ANALYTICAL SOLUTIONS OF THE IVANCEVIC OPTION PRICING MODEL WITH A NONZERO ADAPTIVE MARKET POTENTIAL

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Abstract: The Ivancevic option pricing model (IOPM) is a nonlinear adaptive-wave alternative for the classical Black-Scholes option pricing model, representing a controlled Brownian motion in an adaptive setting relating to nonlinear Schrödinger equation. Despite the applicability and usefulness of the IOPM, analytical solutions of such model are barely found in literature. Therefore, this paper obtains analytical solutions of the IOPM by means of a proposed semi-analytical method referred to as projected differential transform method (PDTM). Cases of nonzero adaptive market potential are considered. The proposed method is proven to be direct, and effective as the obtained solutions tend rapidly to their exact forms.

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1. Introduction

In mathematical finance, the classical Black-Scholes model serves as hallmark financial model; it describes the time-evolution of the market value of financial equity such as stock option [1, 2, 3]. The basic assumptions under which this classical arbitrage pricing theory is formulated include the following: the asset price S (or the underlying asset) following a geometric Brownian motion (GBM), the drift parameter, μ and the volatility rate, σ are assumed constants, lack of arbitrage opportunities (no risk-free profit), frictionless and competitive markets [4, 5, 6]. Thus, the stock price $S = S(t)$, at time t, $(0 \leq t \leq T)$ follows the stochastic differential equation (SDE):

$$dS = S(\mu dt + \sigma dW_t), \ S \in [0, \infty) \tag{1.1}$$

where $\mu, \sigma > 0$, and W_t are mean rate of return of S, the volatility, and a standard Brownian motion respectively.

So, for an option value $u = u(S, t)$, the Black-Scholes partial differential equation (PDE) associated to (1.1) can be expressed as:

$$\frac{\partial u}{\partial t} + rS\frac{\partial u}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 u}{\partial S^2} - ru = 0 \tag{1.2}$$

with $u(0, t) = 0$, $u(S, t) \to 0$ as $S \to \infty$, $u(S, T) = max(S - E, 0)$, E is a constant and

$$S(t) = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}, \ \ S_0 = S(0). \tag{1.3}$$

In literature, detailed and extensive work on the importance of (1.2) with respect to exact, analytical, approximate or numerical methods of solutions have been captured [7, 8, 9, 10].

Recently, Vukovic [11], established the interconnectedness of the Schrödinger and the Black-Scholes equations via the tools of quantum physics in the sense of Hamiltonian operator. It was noted that while the Black-Scholes Hamiltonian was anti-Hermitian causing the eigenvalues to be complex, the Schrödinger Hamiltonian was Hermitian. It was further showed that the Black-Scholes equation can be derived from the Schrödinger equation via the application of quantum mechanics tools [12, 13]. The facts incorporated include the points that: the Schrödinger equation requires a complex state function while the Black-Scholes equation is a real PDE that yields a real valued expression for the option price at all time.

The Black-Scholes model (1.2) can be applied to a reasonable number of one dimensional option models ascribed to u and S, say for puts/calls and stocks/dividends respectively [2]. As noted in [14, 15], one could consider the associated

probability density function (PDF) resulting from the backward Fokker-Planck equation using the classical Kolmogorov probability method instead of the market value of an option obtained via the Black-Scholes equation.

2. The Ivancevic Option Pricing Model (IOPM) [16]

As an alternative method for obtaining the same PDF for the market value of a stock option, Ivancevic [17] applied the quantum-probability formation as a solution to a time-dependent Schrödinger equation (linear or nonlinear) for the evolution of the complex-valued wave function, and proposed an adaptive, wave-form nonlinear model [18, 19]. Henceforth, such nonlinear adaptive model is referred to as Ivancevic option pricing model as follows:

$$i\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2\frac{\partial^2 w}{\partial S^2} + \beta|w|^2 w = 0, \ \ i^2 = -1 \tag{2.1}$$

where $w = w(S, t)$ denotes the option pricing wave-function at time t , $|w|^2 = |w(S,t)|^2$ represents the PDF for the option price with regard to stock price and time, σ represents a constant or stochastic process as the dispersion frequency volatility coefficient while β is referred to as the Landau coefficient representing adaptive market potential. The model (2.1) becomes linear if $\beta = 0$. In this work, a case of non-zero adaptive market potential ($\beta \neq 0$) will be considered in terms of analytical solutions using a proposed semi-analytical method referred to as projected differential transformation method (PDTM).

