

A Dilated Trigonometrically Equipped Algorithm to Compute Periodic Vibrations through Block Milne's Implementation

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Abstract—This paper intends to investigate the use of a dilated trigonometrically equipped algorithm to compute periodic vibrations in block Milne's implementation. The block-Milne implementation is established by developing a block variable-step-size predictor-corrector method of Adam's family using a dilated trigonometrically equipped algorithm. The execution is carried out using a block variable-step-size predictor-corrector method. This system has significant advantages that include the varying step-size and finding out the convergence-criteria and error control. Convergence-criteria and operational mode are discussed to showcase the accuracy and effectuality of the proposed approach.

Keywords—dilated trigonometrically equipped algorithm; block-Milne's device; convergence-criteria; max errors; principal-local-truncation-error

NOMENCLATURE

BMD: errors in BMD for computing test problems 1, 2 and 3.
 M_{employed} : approach employed.
 $\text{Max}_{\text{errors}}$: the magnitude of the max errors of BMD.
 C_{criteria} : convergence-criteria.
 TSDM: errors in TSDM (trigonometrically-fitted Second derivative method) for numerical tested problem 1, [12].
 BHMTB: errors in BHMTB (block hybrid method with trigonometric basis) for numerical tested problem 1 [10].
 BHT: errors in BHT (block hybrid trigonometrically fitted of $\delta = 10^{-6}$) for numerical tested problem 1, 2 and 3 [13].
 BHTRKNM: errors in BHTRKNM (block hybrid trigonometrically fitted Runge-Kutta-Nystrom method of $\delta = 10^{-6}$) for numerical tested problem 1, 2 and 3 [14].
 BHTFM: errors in BHTFM (block hybrid trigonometrically fitted method) for numerical tested problem 2 [11]

I. INTRODUCTION

The dilated trigonometrically equipped algorithm is virtually among the utilitarian procedures which constitute block-Milne's implementation for approximating solutions of periodic vibrations. A trigonometrically equipped algorithm is more effective compared to non-fitted methods especially when the outcome exhibits periodical vibrations [1, 2]. The Block

Milne's implementation is of significant importance for documenting block variable-step-size predictor-corrector methods, convergence-criteria and check errors [3-6]. This composition aims at approximating the solution of periodic vibrations of class: [1, 6]

$$g'' = f(v, g), g(v_0) = g_0, g'(v_0) = g_0' \quad (1)$$

for $v \in [v_0, X]$

where $f: R \times R^c \rightarrow R^c$, c is the dimension of the physical system. Presuming $f \in R$ equals to the differentiability to sufficient degree on $u \in [v_0, X]$ and meets a worldwide Lipchitz precondition, i.e., $L \geq 0 \exists$

$$|f(v, g) - f(v, \bar{g})| \leq L|g - \bar{g}|, \forall g, \bar{g} \in R.$$

Underneath this presumption, (1) ascertains the universal and singularity place as $v \in [u, \rho]$ which is similarly considered to meet the Weierstrass theorem [6-9]. In particular, periodic vibrations often spring up in areas of scientific knowledge like Newtonian mechanics, uranology, quantum theory, control theory, electric circuits and biological science. Various approximation methods have been established. Clear cut methods instituted on trigonometrically equipped algorithms may be realized [1, 10-18]. Their execution is carried out while adopting fixed step-size strategies [1, 10-22]. The motive governing a trigonometrically equipped algorithm is inherent in the fact that if the frequency is acknowledged in advance, this method turns more beneficial compared to the multinomial established methods [1, 10-18].

In addition, a block Milne's implementation is primarily employed for changing step-size, fixing convergence-criteria and check ciphered errors [3-6, 23-26]. The main target of this paper is to implement a dilated trigonometrically equipped algorithm for ciphering periodic vibrations where the frequency of the outcome is anticipated in advance. This approach possesses advantages like varying step-size, deciding the convergence criteria and error control [7-8, 24-25, 27-29]. The dilated trigonometrically equipped algorithm has extra advantages which includes reduced computation costs and

improved accuracy. Milne’s implementation is examined as a broad prospect of the predictor-corrector method on account of the computational benefits. This approach employs components like Adams type class, block predictor-corrector pair of similar order and principal local truncation errors as mentioned in [7-8, 24-25, 27-29].

