An Auslader-Reiten principle and a lifting problem over commutative DG algebras

Maiko Ono

July 2018

Graduate School of Natural Science and Technology
(Doctor's Course)
OKAYAMA UNIVERSITY

Contents

Preface

Homological algebra has its roots in algebraic topology. In the 1950s, H. Cartan and S. Eilenberg developed homological algebra as a fundamental common tool in various fields. The homological approach to commutative ring theory was first introduced in a research on generators and their relations of a finitely generated module by D. Herbert in 1890. He showed that each finitely graded module of a polynomial ring over a field has a finite graded free resolution. Nowadays it is known as the Herbert's syzygy theorem. In 1950s, commutative ring theory came to a turning point. A lot of researchers (e.g., M. Auslander, D. Buchsbaum, D. Rees, D. G. Northcott, J.-P. Serre, etc.) started to investigate commutative rings by using homological methods. On account of their studies, commutative ring theory made great progress. In particular, one of the most significant theorems, which was proved by J.-P. Serre, is a characterization of regular local rings by the finiteness of the global dimension of a commutative Noetherian local ring. This is a revolutionary result that represents the interplay between ideal theoretic aspects and homological algebraic aspects of regular local rings. Since then, homological algebra has contributed to the development of commutative ring theory. Moreover many various conjectures which are stated by notations of homological algebra has proposed in commutative ring theory. (the Nakayama conjecture, the Auslander-Reiten conjecture, the Huneke-Wiegand conjecture, etc.)

On the other hand, A. Grothendieck and J-L. Verdier established derived categories in 1960s. By virtue of the introduction of derived categories, it is possible to uniformly treat the classical homological algebra. Nowadays, we often use them as useful tools to simply and deeply understand our research objects. In this thesis, we work on our problems by using (categorical) homological methods.

This doctoral thesis is composed of the following two individual themes.

In Chpater 1, we generalized the Auslander-Reiten (AR) duality theorem in the derived category of a commutative Noetherian ring. In 1975, Auslander-Reiten theory was established by M. Auslander and I. Reiten. Auslnader-Reiten theory gives us the fundamental structure of a category of maximal Cohen-Macaulay modules. The AR duality plays a central role of Auslander-Reiten theory. In fact, we see that the AR duality induces the existence of Auslander-Reiten sequence in the category of maximal Cohen-Macaulay modules over a Cohen-Macaulay local ring with an isolated singularity. In recent years, O. Iyama and M. Wemyss [20] have generalized the AR duality to the the case where a ring is a Cohen-Macaulay local that has singularities with one dimensional singular loci. Then we consider a problem as follows; what is the general principle behind the AR duality theorem and the generalized AR duality due to Iyama and Wemyss? We attempted to find a more general form of the theorem which leads us to them. Finally we give a principle in the derived category of modules over a commutative Noetherian ring which we call an AR principle. By applying our AR principle to certain cases, we see that it implies naturally not only the classical AR duality but the Iyama-Wemyss's duality. As an application of our AR principle, we give a partial answer to a conjecture which is called the Auslander-Reiten(AR) conjecture. The AR conjecture can be stated as follows;

Let R be a commutative Noetherian ring and N a finitely generated R-module. If $\mathrm{Ext}^i_R(N,N) = \mathrm{Ext}^i_R(N,R) = 0$ for all $i > 0$, then N is projective.

In Chapter 2, we investigate a lifting problem for differential graded (DG) modules over a differential graded (DG) algebra. First of all, we introduce our motivation for this research. M. Auslander, S. Ding and Ø. Solberg [3] showed that the AR conjecture holds for complete intersections. They showed the result by using lifting theory for finitely generated modules over a complete intersection. To prove that the AR conjecture holds for a commutative Noetherian ring R, we may assume that R is a complete local ring. It is known from the Cohen's structure theorem that there is a surjective ring homomorphism $S \to R$ where S is a regular local ring. If we can prove that a finitely generated R-module which satisfies the assumption of the AR conjecture is liftable to S, then the AR conjecture is completely resolved. (Here, a finitely generated R-module N is said to be liftable to S if there is a finitely generated S-module M such that (1) $N \cong R \otimes_S M$ and (2) $\mathrm{Tor}^S_i(R, M) = 0$ for all $i > 0$.) However, it is difficult to investigate such a lifting problem. So, we employ our strategy that we approach the lifting problem by treating it as a problem for differential graded (DG) modules. It is known that it is possible to construct a commutative DG S-algebra $S\langle X_1, X_2, \cdots | dX_1 = x_1, dX_2 = x_2, \cdots \rangle$ that is quasi-isomorphic to the ring R as DG S-algebras. See [29]. The commutative DG S-algebra $R' = S\langle X_1, X_2, \cdots | dX_1 = x_1, dX_2 = x_2, \cdots \rangle$ resembles a "DG" polynomial ring over S with variables $X_1, X_2, \cdots$. Now we pose a conjecture;

If a semi-free DG R'-module N satisfies the condition $\mathrm{Ext}^i_{R'}(N, N) = 0$ and $\mathrm{Ext}^i_{R'}(N, R') = 0$ for $i > 0$, then N is liftable to S.

In this thesis, we consider a lifting problem in the situation $A \to B = A\langle X | dX = t\rangle$ where A is a commutative DG algebra and B is an extended DG R-algebra of A by the adjunction of one variable X which kills the cycle t in A. Recently, S. Nasseh and S. Sather-Wagstaff [22], and S. Nasseh and Y. Yoshino[23] studied the lifting or weak lifting problems for $A \to A\langle X | dX = t\rangle$ where the degree of X, which is denoted by $|X|$, is odd. In this case, $A\langle X | dX = t\rangle$ is just a Koszul complex over A. We remark that in the case where $|X|$ is even, $A\langle X | dX = t\rangle$ has the structure of a free algebra over A with a divided variable X.

Secondly, we introduce our main results which are sated in Chapter 2. In the rest of the preface, let A be a commutative DG algebra over a commutative ring and B be a extended DG algebra $A\langle X | dX = t\rangle$ of A where $|X|$ is positive even. For a semi-free DG B-module N, we describe a obstruction $[\Delta_N]$ to be a liftable to A as an element of $\mathrm{Ext}^{|X|+1}_B(N, N)$. In particular, we characterize the liftability for a semi-free DG B-module N which is bounded below in terms of the condition whether $[\Delta_N]$ is zero or non-zero. Moreover, we prove that if a semi-free DG B-module N is liftable to A and $\mathrm{Ext}^{|X|}_B(N, N) = 0$, then a lifting of N is determined up to DG A-isomorphisms. Finally, we present an example for liftings in the situation $R \to R\langle X, Y | dX = x, dY = y\rangle$ where R is a commutative complete local ring and $R\langle X, Y | dX = x, dY = y\rangle$ is an extended DG algebra that is obtained by the adjunction of variables X and Y of degree 1 and degree 2, respectively.

Acknowledgments.

I am deeply grateful to my supervisor Professor Yuji Yoshino for his kind guidance and warm encouragement, and for giving me stimulating research environments while carrying out my doctoral work. I would like to express my gratitude to Professor Mitsuyasu Hashimoto, Professor Tokuji Araya, Professor Ryo Takahashi, Professor Kei-ichiro Iima, Professor Hiroyuki Minamoto, Professor Nasseh Saeed and Dr. Tsutomo Nakamura for their variable suggestions and dissuasions. I also would like thank to Professor Hai-Long Dao for giving me an opportunity to visit Kansas University on February, 2017 and discuss mathematical problems.

Finally, I wish to thank to my family for their support and warm encouragement.

1 An Auslander-Reiten principle in derived categories

The contents of this chapter are entirely contained in the author's paper [25] with Y.Yoshino.

In Chapter 1, we give a principle in derived categories, which lies behind the classical Auslander-Reiten duality and its generalized version by Iyama and Wemyss. We apply the principle to show the validity of the Auslander-Reiten conjecture over a Gorenstein ring in the case where the ring has dimension larger than two and the singular locus has at most one dimension.

1.1 Introduction

Throughout this chapter, all rings are assumed to be commutative Noetherian rings.

Let R be a Cohen-Macaulay local ring of Krull dimension d with canonical module ω and let M, N be maximal Cohen-Macaulay modules over R. Assume that R has only an isolated singularity. Then we have an isomorphism

$$\underline{\mathrm{Hom}}_R(M, N)^\vee \cong \mathrm{Ext}^1_R(N, \tau M), \tag{1.0.1}$$

where $\tau M = \mathrm{Hom}_R(\Omega^d(\mathrm{Tr}M), \omega)$. For the definition of $\mathrm{Tr}M$, see the paragraph preceding Corollary 1.7. The isomorphism is known as Auslander-Reiten duality, or simply AR duality. For the proof of (1.0.1) the reader should refer to [5, Proposition 1.1].

The AR duality plays a crucial role in the theory of maximal Cohen-Macaulay modules. In fact, one can derive from (1.0.1) the existence of Auslander-Reiten sequence in the category of maximal Cohen-Macaulay modules over an isolated singularity. See [30, Theorem 3.2]. Further assuming that R is Gorenstein, it assures us that the stable category of the category of maximal Cohen-Macaulay modules has $(d-1)$-Calabi-Yau property. See [21, Theorem 8.3].

Recently, Iyama and Wemyss have generalized the AR duality to rings whose singular locus has at most one dimension. See [20, Theorem 3.1].

The purpose of this chapter is to propose a general principle behind the AR duality, by which we mean a general theorem for modules or chain complexes of modules in a kind of general form that encompasses the classical

AR duality and its generalization. In the end we have reached the following conclusion to this aim of building the principle, which we dare call the AR principle :

Theorem[**AR Principle**](Theorem 1.3). *Let R be a commutative Noetherian ring and let W be a specialization-closed subset of* $\mathrm{Spec}(R)$*. Given a bounded complex I of injective R-modules with $I^i = 0$ for all $i > n$ and a complex X such that the support of $H^i(X)$ is contained in W for all $i < 0$, the natural map $\Gamma_W I \to I$ induces isomorphisms*

$$\mathrm{Ext}^i_R(X, \Gamma_W I) \xrightarrow{\cong} \mathrm{Ext}^i_R(X, I) \qquad \textit{for } i > n.$$

This result is proved in §2. We emphasize that this theorem is similar to a version of the local duality theorem; see Remark 1.4.

In §3 we apply the AR principle to deduce the formula (1.0.1). See Corollary 1.7. In fact, we consider the case where $(R, \mathfrak{m})$ is a local ring, $W = \{\mathfrak{m}\}$, and I is a dualizing complex of R. Then it naturally induces Theorem 1.6 below, which is also regarded as a generalization of the original AR duality (1.0.1).

In almost the same circumstances above but $W = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim R/\mathfrak{p} \leqq 1\}$, we deduce from AR principle the generalization of AR duality due to Iyama and Wemyss. This will be explained in detail in §4. See Theorem 1.10 and Corollary 1.13 in particular.

In §5 we discuss the Auslander-Reiten conjecture for modules over Gorenstein rings. The Auslander-Reiten conjecture (abbreviated to ARC) can be stated in its most general form as follows:

(ARC) Let R be a commutative Noetherian ring and M a finitely generated R-module. If $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$, then M is projective.

This conjecture is a source of the generalized Nakayama conjecture, and related to other conjectures such as Nakayama and Tachikawa conjecture. Initially Auslander and Reiten [4] asked it for non-commutative Artinian algebras, but later Auslander, Ding and Solberg [3] have set up it for commutative Noetherian rings, and shown (ARC) holds for complete intersections. In recent years, there has been several studies on (ARC), and it is proved affirmatively in several cases such as

- Artinian Gorenstein local rings with radical cube zero, by Huneke, Şega and Vraciu [19].
- Gorenstein local rings with codimension at most four, by Şega [27].
- Gorenstein rings with only an isolated singularity and Krull dimension not less than 2, by Araya [1].

By virtue of our AR principle we can prove a more stronger result than (ARC) in some cases. Actually Corollary 5.5 below forces the following:

Theorem (See Corollary 1.18). Let R be a Gorenstein local ring of dimension d that is larger than 2. Assume that M is a maximal Cohen-Macaulay R-module whose non-free locus has dimension $\leqq 1$, i.e. $M_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$-free for any $\mathfrak{p} \in \operatorname{Spec}(R)$ with $\dim R/\mathfrak{p} > 1$. Furthermore we assume that

$$\operatorname{Ext}_R^{d-1}(M, M) = 0 = \operatorname{Ext}_R^{d-2}(M, M).$$

Then M is a free R-module.

1.2 AR principle in derived category

Let R be a commutative Noetherian ring. We denote by $\mathcal{D} = D(R)$ the full derived category of R. Note that the objects of $\mathcal{D}$ are chain complexes over R, which we denote by the cohomological notation such as

$$X = (\cdots \to X^{n-1} \to X^n \to X^{n+1} \to \cdots).$$

It should be noted that $\mathcal{D}$ has a structure of triangulated category with shift functor, denoted by $X \mapsto X[1]$.

Recall that a full subcategory $\mathcal{L}$ of $\mathcal{D}$ is called a *localizing* subcategory if it is a triangulated subcategory and it is closed under direct sums and direct summands. By Bousfield theorem [24, Theorem 2.6], the natural inclusion $i : \mathcal{L} \hookrightarrow \mathcal{D}$ has a right adjoint functor $\gamma : \mathcal{D} \to \mathcal{L}$, i.e.,

$$\operatorname{Hom}_{\mathcal{D}}(iX, Y) \cong \operatorname{Hom}_{\mathcal{L}}(X, \gamma Y),$$

for all $X \in \mathcal{L}$ and $Y \in \mathcal{D}$.

