Applied Mathematical Modelling with New Parameters and Applications to Some Real Life Problems

by

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submitted in accordance with the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in the subject

APPLIED MATHEMATICS

in the

Department of Mathematical Sciences

at the

University of South Africa

Supervisor: Prof. Emile Franc Doungmo Goufo

(September 2018)

Preface

This study was carried out in the Department of Mathematical Sciences, University of South Africa, Florida science campus, South Africa, from January 2015 to May 2018, under the supervision of Professor Emile Franc DOUNGMO GOUFO. This study is the original work of the researcher and has not been submitted in any form for any degree or diploma at any tertiary institution. Where use has been made of works by other authors, they have been duly acknowledged.

Abstract

Some Epidemic models with fractional derivatives were proved to be well-defined, well-posed and more accurate [34, 51, 116], compared to models with the conventional derivative. An Ebola epidemic model with non-linear transmission is fully analyzed. The model is expressed with the conventional time derivative with a new parameter included, which happens to be fractional (that derivative is called the $\beta-$derivative). We proved that the model is well-defined and well-posed. Moreover, conditions for boundedness and dissipativity of the trajectories are established. Exploiting the generalized Routh-Hurwitz Criteria, existence and stability analysis of equilibrium points for the Ebola model are performed to show that they are strongly dependent on the non-linear transmission. In particular, conditions for existence and stability of a unique endemic equilibrium to the Ebola system are given. Numerical simulations are provided for particular expressions of the non-linear transmission, with model's parameters taking different values. The resulting simulations are in concordance with the usual threshold behavior. The results obtained here may be significant for the fight and prevention against Ebola haemorrhagic fever that has so far exterminated hundreds of families and is still affecting many people in West-Africa and other parts of the world.

The full comprehension and handling of the phenomenon of shattering, sometime happening during the process of polymer chain degradation [129, 142], remains unsolved when using the traditional evolution equations describing the degradation. This traditional model has been proved to be very hard to handle as it involves evolution of two intertwined quantities. Moreover, the explicit form of its solution is, in general, impossible to obtain. We explore the possibility of generalizing evolution equation modeling the polymer chain degradation and analyze the model with the conventional time derivative with a new parameter. We consider the general case where the breakup rate depends on the size of the chain breaking up. In the process, the alternative version of Sumudu integral transform is used to provide an explicit form of the general solution representing the evolution of polymer sizes distribution. In particular, we show that this evolution exhibits existence of complex periodic properties due to the presence of cosine and sine functions governing the solutions. Numerical simulations are performed for some particular cases and prove that the system describing the polymer chain degradation contains complex and simple harmonic poles whose effects are given by these functions or a combination of them. This result may be crucial in the ongoing research to better handle and explain the phenomenon of shattering.

Lastly, it has become a conjecture that power series like Mittag-Leffler functions and their variants naturally govern solutions to most of generalized fractional evolution models such as kinetic, diffusion or relaxation equations. The question is to say whether or not this is always true! Whence, three generalized evolution equations with an additional fractional parameter are solved analytically with conventional techniques. These are processes related to stationary state system, relaxation and diffusion. In the analysis, we exploit the Sumudu transform to show that investigation on the stationary state system leads to results of invariability. However, unlike other models, the generalized diffusion and relaxation models are proven not to be governed by Mittag-Leffler functions or any of their variants, but rather by a parameterized exponential function, new in the literature, more accurate and easier to handle. Graphical representations are performed and also show how that parameter, called β, can be used to control the stationarity of such generalized models.

Key terms: Conventional derivative with a new parameter; Ebola epidemic model; non-linear incidence; existence; stability; depolymerization; replicated fractional poles; simple and complex harmonic motion; shattering; generalized evolution models; exponential with a parameter; Sumudu transform; Mittag-Leffler functions.

Declaration 1 - Plagiarism

Student number: 3044-548-5

I declare that "Applied Mathematical Modeling with New Parameters and Applications to Some Real Life Problems" is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

I further declare that I have not previously submitted this work, or part of it, for examination at Unisa for another qualification or at any other higher education institution.

	September 2018
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Declaration 2 - Publications

Papers published

- Complex harmonic poles in the evolution of macromolecules depolymerization, Journal Of Computational Analysis And Applications, Vol. 25, No 8, PP 1490-1503, 2018.
- Stability analysis of epidemic models of Ebola hemorrhagic fever with non-linear transmission, The Journal of Nonlinear Science and Applications, 2016, Vol. 9, No 6, pp 4191-4205, 2016.
- Positivity and contractivity in the dynamics of clusters' splitting with derivative of fractional order, Central European Journal of Mathematics, Vol.13, No 1, 351–362, 2015.

Papers in preparation

- Control parameter & solutions to generalized evolution equations of stationarity, relaxation and diffusion.
- An application of Caputo-Fabrizio operator to replicator-mutator dynamics: Bifurcation, chaotic limit cycles and control.

September 2018

SIGNATURE
(Mrs Stella B. Mugisha)

DATE

Dedication

I dedicate this thesis to the memories of my dad, Charles Rwamura Mawenu Abooki and my daughter, Jane Karungi Mugisha.

Acknowledgements

I wish to thank God, the Almighty Father, for his guidance and for seeing me through this journey. A special thought to the memory of my grandfather, Raphael Bamujerra Bitamazire, Apuuli and my mom, Norah Bacooco Rwabwogo, Amooti, whom I believe have become my guardian angels.

I would like to express my sincere gratitude to my supervisor, Professor Emile Franc DOUNGMO GOUFO, for his time, fruitful discussions, critical evaluations, remarks, orientations, constructive criticisms and his availability.

Many thanks to my beloved children, Mark Katonera Mugisha, Andrew Muhumuza Mugisha, Matthew Magezi Mugisha and his wife Dr Elsie Kibuka-Mugisha for their love, prayers, support and continuous encouragement. I also wish to thank the rest of my family, especially my husband, Prof Xavier Rujagaata Mugisha and my cousin Dr Alice-Stella Kwizera for being there for me through their prayers, love and encouragement.

I am grateful to the Department of Mathematical Sciences at UNISA, for their support in allowing me to go for sabbatical leave. I am also grateful to Dr Oukouomi Noutchie, my MSc supervisor, for his encouragement in furthering my studies. I give my special thanks to Mr Williams Chukwu, who is like a son to me, for his assistance, my friends, Prof John Kahimbaara Akiiki, Mrs Toppie Mukasa-Lwanga, Dr DR Mabule, Ms Princess Moeng and other friends not mentioned for their advice.

Finally, I am grateful to the University of South Africa (UNISA) for the financial assistance I received through the staff bursary scheme.

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Chapter 1

Introduction

In the field of Applied Mathematical Modelling, concepts of rate of change and variation in natural phenomena are generally used to build the involved differential equations that govern the processes. In many instances of applied sciences, a dynamical system which evolves is described by a concentration function $(t, \xi) \to u(t, \xi)$, where t is the time and ξ is an element of some state space Ω, that identifies individuals uniquely. The function u could be interpreted as the probability (density function) of finding an individual which at the time t, enjoys the property ξ.

The derivative of a function at a particular point describes the rate of change of the function near that point and the process of finding the derivative is called differentiation. The concept of differentiation was pioneered by Isaac Newton (1643–1727) and Gottfried Leibniz (1646–1716) centuries ago, making them the fathers of modern calculus. The definition of Newtonian derivative also known as local differentiation is given as follows:

Definition 1.0.1. *Let u be a function defined in a closed interval $[a, b]$, then the local derivative of the function $u(x)$, written as $u'(x)$ or $\frac{du}{dx}(x)$ is given by*

$$\lim_{h \longrightarrow 0} \frac{u(x+h) - u(x)}{h}.$$

The concept of differentiation was first applied to general physics and later on, to other fields of Applied Sciences. However, with the passage of time, the Newtonian concept of differentiation can no longer satisfy all the complexity of today's life application. For example, a couple of complex phenomena with dynamic features that exist in certain areas of Sciences and Engineering are still totally or partially unexplained by the existing traditional methods (see [16, 54, 51, 76, 86]) and remain open problems. This trend has called for an increasing volition among researchers, trying new approaches, by extending or expanding classical models and investigating them with various and different techniques, thereby establishing broader outlooks of the phenomena under investigation.

The notion of differentiation has been widely used and tailored to solve many applications [16, 12, 13, 21, 33, 44, 54, 51, 76, 86, 89, 116, 122]. However, one of the greatest attempts that enhance mathematical models and methods is the development of the concept of differentiation using a fractional order. There is a growing interest globally, in extending the analysis that involves normal calculus with integer orders to the one with non-integer orders, which may be real or complex. It is important to mention that their applications have attracted a great number of attentions in different applications in the past few years [36, 110, 115, 106]. Despite its three centuries of existence, the usage and application of fractional differentiation is still unpopular amongst scientists in many parts of the world, which include Sub-Saharan Africa.

Differential equations involving fractional derivative is a useful tool for describing non-linear phenomena in many branches of Science and Engineering. Using differential equations involving fractional derivative have gained applications in a wide range of fields including acoustic dissipation, epidemiology, continuous time random walk, biomedical engineering, fractional signal and image processing, control theory, Levy statistics, fractional phase-locked loops, fractional Brownian motion, porous media, fractional filters motion and non-local phenomena. Moreover, it has been proved that it provides a better description of the phenomenon under investigation than models with the conventional integer-order derivative [34, 51, 116, 13, 89]. The literature contains various definitions of fractional derivatives. A new fractional derivative with no singular kernel was recently proposed by Caputo and Fabrizio [37], but is still under investigation. The old and classical Caputo fractional derivative [36] is the most used one for modelling real world problems [34, 54]. This derivative has its own shortcomings: it does not obey the traditional chain rule, which is one of the key elements of the match asymptotic method [16, 17, 86, 125].

The β–derivative

Until today, the match asymptotic method has not been used to solve any kind of fractional differential equations because of the shortcoming. Hence, the conformable fractional derivative was proposed [87], which is theoretically much easier to handle because it obeys the chain rule. It also exhibits a huge failure in the sense that the fractional derivative of any differentiable function at the point zero is zero.

In response to this problem, Atangana and Doungmo Goufo [17, 16] recently proposed and developed a modified version of derivative with a new parameter, named the Atangana-Goufo derivative or simply the β–derivative, which is an extension of the conventional first order derivative. The β–derivative may not be considered as a fractional derivative but as a derivative with a fractional parameter. It is defined as follows:

$$ {}_0^A D_t^{\beta} u(t) = \begin{cases} \lim\limits_{\varepsilon \to 0} \frac{u\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-u(t)}{\varepsilon} & \text{for all } t \geq 0, \quad 0 < \beta \leq 1 \\ \\ u(t) & \text{for all } t \geq 0, \; \beta = 0, \end{cases} \tag{1.0.1} $$

where u is a function such that $u : [0, \; \infty) \to \mathbb{R}$ and Γ the gamma-function

$$ \Gamma(\zeta) = \int_0^{\infty} t^{\zeta-1} e^{-t} dt. $$

If the above limit exists then, u is said to be $\beta-$differentiable. Note that for $\beta = 1$, we have ${}_0^A D_t^{\beta} u(t) = \frac{d}{dt} u(t)$. Moreover, unlike other fractional derivatives, the β–derivative of a function can be locally defined at a certain point, the same way like first order derivative does. For a general order, let us say $m\beta$, the $m\beta$–derivative of u is defined as

$$ {}_0^A D_t^{m\beta} u(t) = {}_0^A D_t^{\beta} \left({}_0^A D_t^{(m-1)\beta} u(t) \right), \quad \text{for all } t \geq 0, \; m \in \mathbb{N}, \; 0 < \beta \leq 1. \tag{1.0.2} $$

Observe that the $m\beta$–derivative of a given function provides information about the previous $m-1$–derivatives of the same function. For instance, we have

$$ \begin{aligned} {}_0^A D_t^{2\beta} u(t) &= {}_0^A D_t^{\beta} \left({}_0^A D_t^{\beta} u(t) \right) \\ &= \left(t + \frac{1}{\Gamma(\beta)} \right)^{1-\beta} \left[(1-\beta) \left(t + \frac{1}{\Gamma(\beta)} \right)^{-\beta} u' + \left(t + \frac{1}{\Gamma(\beta)} \right)^{1-\beta} u'' \right]. \end{aligned} \tag{1.0.3} $$

This gives the β–derivative a unique property of memory, which was not provided for by any other derivative. It is also easy to verify that for $\beta = 1$, we obtain the second derivative of u. In addition to the fact that β–derivative cuts off the weaknesses noticed for other derivatives, its aim was, first of all, to extend the well-known match asymptotic method to the scope of the fractional differential equation, and also to describe the boundary layers problems within the framework of fractional calculus. More properties and details on this new derivative are given below, in the introduction section of Chapter 2. Interested readers can also consult the newly published book [16], the article [17] and the references therein.

Background of the study

In this thesis, the β–derivative is used to generate some new models that shall be investigated. Also three key-areas are our main interests, which include the following:

- Applied analysis with stability results for differential equations with a new parameter
- Mathematical epidemiology with application to Ebola haemorrhagic fever with non-linear transmission
- Biophysics with application to an evolution model of macromolecules' depolymerization.

More details are given as follows

1.0.1 Applied analysis: Stability results for differential equations using the β-derivative

In this study, the β-derivative will generate new types of differential equations like those used in modeling some epidemic diseases. Due to the fact that stability in an epidemic situation is very important, establishing stability results of the newly generated differential equations would be the first thing to do before analyzing the epidemic models expressed with the local β-derivative. Then, conditions guaranteeing the stability in some modeled epidemic situations with the β-derivative concept will be given and applied to a model of Ebola haemorrhagic fever.

1.0.2 Mathematical epidemiology: Ebola haemorrhagic fever and non-linear transmission

In recent decades, many authors have paid special attention to the modeling of real world phenomena with the concept of fractional order derivatives, which are more reliable because it provides better predictions compared to conventional models of integer order derivative [34, 51, 116, 13, 89]. A concrete proof was given in [116]. It demonstrated that some epidemic models that are based on variation and modeled with conventional derivative were unable to reproduce the statistical data collected in a real outbreak, with enough degree of accuracy. As example, the application of half-order derivatives and integrals can be found in [110, 106, 115]. When they were compared to classical models, they have been proved to be more useful and reliable in the formulation of some electrochemical problems. For more examples, interested reader can consult the works

[17, 51, 89, 33, 122, 21, 12, 44] that have successfully generalized classical derivatives to derivatives of fractional order.

In the field of mathematical epidemiology, Doungmo Goufo et al. [51] have produced some interesting and useful properties of Kermack-McKendrick epidemic model, with non-linear incidence, modelled with fractional order derivative. It should be noted that the Kermack-McKendrick epidemic model is the basis on which many other multi-compartmental models have been developed. The results obtained therein sustain the legitimation of epidemic models with fractional order derivative and may help analyze more complex models in the field. For example, the outbreak of Ebola haemorrhagic fever that recently occurred in some West African countries infected over 30,000 people and killed up to 15,000 people around the world, and these numbers are still rising.

Ebola haemorrhagic fever is caused by genus Ebola virus, a member of the family of filoviridea. Its other siblings are genus Marburgvirus and the genus Cuevavirus. There are three distinct species of the genus Ebolavirus, namely Bun Dibugyoe Bolavirus (BDBV), Zaire Ebolavirus (EBOV), Sudan Ebolavirus (SUDV), which are believed to be largely responsible for the Ebola outbreaks in Africa in general and the 2014 fatal outbreak in West Africa in particular. Ebola virus is an unusual but fatal virus that, when spreading throughout the body, damages the immune system and organs. Ultimately, it causes levels of blood-clotting cells to drop [24]. This causes uncontrollable bleeding inside and outside the body [77] to yield a severe hemorrhagic disease characterized by initial fever and malaise followed by shock, gastrointestinal bleeding symptoms, to end by multi-organ system failure.

In Africa, the transmission of Ebola virus is believed to be non-linear and occurred in various ways. Most of the infections among human beings are caused by the handling of infected fruits or meats by bats, macaques, baboons, vervets, monkeys, chimpanzees, gorillas, forest antelope and porcupines. They are sometime found dead or sick in the scrub land or forest. Ebola virus is known to be contagious and it is transmitted from one person to another through human-to-human, human-to-animal or fruit-to-human bi-relations. The usual infection from human to human results from direct contact (through broken skin or mucous membranes) with the blood, secretions, organs or other bodily fluids of infected person. Transmission of Ebola disease can also occur by indirect contact with environments that are contaminated with such fluids [50, 68, 66, 90, 137], especially at the burial of Ebola victims. Literature concerning Ebola's cure, vaccine, species variety and dynamics is still limited and far from being complete. Therefore, it is urgently necessary to conduct various research and explore new methods and techniques, that can help to better understand the outbreak process and educate people about the real dynamic of Ebolavirus, its transmission's mode and ways to avoid or minimize its spread. There is no confirmed cure for the disease yet, and the true and real dynamic of the virus is not yet fully comprehended. It is then reasonable to apply the β-derivative to the disease and establish a broader outlook of the real nature of this killing disease that has become a nightmare to all the nations. Furthermore, the development and application of fractional calculus to mathematical epidemiology is still relatively new.

As already mentioned, some Epidemic models with fractional derivatives were proved to be well-defined, well-posed and more accurate compared to models with the conventional derivative. That is why in this thesis, an Ebola epidemic model with non-linear transmission is considered. The model is expressed by means of the β-derivative and is proven to be well-defined and well-posed. Moreover, conditions for boundedness and dissipativity of the trajectories are established. Exploiting the generalized Routh-Hurwitz Criteria, existence and stability analysis of equilibrium points for Ebola model are performed to show that they are strongly dependent on the non-linear transmission. In particular, conditions for existence and stability of a unique endemic equilibrium to the Ebola system are given. Finally, numerical simulations are provided for particular expressions of the non-linear transmission set with some fixed values of the parameters involved in the dynamics. The resulting simulations prove to be in concordance with the usual threshold behavior. These results are significant and may be substantial in the fight and prevention against Ebola haemorrhagic fever that has so far destroyed many families in the world.

1.0.3 Biophysics: Evolution model for macromolecules depolymerization

The evolution of the sizes distribution occurring during polymer chain degradation is well known [56, 52, 142] to be described by the following integrodifferential equation

$$\frac{\partial}{\partial t} g(x,t) = -g(x,t)\int_0^x H(y,x-y)dy + 2\int_x^{\infty} g(y,t)H(x,y-x)dy, \qquad x,\ t>0, \quad (1.0.4)$$

where $g(x,t)$ represents the density of x-groups (i.e. groups of size x) at time t and $H(x,y)$ gives the average fragmentation rate, that is, the average number at which clusters of size $x+y$ undergo splitting to form an x-group and a y-group. Expressing the solution of equation (1.0.4) in its explicit form is very hard since fragmentation (or polymer chain degradation) processes, as explained in the previous section, are difficult to analyse as they involve evolution of two intertwined quantities:

- the distribution of mass among the particles in the ensemble,
- the number of particles in it.

Although equation (1.0.4) is linear, it displays non-linear features such as "shattering" phenomena which they cannot fully explain [55, 52, 140]. Then, in order to have a broader understanding of the evolution of polymer chain degradation and the phenomenon of shattering, we have explored the possibility of extending its analysis by considering the β-derivative, which has yielded the following integrodifferential equation:

$$ {}_{0}^{A}D_{t}^{\beta} g(x,t) = -g(x,t)\int_{0}^{x} H_{\beta}(y,x-y)dy + 2\int_{x}^{\infty} g(y,t)H_{\beta}(x,y-x)dy, \qquad x,\ t>0. \tag{1.0.5} $$

subject to the initial condition

$$ g(x,0) = g_{0}(x), \qquad x>0 \tag{1.0.6} $$

where $g(x,t)$ and $H_{\beta}(x,y)$ are defined as above. These models will be analyzed using various techniques as mentioned in section 3.2. Most of those techniques are innovative and significant simply because, as we already mentioned above, that the full comprehension and handling of a bizarre phenomenon like shattering is still an open problem. This process sometime happens during the polymer chain degradation [129, 142] and uses the traditional evolution equation to describe the degradation. Such a traditional model has been proven to be very hard to handle because it involves evolution of two intertwined quantities. Moreover, the explicit form of its solution is, in general, impossible to obtain. That is why this thesis tries to explore the possibility of generalizing evolution equation modeling the polymer chain degradation and analyze the model with β-derivative. We consider the general case where the breakup rate depends on the size of the chain breaking up. In the process, the alternative version of Sumudu integral transform is used to provide an explicit form of the general solution representing the evolution of polymer sizes distribution. In particular, we show that this evolution exhibits existence of complex periodic properties due to the presence of cosine and sine functions governing the solutions. Numerical simulations are performed for some particular cases and proves that such a system describing the polymer chain degradation contains complex and simple harmonic poles whose effects are given by these functions or a combination of them. As we will see at the end of this study, our result may be crucial in the ongoing research to better handle and explain the phenomenon of shattering.

1.1 Breakdown of the thesis

Recall that one of the objectives of this thesis is to establish additional useful properties of the new β-derivative that are necessary in analyzing the mathematical models of real life phenomena under investigation. Those phenomena include the Ebola virus dynamics and the process of polymers/biopolymers chain degradation. Hence, Chapter 2 will provide a comprehensive definition, history and properties of the derivative with new parameter. Even though some of the methods applied in this thesis are relatively well known, most of our analysis required less familiar techniques and results. Therefore in Chapter 3, a discussion of these subsidiary methods, techniques and results is given. One of the most important results here, is the establishment, in Chapter 4, of the stability results for differential equations with the β-derivative. This is the first thing to do before analyzing any epidemic model expressed with the β-derivative. Recall that the same stability results are already known for models with the conventional integer Calculus [130, 80, 43]. In fractional calculus, it was proved only for models with the (old) Caputo fractional derivative [102, 39, 82] making such models easy to analyze. However this has never been done for the models with β-derivative and the relevance of this property is huge since it allows us to find conditions on the eigenvalues of the Jacobian matrix that shall make the disease free equilibrium of the whole system stable, globally stable or asymptotically stable. This property will also open doors to more complex investigations on stability for other systems modeled with the derivative with the new parameter: The β-derivative.

Note that we wrote “(old) Caputo” because Caputo and his collaborator Fabrizio recently introduces a new fractional derivative, but this time with no singular kernel at $t = \tau$ as it is the case on the old definition, (see[37, 47, 49, 98]).

Another objective is to use the new β-derivative to model and analyze a current disease: Ebola.

To prove that the model is wellposed and establish conditions for boundedness and dissipativity of the trajectories.

To make use of relevant mathematical tools to establish an expression of the basic reproduction ratio, conditions for existence of a unique disease free and/or endemic equilibrium point.

To make use of the stability results obtained previously to investigate the stability of the possible unique disease free and/or endemic equilibrium point for Ebola epidemic model.

To study the possibility of existence of a global equilibrium point for the whole metpopulation system and analyse its stability.

To perform numerical simulations and interpret them.

Chapter 5 deals with the proposition of an adequate model of macromolecules depolymerization using the new β-derivative. The objectives are as follows:

To define the suitable Banach space in which the depolymerization model can be investigated adequately.

To provide an explicit form of the general solution representing the evolution of polymer sizes distribution.

To perform numerical simulations and interpret them.

Lastly in Chapter 6, we summarize number of applications related to solutions of generalized evolution equations of stationarity, relaxation and diffusion. Those three generalized evolution equations with additional fractional parameter are solved analytically with conventional techniques described in Chapter 3. In the analysis, operators like the Sumudu transform are exploited to show that investigation on the stationary state system leads to results of invariability. However, the generalized models related to diffusion and relaxation are proven not to be governed by Mittag-Leffler functions or any of their variants, but rather by a parameterized exponential function, which is fully defined and presented in the chapter, new in the literature, more accurate and easier to handle. The conclusion follows in Chapter 7.

Chapter 2

Derivative with new parameter: History and properties

2.1 Introduction

It is well-known and it has been quoted very intensively that Fractional Calculus has its origin in the letter by the French mathematician Guillaume François Antoine de L'Hôpital (1661-1704) to the German mathematician and philosopher Gottfried Wilhelm Leibniz (1646-1716), in which de L'Hôpital posed an important question with regard to the order of the derivative and in particular, what the derivative of order $\frac{1}{2}$ might be. According to de Oliveira and Machado [109],

> In a prophetic answer, Leibniz foresaw the beginning of the area that nowadays is named fractional calculus. In fact fractional calculus is as old as the traditional calculus proposed independently by Newton and Leibniz.

According to many sources, but noted here as mentioned by Katugampola in his paper [83], Abel (1823) was the first to apply the fractional calculus in the form of the semi-derivative in the solution to his tautocrone problem.

Background to Fractional Calculus

In ordinary calculus, we all know that the derivative is associated with the tangent of the function at a point of evaluation and this provides an important geometric interpretation of the first-order derivative. According to Katugampola [83], Fourier proposed the first definition of a derivative of arbitrary order and used an integral representation to define the derivative. As recorded in the same article [83], "Liouville (1832) suggested a definition based on the formula for differentiating the exponential function." The version by Liouville for the integration of non-integer order, was a second definition by himself in terms of an integral. According to Ishteva *et.al.*[126], the fractional derivatives provide an excellent tool for describing hereditary and memory properties of various materials and processes, like polymers and related dynamics. Thus, Caputo and Fabrizio [37] affirm that "Fractional derivatives are memory operators which usually represent dissipation of energy or damage in the medium as in the case of inelastic media or re-assessment of the porosity in the diffusion in porous media". Lastly, Cauchy's formula for repeated integration is well known to form the basis of the formula for fractional integration [106, 109].

Definitions of fractional derivatives

There are many different definitions of fractional derivatives with several examples listed by de Oliveira [109] and that include the Liouville, Riemann-Liouville, Caputo, Grünwald-Letnikov, Weyl, Marchaud, Hadamard Chen, Davidson-Essex, Coimbra, Canavati, Jumarie, Reisz, Cossar, Modified Riemann-Liouville, Osler and k-fractional Hilfer fractional derivatives. Only a few of these are commonly used in the current study of fractional Calculus and these are listed below together with their definitions.
These definitions are listed courtesy of Li *et.al.* [95] as follows:
Grünwald-Letnikov:
The Grünwald-Letnikov fractional derivative with fractional order α of $x(t) \in C^m[0,t]$ (the space of continuous functions on $[0,t]$ that have continuous first m derivatives) is defined as follows:

$$^{GL}D^{\alpha}_{0,t}x(t) = \sum_{k=0}^{m-1} \frac{x^k(0)t^{-a+k}}{\Gamma(-a+k+1)} + \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau) d\tau, \qquad (2.1.1)$$

with, $m-1 \leq \alpha < m$ where $m \in Z^+$, and Γ the well-known Gamma-function defined below in (2.1.5).
The original expression is by a limit function, but this definition is not particularly useable for analysis.