Definition 1: *d – block, b – stage method*. Whenever *k* refers to block size and *h* as pace size, then block size in time is *bh*. Let *m = 0,1,2, …* form the block number and let *t = mb*, then the *d – block, b – point* method can be composed in the next class:

$$Y_t = \sum_{\mu=1}^d A_{\mu} Y_{t-\mu} + h \sum_{\mu=0}^d B_{\mu} F_{t-\mu}. \tag{2}$$

For $Y_t = [y_{t+1}, \dots, y_{t+i}, \dots, y_{t+r}]^T, F_t = [f_{t+1}, \dots, f_{t+i}, \dots, f_{t+r}]^T$

A_{μ} and B_{μ} are $b \times b$ constants matrices [6, 17, 30]. Hence, taking off from the over account, a block method has computational benefits. For each practical real-life program, the terminate product is assessed to a greater extent at the same time. Utilizing these methods can permit faster outcomes of the problem which can be handled to generate the sought after accuracy [10-12, 16-17].

II. METHODS

Block-Milne’s device is a combination of Adams-Bashforth *g – step* (predictor) method and Adams-Moulton *g – 1 – step* (corrector) method of the same order. This combination can be of the form:

$$g(v) = \sum_{l=0}^k \alpha_l g_{t-l} + h^2 \sum_{l=0}^k \beta_l(\delta) f_{t-l}, \tag{3}$$

$$g(v) = \sum_{l=0}^k \alpha_l g_{t-l} + h^2 \sum_{l=1}^k \beta_l^*(\delta) f_{t+l}. \tag{4}$$

Equations (3) and (4) represent the Adams family of block Milne’s device with $\delta = wh$, $\beta_n(\delta), n = 0, 1, 2$ comprising invariants that depend on the varying step-size and frequency. Observing that g_{t+l} is the numeric approximate of the analytical results $g(v_{t+l})$ i.e. $g(v_{t+l}) \approx g_{t+l}$, and $f_{t+n} \approx f(v, g_{t+n})$ owning $n = 0, 1, 2$. To arrive at (3) and (4), the trigonometrically fitted method is written as the dilated trigonometrically equipped algorithm in which runs by looking forward to approximate the analytical result $g(v)$ on distinct time intervals of $[v_t, v_{t-j}]$ through with the interpolating subprogram of (5):

$$g(v) = \alpha_0 + \alpha_1 v + \alpha_2 v^2 + \alpha_3 \sin(wv) + \alpha_4 \cos(wv) \tag{5}$$

Revising (4) produces the dilated trigonometrically equipped algorithm as:

$$g(v) = \alpha_0 + \alpha_1 \left(\frac{v-v_n}{h}\right) + \alpha_2 \left(\frac{v-v_n}{h}\right)^2 + \alpha_3 \left(w \left(\frac{v-v_n}{h}\right) - \frac{w^3}{6} \left(\frac{v-v_n}{h}\right)^3\right) + \alpha_4 \left(1 - \frac{w^2}{2} \left(\frac{v-v_t}{h}\right)^2 + \frac{w^4}{24} \left(\frac{v-v_t}{h}\right)^4\right), \tag{6}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and α_4 are invariants needed to be determined in a especial direction. Presume the precondition that (6) matches with the accuracy output at definite length of time v_t, v_{t-n} to get the approximant of (7):

$$y(v_t) \approx y_t, \quad y(v_{t-n}) \approx y_{t-n}. \tag{7}$$

Demanding that the interpolating function (6) conforms to (1) at the levels $v_{i+k}, k = 0, 1, 2$ we get the next approximates as

$$g'(v_{t+k}) \approx f_{t+k}, \quad g''(v_{t+k}) \approx f_{t+k}, \quad k = 0, 1, 2 \tag{8}$$