3. The Overview of the PDT Method [20, 21]

In this section, an overview of the PDTM is being presented.

3.1. A Note on Some Basic Theorems of the PDTM

In consideration, let $u(x, t)$ be an analytic function at (x , t) in a domain D , then in considering the Taylor series of $u(x, t)$, we give regard to some variables $s_v = t$ instead of all the variables as seen in the classical DTM. Thus, the projected DTM of $u(x, t)$ with respect to t at t is defined and denoted by:

$$U(x, h) = \frac{1}{h!}\left[\frac{\partial^h u(x,t)}{\partial t^h}\right]_{t=t_*} \tag{3.1}$$

such that

$$U(x, h) = \sum_{h=0} U(x, h)(t - t\ )^h \tag{3.2}$$

where (3.2) is referred to as the projected differential inverse transform (PDIT) of with respect to the time parameter t.

3.2. Some Basic Properties and Theorems of the PDTM

(a) If $m(x, t) = \alpha m_a(x, t) + \beta m_b(x, t)$, then $M(x, h) = \alpha M_a(x, h) + \beta M_b(x, h)$.
(b) If $m(x, t) = \alpha \frac{\partial^n m_*(x,t)}{\partial t^n}$, then $M(x, h) = \alpha \frac{(h+n)!}{h!} M\ (x, h + n)$.
(c) If $m(x, t) = \alpha \frac{\partial m_*(x,t)}{\partial t}$, then $M(x, h) = \alpha \frac{(h+1)!}{h!} M\ (x, h + 1)$.
(d) If $m(x, t) = p(x) \frac{\partial^n m_*(x,t)}{\partial x^n}$, then $M(x, h) = p(x) \frac{\partial^n M_*(x,h)}{\partial x^n}$.
(e) If $m(x, t) = p(x) m^2(x, t)$, then $M(x, h) = p(x) \sum_{r=0}^h M\ (x, r) M\ (x, h - r)$.
(f) If $p(x, y) = x^r y^{r^*}$, then $P(k, h) = \delta(k - r, h - r\ ) = \delta(k - r)\delta(h - r\ )$,
where δ is the Kronecker delta. Thus, $u(x, t) = \sum_{h=0} U(x, h) t^h$.

4. The PDTM and the Ivancevic Option Pricing Model

In this subsection, the PDTM will be applied to the model equation (2.1) as follows. According to [16], (2.1) will be subjected to a plane function, $w(S, 0) = f(S)$ as an initial condition, thereby yielding the following initial value problem (IVP):

$$\begin{cases} i\frac{\partial w}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 w}{\partial S^2} + \beta |w|^2 w = 0, \\ w(S, 0) = f(S), \end{cases} \tag{4.1}$$

where $w = w(S, t)$ denotes the option pricing wave function at time t , σ represents a constant or stochastic process as the dispersion frequency volatility coefficient while β is referred to as the Landau coefficient representing adaptive market potential.
For simplicity, we re-expressed (4.1) as:

$$\begin{cases} \frac{\partial w}{\partial t} = i\left(\frac{1}{2}\sigma^2 \frac{\partial^2 w}{\partial S^2} + \beta |w|^2 w\right), \\ w(S, 0) = f(S). \end{cases} \tag{4.2}$$

Transforming (4.2) using PDTM yields:

$$\begin{cases} (k + 1)W(S, k + 1) = i\left(\frac{1}{2}\sigma^2 W_{SS}(S, k) + \beta H(s)\right), \\ W(S, 0) = f(S). \end{cases} \tag{4.3}$$

where $\overline{W}(\cdot)$ denotes the complex conjugate of $W(\cdot)$, and

$$H(s) = \sum_{r=0}^{k} \sum_{n=0}^{k-r} \overline{W}(S,r)W(S,n)W(S,k-r-n).$$

This implies that:

$$\begin{cases} W(S,k+1) = \frac{i}{(k+1)}\left(\frac{1}{2}\sigma^2 W_{SS}(S,k) + \beta H(s)\right), \\ W(S,0) = w(S,0). \end{cases} \quad (4.4)$$

