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Abstract Computation of Nash equilibria of bimatrix games is studied from the viewpoint of identifying polynomially solvable cases with special attention paid to symmetric random games. An experiment is conducted on a sample of 500 randomly generated symmetric games with matrix size 12 and 15. Distribution of support size and Nash equilibria are used to formulate a conjecture: for finding a symmetric NEP it is enough to check supports up to size 4 whereas for non-symmetric and all NEP's this number is 3 and 2, respectively. If true, this enables us to use a Las Vegas algorithm that finds a Nash equilibrium in polynomial time with high probability.

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1 Introduction

Bimatrix games have been in focus since the early days of game theory. This is due to the fact that they are the most simple yet complex enough class of games. Many small textbook games (prisoners' dilemma, chicken, battle of sexes etc.) are bimatrix games that convey conceptual messages that contribute to better understand the nature of conflict and/or cooperation. Being a special case of mixed extension of finite games they are guaranteed to have at least one Nash equilibrium (NEP). The problem of computing a NEP is a real challenge in computational game theory and has been considered one of the central problems in computational complexity, Papadimitriou (1994). It is not even settled whether the class called PPAD containing this problem along with other difficult problems, is a distinct class somewhere between P and NP or it belongs to either P, i.e. polynomially solvable, or to NP i.e. needs exponential-time to solve. All algorithms known to-date to solve (find a NEP) the bimatrix game are exponential, including the famous Lemke-Howson(1964) algorithm. Apart from identifying certain polynomially solvable special cases no significant progress has been made towards settling the position of the bimatrix game. Meanwhile attention has turned to approximation and random games. A sketchy overview of some of the results from the literature is the subject of the first part of this paper. The second part is devoted to random symmetric games. To our knowledge this is the first time when the experimental approach is used for the study of symmetric random games. Empirical distribution of support sizes and NEP's are studied in order to set up conjectures about the efficiency of a Las Vegas algorithm. The main inspirational source is the work of Bárány et al. (2005) where it is proved that it is enough to check for equilibrium up to support size 2 only to find a NEP with high probability. We raised the question whether this nice behavior also holds for symmetric games. Symmetry arises naturally in many classes of games. Social dilemmas and evolutionary games stand out as most important. One has to be cautious since symmetric games in general and random symmetric games in particular behave differently from their general counterparts, see e.g. Stanford (1996). We determined all extreme NEP's and their supports of 500 randomly generated matrices of size 12 and 15. The entries of the matrix were independently drawn from a discrete uniform distribution on the interval $[0, 100]$. The most important conclusion is a conjecture: for finding a symmetric NEP it is enough to check supports of size 4 whereas for non-symmetric and all NEP's this number is 3 and 2, respectively. This means that if we do not care about the symmetry of the NEP we are going to find, the method of Bárány et al. (2005) works in its original form. To support the conjecture about symmetric solutions a proof is given for a subclass of games.

We show an example for the limitation on the naive version of a Las Vegas algorithm for random games where entries are drawn from normal distributions with different means but identical, small variance. We also give an example demonstrating that if we only want to find an approximate NEP with high probability, error terms can be significantly reduced if we know the distribution of the entries of the matrix.

The paper is organized as follows. Section 2 contains the necessary preliminaries and definitions. In Section 3 a few classes of polynomially solvable games are identified, among them a new one. Section 4 is a brief overview of approximate equilibria. Section 5 is about random games with special emphasis on symmetric games. Section 6 concludes. Figures are collected in the Appendix.

2. Preliminaries

A general bimatrix game is given by two $m \times n$ matrices A and B. The players get payoffs a_{ij}, b_{ij} if the row player plays her (pure) strategy i and the column player plays her (pure) strategy j. The mixed extension of this game, in normal form, is $G = \{X, Y, xAy, xBy\}$ where X, Y are simplices of probability vectors of proper dimension and xAy, xBy are the expected payoffs. Unless otherwise stated, when we speak of a bimatrix game (A, B), we always mean the mixed extension. The NEP of a game (A, B) is a pair of strategies $(x^*, y^*), x^* \in X, y^* \in Y$ to satisfy

$$\begin{aligned} xAy^* &\leq x^*Ay^* \quad \text{for all } x \in X, \\ x^*By &\leq x^*By^* \quad \text{for all } y \in Y. \end{aligned}$$

By Nash's fundamental theorem Nash (1950) a NEP always exists. If $B = A^T$, then the game is called symmetric, if $B = -A$, then it is zero-sum. It was also proved by Nash (1950), that symmetric games always have at least one symmetric NEP where $x^* = y^*$.

Various characterizations have been developed through the years for NEP's. Since NEP's are not affected by adding a constant to the matrices, we may assume that $A, B \geq 0$ or even $A, B > 0$. The latter will be assumed unless otherwise stated. We will denote a vector of all 1's by e and use the same notation for column and row vectors if it does not cause any confusion.

Characterization 1 (Inequality system) For a pair (x^*, y^*) to be a NEP of a bimatrix game (A, B) it is necessary and sufficient that there exist nonnegative numbers α^*, β^* such that $(x^*, y^*, \alpha^*, \beta^*)$ satisfies the system

$$\begin{aligned} xAy - \alpha &= 0 \\ xBy - \beta &= 0 \\ Ay - \alpha e &\leq 0 \\ xB - \beta e &\leq 0 \\ ex &= 1, ey = 1 \\ x &\geq 0, y \geq 0, \alpha \geq 0, \beta \geq 0. \end{aligned} \tag{1}$$

This takes a more simple form if the game is symmetric, $B = A^T$ and we are only interested in symmetric solutions

$$\begin{aligned} xAx - \alpha &= 0 \\ Ax - \alpha e &\leq 0 \\ ex &= 1 \\ x &\geq 0, \alpha \geq 0. \end{aligned}$$

Characterization 2 (Linear complementarity) Consider the following linear complementarity problem (LCP):

$$
\begin{aligned}
e\alpha - Ay &\geq 0 \\
e\beta - B^T x &\geq 0 \\
x(e\alpha - Ay) &= 0 \\
y(e\beta - B^T x) &= 0 \\
x &\geq 0, y \geq 0, \alpha \geq 0, \beta \geq 0.
\end{aligned}
\tag{2}
$$

If (x^*, y^*) is a NEP of the bimatrix game (A, B), then $x = x^*, y = y^*, \alpha = x^* A y^*, \beta = x^* B y^*$is a solution of the LCP. Conversely, if (x, y, α, β) is a solution of the LCP, then $x^* = \frac{1}{\beta} x, y^* = \frac{1}{\alpha} y$ is a NEP of (A, B).

In the symmetric case (2) takes the form

$$
\begin{aligned}
e\alpha - Ax &\geq 0 \\
x(e\alpha - Ax) &= 0 \\
x &\geq 0, \alpha \geq 0.
\end{aligned}
$$

Characterization 3 (Quadratic programming, Mangasarian and Stone 1964) For a pair (x^*, y^*) to be a NEP of a bimatrix game (A, B) it is necessary and sufficient that there exist nonnegative numbers α^*, β^* such that $(x^*, y^*, \alpha^*, \beta^*)$ is an optimal solution of the quadratic problem

$$
\begin{aligned}
\text{maximize } Q(x, y, \alpha, \beta) &= x(A + B)y - \alpha - \beta \\
\text{subject to} \qquad Ay - \alpha e &\leq 0 \\
xB - \beta e &\leq 0 \\
ex &= 1, ey = 1 \\
x &\geq 0, y \geq 0, \alpha \geq 0, \beta \geq 0,
\end{aligned}
\tag{3}
$$

and the optimal objective function value is 0.

