



Optimal taxation with endogenous fertility and health-damaging emissions

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Abstract: Output is produced from labor and capital by technologies that differ in their emission intensity and relative capital intensity. Aggregate emissions decrease every individual's health, but each individual can invest its own health. Population grows by the difference of fertility and exogenous mortality. Labor is used in production or child rearing. I construct a differential game where the benevolent government is a leader that determines taxes and subsidies, while the representative family is a follower that saves in capital and decides on its number of children. The main results are as follows. Without government intervention, population increases or decreases indefinitely. Capital should be taxed, if dirty technology, and subsidized, if clean technology is relatively capital intensive. Child rearing should be taxed, if dirty technology is relatively capital intensive or mildly labor intensive.

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1. INTRODUCTION

I examine optimal environmental policy in a model where firms produce output from labor and capital according to dirty and clean technologies with constant returns to scale (section 2) and families decide on their consumption, saving, fertility and health care (section 3). I construct a *Stackelberg differential game* (cf. Basar and Olsder 1989), where the representative *dynastic family* is the *follower* that consumes, rear children, saves in capital and invests in its health, taking the taxes and aggregate health-damaging emissions as exogenous (section 3), and the *government* is the *leader* that determines taxes observing the dependence of mortality on emitting production. (section 4).

2. THE ECONOMY AS A WHOLE

The growth rate of population L is equal to the fertility rate f , which is set by the families, minus the exogenously determined mortality rate m :

$$\frac{\dot{L}}{L} = \frac{dL}{dt} \frac{1}{L} = f - m, \quad L(0) = L_0, \quad (1)$$

where t is time. I normalize units so that one unit of labor is needed to rear one newborn and labor devoted to child rearing is equal to fertility fL .

There is only one good that is used in consumption, investment and health care. It is produced from labor and capital by two alternative technologies (or sectors): one *dirty* (subscript or superscript d) that emits in one-to-one proportion to its output Y_d ; and the other *clean* (subscript

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or superscript c) that emits in fixed proportion $\gamma \in (0, 1)$ to its output Y_c . Then, aggregate emissions S are

$$S = Y_d + \gamma Y_c \quad \text{with } 0 < \gamma < 1. \quad (2)$$

Technologies $j \in \{c, d\}$ produce the good according to

$$Y_j = F^j(K_j, L_j), \quad F_K^j > 0, \quad F_L^j > 0, \quad F_{KK}^j < 0, \quad F_{LL}^j < 0, \quad F_{KL}^j > 0, \quad F^j(K_j, L_j) \text{ linearly homogeneous}, \quad (3)$$

where the subscripts K and L denote the partial derivatives of the function F^j with respect to capital K_j and labor L_j , correspondingly. At any time t , capital K can be transferred between the technologies $j \in \{c, d\}$, but labor (= population) L between those and child rearing fL :

$$K \geq K_c + K_d, \quad L \geq L_c + L_d + fL. \quad (4)$$

To ensure a unique solution, I assume the difference between the emissions-intensity of the sectors large enough,

$$\gamma < \min\left(1, \frac{\xi_c}{\xi_d}\right) \frac{F^d(\xi_d, 1)}{F^c(\xi_c, 1)} < 1, \quad (5)$$

where $\xi_c \doteq K_c/L_c$ and $\xi_d \doteq K_d/L_d$ are capital intensities in the clean and dirty sectors, respectively.

In Appendix A, I show that if production is carried out by competitive firms, then output per head, $y \doteq (Y_c + Y_d)/L$, and emissions per head, s , are functions of capital per head, $k \doteq K/L$ and the fertility rate f as follows:

$$Y/L = y = (1 - f)w + kr, \quad S/L = s(k, f), \quad s_k \doteq \frac{\partial s}{\partial k} > 0 \Leftrightarrow s_f \doteq \frac{\partial s}{\partial f} > 0 \Leftrightarrow \xi_d > \xi_c, \quad (6)$$

where the wage w , the interest rate r , the sectorial factor intensities ξ_d and ξ_c and the partial derivatives s_k and s_f are constants. In other words:

Proposition 1. If and only if the dirty sector is more capital intensive than the clean sector, $\xi_d > \xi_c$, emissions s increase with higher capital per head, k , or with a higher fertility rate f .