For a chain complex X we define the small support $\operatorname{supp}(X)$ to be the set of prime ideals $\mathfrak{p}$ such that $X \,{}^{\mathrm{L}}\!\otimes_R \kappa(\mathfrak{p}) \neq 0$, where $\kappa(\mathfrak{p}) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. (Cf.,

[13].) The small support of a full subcategory $\mathcal{L}$ of $\mathcal{D}$ is the union of all the small supports of objects of $\mathcal{L}$, so that

$$\mathrm{supp}(\mathcal{L}) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid X \overset{\mathrm{L}}{\otimes}_R \kappa(\mathfrak{p}) \neq 0 \text{ for some } X \in \mathcal{L}\}.$$

For any subset $W \subseteq \mathrm{Spec}(R)$, the full subcategory

$$\mathcal{L}_W = \{X \in \mathcal{D} \mid \mathrm{supp}(X) \subseteq W\}$$

is a localizing subcategory of $\mathcal{D}$. The correspondences $\mathcal{L} \mapsto \mathrm{supp}(\mathcal{L})$ and $W \mapsto \mathcal{L}_W$ yield a bijection between the set of localizing subcategories of $\mathcal{D}$ and the power set of $\mathrm{Spec}(R)$. This was proved by A.Neeman [24, Theorem 2.8].

We say that a localizing subcategory $\mathcal{L}_W$ is *smashing* if W is a specialization-closed subset of $\mathrm{Spec}(R)$, and in this case, the functor $\gamma : \mathcal{D} \to \mathcal{L}_W$ is nothing but the local cohomology functor $\mathrm{R}\Gamma_W$. See [24, Theorem 3.3].

Remark 1.1. The big support $\mathrm{Supp}(X)$ of a chain complex $X \in \mathcal{D}$ is the set of prime ideals $\mathfrak{p}$ of R with the property $X \overset{\mathrm{L}}{\otimes}_R R_{\mathfrak{p}} \neq 0$, or equivalently $H(X)_{\mathfrak{p}} \neq 0$. In general it holds

$$\mathrm{supp}(X) \subseteq \mathrm{Supp}(X),$$

for all $X \in \mathcal{D}$. If X belongs to $\mathcal{D}^-_{fg}(R)$, by which we denote a full subcategory of $\mathcal{D}$ consisting of right bounded complexes with finite cohomologies, then we have $\mathrm{supp}(X) = \mathrm{Supp}(X)$ which is a closed subset of $\mathrm{Spec}(R)$. Given a specialization-closed subset W of $\mathrm{Spec}(R)$, a complex X is an object of $\mathcal{L}_W$ if and only if the big support of X is contained in W if and only if $\mathrm{Supp}H^i(X) \subseteq W$ for all $i \in \mathbb{Z}$. See [11].

Definition 1.2. Let $X \in \mathcal{D}$ be a chain complex;

$$\cdots \longrightarrow X^n \xrightarrow{d^n} X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \longrightarrow \cdots .$$

For an integer n we define the truncations $\sigma_{>n}X$ and $\sigma_{\leqq n}X$ as follows:

$$\sigma_{>n}X = \left(\cdots \to 0 \to \mathrm{Im}\ d^n \to X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \to \cdots\right)$$
$$\sigma_{\leqq n}X = \left(\cdots \to X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \to \mathrm{Ker}\ d^n \to 0 \to \cdots\right)$$

See [18, Chapter 1; §7] for more detail. Note that there is an exact triangle in $\mathcal{D}$;

$$\sigma_{\leqq n}X \longrightarrow X \longrightarrow \sigma_{>n}X \longrightarrow \sigma_{\leqq n}X[1].$$

Now the following theorem is a main theorem of this part, which we call AR principle. Actually this is an equivalent version of the theorem in the Introduction.

Theorem 1.3. *Let X, I be chain complexes in $\mathcal{D}$ and let $\mathcal{L}$ be a smashing subcategory of $\mathcal{D}$ with $\gamma : \mathcal{D} \to \mathcal{L}$ a right adjoint functor to the natural embedding $\iota : \mathcal{L} \hookrightarrow \mathcal{D}$. We assume the following conditions hold for some integer n;*

1. *I is a bounded injective complex, and right bounded at most in degree n.*

2. *$\sigma_{\leqq -1} X \in \mathcal{L}$.*

Then the natural map $\gamma I \to I$ induces an isomorphism;

$$\sigma_{>n}\mathrm{RHom}_R(X, \gamma I) \cong \sigma_{>n}\mathrm{RHom}_R(X, I).$$

Proof. Since γ is right adjoint to ι, we have a counit morphism $\iota\gamma I \to I$ in $\mathcal{D}$, which induces the morphism

$$\mathrm{RHom}_R(X, \gamma I) \to \mathrm{RHom}_R(X, I).$$

To prove the theorem it is enough to shown that this morphism induces isomorphisms

$$H^i(\mathrm{RHom}_R(X, \gamma I)) \cong H^i(\mathrm{RHom}_R(X, I))$$

for $i > n$.

Note that $H^i(\mathrm{RHom}_R(X, I)) \cong \mathrm{Hom}_{\mathcal{D}}(X, I[i])$, where $[i]$ denotes the i iterations of the shift functor $[1]$ in the triangulated category $\mathcal{D}$. Therefore, noting that I is a bounded injective complex, we see that an element f of $H^i(\mathrm{RHom}_R(X, I))$ is a homotopy equivalence class of a chain map $X \to I[i]$:

$$\begin{CD}
\cdots \to X^{-i-1} @>>> X^{-i} @>>> \cdots @>>> X^{-i+n} @>>> X^{-i+n+1} \to \cdots \\
@V{f^{-i-1}}VV @V{f^{-i}}VV @. @V{f^{-i+n}}VV @V{0}VV \\
\cdots \to I^{-1} @>>> I^{0} @>>> \cdots @>>> I^{n} @>>> 0
\end{CD}$$

Since $-i + n < 0$, we have

$$\mathrm{Hom}_{\mathcal{D}}(X, I[i]) \cong \mathrm{Hom}_{\mathcal{D}}(\sigma_{\leqq -1} X, I[i]). \tag{1.3.1}$$

Now since $\mathcal{L}$ is smashing, it forces that γ is of the form $\mathrm{R}\Gamma_W$ for a specialization-closed subset W of $\mathrm{Spec}(R)$. Thus γI is a subcomplex of I and each term of γI is also an injective module. As a consequence γI satisfies the same condition as I. Therefore similar argument as above shows the isomorphism

$$\mathrm{Hom}_{\mathcal{D}}(X, \gamma I[i]) \cong \mathrm{Hom}_{\mathcal{D}}(\sigma_{\leqq -1}X, \gamma I[i]). \tag{1.3.2}$$

The right-hand sides in the equations (1.3.1),(1.3.2) are naturally isomorphic each other, since $\sigma_{\leqq -1}X \in \mathcal{L}$. This completes the proof. ∎

Theorem [AR Principle] stated in the introduction is a direct restatement of Theorem 2.3. In fact, if W is a specialization-closed subset of $\mathrm{Spec}(R)$ and if $\mathcal{L} = \mathcal{L}_W$, then it follows that $\gamma = \mathrm{R}\Gamma_W$, and the condition (2) in Theorem 2.3 is equivalent to that $\mathrm{Supp} H^i(X) \subseteq W$ for $i < 0$, by Remark 2.1.

Remark 1.4. We adopted such description of the AR principle as in Theorem 1.3, because of its similarity to the generalized version of local duality, that can be stated as follows:

Let X, I be complexes in $\mathcal{D}$ and let $\mathcal{L}$ be a smashing subcategory of $\mathcal{D}$ with $\gamma : \mathcal{D} \to \mathcal{L}$ being as above. We assume the following conditions hold;

1. *I is a bounded injective complex.*
2. *$X \in \mathcal{D}^-_{fg}(R)$.*

Then we have an isomorphism in $\mathcal{D}$;

$$\mathrm{RHom}_R(X, \gamma I) \cong \gamma \mathrm{RHom}_R(X, I).$$

This version of local duality theorem was proposed by Hartshorne [18, Chapter V, Theorem 6.2] and later generalized by Foxby[13, Proposition 6.1].

Question 1.5. In Theorem 1.3 and Remark 1.4, do the conclusions hold true if $\mathcal{L}$ is not necessarily smashing but just localizing?

1.3 The case of isolated singularity

Now in this section we assume that $(R, \mathfrak{m})$ is a local ring of dimension d. We apply the AR principle to the following setting;

- $W_0 = \{\mathfrak{m}\}$,
- $\mathcal{L} = \mathcal{L}_{W_0} = \{X \in \mathcal{D} \mid \mathrm{supp} X \subseteq \{\mathfrak{m}\}\}$, and
- I is a dualizing complex of R.

We normalize I so that it is of the form;

$$0 \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \cdots \longrightarrow I^d \longrightarrow 0,$$

where $I^i = \bigoplus_{\dim R/\mathfrak{p}=d-i} E_R(R/\mathfrak{p})$ for each i. (Cf. [18] or [28].) In this case, since $\gamma = \mathrm{R}\Gamma_{\mathfrak{m}}$, we have $\gamma I = E[-d]$ where $E = E_R(R/\mathfrak{m})$ is the injective hull of $R/\mathfrak{m}$.

For a chain complex $X \in \mathcal{D}$ we denote

$$X^\vee = \mathrm{RHom}_R(X, E), \quad X^\dagger = \mathrm{RHom}_R(X, I),$$

which are respectively called the Matlis dual and the canonical dual (or Grothendieck dual) of X. To all such situations, Theorem 1.3 can be applied directly and we get the following theorem.

Theorem 1.6. *Let* $(R, \mathfrak{m})$ *is a local ring of dimension* d *as above. We assume* $X \in \mathcal{D}$ *satisfies that* $\mathrm{supp}(\sigma_{\leqq -1} X) \subseteq \{\mathfrak{m}\}$. *Then we have an isomorphism*

$$\sigma_{>0}(X^\vee) \cong \sigma_{>d}(X^\dagger)[d].$$

By Remark 2.1, the theorem can be stated in the following way:

If $\mathrm{Supp} H^i(X) \subseteq \{\mathfrak{m}\}$ *for all* $i \leqq -1$, *then* $\mathrm{Ext}^j_R(X, E) \cong \mathrm{Ext}^{j+d}_R(X, I)$ *for all* $j > 0$.

Now assume that $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring which possesses canonical module ω. Note in this case that the dualizing complex I is a minimal injective resolution of ω. We denote the category of maximal Cohen-Macaulay modules over R by $\mathrm{CM}(R)$. For a finitely generated R-module M we write as $\mathrm{NF}(M)$ the non-free locus of M, i.e.,

$$\mathrm{NF}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not } R_{\mathfrak{p}}\text{-free}\}.$$

It is known and easily proved that $\mathrm{NF}(M)$ is a closed subset of $\mathrm{Spec}(R)$ whenever M is finitely generated, since $\mathrm{NF}(M) = \mathrm{Supp}\,\mathrm{Ext}^1_R(M, \Omega M)$.

We need to recall the definition of the (Auslander) transpose for the corollary below. Let $F_1 \xrightarrow{\partial} F_0 \to M \to 0$ be a minimal free presentation of a finitely generated R-module M. Then the transpose $\mathrm{Tr}M$ is defined as $\mathrm{Coker}(\mathrm{Hom}(\partial, R))$.

Theorem 1.6 implies the following result that generalizes a little the Auslander-Reiten duality mentioned in the beginning of this chapter.