Riemann-Liouville:
Though literature on various versions of fractional derivatives is vast, the most popular remain the definitions in the sense of Riemann-Liouville and Caputo [36, 47, 57, 37]. The Riemann–Liouville fractional derivative (RLFD) was named after the work of Bernhard Riemann and Joseph Liouville who in late 1832, were the first to explore the possibility

of fractional calculus. Their idea was to first define the fractional integral

$$D_{0,t}^{-\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(\tau)}{(t-\tau)^{1-\alpha}} d\tau \tag{2.1.2}$$

based on Euler transform when applied to analytic function and Cauchy's formula for calculating iterated integrals. This yielded the Riemann-Liouville fractional derivative with fractional order α of $x(t)$ defined as follows:

$$^{RL}D_{0,t}^{\alpha}x(t) = \frac{d^m}{dt^m} D_{0,t}^{-(m-\alpha)}x(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t (t-\tau)^{m-\alpha-1} x(\tau) d\tau, \tag{2.1.3}$$

with, $m-1 \leq \alpha < m$ where $m \in Z^+$.

Caputo:
The Caputo fractional derivative with fractional order α of $x(t)$ is defined as follows:

$$^{C}D_{0,t}^{\alpha}x(t) = D_{0,t}^{-(m-\alpha)} \frac{d^m}{dt^m} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} x(\tau) d\tau, \tag{2.1.4}$$

with, $m-1 \leq \alpha < m$ where $m \in Z^+$.
To deal in particular with the periodic functions, Weyl introduced the Weyl fractional derivative, a definition of which can be found in numerous references such as [106, 109].

Properties of the fractional derivatives

As mentioned before, fractional derivatives are excellent at describing memory properties and these effects are generally not addressed in integer-order derivatives. The Caputo fractional derivative has initial conditions that are expressed as initial values of integer-order derivatives and for this reason the Caputo fractional derivative and its extensions is widely used to model physical phenomena and problems. However, the Riemann-Liouville fractional derivative, according to Rahimy [118], "requires initial conditions expressed in terms of initial values of fractional derivatives of the unknown function".
For $1 < \alpha \in (m-1, m)$, $m \in Z^+$, there is a formula that relates the Caputo with the Riemann-Liouville. This is listed by a number of references, like [95, 106, 89] and it reads as:

$$^{C}D_{0,t}^{\alpha}x(t) = ^{RL}D_{0,t}^{\alpha}x(t) - \sum_{k=0}^{m-1} \frac{t^k x^k(0)}{k!}.$$

Since by the Caputo definition the function $x(t)$ needs to be differentiable in the integrand, which means there are realistically less functions for which fractional derivatives can be derived in the Caputo sense than in the Riemann-Liouville sense. The Caputo fractional derivative is thus only defined for differentiable functions. The Riemann-Liouville fractional derivative of a constant is not zero while it is for the Caputo fractional derivative. The Caputo fractional derivative, although it is more restrictive than the Riemann-Liouville fractional derivative, is more suitable in treating problems which involves fractional differential equations [109]. The Gamma-function is very important in fractional calculus. It is defined as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \tag{2.1.5}$$

where obviously $t^{z-1} = \mathrm{e}^{(z-1)\ln t}$.
This integral is convergent for all complex $z \in \mathbb{C}$, $(\Re(z) > 0)$. The integral in the definition is called the Euler integral of the second kind.

2.1.1 Methods for evaluating fractional differential equations

As described by Miller and Ross or by Oldham and Spanier in their respective books, [106],[111], some methods for the evaluation of fractional differential equations do not work for arbitrary real order. According to [111], you can use an iteration method that allows the solution of fractional differential equations of arbitrary real order, but only for simple equations is this effective.
The Mittag-Leffler function defined as $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$ is a function extensively used throughout the literature in the formulation of solutions of fractional differential equations.
A method for evaluation of fractional differential equations, using the Laplace transform technique, is free from many of the disadvantages of other methods, like the iteration method. The Laplace transform method is suitable for the evaluation of a wide class of initial value problems with fractional differential equations. According to [111], the solution method is based on the formula of the Laplace transform of the two-parameter Mittag-Leffler function, $E_{\alpha,\beta}$.
This method using the Laplace transform is widely used in the applied sciences like engineering and with problems in physics, chemistry, financing and even banking models.
As mentioned earlier, the Mittag-Leffler functions play a very important role in the solution of fractional differential equations. The two-parameter, or generalized Mittag-Leffler function is defined as;

$$E_{\alpha,\beta} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$$

Rida and Arafa [119] developed a generalization of the Mittag-Leffler function method which is based on the Caputo fractional derivative for the solution of fractional order linear differential equations. It was discovered that this technique is powerful and efficient in determining analytical solutions of a large class of fractional order linear differential equations.

A number of analytical and numerical methods exist for the solution of non-linear fractional differential equations where there is no analytical solutions. Some of these methods are; the Adomian decomposition method (ADM), the Variational iteration method (VIM), the Explicit numerical method (ENM) and the Homotopy analysis method (HAM), which is a general analytical method used for non-linear problems.

2.1.2 Derivative with non-singular kernel and other definitions

There are many versions and extensions of fractional derivatives. The most common in use are the Riemann–Liouville and the Caputo derivatives. The new Caputo-Fabrizio fractional derivative (CFFD), also called the derivative with non-singular kernel is just an extension of the old Caputo fractional derivative where the kernel of the integral has been reformulated. The new CFFD is defined in [37] as;

$$ {}^{cf}D_t^{\alpha} f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t \dot{f}(\tau) \exp[-\frac{\alpha(t-\tau)}{1-\alpha}] d\tau, \tag{2.1.6}$$

where $\alpha \in [0,1]$, $a \in [-\infty, t]$, $f \in H^1(a,b)$ with $H^1(a,b)$ the Sobolev space given by its generalized version

$$ H^n(a,b) = \{f : f, \frac{d}{dt} f, \cdots, D_t^n f \in L^2(a,b)\}, $$

here $L^2(a,b)\}$ is the space of square integrable function on (a,b) and $M(\alpha)$ is a normalisation constant such that $M(0) = M(\infty) = 1$.

When f does not belong to H^1, the above formula is re-formulated for $f \in L^1(-\infty, b)$ and for $\alpha \in [0,1]$ to read as

$$ {}^{cf}D_t^{\alpha} f(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_a^t \dot{f}(\tau) \exp[-\frac{(t-\tau)}{\alpha}] d\tau, \tag{2.1.7}$$

where $f \in L^1(-\infty, b)$ and $M(\alpha)$ is a normalisation constant such that $M(0) = M(\infty) = 1$. It is clear that compared to the old Caputo, the kernel does not have singularity at $t = \tau$ as is the case with the old Caputo.

This derivative with non-singular kernel would better describe the evolution of systems with hereditary/memory effect.

Some of other versions according to the references [37, 47, 49, 98, 13, 36, 81, 22, 87, 1] are detailed as follows:

1. It happens that the derivatives defined in Section 2.1 are particularly suitable to describe physical phenomena, related to fatigue, damage and electromagnetic hysteresis, but are incapable of properly describing some behavior observed in the systems with huge heterogeneities. That is the main reason why the CFFD was proposed. Considering (2.1.6) and without loss of generality we can put $a = 0$ to have

$$ {}^{cf}D_t^{\alpha}u(t) = \frac{M(\alpha)}{(1-\alpha)} \int_0^t \dot{u}(\tau) \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau, \tag{2.1.8} $$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. But, for the function that does not belong to $H^1(a;b)$, we defined its Caputo-Fabrizio fractional as

$$ {}^{cf}D_t^{\alpha}u(t) = \frac{\alpha M(\alpha)}{(1-\alpha)} \int_0^t (u(t) - u(\tau)) \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau. \tag{2.1.9} $$

The definition of the CFFD was improved by Losada and Nieto [98] to become

$$ {}^{cf}D_t^{\alpha}u(t) = \frac{(2-\alpha)M(\alpha)}{2(1-\alpha)} \int_0^t \dot{u}(\tau) \exp\left(-\frac{\alpha(t-\tau)}{1-\alpha}\right) d\tau. \tag{2.1.10} $$

Unlike the classical version of Caputo fractional order derivative [36, 117], the new CFFD has no singular kernel due to the substitution of the kernel $\frac{1}{(t-\tau)^{\alpha}}$ appearing in the classical definition. Moreover the CFFD satisfies the following relations for any suitable function u:

$$ \lim_{\alpha \to 1} {}^{cf}D_t^{\alpha}u(t) = \dot{u}(t) \tag{2.1.11} $$

and

$$ \lim_{\alpha \to 0} {}^{cf}D_t^{\alpha}u(t) = u(t) - u(a), \tag{2.1.12} $$

where a is the starting point of the integro-differentiation.

The fractional integral (anti-derivative) associated to the CFFD was proposed as well by Losada and Nieto [98] and proved to be:

$$ {}^{cf}I_t^{\alpha}u(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}u(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t u(\tau)\, d\tau, \tag{2.1.13} $$

$\alpha \in [0, 1]$ $\;t \geq 0$. This anti-derivative is seen as kind of an average between function

u and its integral of order one. The Laplace transform of the CFFD is given by

$$\mathcal{L}\left({}^{cf}D_t^{\alpha}u(t),s\right)=\frac{s\tilde{u}(x,s)-u_{0}(x)}{s+\alpha(1-s)} \tag{2.1.14}$$

where $\tilde{u}(x,s)$ is the Laplace transform $\mathcal{L}(u(x,t),s)$ of $u(x,t)$.

Other versions are:

2. The modified Liouville fractional derivative of a function f is defined as

$$D_x^{\alpha}\left(f(x)\right)=\frac{1}{\Gamma\left(n-\alpha\right)}\left(\frac{d}{dx}\right)^{n}\int_0^x\left(x-t\right)^{n-\alpha-1}\left(f\left(t\right)-f(0)\right)dt,\ n-1<\alpha\leq n \tag{2.1.15}$$

3. The local fractional derivative of a function says f is defined as

$$L_x^{\alpha}\left(f(x)\right)=\lim_{x\to x_0}\frac{\Gamma\left(1+\alpha\right)\left(f\left(x\right)-f(x_0)\right)}{\left(x-x_0\right)^{\alpha}} \tag{2.1.16}$$

4. The conformable fractional derivative of a function says f is given as

$$T_{\alpha}\left(f(x)\right)=\lim_{\varepsilon\to 0}\frac{f\left(x+\varepsilon x^{1-\alpha}\right)-f(x)}{\varepsilon} \tag{2.1.17}$$

5. The modified conformable fractional derivative of a given function f defined in the interval (a, b) is given as

$$T_{\alpha}\left(f(x)\right)=\lim_{\varepsilon\to 0}\frac{f\left(x+\varepsilon(x-a)^{1-\alpha}\right)-f(x)}{\varepsilon}\ . \tag{2.1.18}$$

2.1.3 Further development in the literature of fractional derivatives

Recently, Doungmo Goufo and Atangana in their article [53, 47] formulated a fractional derivative operator without singular kernel, which is an analogue of the well-known Riemann-Liouville fractional derivative with singular kernel. They proposed the New Riemann-Liouville fractional order derivative (NRLFD) given for $\alpha\in[0,1]$ by

$${}_a\mathfrak{D}_t^{\alpha}f(t)=\frac{M(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f\left(\tau\right)exp\left(-\frac{\alpha}{1-\alpha}(t-\tau)\right)d\tau. \tag{2.1.19}$$

Again, the NRLFD is without any singularity at $t = \tau$ in comparison to the classical Riemann–Liouville fractional order derivative (2.1.3) and also it verifies

$$\lim_{\alpha \to 1} {}_a\mathfrak{D}_t^\alpha f(t) = \dot{f}(t) \tag{2.1.20}$$

and

$$\lim_{\alpha \to 0} {}_a\mathfrak{D}_t^\alpha f(t) = f(t). \tag{2.1.21}$$

In order to address the issue of locality that exists in the above definitions of fractional derivatives, non-local definitions were proposed and generalized [14, 57] as follows: Let f be a function in $H^1(a;b)$; $b > a$; $\alpha \in [0;1], \beta \in (0,+\infty)$ then, the Caputo-sense one-parameter and non-local fractional derivative of order α is given by:

$${}^{ab}D_t^\alpha f(t) = \frac{M(\alpha)}{(1-\alpha)} \int_a^t \dot{f}(\tau)\ E_\alpha\left[-\frac{\alpha(t-\tau)^\alpha}{1-\alpha}\right] d\tau ={}_a^{abc} D_t^\alpha f(t), \tag{2.1.22}$$

where $M(\alpha)$ is the same type of normalization function defined in (2.1.6) and E_α the one-parameter Mittag-Leffler function.

The Caputo-sense two-parameter and non-local fractional derivative of order α knowing β as a parameter is given by:

$${}^{gc}D_t^{\alpha,\beta} f(t) = \frac{\beta W(\alpha,\beta)}{(\beta-\alpha)} \int_a^t \dot{f}(\tau)\ (t-\tau)^{\beta-1} E_{\alpha,\beta}\left[-\frac{\alpha\beta(t-\tau)^\alpha}{\beta-\alpha}\right] d\tau, \tag{2.1.23}$$

where $W(\alpha,\beta)$ is a two-variable normalization function such that $W(0,1) = W(1,1) = 1$, and $E_{\alpha,\beta}$ the two-parameter Mittag-Leffler function.

The Laplace transform of the definition (2.1.23) is given by

$$\mathcal{L}\left({}^{gc}D_t^\alpha f(t), s\right) = \frac{M(\alpha)}{(1-\alpha)} \frac{s^\alpha \tilde{f}(x,s) - s^{\alpha-1} f(0)}{s^\alpha + \frac{\alpha}{1-\alpha}}, \tag{2.1.24}$$

where $\tilde{f} = \mathcal{L}(f(t), s)$.

These definitions are reputed to be very useful in describing many complex problems in thermal sciences. The authors also assert that the kernel's non-locality allows a better description of memory within the structure and media with different scales [14, 53, 47]. The non-locality of this definition stems from the use of the generalized Mittag-Leffler function, which is considered non-local, in the formulations of the fractional derivative above. The definitions above employed the fact that the Mittag-Leffler function is a generalisation of the exponential function [65] and used it to re-formulate the new

CFFD and the Riemann-Liouville fractional derivative.

In the 2011 article Wang and Wei [99], the authors used the Schaefer fixed point theorem to establish the existence and uniqueness results for fractional differential equations boundary value problems. Alqahtani [6] employs a fixed point theorem to prove the existence and uniqueness of the non-linear Nagumo equation. There is also the article by Hristov [78] where he uses the Caputo-Fabrizio time fractional derivative to model the transient heat diffusion for homogeneous rigid heat conductors.

The use of the old Caputo fractional derivative to model linear evolution equations and the establishment of the well-posedness of those models have been done before, notably by Bazhlekova, with her article [26] outlining some of the research in approximation properties.

These works and other research shall be further investigated and accounted for more extensively in the thesis. It is also important to mention that there exist in the literature more and diversified versions of fractional derivatives, but, most of which are dominated by the most popular, namely Riemann-Liouville and Caputo derivatives, which were defined in the previous section.

Each version has its domain of applicability in Applied Sciences and Engineering, but presents some advantages and disadvantages [13, 52, 51, 115, 122]. Not all of them satisfy the common properties of the standard concept of derivative, and therefore, some limitations may happen depending on the complexity of the natural phenomenon under investigation. Almost all these derivatives lead to analytic and power series functions, not easy to manipulate.

- For instance, one of the major challenges with existing definitions of fractional derivatives is their difficulty to explicitly provide the variation of the functions. Moreover, systems using fractional derivative are not easy to handle analytically. A clear example is hypergeometric functions [13, 40]. Indeed, the hypergeometric function and its generalization encompass an extensive class of analytical functions as shown in the following representation of the generalized hypergeometric series

  $$ {}_pH_q(a_1, \cdots, a_p, b_1, \cdots, b_q; z) = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\cdots\Gamma(a_p+k)z^k}{\Gamma(b_1+k)\cdots\Gamma(b_q+k)k!}. $$

  Recall that b_i should not be non positive integers.

- The second problem is that, usually when investigating a natural phenomenon, one uses real observed data to make analysis and simulations. Then, one compares the simulations to the observed phenomenon in order to draw conclusions and provide some recommendations. This involves plotting and when the function is in the form of power series, one can only get an approximation result. Note that the higher the number of terms for the series, the more accurate the results.

The last two definitions listed above in the previous section, namely the conformable fractional derivative (2.1.17) and modified conformable fractional derivative (2.1.18), seem to satisfy some common properties of the standard concept of derivative, but as we said before they have some weaknesses expressed by the fact that their fractional derivative of any differentiable function at the point zero is zero. This cannot allow them to be used in modeling real world problems. That is how Atangana and Doungmo Goufo [17, 16] developed the β–derivative (1.0.1), a suitable fractional derivative already defined in Chapter 1 and that helps us filling in the lack caused by other fractional derivatives. In the next lines, additional properties of the β–derivative are provided.

Note that, contrary to the definition (2.1.17) here above, the definition of the β–derivative (1.0.1) does not depend on the interval on which the function is defined. Moreover, unlike the modified conformable fractional derivative (2.1.18), the β–derivative of a suitable differentiable function at a point zero is different to zero.

Theorem 2.1.1. *Assuming that, a given function say* $g\ :[a,\ \infty)\to\mathbb{R}$ *is* $\beta-$*differentiable at a given point say* $t_0\geq a,\ \ \beta\in(0,\ 1]$, *then,* g *is also continuous at* t_0.

Proof. [17, Theorem 2] or [16, Theorem 2.3.1]. ■

Theorem 2.1.2. *Assuming that* f *is* $\beta-$*differentiable on an open interval* $(a,\ b)$ *then*

1. *If* ${}_0^AD_t^\beta f(t)<0$ *for all* $t\in(a,\ b)$ *then* f *is decreasing on* $(a,\ b)$;
2. *If* ${}_0^AD_t^\beta f(t)>0$ *for all* $t\in(a,\ b)$ *then* f *is increasing on* $(a,\ b)$;
3. *If* ${}_0^AD_t^\beta f(t)=0$ *for all* $t\in(a,\ b)$ *then* f *is constant on* $(a,\ b)$.

Theorem 2.1.3. *Assuming that,* $g\neq0$ *and* f *are two functions* $\beta-$*differentiable with* $\beta\in(0,1]$ *then, the following relations can be satisfied*

1. ${}_0^AD_t^\beta\left(af\left(t\right)+bg(t)\right)=a{}_0^AD_t^\beta\left(f\left(t\right)\right)+b{}_0^AD_t^\beta\left(f\left(t\right)\right)$ *for all* a *and* b *real number;*
2. ${}_0^AD_t^\beta\left(c\right)=0$ *for* c *any given constant ;*
3. ${}_0^AD_t^\beta\left(f\left(t\right)g(t)\right)=g\left(t\right){}_0^AD_t^\beta\left(f\left(t\right)\right)+f\left(t\right){}_0^AD_t^\beta\left(g\left(t\right)\right)$;
4. ${}_0^AD_t^\beta\left(\frac{f(t)}{g(t)}\right)=\frac{g(t){}_0^AD_t^\beta(f(t))-f(t){}_0^AD_t^\beta(g(t))}{g^2(t)}$.

2.1.4 Partial derivative with new parameter and properties

Definition 2.1.4 ([17, 16]). *Let u be a function of two variable x and y, then, the β-derivative of u with respect to x is defined as follow:*

$$ {}_0^A D_x^\beta u(x,y) = \lim_{\varepsilon \to 0} \frac{u\left(x + \varepsilon\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}, y\right) - u(x,y)}{\varepsilon}, \tag{2.1.25} $$

for $0 < \beta \leq 1$.

Definition 2.1.5 ([17, 16]). *The β–Laplace operator in two dimensions of a function u is given by*

$$ {}_0^A \Delta_x^\beta u(x,y) = \frac{\partial^{2\beta} u(x,y)}{\partial x^{2\beta}} + \frac{\partial^{2\beta} u(x,y)}{\partial y^{2\beta}} \tag{2.1.26} $$

for $0 < \beta \leq 1$, where x and y are the standard cartesian coordinates of the xy–plane.

Definition 2.1.6 ([17, 16]). *From equality* (2.1.26)*, the mixed $(\beta;\gamma)$–Laplace transform is expressed as:*

$$ {}_0^A \Delta_x^{\beta,\gamma} u(x,y) = \frac{\partial^{2\beta} u(x,y)}{\partial x^{2\beta}} + \frac{\partial^{2\gamma} u(x,y)}{\partial y^{2\gamma}} \tag{2.1.27} $$

Theorem 2.1.7 (**Clairauts theorem for partial β–derivatives**:)**.** *Assume that $u(x;y)$ is function such that $\partial_x^\beta[\partial_y^\gamma u(x;y)]$ and $\partial_y^\gamma[\partial_x^\beta u(x;y)]$ exist and are continuous over a domain $Y \subseteq \mathbb{R}^2$ then,*

$$ \partial_x^\beta[\partial_y^\gamma u(x;y)] = \partial_y^\gamma[\partial_x^\beta u(x;y)]. \tag{2.1.28} $$

Proof. [16, Theorem 2.5.1] ■

Furthermore, The β–Laplace operator satisfies the following properties: For $a, b \in \mathbb{R}$,

- ${}_0^A\Delta_.^\beta(au + bv) = a{}_0^A\Delta_.^\beta u + b{}_0^A\Delta_.^\beta v$;
- ${}_0^A\Delta_.^\beta \times (aU + bV) = a{}_0^A\Delta_.^\beta \times U + b{}_0^A\Delta_.^\beta \times V$;
- ${}_0^A\Delta_.^\beta({}_0^A\Delta_.^\beta \times U) = 0$;
- ${}_0^A\Delta_.^\beta \times ({}_0^A\Delta_.^\beta U) = 0$.

Chapter 3

Preliminary and auxiliary mathematical results

3.1 Introduction

In this chapter, important and most used mathematical results, seen as preliminaries for the following analysis and which are going to be used throughout this thesis are summarized. Some of them are well-known while others are recent and really innovative. Relevant references have been provided for any reader who needs to have more details.

3.1.1 Kermack-McKendrick epidemic models with fractional derivative

The Kermack-McKendrick epidemic model which was developed in the late 1920s is widely seen as one of the first compartmental model in mathematical epidemiology. We owe it to the work by Kermack and McKendrick [84, 85]. In this model, a population of size $N(t)$ is divided into different classes, disjoint and based on their disease status. At time t, $S = S(t)$ is the part of population representing individuals susceptible to a disease, $I = I(t)$ is the part of population representing infectious individuals, $R = R(t)$ is the part representing individuals that recovered from the disease. The following hypotheses about the transmission process of the infectious disease and its host population are usually considered during the mathematical analysis [51, 84, 85].

- The transfer rates from a compartment to another is supposed to be proportional to the population size of the compartment. For example, the transfer rate from S to I, the (non-linear) incidence rate can be written as $\beta g(I)S(t)$, where β is some rate constant and the function g characterizing the non-linearity is assumed to be at least $C^3(0, N_0]$ with $g(0) = 0$ and $g(I) > 0$ for $0 < I \leq N_0$, with $N_0 = N(0)$ the initial total population. Note that the classical mass balance incidence has $g(I) \equiv I$ and β is called the transmission coefficient.

- The ensemble of individuals in the host population is well mixed and homogenous so that the Law of Mass Action holds: the number of contacts between hosts from different compartments depends only on the number of hosts in each compartment. For instance, the recovery rate, that is the number of individuals recovering from the disease per unit time, can be expressed as μI, with μ the recovering rate.
- The transmission mode is supposed to be horizontal and happens through direct contact between hosts.
- There is no latent period and all the infected individual hosts become infectious following an infection.
- There is no reinfection or loss of immunity and then, no transfer from the compartment R back to S.

3.1.2 The generalized Routh-Hurwitz Criteria

The generalized Routh-Hurwitz Criteria [4, 64, 79, 120] are fundamental mathematical tests that provide necessary and sufficient conditions for all the roots of the characteristic polynomial (with real coefficients) to lie in the left half of the complex plane. The name originates from the German mathematician Adolf Hurwitz [79] and the English mathematician Edward John Routh [120] who independently contributed to the establishment of the criteria. Routh-Hurwitz Criteria are formulated as follows;

Theorem 3.1.1 (**Routh-Hurwitz Criteria**:). *Let*

$$P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

be a given polynomial with real constants coefficients $a_i, \ \ i = 1, \cdots, n$, *now, define the* n *Hurwitz matrices using the coefficients* a_i *of the characteristic polynomial:*

$$H_1 = (a_1), \ H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \ H_3 = \begin{pmatrix} a_1 & 1 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{pmatrix},$$

and

$$H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 1 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{pmatrix},$$

where $a_j = 0$ *for* $j > n$. *All the roots of the polynomial* $P(\lambda)$ *are negative or have negative real part if and only if the determinants of all Hurwitz matrices are positives:*

$$\det H_j > 0, \quad j = 1, \cdots, n.$$

Remark 3.1.2. *For degree* $n = 2$, *the Routh-Hurwitz Criteria simplify to*

$$\det H_1 = a_1 > 0$$

and

$$\det H_2 = det \begin{pmatrix} a_1 & 1 \\ 0 & a_2 \end{pmatrix} = a_1 a_2 > 0$$

or $a_1 > 0$ *and* $a_2 > 0$.
For polynomial with higher degrees $n = 3, 4$ *and* 5, *the above criteria are summarized as follows:*

- $n = 3:$ $a_1 > 0,\ a_3 > 0$ *and* $a_1 a_2 > a_3$;
- $n = 4:$ $a_1 > 0,\ a_3 > 0,\ a_4 > 0,$ *and* $a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$;
- $n = 5:$ $a_i > 0,\ i = 1, 2, 3, 4, 5$ $a_1 a_2 a_3 > a_3^2 + a_1^2 a_4$ *and*

$$(a_1 a_4 - a_5)(a_1 a_2 a_3 - a_3^2 - a_1^2 a_4) > a_5 (a_1 a_2 - a_3)^2 + a_1 a_5^2.$$

These criteria are all proved in [64].

3.1.3 Preliminary stability results

Stability analysis for equilibrium solutions [130, 80, 43]
Consider an open set D in the phase space $\mathbb{R}^n$ and a function $u \in C^1(D \to \mathbb{R}^n)$, called a *vector field*. A system of differential equations can be defined as follows:

$$x' = u(x). \tag{3.1.1}$$

A *solution* to (3.1.1) in an interval $I \subset \mathbb{R}$ is a differentiable function $\varphi : I \to \mathbb{R}^n$ such that

$$\varphi'(t) = u(\varphi(t)).$$

When the vector field $u(x)$ is smooth (C^1), the fundamental theory of differential equations ensures that, for each initial point $x_0 \in D$, a unique solution $x(t,\ x_0)$ exists in an interval $I = (-\varrho, \varrho)$ such that $x(0, x_0) = x_0$. We say that such a solution starts from the initial point x_0. A solution can be extended to its maximal interval of existence. If a solution $x(t,\ x_0)$ remains in a compact subset of D during its maximal interval of existence, then it exists for all $t \in \mathbb{R}$.

The orbit of a solution $x(t, x_0)$ is given by the set

$$\{\xi(x_0) = x(t,\ x_0) : t \in \mathbb{R}.\}$$

A solution $x(t)$ is called an *equilibrium* or steady-state, if it is a constant for all t, i.e. $x(t) = \bar{x}$ for $t \in \mathbb{R}$. In this case, $\bar{x}$ satisfies $f(\bar{x}) = 0$ since $x'(t) \equiv 0$.