This is the Rybczynski (1956) effect. If the supply of capital k increases, or if the supply of labor $1-f$ decreases (i.e. the fertility rate f decreases), then the capital-intensive sector expands and the labor intensive-sector contracts.

The government finances a subsidy b to child rearing fL by a tax x on capital K and a poll tax τ . Its budget is $b f L = x K + \tau L$, or in per head terms (cf. $k \doteq K/L$)

$$b f = x k + \tau. \quad (7)$$

3. THE REPRESENTATIVE FAMILY

3.1 Utility

Economy-wide emissions S are a public good that harms every individual's health simultaneously. On the other hand, each individual can improve its health by its spending h on health care. Let the level of health without emissions and health care be constant ϑ . Then, by a proper choice of units, the measure of an individual's health is $\vartheta - S + h$. I introduce this health into Becker's (1981) family-optimization model as follows. At any moment of time, t , an individual derives utility u from its consumption per head, $c \doteq C/L$, and the fertility rate in its family, f , and its health $\vartheta - S + h$ according to

$$u(t) \doteq \ln c(t) + \alpha \ln f(t) + \beta \ln[\vartheta - S + h(t)], \quad (8)$$

where constants $\alpha > 0$ and $\beta > 0$ are the relative weights of the fertility rate and health, correspondingly.

With the mortality rate m , an individual's probability of dying in a short time dt is equal to $m dt$. Thus, e^{-mt} is the probability that an individual will survive beyond the period $[0, t]$, and $e^{-mt} u(t)$ the individual's expected utility at time t , where $u(t)$ is periodic utility (8). Let ρ be the constant rate of time preference for an individual who could live forever. The representative family member's expected utility for period $t \in [0, \infty)$ is

$$U \doteq \int_0^{\infty} u(t) e^{-(\rho+m)t} dt \quad \text{with (8),} \quad \rho > 0. \quad (9)$$

3.2 Saving

Because capital K is the only asset in the model, it evolves according to private saving,

$$\dot{K} = yL + b f L - x K - \tau L - h L - c L - \mu K, \quad K(0) = K_0, \quad (10)$$

where yL is output, b the subsidy to child rearing fL , x the tax on capital K , τ the poll tax, cL consumption, hL total spending on health care and $\mu > 0$ the depreciation rate of capital K . Noting $k = K/L$, (1) and (6), the constraint (10) can be converted into per capita form as follows:

$$\dot{k} = (1-f)w + kr + bf - \tau - h - c + (m-f-x-\mu)k, \quad k(0) = k_0. \quad (11)$$

3.3 Optimization

The family maximizes its utility (9) by its consumption per head, c , fertility rate f and health care h subject to the accumulation of capital, (11), given total emissions S and the taxes (x, b, τ) . The Hamiltonian of this problem is

$$\begin{aligned} \Phi &\doteq \ln c + \alpha \ln f + \beta \ln(\vartheta - S + h) + \\ &\phi[(1-f)w + kr + bf - \tau - h - c + (m-f-x-\mu)k], \end{aligned} \quad (12)$$

where the co-state variable $\phi(t)$ evolves according to

$$\begin{aligned} \dot{\phi} &\doteq \frac{d\phi}{dt} = (\rho + m)\phi - \frac{\partial \Phi}{\partial k} = (\rho + \mu + x + f - r)\phi, \\ \lim_{t \rightarrow \infty} \phi k e^{-(\rho+m)t} &= 0. \end{aligned} \quad (13)$$