In the symmetric case

$$
\begin{aligned}
\text{maximize } Q(x, \alpha) &= x(A + A^T)x - \alpha. \\
\text{subject to} \qquad Ax - \alpha e &\leq 0 \\
ex &= 1 \\
x &\geq 0, \alpha \geq 0.
\end{aligned}
$$

The problem of computing NEP's has been of great interest ever since the early days of game theory for both game theorists and theoretical computer scientists. From Characterization 1 we can construct an algorithm that finds a NEP by "brute force". Denote the support of a strategy x by $Su(x)$. $Su(x)$ is the set of indices of the positive components of x. If we know the supports $Su(x^*)$ and $Su(y^*)$ of a NEP, then we can compute the exact NEP in polynomial time. This is so because an equilibrium strategy of the column player equalizes the payoff that the row player gets. The same holds for the row player. Then we have a linear program that is known to be polynomially solvable. Consequently, by going through all the finitely many possible pairs of supports, we are guaranteed to find a NEP. In the symmetric case it is enough to check "only" 2^n supports, preferably in a systematic way. This exhaustive search can be efficient if we know beforehand that there is a NEP with support $k \ll n$. If $k \leq 2$ we do not even have to bother with LP's, a much simpler algorithm will do.

The first elegant algorithm for general bimatrix games that finds a NEP is due to Lemke and Howson (1964) and is based on complementary pivoting. It turned out very soon that their algorithm is not efficient in the sense that it can take exponentially many steps to reach a solution, Savani and von Stengel (2004). Moreover, not every (extreme) NEP is reachable by the algorithm. All algorithms for finding a NEP for the bimatrix game known to date are exponential-time and it is not known whether there is one with polynomial runtime. Many NEP related problems have been shown to belong to the NP-class (see Gilboa and Zemel (1989)) but finding a NEP for the general bimatrix game is not among them. It is widely believed that it belongs to a special complexity class called PPAD ("Polynomial Parity Arguments on Directed Graphs") first defined by Papadimitriou (1994) containing such well-known problems as e.g. Brouwer's and Kakutani's fixed point problem, Arrow and Debreu's economic equilibrium problem, envy-free cake cutting etc. PPAD is somewhere between P and NP. There are strong arguments for PPAD being a distinct class between P and NP but there is no proof available as of now. An important feature that points towards bimatrix games lying outside of NP is the fact that bimatrix problems are known to have solutions (NEP's) while in NP one must count with the possibility that there is no solution to the problem.

In this respect there is not much difference between general and symmetric bimatrix games since there are simple symmetrization techniques available. Let (A, B) be a bimatrix game with $m \times n$ positive matrices. Consider the symmetric bimatrix game (C, C^T) where

$$C = \begin{bmatrix} 0 & A \\ B^T & 0 \end{bmatrix} . \tag{4}$$

As first proposed by Griesmer et al (1963) and also discussed in Mehta et al (2014), a one-to-one correspondence can be established between the NEP's of (A, B) and certain symmetric NEP's of the symmetric game (C, C^T). In particular, a NEP (x, y) of (A, B) corresponds to the symmetric NEP $(\eta(\frac{1}{v}x, \frac{1}{w}y), \eta(\frac{1}{v}x, \frac{1}{w}y))$ where $v = xAy, w = xBy$ and $\eta(a)$ denotes the normalization of the non-zero, non-negative vector a. Another, somewhat different symmetrization is due to Gale, Kuhn and Tucker (discussed e.g. in Jurg et al (1992).

3. Efficient algorithms for special bimatrix games

It is well known that matrix games ($B = -A$) can be efficiently solved (in polynomial time) by various versions of interior point methods of linear programming (LP) since in this case the quadratic program becomes an LP. Moreover, learning algorithms, such as e.g. fictitious play, converge to a NEP. The coordination game $B = A$ is also "easy", fictitious play is guaranteed to converge to a NEP. One might hope that bimatrix games that are "close" to zero-sum games in a certain sense could be easier treated than general games.

A nice idea to make things more simple is to find a zero-sum game (A', B') that has the same set of NEP's as the bimatrix game (A, B), in other words, the two games are strategically equivalent. Moulin and Vial (1978) identify a class of games whose unique completely mixed NEP cannot be improved upon by coarse correlation. It is however, rather hard to verify wether a game belongs to this class or not.

Another easy-to-check condition is given by Kannan and Theobald (2010). Consider a bimatrix game (A, B), where

$$a_{ij} + b_{ij} = f(i, j) \text{ for all } i, j$$

where f is a "simple" function. E.g. $f(i, j) = u_i + v_j$ for some constants $u_1, ..., u_m, v_1, ..., v_n$. Define now the zero-sum game (A', B') by

$$a'_{ij} = a_{ij} - v_j, \qquad b'_{ij} = b_{ij} - u_i.$$

It can easily be seen that

$$\begin{aligned} xA'y^* - x^*A'y^* &= xAy^* - x^*Ay^*, \\ xB'y^* - x^*B'y^* &= xBy^* - x^*By^*. \end{aligned}$$

Therefore (A', B') has the same set of NEP's as (A, B).

If $f(i, j) = u_i v_j$ for some constants $u_1, ..., u_m, v_1, ..., v_n$, then the rank of $A + B$ is 1, pretty "close" to the case of zero-sum games where the rank of $A + B = 0$ is zero. Kannan and Theobald (2010) thoroughly study "low rank games" i.e. when $rank(A + B) = k$ is fixed (possibly small). Low rank games do not seem to be any simpler as far as the multitude of NEP's is concerned. Even rank 1 games may have arbitrary many NEP's. In particular, as Kannan and Theobald (2010) prove, for any $d \geq 2$ there exists a non-degenarate $d \times d$ game of rank 1 with at least $2d - 1$ NEP's. Interestingly, a polynomial time algorithm was given by Adsul et al. (2011) for finding a NEP for any rank 1 game. Even finding a symmetric NEP of any rank 1 symmetric game can be done in polynomial time, Mehta et al. (2014). On the negative side, Mehta (2014) proved that for games of rank 3 or more, and for symmetric games of rank 6 or more the problem is PPAD-complete, i.e. of the complexity of finding a NEP for a general bimatrix game.

Much better is the situation if we have the rank restriction not on $A+B$ but on A and/or B. In this case low rank implies small support which may make the exhaustive enumeration method a viable choice. Lipton et al. (2003) prove the following theorem that serves as basis for such a solution.

Theorem 1 (Theorem 4 in Lipton et al.(2003)). Let (x^*, y^*) be a NEP. If $rank(B) \leq k$, then there exists a mixed strategy x for the row player with $card(Su(x)) \leq k+1$ such that (x, y^*) is a NEP. Similarly, if $rank(A) \leq k$, then there exists a mixed strategy y for the column player with $card(Su(y)) \leq k+1$ such that (x^*, y) is a NEP. Furthermore, the payoff both players receive in the NEP's (x, y^*) and (x^*, y) is equal to the payoff in the initial NEP (x^*, y^*).

It is clear that if $A + B$ is negative definite, then (3) is polynomially solvable. It seems a good try to make the quadratic program tractable by adding a constant to each entry of $A + B$. Denote by E the matrix of 1's.

Definition 1 A symmetric matrix A is said to be almost positive (negative) definite if there is a constant γ such that $A + \gamma E$ is positive (negative) definite.

Positive (negative) definite matrices are almost positive (negative) definite by simply setting $\gamma = 0$. There exist, however, almost positive (negative) definite matrices that are not positive (negative) definite.