A *periodic solution* $x(t)$ of period $T > 0$, satisfies $x(t + T) = x(t)$ for all $t \in \mathbb{R}$, and its orbit $\xi = \{x(t) : 0 \leq t < T\}$ is a simple closed smooth curve; Periodic orbits are also called closed orbits. For equilibria and periodic solutions, we are often concerned with their *stability*. Intuitively, an equilibrium $\bar{x}$ is stable if any solution starting close to $\bar{x}$ remains close to $\bar{x}$. This is mathematically expressed as shown in the following definition:

Definition 3.1.3. *An equilibrium $\bar{x}$ of system* (3.1.1) *is*

1. *stable, if for each ϵ–neighborhood $N(\bar{x}, \epsilon)$ of $\bar{x}$, there exists a δ -neighborhood $N(\bar{x}, \delta)$ of $\bar{x}$ such that $x_0 \in N(\bar{x}, \delta)$ implies $x(t, x_0) \in N(\bar{x}, \epsilon)$ for all $t \geq 0$;*
2. *asymptotically stable, if $\bar{x}$ is stable and there exists a b -neighborhood $N(\bar{x}, b)$ such that $x_0 \in N(\bar{x}, b)$ implies $x(t, x_0) \to \bar{x}$ as $t \to \infty$.*

In the above definition, an asymptotically stable equilibrium $\bar{x}$ is said to attract points in a neighborhood $N(\bar{x}, b)$. The set of points that are attracted by $\bar{x}$ is an open set and is called the *basin of attraction* of $\bar{x}$.

Stability analysis by linearization

Linearization is one of standard methods in stability analysis. Let $\bar{x} = 0$ be an equilibrium, namely, $u(0) = 0$, and thus $u(x)$ can be written in Taylor expansion as

$$u(x) \ = \ Ax + F(x), \tag{3.1.2}$$

where the matrix

$$A = \frac{\partial u}{\partial x}(0)$$

is the Jacobian matrix of u at 0, and

$$F(x) = u(x) - Ax.$$

Hence, $F(0) = 0$ and $\frac{\partial F}{\partial x}(0) = 0$. In the expansion (3.1.2), Ax is the linearization of u at 0 and $F(x)$ is the higher order term. The linearized system of (3.1.1) at the equilibrium 0 is

$$y' = Ay. \tag{3.1.3}$$

This leads to the standard result on stability given in the following theorem:

Theorem 3.1.4. *Let A and F be given in* (3.1.2). *If $y = 0$ is asymptotically stable for the linearized system* (3.1.3), *then the equilibrium $\bar{x}$ is asymptotically stable for the non-linear system* (3.1.1)

By Theorem 3.1.4, it is sufficient to investigate the asymptotic stability of an equilibrium for the linearized system. Regarding the latter, we have the following result.

Theorem 3.1.5. *The solution $y = 0$ is asymptotically stable for the linear system* (3.1.3) *if all eigenvalues of A has negative real parts.*

To verify that an $n \times n$ matrix has n eigenvalues with negative real parts can be a challenging task when n is large, especially if entries of A contain non-numerical parameters. Here we can go back and recall the algorithm given in Routh-Hurwitz criteria for $n = 2, 3, 4, 5$ as listed in Section 3.1.2.

Stability results in fractional differentiation[102, 39, 82]

Consider the system given by the following linear state -space form with finite inner dimension n :

$$\begin{cases} d^{\beta}x &= Ax + Bu \\ y &= Cx \end{cases}, \ x(0) = x_0, \tag{3.1.4}$$

where d^{β} the fractional derivative of order β, $0 < \beta \leq 1$, $u \in \mathbb{R}$ is the control, $x \in \mathbb{R}$ the state, $y \in \mathbb{R}^p$ the observation and A, B, C are three suitable operators applying on them. Then we have the following definition for internal stability:

Definition 3.1.6. *Consider $\|\cdot\|$ the standard $\mathbb{R}-$norm. The following autonomous* *subsystem of* (3.1.4)*:*

$$d^{\beta}x = Ax, \qquad \textit{with} \quad x(0) = x_0$$

is said to be

- *stable if and only if for all* x_0*, there is* A *such that for all* $t > 0$, $\|x(t)\| \leq A$
- *asymptotically stable if and only if* $\lim_{t\to+\infty} \|x(t)\| = 0$.

Hence we have the classical stability results [102, Theorem 2]

Theorem 3.1.7. *The autonomous* (3.1.4) *is*

- *asymptotically stable if and only if* $|arg(spec(A))| > \frac{\beta\pi}{2}$*, where* $arg(spec(A))$ *represents the argument (arg) of the spectrum (spec) of operator* A*.*
  In this case, the components of the state decay towards 0, exactly like $t^{-\beta}$*.*
- *stable if and only if it is asymptotically stable or those critical eigenvalues which satisfy* $|arg(spec(A))| = \frac{\beta\pi}{2}$ *have a geometric multiplicity of one.*

This result insinuates that stabilities for systems of types (3.1.4) are guaranteed if and only if the roots of some polynomial (i.e. the eigenvalues of the matrix of the dynamics or the poles of transfer matrix) lie outside the closed angular

$$|arg(\lambda)| \leq \frac{\beta\pi}{2}.$$

This obviously generalizes the well known results for the integer case $\beta = 1$, as given in Routh-Hurwitz Criteria.

3.1.4 Generalized Mean Value Theorem

The following result will be important in our analysis:

Theorem 3.1.8. *Let the function* $\mathcal{J} \in C[t_1, t_2]$ *and its fractional derivative* $D_t^{\alpha}\mathcal{J} \in C(t_1, t_2]$ *for* $0 \leq \alpha < 1$*, and* $t_1, t_2 \in \mathbb{R}$ *then we have*

$$\mathcal{J}(t) = \mathcal{J}(t_1) + \frac{1}{\Gamma(\alpha)} D_t^{\alpha}\mathcal{J}(\tau)(t - t_1)^{\alpha} \ \textit{for all} \ t \in (t_1, t_2],$$

where $0 \leq \tau < t$*.*

Proof*.* See the 'Generalized Mean Value Theorem' proved in [108]. ■

3.1.5 The modified Sumudu integral transform

The modified Sumudu integral transform is given in the following definition:

Definition 3.1.9. *Let g be a function defined in $(0, \infty)$, then, we defined the modified Sumudu transform of g as*

$$S_\beta(g(t), u) = \int_0^\infty \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta - \lceil \beta \rceil} \frac{1}{u} e^{-\frac{t}{u}} g(t) dt, \tag{3.1.5}$$

where $\lceil \beta \rceil$ is the smallest integer greater or equal to β. Since $\beta \in (0, 1]$ in this thesis then, $\beta - \lceil \beta \rceil = \beta - 1$.

An important property of the modified Sumudu transform:
If $S(g(t), u)$ is the well known Sumudu transform of g defined in [134] as

$$S(g(t), u) = \int_0^\infty \frac{1}{u} \exp[-\frac{t}{u}] g(t) dt,$$

then, we have the following relation:

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t), u) = \frac{1}{u^n} S(g(t), u) - \sum_{k=0}^{n-1} \frac{1}{u^{n-k}} g^{(k)}(0). \tag{3.1.6}$$

Proof. By definition we have

$$\begin{aligned} S_\beta({}_0^A D_t^\beta g^{n-1}(t), u) &= \int_0^\infty \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} \\ &\cdot \frac{1}{u} \exp\left[-\frac{t}{u}\right] \left(\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} \lim_{\varepsilon \to 0} \frac{g^{n-1}\left(t + \varepsilon \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - g^{n-1}(t)}{\varepsilon} \right) dt \\ &= \int_0^\infty \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} \frac{1}{u} \exp\left[-\frac{t}{u}\right] \left(\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \lim_{\eta \to 0} \frac{g^{n-1}(t+\eta) - g^{n-1}(t)}{\eta} \right) dt \end{aligned} \tag{3.1.7}$$

where we have put $\eta = \varepsilon \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Hence, making use of the well known property of Sumudu transform $S(g(t), u)$ [134], we obtain

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t), u) = S(g^n(t), u) = \frac{1}{u^n} S(g(t), u) - \sum_{k=0}^{n-1} \frac{1}{u^{n-k}} g^{(k)}(0),$$

which concludes the proof. ∎

More properties and information about this new version of Sumudu transform can be found in [10].

3.1.6 Next generation method

The next generation method [43, 130] is one of the fundamental technique in mathematical epidemiology helping the computation of the basic reproduction ratio. The method is standard, classic and can be found in many works including [43, 42, 73, 130] and the references therein.

3.2 Methods of investigation

In this research, various methods in applied sciences, especially in applied functional analysis, applied analysis, mathematical epidemiology and fractional calculus will be used to investigate the problems mentioned above. They are summarized as follows:

- To establish the stability results for differential equations expressed with the β–derivative, we shall consider two operators, one for the β–derivative in the classical derivation's sense (D) and another for the distributions' sense (d). To proceed, we will also consider eigenfunctions of these operators D and d and propose new definitions for the concepts of internal stability and asymptotical stability of the system with β–derivative. Contrary to other systems, like the systems with Caputo fractional derivative whose eigenfunctions are given by Mittag-Leffler functions and their variants, the eigenvalue system with β–derivative for the operator D will yields a special function $\mathcal{E}_\beta$, new in the literature and called the (Atangana-Goufo) beta-exponential function, recently introduced by Doungmo Goufo and Atangana (see [48]) and defined as ($K \in \mathbb{R}$):

$$\mathcal{E}_\beta(t) = Exp\left[-K\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right]. \tag{3.2.1}$$

  Recall that the Mittag-Leffler function and its generalized version are respectively defined by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \tag{3.2.2}$$

and

$$E_{\alpha,\theta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \theta)}, \tag{3.2.3}$$

for the complex argument $z \in \mathbb{C}$ and the parameters $\alpha, \theta \in \mathbb{C}$ with $Re\,\alpha > 0$, $Re\,\theta > 0$. It is obvious that the special function $\mathcal{E}_\beta$, compared to the power series $E_\alpha(z)$, is friendlier and easier to handle. Hence, we have to be able to find a similar function for the operator d and study the link between these two special functions. This shall lead to conditions on the eigenvalues that guarantee the stabilities. With these stability conditions established, our epidemic model of Ebola may easily be investigated by focussing on the behavior of the eigenvalues of the Jacobian matrix that will influence the stability (local, global and asymptotical) of the disease free equilibrium of the whole system (endowed with the local beta derivative). The advantage of using the β–derivative and novel techniques of fractional differentiation is that it will yield wider conditions for the eigenvalues than those usually obtained via the traditional Newton differentiation and hence, leading to a more accurate statistical descriptions and recommendations for Ebola disease.

- After that, the method will consist of using the β–derivative to express and analyze an Ebola epidemic model with non-linear transmission. We shall first prove that the model using beta-derivative and relatively new in the literature, is well-defined, well-posed. The method here will consist of investigating the direction of the vector field of the system on each coordinate plane and see whether this vector field points to the interior of positive orthant $\mathbb{R}^3_+$ or is tangent to the coordinate plane. This will assure that each solution remains positive with positives initial conditions. Moreover, conditions for boundedness and dissipativity of the trajectories are established. We will exploit the generalized Routh-Hurwitz Criteria [4] to study the existence and stability of equilibrium points for Ebola model. This stability analysis might show that equilibrium points for Ebola model with a new parameter are strongly dependent on the non-linear transmission. Finally, making use of Mathematical Software like MatLab and Mathematica, numerical simulations shall be provided for particular expressions of the non-linear transmission.
- Because of the fact that we are dealing with evolution equations, like the model of (bio)polymer chain degradation, we shall first examine the case where the breakup rate is independent of the size of the (bio)polymer chain before considering the general case where the breakup rate depends on the size of the chain breaking up. In the process, the new version of Sumudu integral transform (see [10, 48, 134]),

the regularity of Lebesgue integrable function will be used. We hope to provide an explicit form of the general solution representing the evolution of polymer sizes distribution and interpret it. This will allow us to perform numerical simulations using graphical software like Matlab and Mathematica

Chapter 4

Stability results for differential equations with a new parameter and Applications to an epidemic model of Ebola hemorrhagic fever with non-linear transmission

4.1 Introduction

Due to the complexity of new outbreaks of diseases happening around the world, the development and application of new approaches in mathematical epidemiology has exploded recently. Many authors have paid special attention to the modeling of real world phenomena in a broader outlook like for instance, the inclusion of the concept of fractional order derivatives or simply adding new parameters in the process. It happened that some of such modelling are more reliable and provide better predictions compared to models with conventional (interger order) derivative [34, 51, 116, 89]. A concrete proof was given in [116] with the fact that some epidemic models based on variation with conventional derivative were unable to reproduce the statistical data collected in a real outbreak of some disease with enough degree of accuracy. Other examples are provided in [110, 106, 115] with the application of half-order derivatives and integrals, which, compared to classical models, are proved to be more useful and reliable for the formulation of certain electrochemical problems. For more example the reader can refers to the works [16, 17, 51, 89, 33, 122, 21, 44] that have successfully generalized, in various ways, classical derivatives to derivatives of fractional order.

In the domain of mathematical epidemiology, Doungmo Goufo et al. [51] provided several interesting and useful properties of Kermack-McKendrick epidemic model with non-linear incidence and fractional order derivative. Recall that Kermack-McKendrick epidemic model is considered as the basis from which many other multi-compartmental models were developed. The results obtained therein sustain the legitimation of epidemic models with fractional order derivative and may help analyze more complex models in the field.

Accordingly, the outbreak of Ebola haemorrhagic fever is currently occurring in West African countries and has infected around 28637 people, killed more than 11315 people so far around the world, and these numbers are still rising. Not only the West African region is affected as clearly shown in Fig. 4.1. There is no known and yet confirmed cure for the disease and since the true and real dynamic of the virus is not yet apprehended totally, it is reasonable to apply recent developed concepts to the disease in order to establish a broader outlook on the real nature of this killer disease that has become a nightmare for all the nations. More justifications and motivations are provided in Section 4.2.2 here below.

4.2 Some important notes

4.2.1 Ebola haemorrhagic fever and non-linear transmission

Ebola haemorrhagic fever is caused by Ebola virus, a virus from the family of filoviridea. The genus Ebolavirus counts itself among three members of the Filoviridae family (filovirus), together with the genus Marburgvirus and the genus Cuevavirus. Three distinct species of the Genus Ebolavirus, namely Bundibugyoebolavirus (BDBV), Zaire ebolavirus (EBOV),Sudan ebolavirus (SUDV) are believed to be largely responsible for the Ebola outbreaks in Africa in general and the actual 2014 fatal outbreak occurring in West Africa. Ebola virus is an unusual but fatal virus that, when spreading throughout the body, damages the immune system and organs. Ultimately, it causes levels of blood-clotting cells to drop [24]. This causes uncontrollable bleeding inside and outside the body [77] to yield a severe hemorrhagic disease characterized by initial fever and malaise followed by shock, gastrointestinal bleeding symptoms, to end by multi-organ system failure.

In Africa, the transmission of ebolavirus is believed to be non-linear and happen in various ways. Most of the infections that occur in living beings are possible by the

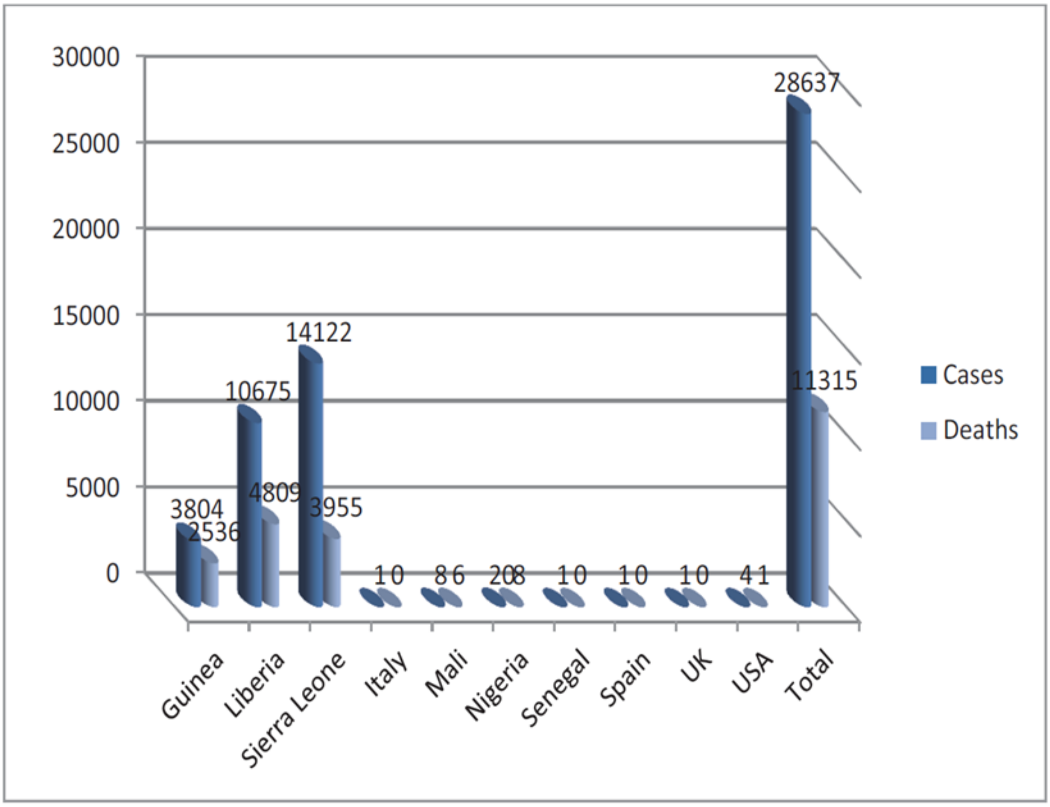

Figure 4.1: Number of Ebola cases and deaths per country[a]

[a]Source:"Ebola Situation report on 7 February 2016". World Health organization. 7 February 2016. Retrieved 8 February 2016. http://apps.who.int/iris/bitstream/10665/147112/1/roadmapsitrep7Jan2016eng.pdf.

handling of infected fruit bats, macaques, baboons, vervets, monkeys, chimpanzees, gorillas, forest antelope and porcupines, sometime found dead or sick in the scrubland or forest. Ebola virus is then transmitted from one person to another through human-to-human, human-to-animal or human-to-fruit birelations. The usual infection results from direct contact (through broken skin or mucous membranes) with the blood, secretions, organs or other bodily fluids of infected people. Transmission of Ebola disease also occurs due to indirect contact with environments contaminated with such fluids [68, 66, 90, 137] or during burial ceremonies in which mourners have direct contact with the body of a deceased person.

The literature concerning Ebola's cure, vaccine, species variety and dynamics is still limited and far from being complete. Therefore, it is urgently necessary to conduct various research and explore new methods and techniques. This will help to better understand the outbreak process and educate people about the real dynamic of Ebolavirus, its transmission's mode and ways to avoid or reduce its spreading. Fig. 4.2 graphically shows the various and most common modes of transmission used by ebola virus to infect human beings and Fig. 4.3 shows some basic prevention about the spread of Ebola virus.

4.2.2 Conventional derivative with new parameter: Justification, motivation

Today, it is widely known that the Newtonian concept of derivative can no longer satisfy all the complexity of the natural occurrences. A couple of complex phenomena and features happening in some areas of sciences or engineering are still (partially) unexplained by the traditional existing methods and remain open problems. Usually in mathematical modeling of a natural phenomenon that changes, the evolution is described by a family of time-parameter operators, that map an initial given state of the system to all subsequent states that takes the system during the evolution.

A widely devotion has been predominantly offered to way of looking at that evolution in which time's change is described as transitions from one state to another. Hence, this is how the theory of semigroups was developed [61, 113], providing mathematicians with very interesting tools to investigate and analyze resulting mathematical models. However, most of the phenomena scientists try to analyze and describe mathematically are complex and very hard to handle. Some of them, like depolymerization, rock fractures and fragmentation processes are difficult to analyze [141] and often involve evolution of two intertwined quantities: the number of particles and the distribution of mass among the particles in the ensemble. Then, though linear, they display non-linear features such as phase transition (called "shattering") causing the appearance of a "dust" of "zero-size" particles with nonzero mass.

Another example is the groundwater flowing within a leaky aquifer. Recall that an aquifer is an underground layer of water-bearing permeable rock or unconsolidated materials (gravel, sand, or silt) from which groundwater can be extracted using a water well. Then, how do we explain accurately the observed movement of water within the leaky aquifer? As an attempt to answer this question, Hantush [71, 72] proposed an equation with the same name and his model has since been used by many hydro-geologists around the world. However, it is necessary to note that the model does not take into account all the non-usual details surrounding the movement of water through a leaky geological

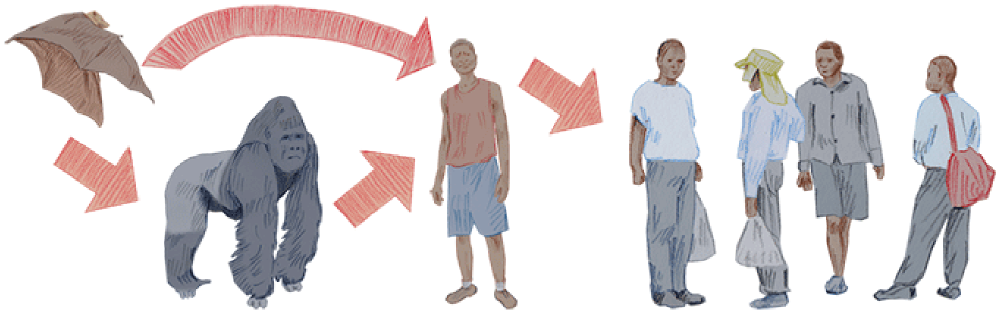

Figure 4.2: Ebola virus transmission modes Source : <http://www.abc.net.au/news/2014-07-30/ebola-virus-explainer/5635028> (Retrieved on 20 February 2016).

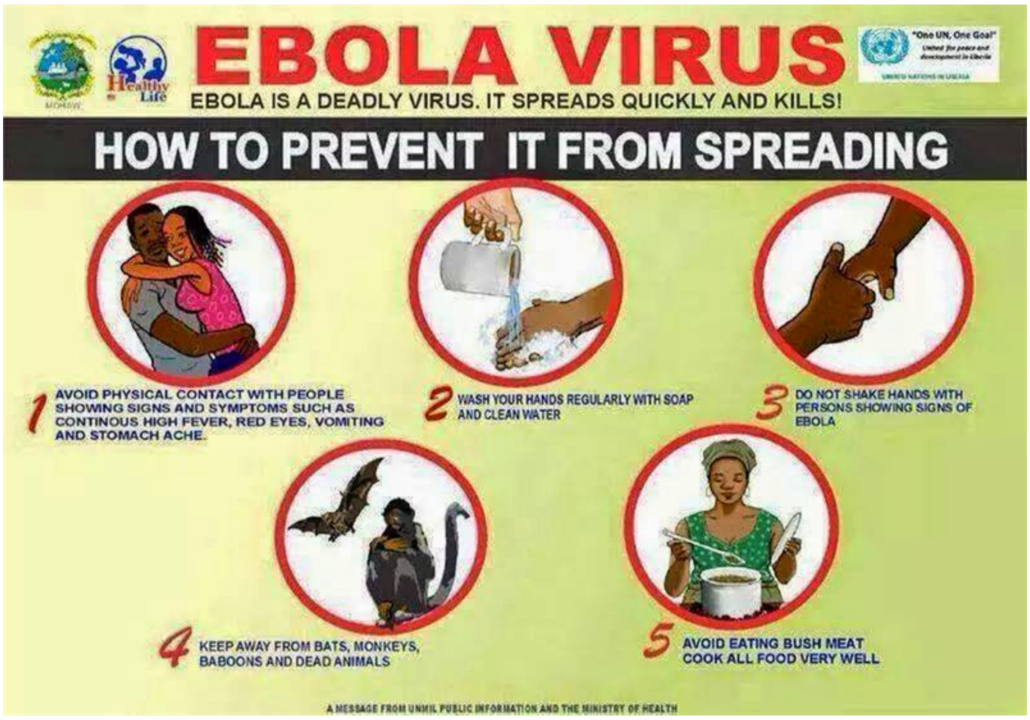

Figure 4.3: Preventing Ebola virus from spreading Source: <http://www.oaupeeps.com/2014/07/ebola-outbreak-causes-transmission.html> (Re on 20 February 2016.)

formation. Indeed, due to the deformation of some aquifers, the Hantush equation is not able to account for the effect of the changes in the mathematical formulation. Hence, all those non-usual features are beyond the usual models' resolutions and need other techniques and methods of modeling with more parameters involved.

Furthermore, time's evolution and changes occurring in some systems do not happen on the same manner after a fixed or constant interval of time and do not follow the same routine as one would expect. For instance, a huge variation can occur in a fraction of second causing a major change that may affect the whole system's state forever. Indeed, it has turned out recently that many phenomena in different fields, including sciences, engineering and technology can be described very successfully by the models using fractional order differential equations [34, 25, 52, 54, 51, 44, 41, 76, 89, 117]. Hence, differential equations with fractional derivative have become a useful tool for describing non-linear phenomena that are involved in many branches of chemistry, engineering, biology, ecology and numerous domains of applied sciences. Many mathematical models, including those in acoustic dissipation, mathematical epidemiology, continuous time random walk, biomedical engineering, fractional signal and image processing, control theory, Levy statistics, fractional phase-locked loops, fractional Brownian, porous media, fractional filters motion and non-local phenomena have proved to provide a better description of the phenomenon under investigation than models with the conventional integer-order derivative [34, 51, 116, 89].

One of the attempts to enhance mathematical models was to introduce the concept of derivative with fractional order. There exists a very large literature on different definitions of fractional derivatives. The most popular ones were already given in Chapter 2, namely the Riemann–Liouville and the Caputo derivatives. Recall that each of them presents some advantages and disadvantages [52, 115, 122]. Not all of them satisfy the common properties of the standard concept of derivative, and therefore, there are some limitations that will not allow them to adequately describe real world problems and phenomena, as already mentioned in Chapter 2. For the sake of more clarity, we recall some of them as follows:

The Riemann–Liouville derivative of a constant is not zero while Caputo's derivative of a constant is zero but demands higher conditions of regularity for differentiability.

To compute the fractional derivative of a function in the Caputo sense, we must first calculate its derivative.

Caputo derivatives are defined only for differentiable functions while functions that have no first order derivative might have fractional derivatives of all orders less than one in the Riemann–Liouville sense.

Guy Jumarie (2005 and 2006) proposed a simple alternative definition to the Riemann–Liouville derivative, the modified one showed above.