Therefore, for $W(S,\zeta) = W(\zeta)$ and, $k = 0,1,2,3,\ldots$, the following are respectively obtained:

$$\begin{cases} W(1) = i\left(\frac{1}{2}\sigma^2 W_{SS}(0) + \beta\overline{W}(0)W^2(0)\right), \\ W(2) = \frac{i}{2}\left(\frac{1}{2}\sigma^2 W_{SS}(1) + \beta(2\overline{W}(0)W(0)W(1) + \overline{W}(1)W^2(0))\right), \end{cases}$$

$$\begin{aligned} W(3) =& \frac{i}{3}\left(\frac{1}{2}\sigma^2 W_{SS}(2) + \beta(2\overline{W}(0)W(0)W(2) + \overline{W}(0)W^2(1) + 2\overline{W}(1)W(0)W(1) \right. \\ & \left. + \overline{W}(1)W^2(0))\right), \\ W(4) =& \frac{i}{4}\left(\frac{1}{2}\sigma^2 W_{SS}(3) + \beta(2\overline{W}(0)W(0)W(3) + 2\overline{W}(0)W(1)W(2) \right. \\ & \left. + 2\overline{W}(1)W(0)W(2) + \overline{W}(1)W^2(1) + 2\overline{W}(2)W(0)W(1) + \overline{W}(3)W^2(0))\right), \\ W(5) =& \frac{i}{5}\left(\frac{1}{2}\sigma^2 W_{SS}(4) + \beta(2\overline{W}(0)W(0)W(4) + 2\overline{W}(0)W(1)W(3) + \overline{W}(0)W^2(2) \right. \\ & + 2\overline{W}(1)W(0)W(3) + 2\overline{W}(1)W(1)W(2) + 2\overline{W}(2)W(0)W(2) \\ & \left. + \overline{W}(2)W^2(1) + 2\overline{W}(3)W(0)W(1) + \overline{W}(4)W^2(0))\right), \end{aligned}$$

and so on.

4.1. Numerical Illustrative Examples

In this subsection, we consider the following cases for numerical computation:

Case A: Suppose $\beta = 2$, $\sigma = \sqrt{2}$ and $f(S) = e^{2iS}$. Then the corresponding Ivancevic option pricing model is:

$$\begin{cases} \frac{\partial w}{\partial t} = i\left(\frac{\partial^2 w}{\partial S^2} + 2|w|^2 w\right), \\ w(S,0) = e^{2iS}. \end{cases} \quad (4.5)$$

So, applying the PDTM with the parameters in case A through (4.4) gives the following:

$$W(0) = e^{2iS}, \quad W(1) = -2ie^{2iS}, \quad W(2) = -2e^{2iS}, \quad W(3) = \frac{4i}{3}e^{2iS},$$

$$W(4) = \frac{2}{3}e^{2iS}, \quad W(5) = -\frac{4i}{15}e^{2iS}, \quad W(6) = -\frac{4}{45}e^{2iS}, \quad W(7) = \frac{8i}{315}e^{2iS},$$

$$W(8) = \frac{2}{315}e^{2iS}, \quad W(9) = -\frac{4i}{2835}e^{2iS}, \quad \text{and so on.}$$

Whence,

$$\begin{aligned}
w(S,t) &= \sum_{n=0} W(n)t^n \\
&= W(0) + W(1)t + W(2)t^2 + W(3)t^3 + W(4)t^4 + \cdots \\
&= e^{2iS}\left(1 - 2it - 2t^2 + \frac{4i}{3}t^3 + \frac{2}{3}t^4 - \frac{4i}{15}t^5 - \frac{4}{45}t^6 + \frac{8i}{315}t^7 + \frac{2}{315}t^8 - \cdots\right) \\
&= e^{2iS}\left(1 + (-2it) + \frac{(-2it)^2}{2!} + \frac{(-2it)^3}{3!} + \frac{(-2it)^4}{4!} + \frac{(-2it)^5}{5!} + \cdots\right) \\
&= e^{2iS}e^{-2it} \\
&= cos2(S-t) + isin2(S-t).
\end{aligned}$$

Showing that

$$w(S,t) = e^{2i(S-t)} \tag{4.6}$$

satisfies (4.5) is obvious and straightforward.