Corollary 1.7. *Let R be a Cohen-Macaulay local ring with canonical module and let $M, N \in \mathrm{CM}(R)$. Assume that $\mathrm{NF}(M) \cap \mathrm{NF}(N) \subseteq \{\mathfrak{m}\}$. Then we have an isomorphism*

$$\underline{\mathrm{Hom}}_R(M,N)^\vee \cong \mathrm{Ext}^1_R(N, \tau M),$$

where $\tau M = [\Omega^d(\mathrm{Tr}M)]^\dagger$

Proof. Setting $X = \mathrm{Tr}M \overset{\mathrm{L}}{\otimes}_R N$, we see that the condition $\mathrm{NF}(M) \cap \mathrm{NF}(N) \subseteq \{\mathfrak{m}\}$ forces that $\mathrm{supp}\ \sigma_{<0}X \subseteq \mathrm{Supp}\ \sigma_{<0}X \subseteq \{\mathfrak{m}\}$. Hence we can apply Theorem 1.6 to X and get an isomorphism

$$H^{d+1}(X^\dagger) \cong H^1(X^\vee) \cong H^{-1}(X)^\vee.$$

It is known that $H^{-1}(X) = \mathrm{Tor}^R_1(\mathrm{Tr}M, N) \cong \underline{\mathrm{Hom}}_R(M,N)$. See [30, Lemma 3.9]. On the other hand, we have

$$H^{d+1}(X^\dagger) \cong \mathrm{Ext}^{d+1}_R(\mathrm{Tr}M, N^\dagger) \cong \mathrm{Ext}^1_R(\Omega^d\mathrm{Tr}M, N^\dagger) \cong \mathrm{Ext}^1_R(N, [\Omega^d\mathrm{Tr}M]^\dagger),$$

since $X^\dagger = \mathrm{RHom}_R(\mathrm{Tr}M \overset{\mathrm{L}}{\otimes}_R N, I) \cong \mathrm{RHom}_R(N, [\mathrm{Tr}M]^\dagger) \cong \mathrm{RHom}_R(\mathrm{Tr}M, N^\dagger)$. ■

Remark 1.8. We remark form Corollary 3.2 that AR duality still holds even if M is a finitely generated R-module but N is not necessarily finitely generated. Suppose that $\mathrm{NF}(M) \cap \mathrm{Supp}(N) \subseteq \{\mathfrak{m}\}$ and $H^i(N^\dagger) = 0$ for $i > 0$. In a similar way to Corollary 3.2, we can show that

$$\underline{\mathrm{Hom}}(M,N)^\vee \cong \mathrm{Ext}^1_R(N, \tau M).$$

Note that if R is a Cohen-Macaulay complete local ring and N is a big Cohen-Macaulay module, it follows that $H^i(N^\dagger) = 0$ for $i > 0$. See [15, Proposition 2.6]. For example, $R = k[[x,y]]$ is a formal power series ring where k is a field and $N = R \oplus E_R(R/(y))$. Assume that M is a finitely generated R-module which is a locally free on the punctured spectrum. Since N is a big Cohen-Macaulay module from [15, Remark 3.3], we obtain that

$$\underline{\mathrm{Hom}}(M, E_R(R/(y)))^\vee \cong \mathrm{Ext}^1_R(E_R(R/(y)), \tau M).$$

1.4 The case of codimension one singular locus

In this section $(R, \mathfrak{m})$ always denotes a local ring of dimension d as before. We consider the following conditions, in which we apply the AR principle 1.3:

- $W_1 = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \dim R/\mathfrak{p} \leqq 1\}$,
- $\mathcal{L} = \mathcal{L}_{W_1} = \{X \in \mathcal{D} \mid \mathrm{supp} X \subseteq W_1\}$, and
- R has a (normalized) dualizing complex I.

In this case, since $\gamma = \mathrm{R}\Gamma_{W_1}$, it follows that γI is a two-term complex;

$$0 \longrightarrow I^{d-1} \xrightarrow{\partial} I^d \longrightarrow 0, \tag{1.8.1}$$

where

$$I^{d-1} = \bigoplus_{\dim R/\mathfrak{p}=1} E_R(R/\mathfrak{p}) =: J, \quad I^d = E_R(R/\mathfrak{m}) =: E.$$

We thus have a triangle in $\mathcal{D}$;

$$E[-d] \longrightarrow \gamma I \longrightarrow J[-d+1] \xrightarrow{\partial[-d+1]} E[-d+1].$$

Now let $X \in \mathcal{D}$ and assume that $\sigma_{\leqq -1} X \in \mathcal{L}$. It follows that there is a triangle in $\mathcal{D}$;

$$X^{\vee}[-d] \longrightarrow \mathrm{RHom}_R(X, \gamma I) \longrightarrow \mathrm{RHom}_R(X, J)[-d+1] \xrightarrow{\mathrm{RHom}(X,\partial)} X^{\vee}[-d+1].$$

On the other hand, Theorem 1.3 says that there are isomorphisms

$$H^{d+i}(X^{\dagger}) \cong H^{d+i}(\mathrm{RHom}_R(X, \gamma I)),$$

for $i > 0$. Combining this isomorphism with the triangle above, we have the following proposition.

Proposition 1.9. *Assume that $X \in \mathcal{D}$ satisfies that $\mathrm{supp}(\sigma_{\leqq -1} X) \subseteq W_1$. Then there is a long exact sequence of R-modules:*

$$\begin{array}{llllll}
 & \mathrm{Hom}_R(H^{-1}(X), J) & \xrightarrow{\mathrm{Hom}_R(H^{-1}(X),\partial)} & H^{-1}(X)^{\vee} & \longrightarrow & H^{d+1}(X^{\dagger}) \\
\longrightarrow & \mathrm{Hom}_R(H^{-2}(X), J) & \xrightarrow{\mathrm{Hom}_R(H^{-2}(X),\partial)} & H^{-2}(X)^{\vee} & \longrightarrow & H^{d+2}(X^{\dagger}) \\
\longrightarrow & \quad\cdots & & & & \\
\longrightarrow & \mathrm{Hom}_R(H^{-i}(X), J) & \xrightarrow{\mathrm{Hom}_R(H^{-i}(X),\partial)} & H^{-i}(X)^{\vee} & \longrightarrow & H^{d+i}(X^{\dagger}) \\
\longrightarrow & \quad\cdots . & & & &
\end{array}$$

This leads us to the following theorem that is more applicable to our computation. Recall that $\mathcal{D}_{fg}(R)$ denotes the full subcategory of $\mathcal{D}$ consisting of all chain complexes whose cohomology modules are finitely generated R-modules.

Theorem 1.10. *Let X be a chain complex in $\mathcal{D}_{fg}(R)$, and assume that* $\mathrm{supp}(\sigma_{\leqq -1}X) \subseteq W_1$*. Then, for any $i > 0$, there is a short exact sequence*

$$0 \longrightarrow H^0_{\mathfrak{m}}(H^{-i}(X))^{\vee} \longrightarrow H^{d+i}(X^{\dagger})^{\wedge} \longrightarrow H^1_{\mathfrak{m}}(H^{-i-1}(X))^{\vee} \longrightarrow 0,$$

and an isomorphism

$$H^0_{\mathfrak{m}}(H^{-i}(X))^{\vee} \cong H^0_{\mathfrak{m}}(H^{d+i}(X^{\dagger})).$$

In the sequence above, $^{\wedge}$ denotes the $\mathfrak{m}$-adic completion.

Note from Remark 1.1 that the assumption for X in Theorem 1.10 is precisely saying that $\mathrm{Supp} H^i(X) \subseteq W_1$ for $i < 0$.

Before proving Theorem 1.10 we note the following lemmas.

Lemma 1.11. *Let $\partial : J \to E$ be the map in (1.8.1) above. Suppose we are given a finitely generated R-module M such that* $\dim M \leqq 1$*. Then the following hold.*

1. *There are isomorphisms of $\widehat{R}$-modules*

$$\mathrm{Ker}(\mathrm{Hom}_R(M, \partial))^{\wedge} \cong H^1_{\mathfrak{m}}(M)^{\vee}, \quad \mathrm{Coker}(\mathrm{Hom}_R(M, \partial))^{\wedge} \cong H^0_{\mathfrak{m}}(M)^{\vee}.$$

2. *$H^1_{\mathfrak{m}}(M)^{\vee}$ is a Cohen-Macaulay $\widehat{R}$-module of dimension one, in particular, it holds that $H^0_{\mathfrak{m}}(H^1_{\mathfrak{m}}(M)^{\vee}) = 0$.*

Proof. (1) Noting that $\dim M \leqq 1$, we have $\mathrm{Hom}_R(M, \bigoplus_{\dim R/\mathfrak{p}=i} E_R(R/\mathfrak{p})) = 0$ for all $i > 1$. It hence follows the equalities

$$\mathrm{Ker}(\mathrm{Hom}_R(M, \partial)) = H^{d-1}(M^{\dagger}) \text{ and } \mathrm{Coker}(\mathrm{Hom}_R(M, \partial)) = H^d(M^{\dagger}).$$

On the other hand the local duality theorem implies that

$$H^{d-1}(M^{\dagger})^{\wedge} \cong H^1_{\mathfrak{m}}(M)^{\vee} \text{ and } H^d(M^{\dagger})^{\wedge} \cong H^0_{\mathfrak{m}}(M)^{\vee}.$$

(2) We may assume $\dim M = 1$. Note that $\overline{M} = M/H^0_{\mathfrak{m}}(M)$ is a one-dimensional Cohen-Macaulay R-module, and $H^1_{\mathfrak{m}}(M) = H^1_{\mathfrak{m}}(\overline{M})$. Replace M by $\overline{M}$, and we may assume that M is a one-dimensional Cohen-Macaulay module. Then it is known that $(M^{\dagger})[d-1] = \mathrm{Ext}_R^{d-1}(M, I)$ is again one-dimensional Cohen-Macaulay, hence so is the completion $\mathrm{Ext}_R^{d-1}(M, I)^{\wedge}$. However it follows from the local duality that $\mathrm{Ext}_R^{d-1}(M, I)^{\wedge} = H^1_{\mathfrak{m}}(M)^{\vee}$. ■

Lemma 1.12. *Let M be a finitely generated module over a local ring $(R, \mathfrak{m})$. Then the equality*

$$H^0_{\mathfrak{m}}(M) \cong H^0_{\mathfrak{m}}(M^{\wedge})$$

holds.

Proof. Note that $H^0_{\mathfrak{m}}(M)$ is a unique submodule N of M such that N is of finite length and M/N has no nontrivial submodule of finite length (or equivalently depth $M/N > 0$). Taking the $\mathfrak{m}$-adic completion for modules in a short exact sequence $0 \to H^0_{\mathfrak{m}}(M) \to M \to \bar{M} \to 0$, and noting that the $\mathfrak{m}$-adic topology on $H^0_{\mathfrak{m}}(M)$ is discrete, we have an exact sequence

$$0 \to H^0_{\mathfrak{m}}(M) \to M^{\wedge} \to \bar{M}^{\wedge} \to 0.$$

Since $\mathrm{depth}\bar{M}^{\wedge} > 0$ as $\mathrm{depth}\bar{M} > 0$, we have the desired equality $H^0_{\mathfrak{m}}(M) = H^0_{\mathfrak{m}}(M^{\wedge})$. ■

Now we proceed to the proof of Theorem 1.10. It follows from Theorem 1.9 that there is an exact sequence:

$$0 \to \mathrm{Coker}(\mathrm{Hom}_R(H^{-i}(X), \partial)) \to H^{d+i}(X^{\dagger}) \to \mathrm{Ker}(\mathrm{Hom}_R(H^{-i-1}(X), \partial)) \to 0,$$

for $i > 0$. Since $H^{-i}(X)$ and $H^{-i-1}(X)$ are finitely generated and their dimensions are at most one, we can apply Lemma 1.11 and get a short exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(H^{-i}(X))^{\vee} \longrightarrow H^{d+i}(X^{\dagger})^{\wedge} \longrightarrow H^1_{\mathfrak{m}}(H^{-i-1}(X))^{\vee} \longrightarrow 0,$$

as in Theorem 1.9. To show the isomorphism in Theorem 1.10, apply the functor $H^0_{\mathfrak{m}}$ to this short exact and it is enough to notice from Lemma 1.11(2) and 1.12 that $H^0_{\mathfrak{m}}(H^1_{\mathfrak{m}}(H^{-i-1}(X))^{\vee} = 0$ and $H^0_{\mathfrak{m}}(H^{d+i}(X^{\dagger})) \cong H^0_{\mathfrak{m}}(H^{d+i}(X^{\dagger})^{\wedge})$. ■

Now let us assume that $(R, \mathfrak{m})$ is Cohen-Macaulay and let $M, N \in \mathrm{CM}(R)$. We apply Theorem 1.10 above to $X = \mathrm{Tr}M \overset{\mathrm{L}}{\otimes}_R N$, and we get a theorem of Iyama and Wemyss [20].

Corollary 1.13. *Let* $(R, \mathfrak{m})$ *be a Cohen-Macaulay local ring and let* $M, N \in \mathrm{CM}(R)$*. We assume that* $\mathrm{NF}(M) \cap \mathrm{NF}(N) \subseteq W_1$*. Then, for each* $i > 0$*, there is a short exact sequence;*

$$0 \to H^0_{\mathfrak{m}}(\underline{\mathrm{Hom}}_R(M, \Omega^{i-1}N))^\vee \to \mathrm{Ext}^i_R(N, \tau M)^\wedge \to H^1_{\mathfrak{m}}(\underline{\mathrm{Hom}}_R(M, \Omega^i N))^\vee \to 0,$$

and an isomorphism;

$$H^0_{\mathfrak{m}}(\underline{\mathrm{Hom}}_R(M, \Omega^{i-1}N))^\vee \cong H^0_{\mathfrak{m}}(\mathrm{Ext}^i_R(N, \tau M)).$$

1.5 A remark on the Auslander-Reiten conjecture

Now in this section we restrict ourselves to consider the case where R is Gorenstein. In this case it is easy to see that the syzygy functor $\Omega : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ is an auto-equivalence. Hence, in particular, one can define the cosyzygy functor Ω^{-1} on $\underline{\mathrm{CM}}(R)$ as the inverse of Ω. We note from [12] and [17, 2.6] that $\underline{\mathrm{CM}}(R)$ is a triangulated category with shift functor $[1] = \Omega^{-1}$. Note that $\underline{\mathrm{Hom}}_R(M, N) \cong \mathrm{Ext}^1_R(M, \Omega^1 N)$ for all $M, N \in \mathrm{CM}(R)$. Note also that, since R is Gorenstein, we have

$$\tau M = [\Omega^d \mathrm{Tr} M]^\dagger \cong \Omega^{d-2}(M^*)^* \cong M[d-2].$$

Therefore Corollary 1.7 implies the fundamental duality.