New fractional derivatives with no singular kernel were recently proposed by many authors including Caputo et al. in [37], Doungmo Goufo [53], and a version with non-local and non-singular kernel was introduced by Atangana and Baleanu [14]. However, Caputo fractional derivative [36], for instance, remains the one mostly used for modelling real world problems in the field [34, 25, 52, 54, 51]. However, this derivative exhibits some limitations like not obeying the traditional chain rule; which chain rule represents one of the key elements of the match asymptotic method [16, 17, 86, 125]. Recall that the match asymptotic method has never been used to solve any kind of fractional differential equations because of the nature and properties of fractional derivatives. Hence, the conformable derivative was proposed [2, 87]. This derivative is theoretically very easier to handle and obeys the chain rule. But it also exhibits a huge failure that is expressed by the fact that the derivative of any differentiable function at the point zero is zero. This does not make any sense in a physical point of view.

Accordingly, a modified new version, the β–derivative was proposed in order to skirt the noticed weakness. The main aim of this new derivative was, first of all, to perform a wider analysis on the well-known match asymptotic method [16, 17, 86, 125] and later extend and describe the boundary layers problems within new parameters. Note that the β–derivative is not considered here as a fractional derivative in the same sense as Riemann–Liouville or Caputo fractional derivative. It is the conventional derivative with a new (fractional) parameter and as such, has been proven to have many applications in applied sciences [16, 17] and mathematical epidemiology [15]. Our goal is to pursue the investigation in the same momentum. Recall as done in Chapter 1 that it is defined as:

Definition 4.2.1. *Let g be a function, such that, $g : [a, \infty) \to \mathbb{R}$ then, the $\beta-$derivative of g is defined as:*

$$
{}_0^A D_t^\beta g(t) = \begin{cases} \lim\limits_{\varepsilon \to 0} \frac{g\left(t+\varepsilon\left(t+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-g(t)}{\varepsilon} & \text{for all } t \geq 0, \quad 0 < \beta \leq 1 \\ g(t) & \text{for all } t \geq 0, \, \beta = 0, \end{cases} \tag{4.2.1}
$$

where Γ is the gamma-function

$$
\Gamma(\zeta) = \int_0^\infty t^{\zeta-1} e^{-1} dt.
$$

4.3 Model formulation with a new parameter

As mentioned here above, the aim of this thesis is to propose new approaches, extend classical models to models with the new derivative and investigate them with various and different techniques in order to establish broader outlooks on the real phenomena they describe. So let us consider a region with a constant overall population $N(t)$ at a given time t, with $N(0)$ noted N_0. The population $N(t)$ is divided into four compatements, namely $S(t)$ the number representing individuals susceptible to catch Ebola, $I(t)$ the number of individuals infected with Ebola, $R(t)$ the number representing people that recover from Ebola and $M(t)$ the number of individual that are believed to have become immunized after Ebola infection and recovery. We assume that all recruitment, occurring at a constant rate Λ, is into the class of susceptible to catch the Ebola fever and that every infected person becomes automatically infectious. Some people of the total population are considered to die due to a non-disease related death at a rate constant μ, so that thus $\frac{1}{\mu}$ can be taken as the average lifetime. In addition, Ebola virus kills infectious people at a rate constant d. We consider the usual non-linear mass balance incidence expressed as $\kappa S g(I)$ to indicate successful transmission of Ebola virus due to non-linear contacts dynamics in the populations by infectious. Here, the function g characterizing the non-linearity is assumed to be at least $C^3(0, N_0]$ with $g(0) = 0$ and $g(I) > 0$ for $0 < I \leq N_0$ and κ is some rate constant. After receiving an effective test treatment or due to personal and yet unknown biological factors, Ebola infectious individuals can spontaneously recover from the disease with a rate constant τ, entering the recovered (immunized) class. Since research about the real dynamics and transmission mode of Ebola virus is still ongoing, we assume that a fraction γR of recovered people $\gamma \leq 1$, after receiving a treatment reduces their risk to get infected again and are believed to be immunized. Thus, a fraction $(1-\gamma)R$ of recovered people go back to susceptible class with a rate constant δ. The transfer diagram describing the above dynamics for Ebola fever is given in Fig. 4.4 and expressed by the system

$$\begin{cases} {}_0^A D_t^\beta S(t) &= \Lambda - \kappa S(t) g(I)(t) + (1-\gamma)\delta R(t) - \mu S(t) \\ {}_0^A D_t^\beta I(t) &= \kappa S(t) g(I)(t) - (\mu + d + \tau) I(t) \\ {}_0^A D_t^\beta R(t) &= \tau I(t) - (\mu + \gamma) R(t) - (1-\gamma)\delta R(t) \\ {}_0^A D_t^\beta M(t) &= \gamma R(t) - \mu M(t), \end{cases} \tag{4.3.1}$$

with initial conditions

$$S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0, \quad M(0) = M_0, \tag{4.3.2}$$

where

$${}_{0}^{A}D_{t}^{\beta}\left(f(t)\right) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon\left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right) - f(t)}{\varepsilon},$$

for all $t \geq 0$ and $0 < \beta \leq 1$.

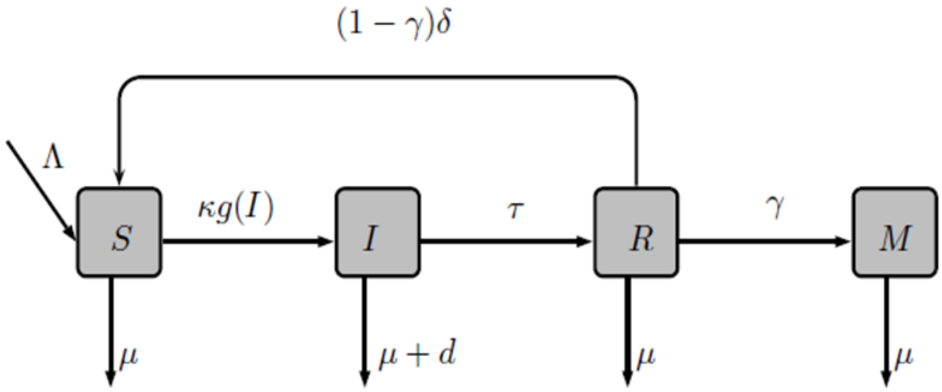

Figure 4.4: Transfer diagram for the dynamics of Ebola fever transmission in West-Africa.

4.4 Mathematical analysis

In this section, the model (4.3.1)-(4.3.2) is analyzed in order to prove its well posedness, study the conditions for the existence of disease free and endemic non-trivial equilibria, provide an expression for the basic reproduction ratio and threshold conditions for asymptotic stability of equilibria.

4.4.1 Positivity of solutions

Proposition 4.4.1. *There exists a unique solution for the initial value problem given (4.3.1)-(4.3.2). Furthermore, if the initial conditions (4.3.2) are non-negative then the corresponding solution* $(S(t), I(t), R(t), M(t))$ *of the Ebola model (4.3.1) is non-negative for all* $t > 0$.

Proof. The proof of the first part follows from Remark 3.2 supported by Theorem 3.1 in [96]. For the second part, we show the positively invariance of the non-negative orthant

$\mathbb{R}^4_+ = \{(S, I, R, M) \in \mathbb{R}^4 : \; S \geq 0, \; I \geq 0, \; R \geq 0, \; M \geq 0\}$. Then, we can investigate the direction of the vector field

$$\begin{pmatrix} \Lambda - \kappa S(t)g(I)(t) + (1-\gamma)\delta R(t) - \mu S(t) \\ \kappa S(t)g(I)(t) - (\mu + d + \tau)I(t) \\ \tau I(t) - (\mu + \gamma)R(t) - (1-\gamma)\delta R(t) \\ \gamma R(t) + \mu M(t), \end{pmatrix}^T \tag{4.4.1}$$

on each coordinate space and see whether the vector field points to the interior of $\mathbb{R}^4_+$ or is tangent to the coordinate space.

On the coordinate space IRM, we have $S = 0$ and

$${}_0^A D_t^\beta S_{|_{S=0}} = \Lambda + (1-\gamma)\delta R \geq 0.$$

On the coordinate space SRM, we have $I = 0$ and

$${}_0^A D_t^\beta I_{|_{I=0}} = 0.$$

On the coordinate space SIM, we have $R = 0$ and

$${}_0^A D_t^\beta R_{|_{R=0}} = \tau I \geq 0.$$

On the coordinate space SIR, we have $M = 0$ and

$${}_0^A D_t^\beta M_{|_{M=0}} = \gamma R \geq 0.$$

Making use of the same arguments as in [51, Property ii] together with Theorem 2.1.2, we conclude the proof by stating that the vector field (4.4.1) either points to the interior of $\mathbb{R}^4_+$ or is tangent to each coordinate space. ■

4.4.2 Boundedness and dissipativity of the trajectories

From the above model (4.3.1), if we add all the equations, we obtain from $N(t) = S(t) + I(t) + R(t) + M(t)$ and Theorem 2.1.3 that

$$D_t^\beta N(t) = \Lambda - \mu N(t) - dI(t).$$

Then, this yields $D_t^\beta N(t) \leq \Lambda - \mu N(t)$. Therefore, making use of the previous section, we have proved the following Proposition

Proposition 4.4.2. $\lim_{t\to+\infty} N(t) \leq \frac{\Lambda}{\mu}$.

Furthermore, we have the following invariance property: If $N(0) \leq \frac{\Lambda}{\mu}$*, then* $N(t) \leq \frac{\Lambda}{\mu}$*, for all* $t \geq 0$.

In particular, the region

$$\Psi_\varepsilon = \left\{(S; I; R; M) \in \mathbb{R}_+^4,\ N(t) \leq \frac{\Lambda}{\mu} + \varepsilon\right\} \tag{4.4.2}$$

is a compact forward and positively-invariant set for the system (4.3.1) with non-negative initial conditions in $\mathbb{R}_+^4$ *and that is absorbing for* $\varepsilon > 0$.

Thus, we will restrict our analysis to this region Ψ_ε for $\varepsilon > 0$.

4.4.3 Existence and stability analysis of equilibrium points

We can consider the systems

$$\left\{\begin{array}{lcl} {}_0^AD_t^\beta S(t) &=& \Lambda - \kappa S(t)g(I)(t) + (1-\gamma)\delta R(t) - \mu S(t) \\ {}_0^AD_t^\beta I(t) &=& \kappa S(t)g(I)(t) - (\mu + d + \tau)I(t) \\ {}_0^AD_t^\beta R(t) &=& \tau I(t) - (\mu+\gamma)R(t) - (1-\gamma)\delta R(t) \end{array}\right. \tag{4.4.3}$$

and

$${}_0^AD_t^\beta N(t) = \Lambda - \mu N(t) - dI(t). \tag{4.4.4}$$

To obtain the equilibrium points of the system (4.4.3)-(4.4.4), let us put

$$\left\{\begin{array}{lcl} 0 = {}_0^AD_t^\beta S(t) &=& \Lambda - \kappa S(t)g(I)(t) + (1-\gamma)\delta R(t) - \mu S(t) \\ 0 = {}_0^AD_t^\beta I(t) &=& \kappa S(t)g(I)(t) - (\mu + d + \tau)I(t) \\ 0 = {}_0^AD_t^\beta R(t) &=& \tau I(t) - (\mu+\gamma)R(t) - (1-\gamma)\delta R(t) \\ 0 = {}_0^AD_t^\beta N(t) &=& \Lambda - \mu N(t) - dI(t). \end{array}\right. \tag{4.4.5}$$

The solutions of this system are $X^o = (\frac{\Lambda}{\mu}, 0, 0, \frac{\Lambda}{\mu})$ and $X^e = (S^e, I^e, R^e, N^e)$, where

$S^e = \frac{(\mu+d+\tau)I^e}{\kappa g(I^e)}$

$R^e = \frac{\tau I^e}{\mu+\gamma+(1-\gamma)\delta}$

$N^e = \frac{\Lambda - dI^e}{\mu}$

and I^e satisfying the equation:

$$\frac{g(I)}{I}\left[1-\left(\frac{(\mu+d+\tau)(\mu+\gamma+(1-\gamma)\delta)-(1-\gamma)\delta\tau}{\Lambda(\mu+\gamma+(1-\gamma)\delta)}\right)I\right]=\frac{\mu(\mu+d+\tau)}{\Lambda\kappa}. \quad (4.4.6)$$

Existence and stability of the disease-free equilibrium (DFE)

X^o is the DFE and to analyze its stability for the system (4.4.3)-(4.4.4), we study the eigenvalues of the Jacobian matrix evaluated at that equilibrium point. Thus, evaluated at X^o, the jacobian obtained from the linearized system (4.4.3)-(4.4.4) is given by:

$$J(X^o)=Df(X^o)=\begin{pmatrix} -\mu & -\kappa\frac{\Lambda}{\mu}g'(0) & (1-\gamma)\delta & 0 \\ 0 & \kappa\frac{\Lambda}{\mu}g'(0)-(\mu+d+\tau) & 0 & 0 \\ 0 & \tau & -(\mu+\gamma)-(1-\gamma)\delta & 0 \\ 0 & -d & 0 & -\mu \end{pmatrix} \quad (4.4.7)$$

Theorem 4.4.3. *Taking into consideration the non-linear incidence function g, the disease free equilibrium of the Ebola disease system (4.4.3)-(4.4.4) always exists and is asymptotically stable if*

$$\frac{\kappa\Lambda g'(0)}{\mu(\mu+d+\tau)}<1$$

Proof. The existence of X^o is obvious. Following the same approach as [51, 102] we know that asymptotical stability the DFE (equilibrium point) X^o for the model (4.4.3)-(4.4.4) is guaranteed if and only if all the four eigenvalues , say $\lambda_{1,2,3,4}$ of $J(X^o)$ lie outside the closed angular sector

$$\alpha\frac{\pi}{2}\geq|arg\lambda_i|, \qquad \text{for } i=1,2,3,4.$$

Hence, it is enough to show that

$$\alpha\frac{\pi}{2}<|arg\lambda_i| \quad (4.4.8)$$

for all $i = 1, 2, 3, 4$. Making use of the characteristic matrix

$$\Delta_J(\lambda) = \begin{bmatrix} \mu+\lambda & \kappa\frac{\Lambda}{\mu}g'(0) & -(1-\gamma)\delta & 0 \\ 0 & -\kappa\frac{\Lambda}{\mu}g'(0)+(\mu+d+\tau)+\lambda & 0 & 0 \\ 0 & -\tau & \mu+\gamma+(1-\gamma)\delta+\lambda & 0 \\ 0 & d & 0 & \mu+\lambda \end{bmatrix} \tag{4.4.9}$$

and the characteristic equation $(\mu+\lambda)^2(\mu+\gamma-(1+\gamma)\delta+\lambda)(-\kappa\frac{\Lambda}{\mu}g'(0)+(\mu+d+\tau)+\lambda) = 0$, we obtain the eigenvalues

$$\lambda_{1,2} = -\mu$$
$$\lambda_3 = -(\mu+\gamma-(1-\gamma)\delta)$$
$$\lambda_4 = \kappa\frac{\Lambda}{\mu}g'(0) - (\mu+d+\tau).$$

λ_4 satisfies the constraint (4.4.8) if $\frac{\kappa\Lambda g'(0)}{\mu} < \mu+d+\tau$ and since $\lambda_{1,2,3}$ obviously satisfy the constraint, the proof is complete. ■

For the Ebola model (4.4.3)-(4.4.4), we usually refers the quantity

$$\mathcal{R}_0 = \frac{\kappa\Lambda g'(0)}{\mu(\mu+d+\tau)} \tag{4.4.10}$$

to as the basic reproduction number and is defined to be the number of secondary Ebola cases that one case will produce in a completely Ebola disease susceptible population. In the biological points of view, Theorem 4.4.3 insinuates that Ebola epidemic disease will dies out if $\mathcal{R}_0 < 1$.

Existence and stability of the endemic equilibrium

As in [16, 17, 97], we can put (4.4.6) in the form

$$\frac{1}{\vartheta} = \frac{\mu(\mu+d+\tau)}{\Lambda\kappa} = \frac{g(I)}{I}\left(1-\frac{I}{\Theta}\right) \equiv h(I), \tag{4.4.11}$$

where $\Theta = \frac{\Lambda(\mu+\gamma+(1-\gamma)\delta)}{(\mu+d+\tau)(\mu+\gamma+(1-\gamma)\delta)-(1-\gamma)\delta\tau}$. Then, the number of solutions in terms of I of equation (4.4.10) is dependent on the non-linear incidence function $g(I)$, especially, $\lim_{I\to 0}\frac{g(I)}{I} \equiv h(0)$ and the sign of $h'(I)$. Moreover, Θ is the maximum possible value that can take I^e and in the classical mass action incidence, where $g(I) = I$, the quantity $\vartheta = \frac{\Lambda\kappa}{\mu(\mu+d+\tau)}$ is viewed as the contact reproduction number. As shown in [51, 74], if we denote by ϑ^* the unique value of ϑ verifying (4.4.11) when I reaches a unique maximum value I_m in $(0,\Theta)$, then conditions of existence of the endemic equilibrium X^e are given in the following theorem:

Theorem 4.4.4. *The Ebola model (4.4.3)-(4.4.4)*

1. *has no endemic equilibrium point if* $h(0) \leq \frac{1}{\vartheta}$ *and* $h'(I) < 0$ *for all* $I \in (0, \Theta)$
2. *has no endemic equilibrium point if* $h(0) = 0,\ h''(I) < 0$ *on* $(0, \Theta]$ *and* $\vartheta < \vartheta^*$
3. *has 1 endemic equilibrium point if* $h(0) > \frac{1}{\vartheta}$ *and* $h'(I) < 0$ *for all* $I \in (0, \Theta)$
4. *has 1 endemic equilibrium point if* $h(0) = 0,\ h''(I) < 0$ *on* $(0, \Theta]$ *and* $\vartheta = \vartheta^*$
5. *has 2 endemic equilibria* I_1^e *and* I_1^e *if* $h(0) = 0,\ h''(I) < 0$ *on* $(0, \Theta]$ *and* $\vartheta > \vartheta^*$, *where* $I_1^e \in (0, I_m)$ *and* $I_2^e \in (I_m, \Theta)$.

Considering the expression of $\mathcal{R}_0$ given in (4.4.10), knowing that $g'(0) \sim \lim_{I\to 0} \frac{g(I)-g(0)}{I-0} \equiv h(0)$ and that $h(I)$ is positive for $I \in (0, \Theta)$, with $h(\Theta) = 0$, then, item 3 of Theorem 4.4.4 together with (4.4.11) yield the following lemma

Corollary 4.4.5. *The Ebola model (4.4.3)-(4.4.4) has a unique endemic equilibrium if* $\mathcal{R}_0 > 1$ *and* $h'(I) < 0$ *for* $I \in (0, \Theta)$.

Next, conditions for the stability of X^e is studied from the linearized system of (4.4.3)-(4.4.4) around the endemic equilibrium $X^e = (S^e, I^e, R^e, N^e)$. The following Jacobian matrix is obtained:

$$J(X^e) = \begin{pmatrix} -\kappa g(I^e) - \mu & -\kappa S^e g'(I^e) & (1-\gamma)\delta & 0 \\ \kappa g(I^e) & \kappa S^e g'(I^e) - (\mu + d + \tau) & 0 & 0 \\ 0 & \tau & -(\mu+\gamma) - (1-\gamma)\delta & 0 \\ 0 & -d & 0 & -\mu \end{pmatrix} \tag{4.4.12}$$

To analyse the eigenvalues $\lambda_i,\ i = 1, 2, 3, 4$, we develop the characteristic equation

$$\begin{vmatrix} \kappa g(I^e) + \mu + \lambda & \kappa S^e g'(I^e) & -(1-\gamma)\delta & 0 \\ -\kappa g(I^e) & -\kappa S^e g'(I^e) + (\mu + d + \tau) + \lambda & 0 & 0 \\ 0 & -\tau & (\mu+\gamma) + (1-\gamma)\delta + \lambda & 0 \\ 0 & d & 0 & \mu + \lambda \end{vmatrix} = 0 \tag{4.4.13}$$

which yields

$$(\mu + \lambda)(\lambda^3 + K_1\lambda^2 + K_2\lambda + K_3) = 0, \tag{4.4.14}$$

where

$$K_1 = \kappa g(I^e) + 2\mu + (1-\gamma)\delta + (\mu + d + \tau)\left(1 - I^e \frac{g'(I^e)}{g(I^e)}\right)$$

$$\begin{aligned} K_2 =& \kappa(\mu + d + \tau) g'(I^e) I^e + (\mu + \gamma + (1-\gamma)\delta)(\kappa g(I^e) + 2\mu) \\ &+ (\kappa g(I^e) + 2\mu + \gamma + (1-\gamma)\delta)(\mu + d + \tau)\left(1 - I^e \frac{g'(I^e)}{g(I^e)}\right) \end{aligned} \quad (4.4.15)$$

$$\begin{aligned} K_3 =& \kappa(\mu + d + \tau)(\mu + \gamma + (1-\gamma)\delta) g'(I^e) I^e - \kappa g(I^e)\tau(1-\gamma)\delta \\ &+ (\kappa g(I^e) + \mu)(\mu + \gamma + (1-\gamma)\delta)(\mu + d + \tau)\left(1 - I^e \frac{g'(I^e)}{g(I^e)}\right). \end{aligned}$$

We see that the coefficients K_1, K_2, and K_3 are dependent on the non-linear incidence $g(I)$ Hence, since $\lambda = -\mu$ is already an eigenvalue which is non-positive, the stability of the endemic equilibrium X^e is fully determined by analyzing the roots of

$$P(\lambda) = \lambda^3 + K_1\lambda^2 + K_2\lambda + K_3 = 0$$

given in (4.4.14). Let us denote by Δ_P the discriminant of the polynomial $P(\lambda)$ then, making use of the Routh-Hurwitz Criteria generalized in [4], we state the following the Corollary:

Corollary 4.4.6. *The positive endemic equilibrium X^e of the Ebola model (4.4.3)-(4.4.4) is asymptotically stable if one of the following conditions is satisfied:*

1. $K_1 \geq 0,\ K_2 \geq 0,\ K_3 > 0,\ \Delta_P < 0,$ *and* $0 < \beta \leq \frac{2}{3}$.
2. $K_1 < 0,\ K_2 < 0,\ \Delta_P < 0,$ *and* $\frac{2}{3} < \beta \leq 1$.
3. $K_1 > 0,\ K_3 > 0,\ K_1K_2 > K_3,$ *and* $\Delta_P > 0$.

4.5 Numerical simulations

Let us consider the non-linear incidence function $g(I) = \frac{I^p}{1+rI^q}, \quad p, q > 0, \quad r \geq 0$. We restrict ourselves to the case $r = 0$, to have $g(I) = I^p$. We use the implementation code of the predictor-corrector PECE method of Adams-Bashforth-Moulton type described in [45] to perform numerical simulations for the Ebola model (4.4.3)-(4.4.4). We will consider different values for β in order to appreciate the accuracy of the method employed in this chapter. The table below presents the description and estimated values of the evolved parameters.

Parameters' symbols	Description	Estimation and range[b]
Λ	Recruitment rate by succeptible people in the region	55 $(\text{day})^{-1}$
κ	Transmission coefficient	Not constant
γ	Proportion of recovered individuals that become imunized	0,04
δ	rate at which recovered people go back to susceptible class	0,06
μ	Non-Ebola-disease related death rate	0,01
d	Ebola related death rate	0,7
τ	Recovery rate from Ebola	0,1
p	Symbolizing the non-linear incidence	2

[b]Sources:

"Liberia Ebola SitRep no. 236". 8 February 2016. Retrieved 9 February 2016 http://www.mohsw.gov.lr/documents/Sitrep-20236-20Jan-206th-202014.pdf

"Ebola Situation report on 7 February 2016". World Health organization. 7 February 2016. Retrieved 8 February 2016. http://apps.who.int/iris/bitstream/10665/147112/1/roadmapsitrep7Jan2016eng.pdf

The approximation for solutions $S(t), I(t), R(t)$ and $N(t)$ are presented in Figs. 4.5–4.6 respectively. In each case two different values of β, namely $\beta = 0.93$ and 1 are considered. It appears that numerical results show that the Ebola model (4.4.3)-(4.4.4), using the new β-derivative, exhibits the traditional threshold behaviour.

In Fig. 4.5, we have considered for the non-linear incidence, the transmission coefficient $\kappa = 0.01$ and $p = 2$. Then trajectory of the Ebola model (4.4.3)-(4.4.4) converges to the disease-free equilibrium, which is approximatively at (5500, 0, 0, 5500) with the above given parameters. We also note that the behavior of the system remains similar for close values of the derivative parameter β.

In Fig. 4.6, we have taken the transmission coefficient $\kappa = 0.01$ and $p = 2$. Making use of the involved parameter in the table above, the dynamics shows that there exists one positive endemic equilibrium point, approximately at (11.11, 7.29, 6.78, 4989.70) satisfying the condition 3 of Theorem 4.4.4. Again a similar behavior of the model appears for close values of β.

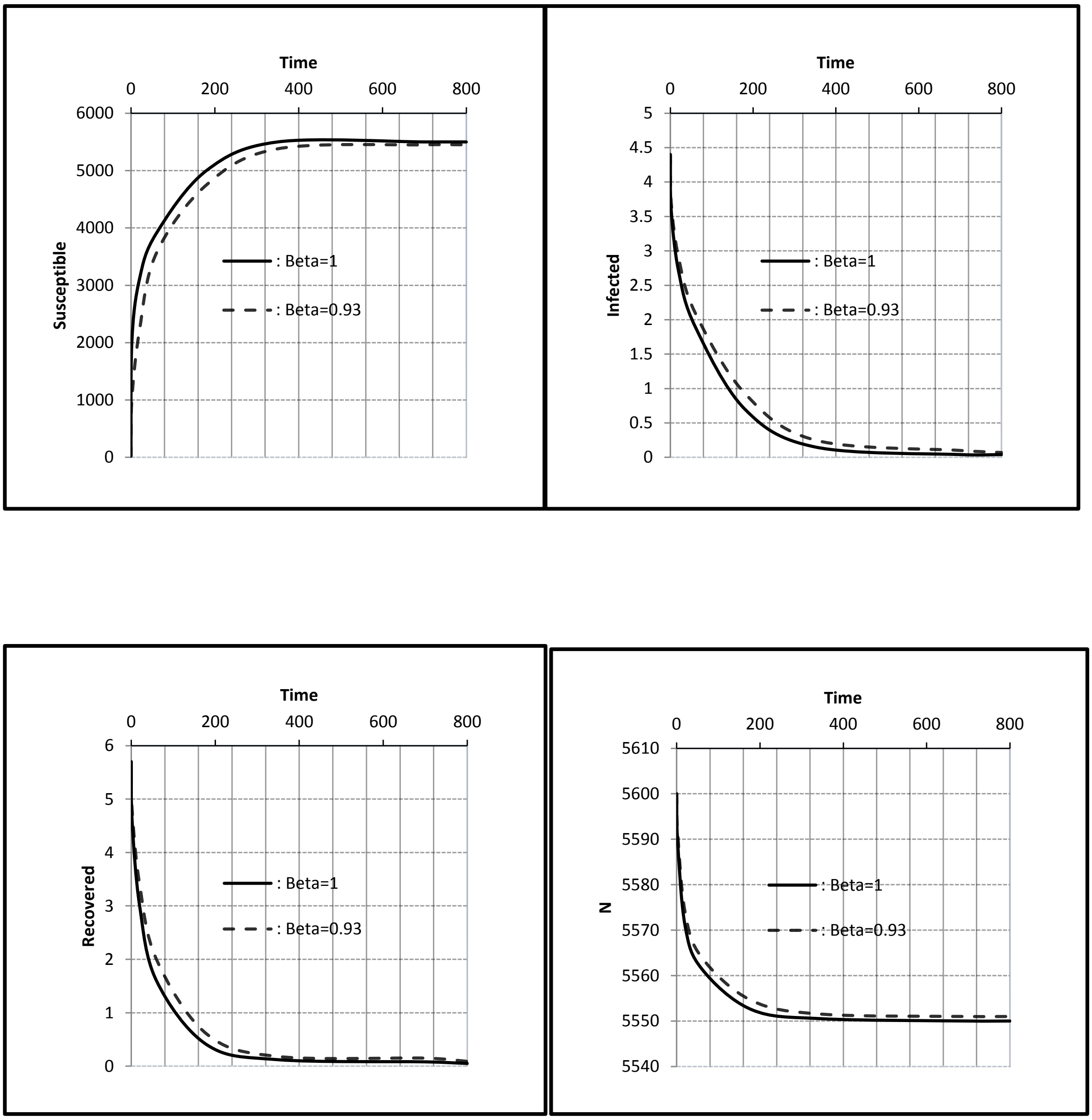

Figure 4.5: The dynamics of Ebola model (4.4.3)-(4.4.4) for $\beta = 1$ and 0.93, when $\mathcal{R}_0 \leq 1$.