Case B: Suppose $\beta = 6$, $\sigma = \sqrt{2}$ and $f(S) = 1$. Then the corresponding Ivancevic option pricing model is:

$$\begin{cases} \frac{\partial w}{\partial t} = i\left(\frac{\partial^2 w}{\partial S^2} + 6|w|^2 w\right), \\ w(S,0) = 1 \end{cases} \tag{4.7}$$

with an exact solution:

$$w(S,t) = e^{6it} \tag{4.8}$$

So, applying the PDTM with the parameters in case A through (4.4) gives the following:

$$W(0) = 1, \quad W(1) = 6i, \quad W(2) = -18, \quad W(3) = -36i, \quad W(4) = 54,$$

Table 1: Solutions of Case A (real parts).

t	$Re(w_{ex})$	$Re(w_{PDTM})$	$Rel.\ error$
0.5	0.540302	0.5393051	0.001845079
0.6	0.696707	0.6955375	0.001678611
0.7	0.825336	0.8240410	0.001569058
0.8	0.921061	0.9196927	0.001485569
0.9	0.980067	0.9786790	0.001416230
1.0	1.000000	0.9986486	0.001351400
1.1	0.980067	0.9788051	0.001287565
1.2	0.921061	0.9199398	0.001217292
1.3	0.825336	0.8243994	0.001134811
1.4	0.696707	0.6959928	0.001025108
1.5	0.540302	0.5398392	0.000856558

$$W(5) = \frac{324i}{5}, \quad W(6) = -\frac{324}{5}, \quad W(7) = -\frac{1944i}{35}, \quad W(8) = \frac{1458}{35},$$

$$W(9) = \frac{972i}{35}, \quad W(10) = -\frac{2916}{175}, \quad W(11) = -\frac{17496i}{1925}, \quad W(12) = \frac{8748}{1925},$$

and so on. Whence,

$$
\begin{aligned}
w(S,t) &= \sum_{n=0} W(n)t^n \\
&= W(0) + W(1)t + W(2)t^2 + W(3)t^3 + W(4)t^4 + \cdots \\
&= \left(1 - 18t^2 + 54t^4 - \frac{324}{5}t^6 + \frac{1458}{35}t^8 - \frac{2916}{175}t^{10} + \frac{8748}{1925}t^{12} + \cdots\right) \\
&\quad + i\left(6t - 36t^3 + \frac{324}{5}t^5 - \frac{1944}{35}t^7 + \frac{972}{35}t^9 - \frac{17946}{1925}t^{11} + \frac{52488}{25025}t^{13} + \cdots\right).
\end{aligned}
$$

Remark: The results of Case A and Case B are presented in Tables 1-4 below. We use $Re(w_{ex})$, $Re(w_{PDTM})$, $Im(w_{ex})$, $Im(w_{PDTM})$, and $Rel.\ error$ to denote real part of the exact solution, real part of the approximate solution, imaginary part of the exact solution, imaginary part of the approximate solution, and relative error respectively. In Figure 1 and Figure 2, the real, and the imaginary parts associated to the solution of Case A are graphically presented respectively.

Table 2: Solutions of Case A (imaginary parts).

t	$Im(w_{ex})$	$Im(w_{PDTM})$	$Rel.\ error$
1.1	0.198669	0.198090	0.002915
1.2	0.389418	0.388600	0.002101
1.3	0.564642	0.563618	0.001814
1.4	0.717356	0.716166	0.001660
1.5	0.841471	0.840162	0.001555
1.6	0.932039	0.930665	0.001475
1.7	0.985450	0.984064	0.001406
1.8	0.999574	0.998232	0.001343
1.9	0.973848	0.972604	0.001278
2.0	0.909297	0.908201	0.001206

Table 3: Solutions of Case B (real parts).