Linking (7) and (8) will lead to a fivefold system of equations which gives birth to $Av=d$. Computing the systems of equation adopting Mathematica 9 kernel 64 to obtain $v_k, k = 0, 1, 2, 3, 4$ and subbing the measures of v_k 's into (6) will yield the continuous block Milne’s device. Assessing the continuous block Milne’s device at some preferred points of $v_{t+k}, k = 1, 2, 3$ will invent the Milne’s device as

$$g(v) = g_l + g_{l-1} + h^2(\beta_1(w, v)f_{l+1} + \beta_2(w, v)f_{l+2} + \beta_3(w, v)f_{l+3}) \tag{9}$$

where w is the frequency, and $\beta_1(w, v), \beta_2(w, v)$, and $\beta_3(w, v)$ are uninterrupted invariants [5, 10-14].

A. Forming Convergence Criteria for Block Milne’s Device

In launching operation of block-Milne’s device, Adams-Bashforth *g – step* and Adams-Moulton *g – 1 – step* approaches are employed in a predictor-corrector approach [3-6, 23-25]. Mixed block-Milne’s device proves the feasibility to determine principal-local-truncation-error in predictor-corrector approach in absence of finding higher differential coefficient, $g(v)$. Taking for granted that $\tilde{q} = \bar{q}$, where \bar{q} and \tilde{q} showcase the order of the predictor and corrector. Right now, in approach of order \tilde{q} , the investigation of the block-Adams-Bashforth *g – step* brings forth principal-local-truncation-errors.

$$\tilde{C}_{q+5}^{[1]} h^{q+5} g^{(q+5)}(\bar{v}_n) = g(v_{l+1}) - g_{l+1}^{[i_1]} + O(h^{q+6}),$$

$$\tilde{C}_{q+5}^{[2]} h^{q+5} y^{(q+5)}(\bar{v}_n) = y(v_{l+2}) - g_{l+2}^{[i_2]} + O(h^{q+6}) \tag{10}$$

$$\tilde{C}_{q+5}^{[3]} h^{p+5} g^{(p+5)}(\bar{v}_n) = g(v_{l+3}) - g_{l+3}^{[i_3]} + O(h^{q+6}).$$

A corresponding break down of block-Adams-Moulton *g – 1 – step* brings about principal-local-truncation-errors:

$$\bar{W}_{q+5}^{[1]} h^{q+5} g^{(q+5)}(\bar{v}_n) = g(v_{l+1}) - g_{l+1}^{[r_1]} + O(h^{q+6}),$$

$$\bar{W}_{q+5}^{[2]} h^{q+5} g^{(q+5)}(\bar{v}_n) = g(v_{l+2}) - g_{l+2}^{[r_2]} + O(h^{q+6}) \tag{11}$$

$$\bar{W}_{q+5}^{[3]} h^{q+5} g^{(q+5)}(\bar{v}_n) = g(v_{l+3}) - g_{l+3}^{[r_3]} + O(h^{q+6}),$$

where $\tilde{C}_{q+5}^{[1]}, \tilde{C}_{q+5}^{[2]}, \tilde{C}_{q+5}^{[3]}, \bar{W}_{q+5}^{[1]}, \bar{W}_{q+5}^{[2]}$ and $\bar{W}_{q+5}^{[3]}$ are existing as independent entities of the step-size h and $g(v)$ behaves as the result to the differential coefficient fulfilling the initial consideration $g(v_n) \approx g_n$ [3-6, 23-25]. To move ahead, the precondition for small assesses of h is reached as $g^{(5)}(\bar{v}_n) \approx g^{(5)}(v_n)$, and the dominance of block-Milne’s device banks right away on this condition. Further simplification of the principal local truncation errors of (10) and (11) while neglecting terms of $O(h^{q+6})$ order, makes it easy to achieve computation of principal- local-truncation-errors of block-Milne’s device as:

$$\bar{W}_{q+5}^{[1]} h^{q+5} g^{(q+5)}(\bar{v}_n) \approx \frac{\bar{W}_{q+5}^{[1]}}{\tilde{C}_{q+5}^{[1]} - \bar{W}_{q+5}^{[1]}} [g_{n+l}^{[i_1]} - g_{n+l}^{[r_1]}] < \omega_1,$$

$$\bar{W}_{q+5}^{[2]} h^{q+5} g^{(q+5)}(\bar{v}_n) \approx \frac{\bar{W}_{q+5}^{[2]}}{\tilde{C}_{q+5}^{[2]} - \bar{W}_{q+5}^{[2]}} [g_{n+l}^{[i_2]} - g_{n+l}^{[r_2]}] < \omega_2, \tag{12}$$

$$\bar{W}_{q+5}^{[3]} h^{q+5} g^{(q+5)}(\bar{v}_n) \approx \frac{\bar{W}_{q+5}^{[3]}}{\bar{c}_{q+5}^{[3]} - \bar{W}_{q+5}^{[3]}} [g_{n+l}^{[i_3]} - g_{n+l}^{[r_3]}] < \omega_3.$$

Mentioning the affirmations that $g_{n+l}^{[i_1]} \neq g_{n+l}^{[r_1]}$, $g_{n+l}^{[i_2]} \neq g_{n+l}^{[r_2]}$ and $g_{n+l}^{[i_3]} \neq g_{n+l}^{[r_3]}$ are the predicted and corrected approximates which are given by the block Milne's device of order p, while $\bar{W}_{q+5}^{[j]} h^{q+5} g^{(q+5)}(\bar{v}_n)$, $j = 1, 2, 3$ are the principal-local-truncation-errors. ω_1, ω_2 and ω_3 are the boundaries of the convergence-criteria of block-Milne's device. Moreover, the estimates of the principal local truncation error (12) are utilized to determine whether to go for the results of the current step or to reconstruct the step with a slighter variable-step-size. The step is established on a trial run as defined by (12) [3-6, 23-25]. The principal-local-truncation-error (12) is the convergence-criteria of block-Milne's, device distinctly seen as block-Milne's device (estimate) for conforming to convergence.

III. NUMERICAL EXAMPLES

Three selected numerical test problems were viewed and solved utilizing the proposed approach (denoted as BMD in this paper) at different convergence criteria of $10^{-3}, 10^{-5}, 10^{-7}, 10^{-9}$ and 10^{-11} [10-14]. Programming codes on block Milne's are compiled employing Mathematica 9 kernel 64. These codes are executed in a block by block mode together with the block Milne's device.

Numerical test problem 1. Consider the inhomogeneous IVP:

$$y''(v) = -100y + 99\sin(v), \quad y(0) = 1, \quad y'(0) = 11, \quad 0 \leq v \leq 1000.$$

Solution:

$$y(v) = \cos(10v) + \sin(10v) + \sin(v)$$

Numerical test problem 2. Consider the nonlinear Duffing equation: $y'' + y + y^3 = B\cos(\Omega v)$, $y(0) = C_0$, $y'(0) = 0$.

Solution:

$$y(v) = C_1 \cos(\Omega v) + C_2 \cos(3\Omega v) + C_3 \cos(5\Omega v) + C_4 \cos(7\Omega v) \text{ where } \Omega=1.0, B=2 \times 10^{-3}, C_0=0.200426728069, C_1=0.20017947753, C_2=0.246946143 \times 10^{-3}, C_3=0.304016 \times 10^{-6}, C_4=0.374 \times 10^{-9}. \text{ Choose } w=1.01$$

Numerical test problem 3. Consider the harmonic oscillator with frequency Ω and small perturbation δ .

$$y'' + \delta y' + \Omega^2 y = 0, \quad y(0) = 0, \quad y'(0) = -\frac{\delta}{2}, \quad 0 \leq v \leq 1000.$$

$$\text{Solution: } y(x) = e^{\left(\frac{\delta}{2}\right)v} \cos\left(\Omega^2 - \frac{\delta^2}{4}\right), \text{ where } \Omega = 1 \text{ and } \delta = 10^{-6}.$$

IV. RESULTS

The numerical results of the implementation of the dilated trigonometrically equipped algorithm on block Milne's device for ciphering periodic vibrations are exhibited in Table I.

TABLE I. MILNE'S IMPLEMENTATION

M _{employed}	Max _{errors}	C _{criteria}
TSDM	1.7e-03	10 ⁻³
BHTFM	1.2e-03	
BHT	1.9e-03	
BHTRKKNM	2.14e-03	
BMD	9.15202e-04	10 ⁻³
BMD	2.79178e-04	
BMD	1.83035e-04	
TSDM	2.7e-05	10 ⁻⁵
BHTFM	1.4e-05	
BHMTB	3.9e-05	
BHTRKKNM	5.98e-05	
BHTRKKNM	2.06e-05	
BMD	1.05006e-07	10 ⁻⁵
BMD	3.75372e-08	
BMD	3.34795e-06	
TSDM	1.0e-07	10 ⁻⁷
BHTFM	1.5e-07	
BHMTB	1.4e-07	
BMD	1.05933e-10	10 ⁻⁷
BMD	3.76048e-11	
BMD	3.4074e-09	
TSDM	6.3e-09	10 ⁻⁹
BHTFM	8.7e-09	
BHTFM	1.1e-09	
BHTRKKNM	4.67e-09	
BMD	1.07025e-13	10 ⁻⁹
BMD	3.9968e-14	
BMD	3.40017e-12	
BHT	9.7e-11	10 ⁻¹¹
BHT	6.7e-11	
BMD	4.88498e-15	10 ⁻¹¹
BMD	1.04361e-14	
BMD	3.57492e-14	
BHT	4.3e-13	10 ⁻¹³
BMD	1.9984e-15	10 ⁻¹³
BMD	5.10703e-15	
BMD	2.39808e-14	
TSDM	3.3e-03	10 ⁻³
BHTFM	1.3e-03	
BMD	3.60282e-06	10 ⁻³
BMD	7.81099e-06	
BMD	1.31395e-04	
TSDM	6.4e-05	10 ⁻⁵
BHTFM	5.6e-05	
BHT	7.7e-05	
BHTRKKNM	7.52e-05	
BMD	3.11695e-10	10 ⁻⁵
BMD	7.32379e-10	
BMD	1.20413e-08	
TSDM	1.0e-07	10 ⁻⁷
BHTFM	1.4e-07	
BHTRKKNM	1.34e-07	
BMD	3.14471e-14	10 ⁻⁷
BMD	7.38853e-14	
BMD	1.20062e-12	
BHTRKKNM	8.11e-09	10 ⁻⁹
BMD	1.38778e-16	10 ⁻⁹
BMD	3.60822e-16	
BMD	1.66533e-15	
BHT	1.23e-11	10 ⁻¹¹
BHTRKKNM	7.13e-11	
BMD	4.06231e-13	10 ⁻¹¹
BMD	9.65783e-13	
BMD	4.14513e-12	

V. CONCLUSION

Computed results demonstrated that the BMD is attained with the aid of the convergence-criteria. These convergence criteria decide whether the result is accepted or the algorithm is repeated. The results also establish that the performance of the BMD allows improved maximum errors compared with the TSDM, BHMTM, BHT, BHTRKNM and BHTFM at all examined convergence criteria of 10^{-3} , 10^{-5} , 10^{-7} , 10^{-9} , 10^{-11} and 10^{-13} [10-14]. Thus, it can be concluded that the devised method is suitable for working out periodic vibrations dealing with non-stiff and stiff ODEs. BMD performs better when compared to the existing methods for reasons pointed out above. Further work will deal with carrying out the block Milne's device on dilated exponentially fitted algorithm.

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