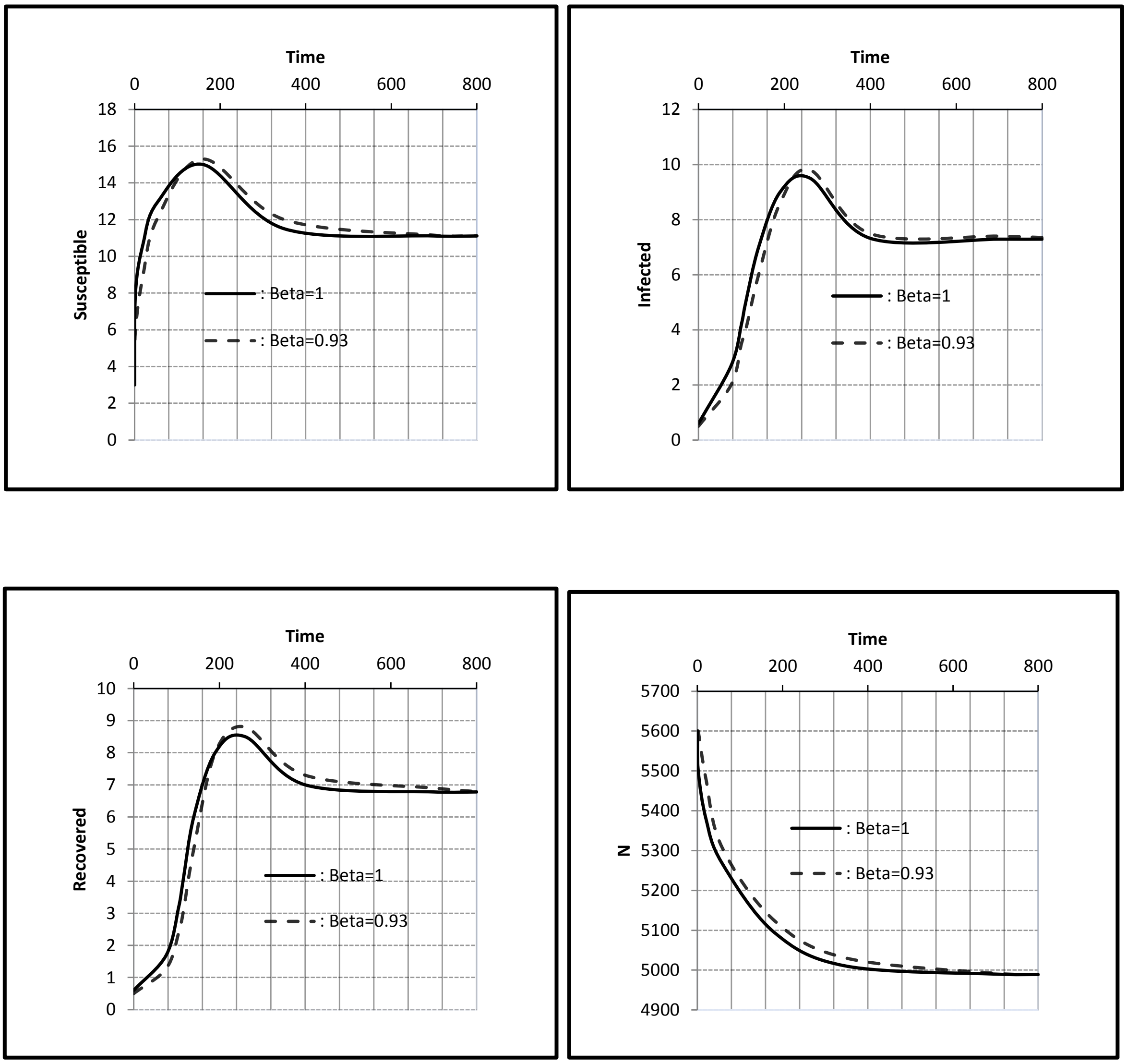

Figure 4.6: The dynamics of Ebola model (4.4.3)-(4.4.4) for $\beta = 1$ and 0.93, when $\mathcal{R}_0 > 1$.

Chapter 5

Evolution of macromolecules depolymerization model with a new parameter

5.1 Introduction

Depolymerization is the process where polymers or biopolymers are converted into monomers or mixtures of monomers. Polymers range from familiar synthetic plastics such as polystyrene (also called styrofoam) to natural biopolymers such as DNA and proteins that are fundamental to biological structure and function. Historically, products arising from the linkage of repeating units by covalent chemical bonds have been the primary focus of polymer science; emerging important areas of the science now focus on non-covalent links. Polyisoprene of latex rubber and the polystyrene of styrofoam are examples of polymeric natural/biological and synthetic polymers, respectively. In biological contexts, essentially all biological macromolecules, i.e. proteins (polyamides), nucleic acids (polynucleotides), and polysaccharides are purely polymeric, composed in large part of polymeric components, for instance, isoprenylated/lipid-modified glycoproteins, where small lipidic molecule and oligosaccharide modifications occur on the polyamide backbone of the protein.

Note that the depolymerization process is a particular form of fragmentation process that can be combined with its opposite process, the coagulation process. So to have a clear and good insight about those processes, we have the following definitions and related concepts.

5.1.1 Pure fragmentation

Fragmentation processes can be observed in natural sciences and engineering. To provide just a few examples we mention the study of stellar fragments in astrophysics, rock fracture, degradation of large polymer chains, DNA fragmentation, evolution of phytoplankton aggregates, liquid droplet breakup or breakup of solid drugs in organisms. Though mathematical study of fragmentation processes can be traced back to papers by Melzak [104] (from the analytical point of view) and Filippov [63] (from the probabilistic one), it was not until the 1980s that a systematic investigation of them was undertaken, mainly by Ziff and his students, e.g. [141, 142], who provided explicit solutions to a large class of fragmentation equations of the form

$$\frac{\partial}{\partial t}u(t,x) = -a(x)u(t,x) + \int_x^\infty a(y)b(x|y)u(t,y)dy, \quad x \geq 0, t > 0, \tag{5.1.1}$$

with power law fragmentation rates $a(x) = x^\alpha, \alpha \in \mathbb{R}$ and where $b(x|y)$, the distribution of particle masses x spawned by the fragmentation of a particle of mass $y > x$, also was given by a power law

$$b(x|y) = (\nu + 2)\frac{x^\nu}{y^{\nu+1}}, \tag{5.1.2}$$

with $\nu \in (-2, 0]$ (see also [54, 140] for a more detailed discussion of this case). Here $u(t,x)$ is the density of particles having mass x at time t. Later a comprehensive probabilistic theory of fragmentation processes was developed by Bertoin and Haas, see e.g. [30, 31, 69, 70], while a development of functional-analytic methods and, in particular, of the semigroup theory, helped to put many earlier phenomenological results on a firm mathematical ground, see e.g. [23, 35]. Fragmentation processes are difficult to analyze as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it. That is why, though linear, they display non-linear features such as phase transition which, in this case, is called shattering and consists in the formation of a 'dust' of particles of zero size carrying, nevertheless, a non-zero mass. Quantitatively we can identify this process by disappearance of mass from the system even though it is conserved in each fragmentation event. Probabilistically, shattering is an example of an explosive, or dishonest Markov process, see e.g. [7, 107] and from this point of view it has been exhaustively analyzed in [30, 31, 63, 69, 70]. In natural sciences shattering was rediscovered in [140] where the loss of mass was noticed by analyzing explicit solutions of fragmentation equations with power-law fragmentation rates. In a general case shattering was explained analytically by linking it to the characterization of the generator of the dynamical system associated with the fragmentation process; these results were compared with the probabilistic approach. If u is a solution to (5.1.1), the total mass of the ensemble at a time t is given by the first moment of u; that is, $M(t) = \int_0^\infty xu(t,x)dx$. From the physical point of view the total mass of fragmenting particles cannot increase, thus fragmentation equations are usually investigated in the space

$$X_1 := L_1(\mathbb{R}_+, xdx) = \left\{u;\ \int_0^\infty |u(x)|xdx < +\infty\right\}. \qquad (5.1.3)$$

The reason for this is that the process in this space should be dissipative which typically results in simpler analysis. However, as we mentioned earlier, fragmentation events result in an increase of number of particles in the system, which is not tracked by the norm in X_1. Apart from an inherent interest in knowing how the number of particles evolves, there is also a practical angle to this question: fragmentation events are often coupled with, in some sense reverse to, coagulation processes which are most easily analyzed in the finite number of particles space:

$$X_0 := L_1(\mathbb{R}_+, dx) = \left\{u;\ \int_0^\infty |u(x)|dx < +\infty\right\}. \qquad (5.1.4)$$

Hence, analysis of the combined fragmentation-coagulation equation requires well posedness of the fragmentation equation in

$$X_{0,1} := L_1(\mathbb{R}_+, (1+x)dx) = \left\{u;\ \int_0^\infty |u(x)|(1+x)dx < +\infty\right\}. \qquad (5.1.5)$$

Nonlocal fragmentation models are investigated in detail in Chapter 6. A special emphasis is placed on the honesty of these models. We recover some fundamental properties from local fragmentation models.

5.1.2 Coagulation fragmentation equations (CFE)

Coagulation-fragmentation processes describe the evolution of systems in which particles react in either fusing together or breaking apart. The first pure coagulation equation was derived in the early part of the twentieth century by Smoluchowski [131, 132] who applied the theory of Brownian motion to the problem of the collision of hard, non-interacting spheres which are thermally agitated in a continuum. The problem was considered as a diffusion process and the approach resulted in a discrete model involving an infinite set of non-linear differential equations. In the late 1920s, Muller extended the results of

Smoluchowski by considering a continuous mass density function. As a result, this was probably the first instance in which the pure coagulation was considered as a continuous problem and modelled as an integro-differential equation. The fragmentation equation was introduced into the models of evolving systems in the 1950s. The coagulation-fragmentation equation was first derived by Melzak [104] in 1957. The equation was formulated in such a way as to ensure that mass was a conserved quantity. The equation had the form

$$
\begin{aligned}
\frac{\partial}{\partial t}u(t,x) = &- u(t,x)\int_0^x \frac{y}{x}\gamma(x,y)dy + \int_x^\infty \gamma(y,x)u(t,y)dy \\
&+ \frac{1}{2}\int_0^x k(x-y,y)u(t,x-y)u(t,y)dy, \\
&- u(t,x)\int_0^\infty k(x,y)u(t,y)dy,
\end{aligned}
\tag{5.1.6}
$$

where $u(t,x)$ represented the density of particles of mass x at time t. We recall that in the continuous version it is assumed that the number of particles is large enough to justify the use of a density function $u(t,x)$. The product $u(t,x)dx$ is then the average number of particles with mass in the interval $(x, x+dx)$ at time t. The fragmentation kernel, $\gamma(x,y)$, describes the rate at which particles of mass y are produced from the fragmentation of particles of size x and the coagulation kernel, $k(x,y)$, describes the rate at which particles of mass x coalesce with particles of mass y. The fragmentation kernel γ, introduced above, is often referred to as the *multiple* fragmentation kernel as this model allows for particles to split into many pieces at each fragmentation process. In his work, Melzak assumed that (i) $k(x,y)$ is a continuous, symmetric, non-negative, bounded function; (ii) $\gamma(x,y)$ is a continuous, non-negative, bounded function. Furthermore, solutions $u(t,x)$ to the coagulation-fragmentation equation were sought in the form

$$
u(t,x) = \sum_{n=0}^{\infty} a_n(x)t^n, \tag{5.1.7}
$$

for some sequence of functions $a_n, \quad n = 0, 1, ...$ Under these assumptions, global existence and uniqueness of continuous, non-negative, bounded solutions to (5.1.6) were established. Melzak also obtained results on the solution of the coagulation fragmentation equation for the case in which the kernels, γ and k, are time-dependent. Various results on the existence, uniqueness and asymptotic behavior of solutions have been established under appropriate hypothesis on the kernels and many distinct approaches have been used to obtain them. The case where the fragmentation and coagulation kernels are both constant has been analyzed via semigroup techniques by Aizenman and

Bak [5]. Asymptotic analysis of coagulation-fragmentation equations may be found in [62], [94] and [128]. Philippe Laurençot [94] investigated a model for the dynamics of a system of particles undergoing simultaneously coalescence and break-up. The equation describing his model was similar to (5.1.6). He showed existence of solutions to the corresponding evolution integral partial differential equation for product-type coagulation kernels with a weak fragmentation. Further information on the development of the coagulation-fragmentation equation may be found in the comprehensive review article by Drake [46]. Although fragmentation equations are often studied in a form involving a single multiple fragmentation kernel, it is also possible to write the fragmentation operator in terms of rate functions. We define the rate functions a and b, via

$$a(x) := \int_0^x \frac{y}{x}\gamma(x,y)dy \tag{5.1.8}$$

and

$$b(x|y) := \frac{\gamma(y,x)}{a(y)} \tag{5.1.9}$$

respectively where (5.1.8) describes the overall rate of break-up of an x-particle and (5.1.9) denotes the distribution of particles of size x formed during the break-up of larger particles of size y. This formulation coincides with the fragmentation problem derived by McGrady and Ziff [141, 142, 140] and will be used in the thesis. Note that $b(x|y) = 0$ for $y < x$ as it is not physically possible for a solid of size $x > y$ to be produced during the break-up of a y-sized solid.

5.1.3 Phytoplankton aggregates

Phytoplankton is a generic name for a great variety of micro-organisms (algae) that live in the ocean and in lakes. Phytoplankton populations are large contributors to the production in the ocean. They are, in particular, the main food available to the early larval stages of many fish species, including the anchovy. An important observation is that phytoplankton cells tend to form aggregates; that is, groups of cells living together. In phytoplankton dynamics, a system of particles called TEP (meaning Transparent Exopolymer Particles) play a major role. They are a by-product of the growth of phytoplankton and their stickiness causes that cells will remain together upon contact [38]. On the other hand, the low level of concentration of TEP results in fragmentation of the aggregate due to external causes, like currents or turbulence on one hand, and internal unspecified forces of biotic nature on the other. The distribution of aggregates can be studied at different levels. Individual-based models, which can be thought of as providing 'microscopic' properties, track the random motion and division of individual particles [121]. A 'macroscopic' description known to ecologists by advection-diffusion -reaction equations works with approximations of densities by empirical concentrations of particles [100] and is heavily used in simulations [3]. The model which we study in this thesis was considered by Arino and Rudnicki [9]. It can be looked at as lying somewhere in between, on a 'mesoscopic' scale, in that it describes the role played by the phytoplankton aggregates of cells which are treated as individual building blocks of the system. The aggregates are structured by size and the phytoplankton consists of aggregates of all possible sizes. The aggregate size can change due to splitting, death, growth or combining of aggregates into bigger ones. To include the effects of cell division, we incorporate the McKendrick-von Foerster renewal condition. The resulting model consists of a partial differential equation with two integral terms responsible for the fragmentation and coagulation processes, the McKendrick-von Foerster renewal boundary condition and the initial condition.

Note that the same justifications like those given in the previous chapter hold to explain the use of this innovative concept of derivative with an additional parameter. Recall Some of depolymerization processes are difficult to analyze [55, 141] and often involve evolution of two intertwined quantities: the number of particles and the distribution of mass among the particles in the ensemble [58, 86, 125]. Hence to contribute in the ongoing investigation to address the issue we consider the following kinetic differential equation.

5.1.4 The kinetic equation

Recall as already mentioned in Chapter 1 that the evolution of the sizes distribution occurring during polymer chain degradation is well known [56, 58, 142] to be described by the following integrodifferential equation

$$\frac{\partial}{\partial t}g(x,t) = -g(x,t)\int_0^x H(y,x-y)dy + 2\int_x^\infty g(y,t)H(x,y-x)dy, \qquad x \in \mathbb{R},\ t>0. \tag{5.1.10}$$

We have also mentioned that expressing the solution of equation (5.1.10) in its explicit form is very hard since fragmentation (or polymer chain degradation) processes, as explained in the previous section, are difficult to analyse as they involve evolution of two intertwined quantities: the distribution of mass among the particles in the ensemble and the number of particles in it. That is why, though linear, they display non-linear features such as “shattering” phenomena which they cannot fully explain [55, 58, 141]. Then, to explore the possibility of extending the analysis by considering the β–derivative defined in the previous section, we have considered the following integrodifferential equation:

$$
{}_0^A D_t^\beta g(x,t) = -g(x,t)\int_0^x H_\beta(y, x-y)dy + 2\int_x^\infty g(y,t) H_\beta(x, y-x)dy, \qquad x \in \mathbb{R},\ t > 0. \tag{5.1.11}
$$

subject to the initial condition

$$
g(x,0) = g_0(x), \qquad x > 0 \tag{5.1.12}
$$

where $g(x,t)$ represents the density of x-groups (i.e. groups of size x) at time t and $H_\beta(x,y)$ gives the average fragmentation rate, that is, the average number at which clusters of size $x+y$ undergo splitting to form an x-group and a y-group.

Recall that there is a growing problem about the choice of the type of fractional derivative to use among the large number of its existing versions. We already mentioned the incapacity of most of them to explicitly provide the variation of the functions. Moreover, many models using fractional derivatives are not easy to handle analytically. The β–derivative, we hope in this particular case, will allow us to palliate some insufficiencies caused by other fractional derivatives.

5.2 Solutions to the model

Note that these above models (5.1.10) and (5.1.11) are well applicable in many branches of natural sciences, including physics, chemistry, engineering, biology, ecology, just to name a few, and in numerous domains of applied sciences, such as the rock fractures and break of droplets. Various types of fragmentation equations have been comprehensively analyzed in numerous works (see, e.g., [56, 133, 141]). In the domain of polymer science, the fragmentation dynamics has also been of considerable interest, since degradation of bonds or depolymerisation results in fragmentation, see [32, 101, 142]. In [101], the authors used statistical arguments to find and analyze the size distribution of the model. The authors in [32] analysed the model in combination with the inverse process, that is, the coagulation process, and provided a similar result for the size distribution. However, the classical fragmentation model (5.1.10) has been proved to be unable to fully describe some bizarre phenomena observed in such a degradation process, like for instance shattering as described above and also in [55, 101, 142, 141]. Recall that shattering is a phenomenon seen as an explosive or dishonest Markov process, see e.g. [7, 107] and has been associated with an infinite cascade of breakup events creating a 'dust' of particles of zero size which, however, carry non-zero mass. Hence, to have explicit solutions to the model, we consider the case where the breakup rate depends on the size of the chain breaking and takes the form

$$H_\beta(x,y) = (x+y)^\nu, \quad \nu \in \mathbb{R}. \tag{5.2.1}$$

Substituting in equation (5.1.11) yields

$$D_t^\beta(g(x,t)) = -x^{\nu+1}g(x,t) + 2\int_x^\infty y^\nu g(y,t)dy, \qquad 0 \le \beta \le 1. \tag{5.2.2}$$

Taking the modified Sumudu transform S_β (see the Appendix below) of both sides of equation (5.2.2) yields

$$S_\beta\left(D_t^\beta g(x,t), r\right) = -x^{\nu+1}G_s^\beta(x,r) + 2\int_x^\infty y^\nu G_s^\beta(y,r)dy,$$

where $G_s^\beta(x,r)$ represents the modified Sumudu transform $S_\beta(g(x,t),r)$ of $g(x,t)$. Using the relation (3.1.6, we obtain

$$r^{-2}(G_s^\beta(x,r) - g_0(x)) = -x^{\nu+1}G_s^\beta(x,r) + 2\int_x^\infty y^\nu G_s^\beta(y,r)dy,$$

rearranged to have

$$\left(1 + x^{\nu+1}r^2\right)G_s^\beta(x,r) - 2r^2\int_x^\infty y^\nu G_s^\beta(y,r)dy = g_0(x). \tag{5.2.3}$$

Next, it is important to mention that considering the differential equation (5.2.2), it is implicitly required that the function $\xi \longrightarrow g(\xi,t)$ is integrable, in the sense of Lebesgue, on any interval $[\epsilon,\infty)$ for $\epsilon > 0$ and almost every $\xi > 0$. Obviously, the same assertion applies to the functions $\xi \longrightarrow g_0(\xi)$ and $\xi \longrightarrow G_s^\beta(\xi,r), \quad 0 \le \beta \le 1$.

This allows us to put

$$Z(x,r) = -2r^2 \int_x^{\infty} y^{\nu} G_s^{\beta}(y,r)dy, \tag{5.2.4}$$

knowing that the integrand will be integrable over any interval $[\epsilon, \infty)$ and the integral will be absolutely continuous at each $x > 0$. The substitution of $Z(x,r)$ into (5.2.3) yields the partial differential equation

$$\left(\frac{1 + x^{\nu+1}r^2}{1 + r^2 x^{\nu}}\right) \partial_x Z(x,r) + Z(x,r) = g_0(x). \tag{5.2.5}$$

Choosing the constant in the general solution so as to have solutions converging to zero at ∞, we obtain its solution given as

$$Z(x,r) = 2r^2 e^{-\sigma_{r,\nu}(x)} \int_x^{\infty} \frac{\xi^{\nu} g_0(\xi)}{1 + r^2 \xi^{\nu+1}} e^{\sigma_{r,\nu}(\xi)} d\xi,$$

where

$$\sigma_{r,\nu}(x) = \int_0^x \frac{2r^2 \xi^{\nu}}{1 + r^2 \xi^{\nu+1}} d\xi = \ln \left(1 + r^2 x^{\nu+1}\right)^{\frac{2}{\nu+1}}. \tag{5.2.6}$$

Thus, substituting $Z(x,r)$ into (5.2.4) yields the solution of (5.2.3) given as

$$\begin{aligned} G_s^{\beta}(x,r) &= \frac{-1}{x^{\nu}} \left(\frac{2r^2 x^{\nu}}{1 + r^2 x^{\nu+1}} e^{-\sigma_{r,\nu}(x)}\right) \int_{\infty}^{x} \frac{\xi^{\nu} g_0(\xi)}{1 + r^2 \xi^{\nu+1}} e^{\sigma_{r,\nu}(\xi)} d\xi + \frac{g_0(x)}{1 + r^2 x^{\nu+1}} \\ &= \frac{g_0(x)}{1 + r^2 x^{\nu+1}} - \frac{2r^2}{\left(1 + r^2 x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}} \int_{\infty}^{x} \xi^{\nu} \left(1 + r^2 \xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1} g_0(\xi) d\xi. \end{aligned} \tag{5.2.7}$$

Applying the inverse of the modified Sumudu transform, which coincides with the inverse Sumudu transform, we are finally lead to the solution of the model (5.2.2), given by

$$
\begin{aligned}
g(x,t) &= S_\beta^{-1}(G_s^\beta(x,r),\, t) \\
&= g_0(x) S_\beta^{-1}\left(\frac{1}{1+r^2x^{\nu+1}},\, t\right) - 2\int_\infty^x \xi^\nu g_0(\xi) S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},\, t\right) d\xi . \\
&= g_0(x)\cos(t\sqrt{x^{\nu+1}}) - 2\int_\infty^x \xi^\nu g_0(\xi) S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},\, t\right) d\xi
\end{aligned}
\tag{5.2.8}
$$

Remark 5.2.1. *The expression $g(x,t)$ in (5.2.8) is well-defined only if the integral*

$$
\int_\infty^x \xi^\nu g_0(\xi) S_\beta^{-1}\left(\frac{r^2\left(1+r^2\xi^{\nu+1}\right)^{\frac{2}{\nu+1}-1}}{\left(1+r^2x^{\nu+1}\right)^{\frac{2}{\nu+1}+1}},\, t\right) d\xi
$$

converges.

We are now capable of taking some specific values of ν to see the exact expression of the solution.

- For $\nu = 1$, expression (5.2.8) becomes

$$
\begin{aligned}
g(x,t) &= g_0(x) S_\beta^{-1}\left(\frac{1}{1+r^2x^2},\, t\right) - 2\int_\infty^x \xi g_0(\xi) S_\beta^{-1}\left(\frac{r^2}{\left(1+r^2x^2\right)^2},\, t\right) d\xi \\
&= g_0(x)\cos(xt) - \frac{t\sin(xt)}{x}\int_\infty^x \xi g_0(\xi) d\xi .
\end{aligned}
\tag{5.2.9}
$$

- For $\nu = -3$, expression (5.2.8) becomes

$$
\begin{aligned}
g(x,t) &= g_0(x) S_\beta^{-1}\left(\frac{1}{1+r^2x^{-2}},\, t\right) - 2\int_\infty^x \xi g_0(\xi) S_\beta^{-1}\left(r^2\left(1+r^2\xi^{-2}\right)^{-2},\, t\right) d\xi \\
&= g_0(x)\cos\frac{t}{x} - 2\int_\infty^x \xi g_0(\xi)\frac{\xi t\sin\frac{t}{\xi}}{2} d\xi \\
&= g_0(x)\cos\frac{t}{x} - \int_\infty^x t\xi^2 g_0(\xi)\sin\frac{t}{\xi} d\xi .
\end{aligned}
\tag{5.2.10}
$$

5.2.1 Numerical Approximations

Explicit forms of the solutions in some particular cases showed that the dynamics of this evolution exhibits complex periodic properties due to the presence of cosine and sine functions, as shown by numerical approximations, in Figure 5.1 to Figure 5.6, plotted for a positive value ($\nu = 1$) and a negative value ($\nu = -3$) of ν. Figure 5.1 to Figure 5.3 represent the solution for $\nu = 1$ with initial condition $g_0(x) = 1/x^3$: Figure 5.1 is the $2-$D surface plot while Figure 5.2 and Figure 5.6 are respectively its cross section and longitudinal section drawn for some specific values of the size x and time t. A similar reasoning applies to Figure 5.4 to Figure 5.6, but this time with $\nu = -3$. This infers existence of complex and simple harmonic poles in the dynamics of polymer chain degradation whose effects are characterized by these functions or a combination of them. This work improved the preceding one with the inclusion of a more general expression of the breakup rate derivative and $\beta-$derivative. This work might be a breakthrough that may lead to a better understanding of bizarre phenomena happening in some dynamics such as the phenomenon of shattering.

