] + E_0\frac{\sqrt{r}}{1+r} \quad (34)$$

And $E_0 = (y_0 + \bar{y})\left(e^{\sqrt{\frac{h}{c}}T} + e^{-\sqrt{\frac{h}{c}}T}\right) - \left(\frac{c+B}{c}\right)T + \frac{1}{2}\sqrt{\frac{c}{h}}In\left(e^{-2\sqrt{\frac{h}{c}}T} - 1\right)^2 \quad (35)$

The expected inventory level is then calculated as:

$$E(x) = \frac{1}{2}\sqrt{\frac{c}{h}}\left[\left(\frac{c+B}{c}\right)\frac{\sqrt{r}}{r+1}Inr - 2\sqrt{\frac{h}{c}}\bar{y} + \frac{\sqrt{r}}{r+1}In(r-1)^2\right] + \frac{\sqrt{r}}{r+1}\left[(y_0 + \bar{y})\left(e^{\sqrt{\frac{h}{c}}T} + e^{-\sqrt{\frac{h}{c}}T}\right) - \left(\frac{c+B}{c}\right)T + \frac{1}{2}\sqrt{\frac{h}{c}}In\left(e^{-2\sqrt{\frac{h}{c}}T} - 1\right)^2\right]. \quad (36)$$

V. Illustrative Examples On Stochastic Venture/Inventory Model

Case1: when the demand rate is constant $(R(t) = R_0 = const)$

Given that

$\bar{v} = 200, y_0 = 200, \bar{y} = 100, B = 50, c = 40, h = 20, T = 20,$

$\bar{v} = 250,\ y_0 = 250, \bar{y} = 150,\ B = 40,\ c = 60, h = 30,\ T = 20$

$\bar{v} = 75,\ y_0 = 75, \bar{y} = 50,\ B = 30,\ c = 50, h = 25,\ T = 20$

Table1 showing the expected optimal inventory level E(x) and current value E(v)

Expected optimal inventory level E(x)				Expected optimal current value E(v)		
T	$E_a(x)$	$E_b(x)$	$E_c(x)$	$E_a(v)$	$E_b(v)$	$E_c(v)$
0	200	250	75	1000	1000	2250
3	112	162	53	5600	6479	1590
6	101	152	51	5072	6058	1511
9	100	150	50	5009	6007	1501
12	100	150	50	5001	6001	1500
15	100	150	50	5001	6002	1501
18	100	150	50	5011	6005	1503
20	101	151	50	5044	6019	1513

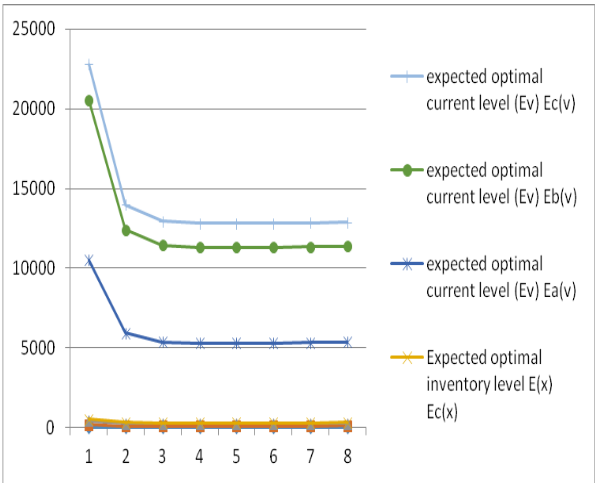


Fig.1 shows the behavior of constant Demand rates

As demand kept constant, both the expected optimal inventory level E(x) and the Expected optimal current value E(v) continues decline and later stabilized to a parallel level.

CASE II: When the Demand rate is a time varying rate. i.e $R(t) = \bar{v} - \frac{2r\sqrt{r}}{r-1}$

Given that:

$\bar{v} = 200, y_0 = 200, \bar{y} = 100, B = 50, c = 40, h = 20, T = 20$

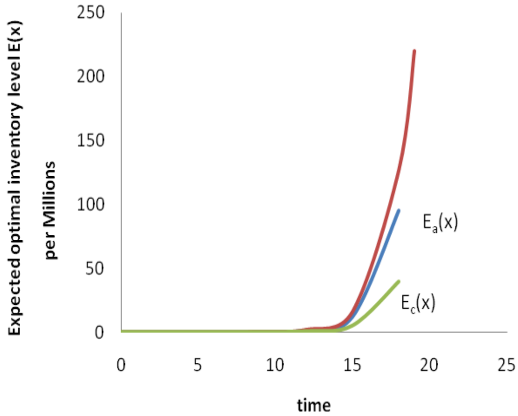
$\bar{v} = 250,\ y_0 = 250, \bar{y} = 150,\ B = 40,\ c = 60, h = 30,\ T = 20$

$\bar{v} = 75,\ y_0 = 75, \bar{y} = 50,\ B = 30,\ c = 50, h = 25,\ T = 20$

Table 2 below showing the expected optimal
Current value (Ev)

T	$E_a(x)$	$E_b(x)$	$E_c(x)$
0	199.999	249.9999	74.999
3	2402.64	3186.8575	992.768
6	20777.41	27686.548	8648.919
9	174062.35	232066.48	72517.635
12	1452770.04	1937010.2	605312.45
15	12109814.94	16146404	5045747.4
18	95465783.43	127287701	39777398.07
19		219941228	
20			

Fig.2 shows the behavior of varied Demand rates



Here, as the demand keep increasing so also both the expected optimal inventory level E(x) and the Expected optimal current value E(v) increases

VI. Conclusion

We have examined the stochastic optimal control of two stochastic dynamical models on Landmark University divisional ventures and investment model where both the inventory level and production are seeing clearly to be stochastic in nature and demand rate is equally seeing to be deterministic in nature.

The numerical illustrative examples used for stochastic production model were to displays the optimal expected inventory level E(x) against time (t) and the expected current values E(v) against time (t).The cases a when demand is constant i.e. $(R(t) = R_0 = const)$ and when the demand rate is a time varying rate i.e. $R(t) = (\bar{v} - \frac{2r\sqrt{r}}{r-1})$ are the two cases considered.

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