| t	$Re(w_{ex})$	$Re(w_{PDTM})$	$Rel.\ error$
0.01	0.059964	0.0599640	0.000000000
0.02	0.992809	0.9928086	4.02897E-07
0.03	0.983844	0.9838437	3.04926E-07
0.04	0.971338	0.9713380	0.000000000
0.05	0.955336	0.9553365	5.23376E-07
0.06	0.935897	0.9358968	2.13699E-07
0.07	0.913089	0.9130889	1.09518E-07
0.08	0.886995	0.8869949	1.12740E-07
0.09	0.857709	0.8577087	3.49769E-07
0.10	0.825336	0.8253356	4.84651E-07
0.11	0.789992	0.7899922	2.53167E-07
0.12	0.751806	0.7518057	3.99039E-07
0.13	0.710914	0.7109135	7.03320E-07
0.14	0.667463	0.6674628	2.99642E-07
0.15	0.621610	0.6216100	0.000000000
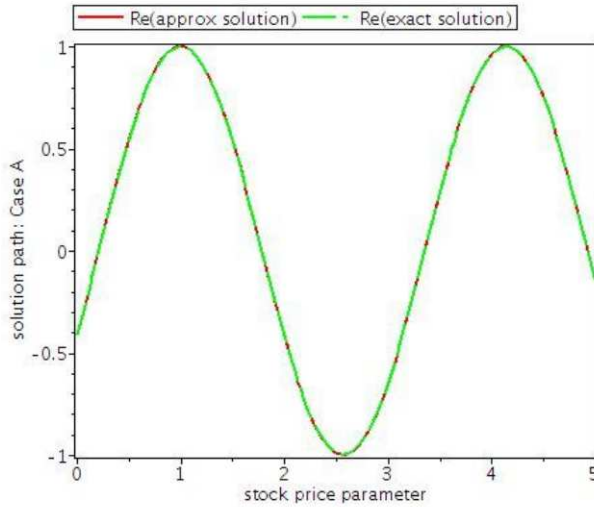
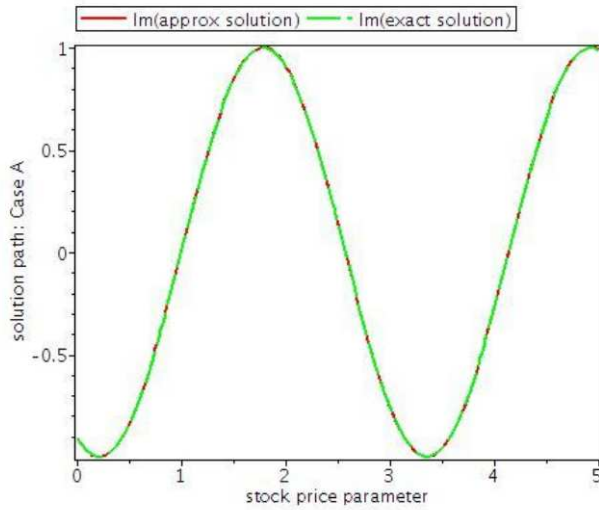Figure 1: Solution graph of real part content of Case A

Figure 2: Solution graph of imaginary part content of Case A

5. Concluding Remarks

In this paper, we considered the Ivancevic option pricing model (IOPM). This nonlinear adaptive-wave model serves as alternative for the classical Black-

Table 4: Solutions of Case B (imaginary parts).

t	$Re(w_{ex})$	$Re(w_{PDTM})$	$Rel.\ error$
0.01	0.059964	0.059964	0.000000
0.02	0.119712	0.119712	1.67E-06
0.03	0.179030	0.179030	2.23E-06
0.04	0.237703	0.237703	1.68E-06
0.05	0.295520	0.295520	6.77E-07
0.06	0.352274	0.352274	5.68E-07
0.07	0.407760	0.407761	1.23E-06
0.08	0.461779	0.461779	4.33E-07
0.09	0.514136	0.514136	0.000000
0.10	0.564642	0.564643	8.86E-07
0.11	0.613117	0.613117	1.63E-07
0.12	0.659385	0.659385	4.55E-07
0.13	0.703279	0.703279	5.69E-07
0.14	0.744643	0.744643	1.34E-07
0.15	0.783327	0.783327	1.28E-07

Scholes option pricing model based on a controlled Brownian motion in an adaptive setting relating to nonlinear Schrdinger equation. By considering cases of nonzero adaptive market potential, analytical solutions of the IOPM by means of a proposed semi-analytical method (projected differential transform method) were obtained. Based on the results, the proposed method is proven to be direct, and effective as the obtained solutions tend rapidly to their exact forms without any form of linearization, perturbation, or discretization.

Acknowledgements

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