Corollary 1.14. *Let* R *be a Gorenstein local ring of dimension* d*. Assume that* $M, N \in \mathrm{CM}(R)$ *satisfy* $\mathrm{NF}(M) \cap \mathrm{NF}(N) \subseteq \{\mathfrak{m}\}$*. Then there is a functorial isomorphism*

$$\underline{\mathrm{Hom}}_R(M, N)^\vee \cong \underline{\mathrm{Hom}}_R(N, M[d-1]).$$

Note that this is the case for any M and N if R has at most an isolated singularity. On the other hand Theorem 1.10 implies the following:

Corollary 1.15. *Let* R *be a Gorenstein local ring of dimension* d*. Assume that* $M, N \in \mathrm{CM}(R)$ *satisfy* $\mathrm{NF}(M) \cap \mathrm{NF}(N) \subseteq W_1$*. Then there is a short exact sequence;*

$$0 \to H^0_{\mathfrak{m}}(\underline{\mathrm{Hom}}_R(M, N[d-1]))^\vee \to \underline{\mathrm{Hom}}_R(N, M)^\wedge \to H^1_{\mathfrak{m}}(\underline{\mathrm{Hom}}_R(M, N[d-2]))^\vee \to 0.$$

Araya [1] shows that Corollary 1.14 implies the Auslander-Reiten conjecture for Gorenstein rings with isolated singularity of dimension not less than 2. Since the ring R is Gorenstein, noting that a module M is a maximal Cohen-Macaulay module if and only if $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$, we can state the AR conjecture for Gorenstein rings as follows:

Conjecture 1.16. Let R be a Gorenstein local ring as above and let M be in $\mathrm{CM}(R)$. If $\mathrm{Ext}^i_R(M, M) = 0$ for all $i > 0$, then M is a free R-module.

In fact, the assumption of the conjecture is equivalent to the conditions $\underline{\mathrm{Hom}}_R(M, M[i]) = 0$ for $i > 0$. On the other hand M is free if and only if $\underline{\mathrm{Hom}}_R(M, M) = 0$. Therefore it is restated in the following form:

– If $\underline{\mathrm{Hom}}_R(M, M[i]) = 0$ for $i > 0$, then $\underline{\mathrm{Hom}}_R(M, M) = 0$.

By virtue of Corollary 1.14, the conjecture is trivially true if R is an isolated singularity and $d \geqq 2$. This is what Araya proved in his paper [1].

In contrast to this, we can prove the following theorem by using Corollary 1.15.

Theorem 1.17. *Let* $(R, \mathfrak{m})$ *be a Gorenstein local ring of dimension* d *and let* $M, N \in \mathrm{CM}(R)$. *Assume the following conditions:*

1. $\mathrm{NF}(M) \cap \mathrm{NF}(N) \subseteq W_1$,
2. depth $\underline{\mathrm{Hom}}_R(M, N[d-1]) > 0$,
3. depth $\underline{\mathrm{Hom}}_R(M, N[d-2]) > 1$.

Then we have $\underline{\mathrm{Hom}}(N, M) = 0$.

Note in the theorem that we adopt the convention that the depth of the zero module is $+\infty$, so that the conditions (2)(3) contain the case when $\underline{\mathrm{Hom}}_R(M, N[d-1]) = \underline{\mathrm{Hom}}_R(M, N[d-2]) = 0$.

The proof of Theorem 1.17 is straightforward from Corollary 1.15. In fact the assumptions for M, N in the theorem imply the vanishing of the both ends in the short exact sequence in Corollary 1.15, hence we have $\underline{\mathrm{Hom}}(N, M) = 0$.

The following is a direct consequence of Theorem 1.17.

Corollary 1.18. *Let* $(R, \mathfrak{m})$ *be a Gorenstein local ring of dimension* d *and let* $M \in \mathrm{CM}(R)$. *Assume that* $\mathrm{NF}(M) \subseteq W_1$. *Furthermore assume*

$$\text{depth } \underline{\mathrm{Hom}}_R(M, M[d-1]) > 0, \quad \text{depth } \underline{\mathrm{Hom}}_R(M, M[d-2]) > 1.$$

Then M *is a free* R*-module.*

This result assures us that the AR conjecture 1.16 holds true if $\mathrm{NF}(M) \subseteq W_1$ and $d \geqq 3$. This is automatically the case, for example, whenever R is a normal Gorenstein local domain of dimension 3.

2 A lifting problem for DG modules

The contents of this chapter are entirely contained in the author's paper [26] with Y.Yoshino.

Let $B = A\langle X|dX = t\rangle$ be an extended DG algebra by the adjunction of a variable of positive even degree n, and let N be a semi-free DG B-module that is assumed to be bounded below as a graded module. We prove in this paper that N is liftable to A if $\mathrm{Ext}_B^{n+1}(N, N) = 0$. Furthermore such a lifting is unique up to DG isomorphisms if $\mathrm{Ext}_B^{n}(N, N) = 0$.

2.1 Introduction

Lifting problems of algebraic structures appear in various phases of algebra theory. In fact many authors have studied variants of liftings in their own fields such as modular representation theory, deformation theory and commutative ring theory etc. From the particular view point of ring theory, the lifting problem and its weak variant, called weak lifting problem, was systematically investigated firstly by M. Auslander, S. Ding and Ø. Solberg [3]. The second author of the present paper extended the lifting problems to chain complexes and developed a theory of weak liftings for complexes in [31]. On the other hand in the papers [22, 23], the lifting or weak lifting problems were generalized into the corresponding problems for DG modules, however they only considered the cases of Koszul complexes that are DG algebra extensions by adding one variable of odd degree. In contrast, our main target in the present paper is the lifting problem for DG algebra extension obtained by adding a variable of positive *even* degree.

Let A be a commutative DG algebra over a commutative ring R, and X be a variable of degree $n = |X|$. Then we can consider the extended DG algebra $A\langle X|dX = t\rangle$ that is obtained by the adjunction of variable X with relation $dX = t$, where t is a cycle in A of degree $n-1$. Note that if n is odd, then $A\langle X|dX = t\rangle = A \oplus XA$ as a right A-module, which is somewhat similar to a Koszul complex. In contrast, if n is even, $A\langle X|dX = t\rangle = \bigoplus_{i\geq 0} X^{(i)}A$ is a free algebra over A with divided variable X that resembles a polynomial ring. In each case there is a natural DG algebra homomorphism $A \to A\langle X|dX = t\rangle$. See §2 below for more detail.

In general, let $A \to B$ be a DG algebra homomorphism. Then a DG B-module N is said to be *liftable* to A if there is a DG A-module M with the property $N \cong B \otimes_A M$ as DG B-modules. In such a case M is called a *lifting* of N. We are curious about the lifting problem for the particular case that $B = A\langle X | dX = t\rangle$. The both papers [22, 23] treated the lifting problem in such cases but with the assumption that $|X|$ is odd. They actually showed that the vanishing of $\mathrm{Ext}_B^{|X|+1}(N, N)$ implies the weak liftability of N.

We consider the lifting problem for $A \to B = A\langle X | dX = t\rangle$ in the case that $|X|$ is positive and even. Surprisingly enough we are able to show in this paper that the vanishing of $\mathrm{Ext}_B^{|X|+1}(N, N)$ implies the liftablity (not weak liftability) and moreover the vanishing of $\mathrm{Ext}_B^{|X|}(N, N)$ implies the uniqueness of such a lifting. The following is our main theorem of this chapter that answers the question raised in [23, Remark 3.8].

Theorem. (Theorem 2.17 and Theorem 2.19) *Let A be a DG R-algebra, where R is a commutative ring. Let $B = A\langle X | dX = t\rangle$ denote a DG R-algebra obtained from A by the adjunction of a variable X of positive even degree. Further assume that N is a semi-free DG B-module.*

(1) *Under the assumption that N is bounded below as a graded R-module, if $\mathrm{Ext}_B^{|X|+1}(N, N) = 0$, then N is liftable to A.*

(2) *If N is liftable to A and if $\mathrm{Ext}_B^{|X|}(N, N) = 0$, then a lifting of N is unique up to DG A-isomorphisms.*

In §2 we prepare the necessary definitions and notations for DG algebras and DG modules that will be used in this chapter. In §3 we introduce the notion of j-operator and give several useful properties of j-operators. §4 is the main body of the present chapter, where we prove the main theorem above. §5 is devoted to giving an example for liftings. In fact, we present it in the situation $S \to S\langle X, Y | dX = x, dY = y\rangle$ where S is a commutative complete local ring and $S\langle X, Y | dX = x, dY = y\rangle$ is an extended DG algebra that is obtained by the adjunction of variables X and Y of degree 1 and degree 2 respectively.

2.2 Preliminary on DG algebras and DG modules

We summarize some definitions and notations that will be used in Chapter 2. Throughout this chapter, R always denotes a commutative ring. Basically all modules considered in this chapter are meant to be R-modules and all algebras are R-algebras.

Let $A = \bigoplus_{n\geq 0} A_n$ be a non-negatively graded R-algebra equipped with a graded R-linear homomorphism $d^A : A \to A$ of degree -1. Then $A = \left(\bigoplus_{n\geq 0} A_n,\ d^A\right)$ is called a *(commutative) differential graded R-algebra*, or a *DG R-algebra* for short, if it satisfies the following conditions:

1. For homogeneous elements a and b of A, $ab = (-1)^{|a||b|}ba$ where $|a|$ denotes the degree of a. Moreover if $|a|$ is odd, then $a^2 = 0$.

2. The graded R-algebra A has a differential structure, by which we mean that $\left(d^A\right)^2 = 0$.

3. The differential d^A satisfies the derivation property; $d^A(ab) = d^A(a)b + (-1)^{|a|}ad^A(b)$ for homogeneous elements a and b of A.

Note that all DG algebras considered in this chapter are non-negatively graded R-algebras. We often denote by $A^\natural$ the underlying graded R-algebra for a DG R-algebra A.

Let $f : A \to B$ be a graded R-algebra homomorphism between DG R-algebras. By definition f is a *DG algebra homomorphism* if it is a chain map, i.e., $d^B f = f d^A$.

Let A be a DG R-algebra and $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a graded left A-module equipped with a graded R-linear map $\partial^M : M \to M$ of degree -1. Then $M = \left(\bigoplus_{n\in\mathbb{Z}} M_n, \partial^M\right)$ is called a left *differential graded A-module*, or a *DG A-module* for short, if it satisfies the following conditions:

1. The graded module M has a differential structure, i.e., $\left(\partial^M\right)^2 = 0$.

2. The differential ∂^M satisfies the derivation property over A, i.e., $\partial^M(am) = d^A(a)m + (-1)^{|a|}a\partial^M(m)$ for $a \in A$ and $m \in M$.

Note that every left DG A-module M can be regarded as a right DG A-module by defining the right action as $ma = (-1)^{|a||m|}am$ for $a \in A$ and $m \in M$. Similarly to the case of DG algebras, $M^\natural$ denotes the underlying graded $A^\natural$-module for a DG A-module M.

Let M and N be DG A-modules. Then the graded tensor product $M^\natural \otimes_{A^\natural} N^\natural$ of graded modules has the differential mapping defined by

$$\partial^{M\otimes_A N}(m\otimes n) = \partial^M(m)\otimes n + (-1)^{|m|} m\otimes \partial^N(n) \quad \text{for } \ m \in M \ \text{ and } \ n \in N.$$

The tensor product of DG A-modules is denoted by $M \otimes_A N$, by which we mean the DG A-module $\left(M^\natural \otimes_{A^\natural} N^\natural, \partial^{M\otimes_A N}\right)$.

If $A \to B$ is a DG algebra homomorphism, and if M is a DG A-module, then $B \otimes_A M$ is regarded as a DG B-module via action $b(b' \otimes m) = bb' \otimes m$ for $b, b' \in B$ and $m \in M$.

Definition 2.1. Let $A \to B$ be a DG algebra homomorphism. A DG B-module N is called *liftable* to A if there exists a DG A-module M such that N is isomorphic to $B \otimes_A M$ as DG B-modules. In this case, M is called a *lifting* of N.

Let A be a DG R-algebra and let M, N be DG A-modules.

A graded R-module homomorphism $f : M \to N$ of degree r ($r \in \mathbb{Z}$) is, by definition, an R-linear mapping from M to N with $f(M_n) \subseteq N_{n+r}$ for all $n \in \mathbb{Z}$. In such a case we denote $|f| = r$. The set of all graded R-module homomorphisms of degree r is denoted by $\mathrm{Hom}_R(M,N)_r$. Then $\mathrm{Hom}_R(M,N) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_R(M,N)_r$ is naturally a graded R-module. A graded R-module homomorphism $f \in \mathrm{Hom}_R(M,N)_r$ is called *A-linear* if it satisfies $f(am) = (-1)^{r|a|}af(m)$ for $a \in A$ and $m \in M$. We denote by $\mathrm{Hom}_A(M,N)_r$ the set of all A-linear homomorphisms of degree r. Then $\mathrm{Hom}_A(M,N) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_A(M,N)_r$ has a structure of graded A-module, on which we can define the differential as follows:

$$\partial^{\mathrm{Hom}_A(M,N)}(f) = \partial^N f - (-1)^{|f|} f \partial^M.$$

In such a way we have defined the DG A-module $\mathrm{Hom}_A(M,N)$.

By definition, a *DG A-homomorphism* $f : M \to N$ is an A-linear homomorphism of degree 0 that is a cycle as an element of $\mathrm{Hom}_A(M,N)$. A DG A-homomorphism $f : M \to N$ gives a *DG A-isomorphism* if f is invertible as a graded A-linear homomorphism. On the other hand a DG A-homomorphism $f : M \to N$ is called a *quasi-isomorphism* if the homology mapping $H(f) : H(M) \to H(N)$ is an isomorphism of graded R-modules.