Example 1 Consider the matrix

$$A = \begin{bmatrix} -2 & -5 \\ -5 & -10 \end{bmatrix}$$

which is indefinite. If we add $\gamma = -4$ to each entry, then the matrix

$$A' = \begin{bmatrix} -6 & -9 \\ -9 & -14 \end{bmatrix}$$

is negative definite, i.e. A is almost negative definite.

It is yet to be explored how almost positive (negative) definite matrices can be characterized in order to recognize and use them in solving the quadratic program (3). A small step in this direction is the following theorem. Let

$$A = \begin{bmatrix} -a & -b \\ -b & -d \end{bmatrix}$$

be an indefinite matrix, $a, b, d > 0$.

Theorem 2 For A to be almost negative definite it is necessary and sufficient that $a + d > 2b$.

Proof Since A is indefinite $\det A = ad - b^2 < 0$. Add now a constant x to each entry of A to get A'

$$A' = \begin{bmatrix} -a+x & -b+x \\ -b+x & -d+x \end{bmatrix}.$$

For A' to be negative definite it is necessary and sufficient that either

(i) $-a+x<0$ and $\det A'=(x-a)(x-d)-(x-b)^2>0$, or

(ii) $-a+x>0$ and $\det A'=(x-a)(x-d)-(x-b)^2<0$.

Consider case (i). After rearrangement $(a+x)(d+x)-(b+x)^2>0$ becomes

$$ad-b^2-(a+d-2b)x>0. \tag{5}$$

Sufficiency. If $a+d>2b$, then from (5) we get that for any

$$x<\frac{ad-b^2}{a+d-2b}<0$$

the conditions of (i) hold.

Necessity. If $a+d-2b=0$, then obviously (5) cannot hold. If $a+d-2b<0$, then if both conditions of (i) held, then we would have

$$\frac{b^2-ad}{2b-a-d}<x<a$$

which is impossible since this would imply $(a-b)^2<0$.

The proof for case (ii) goes similarly.□

Corollary 1 If for a symmetric bimatrix game (A,A^T) the matrix $A+A^T$ is almost negative definite, then the game has a unique symmetric NEP.

It is not clear whether Theorem 2 can be generalized to $n\times n$ matrices. It seems that having at most one negative (positive) eigenvalue is a necessary condition.

4. Finding approximate equilibria

Knowing that finding an exact NEP of a general bimatrix game is hard, at least all known algorithms run in exponential time, an ever growing attention has been paid to finding approximate equilibria in polynomial (or less ambitiously in subexponential) time. There are, however, various definitions of approximate equilibria. The following is the most simple.

Definition 2 (ϵ-NEP) For any $\epsilon>0$ a strategy profile (x,y) is an ϵ-NEP of the $m\times n$ bimatrix game (A,B), if for any pure strategy i of the row player $e_iAy\le xAy+\epsilon$ and for any pure strategy j of the column player $xBe_j\le xBy+\epsilon$.

In an ϵ-NEP no player could increase her payoff more than ϵ by unilaterally changing her strategy. A stronger concept is the ϵ-well supported NEP.

Definition 3 For any $\epsilon>0$ a strategy profile (x,y) is an ϵ-well supported NEP of the bimatrix game (A,B), if

(i) for any pure strategy i of the row player

$$x_i > 0 \Longrightarrow e_i A y \geq e_k A y - \epsilon \quad \text{for all } k = 1, ..., m,$$

(ii) for any pure strategy j of the column player

$$y_j > 0 \Longrightarrow x B e_j \geq x B e_l - \epsilon \quad \text{for all } l = 1, ..., n.$$

The interpretation of an ϵ-well supported NEP is straightforward: each player plays only approximately best-response pure strategies with positive probability. Every ϵ-well supported NEP is also an ϵ-NEP but the converse need not be true.

When speaking of an algorithm running in "polynomial time" we mean that the running time is a polynomial function of the length of the binary coding of the problem data and $\frac{1}{\epsilon}$. Since the error term ϵ is additive, to evaluate and compare the efficiency of algorithms we have to normalize A and B by adding constants and multiplying by positive numbers. The accepted standard is the $[0, 1]$ normalization meaning that all entries of both matrices are in the interval $[0, 1]$ and at least one entry in both matrices has value 0, and there is another with value 1.

The currently best polynomial algorithm is due to Tsaknakis and Spirakis (2008) with $\epsilon \approx 0,3393$. This bound is slightly better for symmetric games. For any $\delta > 0$, there is a polynomial algorithm with error term $\epsilon = \frac{1}{3} + \delta$ as proven by Kontogiannis and Spirakis (2011).

It is unknown whether there exists a polynomial-time algorithm for finding an approximate NEP of a general bimatrix game. Subexponential-time algorithm do exist, however. The first one was given by Lipton at al. (2003) and later another one by Tsaknakis and Spirakis (2010). The former is based on the "sampling method". Key to the idea is the k-uniform mixed strategy. x is a k-uniform strategy if it is the uniform distribution on a multiset S of pure strategies with $card(S) = k$. The main result of Lipton et al. (2003) is, somewhat simplified, the following theorem.

Theorem 3 For a $[0, 1]$-normalized $n \times n$ bimatrix game (A, B) for any $\epsilon > 0$ there exists for every $k \geq \frac{12 \ln n}{\epsilon^2}$ a k-uniform ϵ-NEP (x', y').

Thus to find an ϵ-NEP it is enough to exhaustively check all multisets of cardinality k, the least integer greater than $\frac{12 \ln n}{\epsilon^2}$. For each multiset, checking for equilibrium can be done in polynomial time. Since there are $\binom{n+k-1}{k}^2$ pairs of multisets to look at, we have a quasipolynomial $n^{O(\ln n)}$ algorithm.

For the less ambitious goal of approximating the payoffs in an actual NEP, the necessary support size can be made independent of n.

Theorem 4 (Lipton et al. (2003)) For a $[0, 1]$-normalized $n \times n$ bimatrix game (A, B), given any NEP (x^*, y^*) and any $\epsilon > 0$, there exists for every $k \geq \frac{5}{\epsilon^2}$, a pair of k-uniform strategies (x, y), such that

$$\begin{aligned} \|xAy - x^*Ay^*\| &< \epsilon, \\ \|xBy - x^*By^*\| &< \epsilon. \end{aligned}$$

For games of special structure polynomial algorithms do exist. The comprehensive review of Ortiz and Irfan (2017) is a good guide through the jungle of recent results.

Another line of research concerns relative ϵ-NEP's, a strategy profile in which the payoff of each player is at least $(1-\epsilon)$ times that of the best-response strategy. We refer to the paper of Feder et al. (2007) for results about relative ϵ-NEP's.

5. Random bimatrix games

In random games the entries of the payoff matrices of a bimatrix game (A, B) are not fixed but are determined by chance governed by a known probability distribution. They may have special features when compared to their deterministic counterparts, mostly they are more tractable. This is no surprise since e.g. when it comes to computational complexity, worst-case analysis usually relies on special, sometimes pathological instances very unlikely to occur in practice or when chance enters the picture.

Seeing the disappointing behavior of NEP's in deterministic games anything more tractable in the realm of random games has to be appreciated. As we saw in the previous section, a step back in precision, i.e. being content with some sort of approximation, makes life easier, more efficient algorithms (even polynomial-time) can be devised if we give up exactness and allow for some error in the payoffs and/or equilibrium strategies. In random games there is even more room for gaining some leverage. We accept failure to get an answer to a problem (e.g. finding a NEP) if this can only happen with "low probability", or the success with "high probability". In precise terms, this means that the probability of failure (success) goes to 0 (1) when the size of the problem (the number of rows and/or columns of the matrices A,B) goes to ∞.

