

## The GSP as a technical and psychological-symbolic tool: The case of a lateral entry teacher

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Este es un estudio preliminar a un experimento de enseñanza que fue llevado a cabo durante un semestre académico. Los participantes fueron un estudiante de educación matemática, una aspirante a obtener su licencia para enseñar en la escuela media a través de su práctica de enseñanza y cursos universitarios, y dos maestras, ya tituladas y enseñando en la escuela secundaria. El propósito del experimento fue tratar de entender cómo maestros, que poseen cierto conocimiento de objetos geométricos, utilizan el programa *Geometer's Sketchpad* (GSP) para expandir y consolidar dicho conocimiento. El estudio utilizó tareas semi-estructuradas especialmente diseñadas para el GSP con la intención de motivar la triada comunicativa entre el profesor, el GSP y el investigador, además de dar al profesor la libertad de explorar situaciones geométricas para generar conjeturas, investigarlas y luego probarlas. En este artículo analizamos cómo la profesora aspirante a obtener su licencia para enseñar (aquí denominada con el pseudónimo de Susan) usa el GSP para resolver una tarea cuyo objetivo fue el de investigar varias propiedades de los trapecios isósceles. El análisis muestra que Susan usó el GSP no solamente como una herramienta técnica, sino que la constituyó en una herramienta simbólica en el proceso de conceptualizar algunas de las propiedades de trapecios isósceles y la demostración geométrica de ellas.

One academic semester long pilot teaching-experiment on the learning of geometry was conducted with one pre-service teacher, one lateral entry teacher, and two in-service teachers. The purpose of the pilot teaching-experiment was to understand *how* learners, who already have some knowledge of geometric objects, are able to reorganize and broaden their knowledge when using the *Geometer's Sketchpad* (GSP). The guiding

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principle of the teaching-experiment was to use semi-structured tasks especially designed for the GSP to foster the three-way-interaction among the student-teacher, the GSP, and the teacher-interviewer. The objective of each task was to allow each participating student-teacher the freedom to explore geometric situations, to make conjectures, and to prove them. In this article, we analyze *how* the lateral entry teacher, here with the pseudonym of Susan, solved a task which purpose was to investigate isosceles trapezoids and some of the properties. The analysis indicates that Susan used the GSP as a technical tool but also constituted it into a psychological-symbolic tool in the process of conceptualizing and proving some of the properties of isosceles trapezoids.

### The GSP as a mediational tool

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Several research studies (among others Mariotti, 2000; Jiang, 2002; Christou, Mousoulides, Pittalis, and Pitta-Pantazi, 2004; De Villiers, 2004) have demonstrated that dynamic geometry environments *mediate* between the learners' physical actions and their conceptualizations. Now, the first question in front of us is if the GSP is *per se* a mechanical tool mediating and guiding the thinking of the user or/and if the user comes to act on the tool according to her/his own conceptualizations. The other question is if the GSP changes the nature of the interaction between the teacher and the students making it triadic and more meaningful. The analysis presented here indicates that the GSP fosters a triadic communication among the learner, the tool and the teacher-interviewer, and that such communication greatly enhances the learner's conceptualization of geometric objects. We first argue, from a theoretical perspective, that the GSP, like any other dynamic geometry environment, is not only a technical tool but it is also a symbolic or psychological tool. Through the analysis, then, we document that with the aid of the teacher-interviewer, Susan transformed this tool into a psychological-symbolic tool to serve her conceptualizing goals, and that in the process, her level of deduction evolved from incomplete and tacit to more explicit and better argued.

The concepts of *tool* and *tool mediation* are central to the vygotskian perspective on cognitive development. Vygotsky considers two kinds of tools: technical (or material) tools and psychological (or symbolic) tools. Technical tools are directed to produce a set of changes on the object(s) they are applied to. In contrast, psychological tools direct the mind and behavior of the individuals (Kozulin, 1990). What kind of tool could be a dynamic geometry environment like the GSP? The GSP –or any other dynamic geometry

environment— can be considered as an amalgamation of technical and psychological tools, because it intertwines kinesthetic, visual, and conceptual activity and mediates the transformation of physical activity into conceptual activity. That is, any dynamic geometry environment has a symbolic mediatory purpose because it facilitates, for the learner, the emergence of conceptual geometric activity from his/her kinesthetic-visual activity. In other words, the GSP is a psychological-symbolic tool that induces the learner both to observe the variants and invariants of successive drawings as they are generated in the dragging process and to gain insight into the *generalized figure*, which is nothing else than the class of all drawings with the same structural properties regardless of their size and location in space (Sáenz-Ludlow and Athanasopoulou, 2008).

Vygostky points out that the principle from “action to thought” should be applied not only to the development of intelligence, but also to the functioning of intelligence (Wertsch, 1985). This principle implies, in our particular case, the use of the GSP, that there is a considerable distance between *learning how* to operate the tool to make drawings on the screen and *becoming aware* of the geometric structure of those drawings (i.e., becoming aware of the geometric figure). Drawing from Plato, Laborde (1993) made the differentiation *between drawing and figure*; *drawing* refers to the material entities on the screen (i.e., drawings as products of dynamic geometry environments) and *figure* refers to the theoretical object. Jones (2000) retakes this differentiation and explains that in terms of dynamic geometry packages, a *drawing* can be a juxtaposition of geometrical objects resembling closely the intended construction (something that can be made to “look right”). Although, *drawings* may *look like* an intended *figure* they could lack the right relationships among the constitutive elements. This can also be restated by saying that a *figure* captures the relationships between its constituent geometric objects in such a way that any drawing of the *figure* is invariant when any basic geometric object used in its construction is dragged. In other words, when the *drawing* passes the *dragging test* it becomes a concrete instantiation of the *figure* as a concept.

Something similar happens between oral and written language. A meaning could be conveyed orally, even when syntactical elements of the language are not used rigorously because all kind of gestures and voice modulation could also become meditational means. However, in written language, the same meaning cannot be conveyed if the structural elements of the language are not rigorously used. Thus, in an analogous manner, we could say that meanings in written natural language, to be conveyed properly, have to pass the *writing test*.

Hence, constructing a *drawing* (i.e., an icon of a geometric figure) using the capabilities of the GSP does not necessarily entail the understanding of the structural properties of the geometric elements of the *figure* and the relationships of the actual *figure* with other geometric figures. However, certain drawings in the GSP could be *semi-structured* in the sense that, when dragged induce not only the generalization of a particular geometric figure, but also the generalization of several geometric figures. An example of a semi-structured drawing is given in Figure 1 (see this figure in the section about methodology) because, when dragged, it produces not only drawings of trapezoids but also drawings of different quadrilaterals. These *semi-structured drawings* can also be used as instructional tasks to determine which relationships among certain elements of these drawings would produce drawings of only one particular geometric figure.

Thus, conceptualizing a *figure* is, in essence, a process of de-contextualization made, possible by the dragging capability of dynamic geometric environments, among those the GSP. The passage from action to concept or in our particular case, the passage from *drawing* to *generalized figure*, is equivalent to the passage from the concrete and particular to the general and abstract. This passage is nothing else than an instance of the vygotskian *principle of de-contextualization through mediational means* (Werstch, 1985).

De-contextualization is a process whereby the meaning of signs (in this case drawings as iconic signs) becomes less and less dependent on spatio-temporal contexts (size and space location of a sequence of drawings) and more and more dependent on the structural properties of the constituent elements of those drawings (i.e., the conceptualization of a *generalized figure* as a more structured sign or symbol, as Peirce calls them).

Dragging actions in the GSP generate a sequence of geometric drawings that facilitate the emergence of conjectures and insights into the formation of geometric arguments to accept or reject those conjectures. Thus, the GSP as a *technical* and *psychological-symbolic* tool facilitates not only the passage from iconic drawings to geometric figures (as symbolic geometric objects), but it also mediates the emergence of mental processes to construct geometric arguments. In the process, the GSP facilitates a *cognitive continuity* that goes from the perception of iconic drawings to the conceptualization of symbolic figures and to the emergence of geometric arguments. This continuity is accompanied with emerging processes of abduction, induction, and deduction. The passage from *perceiving-and-seeing* to *conjecturing-and-proving* is essential to the teaching-learning of geometry. The GSP mediates the development of the geometric reasoning of the learners, which is one of the main goals for the teaching of geometry espoused

by the National Council of Teachers of Mathematics (NCTM Principles and Standards, 2000).

In addition, the GSP, as a *technical and psychological-symbolic tool*, not only mediates between the learner and the computer, but it also mediates the communication between the teacher and the learner. The three-way interaction between learner, computer and teacher, allows the sharing of interpretations and the discussion of ideas and it provides the teacher with opportunities to challenge students' interpretations or misinterpretations (usually called errors). Errors or misinterpretations are part and parcel of processes of learning. Leont'ev and Brousseau call our attention on this issue. They consider errors and interactions with others as essential in the continuity of sense-making in the learning process.

“Man learns from errors and still more from the successes of other people” (Leont'ev, 1970, p. 123). “Errors are not erratic or unexpected; the error is a component of the meaning of the acquired piece of knowledge” (Brousseau, 1997, p. 82).

In fact, errors and interactions with others are stopping-to-reflect opportunities in the refining process of knowing. The GSP, as a mediating technical and psychological-symbolic tool, re-defines the channels for discussion and communication between teacher, learners and computer, and it provides opportunities for the students to modify their interpretations and misinterpretations of geometric objects. In this interaction, learners come to refine their geometric knowledge and to become aware of their own mental actions. The re-definition of the channels of communication also implies a shift from the teacher's monologue to the student-teacher dialogue and the concomitant shift of authority from “*domination and obedience to negotiation and consent*” (Amit and Fried, 2005, p. 164). This new way of looking at the student-teacher communication also shifts the conceptual agency of the teacher to the conceptual agency of the student by transforming students' participation on their own learning from *passive* to *active*. That is, the GSP makes possible a teaching-learning environment that is student-centered. Students' active participation on their own learning, contributes to transform their geometric reasoning into a habit of mind, and like any habit, it must be developed through a consistent way of reasoning in many contexts and from the earliest grades of their mathematical schooling. This is also another goal for the teaching of geometry espoused by the National Council of Teachers of Mathematics (Principles and Standards of School Mathematics, NCTM, 2000).

In summary, the GSP could be used as an effective technical and psychological-symbolic tool with the potential to promote student-centered tea-

ching-learning styles focused on the development of independent learners that come to refine their understanding and to trust the authority of their own reasoning.

## Literature review

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Several research studies have concentrated on elementary and secondary school pre-service teachers to give them the opportunity to experiment by themselves ways of learning geometry using dynamic environments. Among these studies are those conducted by Mariotti, 2000; Jiang, 2002; Christou, Mousoulides, Pittalis and Pitta-Pantazi, 2004; and De Villiers, 2004. Other studies have concentrated on the geometric progress of individual students. Among these studies is that of Choi-Koh (1999).

Christou, Mousoulides, Pittalis and Pitta-Pantazi (2004) did a study with three pre-service primary school teachers. They used the GSP as a mediatory tool for the exploration of plane geometric figures. In this study, pre-service teachers explored, recognized and analyzed the properties of kites and then used these properties to construct this figure in different ways. Two of the three teachers used the property of perpendicularity of its diagonals and the third used the property of two pairs of adjacent equal sides. The researches extended the task to investigate the kind of quadrilaterals formed by connecting the midpoints of the adjacent sides of kites. The pre-service teachers conjectured that the mid-point quadrilateral was a rectangle and also were able to prove it. This study indicated that both, the GSP and appropriate guiding questions on the part of the teacher, can propel students' conceptualizations of conjectures and their proofs.

Jiang (2002) interviewed two secondary school mathematics pre-service teachers using the GSP environment and emphasizing the making of conjectures and the process of proving or disproving them. These teachers came to the realization that their students could also have a conceptual understanding on geometry, as they did, if they were given the opportunity to use this tool to explore geometric situations.

De Villiers (2004) also conducted a study with pre-service secondary-school mathematics teachers and used the GSP as a mediatory tool. In his study, pre-service teachers started with different definitions of isosceles trapezoids, proved their properties and constructed them in different ways, according to the definition. Pre-service teachers compared different constructions and manifested their preference for the axis-of-symmetry definition. Then, they were also asked to define isosceles trapezoids and explore their

properties. Because of the dragging capability of the GSP, students were able to construct generalizations as well as specializations dragging vertices or line segments. The researcher indicated both, the use of the GSP and the use of structured activities, at corresponding Van Hiele levels, provide valuable contexts for discoveries, explanations, justifications and proofs that could be incorporated in the high-school geometry curriculum.

Choi-Koh (1999) investigated the development of a secondary-school student as he passed from the intuitive to the deduction Van Hiele levels using the GSP as a mediatory tool. This researcher indicated that by means of the GSP environment and open-ended questions on the part of the teacher, the student passed from intuitive level to the analytical level, then to the inductive level, to finally achieve the deductive level. According to the report, the student used the GSP as a tool not only for verification purposes, but also for the construction of counterexamples. The report also indicated that the geometry software provided an interactive environment that mediated and shaped the student's constructions. Such an environment also fostered the student's ability to make conjectures and the improvement of his geometric thinking as he tried to solve open-ended problems prepared by the researcher.

The research studies mentioned above indicate the value of the GSP as instructional tool to establish a communication between the student and the teacher. However, for this interaction to be effective in the teaching-learning process of geometry, it is also necessary both, the guiding role of the teacher and well structured geometric tasks, appropriate to this dynamic environment and to the geometric level of the students. Mariotti (2000), for example, emphasizes the role of the teacher as fundamental to enhance the mathematical discussion to guide the evolution of student's construction of geometric meanings in a dynamic environment. Choi-Koh (1999) emphasizes the importance of the appropriateness of the tasks for the GSP environment to link both, students' intuitions and their ability to analyze geometric objects in order to generate conjectures and to construct geometric proofs.

## Methodology

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### ***Teaching-experiment***

The teaching-experiment methodology is based on long-term interaction between teacher and students to follow their conceptual evolution and cognitive manifestations (Steffe, 1980; Cobb and Steffe, 1983). This pilot teaching-experiment focused on teachers' conceptions of geometric objects,

the construction of those objects using the GSP, their ability to make conjectures and to prove them. Two researchers conducted the experiment; one was the teacher-interviewer, and the other, the active participant observer. We followed two guiding principles: (a) the use of semi-structured tasks for the GSP environment; and (b) the indirect but guiding role of the teacher-interviewer by means of open-ended questions that gave learners the freedom to explore, to make conjectures, to investigate them and to prove them. All the tasks for the pilot study were prepared before hand and some of them were re-designed according to the geometric needs of the students.

Given the central role that congruence and similarity concepts play in euclidean geometry, special attention was paid to congruence and similarity of triangles at the beginning of the study, to make sure that the participating teachers understood the criteria for congruence and similarity and their implications.

This pilot teaching-experiment lasted one academic semester with one pre-service middle school teacher, one lateral entry middle school teacher and two in-service high school teachers. Twenty one (21) one-to-one interviews were conducted; teachers were interviewed on a weekly basis and each interview lasted, on the average, 90 minutes. Here we analyze one interview with the middle school lateral entry teacher (with the pseudonym of Susan) that lasted 120 minutes.

### ***A semi-structured task***

In the GSP dynamic environment, the given task presented the semi-structured drawing ABCD with ABIICD (Figure 1).

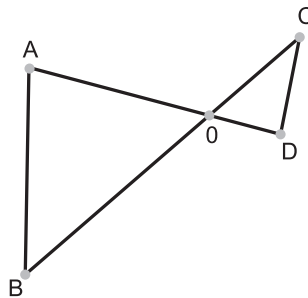


Figure 1. Semi-structured drawing ABCD with ABIIC

Dragging vertices or sides of this drawing leads to the formation of drawings of different quadrilaterals. The purposes of the task were (a) to *drag* vertices and sides of the given drawing to observe *which* kinds of quadrilaterals could be observed; (b) to investigate *how* the given drawing could be trans-



formed into an isosceles trapezoid; (c) to investigate and prove properties of isosceles trapezoids; and (d) to investigate *how* the knowledge of these properties could be useful in the transformation of the given drawing into an isosceles trapezoid. Here we analyze only the part of the interview that is relevant to isosceles trapezoids.

## Analysis

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### ***Initial state of Susan's knowledge about isosceles trapezoids***

Susan recognized the obvious triangles OAB and OCD in Figure 1, and also a trapezoid if one were to connect the segments AC and BD. She knew that a quadrilateral with only one pair of parallel sides was a trapezoid. The following question asked was about what types of trapezoids she knew, and she answered "*isosceles trapezoids*". She also took as well known facts that isosceles trapezoids have the pair of non-parallel sides congruent, its diagonals congruent and the two pairs of base angles congruent, and she used these facts, as if they were definitions. She did not differentiate between the definition and the properties of isosceles trapezoids. It is also important to note that Susan knew how to use the measuring capability of the GSP and she had the tendency to give justifications congruence of segments by measuring their lengths and congruence of triangles by measuring their areas. She seemed to have not used the GSP for purposes of exploration. It could be said that Susan was at level 1 of the Van Hiele model (i.e., level of analysis). At this level, the learner is able to recognize some properties of the figure, although the interrelationship between the definition of a figure and its properties appears not to be there and neither is the conceptual organization of those properties to draw conclusions about similarities and differences of a given figure with respect to other geometric figures.

### ***Constructing isosceles trapezoids from a semi-structured drawing***

The teacher-interviewer probed Susan's knowledge about isosceles trapezoids and challenged her to construct an isosceles trapezoid using the given drawing (Figure 1). Her very first reaction was to measure the non-parallel sides AC and BD but immediately she switched into the equality of diagonals and measured them. Although, Susan described at once two properties of isosceles trapezoids (congruent diagonals and congruent non-parallel sides), she was not able to apply any of them to transform the given trapezoid into one that were isosceles. In the beginning, she dragged different vertices and segments unsuccessfully. She soon abandoned this strategy and decided to construct an isosceles trapezoid from scratch, using

what she called the “*mirror method*”. She said that she used this method for constructing isosceles trapezoids in a project she did at the school. She took some moments to bring the construction to memory.

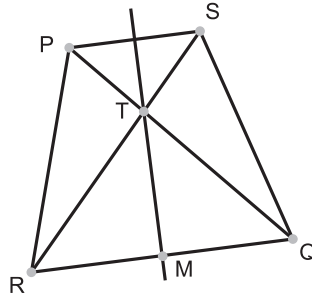


Figure 2. Two line segments with a common vertex,  $PM$  and  $RM$

Then, Susan constructed her own figure. She started from two line segments with a common vertex,  $\overline{PR}$  and  $\overline{RM}$  (Figure 2). Then she constructed a perpendicular line on  $\overline{RM}$  at point  $M$  and using the GSP she “mirrored” these two segments over this perpendicular line. She completed the construction of the isosceles trapezoid joining points  $P$  and  $S$  (Figure 2). She continued constructing the diagonals and she *measured* them. Tabulating the measures on the screen, she dragged the vertices of her isosceles trapezoid to verify that the diagonals were always equal. This, of course, was the case because she had constructed a robust figure. However, she had no confidence in the properties of the figure although she made the assertion since the beginning that the diagonals of isosceles trapezoid were congruent. Not being able to construct isosceles trapezoid using what she knew about them, indicates that her knowledge was based on rote memory. Skemp’s (1987) would describe this knowledge as instrumental but not relational. This instrumental knowledge led us to hypothesize that she was at the level of analysis (level 1) of the Van Hiele model.

Regardless of being able to use the “*mirror method*” to draw isosceles trapezoids, Susan was not aware of the geometric principals behind this construction (*i.e.*, the bases of isosceles trapezoids have a *common perpendicular bisector* and this perpendicular bisector could be used as a *mirror* or axis of symmetry). If she had known this property and its meaning, she could have constructed the “*mirror*” she needed by constructing the common perpendicular bisector to the two bases of the trapezoid into Figure 1 and transform it into an isosceles trapezoid. This illustrates that manipulating *drawings* on the screen does not necessarily mean that the properties of a geometric *figure* are understood as the essential characteristics of that figure. It also illustrates that the GSP can be used *only* as a technical tool

rather than as a psychological-symbolic tool to put the *drawings* under the conceptualizing lens of the user.

### ***Proving that when trapezoids are isosceles, they have congruent diagonals***

When Susan verified the congruence of the diagonals by dragging any vertex of the isosceles trapezoid she constructed (Figure 2), the teacher-interviewer pushed her to jump into the following step to prove the congruence of them. That is, Susan had to prove that “if a quadrilateral is an isosceles trapezoid, then its diagonals are congruent”. She expressed her thoughts as follows.

45. **Susan:** *Ok and I am going to color the triangles PTR and STQ just to color them. So these two triangles –triangles PTR and STQ– are congruent [she gives no justification]<sup>3</sup>. Ok so that means again I know that the length of RT would be equal to the length of TQ and the diagonal RS can be RT plus TS, and the diagonal PQ can be PT plus TQ. RT is equal to TQ or the length of RT equal to the length of TQ, and the length of ST is equal to PT. So, that would make those two diagonals [RS and TQ] congruent (Figure 3).*

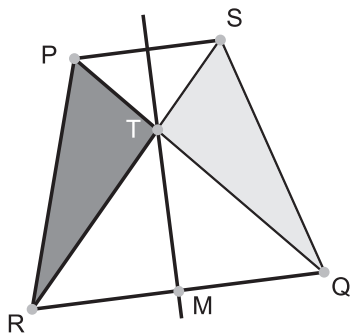


Figure 3. Those two diagonals [RS and TQ] congruent

Obviously, Susan did not prove the congruence of the triangles she focused on. In reality, there were not enough elements to argue the congruence. In addition, she did not use the fact that the trapezoid was isosceles. Probably, the assertion that triangles PTR and STQ were congruent was based on her assumption, that when the trapezoid is isosceles, the point of intersection of the diagonals (point T) lies on the perpendicular bisector of the two bases. In order to use this assumption in the proof, it is necessary to construct a geometric argument to prove that this is the case in isosceles trapezoids. It also

3 Brackets are used for clarification.

seems interesting that although the question was to prove that the diagonals were congruent, Susan concentrated on the parts of the diagonals determined by their point of intersection  $T$ . When she was asked for the justification of the congruence of the triangles  $PTR$  and  $STQ$ , she proposed to measure their areas without being aware that two triangles could have the same area and still not be congruent. This is another instance in which Susan's persistence on measuring as a way of "proving" indicates her use of the GSP only as a technical tool rather than as a psychological-symbolic tool to achieve particular geometric objectives.

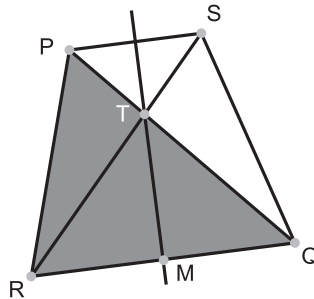


Figure 4

Then the teacher-interviewer requested a proof in which measurement were not involved. Susan continued the analysis of the figure by considering point  $M$  as the midpoint of  $RQ$  (the midpoint created by the mirror) and making  $TM$  the perpendicular bisector of  $RQ$  in order to conclude that  $TQ$  was congruent to  $TR$ . When the teacher-interviewer asked her *how* she knew that point  $T$  (the intersection of diagonals) was on the perpendicular bisector, Susan was confused. For this reason, the teacher-interviewer decided to ask her *how* many triangles she could see in her drawing and to show them. Susan needed to understand that she had to compare the triangles  $PRQ$  and  $SQR$  because the sides  $PQ$  and  $SR$  were the diagonals of the isosceles trapezoid.

88. **Susan:** *Ok, so I am looking at  $PRQ$  and  $SRQ$ ... it's cool. Oh! I get it; the light bulb just went on in my head. All right, so  $\triangle PRQ$  and  $\triangle SQR$  are congruent by side-angle-side (SAS).  $PR = SQ$ ,  $\angle PRQ = \angle SQR$ , and  $RQ = RQ$ . Thus, the sides  $PQ$  and  $SR$  are congruent.*

We should observe that Susan did not explicitly argued that  $PR = SQ$  because they are the non-parallel sides of a trapezoid that was isosceles. She asserted that  $\angle PRQ = \angle SQR$  because she also took as a definition of isosceles trapezoids those trapezoids which simultaneously have base angles congruent and non-parallel sides congruent. This was probably an inference of her mirror-construction of isosceles trapezoids. She did not specify either

that the sides PQ and SR, in this case also diagonals of the trapezoid, were congruent as an implication of the congruence of the triangles. In other words, her proof was encrypted and incomplete. This may be due to her lack of differentiation between definition and properties as well as between the necessary elements for congruence and the implications of that congruence. Therefore, the above proof indicates that Susan was at the informal level of deduction. In what follows, the teacher-interviewer continued to challenge her deductive reasoning.

Since Susan's first idea was to compare the small triangles PTR and STQ, the teacher-interviewer took the opportunity to ask her to prove the congruence of these triangles. Susan considered the congruence of angles RPQ and RSQ as an implication of the congruence of the triangles PRQ and SRQ that she had just completed. She asserted that the vertical angles PTR and STQ were congruent and that the sides PR and SQ are congruent because the trapezoid was isosceles. Then she said that these elements were not enough to apply the ASA criterion. It was clear that she was not able to make the connection with the sum of the angles of a triangle being  $180^\circ$  in order to use the angle-side-angle criterion of congruence. Her instrumental understanding came up one more time. Once the teacher-interviewer helped Susan to see the relationship among the angles, she wrote the proof of the congruence of these triangles. She also wrote that  $TR=TQ$  and  $TP=TS$  as implications of this congruence. Taking into account the equality of these sides, she concluded that point T, intersection of the diagonals, was on the perpendicular bisector of the bases in isosceles trapezoids. Therefore, Susan had proven that *"if a trapezoid is isosceles, the point of intersection of the diagonals lies on the perpendicular bisector to both bases."*

The teacher-interviewer prompted Susan to summarize the properties of isosceles trapezoids she knew up to now.

424. **Interviewer:** Now, what do we know about the properties of isosceles trapezoids?
425. **Susan:** With the...
426. **Interviewer:** A property or properties of isosceles trapezoids.
427. **Susan:** The whole...
428. **Interviewer:** From the definition to the properties, what do you know?
429. **Susan:** In order for a trapezoid to be isosceles, the diagonals have to be equal and also their intersection should lie on the perpendicular bisector of the bases.

The way Susan worded her conclusions (line 429) indicates she did not clearly recognize that in the statements she proved, the given (*i.e.*, the

sufficient condition) was that the trapezoid was isosceles, and what needed to be proved (i.e., the necessary conditions) were: (a) that the diagonals were congruent, and (b) that the point of intersection of the diagonals lies on the perpendicular bisector of both bases. This indicates, once again, that although Susan was making some progress she was still at the analysis level (level 1) of the Van Hiele model.

In what follows, the teacher-interviewer asked Susan to prove the converses of the above propositions. First the teacher-interviewer asked Susan to prove that “*if the diagonals of a trapezoid are congruent, then the trapezoid is isosceles*”. Susan wrote the statement on the GSP and proceeded to think about the proof.

***Proving that if the diagonals in a trapezoid are congruent, it is an isosceles trapezoid***

The teacher-interviewer wanted Susan to differentiate between the sufficient and necessary conditions of a geometric proposition. The teacher-interviewer inferred that a good way to do it was to ask her to prove the converse of the proposition she had proved (i.e., if the trapezoid is isosceles, then the diagonals are congruent). Susan remained surprised and silent for a moment. It seems as if such a proposition was something strange in Susan’s mind because since the beginning she also took as definition of isosceles trapezoids those trapezoids with congruent diagonals. While she worked on this proof, she was fluctuating between the levels of analysis, informal deduction, and deduction in the Van Hiele model. She struggled trying to understand what she had to prove and finally she decided to construct a new figure with two equal diagonals forming a quadrilateral. Then she realized that such a quadrilateral was not a trapezoid. At that point the teacher-interviewer clarified for her that the *given* was that the *diagonals in a trapezoid were congruent* and therefore she had to prove that under this condition the trapezoid would be isosceles. Susan abandoned the quadrilateral with the equal diagonals and made the drawing in Figure 5.

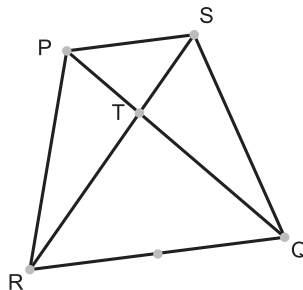


Figure 5. A monologue comparing the big triangles PQR and SQR

Looking at Figure 5 she started a monologue comparing the big triangles PQR and SRQ and explained to herself that the triangles had only two sides congruent and there was no information about angles. Susan could not figure out how to prove it and she was repeating that “if the diagonals in a trapezoid are equal, then the trapezoid would be isosceles”. Then the teacher-interviewer guided her to figure out that what she needed was to construct *right triangles* to overcome the difficulty of the lack of angles. This was a crucial hint that Susan played with and thought for a while. Finally, she constructed the appropriate perpendicular lines forming the rectangle RVUQ which simultaneously determined the right triangles RVS and QUP as well as the right triangles RVP and QUS (Figure 6).

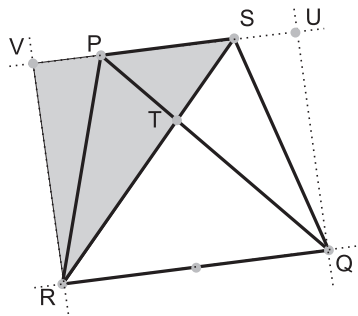


Figure 6. The appropriate perpendicular lines forming the rectangle RVUQ

After this construction, she proved that triangle RVS is congruent to triangle QUP using the congruence of the diagonals ( $RS=PQ$ ) and the congruence of the sides  $RV$  and  $QU$  ( $RV=QU$ ) due to the fact that perpendicular segments between parallel lines are equal. She stated that one of the implications of this congruence was the congruence of  $VS$  and  $UP$  ( $VS=UP$ ). She used this implication to prove that the right triangles  $RVP$  and  $QUS$  were congruent. She took the time to justify the equality of the segments  $VP$  and  $SU$ . She wrote, “ $VP=SU$  since  $VS=VP+PS$  and  $PU=PS+SU$  and  $VS=PU$ ”. Then she noticed that she had enough elements to justify the congruence of the other pair of right triangles  $RVP$  and  $QUS$  because “ $VR=UQ$  and  $VP=SU$ ”. As an implication of this congruence, she concluded that  $PR$  was congruent to  $SQ$  and, therefore, the trapezoid was an isosceles trapezoid because the non-parallel sides were congruent. After the hint was given to Susan to use an auxiliary construction to introduce right triangles, she took off with the proof. Her reasoning in this proof indicates that, with the assistance of the teacher-interviewer, she could move up to the deductive level (level 3) of the Van Hiele model.