A DG A-module F is said to be *semi-free* if $F^\natural$ possesses a graded $A^\natural$-free basis E which decomposes as a disjoint union $E = \bigsqcup_{i\geq 0} E_i$ of subsets indexed by natural numbers and satisfies $\partial^M(E_i) \subseteq \sum_{j<i} AE_j$ for $i \geq 0$. A *semi-free resolution* of a DG A-module M is a DG A-homomorphism $F \to M$ from a semi-free DG A-module F to M, which is a quasi-isomorphism. It is known that any DG A-module has a semi-free resolution. See [7, Theorem 8.3.2].

Given a DG A-module M, and taking a semi-free resolution $F_M \to M$, one can define the ith extension module by

$$\mathrm{Ext}^i_A(M, N) := H_{-i}\left(\mathrm{Hom}_A(F_M, N)\right),$$

which is known to be independent of the choice of a semi-free resolution of M over A. See [1, Proposition 1.3.2.].

There is a well-known way of constructing of a DG algebra that kills a cycle by adjunction of a variable. See [1],[16] and [29] for details. To make it more explicit, let A be a DG R-algebra and take a homogeneous cycle t in A. We are able to construct an extended DG R-algebra B of A by the adjunction of a variable X with $|X| = |t| + 1$ which kills the cycle t in the following way. In both cases, we denote B by $A\langle X | dX = t\rangle$.

(1) If $|X|$ is odd, then $B = A \oplus XA$ with algebra structure $X^2 = 0$ in which the differential is defined by $d^B(a + Xb) = d^A(a) + tb - Xd^A(b)$.

(2) If $|X|$ is even, then $B = \bigoplus_{i\geq 0} X^{(i)}A$ which is an algebra with divided powers of variable X. Namely it has the multiplication structure $X^{(i)}X^{(j)} = \binom{i+j}{i} X^{(i+j)}$ for $i, j \in \mathbb{Z}_{\geq 0}$ with $|X^{(i)}| = i|X|$. (Here we use the convention $X^{(0)} = 1, X^{(1)} = X$.) Adding to the derivation property, the differential on B is simply defined by the rule $d^B(X^{(i)}) = X^{(i-1)}t$ for $i > 0$, hence it is given as follows for general elements:

$$d^B\left(\sum_{i=0}^{n} X^{(i)}a_i\right) = \sum_{i=0}^{n-1} X^{(i)}\left\{d^A(a_i) + ta_{i+1}\right\} + X^{(n)}d^A(a_n).$$

In each case a natural map $A \to B = A\langle X|dX = t\rangle$ is a DG algebra homomorphism. As we have mentioned in the introduction, we are interested in DG B-modules N that are liftable to A, particularly in the case (2) above, that is, when $|X|$ is even. In fact, S. Nasseh and Y. Yoshino have studied a liftable condition, or more generally a weak liftable condition, for DG B-modules in the case where $|X|$ is odd. See [23, Theorem 3.6] for more detail.

2.3 the j-operator

As in the end of the previous section, A is a DG R-algebra in which we take a cycle t, that is, $d^At = 0$. Our specific assumption here is that $|t|$ is an odd non-negative integer. We denote by $B = A\langle X | dX = t\rangle$ the extended DG algebra of A by the adjunction of variable X that kills the cycle t. Since $|X|$ is even, note that

$$B^\natural = \bigoplus_{i\geq 0} X^{(i)} A^\natural, \tag{2.1.1}$$

where the right hand side is a direct sum of right A-modules.

Let N be a DG B-module, and we assume the following conventional assumption:

$$\text{There is a graded } A^\natural\text{-module } M \text{ satisfying } N^\natural = B^\natural \otimes_{A^\natural} M. \tag{2.1.2}$$

Note that if N is a semi-free DG B-module, then, since $N^\natural$ is a free $B^\natural$-module, it is always under such a circumstance. By virtue of the decomposition (2.1.1), we may write $N^\natural$ as follows under the assumption (2.1.2):

$$N^\natural = \bigoplus_{i\geq 0} X^{(i)} M. \tag{2.1.3}$$

Note that there are equalities of R-modules

$$N_n = \bigoplus_{i\geq 0,\ k+i|X|=n} X^{(i)} M_k,$$

for all $n \in \mathbb{Z}$.

Now let r be an integer and let $f \in \mathrm{Hom}_R(N,N)_r$. Recall that f is R-linear with $f(N_n) \subseteq N_{n+r}$ for all $n \in \mathbb{Z}$. Given such an f, we consider the restriction of f on M, i.e., $f|_M \in \mathrm{Hom}_R(M,N)_r$. Along the decomposition (2.1.3), one can decompose $f|_M$ into the following form:

$$f|_M = \sum_{i\geq 0} X^{(i)} f_i, \tag{2.1.4}$$

where each $f_i \in \mathrm{Hom}_R(M,M)_{r-i|X|}$. Actually, for $m \in M$, there is a unique decomposition $f(m) = \sum_i X^{(i)} m_i$ with $m_i \in M$ along (2.1.3). Then f_i is defined by $f_i(m) = m_i$. Note that the decomposition (2.1.4) is unique as long as we work under the fixed setting (2.1.3). We call the equality (2.1.4) *the expansion of* $f|_M$ and often call f_0 *the constant term of* $f|_M$.

Taking the expansion of $f|_M$ as in (2.1.4), we consider the graded R-linear homomorphism

$$\varphi = \sum_{i\geq 0} X^{(i)} f_{i+1},$$

which belongs to $\mathrm{Hom}_R(M,N)_{r-|X|}$. This R-linear mapping φ can be extended to an R-linear mapping $j(f)$ on N by setting $j(f)(X^{(i)}m_i) = X^{(i)}\varphi(m_i)$ for each $i \geq 0$ and $m_i \in M$. In such a way we obtain $j(f) \in \mathrm{Hom}_R(N,N)_{r-|X|}$.

Summing up the argument above we get the mapping $j : \mathrm{Hom}_R(N,N)_r \to \mathrm{Hom}_R(N,N)_{r-|X|}$ for all $r \in \mathbb{Z}$, which we call *the j-operator* on $\mathrm{Hom}_R(N,N)$. For the later use we remark that the actual computation of $j(f)$ is carried out in the following way;

$$\begin{aligned} j(f)(X^{(n)}m) &= X^{(n)}j(f)(m) \\ &= X^{(n)}\sum_{i\geq 0} X^{(i)}f_{i+1}(m) \\ &= \sum_{i\geq 0} X^{(n+i)}\binom{n+i}{i}f_{i+1}(m) \end{aligned} \tag{2.1.5}$$

for $n \geq 0$ and $m \in M$.

Theorem and Definition 2.2. Under the assumption (2.1.2) we can define a graded R-linear mapping $j : \mathrm{Hom}_R(N,N) \to \mathrm{Hom}_R(N,N)$ of degree $-|X|$, which we call *the j-operator* on $\mathrm{Hom}_R(N,N)$. For any $f \in \mathrm{Hom}_R(N,N)_r$, taking the expansion (2.1.4) of $f|_M$ along the decomposition (2.1.3), $j(f)$ maps $X^{(n)}m$ to $\sum_{i\geq 0} X^{(n+i)}\binom{n+i}{i}f_{i+1}(m)$ as in (2.1.5).

Remark 2.3. The notion of j-operator was first introduced by J. Tate in the paper [29] and extensively used by T.H. Gulliksen and G. Levin [16].

In the rest of this chapter we always assume the condition (2.1.2) for a DG B-module N.

Definition 2.4. We denote by $\mathcal{E}$ the set of all B-linear homomorphisms on N, i.e., $\mathcal{E} = \mathrm{Hom}_B(N,N)$. Note that $\mathcal{E} \subseteq \mathrm{Hom}_R(N,N)$ and that a homogenous element $f \in \mathrm{Hom}_R(N,N)$ belongs to $\mathcal{E}$ if and only if $f(bn) = (-1)^{|b||f|}bf(n)$ for $b \in B$ and $n \in N$.

We say that a graded R-linear mapping $\delta \in \mathrm{Hom}_R(N,N)$ is a *B-derivation* if it satisfies $|\delta| = -1$ (i.e., $\delta \in \mathrm{Hom}_R(N,N)_{-1}$) and $\delta(bn) = d^B(b)n + (-1)^{|b||\delta|}b\delta(n)$ for $b \in B$ and $n \in N$. We denote by $\mathcal{D}$ the set of all B-derivations on N.

Remark 2.5.

1. Assume that an R-linear mapping $\delta : N \to N$ satisfies the derivation property $\delta(bn) = d^B(b)n + (-1)^{|b||\delta|} b\delta(n)$. Since $|d^B(b)n| = |b| + |n| - 1$ and $|b\delta(n)| = |b|+|n|+|\delta|$, if $|\delta| \neq -1$ then δ is never a graded mapping.

2. If $\delta \in \mathcal{D}$ then the actual computation for δ is carried out by the following rule:
$$\delta\left(\sum_{i\geq 0} X^{(i)} m_i\right) = \sum_{i\geq 0} X^{(i)} \left\{tm_{i+1} + \delta(m_i)\right\}.$$

3. If $\delta, \delta' \in \mathcal{D}$ then it is easy to see that $\delta - \delta'$ is in fact B-linear, hence $\delta - \delta' \in \mathcal{E}$.

Note that both $\mathcal{E}$ and $\mathcal{D}$ are graded R-submodules of $\mathrm{Hom}_R(N, N)$.

Lemma 2.6. *Under the assumption (2.1.2), any A-linear homomorphism $\alpha : M \to M$ is uniquely extended to $\tilde{\alpha} \in \mathcal{E}$ such that the constant term of the expansion of $\tilde{\alpha}|_M$ equals α. Similarly any A-derivation $\beta : M \to M$ is uniquely extended to $\tilde{\beta} \in \mathcal{D}$ such that the constant term of the expansion of $\tilde{\beta}|_M$ equals β.*

In both cases, we have $j(\tilde{\alpha}) = 0$ and $j(\tilde{\beta}) = 0$.

Proof. In each case the extension is obtained by making the tensor product with B over A:
$$\tilde{\alpha} = B \otimes_A \alpha, \quad \tilde{\beta} = B \otimes_A \beta.$$
More precisely, any element $n \in N$ is written as $n = \sum_{i\geq 0} X^{(i)} m_i$ for $m_i \in M$ along (2.1.3), and taking into account the linearity of $\tilde{\alpha}$ and the derivation property of $\tilde{\beta}$, we can define them by the following equalities:
$$\tilde{\alpha}(n) = \sum_{i\geq 0} X^{(i)} \alpha(m_i), \quad \tilde{\beta}(n) = \sum_{i\geq 0} X^{(i)} \left\{tm_{i+1} + \beta(m_i)\right\}.$$
Their uniqueness follows from the next lemma. ∎

Lemma 2.7. *Assume that $f, g \in \mathcal{E}$ and $\delta, \delta' \in \mathcal{D}$. Then the following assertions hold.*

(1) *$f = g$ if and only if $f|_M = g|_M$.*

(2) $\delta = \delta'$ *if and only if* $\delta|_M = \delta'|_M$.

Proof. (1) Assume $f|_M = g|_M$. For each $n \in N$ we decompose it into the form $n = \sum_{i\geq 0} X^{(i)} m_i$ along the decomposition (2.1.3). Since f and g are B-linear, we have

$$f\left(\sum_{i\geq 0} X^{(i)} m_i\right) = \sum_{i\geq 0} X^{(i)} f(m_i) = \sum_{i\geq 0} X^{(i)} g(m_i) = g\left(\sum_{i\geq 0} X^{(i)} m_i\right),$$

and hence $f = g$.

(2) Assume $\delta|_M = \delta'|_M$. Noting from Remark 2.5(3) that $\delta - \delta'$ is a B-linear homomorphism, we see that $\delta - \delta' = 0$ by virtue of (1). ∎

Lemma 2.8. *The following assertions hold.*

(1) *If* $f \in \mathcal{E}$, *then* $j(f) \in \mathcal{E}$.

(2) *If* $\delta \in \mathcal{D}$ *then* $j(\delta) \in \mathcal{E}$.

(3) *Let* $\delta \in \mathcal{D}$. *Then the constant term* δ_0 *of the expansion of* $\delta|_M$ *is an* A*-derivation on* M.

As a consequence, the j*-operator defines a mapping* $\mathcal{E} \cup \mathcal{D} \to \mathcal{E}$.

Proof. (1) Write $f|_M = \sum_{i\geq 0} X^{(i)} f_i$ as in (2.1.4). Since f is B-linear and noting that $|f| \equiv |f_i| (\mathrm{mod}\ 2)$, we see that

$$f(am) = (-1)^{|a||f|} a f(m) = (-1)^{|a||f|} a \sum_{i\geq 0} X^{(i)} f_i(m) = \sum_{i\geq 0} X^{(i)} (-1)^{|a||f_i|} a f_i(m)$$

for $a \in A$ and $m \in M$. Thus by the uniqueness of expansion it is easy to see that

$$f_i(am) = (-1)^{|a||f_i|} a f_i(m).$$

Namely each f_i is A-linear, and therefore $j(f)|_M = \sum_{i\geq 0} X^{(i)} f_{i+1}$ is A-linear as well. Meanwhile, it follows from the definition of $j(f)$ or (2.1.5) that $j(f)$ commutes with the action of X on N. Thus $j(f)$ is B-linear, and $j(f)$ belongs to $\mathcal{E}$.