Early work concentrated on determining the probability of a pure NEP, first in a zero-sum game, Goldman (1957), then in a general bimatrix game Powers (1990), Stanford (1995) and finally in symmetric bimatrix games Stanford(1996). The ultimate achievement in this respect was the determination of the (limit) distribution of the number of pure NEP's.

Another line of research aimed at determining the distribution or at least the expected value of important characteristics of random games. These results, beyond their value of their own, can also contribute to the ultimate goal: devising algorithms that can find a NEP (or ϵ-NEP) in polynomial time with high probability. A milestone in this direction was the paper of Bárány et.al. (2005). Their main result states that, for finding at least one exact NEP of a general random bimatrix game where entries of the matrices are independently drawn either from a continuous uniform distribution with mean 0 or from a standard normal distribution, a Las Vegas algorithm works efficiently. The Las Vegas algorithm begins with support size 1 and systematically checks for equilibrium for support sizes $2, 3, ..., n$. Finding a NEP is guaranteed but it may take exponential-time. But by Bárány et al.'s result, with high probability, we need not go beyond 2 and the search terminates in polynomial time.

Random games have been a subject of many fine papers lately addressing structural properties such as e.g. the expected number and distribution of NEP's as to their support sizes. Most random bimatrix games under scrutiny are assumed to have matrices whose entries are independently drawn from the same continuous distribution. When it comes to taking a specific distribution the usual choice is the uniform and the normalized Gaussian distributions. Occasionally, the Cauchy distribution attracts some attention as in Roberts (2006). Special attention is paid to the asymptotic behavior when the number of pure strategies goes to infinity. These properties translate to theoretical computational issues as well as practical algorithms. The algorithms guarantee either an exact or approximate NEP.

Much less has been done in the way of conducting experiments to find out how the theory aligns with the experimental data obtained in test problems that only approximate the conditions the theoretical results are based on. Examples of experimental work are Faris and Maier (1987), Fearnley et al (2015). There are natural limits to exactly simulate continuous distributions and infinitely many strategies. One has to settle for finite approximations in both aspects and evaluating the match (or mismatch) of experimental findings and theoretical results.

In our experiment we focused on symmetric games. When participants in a game cannot be distinguished and only the number of players taking a particular course of action counts, symmetry is a salient feature. Typical examples are congestion games and internet games. The theoretical challenge is that results for general games cannot automatically be carried over to symmetric games. A good example is the distribution of pure strategy NEP's in symmetric and in the general case. Another distinguishing feature of symmetric games is that the set of NEP's for any game can be separated into two classes: symmetric and non-symmetric. The existence of symmetric NEP's in a symmetric game is guaranteed by Nash's theorem. Nash (1950), realizing the importance of symmetry, devoted a separate existence theorem to this class of games.

We worked with 500 independently generated random symmetric bimatrix test problems. Because of symmetry it is enough to generate only one matrix. All entries were integers drawn uniformly from the interval $[0, 100]$. This is a scaled-up approximation to the continuous uniform distribution on the unit interval $[0, 1]$. A major difference between the two is that having two identical entries is not a zero-probability event any more and there is a tendency to have (slightly) more NEP's in the discrete case than in the continuous. Distributions, expected values may well be shifted. Nevertheless, tendencies can be identified, qualitative statements and conjectures be formulated.

We studied two sets of experimental data. One is where the number of pure strategies $n = 12$, the other where $n = 15$. We used the solver developed by

Avis et al. (2010) available freely on the internet. The solver determines all extreme NEP's. The size limitation is $n = 15$, this is why we did not go beyond this number. Due to the fact that the number of NEP's grows very fast with the increase of n, even the modest sizes 12 and 15 of the matrices produce several thousands of NEP's, more than enough to draw statistical conclusions.

The raw data of the analysis is compiled in six matrices $P_{sym}, P_{nons}, P_{tot}$, and $R_{sym}, R_{nons}, R_{tot}$ of size 500×12 and 500×15, respectively. An entry p_{ij} of P_{sym} is the number of symmetric NEP's in test problem i with support size j. Similarly, P_{nons} contains non-symmetric and $P_{tot} = P_{sym} + P_{nons}$ all NEP's. Entries of $R_{sym}, R_{nons}, R_{tot}$ are similarly defined. The row sums give the total number of symmetric, nonsymmetric and all NEP's, respectively, for a test problem, the column sums indicate the total number of NEP's of a particular support size.

Empirical distributions of support sizes in the symmetric, nonsymmetric and combined cases for $n = 12$ and $n = 15$ resemble a Poisson distribution though χ^2 tests fail to give convincing evidence. What is common is the unimodal nature of the distributions as can be seen on Figures 1-6 in the Appendix. The only available theoretical result for the distribution of support size for general symmetric bimatrix games is due to Kontogiannis and Spirakis (2009). Their model is based on generating the matrix entries from the standard normal distribution. They show that the total (symmetric and non-symmetric combined) support sizes sharply concentrate around $0,316n$ asymptotically as $n \to \infty$. For $n = 12$ and $n = 15$ this means $3,792$ and $4,74$, respectively. This ties in with empirical data, both empirical distributions peak at support size 4 and 5. As far as the expected number of NEP's $E(n)$ is considered, it grows exponentially but slower than for general non-symmetric bimatrix games, by an asymptotic factor $1,1512$ i.e. $E(n+1) = 1,1512E(n)$ for large enough n. In our experiment the number of NEP's went up from 17644 to 37904 when the problem size was increased by 3. This is larger than expected from the theoretical asymptotic results. No wonder, increasing the size while keeping the range of the discrete random variables constant increases the probability of getting identical elements in the matrix thereby giving better chances for the NEP defining inequalities to hold. Moreover, the difference between the uniform and Gaussian distributions might also be relevant. Unfortunately, no such results are available for the uniform distribution, let alone its discrete version. It is also worth noting that the percentage of symmetric equilibria among all NEP's decreased from $\frac{5436}{17644} = 0,3081$ to $\frac{8978}{37904} = 0,2369$ when n went up from 12 to 15.

Analysis of the matrices P and R columnwise gives us information on the distribution of games having a particular support size. The only support size where theoretical results are available for comparison is size 1. This is the case of pure Nash equilibria. Stanford (1996) proves that the number of symmetric NEP's X occurring in a random symmetric game is asymptotically Poisson if $n \to \infty$ with mean 1. It is assumed that all entries for the game matrix are drawn independently from the same but arbitrary continuous distribution. Interestingly, this is the same limit distribution we get for random general bimatrix games, Stanford (1995) and Powers (1990). If Y denotes the number of asymmetric NEP's, then $\frac{1}{2}Y$ is also Poisson with mean $\frac{1}{2}$. The number of all pure NEP's $X+Y$ has a special distribution determined by Stanford (1996). Based on this distribution, Stanford (1996) calculated the probability that a random symmetric bimatrix game has at least one pure NEP and found it $0,7769$, larger than $0,6321$ obtained for general bimatrix games. We did a χ^2-test for the theoretical limit distributions and our empirical distributions. We found that on usual significance levels we cannot reject the hypothesis that the empirical distribution stems from the theoretical limit distribution.

What is the situation with support sizes more than 1, in particular with support size 2? We do not know any anchor, theoretical or empirical. The following tables summarize the empirical distributions for all support sizes s.