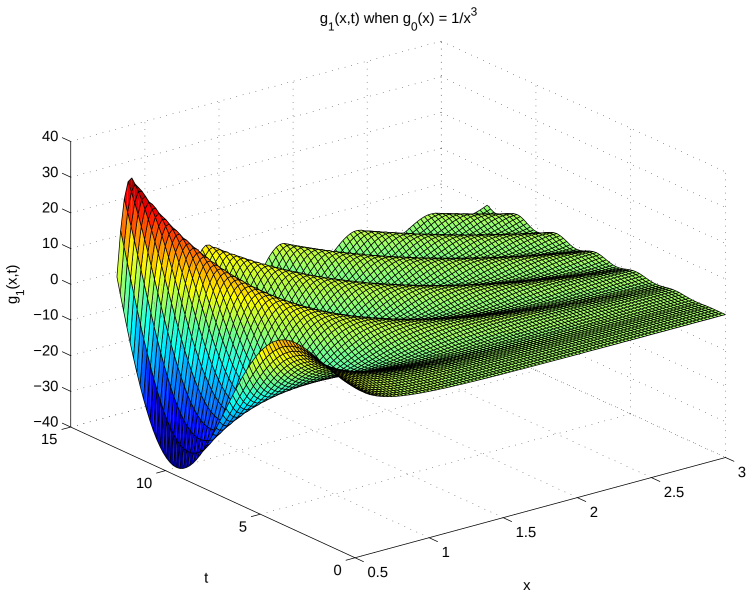

Figure 5.1: $g(x,t)$ when $\nu = 1$ and $g_0(x) = 1/x^3$.

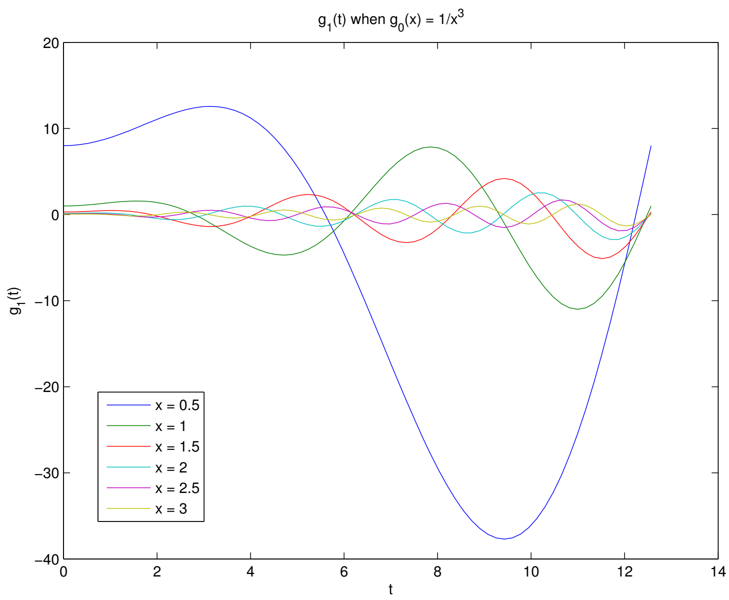

Figure 5.2: $g(x,t)$ as a function of t when $\nu = 1$ and $g_0(x) = 1/x^3$, for a few values of x.

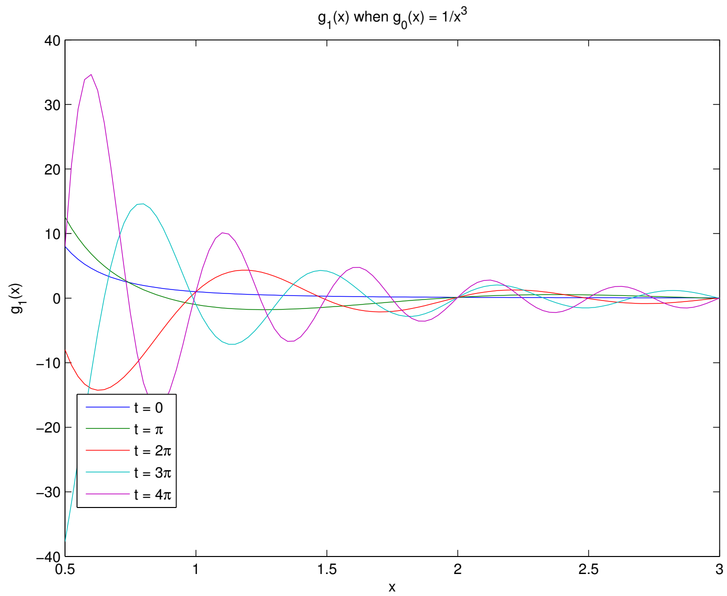

Figure 5.3: $g(x,t)$ as a function of x when $\nu = 1$ and $g_0(x) = 1/x^3$, for a few values of $t:\ 0, \pi, 2\pi, 3\pi, 4\pi$.

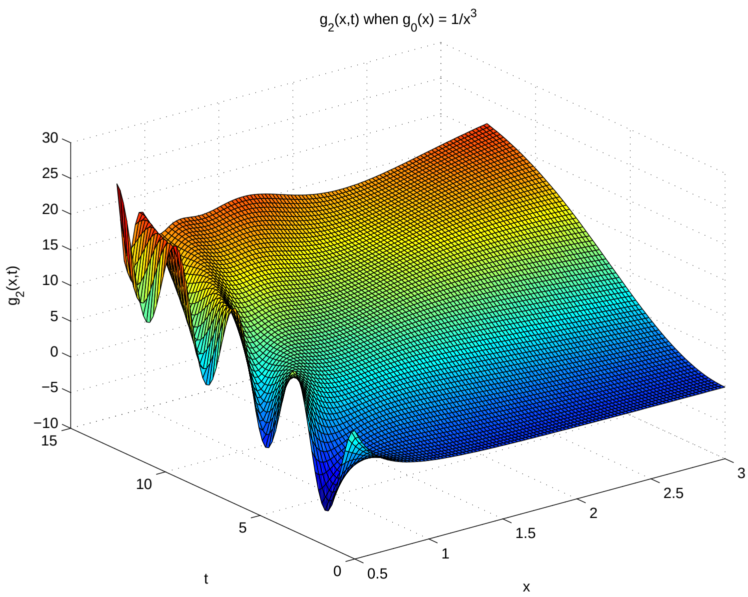

Figure 5.4: $g(x,t)$ when $\nu = -3$ and $g_0(x) = 1/x^3$.

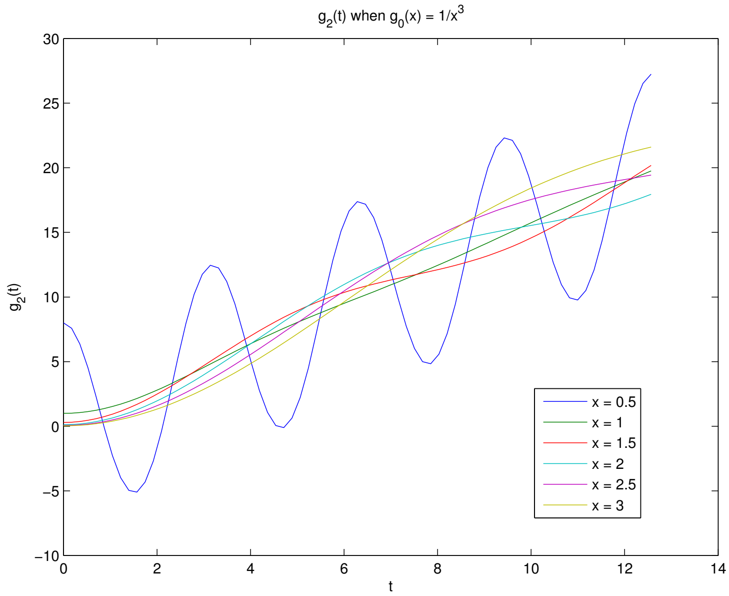

Figure 5.5: $g(x,t)$ as a function of t when $\nu = -3$ and $g_0(x) = 1/x^3$, for a few values of x.

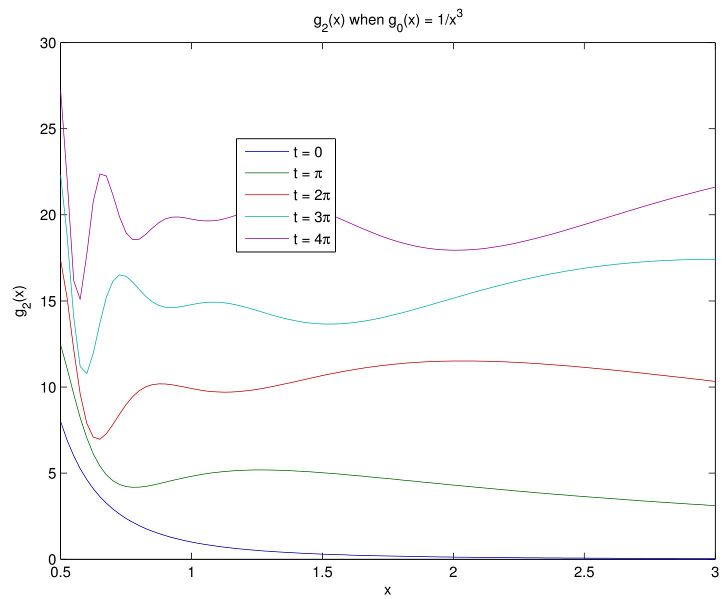

Figure 5.6: $g(x,t)$ as a function of x when $\nu = -3$ and $g_0(x) = 1/x^3$, for a few values of $t:\ 0, \pi, 2\pi, 3\pi, 4\pi$.

Chapter 6

Control parameter & solutions to generalized evolution equations of stationarity, relaxation and diffusion

6.1 Introduction

A growing importance has been put by mathematicians as well as theoretical and applied physicists on the Mittag-Leffler function and its generalized version respectively defined by the power series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \tag{6.1.1}$$

and

$$E_{\alpha,\theta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \theta)}, \tag{6.1.2}$$

for the complex argument $z \in \mathbb{C}$ and the parameters $\alpha, \theta \in \mathbb{C}$ with $Re\, \alpha > 0,\ Re\, \theta > 0$, and where Γ is the Gamma-function,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}.$$

One of the main reasons for the recent surge of interest in these functions is their implication when solving stationary fractional differential systems like

$$D_t^\gamma g(t) = K g(t),$$

where D_t^γ is any of the existing most popular time fractional derivatives of order γ (in the sense of Caputo or Riemann–Liouville and so on).

Diffusion and relaxation models, using derivatives of the fractional order have been intensively analyzed in numerous works [20, 67, 115, 92, 75, 124, 135, 105]. Most of the results always lead to the use of higher transcendental functions like Mittag-Leffler functions to describe the evolution of the natural phenomenon under investigation. For instance, in [20], the authors showed that fractional solution of the diffusion equation obtained from a stochastic Ising model, where they used the Adomian decomposition method to solve the fractional diffusion equation, gives a non-Debye type behavior which can also be represented by the Mittag–Leffler decay function. Furthermore, the Caputo derivative, developed and proposed in 1967 by Michele Caputo [36] has intensively been useful for solving differential equations and models that involve a fractional time derivative [67, 115]. The obtained solution is always governed by a (generalized) Mittag-Leffler function or combination of Mittag-Leffler functions.

Indeed, Mittag-Leffler functions govern the evolution of fractional order integral equations or fractional order differential equations, and especially in the investigations of the fractional generalization of the diffusion and kinetic equations, random walks, Lévy flights, superdiffusive transport and in the study of complex systems. The ordinary and generalized Mittag-Leffler functions interpolate between a purely exponential law and power-law-like behavior of phenomena governed by ordinary kinetic equations and their fractional counterparts, see [88, 91, 92, 75, 124]. Moreover, the Mittag-Leffler relaxation function $E_\alpha(-x)$, (with $\alpha \in [0,1]$ and x a nonnegative real variable, usually standing for the time), is well known to arise in the description of complex relaxation phenomena occurring in complex physical and biophysical systems [135, 105, 127, 28].

Hence, during the last decades in fractional calculus, Mittag-Leffler functions have served as powerful tools to analyze anomalous dynamics and strange kinetics. However, despite this growing importance, their analyticity, their behaviour as holomorphic functions and their dependence upon the parameters are still largely unexplored. A possible valid explanation comes from the fact that there are no many numerical algorithms available to compute the function accurately for all the involved argument and parameters z, α, θ. This causes a huge problem and the only attempt to palliate the difficulty has been to explore the behaviour of $E_{\alpha,\theta}(z)$ for large sets of the involved argument and parameters z, α and θ. The obtained results are nothing but just approximations of the reals ones.

Then, despite the different versions of fractional derivatives, it seems obvious and we can make it as a conjecture that: Mittag-Leffler functions arise naturally in the solution of differential equations using these derivatives of fractional order. That is why the time derivative with a new parameter proposed and explored in this thesis provides a new tool to investigate anomalous dynamics, strange kinetics, diffusion and relaxation models. The obtained results provide us with functions $\mathcal{E}_\beta$ more accurate and easier to handle. To continue we need to recall some important theorems that will be important in this chapter.

Theorem 6.1.1. *Assuming that $u : [a, \infty) \to \mathbb{R}$, be a functions such that, f is differentiable and also $\alpha-$differentiable. Let v be a function defined in the range of u and also differentiable, then we have the following rule*

$$ {}_0^A D_t^\beta \left(vou(x)\right) = \left(t + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} u'(t) v'\left(u(t)\right) \tag{6.1.3}$$

Definition 6.1.2. *Let $u : [a, \infty) \to \mathbb{R}$ is given function, then we propose that the $\beta-$ integral of u is*

$$ {}_a^A I_t^\beta \left(u(t)\right) = \int_a^t \left(\xi + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} u(\xi) d\xi \tag{6.1.4}$$

The above operator is the inverse operator of the proposed $\beta-$derivative. We shall present to underpin this statement by the following theorem.

Theorem 6.1.3. ${}_a^A D_t^\beta \left[{}_a^A I_t^\alpha u(t)\right] = u(t)$ *for all $t \geq 0$ with u a given continuous and differentiable function.*

Theorem 6.1.4.

$$ {}_a^A I_t^\beta \left[{}_a^A D_t^\beta u(t)\right] = u(t) - u(a) \tag{6.1.5}$$

for all $t \geq 0$ with u a given continuous and differentiable function.

6.2 Generalized stationarity with a new Parameter

6.2.1 Generalized time evolution

Mathematical models are usually formulated as initial value problems for dynamical evolution equations written as

$$ \frac{d}{dt} p(t) = A p(t) \tag{6.2.1}$$

where t is the time taken from $\mathbb{R}_+$ and A is an operator in a Banach space. The aim here is to find the state $p(t)$ of the model at a time $t > t_0$ depending on the initial state $p(t_0)$ at the initial time t_0. Many scientists around the world have tried to extend classical models to models with fractional derivative (see [27, 59, 60, 54] and analyse them with various methods in order to provide a broader view on the natural phenomena under investigation. For example, the authors [27] successfully extended the advection-dispersion equation (to the fractional one) by using various techniques including the well-known action of Fourier transform on integer derivatives to rational order. However, generalizing the model (6.2.1), by substituting the time differentiation $\frac{d}{dt}$ with a derivative $\frac{d^\sigma}{dt^\sigma}$ of fractional order $\sigma > 0$ to obtain the following model

$$\frac{d^\sigma}{dt^\sigma}p(t) = Ap(t) \tag{6.2.2}$$

has raised a number of fundamental questions [29, 75, 18, 136, 122, 110] and still today, is dividing the scientific community. The term $\frac{d}{dt}$ is seen as the representation of the rate of accumulation or loss in the system and mainly reflects the basic principle of locality, together with the time translation stationarity. Moreover, we know from the classical calculus that

$$\frac{d}{dx}g(x) = \lim_{t \longrightarrow 0} \frac{g(x) - g(x-t)}{t} = -\lim_{t \longrightarrow 0} \frac{\mathfrak{g}(t)g(x) - g(x)}{t}. \tag{6.2.3}$$

This means that $-\frac{d}{dt}$ defines the infinitesimal generator of the time translations given by

$$\mathfrak{g}(t)g(x) = g(x-t).$$

In a similar manner, fractional derivative D_t^γ of order $0 < \gamma < 1$ can be defined [19, 53, 76, 122, 139] as

$$D_\tau^\gamma(g(\tau)) = -\lim_{t \longrightarrow 0} \frac{\mathfrak{g}_\gamma(t)g(\tau) - g(\tau)}{t}, \tag{6.2.4}$$

where $\mathfrak{g}_\gamma(t)$ represents the fractional time evolution and is considered as the universal attractor semigroups of coarse grained macroscopic time evolutions. For instance, it is shown that [19, 59, 122],

$$D_\tau^\gamma(g(\tau)) = -\frac{1}{\Gamma(-\gamma)} \int_0^\infty \frac{g(\tau - r) - g(\tau)}{r^{\gamma+1}} dr,\ 0 < \gamma < 1, \tag{6.2.5}$$

which is the fractional derivative of $g(t)$ in the sense of Marchaud.

Definition 6.2.1. *A time evolution is a pair* $(\{\mathfrak{T}_\beta(t), 0 \leq t < \infty\}, (X_\beta, \|\cdot\|))$ *with* $\mathfrak{T}_\beta(t) = \mathfrak{T}(t\beta)$ *defining a semigroup of operators* $\{\mathfrak{T}_\beta(t), 0 \leq t < \infty\}$ *mapping the Banach space* $(X_\beta(\mathbb{R}), \|\cdot\|)$ *of functions* $g_\beta(x) = g(x\beta)$ *on* $\mathbb{R}$ *to itself.*

In the expression $\mathfrak{T}_\beta(t)$, the variable $t > 0$ represents a time duration and the variable $x \in \mathbb{R}$ in the expression $g_\beta(x)$ stands for a time instant. The index $\beta > 0$ indicates the units of time. The elements $g_\beta(x) = g(x\beta)$, as functions of the time coordinates $x \in \mathbb{R}$, represent observable states of a given physical system.

6.2.2 Basic settings for time evolutions

- Semigroup: The following conditions define the semigroup:
  $\mathfrak{T}_\beta(t_1)\mathfrak{T}_\beta(t_2)g_\beta(t_0) = \mathfrak{T}_\beta(t_1+t_2)g_\beta(t_0)$
  $\mathfrak{T}_\beta(0)g_\beta(t_0) = g_\beta(t_0)$,

with t_1, $t_2 > 0$, $t_0 \in \mathbb{R}$ and $g_\beta \in X_\beta$.

- Homogeneity of the time argument t: This requires the commutativity with translations [76]
  $\mathfrak{T}_\beta(t_1)\mathcal{T}_\beta(t_2)g_\beta(t_0) = \mathcal{T}_\beta(t_2)\mathfrak{T}_\beta(t_1)g_\beta(t_0)$,

with $t_2 > 0$, t_1, $t_0 \in \mathbb{R}$. Hence, this allows to shift the origin of time and it reflects the basic symmetry of time translation invariance.

- Continuity: We assume that the time evolution is strongly continuous in t such that
  $\lim_{t\longrightarrow 0} \|\mathfrak{T}_\beta(t)g_\beta - g_\beta\| = 0$

for all $g_\beta \in X_\beta$.

- Causality: Operator of the time evolution should be causal so that the function $g_\beta(t_0) = (\mathfrak{T}_\beta(t)f_\beta)(t_0)$ only depends on the values of $f_\beta(x)$ for $x < t_0$.
- Coarse Graining: The time evolution operator $\mathfrak{T}_\beta(t)$ should be establishable using the procedure of a coarse graining. the main idea here is to combine a time average $\frac{1}{t}\int_{x-t}^{x} f_\beta(\xi)d\xi$ when t, $x \longrightarrow \infty$ with a rescaling of x and t.

6.2.3 Beta-stationarity

In this section, we study different types of stationary states arising in models with β time derivative introduced above. The following definition is significant:

Definition 6.2.2. *An observable state $g_\beta(x)$ is said to be strictly stationary or strictly invariant under the time evolution $\mathfrak{T}_\beta(t)$ if the condition*

$$\mathfrak{T}_\beta(t)g_\beta(x) = g_\beta(x) \tag{6.2.6}$$

holds for all $t \geq 0$ and $x \in \mathbb{R}$

To provide more details about the significance of the new type of stationarity, we consider variants of (6.2.6) in the expression of infinitesimal forms of stationarity using the Beta-time derivative and where the generator $\mathcal{G}_\beta$ with $\beta \in (0,1]$ is given by the $\beta-$derivative

${}_0^A D_t^\beta$.

1. The first model to consider is given as

$$\begin{cases} {}_0^A D_t^\beta g(t) = 0, & 0 < \beta \leq 1 \\ g(0) = g_0, \end{cases} \tag{6.2.7}$$

Definition 6.2.3. *Let g be a function defined in $(0, \infty)$, then, we defined the modified-Sumudu transform of g as*

$$S_\beta(g(t), u) = \int_0^\infty \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta - \lceil \beta \rceil} \frac{1}{u} e^{-\frac{t}{u}} g(t) dt, \tag{6.2.8}$$

where $\lceil \beta \rceil$ is the smallest integer greater or equal to β. Since $\beta \in (0, 1]$ in this article then, $\beta - \lceil \beta \rceil = \beta - 1$.

Recall as shown in the previous section the **important property of modified-Sumudu transform** given as follows: If $S(g(t), u)$ is the well known Sumudu transform of g defined in [134] as

$$S(g(t), u) = \int_0^\infty \frac{1}{u} \exp[-\frac{t}{u}] g(t) dt,$$

then, we have the following relation:

$$S_\beta({}_0^A D_t^\beta g^{n-1}(t), u) = \frac{1}{u^n} S(g(t), u) - \sum_{k=0}^{n-1} \frac{1}{u^{n-k}} g^{(k)}(0) \tag{6.2.9}$$

Making use of the relation (6.2.9), the modified-Sumudu transform S_β of the first equation in (6.2.7) yields

$$u^{-2} F_s^\beta(u) - u^{-2} g(0) - u^{-1} g'(0) = 0,$$

where $F_s^\beta(u) = S_\beta(g(t), u)$. Then,
$F_s^\beta(u) = g(0)$ since ${}_0^A D_t^\beta g$ coincides with g' for $\beta = 1$. Taking the inverse modified-Sumudu transform S_β^{-1}, gives the solution

$$g(t) = g_0 S_\beta^{-1}(1) = g_0 = \text{constant} \tag{6.2.10}$$

Hence, we recover the expected result, the fact that this type of stationary states with the $\beta-$derivative correspond to the ones for which the function is constant. Note that there are some new types of stationary states for which a fractional integral rather than the function itself is constant. Since there are many definitions of fractional derivatives, and therefore many other types of stationarity, it is important to know how to make the right choice about which type of fractional derivative will be adequate for describing the generalized phenomenon under investigation.

2. A second type of stationarity using the Beta time derivative is given by the system

$$\begin{cases} {}_0^A D_t^\beta g(t) = K \\ g(0) = g_0 \end{cases} \tag{6.2.11}$$

Taking the modified-Sumudu transform S_β of the first equation in (6.2.11) yields

$$u^{-2}F_s^\beta(u) - u^{-2}g(0) - u^{-1}g'(0) = KS_\beta(1,u),$$

where $F_s^\beta(u) = S_\beta(g(t),u)$. Then,

$$F_s^\beta(u) = \frac{KS_\beta(1,u) + u^{-1}g'(0) - u^{-2}g(0)}{u^{-2}} \tag{6.2.12}$$

Taking the inverse modified-Sumudu transform S_β^{-1}, gives the solution

$$\begin{aligned} g(t) &= S_\beta^{-1}\left[\frac{KS_\beta(1,u) + u^{-1}g'(0) - u^{-2}g(0)}{u^{-2}}\right] \\ &= \frac{K}{\beta}\left[\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \left(\frac{1}{\Gamma(\beta)}\right)^\beta\right] + g_0. \end{aligned} \tag{6.2.13}$$

We recover the solution (6.2.10): $g(t) = g_0$ for $K = 0$.

A graphical representation for the solution (6.2.13) of the model (6.2.11) is given in Fig. 6.1 for $\beta = 0.09,\ 0.20,\ 0.90$ and for $g_0 = 1$ and $K = 2$. The stationarity of the model appears sooner as β decreases as shown in Fig. 6.1 ((a) and (b)) and the stationarity is maintained even when the time interval is extended as it can be seen in Fig. 6.1 ((c) and (d)). This shows how the parameter β can be used to control the stationarity of a model of type (6.2.11).