$(2),(3)$; Write $\delta|_M = \sum_{i\geq 0} X^{(i)}\delta_i$. Since δ is a B-derivation, we have equalities;

$$\begin{aligned}\delta(am) &= d^B(a)m + (-1)^{|a||\delta|}a\delta(m)\\ &= \left\{d^A(a)m + (-1)^{|a||\delta|}a\delta_0(m)\right\} + \sum_{i\geq 1} X^{(i)}(-1)^{|a||\delta|}a\delta_i(m),\end{aligned}$$

for $a \in A$ and $m \in M$. On the other hand, $\delta(am) = \sum_{i\geq 0} X^{(i)}\delta_i(am)$. Comparing these equalities and noting that $|\delta| \equiv |\delta_i| (\text{mod } 2)$ for all $i \geq 0$, we eventually have

$$\delta_0(am) = d^A(a)m + (-1)^{|a||\delta_0|}a\delta_0(m) \text{ and } \delta_i(am) = (-1)^{|a||\delta_i|}a\delta_i(m) \text{ for } i > 0,$$

which imply the desired results in (2) and (3). ■

Proposition 2.9. *The following equalities hold for $f, g \in \mathcal{E}$ and $\delta, \delta' \in \mathcal{D}$.*

(1) $j(fg) = j(f)g + fj(g)$.

(2) $j(f\delta)|_M = j(f)\delta|_M + fj(\delta)|_M$.

(3) $j(\delta f)|_M = j(\delta)f|_M + \delta j(f)|_M$.

(4) $j(\delta\delta')|_M = j(\delta)\delta'|_M + \delta j(\delta')|_M$.

Before proving Proposition 2.9, we should remark that graded R-module homomorphisms $f\delta$, δf and $\delta\delta'$ do not necessarily belong to $\mathcal{E}$ or $\mathcal{D}$, and neither do $j(f\delta), j(\delta f)$ and $j(\delta\delta')$. The equalities in (2)-(4) hold only when they are restricted on M.

Proof. (1) Note from Lemma 2.8 that $j(fg)$, $j(f)g$ and $fj(g)$ are elements of $\mathcal{E}$. By this reason, we have only to show that $j(fg)|_M = j(f)g|_M + fj(g)|_M$ by Lemma 2.7. Taking the expansions as $f|_M = \sum_{i\geq 0} X^{(i)}f_i$ and $g|_M = \sum_{i\geq 0} X^{(i)}g_i$, we have the equalities:

$$fg|_M = f(\sum_{i\geq 0} X^{(i)}g_i) = \sum_{i\geq 0} X^{(i)}fg_i = \sum_{n\geq 0} X^{(n)} \sum_{i=0}^{n} \binom{n}{i} f_i g_{n-i}.$$

Hence it follows from the definition of j-operator that

$$j(fg)|_M = \sum_{n\geq 0} X^{(n)} \sum_{i=0}^{n+1} \binom{n+1}{i} f_i g_{n-i+1}.$$

On the other hand, we have equalities;

$$\begin{aligned}
j(f)g|_M + fj(g)|_M &= \sum_{k\geq 0} j(f)(X^{(k)}g_k) + \sum_{k\geq 0} f(X^{(k)}g_{k+1}) \\
&= \sum_{k\geq 0} X^{(k)}(j(f)(g_k) + f(g_{k+1})) \\
&= \sum_{k\geq 0}\sum_{i\geq 0} X^{(i+k)}\binom{i+k}{i}(f_{i+1}g_k + f_i g_{k+1}) \\
&= \sum_{n\geq 0} X^{(n)}\left\{\sum_{i=1}^{n+1}\binom{n}{i-1} f_i g_{n-i+1} + \sum_{i=0}^{n}\binom{n}{i} f_i g_{n-i+1}\right\}.
\end{aligned}$$

Since $\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i}$ for $0 < i \leq n$, we deduce that $j(fg)|_M = j(f)g|_M + fj(g)|_M$.

(2) Recall from the previous lemma that the constant term δ_0 in the expansion $\delta|_M = \sum_{i\geq 0} X^{(i)}\delta_i$ is an A-derivation on M. Set $\tilde{\delta_0}$ as the extended B-derivation of δ_0 on N defined by means of Lemma 2.6. Then as we noted in Remark 2.5 (3), $\delta - \tilde{\delta_0}$ is B-linear of degree $|\delta| = -1$. Moreover we see that $j(\delta - \tilde{\delta_0}) = j(\delta)$, since $j(\tilde{\delta_0}) = 0$. Thus it follows from the equality (1) of the present lemma that $j((\delta - \tilde{\delta_0})f) = j(\delta - \tilde{\delta_0})f + (\delta - \tilde{\delta_0})j(f) = j(\delta)f + \delta j(f) - \tilde{\delta_0}j(f)$. On the other hand, $j((\delta - \tilde{\delta_0})f) = j(\delta f) - j(\tilde{\delta_0}f)$. Therefore we have that

$$j(\delta f) - j(\tilde{\delta_0}f) = j(\delta)f + \delta j(f) - \tilde{\delta_0}j(f).$$

Hence it is enough to prove the equality in the case where $\delta = \tilde{\delta_0}$, that is,

$$j(\tilde{\delta_0}f)|_M = \tilde{\delta_0}j(f)|_M. \tag{2.9.1}$$

To prove this, take the expansion as $f|_M = \sum_{i\geq 0} X^{(i)}f_i$, and we get

$$\tilde{\delta_0}f|_M = \tilde{\delta_0}\left(\sum_{i\geq 0} X^{(i)}f_i\right) = \sum_{i\geq 0} X^{(i)}(tf_{i+1} + \delta_0 f_i).$$

Then it follows that

$$j(\tilde{\delta_0}f)|_M = \sum_{i\geq 0} X^{(i)}(tf_{i+2} + \delta_0 f_{i+1}),$$

while

$$\tilde{\delta_0} j(f)|_M = \tilde{\delta_0}\left(\sum_{i\geq 0} X^{(i)} f_{i+1}\right) = \sum_{i\geq 0} X^{(i)}(tf_{i+2} + \delta_0 f_{i+1}).$$

This proves (2.9.1).

(3) Similarly to (2), it is sufficient to prove the equality $j(f\tilde{\delta_0})|_M = j(f)\tilde{\delta_0}|_M$. If $f|_M = \sum_{i\geq 0} X^{(i)} f_i$ is the expansion, then we have $f\tilde{\delta_0}|_M = f\delta_0 = \sum_{i\geq 0} X^{(i)} f_i \delta_0$. Hence it follows from the definition of j-operator that

$$j(f\tilde{\delta_0})|_M = \sum_{i\geq 0} X^{(i)} f_{i+1}\delta_0 = j(f)\delta_0 = j(f)\tilde{\delta_0}|_M,$$

as desired.

(4) Let δ_0 be the constant term of the expansion $\delta|_M = \sum_{i\geq 0} X^{(i)}\delta_i$. As in the proof of (2) we take the extension $\tilde{\delta_0}$ of δ_0, and hence it holds that $\delta - \tilde{\delta_0}$ is B-linear, and that $j(\delta - \tilde{\delta_0}) = j(\delta)$. Now applying the equality proved in (2), we have that

$$j((\delta-\tilde{\delta_0})\delta')|_M = j(\delta-\tilde{\delta_0})\delta'|_M + (\delta-\tilde{\delta_0})j(\delta')|_M = j(\delta)\delta'|_M + \delta j(\delta')|_M - \tilde{\delta_0} j(\delta')|_M. \tag{2.9.2}$$

In contrast, we have

$$j((\delta - \tilde{\delta_0})\delta') = j(\delta\delta') - j(\tilde{\delta_0}\delta'). \tag{2.9.3}$$

Combining these equalities, we obtain the equality:

$$j(\delta\delta')|_M = j(\delta)\delta'|_M + \delta j(\delta')|_M + j(\tilde{\delta_0}\delta')|_M - \tilde{\delta_0} j(\delta')|_M.$$

Thus it is enough to prove the following equality:

$$j(\tilde{\delta_0}\delta')|_M = \tilde{\delta_0} j(\delta')|_M. \tag{2.9.4}$$

To prove (2.9.4) let $\delta'|_M = \sum_{i\geq 0} X^{(i)}\delta'_i$ be the expansion, and we have that

$$\tilde{\delta_0}\delta'|_M = \tilde{\delta_0}\sum_{i\geq 0} X^{(i)}\delta'_i = \sum_{i\geq 0} X^{(i)}(t\delta'_{i+1} + \delta_0\delta'_i),$$

therefore it follows

$$j(\tilde{\delta_0}\delta')|_M = \sum_{i\geq 0} X^{(i)}(t\delta'_{i+2} + \delta_0\delta'_{i+1}).$$

On the other hand, we have

$$\tilde{\delta}_0 j(\delta')|_M = \tilde{\delta}_0 \sum_{i\geq 0} X^{(i)}\delta'_{i+1} = \sum_{i\geq 0} X^{(i)}(t\delta'_{i+2} + \delta_0\delta'_{i+1}).$$

It proves the equality (2.9.4). ■

Corollary 2.10. *Let $f \in \mathcal{E}$ and $\delta \in \mathcal{D}$, and assume that f is invertible in $\mathcal{E}$. Then we have equalities*:

(1) $j(f)f^{-1} + fj(f^{-1}) = 0$.

(2) $j(f\delta f^{-1}) = j(f)\delta f^{-1} + fj(\delta)f^{-1} + f\delta j(f^{-1})$.

Proof. (1) It follows from Proposition 2.9 (1) that $j(ff^{-1}) = j(f)f^{-1} + fj(f^{-1})$. On the other hand, it holds $j(ff^{-1}) = j(\mathrm{id}_N) = 0$, hence the equality (1) follows.

(2) First of all we note that both $j(f)\delta f^{-1} + f\delta j(f^{-1})$ and $fj(\delta)f^{-1}$ are B-linear. To verify this fact we remark that the following equalities hold:

$$\begin{aligned}
&(j(f)\delta f^{-1} + f\delta j(f^{-1}))(X^{(n)}m)\\
=&j(f)\delta(X^{(n)}f^{-1}(m)) + f\delta(X^{(n)}j(f^{-1})(m))\\
=&j(f)(tX^{(n-1)}f^{-1}(m)+X^{(n)}\delta f^{-1}(m))+f(tX^{(n-1)}j(f^{-1})(m)+X^{(n)}\delta j(f^{-1})(m))\\
=&tX^{(n-1)}(j(f)f^{-1}(m) + fj(f^{-1}(m))) + X^{(n)}(j(f)\delta f^{-1} + f(m)\delta j(f^{-1})(m))\\
=&X^{(n)}(j(f)\delta f^{-1} + f\delta j(f^{-1}))(m),
\end{aligned}$$

where the last equality holds because of (1). On the other hand, since $f\delta f^{-1}$ belongs to $\mathcal{D}$, $j(f\delta f^{-1})$ is B-linear as well. Therefore it is enough to prove the equality: $j(f\delta f^{-1})|_M = (j(f)\delta f^{-1} + fj(\delta)f^{-1} + f\delta j(f^{-1}))|_M$ by Lemma 2.7. From Proposition 2.9(2), we get

$$j(f^{-1}(f\delta f^{-1}))|_M = j(f^{-1})(f\delta f^{-1})|_M + f^{-1}j(f\delta f^{-1})|_M. \tag{2.10.1}$$

Meanwhile, Proposition 2.9(3) implies that

$$j(f^{-1}(f\delta f^{-1}))|_M = j(\delta f^{-1})|_M = j(\delta)f^{-1}|_M + \delta j(f^{-1})|_M. \tag{2.10.2}$$

Summarizing (2.10.1) and (2.10.2), we see that

$$\begin{aligned}
j(f\delta f^{-1})|_M &= -fj(f^{-1})(f\delta f^{-1})|_M + fj(\delta)f^{-1}|_M + f\delta j(f^{-1})|_M\\
&= j(f)\delta f^{-1}|_M + fj(\delta)f^{-1}|_M + f\delta j(f^{-1})|_M.
\end{aligned}$$

where the last equality holds by virtue of (1) in the present corollary. This completes the proof. ■

2.4 Main results

Now we are able to prove the main theorems of this chapter. See Theorem 2.17 and Theorem 2.19 below.

In the rest of the section, A always denotes a DG R-algebra and $B = A\langle X|dX = t\rangle$ is an extended DG algebra by the adjunction of variable X that kills the cycle $t \in A$, where $|X|$ is a positive even integer. Let N be a DG B-module and we always assume here that N is semi-free. We are interested in the conditions that sufficiently imply the liftability of N to A. Since $N^\natural$ is free as a $B^\natural$-module, the condition (2.1.2) is satisfied, that is, there is a graded $A^\natural$-module M such that $N^\natural \cong B^\natural \otimes_{A^\natural} M$ as graded $B^\natural$-modules.

The differential mapping ∂^N on N belongs to $\mathcal{D}$ which, we recall, is the set of all B-derivations on N. It thus follows from Lemma 2.8 that $j(\partial^N)$ is B-linear, equivalently $j(\partial^N) \in \mathcal{E}$. This specific element of $\mathcal{E}$ will be a key object when we consider the lifting property of N in the following argument. This is the reason why we make the following definition of Δ_N as

$$\Delta_N := j(\partial^N). \tag{2.10.3}$$

Recall again from Lemma 2.8 that Δ_N is a B-linear homomorphism on N such that $|\Delta_N| = -|X| - 1$ is an odd integer.

Remark 2.11. The exact same definition was made by S. Nasseh and Y. Yoshino in the case where $|X|$ is odd. See [23, Convention 2.5].