***Proving the converse of the proposition that when the trapezoid is isosceles, the intersection of its diagonals lies on the perpendicular bisector of the bases***

The teacher-interviewer wanted to know, at this point, how stable Susan's deductive level was. Thus she stated the following proposition and Susan wrote it on the GSP: "if the intersection of the diagonals lies on the perpendicular bisector to one of the bases of a trapezoid, then the trapezoid is an isosceles trapezoid". Susan was made aware that she had proven the converse of this proposition before and was asked to state that proposition. She said: "if a trapezoid is isosceles, the intersection of the diagonals lies on the perpendicular bisector of its bases". This time, Susan stated it correctly. This indicates her awareness of the differentiation between sufficient and necessary conditions. Then, she was asked to prove the converse of that proposition here stated by the teacher-interviewer.

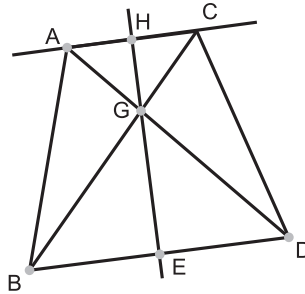


Figure 7. Triangle GEB is congruent to triangle GED

Susan constructed Figure 7 and proved that triangle GEB is congruent to triangle GED giving the following argument. "GE=GE, BE=DE just because it's the midpoint, and then BG=DG they are hypotenuses of these right triangles (Pythagorean theorem)". Then, she proved that  $\triangle AGH$  is congruent to  $\triangle CGH$  giving the following argument.

*$\angle AGH = \angle CGH$  because  $\angle AGH = \angle EGD$  vertical angles,  $\angle CGH = \angle BGE$  vertical angles; but I know that  $\angle BGE = \angle DGE$  as an implication of the congruent triangles GED and GEB. I also know that  $HG = HG$  and angles GHA and GHC are right angles because BD is parallel to AC and EH is perpendicular to BD. Then, the right triangles AGH and CGH are congruent. Then, an implication is that  $GA = GC$ .*

Finally, Susan explained that the point G is equidistant from points A and C, and therefore, G lies on the perpendicular bisector of AC. So  $AG + GD = CG + GB$ , this means that  $AD = CB$ . Therefore, in the trapezoid ABCD, the diagonals AD and CB are congruent, making it an isosceles trapezoid.



The teacher-interviewer continued the testing Susan's deductive level and she requested one more proof of the same proposition. Susan used the copy and paste facility of the GSP and she produced Figure 8. Then she said:

*Knowing that point G lies on the perpendicular bisector of BD, then triangle GBD is an isosceles triangle with congruent base angles. Since AC is parallel to BD, it implies that the alternate interior angles are congruent, and therefore, triangle GAC is isosceles too. Thus G is on the perpendicular bisector of AC. Then, the corresponding sums of the parts of each diagonal are congruent (i.e., the diagonals are congruent) and the trapezoid is isosceles.*

It was not clear if Susan was using the proposition she had proved before or if she used the property of the equality of the diagonals as a definition of isosceles trapezoids. Her proof did not specify the reason.

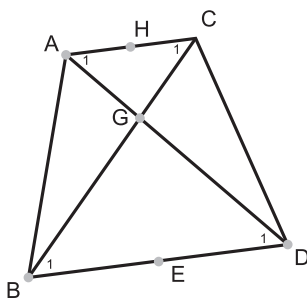


Figure 8. The copy and paste facility of the GSP

It was interesting that Susan offered one more proof of the proposition without a request for doing so. She went further and proved that triangles GBA and GCD were congruent by SAS. At the beginning of the interview, she wanted to prove that these triangles were congruent, but in that occasion she did not have the geometric conditions to support the argument. This time, she gave the following argument: *"I just proved that  $AG=GC$ ,  $GD=GB$  and the vertical angles  $\angle AGB=\angle CGD$ . Then, by SAS, triangle GBA and triangle GDC are congruent. Therefore,  $AB$  is congruent to  $CD$  and the trapezoid is an isosceles trapezoid"*. Here Susan used the definition of isosceles trapezoids as trapezoids with non-parallel sides congruent. When she finished, she smiled and said *"I did it"*.

Susan's last proofs indicate that, with the guidance of the teacher-interviewer, she could move between different Van Hiele levels. It was a continued effort for two hours, but in the process Susan was learning how to use the GSP not only as a technical tool, but also as a psychological-symbolic tool to direct her conceptual activity. The fact that she was now able to co-

rectly state the proposition *“if a trapezoid is isosceles, the intersection of the diagonals lies on the perpendicular bisector of its bases”*, which she had stated incorrectly before, and at the same time she was now able to prove its converse as stated by the teacher-interviewer (*“if the intersection of the diagonals lies on the perpendicular bisector to one of the bases of a trapezoid, then the trapezoid is an isosceles trapezoid”*) indicates that Susan have started to consolidate her deductive level. However, it was not clear whether or not she was aware of the differentiation between definition and properties of isosceles trapezoids; differentiations that would be of great assistance in the development of her geometric thinking.

At the end, the teacher-interviewer asked Susan what she could do, at this point, to transform Figure 1 into an isosceles trapezoid. Susan said *“I would draw the perpendicular bisectors of the bases and drag points or segments to make the intersections of the diagonals go on the perpendicular bisectors.”* Susan has gotten the idea although she was too tired to actually do it. However, her answer indicates that now she was able to use the GSP as a psychological-symbolic tool.

### Concluding remarks

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From the analysis of the interview, we could say that Susan first experienced the GSP geometric environment only as a technical tool to measure and to draw segments. Then, guided by the teacher-interviewer’s indirect hints and questions, she started to use the GSP as a psychological-symbolic tool to deepen her knowledge about the properties of isosceles trapezoids. The GSP provided her with opportunities to make dynamic constructions and it also fostered her ways of thinking and means of arguing geometric proofs. That Susan first used the GSP as a technical tool was illustrated when she argued the congruence of segments and triangles by either measuring their lengths or their areas. However, with the guidance of the teacher-interviewer, she came around to use the GSP as a psychological-symbolic tool to direct her geometric explorations and her ways of arguing geometric proofs about properties of isosceles trapezoids. This indicates that while the GSP is, by nature, a technical tool that facilitates the production of clean drawings (whether or not they satisfy the necessary conditions to pass the dragging test), this tool also has the potential of being transformed, by the learner, into a psychological tool to facilitate the exploration of geometric drawings and the formation of geometric arguments.

At the beginning of the interview, it seems that Susan had only an instrumental understanding of some properties of isosceles trapezoids because

she could not transform the trapezoid in Figure 1 into an isosceles one. Susan had also different definitions of isosceles trapezoids because she considered their properties also as definitions or combined them according to need. For example, one definition she used was that isosceles trapezoids have the non-parallel sides equal and the pairs of base angles congruent. Another definition she used was that isosceles trapezoids are those with equal diagonals. In addition, the property that relates the intersection of the diagonals of isosceles trapezoids and the perpendicular bisector of the bases was unknown to her. The teacher-interviewer guided her to arrive at this property. Midway through the interview, she was able to give a proof of the property that states that *“if a trapezoid is isosceles, the intersection of the diagonals lies on the perpendicular bisector of its bases”*. Since she arrived to this property directed by the guidance of the teacher-interviewer, it was not surprising that she summarized her findings in the following terms: *“In order for a trapezoid to be isosceles, the diagonals have to be equal and also their intersection should lie on the perpendicular bisector of the bases”*. In this statement, Susan indicated two issues about her understanding. First, that she could not separate the two properties of the diagonals of isosceles trapezoids; one referring to the equality of the diagonals, the other referring to the relationship between the intersection of the diagonals and the perpendicular bisectors of the bases. Second, that she was unaware of the sufficient and necessary conditions of the statement she had proved. She failed to recognize that the sufficient condition for her proof was that the trapezoid was isosceles. For this reason, by the end of the interview, the teacher-interviewer carefully stated the converse of the property Susan had proved but had incorrectly stated. The teacher-interviewer asked Susan to prove that *“if the intersection of the diagonals lies on the perpendicular bisector of one of the bases of a trapezoid, then the trapezoid is an isosceles trapezoid”*. After Susan completed the proof of this statement, the teacher-interviewer challenged her to give another proof of the same proposition. Susan’s improvement was manifested when she was able to generate two more proofs for the same statement. This improvement was also the result of a triadic interaction among Susan, the teacher-interviewer and the GSP. In addition, Susan’s later monologues indicated that the GSP mediated the formation of her geometric arguments as she acted and reacted upon her drawings. Thus, she used the GSP as a psychological-symbolic tool.

The GSP also provided the teacher-interviewer with opportunities to make hypotheses about Susan’s line of reasoning and to challenge her through indirect questioning. Without the GSP, it would have been difficult to understand Susan geometric assumptions since, most of the time, her arguments were tacit and incomplete. Appropriate questioning on the part of the teacher-interviewer, mediated by the GSP, helped the lateral entry teacher to

refine her previous knowledge about isosceles trapezoids and to investigate a property of isosceles trapezoids that was unknown to her. The analysis also indicates that, little by little, this lateral entry teacher investigated several of the properties of isosceles trapezoids constructing different drawings and exploring them in the GSP environment. In her proofs, she used her previous knowledge about the congruence of triangles and the relationships among the angles formed between parallel lines intersected by a transversal. Once Susan had a good understanding of the property of isosceles trapezoids relating the intersection of the diagonals to the perpendicular bisector of the bases, she was able to verbalize how she would transform Figure 1 into an isosceles trapezoid.

During the months of the teaching-experiment, the GSP became a useful tool to explore questions and to interact with the participating teachers. That is the communication was triadic among the student-teacher, the GSP and the teacher-interviewer. In other words, the GSP made more meaningful the interaction between the teacher-interviewer and the student-teacher. The effectiveness of this dynamic environment depended not only on the careful designed of geometric tasks, but also on the teacher-interviewer's ability to make hypotheses about the conceptual needs of the student-teacher to anticipate and guide the line of questioning. We concur with Mariotti (2000), Jones (2000) and Arzarello et al. (2002), that educational change on the teaching and learning of geometry happens not only because of the software but also because of the teacher's engagement in helping students attain a level of theoretical thinking (deductive level) in the study of geometry. Nonetheless, this engagement of teachers with students will become a reality only when teachers themselves have gone through geometric learning experiences that involve geometric environments like the GSP and learn to use these environments not only as technical tools, but also as psychological-symbolic tools to establish and achieve geometric goals.

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