Table 1 (symmetric, $n=12$)

$s/freq$	1	2	3	4	5	6	7	8	9	10	11	12
0	176	136	110	105	126	192	287	377	449	485	496	500
1	180	161	148	124	126	132	125	85	34	14	4	0
2	95	106	118	116	102	85	55	20	11	1	0	0
3	38	47	58	68	65	44	20	14	5	0	0	0
4	10	31	29	37	34	25	9	3	1	0	0	0
5	1	10	19	21	21	12	1	1	0	0	0	0
6	0	6	9	17	12	4	2	0	0	0	0	0
7	0	0	3	4	10	2	0	0	0	0	0	0
8	0	3	4	3	2	3	0	0	0	0	0	0
9	0	0	2	2	0	1	1	0	0	0	0	0
10	0	0	0	1	0	0	0	0	0	0	0	0
11	0	0	1	2	0	0	0	0	0	0	0	0
12	0	0	0	1	0	0	0	0	0	0	0	0
$Total$	500	500	500	500	500	500	500	500	500	500	500	500

Table 2 (non-symmetric, $n=12$)

	$s/freq$	1	2	3	4	5	6	7	8	9	10	11	12
	0	299	122	77	93	131	232	336	424	481	498	500	498
	1	0	0	0	0	2	0	0	0	0	0	0	2
	2	159	158	112	85	96	97	80	46	12	2	0	0
	3	1	1	0	0	0	0	0	0	0	0	0	0
	4	35	112	98	87	85	75	36	22	2	0	0	0
	5	0	0	0	0	0	0	0	0	0	0	0	0
	6	6	64	77	70	52	33	18	7	4	0	0	0
	7	0	0	1	0	0	0	0	0	0	0	0	0
	8	0	29	63	53	46	21	14	1	1	0	0	0
	9	0	1	0	1	0	0	0	0	0	0	0	0
y	10	0	4	32	31	28	13	7	0	0	0	0	0
	11	0	0	0	1	0	0	0	0	0	0	0	0
	12	0	6	16	31	27	7	4	0	0	0	0	0
	13	0	1	0	0	0	0	0	0	0	0	0	0
	14	0	2	15	18	13	9	4	0	0	0	0	0
	15	0	0	1	0	0	0	0	0	0	0	0	0
	16	0	0	5	11	84	0	0	0	0	0	0	0
	17	0	0	0	0	0	0	0	0	0	0	0	0
	18	0	0	2	8	6	5	1	0	0	0	0	0
	> 18	0	0	1	11	6	4	0	0	0	0	0	0
	$Total$	500	500	500	500	500	500	500	500	500	500	500	500

Table 3 (total $n = 12$)

$s/freq$	1	2	3	4	5	6	7	8	9	10	11	12
0	90	34	26	33	66	140	244	354	445	483	496	498
1	108	42	25	29	37	61	70	56	27	14	4	2
2	129	67	50	42	30	39	42	25	12	3	0	0
3	86	64	32	23	37	37	33	20	6	0	0	0
4	50	73	40	52	52	46	27	10	2	0	0	0
5	23	52	49	33	26	36	18	19	2	0	0	0
6	11	54	46	32	40	25	13	6	0	0	0	0
7	3	29	38	35	35	16	12	4	2	0	0	0
8	0	30	38	28	27	24	11	5	0	0	0	0
9	0	17	35	27	22	15	3	1	2	0	0	0
10	0	14	34	24	19	10	6	0	2	0	0	0
11	0	5	16	18	24	10	5	0	0	0	0	0
12	0	7	14	25	11	4	4	0	0	0	0	0
13	0	5	16	16	10	6	3	0	0	0	0	0
14	0	3	9	15	8	7	4	0	0	0	0	0
15	0	2	10	12	13	3	1	0	0	0	0	0
16	0	0	7	12	9	3	0	0	0	0	0	0
17	0	1	3	12	8	2	2	0	0	0	0	0
18	0	0	3	5	4	3	1	0	0	0	0	0
> 18	0	1	9	27	22	13	1	0	0	0	0	0
$Total$	500	500	500	500	500	500	500	500	500	500	500	500

Table 4 (symmetric $n = 15$)

$s/freq$	1	2	3	4	5	6	7	8	9
0	166	124	82	73	75	91	147	234	302
1	175	137	116	94	102	108	120	111	112
2	116	117	112	96	97	116	186	74	43
3	36	63	78	79	71	54	61	32	24
4	6	30	42	62	48	50	32	20	7
5	1	14	25	33	25	22	26	14	5
6	0	11	22	18	25	16	11	3	4
7	0	3	11	12	22	17	6	3	3
8	0	1	4	14	10	13	5	3	0
9	0	0	4	7	10	2	1	2	0
10	0	0	2	3	3	1	2	2	0
11	0	0	2	3	5	4	2	2	0
12	0	0	0	2	4	4	0	0	0
13	0	0	0	2	1	0	1	0	0
14	0	0	0	2	1	0	0	0	0
15	0	0	0	0	0	1	0	0	0
16	0	0	0	0	1	0	0	0	0
17	0	0	0	0	0	1	0	0	0
$Total$	500	500	500	500	500	500	500	500	500

Table 5 (non-symmetric $n = 15$)

$s/freq$	1	2	3	4	5	6	7	8	9
0	292	91	26	19	33	73	167	245	352
1	4	0	0	0	0	2	2	2	1
2	160	110	64	41	45	50	67	83	69
3	2	2	1	1	1	2	2	1	2
4	35	129	72	48	46	72	51	55	32
5	1	2	0	2	1	0	0	1	0
6	6	88	82	56	44	55	38	35	10
7	0	0	0	3	3	0	0	0	0
8	0	38	76	52	57	40	31	20	6
9	0	2	1	2	1	0	0	0	0
10	0	20	71	56	41	31	30	13	10
11	0	0	1	0	5	0	0	0	0
12	0	14	35	41	37	37	19	9	2
13	0	0	0	1	1	3	2	0	0
14	0	1	31	51	32	23	16	7	4
15	0	0	0	1	0	1	1	0	0
16	0	3	8	40	32	20	17	5	4
17	0	0	1	0	0	1	1	0	0
> 17	0	0	29	85	121	89	56	24	8
total	500	500	500	500	500	500	500	500	500

Table 6 (total $n = 15$)

$s/freq$	1	2	3	4	5	6	7	8	9
0	91	22	7	8	13	40	88	173	259
1	106	30	8	5	10	21	53	48	68
2	133	52	19	12	21	18	33	43	40
3	76	42	17	11	16	19	28	26	29
4	46	56	35	17	17	24	33	35	28
5	23	48	25	15	16	27	23	27	19
6	19	66	33	26	17	30	20	29	9
7	6	47	30	21	19	20	12	15	10
8	0	43	43	27	24	30	19	16	5
9	0	23	36	23	23	16	16	13	0
10	0	23	37	29	21	28	18	14	3
11	0	13	43	24	24	15	16	4	4
12	0	9	26	26	24	20	11	5	3
13	0	4	27	26	20	14	15	6	3
14	0	8	17	22	20	16	8	4	1
15	0	6	16	24	18	19	13	0	1
16	0	2	22	27	14	13	10	2	2
17	0	1	8	10	19	16	8	2	2
> 17	0	5	51	147	164	114	75	38	14
total	500	500	500	500	500	500	500	500	500