3. **Time relaxation using derivative with a new parameter:** For this type of stationarity, we consider the β relaxation Cauchy problem:

$$\begin{cases} {}_0^A D_t^\beta g(t) = -Kg(t) \\ g(0) = g_0, \end{cases} \tag{6.2.14}$$

where K is a relaxation constant. To solve this system, we make use of the simpler

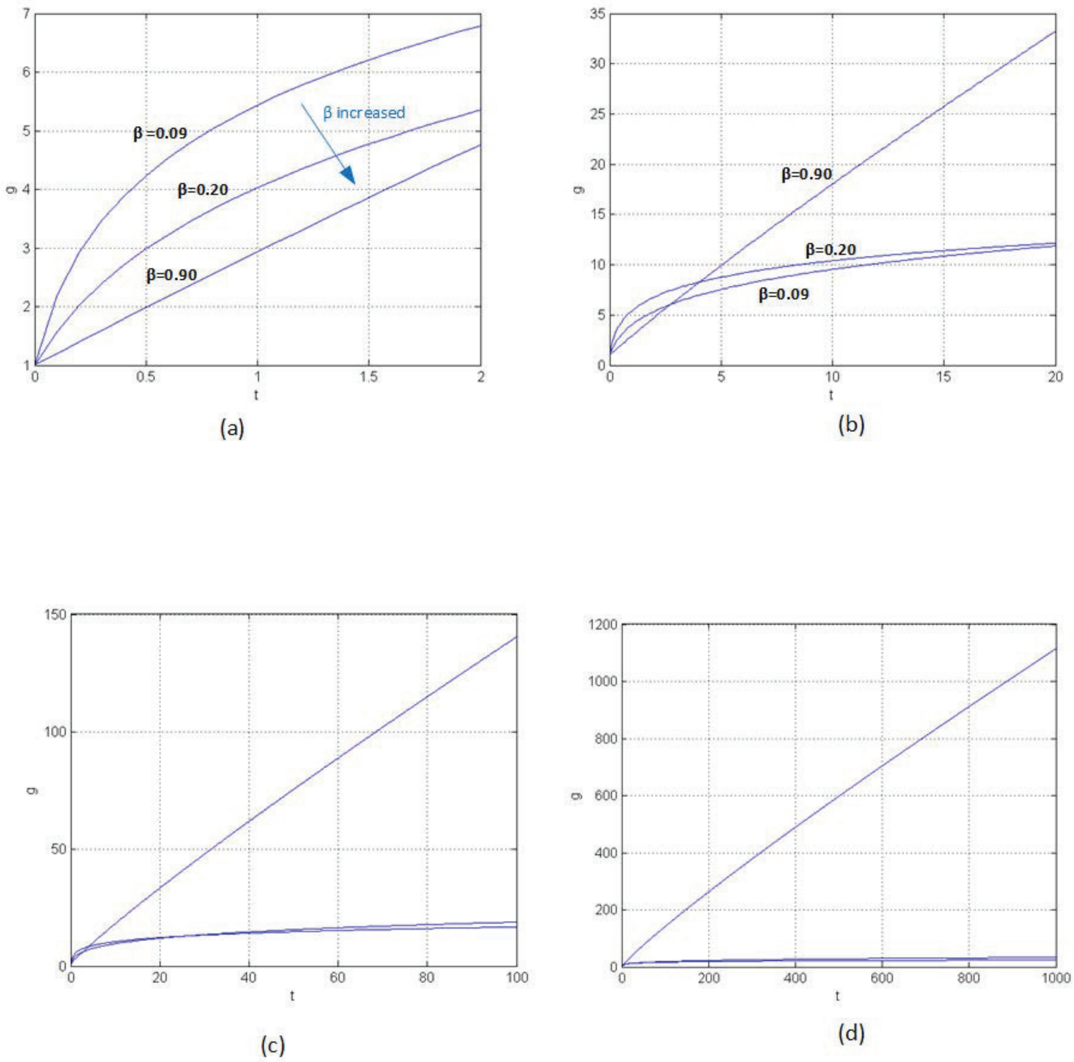

Figure 6.1: A representation of the solution (6.2.13) for the stationarity model (6.2.11) with $g_0 = 1$ and $K = 2$. The model becomes more stationary as β decreases ((a) and (b)) and the stationarity is maintained as time goes on ((c) and (d)).

model

$$\begin{cases} {}_0^A D_t^{\beta} g(t) = -K g(t) \\ g(0) = 1 \end{cases} \tag{6.2.15}$$

The same way as above, we use S_β and its inverse to find out that this eigenvalue problem has the following exponential expression

$$\mathcal{E}_\beta(t) = \exp\left[-K\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right] \tag{6.2.16}$$

as its unique solution. This leads to the solution of the system (6.2.14) is given as

$$g(t) = g_0 \exp\left[-K\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right] = g_0\,\mathcal{E}_\beta(t), \tag{6.2.17}$$

For $K = 0$ we recover the solution (6.2.10) and for $\beta = 1$, we have the well known classical solution

$$g(t) = g_0 e^{-Kt}.$$

A graphical representation for the solution (6.2.17) of the relaxation model (6.2.14) is given in Fig. 6.2 for $\beta = 0.09,\ 0.20,\ 0.90$ and for $g_0 = 1$ and $K = 2$. The stationarity of the model appears to be quicker as β decreases and the stationarity is maintained even when the time interval is extended (Fig. 6.2 ((a) and (b))). This shows how the parameter β can be used to control the stationarity of a relaxation model of type (6.2.14).

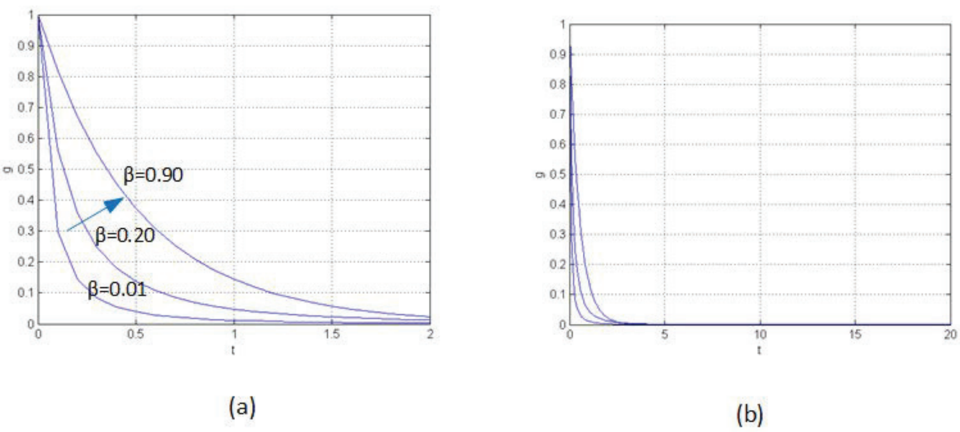

Figure 6.2: A representation of the solution (6.2.17) to the relaxation model (6.2.14) for $g_0 = 1$ and relaxation constant $K = 2$. Again, the stationarity of the model appears sooner with smaller β and is maintained with the time.

4. **Diffusion using derivative with additional fractional parameter:**

$$\begin{cases} {}_0^A D_t^\beta g(t,x) = \Delta g(t,x), \;\; 0<\beta<1, \;\; t>0, \;\; x>0 \\ g(0,x) = f(x), \;\; x>0 \end{cases} \tag{6.2.18}$$

To show existence result for this model, we use the separation of variables technique and set $g(t,x) = T(t)X(x)$. Substitution in (6.2.18) gives

$$X(x){}_0^A D_t^\beta T(t) = T(t)\Delta X(x)$$

or

$$\frac{{}_0^A D_t^\beta T(t)}{T(t)} = \frac{\Delta X(x)}{X(x)}$$

We put $-\lambda = \frac{{}_0^A D_t^\beta T(t)}{T(t)} = \frac{\Delta X(x)}{X(x)}$ to get the eigenvalue system

$$\Delta X(x) = -\lambda X(x), \;\; x>0, \tag{6.2.19}$$

$${}_0^A D_t^\beta T(t) = -\lambda T(t), \;\; t>0. \tag{6.2.20}$$

To solve the eigenvalue system (6.2.19), we use an infinite sequence of pairs $\{\alpha_n, \delta_n\}_{n\in\mathbb{N}}$ with $\{\alpha_n\}$ an increasing sequence such that $\alpha_n \to \infty$ and $\{\delta_n\}$ a family of functions that form a complete orthogonal set in $L^2((x_0,\infty))$. Exploiting α_n defined from (6.2.19), we can find a solution of the eigenvalue problem for the $\beta-$derivative (6.2.20) by putting $\lambda = \alpha_n$, (see [103]). Making use of (6.2.16), the expression:

$$\mathcal{E}_\beta(t) = Exp\left[-\mu\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right] \tag{6.2.21}$$

is the unique solution of the eigenvalue problem

$$\begin{aligned} &{}_0^A D_t^\beta T(t) = -\mu T(t), \;\; t>0 \\ &T(0) = 1. \end{aligned} \tag{6.2.22}$$

Therefore, the solution to (6.2.20) is given as

$$T(t) = \tilde{f}(n) Exp\left[-\lambda\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right]$$

where $\tilde{f}(n)$ is chosen to satisfy the initial condition f. This leads us to a formal solution of the β Cauchy problem (6.2.18) given by

$$g(t,x) = \sum_{n=1}^{\infty} \tilde{f}(n)\mathcal{E}_{\beta}(t)\delta_n(x). \tag{6.2.23}$$

Chapter 7

Conclusion and open problem

The whole research, investigations and analysis in this thesis have been about modeling in Applied Mathematics with new parameters and applying the results to some real life problems that are currently impacting our daily lives.

Indeed, we started by introducing a recent and newly developed concept of differentiation with an additional parameter. That additional parameter was considered to be fractional and the related operator was proven to satisfy the common rules of differentiation.

Therefore, we were able to intensively analyze an Ebola epidemic model with non-linear transmission and have shown that this model, which is itself relatively new in the literature, is well-defined, well-posed. In addition to provide conditions for boundedness and dissipativity of the trajectories for the Ebola model, we also studied existence and stability of equilibrium points to show that they are dependent on the non-linear incidence included in the resulting expression of the basic reproduction $\mathcal{R}_0$. One of the main results here is reflected by conditions for existence and stability of a unique endemic equilibrium point for the Ebola model. Numerical simulations performed for some particular expressions of the non-linear transmission, with coefficients $\kappa = 0.01$, $\kappa = 1$ and power $p = 2$, agree with the obtained results and satisfy the traditional threshold behavior. The work performed in this thesis is pertinent since it generalized the preceding ones with the inclusion of a general expression of the incidence together with a new derivative that extends the conventional one. This is useful and might happen to be capital in the ongoing fight and future prevention again the Ebola virus that has recently shaken the whole world and killed dozens of people in West-Africa.

We have also explored the possibility of using new and alternative methods to generalize evolution equation modeling the polymer chain degradation. In the process, a modified version of the Sumudu transform has been exploited to perform analysis of the system endowed the $\beta-$derivative and where the breakup rate was dependent on the size of the chain breaking up. Explicit forms of the solutions in some particular cases showed that the dynamics of this evolution exhibits complex periodic properties due to the presence of cosine and sine functions, as was shown by numerical approximations, in Figure 5.1 to Figure 5.6, and where a succinct discussion has been done. This infers existence of complex and simple harmonic poles in the dynamics of polymer chain degradation whose effects are characterized by these functions or a combination of them. This work, once more, improves the preceding ones with the inclusion of a more general expression of the breakup rate derivative and $\beta-$derivative. This work might be a breakthrough that may lead to a better understanding of bizarre phenomena happening in some dynamics such as the phenomenon of shattering.

We continued by making use of concepts like Sumudu-transform to be able to exploit that newly developed concept of differentiation with an additional parameter to analyze some type of stationary states and have recovered the classical well-known stationarity results. Then, it appears that, contrary to most of the existing versions of fractional models, solutions to time diffusion and relaxation systems with the additional parameter β, are not governed by the Mittag-Leffler functions, but rather by the parameterized exponential function $\mathcal{E}$ that is defined by (6.2.17) and appears more friendly and easier to handle compared to power series like Mittag-Leffler functions. It also appears that the parameter β is a powerful tool that can be used to control the stationarity of some generalized models. These are pertinent results which modify the conjecture mentioned in Chapter 6, that is, Mittag-Leffler functions do not govern all the models with a fractional derivative or derivative with a fractional parameter. We trust this observation is going to have an impact on modeling within this specific field of mathematics so as to provide a fair and just description of natural phenomenon one tries to analyze. This is the first instance where such results are obtained and will lead to more investigations and innovative results.

Let us finish by mentioning this open problem that may be the topic of our future research: It is about evolution equations and bounded perturbation. Evolution equations using derivatives of fractional order like Caputo's derivative or Riemann-Liouville's derivative have been intensively analyzed in numerous works. But the classical bounded perturbation theorem has been proven not to be in general true for these models, especially for solution operators of evolution equations with fractional order derivative β

less than 1 ($0 < \beta < 1$), as shown by the example in [26, Example 3.1]. Whence, the newly developed concept of differentiation with an additional parameter might be an alternative way of dealing with this issue. We seek to make use of it to show the perturbations by bounded linear operators for linear evolution equations when the derivative order β is less than one. The call is open and the future works will tell us.

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Appendix A

Appendix

A.1 Evolution for transport-convection dynamics with a New Parameter: An alternative method.

Here we use alternative techniques (the two-parameter matrix solution operators) address the well-posedness of the transport-convection models with a new parameter of the type

$$ {}_0^A D_t^\beta p(t,x,n) = -div(\omega(x,n)p(t,x,n)) - a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m), \quad \text{(A.1.1)} $$

where $t > 0,\ 0 < \beta \leq 1,\ x \in \mathbb{R}^3,\ \ n = 1, 2, 3, ...$ and subject to initial conditions

$$ p(0,x,n) = \overset{o}{p}_n(x), \qquad n = 1, 2, 3, ... \quad \text{(A.1.2)} $$

The concepts defined in Chapter 2 are used, especially the derivative with a new parameter ${}_0^A D_t^\beta$. The model (A.6.1) may take the generalised form

$$ \begin{aligned} & {}_0^A D_t^\beta u(x,t) = [\mathbb{A}u(\cdot,t)](x), \quad 0 < \beta \leq 1,\ \ x,\ t > 0 \\ & u(x,0) = \tilde{f}(x)\ ,\ \ x > 0, \end{aligned} \quad \text{(A.1.3)} $$

where $\mathbb{A}$ is a certain differential and (or) integral expression, that can be evaluated at any point $x > 0$ for functions u belonging to a certain subset of the domain of $\mathbb{A}$.

A.2 Two-parameter matrix solution operators

To proceed we can define a Banach space H endowed with the norm $\|\cdot\|_H$, express the model (A.1.3) in the form

$$ \begin{aligned} & {}_0^A D_t^\beta u(t) = Au(t), \quad 0 < \beta \leq 1,\ \ t > 0 \\ & u(0) = f \end{aligned} \quad \text{(A.2.1)} $$

and define the domain

$$D(A) := \{v \in H : Av \in H\} \tag{A.2.2}$$

on which the realization operator A of the expression $\mathbb{A}$ is defined. To study (A.2.1), we can exploit the differential system

$$\begin{aligned} &{}_0^A D_t^\beta u(t) = \mu u(t), \quad 0 < \beta \leq 1, \ \ t > 0, \ \mu \in \mathbb{C} \\ &u(0) = \ f_0. \end{aligned} \tag{A.2.3}$$

It is easy to check that, instead of Mittag-Leffler function or one of its variants, the following expression new in the literature, uniquely solves the model (A.2.3):

$$\mathcal{E}_\beta(t) = f_0 \exp\left[\mu\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right]. \tag{A.2.4}$$

We note that for $\beta = 1$ the following well known classical result holds:

$$u(t) = f_0 e^{\mu t}.$$

Remark A.2.1. *If we set a certain* $T_\beta = T = \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}$, *then the expression* $v(T) = f_0 e^{\mu T}$ *uniquely solves*

$$\begin{aligned} &\partial_T u(T) = \mu u(T), \ \ t > 0, \ \mu \in \mathbb{C} \\ &u(0) = \ f_0, \end{aligned} \tag{A.2.5}$$

where ∂_T *means partial derivative (normal derivative) with respect to* T. *Hence, the expression (A.2.4) uniquely solves (A.2.3) always implies that there exists a function at least in* $C(\mathcal{R}_+, H) \cap C^1(\mathcal{R}_+, H)$ *solving (A.2.5)*

This remark will be very important in our analysis, with a special attention to the expression of T. Next we consider the system of linear differential equations using the β–derivative with constant coefficients:

$$\begin{aligned} &{}_0^A D_t^\beta u_1 = \mu_{11} u_1 + \mu_{12} u_2 + \cdots + \mu_{1n} u_n, \\ &\vdots \\ &{}_0^A D_t^\beta u_n = \mu_{n1} u_1 + \mu_{n2} u_2 + \cdots + \mu_{nn} u_n, \end{aligned} \tag{A.2.6}$$

where $0 < \beta \leq 1, \ \ t > 0, \ \mu \in \mathbb{C}$. The linearity of the operator ${}_0^A D_t^\beta$ allows us to write the system (A.2.6) in the matrix form

$${}_0^A D_t^\beta U(t) = M U(t) \tag{A.2.7}$$

with U is a n–vector whose components are the unknown functions u_i and M is the $n \times n$ matrix $(\mu_{ij})_{1 \leq i,j \leq n}$. Let $U(0) = U_0$ be the initial condition vector for (A.2.7). We extend Peano's idea [114] by stating by analogy to solution (A.2.4) that the system (A.2.7) can be solved using explicitly the formula

$$U(t) = \exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] U_0 \tag{A.2.8}$$

where the matrix exponential

$$\exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp\left[T_\beta M\right] = I + \frac{T_\beta M}{1!} + \frac{T_\beta^2 M^2}{2!} + \cdots \tag{A.2.9}$$

with

$$T_\beta = T_\beta(t) = \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta} \tag{A.2.10}$$

Remark A.2.2. *It is easy to see that the function*

$$\begin{aligned} T_\beta : \quad & \mathbb{R}_+ \longrightarrow \mathbb{R}_+. \\ & t \longmapsto \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}, \quad 0 < \beta \leq 1 \end{aligned}$$

is a topological homeomorphism from $\mathbb{R}_+$ to $\mathbb{R}_+$. Thus, the topological properties of the space $\mathbb{R}$ (endowed with a topology) is preserved when transforming t to $T_\beta(t)$

Now, we consider the space $\mathfrak{M}_n(\mathbb{C})$ of all complex $n \times n$ matrices and endowed with the matrix-norm. By definition we have

$$\exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp\left[T_\beta M\right] = \sum_{k=0}^{\infty} \frac{T_\beta^k M^k}{k!} \tag{A.2.11}$$

for all $M \in \mathfrak{M}_n(\mathbb{C})$ and $0 < \beta \leq 1$. It is well known and not difficult to show that the partial sums of the series (A.2.11) form a Cauchy sequence, and so, the series converges.

Proposition A.2.3. *For any $M \in \mathfrak{M}_n(\mathbb{C})$ and $0 < \beta \leq 1$, the map*

$$\begin{aligned} \mathbb{R}_+ \quad & \longrightarrow \quad \mathfrak{M}_n(\mathbb{C}) \\ t \quad & \longmapsto \quad \exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] \end{aligned} \tag{A.2.12}$$

is continuous.

Proof. The proof follows from the fact that the map $T_\beta \longmapsto \exp\left[T_\beta M\right]$ is continuous in T_β and completed by Remark A.2.2. ■

The following well known results [61] that apply for exponential functions holds

Proposition A.2.4. *For any* $M \in \mathfrak{M}_n(\mathbb{C})$ *and* $0 < \beta \leq 1$,

$$\exp[(T_\beta + S_\beta)M] = \exp[T_\beta M] \cdot \exp[S_\beta M]$$
$$\exp[0M] = I.$$

Hence, the map $T_\beta \longmapsto \exp[T_\beta M]$ is a homomorphism of the additive semigroup $(\mathbb{R}_+, +)$ into a multiplicative semigroup of matrices $(\mathfrak{M}_n, \cdot)$.

Definition A.2.5. *The modified time expressed by* T_β *in (A.2.10) is called the revamped time (or GA-revamped time) corresponding to* t *for the model (A.2.7)*

Remark A.2.6. *Note that* $T_\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *is well defined and increasing for* $0 < \beta \leq 1$ *with*

- $T_\beta(0) = 0$
- $T_1(t) = t$
- $\frac{dT_\beta(t)}{dt} = \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1}$

This means the revamped time always coincide with its corresponding time at the beginning (initial conditions) or when $\beta = 1$ *(coventional first order derivative).*

Definition A.2.7 (**Two-parameter matrix solution operators**). *Let us fix* $\beta \in (0, 1]$ *and* $t \in \mathbb{R}_+$. *The pair* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *where* $T_\beta(t) = \frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}$, *is called the two-parameter matrix solution operator for the system (A.2.7), where* $\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}$ *is the two-parameter family such that*

- $S_\beta(t) = G(T_\beta)$ *with* T_β *the revamped time corresponding to* t.
- $\{G(T_\beta)\}_{T_\beta \geq 0}$, *the one-parameter family defined as*

$$G(T_\beta) = \exp\left[\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) M\right] = \exp[T_\beta M] \tag{A.2.13}$$

and representing a semigroup (in T_β*) generated by the matrix* $M \in \mathfrak{M}_n(\mathbb{C})$,

A.3 Strongly continuous two-parameter solution operators

With the previous definition in mind, we come back to the model (A.2.1):

$$\begin{aligned} {}^{A}_{0}D^{\beta}_{t} u(t) &= Au(t), \quad 0 < \beta \leq 1, \;\; t > 0. \\ u(0) &= f \end{aligned} \tag{A.3.1}$$

If $A : H \to H$ is a bounded linear operator then, we can exploit the Definition A.2.7 to solve the model (A.3.1) together with the exponential series represented in (A.2.11), which is still convergent with respect to the norm in the space of bounded linear operators $\mathcal{B}(H)$. In this case, the pair $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}, T_\beta(t))$ defined in the Definition A.2.7 and that solves (A.3.1) is simply called the two-parameter solution operator for the system (A.3.1). More precisely we have

Theorem A.3.1. *For the system (A.3.1), every uniformly continuous two-parameter solution operator*
$(\{S_\beta(t)\}_{t\geq 0,\,0<\beta\leq 1}; T_\beta(t))$ *on a Banach space* H *induces a solution that is in the form (A.2.13):*

$$u(t) = G(T_\beta)f = \exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)A\right]f, \qquad f\in D(A),$$

for some bounded linear operator A.

Proof. The proof follows from the previous section and the only point to add is that if $A: H\to H$ be a bounded linear operator, then the series

$$\sum_{k=0}^{\infty}\frac{\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta-\Gamma(\beta)^{-\beta}}{\beta}\right)^k A^k}{k!}$$

converges in the used norm for every $t>0$. ■

However, the reality is sometime complex and as mentioned in the introduction, the operator A is, in most of the cases, unbounded. Simple examples are differential operators that are not bounded on the whole space H. Then multiple iterates of operator A appearing in the series (A.2.11) make it impossible to use the series to solve (A.3.1). The main reason is that the common domain of those iterates of A could reduced to the null subspace $\{0\}$. Then, more considerations, in addition to what was developed in the previous section are necessary.

Definition A.3.2 (**Strongly continuous two-parameter solution operators**). *Let us fix* $\beta\in(0,1]$ *and* $t\in\mathbb{R}_+$. *The pair* $(\{S_\beta(t)\}_{t\geq 0,\,0<\beta\leq 1}; T_\beta(t))$ *is said to be a strongly continuous two-parameter solution operator for the system (A.3.1) if the two-parameter family* $\{S_\beta(t)\}_{t\geq 0,\,0<\beta\leq 1}$ *is such that*

- $S_\beta(t) = G(T_\beta)$ *with* T_β *the revamped time corresponding to* t.
- $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ *is a strongly continuous semigroup (in* T_β*) generated by the operator* A*, that is*

(i) $G_A(0) = I$;

(ii) $G_A(T_\beta + S_\beta) = G_A(T_\beta)G_A(S_\beta)$ *for all* $T_\beta, S_\beta \geq 0$;

(iii) $\lim_{T_\beta\to 0^+} G_A(T_\beta)f = f$ *for any* $f\in H$.

Remark A.3.3. *Note that:*

(a) For $\beta=1$, $T_\beta(t)=t$ *and the definition here above coincides with the definition of the classical well known (one-parameter)* C_0*-semigroup.*

(b) If $(\{S_\beta(t)\}_{t\geq 0,\,0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution opera-*

tor for the system (A.3.1) generated by A*, then,*

$$Af = \lim_{t\to 0}\frac{S_\beta(t)f - f}{t} = \lim_{T_\beta\to 0}\frac{G_A(T_\beta)f - f}{T_\beta}, \tag{A.3.2}$$

where the domain of A*,* $D(A)$*, is chosen to be defined as the set of all* $f \in H$ *for which this limit exists.*
The later equality is due to the above Definition A.3.2 and the fact that $T_\beta(t) \to 0$ *as* $t \to 0$*.*

(c) –If $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution operator for the system (A.3.1) generated by* A*, then, for* $f \in D(A)$ *the function* $t \longrightarrow S_\beta(t)f = G_A(T_\beta)f$ *is a classical solution of the fractional Cauchy problem (A.3.1).*
–For $f \in H \setminus D(A)$*, however, the function* $u(t) = S_\beta(t)f$ *is continuous but, in general, not differentiable, nor* $D(A)$*-valued, and, therefore, not a classical solution.*

(d) The strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is bounded in the operator norm over any compact interval of* $\mathbb{R}_+$ *thanks to properties (ii) and (iii) here above and the Banach–Steinhaus theorem which show that any* C_0*-semigroup like* $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ *is bounded in the operator norm over any compact interval of* $\mathbb{R}_+$*.*

(e) If $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *is a strongly continuous two-parameter solution operator for the system (A.3.1) generated by* A*, then, for* $f \in D(A)$ *the function* $T_\beta \longrightarrow G_A(T_\beta)f$ *a classical solution of*

$$\begin{aligned} &\partial_t u(t) = Au(t), \quad t > 0.\\ &u(0) = f \end{aligned} \tag{A.3.3}$$

More precisely, we have the following statement:
Proposition A.3.4. *Let* $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ *be the a strongly continuous two-parameter solution operator for the system (A.3.1) generated by* $(A, D(A))$*. Then* $t \to S_\beta(t)f = G_A(T_\beta)f, \quad f \in D(A)$*, is the only solution of* (A.3.1) *taking values in* $D(A)$*.*

Proof. To prove it we set $u(t) = v(T_\beta) \in D(A)$ for all $t > 0$, where $T_\beta = T_\beta(t)$ is the revamped time corresponding to t, $v \in C(\mathcal{R}_+, H) \cap C^1(\mathcal{R}_+, H)$ and ${}_0^A D_t^\beta u(t) = Au(t), \quad t > 0$. Then, by the Definition (A.3.2), $v(T_\beta)$ satisfies $\partial_t u(t) = Au(t), \quad t > 0$. Let us define the function

$$\begin{aligned} z : \quad (0, T_\beta) &\longrightarrow \quad H\\ S_\beta &\longmapsto \quad G_A(T_\beta - S_\beta)v(S_\beta) \end{aligned}$$

and make use of the well known property of semigroups [61]:

$$\partial_{T_\beta} G_A(T_\beta)v(T_\beta) = AG_A(T_\beta)v(T_\beta) = G_A(T_\beta)Av(T_\beta),$$

to state that z is differentiable and

$$0 = \partial_{S_\beta} z(S_\beta) = G_A(T_\beta - S_\beta)(\partial_{S_\beta} v(S_\beta) - (Av)(S_\beta)). \tag{A.3.4}$$

Thus, z is constant on $(0, T_\beta)$, meaning that for any $\varepsilon, \eta \in (0, T_\beta)$ we have

$$G_A(T_\beta - \varepsilon)v(\varepsilon) = G_A(T_\beta - \eta)v(\eta)$$

which tends to

$$G_A(T_\beta)v(0) = v(T_\beta)$$

as ε tends to 0 and η tends to T_β. This proves that v is defined by the semigroup $\{G_A(T_\beta)\}_{T_\beta \geq 0}$=$\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}$. Hence, by the Definition (A.3.2), u is also defined by the strongly continuous two-parameter solution operator $\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}$, which concludes the proof. ■

It is now clear that for $f \in D(A)$,

$$D_t^\beta S_\beta(t)f = \frac{d}{dT_\beta} G_A(T_\beta)f.$$

Hence, making use of the well know properties of strongly continuous semigroups, we have the following corollary

Corollary A.3.5. *Let $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ be the a strongly continuous two-parameter solution operator for the system (1.0.3) generated by $(A, D(A))$. Then, for $f \in D(A)$, $S_\beta(t)f \in D(A)$ and*

$$D_t^\beta S_\beta(t)f = AS_\beta(t)f = S_\beta(t)Af. \tag{A.3.5}$$

for all $t \geq 0$.

