DOCUMENTS DE TREBALL

DE LA FACULTAT D'ECONOMIA I EMPRESA

Col.lecció d'Economia

E10/243

Multi-sided Böhm-Bawerk assignment markets: the nucleolus and the core-center

Oriol Tejada and Marina Núñez

Department of Actuarial, Financial and Economic Mathematics
Universitat de Barcelona, Av.Diagonal, 690, E-08034 Barcelona, Spain
e-mail: oriol.tejada@ub.edu, mnunez@ub.edu

Agraïments: We would like to thank Carles Rafels for his very helpful comments. The authors acknowledge the support of the Barcelona GSE Research Network and of the Government of Catalonia, of Ministerio de Ciencia e Innovación and FEDER, under grant ECO2008-02344/ECON, and Generalitat de Catalunya, under grant 2009SGR0960. Support from grant Program FPU of Ministerio de Educaci\'{o}n of the Spanish Goverment is also acknowledged.

Tots els treballs publicats en els "Documents de Treball de la Facultat d'Economia i Empresa" han passat i superat un procès d'avaluació anònima i externa.

ISSN: 1136-8365
Dipòsit Legal: B-29452-2010

Multi-sided Böhm-Bawerk assignment markets: the nucleolus and the core-center

Abstract: We show that, contrary to the bilateral case, for multi-sided Böhm-Bawerk assignment markets the nucleolus and the core-center, i.e. the mass center of the core, do not coincide in general. To do so, we prove that both the nucleolus and the core-center of an m-sided Böhm-Bawerk assignment market can be respectively computed from the nucleolus and the core-center of a convex game defined on the set of m sectors. Even more, in the calculus of the nucleolus of this latter game only singletons and coalitions containing all agents but one need to be taken into account. These results simplify the computation of the nucleolus of a multi-sided Böhm-Bawerk assignment market with large number of agents.

Keywords: multi-sided assignment games, core, nucleolus, core-center

JEL Classification: C71, C78

Resum: En aquest treball mostrem que, a diferència del cas bilateral, per als mercats multilaterals d'assignació coneguts amb el nom de Böhm-Bawerk assignment games, el nucleolus i el core-center, i.e. el centre de masses del core, no coincideixen en general. Per a demostrar-ho provem que donat un m-sided Böhm-Bawerk assignment game les dues solucions anteriors podem obtenir-se respectivament del nucleolus i el core-center d'un joc convex definit en el conjunt format pels m sectors. Encara més, provem que per a calcular el nucleolus d'aquest últim joc només les coalicions formades per un jugador o m-1 jugadors són importants. Aquests resultats simplifiquen el càlcul del nucleolus d'un multi-sided Böhm-Bawerk assignment market amb un número molt elevat d'agents.

1 Introduction

The bilateral Böhm-Bawerk horse market (Böhm-Bawerk, 1923) is a model for a two-sided market with no product differentiation, and it is thus a particular case of a bilateral assignment game. The bilateral assignment game was introduced by Shapley and Shubik (1972) as a cooperative game model for a two-sided market with transferable utility. In their paper, the case of the bilateral Böhm-Bawerk horse market is also analyzed.

In the present paper we consider a market with an arbitrary finite number of sectors. One sector consists of a finite number of buyers and each one of the remaining sectors consists of a finite number of sellers. Then each seller offers one unit of a good and each buyer demands one bundle formed by one good of each sector. This market can be studied within the framework of multi-sided assignment games, which are introduced by Quint (1991). Contrary to two-sided assignment games, multi-sided assignment games may have an empty core (Kaneko and Wooders, 1982). Multi-sided assignment games have been studied, among others, by Quint (1991), Stuart (1997), Sherstyuk (1999) and Tejada and Rafels (2010).

The particular case where each buyer places the same valuation on all the bundles is introduced in Tejada (2010) with the name of *multi-sided Böhm-Bawerk assignment market*, extending the bilateral Böhm-Bawerk horse market to multilateral markets. There, an analysis of multi-sided Böhm-Bawerk assignment markets is done and it is shown that the core is nonempty and it is completely determined by the core of a convex game played by the sectors instead of the agents.

For the classical two-sided Böhm-Bawerk game it is well-known that the core is nonempty and reduces to a segment. A study of single-valued solutions for this game is done in Núñez and Rafels (2005), to conclude that, without additional information about the bargaining capabilities of the agents, the classical cooperative theory seems to recommend the midpoint of the core segment. This assertion is supported by the fact that, among other single-valued solutions, the nucleolus (Schmeidler, 1969) coincides with the midpoint of the core segment, that is, with the mass-center of the core. The mass-center of the core was introduced by Gonzalez-Díaz and Sánchez-Rodríguez (2007), with the name of core-center, as a single-valued solution for arbitrary coalitional games.

The aim of the present paper is to analyze the nucleolus and the core-center of multi-sided Böhm-Bawerk assignment markets. We show that both the nucleolus and the core-center of a multi-sided Böhm-Bawerk assignment market can be respectively computed from the nucleolus and the core-center of the associated sectors game, this being a game with many less players. Even more, only singletons and coalitions containing all agents but one need to be taken into account in the calculation of the nucleolus of this latter game. These results simplify the computation of the nucleolus of a multi-sided Böhm-Bawerk assignment market with large number of agents. As a consequence we show that, contrary to the case of two-sided Böhm-Bawerk markets, the nucleolus does not coincide in general with the core-center in the case of multi-sided Böhm-Bawerk assignment markets.

The structure of the paper is as follows. The preliminaries on coalitional games and multi-sided Böhm-Bawerk assignment games are presented in Section 2. In Section 3 we determine which coalitions are to be taken into account for the computation of the nucleolus of a multi-sided Böhm-Bawerk assignment game and we also show that its nucleolus can be obtained from the nucleolus of the related sectors game. Section 4 establishes a parallel result for the core-center. An example is used throughout the paper to illustrate both the model and our results.

2 Preliminaries and notation

A *coalitional game* (a game) is a pair (N, v), where N is the finite set of players and, for all $S \subseteq N$, $v(S) \in \mathbb{R}$ is the worth that coalition S can obtain without the cooperation of agents in $N \setminus S$, being $v(\varnothing) = 0$. Let $|S|$ denote de cardinality of coalition $S \subseteq N$. An *imputation* is a payoff vector $x \in \mathbb{R}^N$, where x_i stands for the payoff to player $i \in N$, that is efficient, $\sum_{i \in N} x_i = v(N)$, and individually rational, $x_i \geq v(\{i\})$ for all $i \in N$. The set of imputations is denoted by $I(v)$. The core of a game is the set of imputations that satisfy coalitional rationality and thus are not blocked by any coalition. Formally, given (N, v), the core is the set $C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N) \text{ and } x(S) \geq v(S) \text{ for all } S \subset N\}$, where as usual $x(S) = \sum_{i \in S} x_i$ and $x(\varnothing) = 0$. A game is *balanced* if the core is nonempty. A *subgame* of (N, v) is any game (N', v') where $N' \subseteq N$ and v' is the restriction of v to the subsets of N'. A game is *totally balanced* if the core of any subgame is nonempty. A game (N, v) is *convex* if for all $i \in N$ and for all $S \subseteq T \subseteq N \setminus \{i\}$ we have $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. The core is an example of set-solution concept. A *single-valued solution* (or point-solution) on a given set of games Γ is a rule α that assigns to each game (N, v) in this set Γ an efficient payoff vector $\alpha(v) \in \mathbb{R}^N$. Examples of single-valued solutions are the nucleolus and the core-center. Each one of these two solutions selects a core allocation that occupies a somehow "central" position in the core. Although for arbitrary coalitional games, these two solutions do not coincide, they do coincide for the particular situation of two-sided Böhm-Bawerk markets.

2.1 Multi-sided assignment games and the Böhm-Bawerk case

An *m-sided assignment problem* (m-SAP) denoted by $(N^1, ..., N^m; A)$, is given by $m \geq 2$ different nonempty finite sets (or types) of agents, $N^1, ..., N^m$, and a nonnegative m-dimensional matrix $A = (a_E)_{E \in \prod_{k=1}^m N^k}$. With some abuse of notation, let it be $N^k = \{1, 2, ..., n_k\}$ for all k, $1 \leq k \leq m$. We shall refer to the i^{th} agent of type k as $i \in N^k$. We name any m-tuple of agents $E \in \prod_{k=1}^m N^k$ an *essential coalition.* Each entry $a_E \geq 0$ represents the profit associated to the essential coalition E. Again with slight abuse of notation, we also use E to denote the set of agents that form the essential coalition. An m-SAP is *square* if $n_1 = ... = n_m$.

A *matching* among $N^1, ..., N^m$ is a set of essential coalitions, $\mu = \{E^r\}_{r=1}^t$ with $t = \min_{1 \leq k \leq m} |N^k|$, such that any agent belongs at most to one coalition in μ. We denote by $\mathcal{M}(N^1, ..., N^m)$ the set of all matchings among $N^1, ..., N^m$. An agent $i \in N^k$, for some $k \in \{1, ..., m\}$, is *unmatched* under μ if it does not belong to any of its essential coalitions. A matching is optimal if it maximizes $\sum_{E \in \mu} a_E$ in $\mathcal{M}(N^1, ..., N^m)$. We denote by $\mathcal{M}_A^*(N^1, ..., N^m)$ the set of all optimal matchings of $(N^1, ..., N^m; A)$.

For each multi-sided assignment problem $(N^1, ..., N^m; A)$, the associated *multi-sided assignment game* (m-SAG) is the cooperative game (N, ω_A) with set of players composed of all agents of all types, $N = \cup_{k=1}^m N^k$, and characteristic function

$$(1) \qquad \omega_A(S) = \max_{\mu \in \mathcal{M}(N^1 \cap S, ..., N^m \cap S)} \left\{ \sum\nolimits_{E \in \mu} a_E \right\}, \text{ for any } S \subseteq N,$$

where the summation over the empty set is zero.

It is known that the core of a multi-sided assignment game, $C(\omega_A)$, coincides with the set of efficient nonnegative vectors $x = (x_{11}, ..., x_{1n_1}; ...; x_{m1}, ..., x_{mn_m})$, with x_{ki} standing for the payoff to agent $i \in N^k$, that satisfy $x(E) \geq a_E$ for all $E \in \prod_{k=1}^m N^k$. As a consequence, the above inequality must be tight if E belongs to some optimal matching, and $x_{ki} = 0$ if agent $i \in N^k$ is unmatched under some optimal matching. Observe that these two latter conditions guarantee the efficiency of the core allocations.

A particular case of multi-sided assignment games are multi-sided Böhm-Bawerk markets, introduced in Tejada (2010). In these markets, each sector

k, for $k \in \{1, 2, \ldots, m-1\}$ is composed of a finite set N^k of sellers, and sector m is composed of a finite set N^m of buyers. Each seller $i_k \in N^k$ has one good of type k to sell, with a reservation price c_{ki_k}. Each buyer $i \in N^m$ wants to buy a bundle formed by one good of each type, with the singularity that, from her point of view, goods of the same type are homogeneous. We denote by w_i the value that buyer i places on an arbitrary bundle $(i_1, \ldots, i_{m-1}) \in \prod_{k=1}^{m-1} N^k$.

Thus, an *m-sided Böhm-Bawerk market* (or problem) can be summarized by a pair $(\mathbf{c}; w)$ where $\mathbf{c} = (c_1, \ldots, c_{m-1}) \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_{m-1}}$ are the sellers' valuations and $w = (w_1, ..., w_{n_m}) \in \mathbb{R}^{N_m}$ are the buyers' valuations.

From now on, in order to simplify the analysis of the model, we will assume that valuations of the sellers of each sector are arranged in a nondecreasing way and valuations of the buyers are arranged in a nonincreasing way, i.e.

(2)

$$c_{k1} \leq c_{k2} \leq ... \leq c_{kn_k}, \text{ for all } k \in \{1, 2, ..., m-1\}, \text{ and } w_1 \geq w_2 \geq ... \geq w_{n_m}.$$

Given an m-sided Böhm-Bawerk problem $(\mathbf{c}; w)$, we denote by $A(\mathbf{c}; w)$ the m-dimensional matrix defined by

$$(3) \qquad a_E = \max\left\{0, w_{i_m} - \sum_{k=1}^{m-1} c_{ki_k}\right\}, \text{ for all } E = (i_1, ..., i_m) \in \prod_{k=1}^{m} N^k.$$

Notice that, by (2), for all $E, E' \in \prod_{k=1}^{m} N^k$,

$$(4) \qquad E \leq E' \Longrightarrow a_E \geq a_{E'}.$$

When no confusion may arise, we write simply A instead of $A(\mathbf{c}; w)$.

Then, $(N, \omega_{A(\mathbf{c};w)})$, where N is composed of all sellers and buyers, is the multi-sided assignment game -see (1)- associated to the multi-sided Böhm-Bawerk market $(\mathbf{c}; w)$, which we call the *multi-sided Böhm-Bawerk assignment game* associated to $(\mathbf{c}; w)$. From Tejada (2010), $(N, \omega_{A(\mathbf{c};w)})$ is a totally balanced game.

For all $i \in \mathbb{N}$, we introduce the notation $D^i = (i, ..., i) \in \mathbb{R}^m$. By (2), the diagonal matching $\mu = \{D^i \mid 1 \leq i \leq n\}$ is an optimal matching, where $n = \min_{1 \leq k \leq m} n_k$. Then, the core $C(\omega_{A(\mathbf{c};w)})$ of $(N, \omega_{A(\mathbf{c};w)})$ coincides with the following set:

$$(5) \quad \left\{ x \in \mathbb{R}_+^{N^1} \times \cdots \times \mathbb{R}_+^{N^m} \;\middle|\; \begin{array}{l} x(D^i) = a_{D^i} \text{ for all } 1 \leq i \leq n, \\ x(E) \geq a_E \text{ for all } E \in \prod_{k=1}^{m} N^k \text{ and} \\ x_{ki} = 0 \text{ for all } i \in N^k, k \in M \text{ and } i > n. \end{array} \right\}.$$

Let us define r as the highest buyer's position that obtains a positive profit when matched with all the sellers in the same position r:

$$r = \max_{1 \leq i \leq n} \{i \mid a_{i...i} > 0\}, \tag{6}$$

with the convention that $r = 0$ if all entries of $A(\mathbf{c}; w)$ are zero. For each $k \in \{1, .., m\}$, agents $i \in N^k$ with $1 \leq i \leq r$ are said to be *active*, while agents $i \in N^k$ with $i > r$ are called *inactive*. It is not difficult to check that any matching formed by essential coalitions with all agents active is optimal.

In Tejada (2010), a new game defined on the set of sectors $M = \{1, ..., m\}$ is associated to each multi-sided Böhm-Bawerk assignment game. The worth in this game of a coalition S of sectors is the profit that in the related market can be obtained by the r^{th} agents of the sectors in S together with the $r + 1^{th}$ agents of the sectors not in S. To this end, for any $S \subseteq M$ let us define $E^S = r\mathbf{1}_S + (r+1)\,\mathbf{1}_{M \backslash S} \in \mathbb{R}^m$, where, for each $T \subseteq M$, $\mathbf{1}_T \in \mathbb{R}^m$ is the vector such that $\mathbf{1}_T(k) = 1$ if $k \in T$ and $\mathbf{1}_T(k) = 0$ if $k \notin T$. It is important to point out that the case where there is no $r + 1^{th}$ agent for some of the sectors in $M \backslash S$ must be treated apart. Observe that, in this case, $E^S \in \mathbb{R}^m$ can still be defined but E^S is not an essential coalition of N, i.e. $E^S \notin \prod_{k=1}^m N^k$. The formal definition of the sectors game is introduced next.

Definition 1 *Given an m-sided Böhm-Bawerk assignment game $(N, \omega_{A(\mathbf{c};w)})$, the associated sectors game $(M, v^M_{\mathbf{c};w})$ is the coalitional game with set of players $M = \{1, 2, ..., m\}$ composed of all sectors and characteristic function defined, for each $S \subseteq M$, by*

$$v^M_{\mathbf{c};w}(S) = \begin{cases} a_{E^S} & \text{if } E^S \in \prod_{k=1}^m N^k \\ 0 & \text{if } E^S \notin \prod_{k=1}^m N^k \end{cases} \tag{7}$$

if $r > 0$ and $v^M_{\mathbf{c};w}(S) = 0$ if $r = 0$.

By definition, whenever $r > 0$ we have $v^M_{\mathbf{c};w}(M) = a_{D^r} > 0$ and $v^M_{\mathbf{c};w}(\varnothing) = 0$. When no confusion may arise we write v^M instead of $v^M_{\mathbf{c};w}$.

It is shown in Tejada (2010) that $(M, v^M_{\mathbf{c};w})$ is a convex game and it is strongly related to $(N, \omega_{A(\mathbf{c};w)})$. To be more precise, each core allocation of the multi-sided Böhm-Bawerk assignment game is uniquely determined by a core allocation of the sectors game. Since payoff vectors of both games correspond to different spaces ($\mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}$ versus $\mathbb{R}^M$), we define a function to map payoffs of the sectors game to payoffs of the multi-sided Böhm-Bawerk game. Given an m-sided Böhm-Bawerk assignment game $(N, \omega_{A(\mathbf{c};w)})$, we introduce the *replica operator* $\mathcal{R}_{\mathbf{c};w} : \mathbb{R}^M \longrightarrow \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}$ defined by $\mathcal{R}(\overline{x}_1, ..., \overline{x}_m) = (x_1, ..., x_m)$, where $x_k = (\overbrace{\overline{x}_k, ..., \overline{x}_k}^{r}, 0, ..., 0) \in \mathbb{R}^{N^k}$ for all $k \in \{1, ..., m\}$. Notice that $\mathcal{R}_{\mathbf{c};w}$ is an injective linear function.

The main result in Tejada (2010) states that if $(N, \omega_{A(\mathbf{c};w)})$ is an m-sided Böhm-Bawerk assignment game and $(M, v^M_{A(\mathbf{c};w)})$ is the associated sectors game, then[1]

$$C(\omega_{A(\mathbf{c};w)}) = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(C\left(v^M_{\mathbf{c};w}\right)), \tag{8}$$

where the *translation vector* $\overrightarrow{t}_{\mathbf{c};w} = (t_{11}, ..., t_{1n_1}; ...; t_{m1}, ..., t_{mn_m}) \in \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}$ is defined by

$$\begin{aligned} t_{ki} &= \max\{0, c_{kr} - c_{ki}\} \text{ for all } 1 \leq k \leq m-1 \text{ and } 1 \leq i \leq n_k, \\ t_{mi} &= \max\{0, w_i - w_r\} \text{ for all } 1 \leq i \leq n_m. \end{aligned} \tag{9}$$

In particular, notice that (8) and (9) imply that, for all $x \in C(\omega_{A(\mathbf{c};w)})$, $k \in M$ and $1 \leq i \leq r$, we have $x_{ki} = x_{kr} + t_{ki}$. Later on in the paper an example is introduced to illustrate the above definitions and results.

In the next two sections we show that an statement analogous to (8) holds for two singled-valued solutions that are tightly linked to the core: the nucleolus and the core-center.

3 The nucleolus

The nucleolus is a single-valued solution for coalitional games that was introduced by Schmeidler (1969). For any imputation x of (N, v) and any coalition $S \subseteq N$ the *excess of coalition S with respect to x* is defined by $e_v(S, x) = v(S) - x(S)$, and it is a measure of the satisfaction of coalition S with respect to the allocation x. Given an imputation x, we define the vector $\lambda(x) \in \mathbb{R}^{2^n-2}$ of excesses of all nonempty coalitions different from N arranged in a non-increasing order, so that those coalitions with a greater complaint occupy the first positions in $\lambda(x)$. That is, $\lambda_k(x) = e_v(S_k, x)$ for all $k \in \{1, ..., 2^n - 2\}$ and $\lambda_k(x) \geq \lambda_j(x)$ if $1 \leq k < j \leq 2^n - 2$, where $\{S_1, ..., S_k, ..., S_{2^n-2}\}$ is the set of all nonempty coalitions of N different from N. The *nucleolus* of the game (N, v) is the imputation $\eta(N, v)$ (we write $\eta(v)$ for short when no confusion regarding the player set can arise) that

[1] Given $A \subseteq \mathbb{R}^k$ and $t \in \mathbb{R}^k$, $t + A = \{y \in \mathbb{R}^k \mid y = t + x, \text{ for some } x \in A\}$.

minimizes $\lambda(x)$ with respect to the lexicographic order[2] over the set of imputations. That is, $\lambda(\eta(v)) \leq_{Lex} \lambda(x)$ for all $x \in I(v)$. It is known that the nucleolus is always a single point and, whenever the core of the game is nonempty, it belongs to the core.

Maschler et al. (1979) give an alternative definition of the nucleolus by means of a finite process that iteratively reduces the set of payoffs to a singleton, called the lexicographic center of the game, that is proved to coincide with the nucleolus.

Let us denote by $\mathcal{C}$ an arbitrary nonempty subset of coalitions of a balanced game (N, v), and consider the algorithm in Maschler et al. (1979) restricted to coalitions in $\mathcal{C}$. This restricted procedure constructs a sequence of coalitions $\Sigma_{\mathcal{C}}^0 \supseteq \Sigma_{\mathcal{C}}^1 \supseteq \cdots \supseteq \Sigma_{\mathcal{C}}^{s+1}$ and a sequence of subsets of payoffs $\mathcal{X}_{\mathcal{C}}^0 \supseteq \mathcal{X}_{\mathcal{C}}^1 \supseteq \cdots \supseteq \mathcal{X}_{\mathcal{C}}^{s+1}$ such that initially $\alpha_{\mathcal{C}}^0 = 0$, $\mathcal{X}_{\mathcal{C}}^0 = C(v)$, $\Sigma_{\mathcal{C}}^0 = \mathcal{C}$ and $\Delta_{\mathcal{C}}^0 = \varnothing$ and, for $t \in \{0, ..., s_{\mathcal{C}}\}$, we define recursively

$$
(10) \qquad
\begin{array}{l}
(a)\ \alpha_{\mathcal{C}}^{t+1} = \min_{x \in \mathcal{X}_{\mathcal{C}}^t} \max_{S \in \Sigma_{\mathcal{C}}^t} e_v(S, x), \\
(b)\ \mathcal{X}_{\mathcal{C}}^{t+1} = \left\{ x \in \mathcal{X}_{\mathcal{C}}^t \mid \max_{S \in \Sigma_{\mathcal{C}}^t} e_v(S, x) = \alpha_{\mathcal{C}}^{t+1} \right\}, \\
(c)\ \Sigma_{t+1}^{\mathcal{C}} = \{ S \in \Sigma_{\mathcal{C}}^t \mid e_v(S, x) \text{ is constant on } x \in \mathcal{X}_{\mathcal{C}}^{t+1} \}, \\
(d)\ \Sigma_{\mathcal{C}}^{t+1} = \Sigma_{\mathcal{C}}^t \backslash \Sigma_{t+1}^{\mathcal{C}} \text{and } \Delta_{\mathcal{C}}^{t+1} = \Delta_{\mathcal{C}}^t \cup \Sigma_{t+1}^{\mathcal{C}},
\end{array}
$$

where $s_{\mathcal{C}}$ is the last index for which $\Sigma^{s_{\mathcal{C}}} \neq \varnothing$. The set $\mathcal{X}^{s_{\mathcal{C}}+1}$ is called the *$\mathcal{C}$-lexicographic center of* (N, v). When no confusion is possible we omit the superscript or subscript $\mathcal{C}$. By Maschler et al. (1979), if we take $\mathcal{C}$ to be the set 2^N of all coalitions, the 2^N-lexicographic center reduces to only one point and it is the nucleolus. For an arbitrary collection $\mathcal{C}$, the procedure is well defined but $\mathcal{X}^{s+1}$ is not necessarily a single point, and even in that case it might not coincide with the nucleolus.

Like in the bilateral case, it is easy to check that in the case of multi-sided assignment games only essential coalitions, $E = (i_1, i_2, \ldots, i_m) \in \prod_{k=1}^m N^k$, and singletons need to be considered in the computation of the nucleolus (Huberman, 1980). We denote the set of essential coalitions and singletons by $\mathcal{E}$.

As it is done in Solymosi and Raghavan (1994) for bilateral assignment games, it can be proved[3] that, for balanced multi-sided assignment games, the $\mathcal{E}$-lexicographic center also reduces to only one point and coincides with the nucleolus. Notice that $|\mathcal{E}| = n_1 \cdots n_m + n$ which is much lower than $|2^N| = 2^n$.

[2]Given $x, y \in \mathbb{R}^n$, we say $x <_{Lex} y$ if there is some $1 \leq i \leq n$ such that $x_i < y_i$ and $x_j = y_j$ for $1 \leq j < i$. Also, we say $x \leq_{Lex} y$ if $x <_{Lex} y$ or $x = y$.

[3]This proof can be provided by the authors under request.

In this section we show that, in the case of m-sided Böhm-Bawerk assignment games, the set of coalitions to be considered in the computation of the nucleolus can be further restricted. To this end the following lemma is needed.

Lemma 1 *Let (N,v) be a balanced game and $\mathcal{C}$ a subset of coalitions of N such that the $\mathcal{C}$-lexicographic center coincides with the nucleolus. Let $\mathcal{F} \subseteq \mathcal{C}$ be a subset of $\mathcal{C}$ such that, for all $S \in \mathcal{C} \backslash \mathcal{F}$, there is $\mathcal{T}_S = \{F_1, ..., F_p\} \subseteq \mathcal{F}$ and $\lambda_1^S, ..., \lambda_p^S, c^S \in \mathbb{R}$ satisfying that, for all $x \in C(v)$,*

(i) $e_v(S,x) \leq e_v(F_l,x)$, for all $l \in \{1,...,p\}$,

(ii) $e_v(S,x) = \lambda_1^S e_v(F_1,x) + ... + \lambda_p^S e_v(F_p,x) + c^S$.

Then, the nucleolus $\eta(v)$ coincides with the $\mathcal{F}$-lexicographic center.

Proof. To simplify the notation, let it be $\mathcal{X}^t$, Σ_t, Σ^t and α^t for $t \in \{0, 1, \ldots, s\}$, the elements of the $\mathcal{C}$-lexicographic center of (N,v), where s is the last index for which $\Sigma^s \neq \varnothing$, and $\mathcal{X}_{\mathcal{F}}^t$, $\Sigma_{t,\mathcal{F}}$, $\Sigma_{\mathcal{F}}^t$ and $\alpha_{\mathcal{F}}^t$ for $t \in \{0, 1, ..., s_{\mathcal{F}}\}$, the corresponding elements of the $\mathcal{F}$-lexicographic center of (N,v), where $s_{\mathcal{F}}$ is the last index for which $\Sigma_{\mathcal{F}}^{s_{\mathcal{F}}} \neq \varnothing$. We claim that, under the conditions of the lemma, we have $s = s_{\mathcal{F}}$ and, for all $t \in \{0, 1, ..., s\}$, $\alpha^t = \alpha_{\mathcal{F}}^t$, $\mathcal{X}^t = \mathcal{X}_{\mathcal{F}}^t$ and $\Sigma^t \cap \mathcal{F} = \Sigma_{\mathcal{F}}^t$.

We prove it by induction on t. The case $t = 0$ is trivial by the definition of step $t = 0$ in (10) together with the fact that $\mathcal{F} \subseteq \mathcal{C}$ and thus $\mathcal{F} \cap \mathcal{C} = \mathcal{F}$. Hence, assume that $\alpha^t = \alpha_{\mathcal{F}}^t$, $\mathcal{X}^t = \mathcal{X}_{\mathcal{F}}^t$ and $\Sigma^t \cap \mathcal{F} = \Sigma_{\mathcal{F}}^t$, for some $t < s$. We shall prove that $\alpha^{t+1} = \alpha_{\mathcal{F}}^{t+1}$, $\mathcal{X}^{t+1} = \mathcal{X}_{\mathcal{F}}^{t+1}$ and $\Sigma^{t+1} \cap \mathcal{F} = \Sigma_{\mathcal{F}}^{t+1}$.

First we claim that for all $S \in \Sigma^t$ there exists $T \in \Sigma^t \cap \mathcal{F}$ such that, for all $x \in \mathcal{X}^t$, $e_v(S,x) \leq e_v(T,x)$. Observe that the inequality holds trivially as an equality if $S \in \mathcal{F}$. Hence, assume that $S \in \mathcal{C} \backslash \mathcal{F}$. By hypothesis *(i)* and *(ii)*, there are $\mathcal{T}_S = \{F_1, ..., F_p\} \subseteq \mathcal{F}$ and $\lambda_1^S, ..., \lambda_p^S, c^S \in \mathbb{R}$ such that $e_v(S,x) \leq e_v(F_l,x)$, for all $l \in \{1, ..., p\}$, and

$$e_v(S,x) = \lambda_1^S e_v(F_1,x) + \cdots + \lambda_p^S e_v(F_p,x) + c^S, \tag{11}$$

for all $x \in \mathcal{X}^t \subseteq C(v)$. If it is the case that $F_l \notin \Sigma^t$ for all $l \in \{1, 2, \ldots, p\}$, then from $\Sigma^t \cap \mathcal{F} = \Sigma_{\mathcal{F}}^t$ we necessarily have $F_1, ..., F_p \in \Delta_{\mathcal{F}}^t$, which by construction of (10) implies that $e_v(F_1,x), ..., e_v(F_p,x)$ are constant on $\mathcal{X}_{\mathcal{F}}^t = \mathcal{X}^t$. Hence, by (11), $e_v(S,x)$ is also constant on $\mathcal{X}^t$, which contradicts $S \in \Sigma^t$. Once the claim is proved, for all $x \in \mathcal{X}^t$ it holds $\max_{S \in \Sigma^t} e(S,x) \leq \max_{S \in \Sigma^t \cap \mathcal{F}} e(S,x)$ and

$$\Sigma^t \neq \varnothing \Leftrightarrow \Sigma^t \cap \mathcal{F} \neq \varnothing. \tag{12}$$

Secondly, for all $x \in \mathcal{X}^t$,

$$\max_{S\in\Sigma^t} e_v(S,x) \leq \max_{S\in\Sigma^t\cap\mathcal{F}} e_v(S,x) = \max_{S\in\Sigma^t_{\mathcal{F}}} e_v(S,x) \leq \max_{S\in\Sigma^t} e_v(S,x),$$

where the equality follows from the induction hypothesis and the last inequality from $\Sigma^t_{\mathcal{F}} = \Sigma^t \cap \mathcal{F} \subseteq \Sigma^t$. Hence,

$$\max_{S\in\Sigma^t} e_v(S,x) = \max_{S\in\Sigma^t_{\mathcal{F}}} e_v(S,x). \tag{13}$$

Thus $\alpha^{t+1} = \min_{x\in\mathcal{X}^t} \max_{S\in\Sigma^t} e_v(S,x) = \min_{x\in\mathcal{X}^t_{\mathcal{F}}} \max_{S\in\Sigma^t_{\mathcal{F}}} e_v(S,x) = \alpha^{t+1}_{\mathcal{F}}$, since $\mathcal{X}^t = \mathcal{X}^t_{\mathcal{F}}$ also by induction hypothesis.

Now, by (13) and $\mathcal{X}^t = \mathcal{X}^t_{\mathcal{F}}$ we obtain $\mathcal{X}^{t+1} = \mathcal{X}^{t+1}_{\mathcal{F}}$. Therefore $\Sigma_{t+1} \cap \mathcal{F} = \Sigma_{t+1,\mathcal{F}}$ and hence $\Sigma^{t+1} \cap \mathcal{F} = \Sigma^{t+1}_{\mathcal{F}}$.

Finally, by (12) we have $s = s_{\mathcal{F}}$. Thus, since the $\mathcal{C}$-lexicographic center of (N,v) coincides with the nucleolus and $\mathcal{X}^{s+1} = \mathcal{X}^{s_{\mathcal{F}}+1}_{\mathcal{F}}$, we have that also the $\mathcal{F}$-lexicographic center of (N,v) coincides with the nucleolus. ■

The above lemma is now applied to the m-sided Böhm-Bawerk assignment game to see that, besides some singletons formed by last active agents of some sectors, only essential coalitions formed by either one (or $m-1$) last active agents of some sectors and $m-1$ (or one) first non-active agents of the remaining sectors need to be taken into account to compute the nucleolus. Formally, given $(N, \omega_{A(\mathbf{c};w)})$ an m-sided Böhm-Bawerk assignment game, let $\mathcal{F}^N = \mathcal{F}^N_{m-1} \cup \mathcal{F}^N_1$ be the subset of coalitions of N defined by

$$\mathcal{F}^N_{m-1} = \left\{ E^S \,\middle|\, S \subseteq M,\ \ |S| = m-1 \text{ and } E^S \in \prod_{k=1}^m N^k \right\} \tag{14}$$

and

$$\mathcal{F}^N_1 = \left\{ E^S \,\middle|\, S \subseteq M, |S| = 1, E^S \in \prod_{k=1}^m N^k \right\} \cup \left\{ \{r \in N^l\} \,\middle|\, E^{\{l\}} \notin \prod_{k=1}^m N^k \right\}, \tag{15}$$

where recall that $E^S = r\mathbf{1}_S + (r+1)\,\mathbf{1}_{M\setminus S}$. Observe that $\mathcal{F}^N_{m-1}$ is composed of all essential coalitions (only if exist) formed by the $r+1^{th}$ agent of one sector and the r^{th} agent of the remaining $m-1$ sectors, whereas $\mathcal{F}^N_1$ is formed by all essential coalitions composed by the r^{th} agent of one sector, let us say $l \in M$, and the $r+1^{th}$ agents of the remaining $m-1$ sectors, whenever these essentials coalitions exist, i.e. $E^{\{l\}} \in \prod_{k=1}^m N^k$, or the singleton formed by the r^{th} agent of sector l otherwise, i.e. when $E^{\{l\}} \notin \prod_{k=1}^{m} N^k$.[4] In particular observe that $|\mathcal{F}_{m-1}^N| \leq m$ and $|\mathcal{F}_1^N| = m$, and hence this time $|\mathcal{F}^N| \leq 2m$ which is much lower than $|\mathcal{E}| = n_1...n_m + n$.

Theorem 2 *Let $(N, \omega_{A(\mathbf{c};w)})$ be an m-sided Böhm-Bawerk assignment game. Then the nucleolus $\eta\left(\omega_{A(\mathbf{c};w)}\right)$ coincides with the $\mathcal{F}^N$-lexicographic center of $(N, \omega_{A(\mathbf{c};w)})$.*

Proof. Consider the $\mathcal{E}$-lexicographic center of (N, ω_A), which is known to coincide with the nucleolus $\eta(\omega_A)$. It can be easily checked that at step $t = 1$ in (10) we obtain $\alpha^1 = 0$, $\mathcal{X}^1 = C(\omega_A)$, $\Sigma_1 = \Delta^1 = \{S \in \mathcal{E} \mid e(S,x) \text{ is constant in } C(\omega_A)\}$ and $\Sigma^1 = \mathcal{E} \backslash \Sigma_1$. Hence, we can start the algorithm of the $\mathcal{E}$-lexicographic center with $\alpha^0 = 0$, $\mathcal{X}^0 = C(\omega_A)$ and

$$(16) \quad \Sigma_0 = \Delta^0 = \{S \in \mathcal{E} \mid e(S,x) \text{ is constant in } C(\omega_A)\} \text{ and } \Sigma^0 = \mathcal{E} \backslash \Sigma_0.$$

Since any essential coalition formed by either only active agents or only inactive agents belongs to some optimal matching, by (5) each such coalition receives a constant payoff in $C(\omega_A)$, and hence, in the above algorithm, Σ^0 is composed of all essential coalitions containing both active agents and inactive agents, and all singletons formed by one active agent (if there exist).

Let $x \in C(\omega_A)$ be an arbitrary core allocation. To prove the theorem we will show that $\mathcal{F}^N$ satisfies the assumptions of Lemma 1, i.e. for each $S \in \Sigma^0 \backslash \mathcal{F}^N$ there is $\mathcal{T}_S = \{F_1, ..., F_p\} \subseteq \mathcal{F}^N$ such that $e_{\omega_A}(S,x) \leq e_{\omega_A}(F_t,x)$ for all $t \in \{1,...,p\}$ and $e_{\omega_A}(S,x) = \lambda_1^S e_{\omega_A}(F_1,x) + ... + \lambda_p^S e_{\omega_A}(F_p,x) + c^S$ for some $\lambda_1^S, ..., \lambda_p^S, c^S \in \mathbb{R}$ which do not depend on x. Thus, let it be $S \in \Sigma^0 \backslash \mathcal{F}^N$. We distinguish two cases, depending on whether S is an essential coalition or a singleton.

Case 1: $S = E = (i_1, ..., i_m) \in \prod_{k=1}^{m} N^k$.

Consider a set of sectors associated to E defined by $S_E = \{k \in M \mid 1 \leq i_k \leq r\}$. By (16), we have $\varnothing \subsetneq S_E \subsetneq M$. Due to the non-symmetrical notation of buyers' and sellers' valuations, we must write separately the case $m \in S_E$ and $m \notin S_E$. Nevertheless, the proof of the latter case is analogous to the proof of the former and hence we assume $m \in S_E$, whereas the case $m \notin S_E$ is left to the reader. Let us also denote by $E' = \sum_{k \in S_E} i_k \mathbf{1}_{\{k\}} + (r+1)\,\mathbf{1}_{M \backslash S_E}$ the essential coalition obtained from E by replacing agents of

[4] We could add a null agent with an arbitrarily high cost if it is a seller, or a null agent with an arbitrary low valuation if it is a buyer, to those sectors $k \in M$ with $n_k = r$, hence ensuring the existence of the $r + 1^{th}$ agent for each sector. In that case, $\mathcal{F}_{m-1}^N = \left\{E^S \mid S \subseteq M,\ |S| = m-1\right\}$ and $\mathcal{F}_1^N = \left\{E^S \mid S \subseteq M,\ |S| = 1\right\}$.

each sector $k \in M\backslash S_E$ by the $r+1^{th}$ agent of the same sector. Since $E' \le E$, by (2) we have $a_E \le a_{E'}$. We start proving that

$$e_{\omega_A}(E,x) \le e_{\omega_A}(E^{S_E},x). \tag{17}$$

Indeed,

$$\begin{aligned}
& e_{\omega_A}(E,x) \\
= \;& a_E - \sum_{k=1}^{m} x_{ki_k} \le a_{E'} - \sum_{k\in S_E} x_{ki_k} - \sum_{k\in M\backslash S_E} x_{ki_k} = a_{E'} - \sum_{k\in S_E} x_{ki_k} \\
= \;& a_{E'} - (x_{mr} + (w_{i_m} - w_r)) - \sum_{k\in S_E\backslash\{m\}} (x_{kr} + (c_{kr} - c_{ki_k})) \\
= \;& \max\left\{0, w_{i_m} - \sum_{k\in S_E\backslash\{m\}} c_{ki_k} - \sum_{k\in M\backslash S_E} c_{k(r+1)}\right\} \\
& - (x_{mr} + (w_{i_m} - w_r)) - \sum_{k\in S_E\backslash\{m\}} (x_{kr} + (c_{kr} - c_{ki_k})) \\
= \;& \max\left\{0, w_r - \sum_{k\in S_E\backslash\{m\}} c_{kr} - \sum_{k\in M\backslash S_E} c_{k(r+1)} + (w_{i_m} - w_r) + \sum_{k\in S_E\backslash\{m\}} (c_{kr} - c_{ki_k})\right\} \\
& - \left((w_{i_m} - w_r) + \sum_{k\in S_E\backslash\{m\}} (c_{kr} - c_{ki_k})\right) - \sum_{k\in S_E} x_{kr} \\
= \;& \max\left\{-\left((w_{i_m} - w_r) + \sum_{k\in S_E\backslash\{m\}} (c_{kr} - c_{ki_k})\right), w_r - \sum_{k\in S_E\backslash\{m\}} c_{kr} - \sum_{k\in M\backslash S_E} c_{k(r+1)}\right\} \\
& - \sum_{k\in S_E} x_{kr} \\
\le \;& \max\left\{0, w_r - \sum_{k\in S_E\backslash\{m\}} c_{kr} - \sum_{k\in M\backslash S_E} c_{k(r+1)}\right\} - \sum_{k\in S_E} x_{kr} \\
= \;& a_{E^{S_E}} - \sum_{k\in S_E} x_{kr} = e_{\omega_A}(E^{S_E},x),
\end{aligned}$$

where the second and the third equalities hold by (5) and (8), the fifth equality holds adding and subtracting $w_r - \sum_{k\in S_E\backslash\{m\}} c_{kr}$ to the second term in the maximum operator, and the last inequality holds by (2). We continue by distinguishing two subcases.

Case 1.1: $a_{E^{S_E}} > 0$.

Since $E \in \prod_{k=1}^m N^k$, it trivially follows $E^{S_E} \in \prod_{k=1}^m N^k$. Recall that by (16), $S_E \subsetneq M$. We now prove that, for each $k' \notin S_E$,

$$e_{\omega_A}(E^{S_E}, x) \leq e_{\omega_A}(E^{M\setminus\{k'\}}, x). \tag{18}$$

Before proving (18) observe that, since $x \in C(\omega_A)$, by (5) we have

$$x(E^M) = \sum\nolimits_{l\in M} x_{lr} = a_{E^M} = w^r - \sum\nolimits_{l\in\{1,\dots,m-1\}} c_{lr}$$

and, for each $k \in M \setminus \{m\}$,

$$x(E^{M\setminus\{k\}}) = \sum\nolimits_{l\in M\setminus\{k\}} x_{lr} \geq a_{E^{M\setminus\{k\}}} \geq w^r - c_{k(r+1)} - \sum\nolimits_{l\in M\setminus\{k,m\}} c_{lr}.$$

Combining the two above expressions we obtain

$$x_{kr} - \left(c_{k(r+1)} - c_{kr}\right) \leq 0, \text{ for all } k \in M \setminus \{m\}. \tag{19}$$

Then, for each $k' \notin S_E$,

$$\begin{aligned}
& e_{\omega_A}(E^{S_E}, x) = a_{E^{S_E}} - \sum_{k\in S_E} x_{kr} = w^r - \sum_{k\in S_E\setminus\{m\}} c_{kr} - \sum_{k\in M\setminus S_E} c_{k(r+1)} - \sum_{k\in S_E} x_{kr} \\
= \; & w^r - c_{k'(r+1)} - \sum_{k\in M\setminus\{k',m\}} c_{kr} - \sum_{k\in M\setminus\{k'\}} x_{kr} + \sum_{k\in(M\setminus\{k'\})\setminus S_E} \left(x_{kr} - \left(c_{k(r+1)} - c_{kr}\right)\right) \\
\leq \; & w^r - c_{k'(r+1)} - \sum_{k\in M\setminus\{k',m\}} c_{kr} - \sum_{k\in M\setminus\{k'\}} x_{kr} \\
\leq \; & \max\left\{0, w^r - c_{k'(r+1)} - \sum_{k\in M\setminus\{k',m\}} c_{kr}\right\} - \sum_{k\in M\setminus\{k'\}} x_{kr} \\
= \; & a_{E^{M\setminus\{k'\}}} - \sum_{k\in M\setminus\{k'\}} x_{kr} = e_{\omega_A}(E^{M\setminus\{k'\}}, x),
\end{aligned}$$

where the second equality follows from the assumption $a_{E^{S_E}} > 0$, the third equality is obtained by adding and subtracting $\sum_{k\in(M\setminus\{k'\})\setminus S_E} (x_{kr} + c_{kr})$ and the first inequality holds by (19). Therefore (18) indeed holds.

Next, since $E = (i_1, ..., i_m) \in \prod_{k=1}^m N^k$, for any $k' \in M\setminus S_E$ we have $i_{k'} \geq r+1$ and thus agent $i_{k'} \in N^{k'}$ exists, which implies $E^{M\setminus\{k'\}}$ is also an essential coalition, i.e. $E^{M\setminus\{k'\}} \in \prod_{k=1}^m N^k$. Therefore, we can consider the following nonempty subset of $\mathcal{F}_{m-1}^N \subseteq \mathcal{F}^N$,

$$\mathcal{T}_S = \left\{E^{M\setminus\{k\}} \mid k \in M\setminus S_E\right\}. \tag{20}$$

Notice that the cardinality of $\mathcal{T}_S$ is the same as that of $M\backslash S_E$. For each $k \in M\backslash S_E$ let $F_k \in \mathcal{T}_S$ denote the associated coalition $M\backslash\{k\}$ of $\mathcal{T}_S$. From (17) and (18), we obtain that $e_{\omega_A}(E,x) \leq e_{\omega_A}(F_k,x)$ for all $F_k \in \mathcal{T}_S$, which implies that property *(i)* of Lemma 1 is satisfied for $S = E$, taking $\mathcal{F} = \mathcal{F}^N$.

Further, we prove that also property *(ii)* of Lemma 1 is satisfied. First of all observe that

$$\begin{aligned}
x(E^{S_E}) &= \frac{1}{|M\backslash S_E|} \sum_{k\in M\backslash S_E} x(E^{S_E}) \\
&= \frac{1}{|M\backslash S_E|} \sum_{k\in M\backslash S_E} \left(x(E^{M\backslash\{k\}}) - x(E^{(M\backslash\{k\})\backslash S_E})\right) \\
&= \frac{1}{|M\backslash S_E|} \left(\sum_{k\in M\backslash S_E} x(E^{M\backslash\{k\}}) - \sum_{k\in M\backslash S_E} x(E^{(M\backslash\{k\})\backslash S_E})\right) \\
&= \frac{1}{|M\backslash S_E|} \sum_{k\in M\backslash S_E} x(E^{M\backslash\{k\}}) - \left(\frac{|M\backslash S_E|-1}{|M\backslash S_E|}\right) x(E^{M\backslash S_E}) \\
&= \frac{1}{|M\backslash S_E|} \sum_{k\in M\backslash S_E} x(E^{M\backslash\{k\}}) - \left(\frac{|M\backslash S_E|-1}{|M\backslash S_E|}\right) (a_{E^M} - x(E^{S_E})),
\end{aligned}$$

where the last equality holds since, by (5), $x(E^{S_E}) + x(E^{M\backslash S_E}) = x(E^M) = a_{E^M}$. Therefore,

$$x(E^{S_E}) = \sum_{k'\in M\backslash S_E} x(E^{M\backslash\{k'\}}) - (|M\backslash S_E| - 1)\, a_{E^M}. \tag{21}$$

and

$$\begin{aligned}
x(E) &= \sum_{k\in S_E} x_{ki_k} + \sum_{k\in M\backslash S_E} x_{ki_k} = \sum_{k\in S_E} x_{ki_k} = x(E^{S_E}) + \sum_{k\in S_E} t_{ki_k} \\
&= \sum_{k'\in M\backslash S_E} x(E^{M\backslash\{k'\}}) - (|M\backslash S_E| - 1)\, a_{E^M} + \sum_{k\in S_E} t_{ki_k}, \qquad (22)
\end{aligned}$$

where the second and third equalities hold by (8) and the last equality holds by (21). To conclude, by (22), the excess $e_{\omega_A}(E,x)$ is an affine combination of the excesses associated to coalitions of $\mathcal{T}_S$:

$$\begin{aligned} e_{\omega_A}(E,x) &= a_E - x(E) \\ &= a_E - \sum_{k\in M\setminus S_E} x(E^{M\setminus\{k\}}) + (|M\setminus S_E|-1)\, a_{E^M} - \sum_{k\in S_E} t_{ki_k} \\ &= \sum_{k\in M\setminus S_E} \Big(a_{E^{M\setminus\{k\}}} - x(E^{M\setminus\{k\}})\Big) \\ &\quad + \overbrace{a_E - \sum_{k\in S_E} t_{ki_k} - \sum_{k\in M\setminus S_E} a_{E^{M\setminus\{k\}}} + (|M\setminus S_E|-1)\, a_{E^M}}^{c^S} \\ &= \sum_{k\in M\setminus S_E} e_{\omega_A}(F_k,x) + c^S, \end{aligned} \tag{23}$$

where the third equality is obtained by adding and subtracting $\sum_{k\in M\setminus S_E} a_{E^{M\setminus\{k\}}}$. Therefore, as we claimed, the two requirements of Lemma 1 applied to $S=E$ (under the assumptions of Case 1.1) are satisfied for all $x\in C(\omega_A)$, taking $\mathcal{F}=\mathcal{F}^N$ and $\mathcal{T}_S$ as in (20).

Case 1.2: $a_{E^{S_E}}=0$.

In this case, consider the following nonempty subset of $\mathcal{F}_1^N \subseteq \mathcal{F}^N$,

$$\begin{aligned} \mathcal{T}_S &= \left\{ \{r\in N^l\} \,\middle|\, l\in S_E \text{ and } E^{\{l\}} \notin \prod_{k=1}^m N^k \right\} \\ &\quad \cup \left\{ E^{\{l\}} \,\middle|\, l\in S_E \text{ and } E^{\{l\}} \in \prod_{k=1}^m N^k \right\}. \end{aligned} \tag{24}$$

For each $l\in S_E$ let $F_l\in\mathcal{T}_S$ denote the associated coalition of $\mathcal{T}_S$. Notice that the cardinality of $\mathcal{T}_S$ is the same as the one of S_E since, for each $l\in S_E$, either $E^{\{l\}}\notin\prod_{k=1}^m N^k$ and we consider the singleton formed by $r\in N^l$ or $E^{\{l\}}\in\prod_{k=1}^m N^k$ and we consider the essential coalition $E^{\{l\}}$. Notice also that in this second case $0\le a_{E^{\{l\}}}\le a_{E^{S_E}}=0$. In any case $w_A(F_l)=0$ for all $F_l\in\mathcal{T}_S$. For each $l\in S_E$,

$$e_{\omega_A}(E^{S_E},x) = -\sum_{k\in S_E} x_{kr} \le -x_{lr} = \omega_A(F_l) - x_{lr} = e_{\omega_A}(F_l,x), \tag{25}$$

and hence

$$e_{\omega_A}(E,x) \le e_{\omega_A}(E^{S_E},x) \le e_{\omega_A}(F_l,x),$$

where the third inequality holds by (17). Hence property *(i)* of Lemma 1 is satisfied for $S = E$, on the assumptions of Case 1.2, taking $\mathcal{F} = \mathcal{F}^N$ and $\mathcal{T}_S$ as in (24). Further, property *(ii)* of Lemma 1 is also satisfied. Indeed,

$$
\begin{aligned}
e_{\omega_A}(E, x) &= a_E - x(E) = a_E - \sum_{l \in S_E} x_{li_l} - \sum_{l \in M \setminus S_E} x_{li_l} \\
&= a_E - x(E^{S_E}) - \sum_{l \in S_E} t_{li_l} \\
&= \sum_{l \in S_E} e_{\omega_A}(F_l, x) + \overbrace{a_E - \sum_{l \in S_E} t_{li_l}}^{c^S}, \qquad (26)
\end{aligned}
$$

where the first equality holds by (8) and (5), and the last equality holds from $w_A(F_l) = 0$ for all $F_l \in \mathcal{T}_S$. Therefore, the two requirements of Lemma 1 applied to $S = E$ (under the assumptions of Case 1.2) are again satisfied, taking $\mathcal{F} = \mathcal{F}^N$ and $\mathcal{T}_S$ as in (24).

Case 2: $S = \{i\}$.

By (16), we can assume $i \in N^l$, for some $l \in M$ and $i \leq r$. Let $\mathcal{T}_S$ be the following singleton of $\mathcal{F}_1^N \subseteq \mathcal{F}^N$,

$$
\mathcal{T}_S = \{F\} = \begin{cases} \{E^{\{l\}}\} & \text{if } E^{\{l\}} \in \prod_{k=1}^m N^k, \\ \{r \in N^l\} & \text{if } E^{\{l\}} \notin \prod_{k=1}^m N^k. \end{cases} \qquad (27)
$$

By (8), we obtain

$$
e_{\omega_A}(\{i\}, x) = -x_{li} = -t_{li} - x_{lr} = e_{\omega_A}(F, x) - \overbrace{\omega_A(F) - t_{li}}^{c^S},
$$

where the last equality holds by adding and subtracting $\omega_A(F)$. Therefore properties *(i)* and *(ii)* of Lemma 1 are satisfied for $S = \{i\}$, taking $\mathcal{F} = \mathcal{F}^N$ and $\mathcal{T}_S$ as in (27).

To sum up, the assumptions of Lemma 1 are satisfied for all $S \in \mathcal{E} \setminus \mathcal{F}^N$, to guarantee that the $\mathcal{F}^N$-lexicographic center coincides with the nucleolus. ■

The result in Theorem 2 simplifies the computation of the nucleolus of a multi-sided Böhm-Bawerk assignment game. Indeed, consider for instance a market situation with eight sellers $S_1, ..., S_8$ each of them owning one unit of a homogenous software good, eight different sellers $H_1, ..., H_8$ each of them owning one unit of a homogenous hardware good and $B_1, ..., B_{10}$ ten potential buyers interested on acquiring a bundle formed exactly by one unit of software and one unit of hardware. Table 1 below shows the valuations of each agent in this three-sided Böhm-Bawerk assignment market, which translates into a 26-person cooperative game. It is straightforward to check that there are five active agents on each side of the market, that is $r = 5$, which is marked in bold.

Table 1

Software (s) sellers	Hardware (h) sellers	Buyers
S_1 values her good at 5	H_1 values her good at 5	B_1 values a s/h pair at 30
S_2 values her good at 5	H_2 values her good at 6	B_2 values a s/h pair at 28
S_3 values her good at 7	H_3 values her good at 8	B_3 values a s/h pair at 26
S_4 values her good at 8	H_4 values her good at 9	B_4 values a s/h pair at 24
S_5 values her good at **10.75**	H_5 values her good at **9.25**	B_5 values a s/h pair at **22**
S_6 values her good at 11	H_6 values her good at 10.5	B_6 values a s/h pair at 21
S_7 values her good at 12	H_7 values her good at 13	B_7 values a s/h pair at 20
S_8 values her good at 13	H_8 values her good at 13	B_8 values a s/h pair at 18
		B_9 values a s/h pair at 17
		B_{10} values a s/h pair at 15

As a result of Theorem 2, in order to calculate the nucleolus of the corresponding coalitional game $(N, \omega_{A(\mathbf{c};w)})$ with 2^{26} coalitions we only have to consider coalitions in $\mathcal{F}^N = \mathcal{F}_1^N \cup \mathcal{F}_2^N$, where $\mathcal{F}_1^N = \{(5,6,6),(6,5,6),(6,6,5)\}$ and $\mathcal{F}_2^N = \{(6,5,5),(5,6,5),(5,5,6)\}$. However, the number of agents is still high, 26, which means that we have to solve several linear programs with 26 variables. The procedure can be simplified further by exploiting the connection between the cores of the multi-sided assignment game and its related sectors game $(M, v^M_{\mathbf{c};w})$. To this end Lemma 1 is applied to the sectors game to show that only singletons and coalitions of size $m-1$ are needed to compute its nucleolus $\eta\left(v^M_{\mathbf{c};w}\right)$. This fact reinforces the idea that the sectors game is a quite special convex game. As a consequence, the relationship between the nucleolus of the m-sided Böhm-Bawerk game $(N, \omega_{A(\mathbf{c};w)})$ and that of its sectors game is established.

Given the corresponding sectors game $(M, v^M_{\mathbf{c};w})$, let us consider the subset of coalitions of M defined by $\mathcal{F}^M = \mathcal{F}_1^M \cup \mathcal{F}_{m-1}^M$, where

$$\mathcal{F}_{m-1}^M = \{S \in M,\ |S| = m-1\}, \tag{28}$$

and

$$\mathcal{F}_1^M = \{S \in M,\ |S| = 1\}. \tag{29}$$

Theorem 3 *Let* $(N, \omega_{A(\mathbf{c};w)})$ *be an* m*-sided Böhm-Bawerk assignment game and let* $(M, v^M_{\mathbf{c};w})$ *be its associated sectors game. Let also* $\eta(\omega_{A(\mathbf{c};w)})$ *and* $\eta\left(v^M_{\mathbf{c};w}\right)$ *be the corresponding nucleolus. Then,*

(a) $\eta\left(v^M_{\mathbf{c};w}\right)$ *coincides with the* $\mathcal{F}^M$*-lexicographic center of* $(M, v^M_{\mathbf{c};w})$*.*

(b) $\eta(\omega_{A(\mathbf{c};w)}) = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}\left(\eta\left(v^M_{\mathbf{c};w}\right)\right)$*.*

Proof. To start proving statement (a) of the theorem, recall the notation $E^S = r\mathbf{1}_S + (r+1)\,\mathbf{1}_{M\setminus S}$ for all $S \subseteq M$ and let us see that (M, v^M), $\mathcal{C} = 2^N$ and $\mathcal{F}^M$ are on the assumptions of Lemma 1. Let it be $\varnothing \subsetneq S \subsetneq M$ and $\overline{x} \in C\left(v^M\right)$ an arbitrary core allocation of the sectors game. If $E^S \in \prod_{k=1}^m N^k$, by Definition 1 and (8), it is straightforward to check that

$$e_{v^M}(S, \overline{x}) = e_{\omega_A}(E^S, x), \tag{30}$$

where $x = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\overline{x})$. As before, let us define for each essential coalition E the set $S_E = \{k \in M \mid 1 \le i_k \le r\}$. The reader can check that $S_{E^S} = S$. We distinguish two cases.

Case a.1: $v^M(S) > 0$.

By Definition 1, $v^M(S) > 0$ implies $E^S \in \prod_{k=1}^m N^k$. Let us consider the nonempty set of coalitions $\mathcal{T}_S = \{M\setminus\{k\} \mid k \in M\setminus S\}$, which is a subset of $\mathcal{F}^M_{m-1}$. Observe that $\mathcal{T}_S$ is in one-to-one correspondence with the set defined in (20). Now, for all $k \in M\setminus S$,

$$e_{v^M}(S, \overline{x}) = e_{\omega_A}(E^S, x) \le e_{\omega_A}(E^{M\setminus\{k\}}, x) = e_{v^M}(M\setminus\{k\}, \overline{x}),$$

where the inequality holds by (18) and both equalities hold by (30). Furthermore, from (23) and making use of (30), we deduce that $e_{v^M}(S, \overline{x}) = \sum_{F\in\mathcal{T}_S} e_{v^M}(F, x) + c^S$, where c^S is defined in (23) . Therefore the two requirements of Lemma 1 applied to S are satisfied, taking $\mathcal{F} = \mathcal{F}^M$ and the collection $\mathcal{T}_S$ above defined.

Case a.2: $v^M(S) = 0$.

Notice that, by Definition 1, either $E^S \in \prod_{k=1}^m N^k$ and $v^M(S) = a_{E^S} = 0$ or $E^S \notin \prod_{k=1}^m N^k$. In either case, let us consider the nonempty set of coalitions $\mathcal{T}_S = \{\{l\} \mid l \in S\}$, which is a subset of $\mathcal{F}^M_1$. Observe that $\mathcal{T}_S$ is in one-to-one correspondence with the set defined in (24). On the one hand, if $E^S \in \prod_{k=1}^m N^k$, by (25) and (30) we easily deduce that $e_{v^M}(S, \overline{x}) \le e_{v^M}(F, \overline{x})$ for all $F \in \mathcal{T}_S$. Furthermore, making use of (30), we deduce that $e_{v^M}(S, \overline{x}) = \sum_{F\in\mathcal{T}_S} e_{v^M}(F, x) + c^S$, where c^S is defined in (26). On the other hand, if

$E^S \notin \prod_{k=1}^m N^k$ we have $e_{v^M}(S,\overline{x}) = -\sum_{k\in S}\overline{x}_k = \sum_{F\in\mathcal{T}_S} e_{v^M}(F,\overline{x})$. Therefore the two requirements of Lemma 1 applied to S are satisfied, taking $\mathcal{F} = \mathcal{F}^M$ and the collection $\mathcal{T}_S$ above defined.

Thus, Lemma 1 guarantees that $\eta\left(v^M_{\mathbf{c};w}\right)$ coincides with the $\mathcal{F}^M$-lexicographic center of $(M, v^M_{\mathbf{c};w})$ and hence we finish the proof of statement (a).

Next we prove statement (b) of the theorem. Let $\mathcal{X}^t_N$, Σ^N_t, Σ^t_N and α^t_N for $t \in \{0,1,\ldots,s_N\}$, be the elements in the algorithm of the $\mathcal{F}^N$-lexicographic center of $(N, \omega_{A(\mathbf{c};w)})$, where s_N is the last index for which $\Sigma^{s_N}_N \neq \varnothing$. Let also $\mathcal{X}^t_M$, Σ^M_t, Σ^t_M and α^t_M for $t \in \{0,1,\ldots,s_M\}$, be the elements in the algorithm of the $\mathcal{F}^M$-lexicographic center of $(M, v^M_{\mathbf{c};w})$, where s_M is the last index for which $\Sigma^{s_M}_M \neq \varnothing$.

Recall the definitions of $\mathcal{F}^N$ and $\mathcal{F}^M$ at (14), (15), (29) and (28), and let us consider the mapping $\Psi : \mathcal{F}^N \rightarrow \mathcal{F}^M$ that assigns each coalition of $\mathcal{F}^N_{m-1}$ to a coalition of $\mathcal{F}^M_{m-1}$ and each coalition of $\mathcal{F}^N_1$ to a coalition of $\mathcal{F}^M_1$ in the following way:

$$\Psi(T) = \begin{cases} S & \text{if } T = E^S \in \mathcal{F}^N \text{ , for some } S \subseteq M, \\ \{l\} & \text{if } T = \{r\} \in \mathcal{F}^N \text{ and } r \in N^l. \end{cases} \tag{31}$$

Observe that, by construction of Ψ and the definitions of $\mathcal{F}^N$ and $\mathcal{F}^M$, Ψ is injective. Moreover, the restriction of Ψ to $\mathcal{F}^N_1$ is bijective, since all $S \subseteq M$ with $|S| = 1$ belong to $\Psi(\mathcal{F}^N)$. When there exists the $r+1^{th}$ agent of each of the m sectors, Ψ is a bijection.

By (30) and Definition 1, for all $T \in \mathcal{F}^N$ and all $x \in C(\omega_A)$,

$$e_{v^M}(\Psi(T),\overline{x}) = e_{\omega_A}(T,x), \tag{32}$$

where $\overline{x} \in C(v^M)$ satisfies $x = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\overline{x})$.

We claim that $\alpha^t_N = \alpha^t_M$, $\mathcal{X}^t_N = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}^t_M)$ and $\Psi\left(\Sigma^t_N\right) \subseteq \Sigma^t_M$, for all $t \in \{0,...,s\}$, and as a consequence $s_M = s_N = s$. We prove it by induction on t. For $t=0$ we only have to prove that $\mathcal{X}^0_N = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}^0_M)$, which holds by (8).

Now assume that $\alpha^t_N = \alpha^t_M$, $\mathcal{X}^t_N = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}^t_M)$ and $\Psi(\Sigma^t_N) \subseteq \Sigma^t_M$, for some $t < s_N$. We prove that $\alpha^{t+1}_N = \alpha^{t+1}_M$, $\mathcal{X}^{t+1}_N = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}^{t+1}_M)$, and $\Psi\left(\Sigma^{t+1}_N\right) \subseteq \Sigma^{t+1}_M$.

In the first place, we claim that, for each $\overline{x} \in \mathcal{X}^t_M$,

$$\max_{T\in\Sigma^t_N} e_{v^M}(\Psi(T),\overline{x}) = \max_{S\in\Sigma^t_M} e_{v^M}(S,\overline{x}). \tag{33}$$

Indeed, by induction hypothesis $\Psi(\Sigma^t_N) \subseteq \Sigma^t_M$, and hence $\max_{T\in\Sigma^t_N} e_{v^M}(\Psi(T),\overline{x}) \leq \max_{S\in\Sigma^t_M} e_{v^M}(S,\overline{x})$. If this latter inequality were strict, there would exist

$S \in \Sigma_M^t \backslash \Psi(\Sigma_N^t)$ such that

$$e_{v^M}(S, \overline{x}) > \max_{T \in \Sigma_N^t} e_{v^M}(\Psi(T), \overline{x}) = \max_{T \in \Sigma_N^t} e_{\omega_A}(T, x), \tag{34}$$

where $x = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\overline{x})$ and the equality holds by (32).

Let us first prove that necessarily $S \in \mathcal{F}^M \backslash \Psi(\mathcal{F}^N)$. Otherwise, suppose that $S \in \Psi(\mathcal{F}^N)$ and let $T \in \mathcal{F}^N$ be such that $S = \Psi(T)$. Since $S \notin \Psi(\Sigma_N^t)$, by construction of (10), $T \in \Delta_N^t$. Thus, $e_{\omega_A}(T, x)$ is constant on $\mathcal{X}_N^t$. But then, since $\mathcal{X}_N^t = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}_M^t)$ by the induction hypothesis and by (32), $e_{v^M}(S, \overline{x})$ is also constant on $\mathcal{X}_M^t$. Therefore $S \in \Delta_M^t$, which contradicts $S \in \Sigma_M^t$.

Once established that $S \in \mathcal{F}^M \backslash \Psi(\mathcal{F}^N)$, we necessarily have $|S| = m - 1$ and $E^S \notin \prod_{k=1}^m N^k$, which implies $v^M(S) = 0$ by Definition 1. Then, for all $k \in S$,

$$e_{v^M}(S, \overline{x}) = 0 - \overline{x}(S) \leq -\overline{x}_k \leq v^M(\{k\}) - \overline{x}_k = e_{v^M}(\{k\}, \overline{x}). \tag{35}$$

Suppose that $\{k\} \notin \Sigma_M^t$ for all $k \in S$. Then, $\{k\} \in \Delta_M^t$ for all $k \in S$, i.e. $e_{v^M}(\{k\}, \overline{x})$ is constant on $\mathcal{X}_M^t$, which implies that $\overline{x}_k$ is also constant on $\mathcal{X}_M^t$. Since $|S| = m - 1$ and $\overline{x}(M) = v^M(M)$, we necessarily have that $\mathcal{X}_M^t$ and $\mathcal{X}_N^t = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}_M^t)$ are composed of a single point, i.e. $t = s_N = s_M$, which contradicts $t < s_N$. Thus it cannot be the case that $\{k\} \notin \Sigma_M^t$ for all $k \in S$. Hence let $\{k\} \in \Sigma_M^t$ for some $k \in S$. By construction of Ψ, there exists $T' = \Psi^{-1}(\{k\}) \in \mathcal{F}^N$. If $T' \notin \Sigma_N^t$, i.e. $T' \in \Delta_N^t$, then $e_{\omega_A}(T', x)$ is constant on $\mathcal{X}_N^t$ and, as above, by the induction hypothesis and by (32) also $e_{v^M}(\{k\}, \overline{x})$ is constant on $\mathcal{X}_M^t$, which contradicts $\{k\} \in \Sigma_M^t$. Therefore, $T' \in \Sigma_N^t$, which together with (32) and (35) implies $e_{v^M}(S, \overline{x}) \leq e_{v^M}(\{k\}, \overline{x}) = e_{\omega_A}(T', x)$, in contradiction with (34). Hence (33) holds, as we claimed.

Once the claim is proved, we show that $\alpha_N^{t+1} = \alpha_M^{t+1}$. Indeed,

$$\alpha_N^{t+1} = \min_{x \in \mathcal{X}_N^t} \max_{T \in \Sigma_N^t} e_{\omega_A}(T, x) = \min_{\overline{x} \in \mathcal{X}_M^t} \max_{T \in \Sigma_N^t} e_{v^M}(\Psi(T), \overline{x}) = \min_{\overline{x} \in \mathcal{X}_M^t} \max_{S \in \Sigma_M^t} e_{v^M}(S, \overline{x}) = \alpha_M^{t+1}, \tag{36}$$

where the second equality holds by (32) and the third equality holds by (33).

Secondly, $\mathcal{X}_N^{t+1} = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}_M^{t+1})$ holds by (33) and (36) since $\mathcal{X}_N^t = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}_M^t)$ by the induction hypothesis.

In the third place, suppose that $\Psi\left(\Sigma_N^{t+1}\right) \not\subseteq \Sigma_M^{t+1}$, i.e. there is $T \in \Sigma_N^{t+1} = \Sigma_N^t \backslash \Sigma_{t+1}^N$ such that $\Psi(T) \notin \Sigma_M^{t+1} = \Sigma_M^t \backslash \Sigma_{t+1}^M$. All this means that $T \in \Sigma_N^t$ and, since $\Psi\left(\Sigma_N^t\right) \subseteq \Sigma_M^t$, also $\Psi(T) \in \Sigma_M^t$. Thus, $\Psi(T) \notin$

Σ_M^{t+1} implies $\Psi(T) \in \Sigma_{t+1}^M \subseteq \Delta_M^{t+1}$, and hence $e_{v^M}(\Psi(T), \overline{x})$ is constant on $\mathcal{X}_M^{t+1}$. We already know that $\mathcal{X}_N^{t+1} = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}_M^{t+1})$. Therefore, by (32), $e_{\omega_A}(T, x)$ is constant on $\mathcal{X}_N^{t+1}$, where $x = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\overline{x})$, which implies $T \in \Delta_N^{t+1}$, and hence we reach a contradiction with $T \in \Sigma_N^{t+1}$. As a consequence, $\Psi\left(\Sigma_N^{t+1}\right) \subseteq \Sigma_M^{t+1}$.

Finally, from the fact that $\mathcal{X}_N^t = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}_M^t)$, we know that $\mathcal{X}_N^t$ reduces to a single point if and only if also $\mathcal{X}_M^t$ reduces to a single point, and thus we conclude that $s_N = s_M = s$ and, from $\mathcal{X}_N^s = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\mathcal{X}_M^s)$, we obtain $\eta(\omega_{A(\mathbf{c};w)}) = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\eta\left(v_{\mathbf{c};w}^M\right))$. ■

Consider again the market in Table 1 and notice that to obtain the nucleolus of the three-sided Böhm-Bawerk assignment game (N, ω_A) we essentially have to compute the nucleolus $\eta\left(v_{\mathbf{c};w}^M\right)$ of the sectors game (M, v^M), which in this case is the three-person game given below:

$$
\begin{array}{lll}
v^M(\{1\}) = a_{566} = 0 & v^M(\{1,2\}) = a_{556} = 1 & \\
v^M(\{2\}) = a_{656} = 0.75 & v^M(\{1,3\}) = a_{565} = 0.75 & v^M(\{1,2,3\}) = a_{555} = 2. \\
v^M(\{3\}) = a_{665} = 0.5 & v^M(\{2,3\}) = a_{655} = 1.75 &
\end{array}
$$

It can be checked that $\eta\left(v^M\right) = (0.1250, 1.0625, 0.8125)$. This can be done by means of the formulae provided in Moulin (1988) to calculate the nucleolus of a three-person game. Then, from part (b) of Theorem 3 we obtain $\eta(w_A)$, as it is shown in the table below, where we write $\eta\left(v^M\right) = \eta$ for short. All this means that we have closed formulae to compute the nucleolus of a three-sided Böhm-Bawerk assignment game, no matter how large the number of agents is.

Table 2

Ag.	t	$\mathcal{R}(\eta)$	$\eta(w_A)$	Ag.	t	$\mathcal{R}(\eta)$	$\eta(w_A)$	Ag.	t	$\mathcal{R}(\eta)$	$\eta(w_A)$
S_1	5.75	0.125	**5.875**	H_1	4.25	1.0625	**5.3125**	B_1	8	0.8125	**8.8125**
S_2	5.75	0.125	**5.875**	H_2	3.25	1.0625	**4.3125**	B_2	6	0.8125	**6.8125**
S_3	3.75	0.125	**3.875**	H_3	1.25	1.0625	**2.3125**	B_3	4	0.8125	**4.8125**
S_4	2.75	0.125	**2.875**	H_4	0.25	1.0625	**1.3125**	B_4	2	0.8125	**2.8125**
S_5	0	0.125	**0.125**	H_5	0	1.0625	**1.0625**	B_5	0	0.8125	**0.8125**
S_6	0	0	**0**	H_6	0	0	**0**	B_6	0	0	**0**
S_7	0	0	**0**	H_7	0	0	**0**	B_7	0	0	**0**
S_8	0	0	**0**	H_8	0	0	**0**	B_8	0	0	**0**
								B_9	0	0	**0**
								B_{10}	0	0	**0**

Let us finally point out that statement (a) in Theorem 3 provides an even better simplification when the sectors game consists of more than three sectors, that is $m > 3$, since it guarantees that in the computation of the nucleolus of the sectors game (M, v^M) not all proper coalitions of M have to be considered, but only those of size 1 and $m-1$.

4 The core center

Gonzalez-Díaz and Sánchez-Rodríguez (2007) study the *core-center* (or mass center of the core) of a coalitional balanced game defined as the the mathematical expectation of the uniform probability distribution over the core. Let $U(A)$ denote the uniform distribution defined over the set A and $E(\mathcal{P})$ the expectation of the probability distribution $\mathcal{P}$. Formally, given an arbitrary balanced game (N, v), the core-center is defined as $\Phi(v) = E[U(C(v))]$.

The nucleolus of a coalitional game has a central position in the core but does not necessarily coincide with its mass center. However, for two-sided Böhm-Bawerk assignment markets the nucleolus coincides with the mass center, since it is the midpoint of the core segment. Thus it is natural to ask whether this property extends to multi-sided Böhm-Bawerk assignment markets. To this end it is necessary to simplify the computation of the core-center, since our markets typically have many agents and there are no easy-to-compute formulae that provide the center of mass of a polytope. With this aim, we prove that, like the nucleolus, the core-center $\Phi(\omega_{A(\mathbf{c};w)})$ of a multi-sided Böhm-Bawerk assignment game $(N, \omega_{A(\mathbf{c};w)})$ and the core-center $\Phi(v^M_{\mathbf{c};w})$ of the corresponding sectors game $(M, v^M_{\mathbf{c};w})$ are related by the injective linear mapping $\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\cdot)$. Our result is proved on the firm basis provided by measure theory (see for instance, Federer, 1969).

Theorem 4 *Let $(N, \omega_{A(\mathbf{c};w)})$ be an m-sided Böhm-Bawerk assignment game and let $(M, v^M_{\mathbf{c};w})$ be the associated sectors game. Let $\Phi(\omega_{A(\mathbf{c};w)})$ and $\Phi\left(v^M_{\mathbf{c};w}\right)$ be the corresponding core-centers. Then, $\Phi(\omega_{A(\mathbf{c};w)}) = \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\Phi\left(v^M_{\mathbf{c};w}\right))$.*

Proof. Let us consider the two metric spaces $(\mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^m}, d^N)$ and $(\mathbb{R}^M, d^M)$, each of them endowed with the corresponding euclidean distance. The dimension $dim(P)$ of a convex polytope P is the dimension of the minimal affine variety in which P is contained. From (8) we know that $C(\omega_A) \subseteq \mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^m}$ and $C(v^M) \subseteq \mathbb{R}^M$ are convex polytopes of the same dimension $k = dim(C(\omega_A)) = dim(C(v^M)) \leq m-1$.

Given an arbitrary metric space (Ω, d), the *diameter* of $B \subseteq \Omega$ is defined by $\delta(B) = sup\{d(x,y) \mid x, y \in B\}$. Let δ^N and δ^M denote the diameters defined on the metric spaces $(\mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^m}, d^N)$ and $(\mathbb{R}^M, d^M)$. We first claim that, for all $B \subseteq C(v^M) \subseteq \mathbb{R}^M$, we have

$$\sqrt{r}\delta^M(B) = \delta^N\left(\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(B)\right), \tag{37}$$

where r is defined in (6). Indeed, if $\overline{x}, \overline{y} \in C(v^M)$ and x, y are the corresponding elements of $C(\omega_A)$ by (8), we have

$$\begin{aligned} d^N(x,y) &= \left(\sum_{k\in M}\sum_{i\in N^k}(x_{ki}-y_{ki})^2\right)^{1/2} = \left(\sum_{k\in M}\sum_{i\in N^k, i\leq r}(\overline{x}_k + t_{ki} - \overline{y}_k - t_{ki})^2\right)^{1/2} \\ &= \left(\sum_{k\in M} r\left(\overline{x}_k - \overline{y}_k\right)^2\right)^{1/2} = \sqrt{r}d^M(\overline{x},\overline{y}). \end{aligned}$$

Let $\mu^N : \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m} \longrightarrow [0,+\infty)$ and $\mu^M : \mathbb{R}^M \longrightarrow [0,+\infty)$ be the Hausdorff outer measures of dimension k that correspond respectively to $(\mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m}, d^N)$ and $(\mathbb{R}^M, d^M)$, where recall that k is the dimension of $C(\omega_A)$ and $C(v^M)$. By definition,

(38)

$$\mu^N(A) = \lim_{\delta\to 0}\left(\inf_{\{B_n\}_{n=1}^{+\infty}}\left\{\sum_{n=1}^{+\infty}\left(\delta^N(B_n)\right)^k \;\middle|\; \begin{array}{l} B_n \subseteq \mathbb{R}^{N^1}\times\cdots\times\mathbb{R}^{N^m},\ A \subseteq \cup_{n=1}^{+\infty} B_n \\ \text{and } \delta^N(B_n) < \delta \text{ for all } n \geq 1 \end{array}\right\}\right)$$

for any $A \subseteq \mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^m}$, and

(39)

$$\mu^M(A) = \lim_{\delta\to 0}\left(\inf_{\{B_n\}_{n=1}^{+\infty}}\left\{\sum_{n=1}^{+\infty}\left(\delta^M(B_n)\right)^k \;\middle|\; \begin{array}{l} B_n \subseteq \mathbb{R}^M,\ A \subseteq \cup_{n=1}^{+\infty} B_n \\ \text{and } \delta^M(B_n) < \delta \text{ for all } n \geq 1 \end{array}\right\}\right)$$

for any $A \subseteq \mathbb{R}^M$. By (8) and (37), and using (38) and (39), for all $B \subseteq C(v^M) \subseteq \mathbb{R}^M$, we have

$$r^{k/2}\mu^M(B) = \mu^N(\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(B)). \tag{40}$$

With some abuse of notation let us also denote by μ^N and μ^M the restrictions of μ^N and μ^M to the borel sets of $(\mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^m}, d^N)$ and $(\mathbb{R}^M, d^M)$ respectively, which are measures by the Carathéodory Extension Theorem.

For any $H \subseteq \mathbb{R}^l$, let $\mathbf{I}_H : \mathbb{R}^l \longrightarrow \mathbb{R}$ be defined by $\mathbf{I}_H(x) = 1$ if $x \in H$ and $\mathbf{I}_H(x) = 0$ if $x \notin H$. By definition of the Lebesgue integral, for all measurable set $B \subseteq C(v^M) \subseteq \mathbb{R}^M$,

(41)

$$r^{k/2}\int \mathbf{I}_B d\mu^M = r^{k/2}\mu^M(B) = \mu^N(\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(B)) = \int \mathbf{I}_{\overrightarrow{t}_{\mathbf{c};w}+\mathcal{R}_{\mathbf{c};w}(B)} d\mu^N,$$

where the second equality holds by (40). Moreover, for any simple function $\overline{s} = \sum_{l=1}^{z} \lambda_l \mathbf{I}_{B_l} : \mathbb{R}^M \to \mathbb{R}$ defined on the measurable sets $B_1, ..., B_z \subseteq C(v^M) \subseteq \mathbb{R}^M$,

$$
\begin{aligned}
r^{k/2} \int \overline{s} d\mu^M &= r^{k/2} \int \sum_{l=1}^{z} \lambda_l \mathbf{I}_{B_l} d\mu^M = \sum_{l=1}^{z} \lambda_l r^{k/2} \int \mathbf{I}_{B_l} d\mu^M \\
&= \sum_{l=1}^{z} \lambda_l \int \mathbf{I}_{\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(B_l)} d\mu^N = \int \sum_{l=1}^{z} \lambda_l \mathbf{I}_{\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(B_l)} d\mu^N \\
&= \int s d\mu^N, \qquad (42)
\end{aligned}
$$

where $s : \sum_{l=1}^{z} \lambda_l \mathbf{I}_{\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(B_l)} : \mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^m} \to \mathbb{R}$ is the corresponding simple function defined on the measurable sets $\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(B_1), ..., \overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(B_z) \subseteq C(\omega_A) \subseteq \mathbb{R}^M$ by the constants λ_1,...,λ_z respectively. For all measurable nonnegative-valued function $\overline{f} : \mathbb{R}^M \to \mathbb{R}$, by the construction of the Lebesgue integral we obtain

$$
\begin{aligned}
r^{k/2} \int_{C(v^M)} \overline{f} d\mu^M &= \sup_{\substack{\overline{s}:\mathbb{R}^M \to \mathbb{R} \\ \overline{s} \text{ simple}}} \left\{ r^{k/2} \int \mathbf{I}_{C(v^M)} \overline{s} d\mu^M \mid 0 \le \overline{s} \le \overline{f} \right\} \\
&= \sup_{\substack{\overline{s}:\mathbb{R}^M \to \mathbb{R} \\ \overline{s} \text{ simple}}} \left\{ \int \mathbf{I}_{\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(C(v^M))} s d\mu^N \mid 0 \le \overline{s} \le \overline{f} \right\} \\
&= \sup_{\substack{s:\mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m} \to \mathbb{R} \\ s \text{ simple}}} \left\{ \int \mathbf{I}_{C(\omega_A)} s d\mu^N \mid 0 \le s \le f \right\} \\
&= \int_{C(\omega_A)} f d\mu^N, \qquad (43)
\end{aligned}
$$

where the second equality holds by (42), $f : \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m} \to \mathbb{R}$ denotes the measurable function that is zero elsewhere except in $C(\omega_A)$, where it is defined as the composition of the inverse of the injective linear mapping $\overrightarrow{t}_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\cdot)$ with $\overline{f}$, and the third equality is explained as follows. By (8), for any simple function $s : \mathbb{R}^{N^1} \times ... \times \mathbb{R}^{N^m} \to \mathbb{R}$ such that $0 \le s \le f$ there is a simple function $\overline{s} : \mathbb{R}^M \to \mathbb{R}$ such that $0 \le \overline{s} \le \overline{f}$ and $\overline{s}(\overline{x}) = s(x)$ for all $\overline{x} \in C(v^M)$ and $x = t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w}(\overline{x}) \in C(\omega_A)$. Indeed, if $s = \sum_{l=1}^{z} \lambda_l \mathbf{I}_{B_l}$ for some measurable sets $B_1, ..., B_l \subseteq \mathbb{R}^n$ we can take $\overline{s} = \sum_{l=1}^{z} \lambda_l \mathbf{I}_{\overline{B}_l}$, where for all $l \in \{1, ..., z\}$ we define $\overline{B}_l = (t_{\mathbf{c};w} + \mathcal{R}_{\mathbf{c};w})^{-1}(B_l \cap C(\omega_A))$.

It is known that the k-dimensional Hausdorff measure agrees with the classical area of an embedded submanifold of $\mathbb{R}^k$, $k \leq m$. Therefore, except for a constant multiplicative factor that coincides with the area of $C(\omega_A)$ and $C(v^M)$, $d\mu^N$ and $d\mu^M$ are the probability density functions of the uniform distributions over $C(\omega_A) \subseteq \mathbb{R}^{N^1} \times \cdots \times \mathbb{R}^{N^m}$ and $C(v^M) \subseteq \mathbb{R}^M$ respectively. Hence, by definition of the core-center, for all $k \in M$ and all $i \in N^k$ such that $1 \leq i \leq r$,

$$\begin{aligned} t_{ki} + \Phi_k\left(v^M\right) &= t_{ki} + \frac{\int_{C(v^M)} \overline{x}_k d\mu^M}{\int_{C(v^M)} d\mu^M} = \frac{r^{k/2} \int_{C(v^M)} (t_{ki} + \overline{x}_k)\, d\mu^M}{r^{k/2} \int_{C(v^M)} d\mu^M} = \frac{\int_{C(\omega_A)} x_{ki} d\mu^N}{\int_{C(\omega_A)} d\mu^N} \\ &= \Phi_{ki}(\omega_{A(\mathbf{c};w)}), \end{aligned}$$

where the second equality holds by linearity of the Lebesgue integral and the third equality holds by (43), using $\overline{f}(\overline{x}) = \overline{x}_k + t_{ki}$. The case $i > r$ is trivial since inactive agents get a null payoff at any core allocation. ■

The above result allows us to compute the core-center of the three-sided Böhm-Bawerk assignment market $(N, \omega_{A_{\mathbf{c};w}})$ of Table 1, since we only need to compute the core-center of the three-player associated sectors game $(M, v^M_{\mathbf{c};w})$. Figure 1 depicts the core of this latter game. Observe that in order to obtain the core-center of $C(v^M_{\mathbf{c};w})$ we need to compute the area of a bidimensional region embedded in $\mathbb{R}^3$. Nevertheless, a well-known result in Measure Theory is that an invertible affine mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ shifts the Lebesgue measure μ of $\mathbb{R}^n$ proportionally to the absolute value of the determinant of f, i.e. $\mu(f(A)) = |det(f)|\mu(A)$ for all measurable set $A \subseteq \mathbb{R}^n$. Hence, for our purpose of computing the center of mass of $C(v^M_{\mathbf{c};w})$ it suffices to calculate the center of mass of the projection of $C(v^M)$ onto the $(\overline{x}_1, \overline{x}_2)$-plane, since $f(\overline{x}_1, \overline{x}_2, \overline{x}_3) = (\overline{x}_1, \overline{x}_2, 2 - \overline{x}_1 - \overline{x}_2 - \overline{x}_3)$ is an invertible affine mapping from $\mathbb{R}^3$ to $\mathbb{R}^3$ with the image of $C(v^M)$ contained in the $\overline{x}_3 = 0$ plane of $\mathbb{R}^3$. Notice that this latter computation can be easily carried out using the standard tools of integral calculus in $\mathbb{R}^2$, and we obtain

$$\Phi\left(v^M\right) = (0.1389, 1.0556, 0.8055).$$

Figure 2 below depicts the projection of $C(v^M)$ onto the $(\overline{x}_1, \overline{x}_2)$-plane, together with the core-center $\Phi(v^M)$ and the nucleolus $\eta(v^M)$ that is obtained at the end of Section 3.

Notice first from $\Phi(v^M) \neq \eta(v^M)$ that in general the core-center of a coalitional game differs from the nucleolus, even in the case of convex games. Moreover, the Shapley value (Shapley, 1972) of the above sectors game is

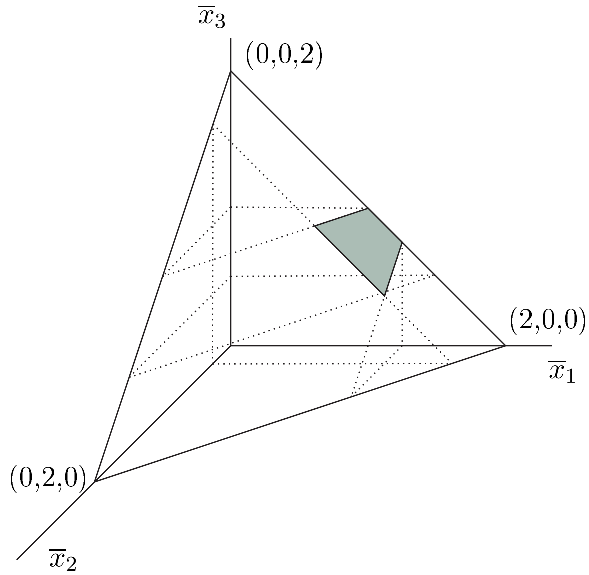

Figure 1: The core of the sectors game associated to the three-sided Böhm-Bawerk assignment game of Table 1

$(0.1667, 1.0417, 0.7917)$. Therefore, although the Shapley value occupies a central position in the core, it is in general also different from the core-center for convex games. Finally, as a consequence of Theorems 2 and 4, from $\Phi(v^M) \neq \eta(v^M)$ we deduce that $\Phi(\omega_A) \neq \eta(\omega_A)$ and thus the nucleolus of a multi-sided Böhm-Bawerk assignment market does not coincide in general with the mass center of the core.

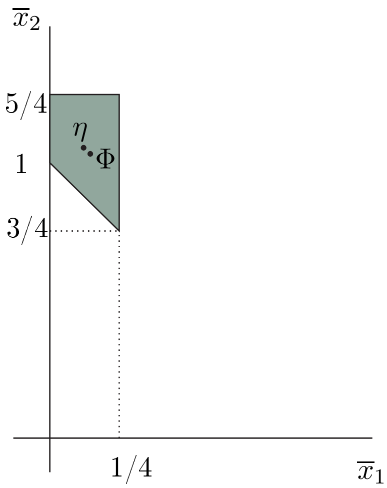

Figure 2: The $(\overline{x}_1, \overline{x}_2)$ projection of the core of the sectors game associated to the three-sided Böhm-Bawerk assignment game of Table 1

References

[1] Böhm-Bawerk E von (1923) Positive theory of capital (translated by W. Smart). G.E. Steckert, New York, (original publication 1891)

[2] Kaneko M, Wooders M (1982) Cores of Partitioning Games. Math Soc Sci 3:313-327.

[3] Federer H (1969). Geometric Measure Theory. Springer-Verlag, New York.

[4] Maschler M, Peleg B and Shapley S (1979) Geometric properties of the kernel, nucleolus, and related solution concepts, Mathematics of Operations Research 4:303Ű338.

[5] Moulin H (1988) Axioms of cooperative decision making. Econometric Society Monographs 15, Cambridge University Press.

[6] Núñez M, Rafels C (2005) The Böhm-Bawerk horse market: a cooperative analysis. Int J Game Theory, 33(1):421-430.

[7] Quint T (1991) The Core of an m-Sided Assignment Game. Games Econ Behav 3:487-503.

[8] Schmeidler D (1969) The nucleolus of a characteristic function game, SIAM Journal of Applied Mathematics 17:1163Ű1170.

[9] Shapley, LS. (1972) Cores of convex games. Int J Game Theory 1: 11-26.

[10] Shapley LS, Shubik M (1972) The assignment game I: The Core. Int J Game Theory 1:111-130.

[11] Sherstyuk, K (1999) Multisided matching games with complementarities. Int J Game Theory 28(4): 489-509.

[12] Stuart, HW Jr. (1997) The supplier-firm-buyer game and its m-sided generalization. Math Soc Sci 34: 21-27.

[13] Tejada O, Rafels C (2010) Symmetrically multilateral-bargained allocations in multi-sided assigment markets. Int J Game Theory 39: 249Ű258.

[14] Tejada O (2010) Multi-sided Böhm-Bawerk assignment markets: the core. Documents de Treball de la Facultat d'Economia i Empresa, E10/238.