Interesting observations can be made if we focus on the possible efficiency of support enumeration (Las Vegas) algorithms . For Las Vegas algorithms to work we need some (probabilistic) guarantee that by considering only small-size supports we can produce a NEP with a desired property (symmetry e.g.). To this end, let us assign to any row of the matrices P and R an integer between 1 and 15 in the following way. Let this number be k if in the particular row the first positive number is in column k. This means that there is a NEP of support size k but there is no NEP of support size $k-1$ or less. Let us call this number just defined minimum-guaranteed support size. Indeed, if we enumerate each support of size k and check whether it is a NEP of a given property, then it is guaranteed that at least one NEP of the desired property is found for a given symmetric bimatrix game We will say that a class of bimatrix games has the Bárány-Vempala-Vetta (BVV) property of degree k if, with high probability (tends to 1 if $n \to \infty$) every game in the class has a minimum-guaranteed support size k. Bárány et al (2005) proved that the class of general bimatrix games has the BVV property of degree 2. Does the class of symmetric bimatrix games (or some subclass thereof) also have the BVV property of small degree? From the matrices P_{sym} and R_{sym} we get the following statistics

Table 7

$s-size$	$n=12$	$rel.freq.$	$n=15$	$rel.freq$
1	324	$0,648$	334	$0,666$
2	112	$0,224$	113	$0,228$
3	40	$0,080$	34	$0,068$
4	18	$0,036$	14	$0,028$
≥ 5	6	$0,012$	5	$0,01$
$Total$	500	1	500	1

We did a χ^2 test where the null-hypothesis was that the relative frequencies in the columns $n = 12$ and $n = 15$ come from the same distribution. We got the statistic $\chi^2 = 2,274$. This is much smaller than $\chi^2_{0,10} = 7,779$, the reference value belonging to the 90% significance level and degree of freedom 4. This suggests that the BVV-property of degree 2 is unlikely to hold since in this case "high probability" would mean $87,2 - 89,4\%$ definitely not high enough keeping in mind that the statistics of support sizes are based on observations of several thousand NEP's. Thus the BVV-property of degree 3 or rather 4 can hold, if any. For degree 3 "high probability" would mean $95,2 - 96,2\%$ whereas for degree 4 this probability is $98,8 - 99\%$. Of course one can "hope" that these probabilities get higher, eventually going to 1 as n grows even in the case of BVV property of degree 2. This seems unlikely in the light of the χ^2 test suggesting that a 25% (from 12 to 15) increase in the size of the game does not lead to a significant increase of the probability of having at least one symmetric NEP of support size no more than 2. We do not know what exact, theoretically well supported distributions are behind these empirical distributions. We know from Stanford (1996), however, that the probability belonging to support size 1 is $1 - e^{-1} = 0,6321$, close to the empirical values $0,648$ and $0,668$ obtainable from Table 7. As indicated before, the fact that these numbers are slightly higher is because we worked with a discrete uniform distribution instead of a continuous one. Though we do not know the theoretical distribution, we think that the BVV-property of degree 4 may hold. The BVV-property of degree 4 does hold for a subclass of random symmetric bimatrix games.

Theorem 5 With high probability, there is a symmetric NEP of the symmetric game (C, C^T) whose support has no more than 4 points if C is defined as in (4).

Proof By Griemer at al.'s symmetrization technique if (p, q) is a NEP of the game (A, B) with payoffs $a = pAq$ and $b = pBq$, then $(\eta(\frac{1}{a}p, \frac{1}{b}q), \eta(\frac{1}{a}p, \frac{1}{b}q))$ is a symmetric NEP of the symmetric game (C, C^T) where C is defined as in (4). As Bárány et al. (2005) showed, with high probability, there is a NEP (p, q) of (A, B) such that the supports of both players are of cardinality no more than 2. This immediately implies, by the construction of C, that the cardinality of the supports of $(\eta(\frac{1}{a}p, \frac{1}{b}q), \eta(\frac{1}{a}p, \frac{1}{b}q))$ is no more than 4. $\square$

For non-symmetric NEP's the following statistics were obtained

$s-size$	$n=12$	$rel.freq.$	$n=15$	$rel.freq.$
1	231	0,462	209	0,666
2	183	0,366	215	0,228
3	66	0,132	66	0,068
≥ 4	20	0,040	10	0,02
$Total$	500	1	500	1

The χ^2 statistic is 10,598, higher than the critical value at any meaningful significance level indicating that we have to reject the hypothesis that the two samples come from the same distribution.

For all NEP's, symmetric and non-symmetric combined, we have the following statistics

$s-size$	$n=12$	$rel.freq.$	$n=15$	$rel.freq.$
1	417	0,834	409	0,83
2	62	0,124	80	0,15
≥ 3	21	0,042	11	0,02
$Total$	500	1	500	1

The χ^2 statistic is 10,141, again pointing towards rejection. The tail of the distribution with higher n is definitely thinner in both the non-symmetrical and the total case. Thin tails mean that it is unlikely that checking support sizes for equilibrium up to 3 and 2 for non-symmetric and all NEP's, respectively is not enough to find at least one NEP.

From these experimental findings we set up the **conjecture:**

For symmetric random bimatrix games the BVV property of degree k holds. For the symmetric case $k = 4$, for the non-symmetric case $k = 3$ and for the overall case $k = 2$.

As a consequence, the Las Vegas algorithm is efficient (runs in polynomial time with high probability) for finding at least one symmetric, non-symmetric and arbitrary NEP, respectively.

Las Vegas algorithms may have a potential to find NEP's for random games with more general bimatrix games than those with entries of identical distributions. There are, however, limitations to considering only small support sizes. Bárány et al (2005) suggest that the range of the Las Vegas algorithm could be enlarged further if the methodology they developed could be extended to random bimatrix games whose entries are independent Gauss variables with non-uniform means. They write: "...add random Gaussians to the entries of the given payoff matrices; an equilibrium of the perturbed game will be an approximate equilibrium of the original game with high probability, given that the variance of the Gaussians is small enough". This would require that e.g. close enough to any completely mixed unique equilibrium point there exists, with high probability, at least one equilibrium with small support (ideally of size 2). The following example shows that unless other restrictive assumptions are made, this is impossible.

Given a square matrix A, the matrix obtained from A by replacing the j-th column with e (a vector of all 1's) is denoted by A_j, whereas the matrix obtained from A by replacing the i-th row with e is denoted by A^i.

Theorem 6 Milchtaich (2006) A necessary and sufficient condition for the existence and uniqueness of a completely mixed equilibrium for the bimatrix game (A, B) is that $\det(A_i)\det(A_j) > 0$ and $\det(B^i)\det(B^j) > 0$ for all $i, j \in \{1, ..., n\}$.

Example 2 Consider a bimatrix game $G = (I, -I)$ where I is the identity matrix of order n. The unique NEP of this game (which happens to be zero-sum) is $x = \frac{1}{n}e, y = \frac{1}{n}e$. Take a random perturbation of G by replacing the entries of the identity matrix with independent Gaussians with mean 1 in the main diagonal and 0 elsewhere and fixed, small enough variance. Since determinants of a matrix A are continuous functions of A, all randomly drawn perturbations of G will satisfy the conditions of Theorem 2 with high probability. Therefore, with high probability, these games will have unique completely mixed NEP's with full-size support, or in other words, it is very unlikely that a random perturbation can be solved efficiently by the Las Vegas algorithm.

Steps towards covering more general games can be taken if we only want to find approximate NEP's. Bárány et al.'s (2005) result about the solvability of general bimatrix games is often quoted as "random bimatrix games are easy". We have seen that this statement is based on the fact that it is enough to enumerate supports of size 2 and then we can be almost sure to have found at least one NEP. Panagopoulou and Spirakis (2014) subscribe to categorizing random bimatrix games "easy" but for another reason. They show that the uniform completely mixed strategy pair is an approximate NEP under very general conditions and the error of approximation goes to 0 as the size of the matrices goes to infinity. Moreover, their results easily carry over to the symmetric case, the main subject of this paper. We will not state their result in its entire generality but focus on symmetric games.