Definition A.3.6. *(**Two-parameter solution operators β-exponentially bounded**)*

- *The strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ for the system (A.3.1) is said to be β-exponentially bounded if there exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|S_\beta(t)\|_H \leq M \exp\left[\omega \left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)\right] \tag{A.3.6}$$

- *If the system (A.3.1) admits a strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ satisfying (A.3.6), then we say that the operator $A \in \mathcal{G}^\beta(M, \omega)$.*
- *$(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ is said to be contractive if*

$$\|S_\beta(t)\|_H \leq 1, \tag{A.3.7}$$

and we say $A \in \mathcal{G}^\beta(1, 0)$.

- *As in [117], we say that the problem (A.3.1) is well-posed if it admits a strongly continuous two-parameter solution operator.*

Let us set

$$\mathcal{G}^{\beta}(\omega) := \bigcup\{\mathcal{G}^{\beta}(M,\omega), M \geq 1\},$$

$$\mathcal{G}^{\beta} := \bigcup\{\mathcal{G}^{\beta}(\omega), \omega \geq 0\}$$

and denote by

$$\mathcal{B}(H) := \mathcal{B}(H;H)$$

the space of all bounded linear operators from H to H.

Remark A.3.7. *The condition (A.3.6) holds if and only if the one parameter family $\{G_A(T_\beta)\}_{T_\beta \geq 0}$ given in the Definition (A.3.2) satisfies*

$$\|G_A(T_\beta)\|_H \leq Me^{\omega T_\beta} \tag{A.3.8}$$

Corollary A.3.8. *The problem (A.3.1) is well-posed if $A \in \mathcal{B}(H)$*

Proof. This is a direct consequence of Theorem A.3.1 and Proposition A.3.4. ■

Next let us recall the following definition:

Definition A.3.9. *The set $\rho(A)$ is called the resolvent set of the operator A and is defined as*

$$\rho(A) = \{\lambda \in \mathbb{R}; \quad \lambda I - A : D(A) \to X \text{ is invertible and } \quad (\lambda I - A)^{-1} \in \mathcal{B}(H)\}. \tag{A.3.9}$$

Then, For $\lambda \in \rho(A)$, the inverse $R(\lambda, A) := (\lambda I - A)^{-1}$ is, by the closed graph theorem, a bounded operator on H and is termed as the resolvent of A at the point λ.

Proposition A.3.10. *If the strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0, 0<\beta\leq 1}; T_\beta(t))$ for the system (A.3.1) is β-exponentially bounded in terms of Definition A.3.6 then, $S_\beta(t)$ is related to its resolvent by the formula*

$$R(\lambda, A)f = \int_0^\infty \left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta-1} \exp\left[-\lambda\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^{\beta} - \Gamma(\beta)^{-\beta}}{\beta}\right)\right] S_\beta(t) f dt, \tag{A.3.10}$$

for $f \in H$ and $Re\lambda > \omega$.

Proof. The proof follows from the Definition A.3.2 where $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ is a strongly continuous semigroup with the operator A as infinitesimal generator and satisfying (A.3.8). Then, from the semigroup theory we have that

$$R(\lambda, A) = \int_0^\infty e^{-\lambda T_\beta} G_A(T_\beta) dT_\beta.$$

Substituting the revamped time T_β and using the Remark A.2.6 lead to the formula. ■

We can therefore propose the following diagram for the system (A.3.1) presenting the relations between the two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ its generator and its resolvent.

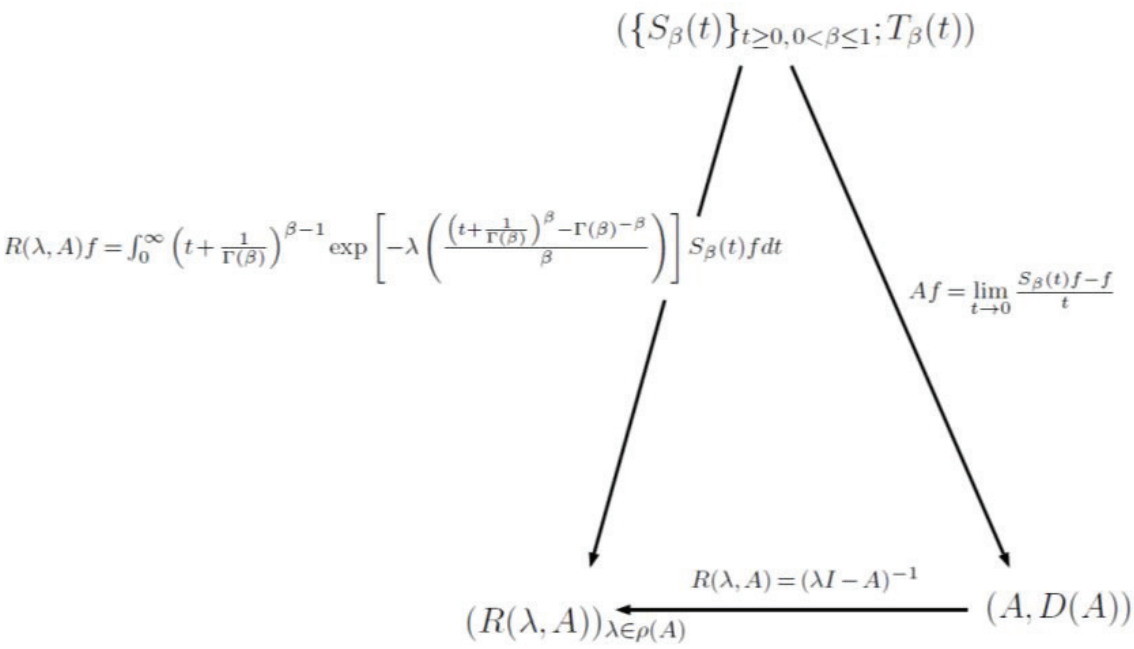

Figure A.1: Relations between the two-parameter solution operator, its generator and its resolvent

A.4 Exponential approximation and application

For dynamical systems (A.3.1) with unbounded operators A, analysis can be done by using the exponential approximation

$$\exp\left[\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) A\right] f = \lim_{p\to\infty}\left[I - \frac{1}{p}\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right) A\right]^{-p} f. \tag{A.4.1}$$

If the above limit exists then, it defines a strongly continuous two-parameter solution operator as given in Definition (A.3.2). Conditions of the existence of the limit (A.4.1) are given by making use of the Hille–Yosida theorem (see [61, Chap II, Section 3]) in the theory of semigroup and completed by the Remark A.3.7. Then we have the following theorem that applies to the model (A.3.1) with the fractional parameter β;

Theorem A.4.1. *$A \in \mathcal{G}^\beta(M, \omega)$ if and only if (a) A is closed and densely defined,* *(b) there exist $M > 0, \omega \in \mathbb{R}$ such that $(\omega, \infty) \in \rho(A)$ and for all $n \geq 1$, $\lambda > \omega$,*

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\lambda - \omega)^n}. \tag{A.4.2}$$

where $\rho(A)$ is the resolvent set of the operator A as defined above.

Proposition A.4.2. *Let $(\{S_\beta(t)\}_{t\geq 0,\, 0<\beta\leq 1}; T_\beta(t))$ be the a strongly continuous two-parameter solution operator for the system (A.3.1) generated by A. Then*

$$S_\beta(t)f = \lim_{p\to\infty}\left[I - \frac{1}{p}\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{\beta}\right)A\right]^{-p} f, \quad \textit{for } f \in H,$$

and the limit is uniform in t on any bounded interval.

Proof. Considering the revamped time corresponding to t, $T_\beta = T_\beta(t)$, we have by definition $S_\beta(t)f = G_A(T_\beta)f$. Since the one parameter family $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ is a C_0-semigroup generated by A, we make use of [61, Corollary III 5.5] to write

$$G_A(T_\beta)f = \lim_{p\to\infty}\left(I - \frac{T_\beta}{p}A\right)^{-p} f, \quad \text{for } f \in H$$

and the proposition is proved. ■

As application, we can approximate the solution for the system (A.3.1), by considering the alternate model given by

$$\begin{aligned} &\frac{u_p\left[k\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{p\beta}\right)\right] - u_p\left[(k-1)\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{p\beta}\right)\right]}{\left(\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{p\beta}\right)} = Au_p\left[k\left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{p\beta}\right)\right] \\ &u_p(0) = f \end{aligned} \tag{A.4.3}$$

for $0 < \beta \leq 1$, $t > 0$. The explicit solution of the problem (A.4.3) is given by

$$u_p(t) = \left[I - \left(\frac{\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta - \Gamma(\beta)^{-\beta}}{p\beta}\right)A\right]^{-p} f$$

which represents an approximation of the solution for the model (A.3.1). Making use of Proposition A.4.2, we see that $\lim_{p\to\infty} u_p(t) = S_\beta(t)f$. Hence, the difference system (A.4.3) is very important in solving the model (A.3.1) since their solutions converge to the solution of (A.3.1) and from Proposition A.3.4, this solution $S_\beta(t)f$ is unique if f is taken from $D(A)$.

A.5 Subordination & prolongation principles with $\beta-$ derivative

In this section, we address the issue of subordination principle for evolution equations with fractional parameters. This principle has been proved only for models with Caputo fractional derivative [25, 117] and the opposite principle has been proved not to be true. Hence, we go farther by also addressing the opposite principle, named here the prolongation principle. Recall that these principles study existence of two-parameter solution operators for problems (1.0.3) with different values of derivative orders. We note that if we have a strongly continuous semigroup $\{G_A(T)\}_{T\geq 0}$ generated by the operator A, we can always identify the Cauchy problem for which it is a solution. This yields the following lemma:

Lemma A.5.1. *Considering the model (1.0.3) and T_β the GA-revamped time corresponding to t. If there is a strongly continuous semigroup (in T_β), say $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ generated by the operator A then, the family $(\{S_\beta(t)\}_{t\geq 0,\,0<\beta\leq 1}; T_\beta(t))$ such that $S_\beta(t) = G(T_\beta)$, is a strongly continuous two-parameter solution operator for the system (1.0.3).*

Theorem A.5.2. *Considering the models (1.0.3) with two different orders β and δ such that $0 < \delta < \beta \leq 1$. Let $\omega \geq 0$ then, $A \in \mathcal{G}^\beta(\omega)$ if and only if $A \in \mathcal{G}^\delta(\omega)$.*

Proof. Suppose $A \in \mathcal{G}^\beta(\omega)$, then, (1.0.3) admits a strongly continuous two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\,0<\beta\leq 1}; T_\beta(t))$ satisfying (A.3.6). Hence, by definition we have $\{S_\beta(t)\}_{t\geq 0,\,0<\beta\leq 1} = \{G_A(T_\beta)\}_{T_\beta\geq 0}$ where T_β is GA-revamped time $\frac{\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta-\Gamma(\beta)^{-\beta}}{\beta}$, corresponding to t and $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ is a strongly continuous semigroup (in T_β) generated by the operator A. Moreover, by Remark A.3.7, we have $G_A(T_\beta)$ satisfying (A.3.8). For $0 < \delta < \beta \leq 1$, let us define $T_\delta = T_\delta(t) = \frac{\left(t+\frac{1}{\Gamma(\delta)}\right)^\delta-\Gamma(\delta)^{-\delta}}{\delta}$, the GA-revamped time (of order δ) corresponding to t, then $\{G_A(T_\delta)\}_{T_\delta\geq 0}$ is also a strongly continuous semigroup (in T_δ) generated by the operator A since $\{G_A(T_\beta)\}_{T_\beta\geq 0}$ is. Moreover, by (A.3.8) we have

$$\|G_A(T_\delta)\|_{\mathcal{X}} \leq Me^{\omega T_\delta}, \tag{A.5.1}$$

and Lemma A.5.1 concludes the first part of the proof, showing the subordination principle for the model (1.0.3).

Conversely, to prove the prolongation principle, we suppose $A \in \mathcal{G}^\delta(\omega)$ and the rest of the proof follows the same steps as above. ■

The following corollary appears as an immediate consequence.

Corollary A.5.3. *Consider any $\beta \in (0, 1)$. Then, there are constants $\omega \geq 0$ and $M \geq 1$ such that the operator A in model (1.0.3) is the infinitesimal generator of a C_0-semigroup $G(t)$ satisfying $\|G(t)\| \leq Me^{\omega t},\ t \geq 0$ if and only if $A \in \mathcal{G}^\beta(M, \omega)$ with the corresponding two-parameter solution operator $(\{S_\beta(t)\}_{t\geq 0,\,0<\beta\leq 1}; T_\beta(t))$ satisfying (A.3.6).*

A.6 Applications to break-up dynamics in transport-convection

Mathematical settings and Model's analysis

In this section we address the well-posedness of the model

$$
{}_0^AD_t^{\beta} p(t,x,n) = -div(\omega(x,n)p(t,x,n)) - a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m), \quad \text{(A.6.1)}
$$

where $t > 0,\ 0 < \beta \leq 1,\ x \in \mathbb{R}^3,\ \ n = 1, 2, 3, ...$ and subject to initial conditions

$$
p(0,x,n) = \overset{o}{p}_n(x), \qquad n = 1, 2, 3, ... \qquad \text{(A.6.2)}
$$

by using the concepts defined here above and setting other suitable conditions. Equation (A.6.1) models the break-up dynamics of moving groups. In terms of the mass size m and the position x, the state of the system is characterized at any moment t by the particle-mass-position distribution $p = p(t, x, m)$, (p is also called the *density* or *concentration* of particles), with $p : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}_+$, the velocity $\omega = \omega(x, m)$ of the transport is supposed to be a known quantity depending on m and x. The average fragmentation rate a_n is the average number at which clusters of size n undergo splitting, $b_{n,m} \geq 0$ is the average number of n-groups produced upon the splitting of m-groups. The space variable x is supposed to vary in the whole of $\mathbb{R}^3 = \Omega$. The function $\overset{o}{p}_n$ represents the density of n-groups at the beginning of observation ($t = 0$) and it is integrable with respect to x over the full space $\mathbb{R}^3$. The necessary assumptions that will be useful in the analysis are introduced in the following sections.

A.6.1 Well-posedness for the break-up part of the model

Since a group of size $m \leq n$ cannot split to form a group of size n, we require $b_{n,m} = 0$ for all $m \leq n$ and

$$
a_1 = 0, \ \sum_{m=1}^{n-1} m b_{m,n} = n, \quad (n = 2, 3, ...), \qquad \text{(A.6.3)}
$$

meaning that a cluster of size one cannot split and the sum of all individuals obtained by break-up of an n-group is equal to n. Because the total number of individuals in a population is not modified by interactions among groups and that the mass is expected to be a conserved quantity, the most appropriate Banach space to work in is the space

$$
\mathcal{X}_1 := \left\{ \mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \|\mathbf{g}\|_1 := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n|g_n(x)|dx < \infty \right\}. \qquad \text{(A.6.4)}
$$

We work in this space because they have many desirable properties, like controlling the norm of their elements which, in our case, represents the total mass (or total number of individuals) of the system and must be finite. Because uniqueness of solutions to the systems of type (A.6.1)-(A.6.2) is proved to be a more difficult problem [55, 112], we restrict our analysis to a smaller class of functions, so we introduce the following class of Banach spaces (of distributions with finite higher moments)

$$\mathcal{X}_r := \left\{ \mathbf{g} = (g_n)_{n=1}^{\infty} : \mathbb{R}^3 \times \mathbb{N} \ni (x,n) \to g_n(x), \|\mathbf{g}\|_r := \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r |g_n(x)| dx < \infty \right\}, \tag{A.6.5}$$

$r \geq 1$, which coincides with $\mathcal{X}_1$ for $r = 1$. We assume that for each $t \geq 0$, the function $(x,n) \longrightarrow p(t,x,n) = p_n(t,x)$ is such that $\mathbf{p} = (p_n(t,x))_{n=1}^{\infty}$ is from the space $\mathcal{X}_r$ with $r \geq 1$. In $\mathcal{X}_r$ we can rewrite (A.6.1)-(A.6.2) in more compact form,

$$\begin{aligned} {}_0^A D_t^{\beta} \mathbf{p} &:= \mathbf{D}\mathbf{p} + \mathbf{F}\mathbf{p}, \\ \mathbf{p}_{|t=0} &= \mathring{\mathbf{p}}, \end{aligned} \tag{A.6.6}$$

where $t > 0, \ 0 < \beta \leq 1, \ x \in \mathbb{R}^3, \ \ n = 1, 2, 3, ...$ Here $\mathbf{p}$ is the vector $(p(t,x,n))_{n\in\mathbb{N}}$, $\mathbf{D}$ the transport expression defined as

$$(p(t,x,n))_{n\in\mathbb{N}} \longrightarrow (-\mathrm{div}(\omega(x,n)p(t,x,n)))_{n=1}^{\infty}, \tag{A.6.7}$$

$\mathring{\mathbf{p}}$ the initial vector $(\mathring{p}_n(x))_{n\in\mathbb{N}}$ which belongs to $\mathcal{X}_r$ and $\mathbf{F}$ the fragmentation expression defined by

$$(\mathbf{F}\mathbf{p})_{n=1}^{\infty} := \left(-a_n p(t,x,n) + \sum_{m=n+1}^{\infty} b_{n,m} a_m p(t,x,m) \right)_{n=1}^{\infty}.$$

In this work, for any subspace $S \subseteq \mathcal{X}_r$, we will denote by S_+ the subset of S defined as $S_+ = \{\mathbf{g} = (g_n)_{n=1}^{\infty} \in S; g_n(x) \geq 0, n \in \mathbb{N}, x \in \mathbb{R}^3\}$. Note that any $\mathbf{g} \in (\mathcal{X}_r)_+$ possesses moments

$$M_q(\mathbf{g}) := \sum_{n=1}^{\infty} n^q g_n$$

of all orders $q \in [0, r]$. In $\mathcal{X}_r$, we define the operators $\mathbf{A}$ and $\mathbf{B}$ by

$$\mathbf{A}\mathbf{g} := (a_n g_n)_{n=1}^{\infty}, \quad D(\mathbf{A}) := \{\mathbf{g} \in \mathcal{X}_r : \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r a_n |g_n(x)| dx < \infty\}; \tag{A.6.8}$$

$$\mathbf{B}\mathbf{g} := (B_n g_n)_{n=1}^{\infty} = \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m g_m \right)_{n=1}^{\infty}, \quad D(\mathbf{B}) := D(\mathbf{A}). \tag{A.6.9}$$

Throughout, we assume that the coefficients a_n and $b_{n,m}$ satisfy the mass conservation conditions (A.6.3). Now let's prove that $\mathbf{B}$ is well defined on $D(\mathbf{A})$ as stated in (A.6.9).

Making use of the condition (A.6.3), we have

$$n^r - \sum_{m=1}^{n-1} m^r b_{m,n} \geq n^r - (n-1)^{r-1} \sum_{m=1}^{n-1} m b_{m,n} = n^r - n(n-1)^{r-1} \geq 0.$$

Hence

$$\sum_{m=1}^{n-1} m^r b_{m,n} \leq n^r \tag{A.6.10}$$

for $r \geq 1$, $n \geq 2$. Note that the equality holds for $r = 1$. For every $\mathbf{g} \in D(\mathbf{A})$, we have then

$$\begin{aligned}
\|\mathbf{Bg}\|_r &= \int_{\mathbb{R}^3} \sum_{n=1}^{\infty} n^r \left(\sum_{m=n+1}^{\infty} b_{n,m} a_m |g_m(x)| \right) dx \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| \left(\sum_{n=1}^{\infty} n^r b_{n,m} \right) dx \\
&= \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| \left(\sum_{n=1}^{m-1} n^r b_{n,m} \right) dx \\
&\leq \int_{\mathbb{R}^3} \sum_{m=2}^{\infty} a_m |g_m(x)| m^r dx \\
&= \|\mathbf{Ag}\|_r \\
&< \infty,
\end{aligned}$$

where we have used the inequality (A.6.10). Then $\|\mathbf{Bg}\|_r \leq \|\mathbf{Ag}\|_r$, for all $\mathbf{g} \in D(\mathbf{A})$, so that we can take $D(\mathbf{B}) := D(\mathbf{A})$ and $(\mathbf{A}+\mathbf{B},\ D(\mathbf{A}))$ is well-defined.

A.6.2 Well-posedness for the transport part of the model

Our primary objective in this section is to analyze the solvability of the Cauchy problem for the transport equation

$${}_0^A D_t^\beta p(t,x,n) = -div(\omega(x,n)\, p(t,x,n)), \tag{A.6.11}$$

$$p(0,x,n) = \overset{o}{p}_n(x), \qquad n = 1,2,3,...$$

in the space $\mathcal{X}_r$, where $t > 0,\ 0 < \beta \leq 1,\ x \in \mathbb{R}^3,\ \ n = 1,2,3,...$
To do so we need the following:
Now let us fix $n \in \mathbb{N}$. We consider the function $\omega_n : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $\omega_n(x) = \omega(x,n)$ and $\tilde{\mathcal{D}}_n$ the expression appearing on the right-hand side of the equation (A.6.11). Then

$$\begin{aligned}
\tilde{\mathcal{D}}_n[p(t,x,n)] &:= -div\,(\omega(x,n)\, p(t,x,n)) \\
&= (\nabla \cdot \omega(x,n)) p(t,x,n) + \omega(x,n) \cdot (\nabla p(t,x,n)).
\end{aligned} \tag{A.6.12}$$

We assume that ω_n is divergence free and globally Lipschitz continuous. Then $div\,\omega_n(x) := \nabla \cdot \omega(x,n) = 0$ and (A.6.12) becomes

$$\tilde{\mathcal{D}}_n[p(t,x,n)] := \omega(x,n) \cdot (\nabla p(t,x,n)). \tag{A.6.13}$$

We note that the operators on the right-hand side of (A.6.6) have the property that one of the variables is a parameter and, for each value of this parameter, the operator has a certain desirable property (like being the generator of a semigroup) with respect to the other variable. Thus we need to work with parameter-dependent operators that can be "glued" together in such a way that the resulting operator inherits the properties of the individual components. Let us provide a framework for such a technique called the method of semigroups with a parameter [55, 112]. Let us consider the space $\mathcal{X} := L_p(S,X)$ where $1 \le p < \infty$, (S,m) is a measure space and X a Banach space. Let us suppose that we are given a family of operators $\{(A_s, D(A_s))\}_{s\in S}$ in X and define the operator $(\mathbb{A}, D(\mathbb{A}))$ acting in $\mathcal{X}$ according to the following formulae,

$$\mathcal{D}(\mathbb{A}) := \{g \in \mathcal{X}; g(s) \in D(A_s) \text{ for almost every } s \in S, \ \ \mathbb{A}g \in \mathcal{X}\}, \tag{A.6.14}$$

and, for $g \in \mathcal{D}(\mathbb{A})$,

$$(\mathbb{A}g)(s) := A_s g(s), \tag{A.6.15}$$

for every $s \in S$.

We set

$$X_x := L_1(\mathbb{R}^3, dx) := \{\psi : \|\psi\| = \int_{\mathbb{R}^3} |\psi(x)| dx < \infty\}$$

and define in X_x the operators $(\mathcal{D}_n, D(\mathcal{D}_n))$ as

$$\begin{aligned} \mathcal{D}_n p_n = \tilde{\mathcal{D}}_n p_n, &\quad \text{with } \tilde{\mathcal{D}}_n p_n \ \text{ represented by } (A.6.13) \\ D(\mathcal{D}_n) := \{p_n \in X_x, &\ \ \mathcal{D}_n p_n \in X_x\}, \quad n \in \mathbb{N}. \end{aligned} \tag{A.6.16}$$

Then, in $\mathcal{X}_r$ we can define for the operator $\mathbf{D}$ (A.6.7) the domain

$$D(\mathbf{D}) = \{\mathbf{p} = (p_n)_{n\in\mathbb{N}} \in \mathcal{X}_r,\, p_n \in D(\mathcal{D}_n) \text{ for almost every } n \in \mathbb{N},\, \mathbf{Dp} \in \mathcal{X}_r\}. \tag{A.6.17}$$

Theorem A.6.1. *Let us fix any $\beta \in (0,1]$. If for each $n \in \mathbb{N}$ the function $\omega_n : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is globally Lipschitz continuous and divergence-free then, the operator $(D(\boldsymbol{D}), \boldsymbol{D})$ is the generator of a contractive strongly continuous two-parameter solution operator for the system (A.6.11).*

Proof. To prove it we apply the subordination principle of Theorem A.5.2, by considering the model (A.6.11) with $\beta = 1$ to have the compact form

$$\partial_t \mathbf{P} = \mathbf{DP}, \tag{A.6.18}$$

subject to the initial condition

$$\mathbf{p}|_{t=0} = \mathring{\mathbf{p}}. \tag{A.6.19}$$

where $\mathbf{D}$ the transport expression defined in (A.6.7). Making use of [112, Theorem 2] or [55, Theorem 3.4.2], it is proved that if the conditions of Theorem A.6.1 are satisfied then, there exists a strongly continuous stochastic (positive and contractive) semigroup generated by $(D(\mathbf{D}), \mathbf{D})$. Hence, $\mathbf{D} \in \mathcal{G}^1(1,0)$ and exploiting the the subordination principle of Theorem A.5.2, we have $\mathbf{D} \in \mathcal{G}^\beta(1,0)$, which prove the theorem. ■

A.6.3 Existence results for the full model

Attention is now shifted to the transport problem with the loss part of the break-up process. We assume that there are two constants $0 < \theta_1$ and θ_2 such that for every $x \in \mathbb{R}^3$,

$$\theta_1 \alpha_n \leq a_n(x) \leq \theta_2 \alpha_n, \tag{A.6.20}$$

with $\alpha_n \in \mathbb{R}_+$ and independent of the state variable x. Then a_n is bounded for each $n \in \mathbb{N}$ and the loss operator $(A_n, D(A_n))$ can be defined in X_x as $A_n(x) = a_n(x)$ with $D(A_n) = X_x = L_1(\mathbb{R}^3)$. The corresponding abstract Cauchy problem for the full model (A.6.1)-(A.6.2) reads as

$$\begin{aligned} {}_0^A D_t^\beta \mathbf{p} &= \mathbf{D}\mathbf{p} + \mathbf{F}\mathbf{p} \\ \mathbf{p}_{|t=0} &= \mathring{\mathbf{p}}. \end{aligned} \tag{A.6.21}$$

The following theorem holds.
Theorem A.6.2. *Assume that (A.6.20) is satisfied for each $n \in \mathbb{N}$.*
There is an extension $(\mathcal{K}, D(\mathcal{K}))$ of $(\boldsymbol{D} + \boldsymbol{F}, D(\boldsymbol{D}) \cap D(\boldsymbol{A}))$ that generates, on $\mathcal{X}_r$, a strongly continuous two-parameter solution operator for the system (A.6.1)-(A.6.2) which is contractive.

Proof. The proof follows the same steps as the proof of Theorem A.6.1 where we apply the subordination principle on the reference [112, Theorem 5] or [55, Theorem 3.5.2]. ■

This concludes, as an application, the well-posedness of an integrodifferential equation modeling convection and break-up processes. It is certain that this whole thesis will inspire more than one author with the introduction of the new concepts. Thus, it emerges to be a breakthrough that might help solving opens problems mentioned throughout this thesis or lead to more complex analysis of evolutions equations often describing phenomena more and more intricate.