The first-order and second-order conditions for the maximization of the Hamiltonian (12) by (c, f, h) are

$$\frac{\partial \Phi}{\partial c} = \frac{1}{c} - \phi = 0, \quad \frac{\partial^2 \Phi}{\partial c^2} = -\frac{1}{c^2} < 0, \quad (14)$$

$$\frac{\partial \Phi}{\partial f} = \frac{\alpha}{f} + \phi(b-w-k) = \frac{\alpha}{f} + \frac{b-w-k}{c} = 0, \quad \frac{\partial^2 \Phi}{\partial f^2} < 0, \quad (15)$$

$$\frac{\partial \Phi}{\partial h} = \frac{\beta}{\vartheta - S + h} - \phi = \frac{\beta}{\vartheta - S + h} - \frac{1}{c} = 0, \quad \frac{\partial^2 \Phi}{\partial h^2} < 0. \quad (16)$$

By (15), I define the *fertility function* [cf. (6)]

$$\begin{aligned} f(c, k, b) &\doteq \frac{\alpha c}{k+w-b}, \quad \frac{\partial f}{\partial c} = \frac{f}{c}, \\ -\frac{\partial f}{\partial b} &= \frac{\partial f}{\partial k} = -\frac{f}{k+w-b}. \end{aligned} \quad (17)$$

By (16), I solve for health care:

$$h = \beta c + S - \vartheta. \quad (18)$$

Noting (13), (14) and (17), I construct the *Euler equation*

$$\begin{aligned} \frac{\dot{c}}{c} &= -\frac{\dot{\phi}}{\phi} = r - \rho - \mu - x - f(c, k, b), \\ \frac{\partial \dot{c}}{\partial c} &= -\frac{\partial f}{\partial c} c = -f, \quad \frac{\partial \dot{c}}{\partial k} = -\frac{\partial f}{\partial k} c = \frac{f c}{k+w-b} = \frac{f^2}{\alpha}. \end{aligned} \quad (19)$$

Plugging (17) and (18) back to (11) and noting (19), I obtain the accumulation of capital per head, k , as follows:

$$\begin{aligned} \dot{k} &= \vartheta - S + w - (1 + \alpha + \beta)c - \tau + (r + m - \mu - x)k, \\ k(0) &= k_0, \quad \frac{\partial \dot{k}}{\partial c} = -1 - \alpha - \beta < 0, \\ \left. \frac{\partial \dot{k}}{\partial k} \right|_{\dot{c}=0} &= (r + m - \mu - x)_{\dot{c}=0} = \rho + f + m. \end{aligned} \quad (20)$$

The system (19) and (20), contains k as a predetermined and c as a non-predetermined variable. Its saddle-point condition is given by [cf. (1)]:

$$\begin{aligned} \frac{\partial \dot{k}}{\partial c} \frac{\partial \dot{c}}{\partial k} - \left. \frac{\partial \dot{k}}{\partial k} \right|_{\dot{c}=0} \frac{\partial \dot{c}}{\partial c} &= -(1 + \alpha + \beta) \frac{f^2}{\alpha} + (\rho + f + m) f = \\ [(\rho + m)\alpha - (1 + \alpha + \beta)f] f / \alpha &> 0 \Leftrightarrow f < \frac{(\rho + m)\alpha}{1 + \beta}. \end{aligned}$$

Thus, to obtain a unique solution, I assume that the fertility rate f is below $\frac{(\rho+m)\alpha}{1+\beta}$.

Finally, from (1) and (19) it follows that

$$\left. \frac{\dot{L}}{L} \right|_{\dot{c}=0, x=0} = f|_{\dot{c}=0, x=0} - m = r - \rho - \mu.$$

Because the interest rate r is constant [cf. (6)], but the family ends up with the steady state with $\dot{c} = 0$, this yields the following result:

Proposition 2. In the case of laissez-faire (i.e. with no taxes $x = 0$), population L increases indefinitely for $r > \rho + \mu$ or decreases indefinitely for $r < \rho + \mu$.

In a two-sector two-input economy where both sectors are subject to constant returns to scale, the wage w the interest rate r are fixed in equilibrium. Thus, there is no mechanism that would keep population stationary.

4. THE GOVERNMENT

4.1 Maximization of social welfare

The government internalizes the external effect of aggregate emissions $S = sL$ on welfare. Inserting the response function of the firms (6) and the family (18) into the family's utility (9), I obtain the social welfare function

$$\int_0^{\infty} [(1 + \beta) \ln c + \alpha \ln f] e^{-(\rho+m)t} dt. \quad (21)$$

Because there is one-to-one correspondence from b to f through the fertility function (17), at each moment of time, I can replace the subsidy b by the fertility rate f as a control variable. Then, the Euler equation (19) becomes

$$\dot{c}/c = r - \rho - \mu - x - f. \quad (22)$$

Inserting aggregate emissions S from (6), health care (18) and the government budget (7) into private capital accumulation (11), I obtain

$$\begin{aligned} \dot{k} &= w + \vartheta - s(k, f)L - (1 + \beta)c + (r + m - \mu)k \\ &\quad - (w + k)f, \quad k(0) = k_0. \end{aligned} \quad (23)$$

The government maximizes social welfare (21) by the capital tax x and the fertility rate f subject to the Euler equation (22) and the evolution of population (1) and capital per head, (23). The Hamiltonian of this problem is

$$\begin{aligned} \Omega &\doteq (1 + \beta) \ln c + \alpha \ln f + \lambda_L (f - m)L \\ &\quad + \lambda_k [w + \vartheta - s(k, f)L - (1 + \beta)c + (r + m - \mu)k \\ &\quad - (w + k)f] + \lambda_c (r - \rho - \mu - x - f)c, \end{aligned} \quad (24)$$

where the co-state variables λ_L , λ_k and λ_c evolve

$$\begin{aligned} \dot{\lambda}_L &= (\rho + m)\lambda_L - \frac{\partial \Omega}{\partial L} = (\rho + 2m - f)\lambda_L + s\lambda_k, \\ \lim_{t \rightarrow \infty} \lambda_L L e^{-(\rho+m)t} &= 0, \end{aligned} \quad (25)$$

$$\begin{aligned} \dot{\lambda}_k &= (\rho + m)\lambda_k - \frac{\partial \Omega}{\partial k} = (\rho + \mu + f - r + s_k L)\lambda_k, \\ \lim_{t \rightarrow \infty} \lambda_k k e^{-(\rho+m)t} &= 0, \end{aligned} \quad (26)$$

$$\begin{aligned} \dot{\lambda}_c &= (\rho + m)\lambda_c - \frac{\partial \Omega}{\partial c} \\ &= (2\rho + m + \mu + x + f - r)\lambda_c + (1 + \beta)(\lambda_k - 1/c), \\ \lim_{t \rightarrow \infty} \lambda_c c e^{-(\rho+m)t} &= 0. \end{aligned} \quad (27)$$

4.2 Optimal capital taxation

I denote the steady state-value of a variable by superscript (*). The control x is *singular* with $\frac{\partial^2 \Omega}{\partial x^2} \equiv 0$. Its optimal value in the steady state, x^* , can be solved by the generalized *Glebsch-Legendre conditions* (cf. Bryson and Ho 1975, pp. 246-270, or Grass et al. 2008, pp. 131-134)

$$(-1)^q \frac{\partial}{\partial x} \left[\frac{d^{2q}}{d\theta^{2q}} \left(\frac{\partial \Omega}{\partial x} \right) \right] \leq 0 \text{ for } q = 0, 1, 2, \dots$$

Given (22), (26) and (27), these conditions hold as follows:

$$\frac{\partial \Omega}{\partial x} = -\lambda_c c = 0 \Leftrightarrow \lambda_c = 0, \quad \frac{\partial^2 \Omega}{\partial x^2} \equiv 0,$$

$$\frac{d}{dt} \left(\frac{\partial \Omega}{\partial x} \right) = -c \dot{\lambda}_c = 1 - (1 + \beta)\lambda_k c = 0$$

$$\Leftrightarrow (1 + \beta)\lambda_k = \frac{1}{c}, \quad \frac{\partial}{\partial x} \left[\frac{d}{dt} \left(\frac{\partial \Omega}{\partial x} \right) \right] \equiv 0,$$

$$\frac{d^2}{dt^2} \left(\frac{\partial \Omega}{\partial x} \right) = -(1 + \beta)(c \dot{\lambda}_k + \lambda_k \dot{c}) = 0,$$

$$\frac{\partial}{\partial x} \left[\frac{d^2}{dt^2} \left(\frac{\partial \Omega}{\partial x} \right) \right] = -(1 + \beta)\lambda_k \frac{\partial \dot{c}}{\partial x} = -\frac{1}{c} \frac{\partial \dot{c}}{\partial x} = 1 > 0.$$

Thus, in the steady state, it holds true that

$$\lambda_c^* = 0, \quad \lambda_k^* = 1/c^*. \quad (28)$$

With a singular control, it is common that the adjustment to the steady state has the *bang-bang property*: the control would jump between its maximum and minimum values. Because this property is extremely inconvenient in economic models, I eliminate it by assuming that the tax rate is chosen from the *set of continuously-evolving controls* (i.e. controls that are continuous in time t). Then, inserting initial values $\lambda_c(0) = 0$ and $\lambda_k(0) = 1/c(0)$ into (27) yields that equation (26) is equivalent to the Euler equation (22) for all $t \geq 0$ through the relations

$$\lambda_c = 0, \quad \lambda_k = 1/c > 0, \quad (29)$$

$$x = r - f - \mu - \rho - \frac{\dot{c}}{c} = r - f - \mu - \rho + \frac{\dot{\lambda}_k}{\lambda_k} = s_k L. \quad (30)$$

The results (29) extend the steady-state optimality conditions (28) for the whole planning period $t \in (0, \infty)$. Equation (30) can be rephrased as follows:

Proposition 3. At any time t , the optimal capital tax is

$$x(t) = s_k L(t) > 0 \Leftrightarrow s_k > 0 \Leftrightarrow \xi_d > \xi_c.$$

Thus, wealth (= capital) must be taxed, $x > 0$, if dirty technology is more capital intensive than clean technology, $\xi_d > \xi_c$, and subsidized, $x < 0$, if vice versa $\xi_d < \xi_c$.

This result can be explained as follows. The capital tax $x > 0$ discourages families to save, decreasing the relative supply of capital, k , in the long run. If the capita-intensive sector is dirty, but the labor-intensive sector clean, $\xi_d > \xi_c$, then labor and capital will move from the dirty to the clean sector (cf. Proposition 1) and emissions will fall. This will decrease the mortality rate, promoting welfare. If the capital-intensive sector is relatively clean $\xi_d < \xi_c$, then the capital subsidy $-x > 0$ leads to the same outcome. Because emissions $S = sL$ are proportional to population L , and because the production functions (3) are linearly homogeneous, then the optimal tax $x > 0$ or the optimal subsidy $-x > 0$ is proportional to population L .

4.3 Optimal population

Plugging (29) into (22) and (25) yields

$$\dot{c}/c = r - \rho - \mu - f - s_k L, \quad (31)$$

$$\dot{\lambda}_L = (\rho + 2m - f)\lambda_L + s/c. \quad (32)$$

In the steady state with $\dot{L} = \dot{k} = \dot{c} = 0$, from (1) and Proposition 3 it follows that

$$m = f^* = r - \rho - \mu - x^* = r - \rho - \mu - s_k L^*.$$

Because the interest rate r is constant [cf. (6)], this and Proposition 3 lead to the following corollary:

Proposition 4. The optimal size of population is given by

$$L^*(m) \doteq \frac{r - \rho - \mu - m}{s_k}, \quad \frac{dL^*}{dm} = -\frac{1}{s_k} < 0 \Leftrightarrow s_k > 0 \\ \Leftrightarrow \xi_d > \xi_c.$$

Thus, the optimal population L^* is an decreasing function of the mortality rate m , if the dirty sector is more capital intensive, $\xi_d > \xi_c$, and an increasing function of that, if the clean sector is more capital intensive, $\xi_d < \xi_c$.

When the dirty sector is more capital intensive, the government sets the tax x in proportion to population L (cf. Proposition 3). This decreases income and fertility f , until population L is fallen low enough. When the clean sector is more capital intensive, the government sets the subsidy $-x$ in proportion to population L . This increases income and fertility f , until population L is risen high enough.

4.4 Optimal fertility policy

Noting (15) and (29), I obtain the first-order and second-order conditions for the maximization of the Hamiltonian (24) by the fertility rate f as follows:

$$0 = \frac{\partial \Omega}{\partial f} = \frac{\alpha}{f} + \lambda_L L - (w + k + s_f L) \lambda_k = \lambda_L L - \frac{b + s_f L}{c} \\ = 0 \Leftrightarrow b = (c \lambda_L - s_f) L, \quad \frac{\partial^2 \Omega}{\partial f^2} = -\frac{\alpha}{f^2} < 0. \quad (33)$$

This result can be rephrased as follows:

Proposition 5. At any time t , the optimal subsidy to child rearing is $b(t) = [c(t) \lambda_L(t) - s_f] L(t)$, where the shadow price for land, $\lambda_L(t)$, evolves according to (25).

Because emissions $S = sL$ are proportional to population L , and because the production functions (3) are linearly homogeneous, then the optimal subsidy b is proportional to population L .

Because all state and all co-state variables are constants in the steady state, by (1), (25), (29) and (31) I obtain

$$f^* = m, \quad \lambda_k^* = \frac{1}{c^*}, \quad \lambda_L^* = -\frac{s \lambda_k^*}{\rho + f^*} = -\frac{s(k^*, f^*)}{(\rho + m) c^*}, \\ r - \mu - s_k L^* = \rho + f^* = \rho + m. \quad (34)$$

From (33) and (34) it follows that

$$-b^* = s_f L^* - c^* \lambda_L^* L^* = \left[s_f + \frac{s(k^*, f^*)}{\rho + m} \right] L^*,$$

which can be rephrased as follows:

Proposition 6. The optimal steady-state tax on child rearing is

$$-b^* = \left[s_f + \frac{s(k^*, f^*)}{\rho + m} \right] L^* > 0 \Leftrightarrow s_f > -\frac{s(k^*, f^*)}{\rho + m}.$$

Thus, in the long run (i.e., in the steady state), child rearing should be taxed, if the dirty sector is capital intensive $s_f > 0$ or moderately labor intensive $s_f \in (-s/(\rho + m), 0]$, and subsidized otherwise. The optimal tax or subsidy is proportional to population L^* .

A tax on child rearing, $-b > 0$, helps to keep population and emissions stationary. Only if the dirty sector

is drastically labor intensive relative to the clean sector, $s_f > -s/(\rho + m)$, it would be better to decrease the size of the dirty sector by attracting people from production into child rearing by a subsidy $b > 0$.

4.5 Saddle-point stability

In the system (1), (23), (31) and (32), where f is determined by (35), population L and capital per head, k , are predetermined, while consumption per head, c , and the co-state variable for population, λ_L , are non-predetermined variables. Consequently, a precondition for the existence of optimal public policy (propositions 2-5) is that the system has two stable and two unstable characteristic roots to perform saddle-point stability (cf. Acemoglu 2009, pp. 271-272). This condition is an empirical matter.

In the following exercise, I show that saddle-point stability is possible. I define the following function by (29) and (33):

$$f = F(L, k, c, \lambda_L) \doteq \frac{\alpha}{(w + k + s_f L)/c - \lambda_L L}, \\ \frac{\partial F}{\partial L} = \left(\lambda_L - \frac{s_f}{c} \right) \frac{f^2}{\alpha}, \quad \frac{\partial F}{\partial k} = -\frac{1}{c} \frac{f^2}{\alpha}, \quad \frac{\partial F}{\partial \lambda_L} = L \frac{f^2}{\alpha}, \\ \frac{\partial F}{\partial c} = (w + k + s_f L) \frac{f^2}{\alpha c^2}. \quad (35)$$

If consumption were 80%, capital 300% and wages 60% of output y and if there were no subsidy to child rearing so far, $b = 0$, I can approximate the parameter α by (17):

$$\frac{f}{\alpha} = \frac{c}{k + w} \approx \frac{80}{300 + 60} = \frac{1}{4}.$$

Noting this and (34), I obtain the steady-state values for the partial derivatives in (35):

$$F_L \doteq \frac{\partial F}{\partial L} = -\frac{f^2}{\alpha} \left[s_f + \frac{s(k^*, f^*)}{\rho + m} \right] < 0, \quad (36)$$

$$F_k \doteq \frac{\partial F}{\partial k} = -\frac{f^2}{\alpha c^*} \approx -\frac{m}{4 \times 0.8} = -\frac{m}{3}, \quad (37)$$

$$F_\lambda \doteq \frac{\partial F}{\partial \lambda_L} = L^* \frac{f^2}{\alpha} = \frac{m}{4} L^*, \quad (38)$$

$$F_c \doteq \frac{\partial F}{\partial c} = \frac{m}{4(c^*)^2} (w + k^* + s_f L^*) > 0. \quad (39)$$

Noting these, the characteristic equation for the system (1), (23), (31) and (32) with (35) in the vicinity of the steady state is

$$J \doteq \begin{vmatrix} F_L L^* - \nu & F_k L^* & F_c L^* & F_\lambda L^* \\ \frac{\partial \dot{k}}{\partial L} & \frac{\partial \dot{k}}{\partial k} - \nu & \frac{\partial \dot{k}}{\partial c} & \frac{\partial \dot{k}}{\partial \lambda_L} \\ -(F_L + s_k) c^* & -F_k c^* & -F_c c^* - \nu & -F_\lambda c^* \\ -F_L \lambda_L^* & -F_k \lambda_L^* & -F_c \lambda_L^* & \rho + m - F_\lambda \lambda_L^* - \nu \end{vmatrix} = 0, \quad (40)$$

where ν is the characteristic root.

In the vicinity of the steady state, from (23) and (34) it follows that

$$\frac{\partial \dot{k}}{\partial k} = r - \mu - s_k L^* - (s_f L^* + w + k^*) F_k \\ = \rho + m - (s_f L^* + w + k^*) F_k. \quad (41)$$

Because consumption per head, c^* , emissions per head, s , capital per head, k^* , and population L^* have equilibrium values in the limit $m \rightarrow 0$, then, given (36)-(41), I obtain

$$\lim_{m \rightarrow 0} F_k = \lim_{m \rightarrow 0} F_c = \lim_{m \rightarrow 0} F_\lambda = 0, \quad \lim_{m \rightarrow 0} \frac{\partial k}{\partial k} = \rho + m,$$

$$\lim_{m \rightarrow 0} J = (F_L L^* - \nu)(\rho + m - \nu)^2 (-F_c c^* - \nu) = 0.$$

This and (39) yield two unstable roots $\nu_1 = \nu_2 = \rho + m > 0$ and one stable root $\nu_3 = -F_c c^* < 0$. Thus, the system is saddle-point stable only if the last root is stable [cf. (36)]:

$$\nu_4 = F_L L^* = -\frac{f^2}{\alpha} \left[s_f + \frac{s(k^*, f^*)}{\rho + m} \right] < 0$$

$$\Leftrightarrow s_f > -\frac{s(k^*, f^*)}{\rho + m}.$$

This result can be rephrased as follows:

Proposition 7. If the mortality rate m is low enough and if furthermore the dirty sector is capital intensive $s_f > 0$ or moderately labor intensive $s_f \in (-s/(\rho + m), 0]$, then optimal public policy is saddle-point stable.

5. CONCLUSIONS

At least when the mortality rate is small enough and the dirty sector is capital intensive or moderately labor intensive, public policy is saddle-point stable. Then, in the laissez-faire case, the size of population decreases with a low and increases with a high mortality rate indefinitely. With government intervention, population becomes stationary and it is possible to maximize welfare by the following strategy. To restrict population growth, child rearing should be taxed, if the dirty sector is capital intensive or moderately labor intensive, otherwise subsidized. If the capita-intensive sector is dirty, but the labor-intensive sector clean, then a capital tax is socially optimal. If the capita-intensive sector is clean, but the labor-intensive sector dirty, then a capital subsidy is socially optimal.

APPENDIX

I define per head variables

$$k \doteq \frac{K}{L}, \quad f \doteq \frac{L_r}{L}, \quad k_j \doteq \frac{K_j}{L} \quad \text{and} \quad l_j \doteq \frac{L_j}{L} \quad \text{for } j \in \{c, d\}. \quad (1)$$

Then, dividing (4) and (3) by population L yields the constraints

$$k \geq k_c + k_d, \quad 1 \geq l_c + l_d + f, \quad y^c \doteq \frac{Y_c}{L} = F^c(k_c, l_c),$$

$$y^d = \frac{Y_d}{L} = F^d(k_d, l_d). \quad (2)$$

Because the sectors are subject to constant returns to scale, there are no pure profits and total factor income is determined by

$$y = (1 - f)w + rk,$$

where $1 - f$ the proportion of labor in production, $(1 - f)w$ total wages per head, k capital per head and rk total interests per head.

In both sectors, the marginal products of labor and capital are equal to the wage w and the interest rate r . Because the production functions F^d and F^c are linearly homogeneous [cf. (3)], this implies

$$F_L^d(\xi_d, 1) = w = F_L^c(\xi_c, 1), \quad F_K^d(\xi_d, 1) = r = F_K^c(\xi_c, 1),$$

$$\xi_c \doteq k_c/l_c, \quad \xi_d \doteq k_d/l_d, \quad (3)$$

where ξ_c and ξ_d are capital intensities in the clean and dirty sectors, respectively. From the four equations (3) it

follows that ξ_c, ξ_d, w and r are all constants. From (.2) it follows that

$$l_c = 1 - f - l_d,$$

$$k = k_d + k_c = \xi_d l_d + \xi_c l_c = (\xi_d - \xi_c)l_d + (1 - f)\xi_c.$$

Solving for l_d and l_d yields

$$l_d = \frac{k - (1 - f)\xi_c}{\xi_d - \xi_c},$$

$$l_c = 1 - f - \frac{k - (1 - f)\xi_c}{\xi_d - \xi_c} = \frac{(1 - f)\xi_d - k}{\xi_d - \xi_c}. \quad (.4)$$

Noting (2), (3) and (.4), we obtain emissions per head as the function

$$s(k, f) \doteq \frac{1}{L}(\gamma Y_c + Y_d) = \gamma y^c + y^d$$

$$= F^d(\xi_d, 1)l_d + \gamma F^c(\xi_c, 1)l_c$$

$$= F^d(\xi_d, 1) \frac{k - (1 - f)\xi_c}{\xi_d - \xi_c} + \gamma F^c(\xi_c, 1) \frac{(1 - f)\xi_d - k}{\xi_d - \xi_c}$$

$$\text{with } s_k \doteq \frac{\partial s}{\partial k} = \frac{F^d(\xi_d, 1) - \gamma F^c(\xi_c, 1)}{\xi_d - \xi_c} > 0 \Leftrightarrow$$

$$s_f \doteq \frac{\partial s}{\partial f} = \frac{\xi_c F^d(\xi_d, 1) - \gamma \xi_d F^c(\xi_c, 1)}{\xi_d - \xi_c} > 0 \Leftrightarrow \xi_d > \xi_c.$$

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