Assume that the game matrix A is positively normalized to $[0,1]$. All elements of the matrix A are independently drawn from a distribution whose expected value is finite and well defined. The distributions need not be identical as in most models but should satisfy the following condition: the expectations of the sum of elements in each row (and each column by symmetry) are assumed to be the same. We adopt the usual definition of an ϵ-NEP (see Definition 2)

Theorem 7 Panagopoulou and Spirakis (2014) Let (A, A^T) be an $n \times n$ random symmetric bimatrix game. Then the completely mixed uniform strategy profile is, with probability at least $1 - \frac{2}{n}$, a $\sqrt{\frac{\ln n}{n}}$-NEP of (A, A^T).

For Theorem 7 to be a strong statement n should be really large. For $n = 12$ e.g. the minimum probability is $\frac{5}{6} = 0,833$ $\epsilon = 0,455$ which is, considering that A is positively normalized to $[0,1]$, a weak statement. This is due to the fact that very little is assumed of the distributions the entries of A are drawn from. If entries of A are independently drawn from the same normal distribution, a rather common assumption, then the error term gets significantly smaller.

Example 3 Let each entry of the game matrix A be independently drawn from the normal distribution $N(\frac{1}{2}, \frac{1}{6})$. The expected value and variance are chosen so that A has entries between 0 and 1 with high probability and thus it is "almost" positively normalized to $[0,1]$ and thus fit for comparison. For the completely mixed strategy pair $(\frac{1}{n}e, \frac{1}{n}e)$ to be an ϵ-NEP of the random symmetric game (A, A^T) the following inequalities should hold

$$\begin{aligned} \frac{1}{n} e_i A e &\leq \frac{1}{n^2} e A e + \epsilon, \\ \frac{1}{n} e A e_j &\leq \frac{1}{n^2} e A e + \epsilon. \end{aligned}$$

or equivalently

$$\begin{aligned} n e_i A e - e A e &\leq n^2 \epsilon, \\ n e A e_j - e A e &\leq n^2 \epsilon. \end{aligned}$$

Each entry ξ of A being an $N(\frac{1}{2}, \frac{1}{6})$ random variable and on the left hand sides of the inequalities there are sums of identical normal variables we have $2n$ identical inequalities

$$n\xi - \eta \leq n^2 \epsilon.$$

$n\xi$ is an $N(\frac{1}{2}n^2, \frac{1}{6}n^{\frac{3}{2}})$ and η an $N(\frac{1}{2}n^2, \frac{1}{6}n)$ random variable. Using the formula for the distribution of the difference of two normal random variables (see e.g.

Weisstein (1995)) we find that the distribution of $n\xi - \eta$ is $N(0, \frac{1}{6}n\sqrt{n+1})$. The probability that all $2n$ inequalities hold is

$$\Pr(n\xi - \eta \leq n^2\epsilon)^{2n} = (\frac{1}{2} + \int_0^{\frac{n}{\sqrt{n+1}}6\epsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt)^{2n}.$$

If we want this probability to be $1 - \frac{2}{n}$ we have, for any fixed n, the equation

$$(1 - \frac{2}{n})^{\frac{1}{2n}} - \frac{1}{2} = \int_0^{\frac{n}{\sqrt{n+1}}6\epsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt.$$

Numerically ε can be determined using the standard normal distribution's table. For $n = 12$ and $n = 15$ we get $\epsilon = 0,1217$ and $0,1151$, respectively, much better than what we would have obtained without the assumption of normality using simply the error term $\sqrt{\frac{\ln n}{n}}$. In fact, for $n = 12$ and $n = 15$ this is $0,455$ and $0,4249$, respectively. It is clear that $\epsilon \to 0$ as $n \to \infty$.

6 Conclusion

The computational complexity of finding a Nash equilibrium (NEP) in general bimatrix games and symmetric and/or all NEP's in symmetric bimatrix games was studied for various classes of games. A new class of games was identified that can be solved polynomially. An experiment with the sample size of 500 was conducted for random symmetric games of size 12 and 15. Random entries of the matrices were drawn from a discrete uniform distribution over the interval [0, 100]. Empirical distributions of support sizes of all extreme NEP's were studied mainly to test the hypothesis that a 25% increase in size increases the critical support size i.e. the size of supports with the property that checking all supports of maximum this size for equilibrium is enough to find a NEP with high probability. This hypothesis was rejected supporting the conjecture that critical support sizes are small, 4 for symmetric, 3 for non-symmetric and 2 for all NEP's and thus the Las Vegas algorithm of Bárány et al. (2005) works for symmetric games as well.

Further research may go in various directions. First and foremost, the conjecture about small critical sizes for the Las Vegas algorithm should be proved or disproved and boundaries set within which this kind of method works. Extending the range of problems where approximate NEP's can be found in polynomial time with high probability is also a challenge. It would also be interesting to see what is the expected number of iterations of the Lemke-Howson algorithm over random bimatrix games and whether it grows exponentially or not.

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Appendix

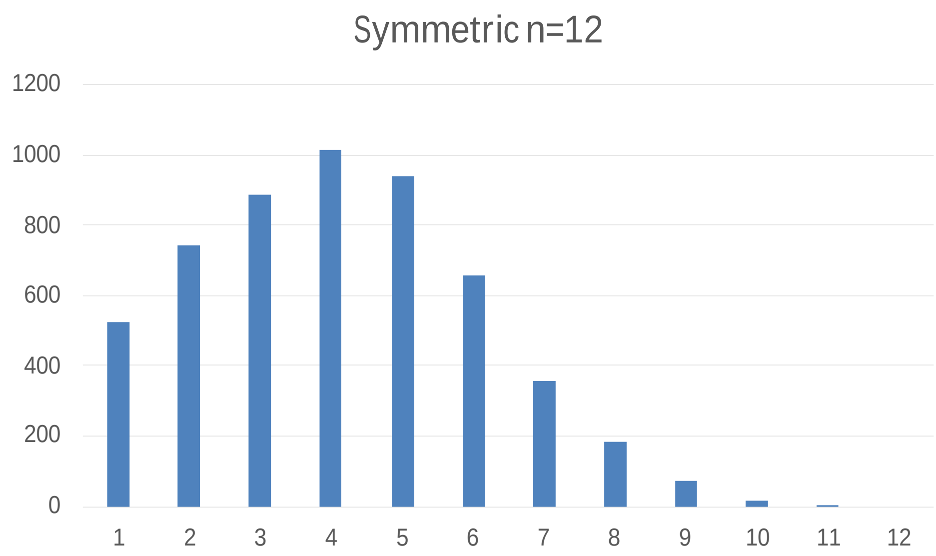
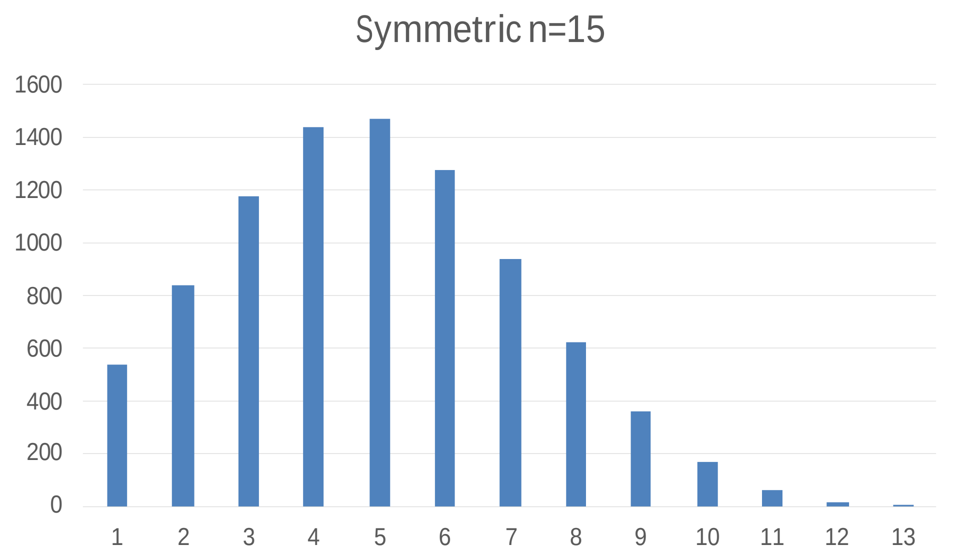