As we see in the next lemma, Δ_N defines an element of $\mathrm{Ext}_B^{|X|+1}(N, N)$, which will turn out to be an obstruction for the lifting of N to A.

Lemma 2.12. *It holds that $\Delta_N \partial^N = -\partial^N \Delta_N$. Hence Δ_N is a cycle of degree $-|X| - 1$ in $\mathcal{E} = \mathrm{Hom}_B(N, N)$.*

Proof. Noting that $\left(\partial^N\right)^2 = 0$, we have from Proposition 2.9 that $0 = j(\partial^N \partial^N)|_M = j(\partial^N)\partial^N|_M + \partial^N j(\partial^N)|_M$. On the other hand it is easily seen that $j(\partial^N)\partial^N + \partial^N j(\partial^N)$ is B-linear. Hence it follows from Lemma 2.7 that $j(\partial^N)\partial^N + \partial^N j(\partial^N) = 0$. ∎

In the proof of our main theorems, we shall need some argument on automorphisms on the DG B-module N. The following lemma is a preliminary for that purpose.

Lemma 2.13. *Let $\varphi : N \to N$ be a graded B-linear endomorphism of degree 0. As before we assume that the expansion is given as $\varphi|_M = \sum_{i \geq 0} X^{(i)} \varphi_i$. If φ is a B-linear automorphism on N, then the constant term φ_0 is an A-linear automorphism on M.*

Proof. Take a graded B-linear endomorphism ψ such that $\varphi\psi = \mathrm{id}_N = \psi\varphi$. Writing $\psi|_M = \sum_{n \geq 0} X^{(n)} \psi_n$ as well, we see that $\varphi\psi|_M = \mathrm{id}_M$ implies that the constant term $\varphi_0\psi_0$ of $\varphi\psi|_M$ is equal to id_M. Similarly $\psi_0\varphi_0 = \mathrm{id}_M$. Therefore φ_0 is an A-linear automorphism on M. ∎

A DG module L, or more generally a graded module $L = \bigoplus_{i \in \mathbb{Z}} L_i$, is said to be *bounded below* if $L_{-i} = 0$ for all suffieceintly large integers i. A graded endomorphism f on a graded module L is said to be *locally nilpotent* if, for any $x \in L$, there is an integer $n_x \geq 0$ such that $f^{n_x}(x) = 0$, where f^{n_x} denotes the n_x times iterated composition of f.

The converse of Lemma 2.13 holds in several cases. The following is one of such cases.

Lemma 2.14. *Adding to the assumption (2.1.2) we further assume that N is bounded below. Let $\varphi : N \to N$ be a graded B-linear endomorphism of degree 0 with expansion $\varphi|_M = \sum_{i \geq 0} X^{(i)} \varphi_i$. Assume that the constant term φ_0 is an A-linear automorphism on M. Then φ is a B-linear automorphism on N.*

Proof. Note that φ is an automorphism if and only if so is $(B \otimes_A \varphi_0^{-1})\varphi$. Hence we may assume $\varphi_0 = \mathrm{id}_M$. Setting $f = \varphi - \mathrm{id}_N$, we are going to prove that f is locally nilpotent. For this we note that $f(M) \subseteq \bigoplus_{i \geq 1} X^{(i)} M$. Then, since f is B-linear, we can show by induction on $n > 0$ that

$$f^n(M) \subseteq \bigoplus_{i \geq n} X^{(i)} M.$$

Since f has degree 0 as well as φ, the graded piece M_r of M of degree r is mapped by f^n into

$$\left(\bigoplus_{i \geq n} X^{(i)} M \right)_r = \bigoplus_{i \geq n} X^{(i)} M_{r - i|X|}.$$

Since $M \subseteq N$ is a graded A-submodule, we remark that M is also bounded below. For a given integer r, we can take an integer n that is large enough so that $M_{r-i|X|} = 0$ for all $i \geq n$, since $|X| > 0$. We thus have from the above that $f^n(M_r) = 0$. This shows that f is locally nilpotent as desired.

Then $\sum_{i=0}^{\infty}(-1)^i f^i = \mathrm{id}_N - f + f^2 - f^3 + \cdots + (-1)^n f^n + \cdots$ is a well-defined B-linear homomorphism on N, and in fact it is an inverse of $\varphi = \mathrm{id}_N + f$. ■

The following is a key to prove one of the main theorems.

Proposition 2.15. *Let f be a graded B-linear endomorphism of degree $-|X|$ on N and g_0 be a graded A-linear homomorphism of degree 0 on M. Then there is a graded B-linear endomorphism g of degree 0 on N satisfying that*

$$j(g) = gf \text{ and } g_0 \text{ is the constant term of } g.$$

Proof. Take the expansion of f as $f|_M = \sum_{n \geq 0} X^{(n)} f_n$. Note here that each f_n is a graded A-linear endomorphism on M of degree $-|X|(n+1)$ for $n \geq 0$. Setting $g|_M = \sum_{n \geq 0} X^{(n)} g_n$, we want to determine each g_n so that g satisfies the desired conditions.

We recall that $gf|_M = \sum_{n \geq 0} X^{(n)} \sum_{0 \leq i \leq n} \binom{n}{i} g_i f_{n-i}$ from the equality in the proof of Proposiotn 2.9 and that $j(g)|_M = \sum_{n \geq 0} X^{(n)} g_{n+1}$. Comparing these equalities, we obtain the following equations for g_n $(n \geq 0)$ to satisfy the required conditions;

$$g_{n+1} = \sum_{i=0}^{n} \binom{n}{i} g_i f_{n-i} \text{ for all } n \geq 0.$$

Starting from g_0 and using these equalities, we can determine the graded A-linear homomorphism g_n by the induction on $n \geq 0$. Thus define g as a linear extension of $g|_M$ to N, that is, $g = B \otimes_A g|_M$. This is a B-linear endomorphism on N of degree $-|X|$, and satisfies all the desired conditions. ■

Lemma 2.16. *Suppose that $\Delta_N = 0$ as an element of $\mathcal{E}$. Then the graded A-module M has structure of DG A-module and $N = B \otimes_A M$ holds as an equality of DG B-modules.*

Proof. In the expansion $\partial^N|_M = \sum_{i \geq 0} X^{(i)} \alpha_i$, that $\Delta_N = 0$ implies that $\alpha_i = 0$ for $i > 0$. Therefore $\partial^N|_M = \alpha_0$ is an A-derivation on M and (M, α_0) defines a DG A-module. Moreover we have $\partial^N = B \otimes_A \alpha_0$ that equals $\tilde{\alpha}$ in the notation of Lemma 2.6. Thus $N = B \otimes_A M$ as DG B-modules. ■

Now we are ready to prove the main theorem. Note from Lemma 2.12 that Δ_N defines a cohomology class in $\mathrm{Ext}_B^{|X|+1}(N,N)$, which we denote by $[\Delta_N]$. As we show in the following theorem the class $[\Delta_N]$ gives a precise obstruction for N to be liftable.

Theorem 2.17. *As before let N be a semi-free DG B-module, and assume that N is bounded below. Then $[\Delta_N] = 0$ as an element of $\mathrm{Ext}_B^{|X|+1}(N,N)$ if and only if N is liftable to A.*

Proof. First of all we recall that N is liftable if and only if there is an A-derivation ∂^M on M of degree -1 that makes (M, ∂^M) a DG A-module and there is a DG B-isomorphism $\varphi : N \to B \otimes_A M$. In such a case φ is a graded B-linear isomorphism of degree 0 that commutes with differentials, i.e., $(B \otimes_A \partial^M)\varphi = \varphi\partial^N$ or equivalently

$$B \otimes_A \partial^M = \varphi\partial^N\varphi^{-1}. \tag{2.17.1}$$

Now assume that N is liftable. Then there is such a DG B-isomorphism φ. Applying the j-operator on (2.17.1) and using Corollay 2.10 (2), we have that

$$0 = j(B \otimes_A \partial^M) = j(\varphi)\partial^N\varphi^{-1} + \varphi j(\partial^N)\varphi^{-1} + \varphi\partial^N j(\varphi^{-1}).$$

It thus follows that

$$j(\partial^N) = -\varphi^{-1}j(\varphi)\partial^N - \partial^N j(\varphi^{-1})\varphi.$$

Here we note form Corollary 2.10 (1) that $\varphi^{-1}j(\varphi) = -j(\varphi^{-1})\varphi$. Therefore if we set $f = \varphi^{-1}j(\varphi)$, then we see that $|f| = -|X|$ is even and $\Delta_N = j(\partial^N) = \partial^N f - f\partial^N$. The last equality shows $[\Delta_N] = 0$ in $\mathrm{Ext}_B^{|X|+1}(N,N)$.

Conversely assume that $[\Delta_N] = 0$. Then there is a graded B-linear endomomorphism γ on N of degree $-|X|$, which satisfies the equality

$$\Delta_N = \partial^N\gamma - \gamma\partial^N. \tag{2.17.2}$$

We note that $|\Delta_N|$ is odd and $|\gamma|$ is even. It follows from Proposition 2.15 that there is a B-linear endomorphism φ on N of degree 0 such that $\varphi_0 = \mathrm{id}_M$ and

$$j(\varphi) = \varphi\gamma. \tag{2.17.3}$$

We should note from Lemma 2.14 that such φ is a B-linear automorphism on N. Define an alternative differential ∂'^N on N by

$$\partial'^N = \varphi \partial^N \varphi^{-1}.$$

Then it follows that $\varphi : (N, \partial^N) \to (N, \partial'^N)$ is a DG B-isomorphism.

Since the equality $j(\varphi^{-1})\varphi + \varphi^{-1} j(\varphi) = 0$ holds by Corollary 2.10 (1), we see from (2.17.3) that

$$j(\varphi^{-1}) = -\gamma \varphi^{-1}. \tag{2.17.4}$$

Thus we conclude that

$$j(\partial'^N) = j(\varphi \partial^N \varphi^{-1}) = \varphi(\gamma \partial^N + \Delta_N - \partial^N \gamma)\varphi^{-1} = 0,$$

which means that (N, ∂'^N) equals $B \otimes_A M$ with M having DG A-module structure defined by $\partial^M = \partial'^N|_M$. See Lemma 2.16. Hence $(N, \partial^N) \cong B \otimes_A M$ as DG B-modules. This proves that N is liftable to A.

■

In the rest of this section we consider the uniqueness of liftings. The following lemma will be necessary for this purpose.

Lemma 2.18. *Let M and M' be DG A-modules, and let $\varphi : B \otimes_A M \to B \otimes_A M'$ be a graded B-linear homomorphism of degree 0. Assume we have an expansion $\varphi|_M = \sum_{i \geq 0} X^{(i)} \varphi_i$, where each $\varphi_i : M \to M$ is a graded A-linear homomorphism. Then the following statements hold :*

(1) *φ is a cycle in $\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M')$ if and only if the following equalities hold for $i \geq 0$:*

$$\varphi_i \partial^M = t \varphi_{i+1} + \partial^{M'} \varphi_i.$$

(2) *φ is a boundary in $\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M')$ if and only if there is a graded B-linear homomorphism γ of degree 1 such that γ has an expansion $\gamma|_M = \sum_{i \geq 0} X^{(i)} \gamma_i$, and there are equalities for $i \geq 0$:*

$$\varphi_i = \gamma_i \partial^M + \partial^{M'} \gamma_i + t \gamma_{i+1}.$$

Proof. A direct computation implies that

$$\varphi\tilde{\partial}^M|_M = \varphi\partial^M = \sum_{i\geq 0} X^{(i)}\varphi_i\partial^M, \tag{2.18.1}$$

where $\tilde{\partial^M}$ is the extended derivation of ∂^M to $B\otimes_A M$ by means of Lemma 2.6. On the other hand,

$$\tilde{\partial}^{M'}\varphi|_M = \tilde{\partial}^{M'}\left(\sum_{i\geq 0} X^{(i)}\varphi_i\right) = \sum_{i\geq 0} X^{(i)}\left\{t\varphi_{i+1} + \partial^{M'}\varphi_i\right\}. \tag{2.18.2}$$

(1) Since $\varphi\tilde{\partial}^M - \tilde{\partial}^{M'}\varphi$ is in $\mathcal{E}$, the cycle condition $\varphi\tilde{\partial}^M - \tilde{\partial}^{M'}\varphi = 0$ is equivalent to that $\varphi\tilde{\partial}^M|_M - \tilde{\partial}^{M'}\varphi|_M = 0$ by Lemma 2.7. Therefore the right hand sides of (2.18.1) and (2.18.2) are equal.

(2) φ is a boundary if and only if there exists a graded B-linear homomorphism γ of degree 1 such that $\varphi = \gamma\tilde{\partial}^M + \tilde{\partial}^{M'}\gamma$. Because φ and $\gamma\tilde{\partial}^M + \tilde{\partial}^{M'}\gamma$ belong to $\mathcal{E}$, this is equivalent to that $\varphi|_M = (\gamma\tilde{\partial}^M + \tilde{\partial}^{M'}\gamma)|_M$ by Lemma 2.7. Then by the same argument as in (1) using (2.18.1) and (2.18.2), we can show the desired equalities. ∎

Theorem 2.19. *Let N be a semi-free DG B-module as before, and assume that N is liftable to A. If $\mathrm{Ext}_B^{|X|}(N,N) = 0$, then a lifting of N is unique up to DG isomorphisms over A.*

Proof. Assume that there are a couple of liftings (M,∂^M) and $(M',\partial^{M'})$ of N. Then there is a DG B-isomorphism $\varphi : (B\otimes_A M, \tilde{\partial}^M) \to (B\otimes_A M', \tilde{\partial}^{M'})$, where $\tilde{\partial}^M$, $\tilde{\partial}^{M'}$ are the extended differentials of ∂^M, $\partial^{M'}$ respectively. (See Lemma 2.6.) We take an expansion $\varphi|_M = \sum_{i\geq 0} X^{(i)}\varphi_i$. Since φ is a cycle of degree 0 in $\mathrm{Hom}_B(B\otimes_A M, B\otimes_A M')$, Lemma 2.18 implies the equality $\varphi_n\partial^M - \partial^{M'}\varphi_n = t\varphi_{n+1}$ holds for each $n \geq 0$. In particular, we have

$$\varphi_0\partial^M - \partial^{M'}\varphi_0 = t\varphi_1. \tag{2.19.1}$$

Since there is an equality $\varphi\tilde{\partial}^M = \tilde{\partial}^{M'}\varphi$, and since $j(\tilde{\partial}^M) = 0 = j(\tilde{\partial}^{M'})$, Proposition 2.9 leads that $j(\varphi)\tilde{\partial}^M|_M = \tilde{\partial}^{M'}j(\varphi)|_M$. It hence follows that

$$j(\varphi)\tilde{\partial}^M = \tilde{\partial}^{M'}j(\varphi),$$

because $j(\varphi)\tilde{\partial}^M - \tilde{\partial}^{M'}j(\varphi)$ is B-linear. Thus $j(\varphi)$ is a cycle of degree $-|X|$ in $\mathrm{Hom}_B(B\otimes_A M, B\otimes_A M')$, and it defines the element $[j(\varphi)]$ of the homology module $H_{-|X|}(\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M'))$. Regarding N as a semi-free resolution of $B \otimes_A M$ and $B \otimes_A M'$, we see that $\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M')$ is quasi-isomorphic to $\mathrm{Hom}_B(N, N)$. Since we assume $\mathrm{Ext}_B^{|X|}(N, N) = 0$, we have $[j(\varphi)] = 0$. Hence there is a graded B-linear homomorphism $\gamma : B \otimes_A M \to B \otimes_A M'$ of odd degree $-|X| + 1$ such that $j(\varphi) = \gamma\tilde{\partial}^M + \tilde{\partial}^{M'}\gamma$. Write $\gamma|_M = \sum_{i \geq 0} X^{(i)}\gamma_i$, and we get from Lemma 2.18 that $\varphi_{n+1} = \gamma_n\partial^M + \partial^{M'}\gamma_n + t\gamma_{n+1}$ for $n \geq 0$. In particular we have

$$\varphi_1 = \gamma_0\partial^M + \partial^{M'}\gamma_0 + t\gamma_1. \tag{2.19.2}$$

Note that $t^2 = 0$, because $|t|$ is odd. Then we obtain from (2.19.1) and (2.19.2) the equality

$$(\varphi_0 - t\gamma_0)\partial^M = \partial^{M'}(\varphi_0 - t\gamma_0).$$

Namely $\varphi_0 - t\gamma_0 : (M, \partial^M) \to (M', \partial^{M'})$ is a DG A-homomorphism. Since φ is a graded B-linear isomorphism, φ_0 is an A-linear isomorphism as well by Lemma 2.13. Then it follows that $\varphi_0 - t\gamma_0 : M \to M'$ is an A-linear isomorphism, since $\varphi_0^{-1} + t\varphi_0^{-1}\gamma_0\varphi_0^{-1}$ gives its inverse. Therefore $\varphi_0 - t\gamma_0 : M \to M'$ is a DG isomorphism over A. This completes the proof. ∎

2.5 An example of liftings

At the end of this thesis, we give an example of liftings (Example 2.23). Before presenting it, we need to show the next lemma.

Lemma 2.20. *Let A be a DG R-algebra and t is a cycle of odd degree in A, and let $B = A\langle X | dX = t\rangle$ be an extended DG algebra by the adjunction of variable X that kills t. Let N be a DG B-module. Assume that N is liftable to A with a lifting M. If $\mathrm{Ext}_B^n(N, N) = 0$ and $\mathrm{Ext}_B^{n+|X|-1}(N, N) = 0$ for some integer n, then $\mathrm{Ext}_A^n(M, M) = 0$.*

Proof. We assume that a DG A-module M is a lifting of N. Note that we may assume that M is semi-free. Take a semi-free resolution $f : F_M \to M$ over A. It is known from [7, Proposition 11.1.6] that $B \otimes_A f : B \otimes_A F_M \to B \otimes_A M$ is a quasi-isomorphism. Hence $B \otimes_A f$ gives a semi-free resolution of $B \otimes_A M$ over B. because $B \otimes_A F_M$ is a semi-free DG B-module. From the assumption of the liftabilty of N, we see that $B \otimes_A f$ induces a semi-free resolution of N over B. Since $B \otimes_A F_M$ is a semi-free DG B-module, we have

$$H_{-i}(\mathrm{Hom}_B(B \otimes_A F_M, N)) \cong H_{-i}(\mathrm{Hom}_B(B \otimes_A F_M, B \otimes_A F_M)) \qquad (2.20.1)$$

for any integer i. See [7, Lemma 9.3.5]. On the other hand, $\mathrm{Ext}^i_B(N, N)$ is defined by the left-hand sides in the isomorphism (2.20.1). Therefore we have $\mathrm{Ext}^i_B(N, N) \cong H_{-i}(\mathrm{Hom}_B(B \otimes_A F_M, B \otimes_A F_M))$ for all i. It follows from a similar argument as above that $\mathrm{Ext}^i_A(M, M) \cong H_{-i}(\mathrm{Hom}_A(F_M, F_M))$ for all i. Hence we may assume that M is a semi-free DG A-module and $(N, \partial^N) = (B \otimes_A M, \widetilde{\partial}^M)$ where $\widetilde{\partial}^M$ is the extended differential of ∂^M defined by means of Lemma 2.6.

In order to prove that $\mathrm{Ext}^n_A(M, M) = 0$, let $\varphi \in \mathrm{Hom}_A(M, M)$ be a cycle of degree $-n$, that is $\partial^M \varphi - (-1)^n \varphi \partial^M = 0$. Set $\widetilde{\varphi}$ as the extended B-linear homomorphism of φ on $B \otimes_A M$. See Lemma 2.6. Remark that $\widetilde{\varphi}$ is a cycle of degree $-n$ in $\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M)$, since $\widetilde{\varphi}\widetilde{\partial}^M - (-1)^n \widetilde{\partial}^M \widetilde{\varphi}$ is B-linear and the equalities $\widetilde{\varphi}\widetilde{\partial}^M|_M - (-1)^n \widetilde{\partial}^M \widetilde{\varphi}|_M = \varphi \partial^M - (-1)^n \partial^M \varphi = 0$ hold. Then $\widetilde{\varphi}$ defines the homology class $[\widetilde{\varphi}]$ in $H_{-n}(\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M))$. Since we assume that $\mathrm{Ext}^n_B(B \otimes_A M, B \otimes_A M) = 0$, there is a B-linear homomorphism $\gamma : (B \otimes_A M, \tilde{\partial}^M) \to (B \otimes_A M, \tilde{\partial}^M)$ of degree $-n+1$ such that

$$\widetilde{\varphi} = \tilde{\partial}^M \gamma - (-1)^{n-1} \gamma \tilde{\partial}^M. \qquad (2.20.2)$$

We take an expansion $\gamma|_M = \sum_{i \geq 0} X^{(i)} \gamma_i$. It follows from Lemma 2.18 that

$$\varphi = \gamma_0 \partial^M - (-1)^{n-1}(\partial^M \gamma_0 + t\gamma_1). \qquad (2.20.3)$$

The equality (2.20.2) implies $\widetilde{\partial}^M j(\gamma)|_M - (-1)^{n-1} j(\gamma) \widetilde{\partial}^M|_M = 0$, because $j(\widetilde{\varphi}) = 0$ and $j(\widetilde{\partial}^M) = 0$. Since $j(\gamma)$ belongs $\mathcal{E}$ from Propotision 2.8 (1), we see that $\widetilde{\partial}^M j(\gamma) - (-1)^{n-1} j(\gamma) \widetilde{\partial}^M$ is B-linear. Hence $\widetilde{\partial}^M j(\gamma) - (-1)^{n-1} j(\gamma) \widetilde{\partial}^M = 0$. Then it means that $j(\gamma)$ is a cycle of degree $-n - |X| + 1$ in $\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M)$. Thus $j(\gamma)$ defines the element $[j(\gamma)]$ in $H_{-n-|X|+1}(\mathrm{Hom}_B(B \otimes_A M, B \otimes_A M))$. It follows from our assumption $\mathrm{Ext}^{n+|X|-1}_B(B \otimes_A M, B \otimes_A M) = 0$ that there is a B-linear homomorphism ζ of degree $-n - |X| + 2$ such that

$$j(\gamma) = \tilde{\partial}^M \zeta - (-1)^{n+|X|-2} \zeta \tilde{\partial}^M.$$

We have an expansion $\zeta|_M = \sum_{i\geq 0} X^{(i)}\zeta_i$. It follows from Proposition 2.18(2) that $\gamma_{i+1} = \zeta_i\partial^M - (-1)^{n+|X|-2}(\partial^M\zeta_i + t\zeta_{i+1})$ for $i \geq 0$. In particular, we get

$$\gamma_1 = \zeta_0\partial^M - (-1)^n(\partial^M\zeta_0 + t\zeta_1), \tag{2.20.4}$$

because $|X|$ is even. Summarizing equalities (2.20.3) and (2.20.4), we have

$$\begin{aligned}
\varphi &= \gamma_0\partial^M - (-1)^{n-1}(\partial^M\gamma_0 + t\gamma_1) \\
&= \gamma_0\partial^M - (-1)^{n-1}[\partial^M\gamma_0 + t\{\zeta_0\partial^M - (-1)^n(\partial^M\zeta_0 + t\zeta_1)\}] \\
&= \{\gamma_0 - (-1)^{n-1}t\zeta_0\}\partial^M - (-1)^{n-1}\partial^M\{\gamma_0 - (-1)^{n-1}t\zeta_0\}.
\end{aligned}$$

The last equality holds, since t is a cycle of odd degree. Namely we see that φ is a boundary in $\mathrm{Hom}_A(M, M)$. Therefore we conclude that $\mathrm{Ext}_A^n(M, M) = 0$. ■

In the rest of this section, we use the following notations.

Notation 2.21. Let R be a commutative ring and x, y be elements in R. Assume that the equalities of ideals $\mathrm{Ann}_R(x) = yR$ and $\mathrm{Ann}_R(y) = xR$ hold. Define A to be the extended DG R-algebra obtained from R by the adjunction of the variable Y of degree 1 to kill the cycle y, that is,

$$A = R\langle Y | dY = y\rangle.$$

Further we denote by B the extended DG algebra of A by the adjunction of the variable X of degree 2 that kills the cycle xY, that is,

$$B = A\langle X | dX = xY\rangle = R\langle X, Y | dY = y, dX = xY\rangle.$$

Under such circumstances, B gives a DG R-algebra resolution of $S = R/yR$. Equivalently there is a DG R-algebra homomorphism $B \to S$ which is a quasi-isomorphism. See [29].

In this situation, we consider a lifting problem for a DG S-algebra homomorphism $R \to B$.

Corollary 2.22. *We work in the setting in Notation 2.21. Let N be a semi-free DG B-module. We assume that R is (y)-adically complete, and N is bounded below and its semi-basis over B is finite in each degree. If $\mathrm{Ext}_B^2(N, N) = 0$ and $\mathrm{Ext}_B^3(N, N) = 0$ then N is liftable to R.*

Proof. Since $\mathrm{Ext}^3_B(N, N) = 0$ and N is bounded below, Theorem 2.17 implies that N is liftable to A. Thus there is a semi-free DG A-module M such that $N \cong B \otimes_A M$. (We see that a lifting DG A-module of N is uniquely determined up to DG A-isomorphisms, because of Theorem 2.19.) It follows from the assumption $\mathrm{Ext}^2_B(N, N) = 0$ and $\mathrm{Ext}^3_B(N, N) = 0$ and Lemma 2.20 that we have $\mathrm{Ext}^2_A(M, M) = 0$. Claim that M is bounded below and its semi-basis over A is finite in each degree, because N is bounded below and its semi-basis over B is finite in each degree. In this case, it is known that M is liftable to R by [22, Theorem 3.4]. This completes the proof.

■

Example 2.23. We use the same notation as in Notation 2.21. Now we assume that an S-module N satisfies the condition $\mathrm{Ext}^3_S(N, N) = 0$. Then N is regarded as a DG B-module through $B \to S$. Taking a semi-free resolution $F_N \to N$ over B. It is known from [8, (1.6)] or [9, (1.3)] that $\mathrm{Ext}^3_S(N, N) \cong \mathrm{Ext}^3_B(F_N, F_N)$. Therefore our main theorem (Theorem 2.17) forces the existence of a semi-free DG A-module M with the property $F_N \cong B \otimes_A M$ as DG B-modules. Furthermore we assume that $\mathrm{Ext}^2_S(N, N) = 0$. Then we see from Corollary 2.22 that there exists a semi-free DG R-module L such that $F_N \cong B \otimes_R L$ as DG B-modules.

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