Figure 1:



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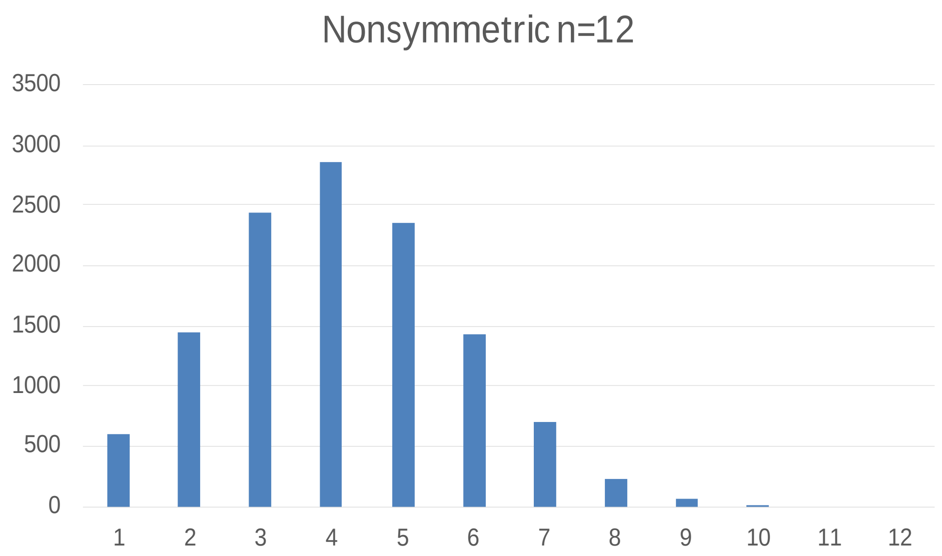

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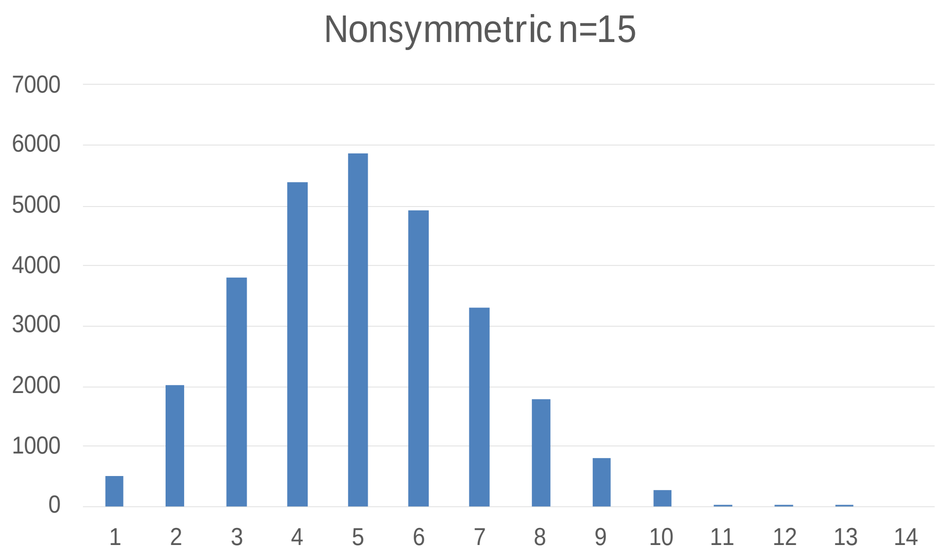

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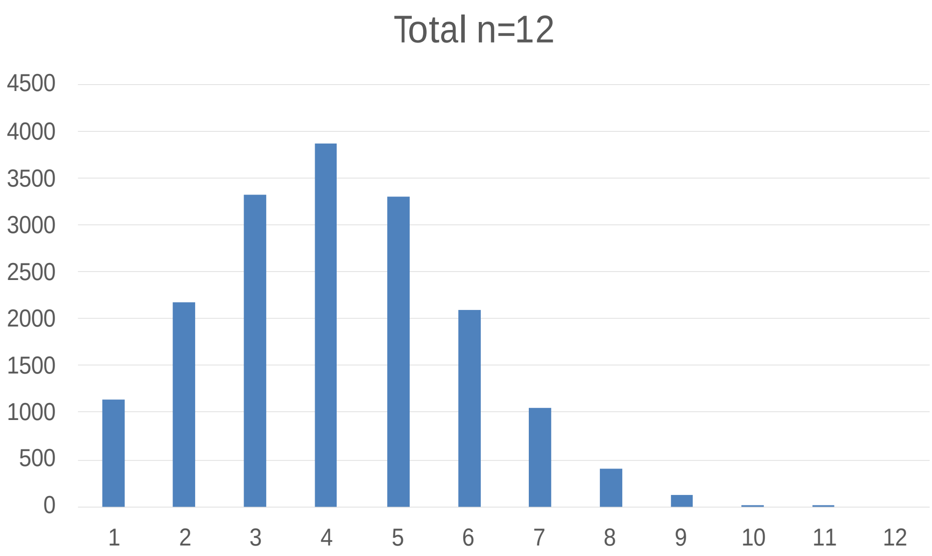

Figure 5:

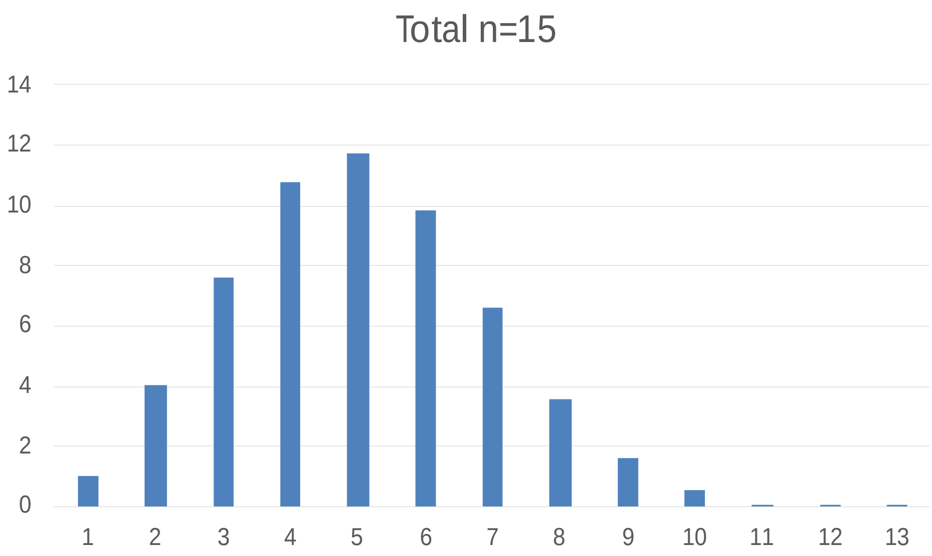

Figure 6: