An existence of trapezoidal waveform of oscillating motion from triangular pressure waveforms: The case of an Oldroyd-B model

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***Abstract*—Oscillating motion of Oldroyd-B flow in a capillary tube at a triangular sine and cosine pressure waveform is considered. An analytical solution of velocity field is obtained for an oscillating laminar flow, which can be used to determine the existence of trapezoidal waveform effect on the flow. When both triangular sine and cosine components are dominate, the oscillating flow can result in significant trapezoidal waveform performance. The dimensionless oscillating frequency, amplitude, and fluid material parameters, are primary factors affecting the flow performance of an oscillating flow in a capillary tube.**

I. Introduction

Theoretical studies on the laminar oscillatory flow from a capillary tube have receive attention in the literature due to their applications in natural systems (respiratory system, circulatory system) as well as engineering systems (pulse combustors, reciprocating pumps). Several sinusoidal waveform profiles in a capillary tube have been studied to determine the oscillating pressure effect on the fluid flow. Richardson and Tyler [1] investigated Newtonian viscous flow in a capillary tube due to sinusoidal pressure waveform and revealed how the oscillating flow could result in a different velocity profile near the wall surface. Since then, diverse researches and studies have been devoted to oscillatory flow performance in capillary tubes. Rahaman and Ramkissoon [2] investigated viscoelastic upper convected Maxwell passed in the pipes and showed how the oscillating flow performance depend on the time-dependent pressure waveform of an arbitrary function. Hariharan et al. [3] reported a theoretical and experimental result of the peristaltic transport of non-Newtonian fluid, modeled as power law and Bingham fluid, in a diverging tube with different wall waveforms: sinusoidal, multi-sinusoidal, triangular, trapezoidal and square waves. Results obtained suggest that square wave has the best pumping characteristics of all the waveforms, and the triangular wave has the worst characteristics. Yin and Ma [4] presented a novel analytical solution for velocity, temperature distributions and a Nusselt number for an oscillating laminar flow in a round pipe driven by a sinusoidal waveform. The result reveals that the dimensionless oscillating frequency, amplitude, and Prandtl number, have the direct effects on the heat transfer performance of an oscillating flow in a capillary tube.

Zhao and Cheng [5] showed that in addition to the sinusoidal waveform, other waveforms exist in an oscillating pipes such as the triangular waveform. The existence of triangular waveforms, made it necessary to further investigate the existence of other waveforms effect of oscillating motion on the flow rate performance of oscillating flow in a capillary tube. Recently, Yin and Ma [6] derived an analytical solution of laminar Newtonian flow with a triangular pressure waveform. Results reveal that the heat transfer coefficient of the oscillating flow depends on the fluid characteristics and oscillating waveform. The triangular waveform of oscillating motion can lead to a greater heat transfer coefficient.

Various rheological models have proposed and available to the literature to describe the rheological behavior of non-Newtonian fluids $[7-9]$. One of this type of rheological model is the rate type fluid model described by Dunn and Rajagopal [7]. Oldroyd-B fluid is one of the subclasses of these fluid models that exhibit strange features such as elastic and memory effects exhibited by most polymer and biological fluid.

The objective of present paper is to establish the existence of trapezoidal waveform from triangular waveform of Oldroyd-B flud of rate type. The oscillating flow with a trapezoidal waveform has not been investigated before. The unanswered questions regarding trapezoidal waveforms, made it necessary to study the effect of oscillating motion on the fluid flow in a capillary tube. In the current investigation, the oscillating flow with a triangular waveform is modeled by using an infinite Fourier series. In order to demonstrate the existence of trapezoidal waveform effect, the triangular sine and cosine waveforms of the oscillating flow is presented. A triangular reduces to a trapezoidal when combined between sine and cosine waveform and hence, trapezoidal waveforms can be treated as a special case of triangular waveform. Using the analytical solution of velocity field, the effects of waveform frequency, waveform amplitude, and material parameter on the flow rate performance of oscillating flow are analyzed.

II. Theoretical modeling

The Cauchy stress tensor $\mathbf{T}$ for an incompressible homogeneous Oldroyd-B fluid is of the form:

$$\mathbf{T}=-p\mathbf{I}+\mathbf{S},\quad \mathbf{S}+\lambda\frac{\delta\mathbf{S}}{\delta t}=\mu\left[\mathbf{A}+\gamma\frac{\delta\mathbf{A}}{\delta t}\right], \tag{1}$$

where

$$\frac{\delta\mathbf{S}}{\delta t}=\frac{d\mathbf{S}}{dt}-\mathbf{LS}-\mathbf{SL}^T,\quad \frac{\delta\mathbf{A}}{\delta t}=\frac{d\mathbf{A}}{dt}-\mathbf{LA}-\mathbf{AL}^T. \tag{2}$$

where $-p\mathbf{I}$ is the indeterminate spherical stress, $\mathbf{S}$ is the extra-stress tensor, λ is the relaxation time, $\frac{d}{dt}$ is the material time differentiation, $\mathbf{L}$ is the velocity gradient, T is the transpose operator, μ is the dynamic viscosity, $\mathbf{A}$ is the first Rivlin-Ericksen tensor and λ and γ are relaxation and the retardation time, respectively.

The flow is incompressible, the balance of the momentum in the absence of body forces is

$$\rho\frac{d\mathbf{u}}{dt}=-\operatorname{grad}p+\operatorname{div}\mathbf{T}. \tag{3}$$

where the velocity is assumed to be of the form:

$$\mathbf{u}=u\left(r,t\right)\mathbf{a}_z, \tag{4}$$

where $\mathbf{a}_z$ is the unit vector in the $z-$direction. Under the above considerations, Eq. (3) gives the following dimensional governing equations

$$\begin{aligned}\rho\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} &= \left(1+\lambda\frac{\partial}{\partial t}\right)\left(-\frac{\partial p}{\partial z}\right)\\ &+\mu\left(1+\gamma\frac{\partial}{\partial t}\right)\\ &\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right).\end{aligned} \tag{5}$$

The boundary conditions corresponding to (5) are:

$$\frac{\partial u}{\partial r} = 0 \text{ when } r=0, \tag{6}$$
$$u = 0 \text{ when } r=r_0. \tag{7}$$

Consider the following non-dimensional quantities

$$\begin{aligned}u^* &= \frac{u}{u_m},\quad t^*=\frac{\upsilon t}{r_0^2},\quad r^*=\frac{r}{r_0},\\ \lambda_1^* &= \frac{\upsilon\lambda_1}{r_0^2},\quad \gamma_3^*=\frac{\upsilon\lambda_3}{r_0^2},\quad A_0=\frac{\mu u_m}{r_0^2},\end{aligned} \tag{8}$$

where u_m is the mean value of the velocity. Using expression (8) in (5) the dimensionless momentum and energy equations (after dropping the $*$ notation):

$$\begin{aligned}\left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t} &= \left(1+\lambda\frac{\partial}{\partial t}\right)\left(-\frac{\partial p}{\partial z}\right)\\ &+\left(1+\gamma\frac{\partial}{\partial t}\right)\\ &\left(\frac{\partial^2 u\left(r,t\right)}{\partial r^2}+\frac{1}{r}\frac{\partial u\left(r,t\right)}{\partial r}\right),\end{aligned} \tag{9}$$

while the boundary conditions in dimensionless form are

$$\frac{\partial u}{\partial r} = 0 \quad \text{when} \quad r=0, \tag{10}$$
$$u = 0 \quad \text{when} \quad r=1. \tag{11}$$

A. Triangular (sine) pressure waveform

An infinite Fourier series describe the triangular sine pressure waveform is considered as the driving force of the investigated oscillating flow. The oscillating Oldroyd-B flow is driven by pressure difference with a trapezoidal sine waveform of amplitude A_0 and frequency ω, i.e.,

$$-\frac{\partial p}{\partial z}=A_0\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{\left(2n-1\right)^2}\sin((2n-1)\,\omega t), \tag{12}$$

Here we need to solve (5) subject to (6) and (7) in the case where pressure gradient is given by (12).

Using (12) into (5), we obtained:

$$\begin{aligned}\left(1+\gamma\frac{\partial}{\partial t}\right)\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r} &= \left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial u}{\partial t}\\ &-\frac{8A_0}{\pi^2}\left(1+\lambda\frac{\partial}{\partial t}\right)\\ &\sum_{n=1}^{\infty}\left[c_n\sin(N\omega t)\right],\end{aligned} \tag{13}$$

where $c_n=\dfrac{\left(-1\right)^{n-1}}{N^2}$ and $N=\left(2n-1\right)$.

Applying the Bessel transform [10], (13) becomes:

$$\begin{aligned}-(1+\gamma\frac{\partial}{\partial t})s_m^2 J_0\left[u_t\left(r,t\right)\right] &= \left(1+\lambda\frac{\partial}{\partial t}\right)\frac{\partial J_0\left[u_t\left(r,t\right)\right]}{\partial t}\\ &-\left[\begin{array}{c}\frac{8A_0}{\pi^2}\left(1+\lambda\frac{\partial}{\partial t}\right)\\ \sum_{n=1}^{\infty}\left[c_n\sin(N\omega t)\right]\end{array}\right]\\ &\times\frac{J_1\left(\lambda_n\right)}{s_m},\end{aligned} \tag{14}$$

where $J_0\left[u_t\left(r,t\right)\right]=\int_0^1 r u_t\left(r,t\right)J_0\left(s_m r\right)dr,\ J_0\left(s_m\right)=0$. Denote $F\left(s_m,t\right)=J_0\left[u_t\left(r,t\right)\right]$, then (14) becomes:

$$\begin{aligned}&F\left(s_m,t\right)+\left(\gamma+\frac{1}{s_m^2}\right)\frac{\partial F\left(s_m,t\right)}{\partial t}\\ &+\frac{\lambda}{s_m^2}\frac{\partial^2 F\left(s_m,t\right)}{\partial t^2}\\ &=\left[\begin{array}{c}A_0\frac{8}{\pi^2}\left(1+\lambda\frac{\partial}{\partial t}\right)\\ \sum_{n=1}^{\infty}\left[c_n\sin(N\omega t)\right]\end{array}\right]\frac{J_1\left(s_m\right)}{s_m^3}.\end{aligned} \tag{15}$$

Assuming the solution of (15) of the form:

$$F\left(s_m,t\right)=\sum_{n=1}^{\infty}\left[A_n\cos\left(N\omega t\right)+B_n\sin\left(N\omega t\right)\right]. \tag{16}$$

then (15) after rearranging yields:

$$\frac{\left[\begin{array}{c}A_n\left(N\pi^2 s_m^3-\lambda N^3\pi^2\omega^2 s_m\right)\\ +B_n\left(N^2\pi^2\omega s_m+\gamma N^2\pi^2\omega s_m^3\right)\\ -256c_n N\gamma\omega J_1\left(s_m\right)\end{array}\right]=0}{s_m^3\pi^2},$$

$$\frac{\left[\begin{array}{c}B_n\left(N^2\pi^2 s_m^3-\lambda N^4\pi^2\omega^2 s_m\right)\\ -A_n\left(N^3\pi^2\omega s_m+\gamma N^3\pi^2\omega s_m^3\right)\\ +256c_n\gamma J_1\left(s_m\right)\left(\lambda_2 N^2\omega^2-1\right)\end{array}\right]}{s_m^3\pi^2}=0. \tag{17}$$

By solving the system (17), the expression for A_n and B_n could be obtain as:

$$A_n = \frac{-256 c_n N \gamma \omega H_1 J_1 (s_m)}{s_m \pi^2 H}, \tag{18}$$

$$B_n = \frac{256 c_n \gamma s_m H_2 J_1 (s_m)}{\pi^2 H}, \tag{19}$$

where

$$\begin{aligned} H &= s_m^4 + N^2\omega^2 - 2\lambda_1 s_m^2 N^2 \omega^2 + \lambda_1^2 N^4 \omega^4 \\ &\quad + 2\lambda_3 N^2 \omega^2 s_m^2 + \lambda_3^2 N^2 \omega^2 s_m^4, \quad (20) \\ H_1 &= 1 + \lambda_1^2 N^2 \omega^2 - \lambda_1 s_m^2 + \lambda_3 s_m^2, \quad (21) \\ H_2 &= 1 + \lambda_1 \lambda_3 N^2 \omega^2. \quad (22) \end{aligned}$$

Hence, the expression for $F(s_m, t)$ can be written as:

$$F(s_m, t) = \frac{8A_0}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{c_n N Q_{nm}}{H} \frac{J_1 (s_m)}{s_m} \right), \tag{23}$$

where $Q_{nm} = -N\omega H_1 \cos(N\omega t) + s_m^2 H_2 \sin(N\omega t)$.

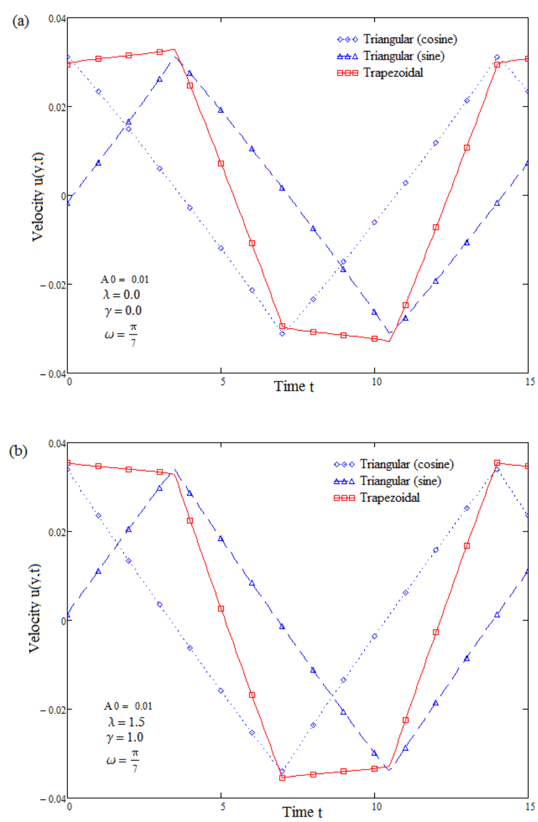

Fig. 1. Waveform effects on velocity distribution at $A_0 = 0.01$ and $\omega = \frac{\pi}{7}$: (a) Newtonian fluid and (b) Oldroyd-B fluid

Applying the inverse Bessel transform

$$u(r, t) = \frac{16A_0}{\pi^2} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \left(\frac{c_n N Q_{nm}}{H} \right) \frac{J_0 (s_m r)}{s_m J_1 (s_m)} \right]. \tag{24}$$

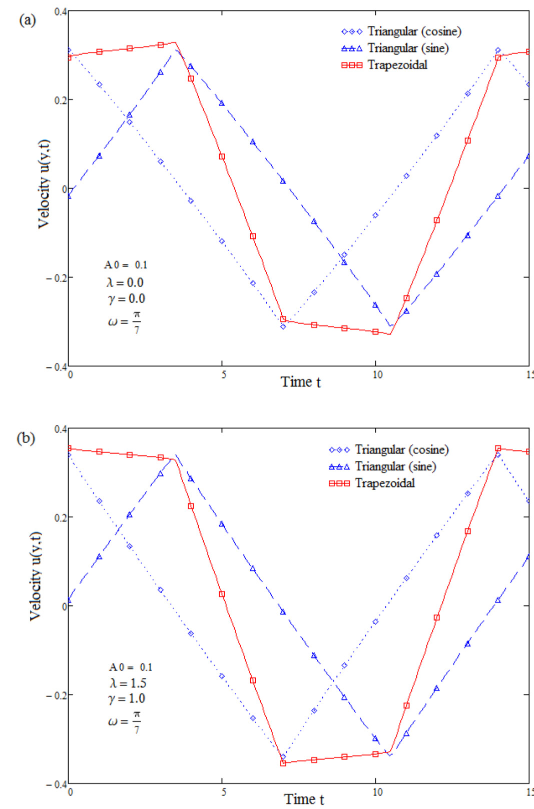

Fig. 2. Waveform effects on velocity distribution at $A_0 = 0.1$ and $\omega = \frac{\pi}{7}$: (a) Newtonian fluid and (b) Oldroyd-B fluid

where J_1 is the Bessel function of first kind of order one and s_m is the eigenvalue of the Bessel function of first kind of order zero.

Making $\lambda = \gamma = 0$, into (20)-(22), the case of Newtonian fluid, (24) becomes identical with that obtained by Yin and Ma [4].

B. Triangular (cosine) pressure waveform

In this case, an infinite Fourier series describe the triangular cosine pressure waveform is considered as the driving force of the investigated oscillating flow:

$$-\frac{\partial p}{\partial z} = A_0 \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\cos((2n+1)\omega t)}{(2n+1)^2}. \tag{25}$$

Using the same methodology in the preceding analysis, we obtain the velocity field as:

$$u(r, t) = \frac{16A_0}{\pi^2} \sum_{m=1}^{\infty} \left[\sum_{n=1}^{\infty} \left(\frac{R_{nm}}{N_1 H} \right) \frac{J_0 (s_m r)}{s_m J_1 (s_m)} \right]. \tag{26}$$

where $N_1 = (2n+1)$ and $R_{nm} = N_1 \omega H_1 \cos(N_1 \omega t) + s_m^2 H_2 \sin(N_1 \omega t)$

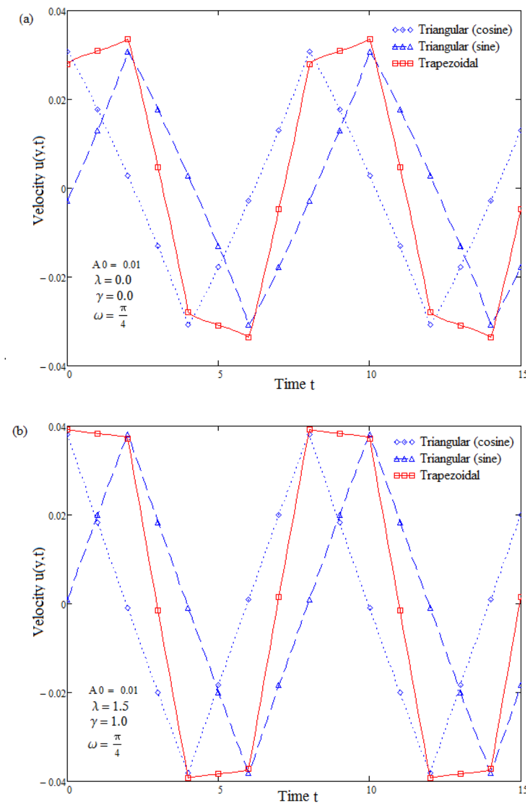

Fig. 3. Waveform effects on velocity distribution at $\omega = \dfrac{\pi}{4}$ and $A_0 = 0.01$: (a) Newtonian fluid and (b) Oldroyd-B fluid

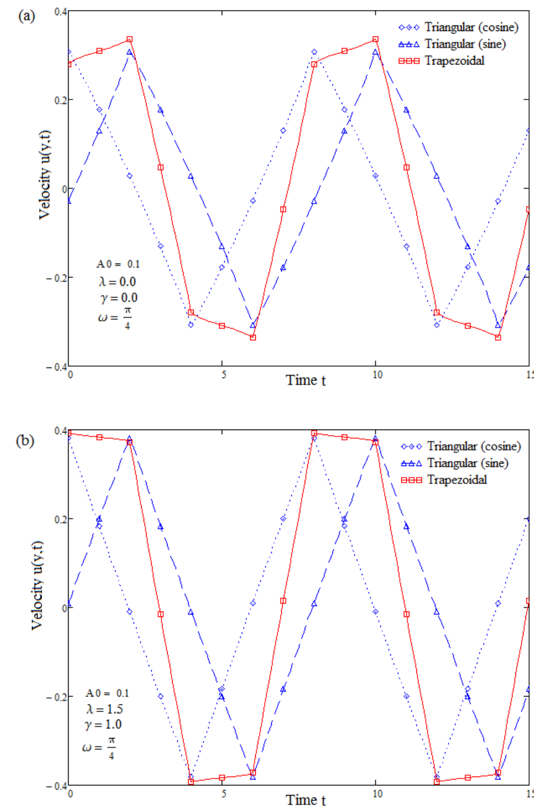

Fig. 4. Waveform effects on velocity distribution at $\omega = \dfrac{\pi}{4}$ and $A_0 = 0.1$: (a) Newtonian fluid and (b) Oldroyd-B fluid

C. Velocity field: A Trapezoidal waveform solution

It is worth point out that the trapezoidal can be deduce as the sum between the triangular (sine) and triangular (cosine), i.e.

$$u_{tra}\left(r,t\right) = u_{tri(\sin)}\left(r,t\right) + u_{tri(\cos)}\left(r,t\right). \qquad (27)$$

where $u_{tri(\sin)}$ is the solution given by (24) and $u_{tri(\cos)}$ is the solution given by (26)

III. Results and Discussion

The transient velocity given by (24), (26) and (27) are plotted in Figs. 1-4 for a Newtonian fluid for which $\lambda = \gamma = 0$ and an Oldroyd-B fluid for which $\lambda \neq 0$ and $\gamma \neq 0$. In all these figures, the trapezoidal waveform profile is generated from existing triangular waveform of the same amplitude and frequency. Also, from these figures, it is observed that the variation for waveform effects with the classical Newtonian and Oldroyd-B fluids are distinct. The Oldroyd-B has the maximum waveform effect as compared with classical Newtonian fluid.

Figs. 1 and 2 illustrates the effect of the pressure oscillating amplitude, A_0, at $\omega = \dfrac{\pi}{7}$, on the transient velocity distribution. As noted, the amplitude of the oscillating pressure significantly affects the transient velocity distribution. When the amplitude of the oscillating pressure is equal to 0.001 as shown in Fig. 1, the peak of the oscillating velocity distribution can reach 0.035, but when the amplitude of the oscillating pressure increases further to 0.1 as shown in Fig. 2, the peak velocity can reach can reach 0.035. These results indicate that maximizing the oscillating amplitude can enhance velocity distribution for both triangular and trapezoidal waveforms.

Figs. 3 and 4 demonstrated the effect of frequency on an oscillating motion. By considering the higher value of the frequency different from the previous figures, i.e.,$\omega = \dfrac{\pi}{4}$. These results indicate that maximizing the oscillating frequency can enhance velocity distribution. Large waveform effects are observed with the value $A_0 = 0.1$ in Fig. 3 as compared with Figs. 3.

IV. Conclusions

This research derived an analytical solution of the velocity distribution of a laminar oscillating flow of Oldroyd-B fluid flow in a capillary tube driven by triangular pressure waveforms. The model of classical Newtonian flow is obtained as a limiting case and compare with the result of Oldroyd-

B fluid oscillating motions. Results show the existence of trapezoidal waveform of transient velocity distribution as a sum of sine and cosine triangular waveforms. For an oscillating flow driven by a triangular and trapezoidal waveforms, when the oscillating frequency increases or when the amplitude increases, the oscillating motion can enhance the velocity distribution. The waveform effect of an Oldroyd-B fluid is similar to the classical Newtonian. However, the Oldroyd-B fluid oscillating motion can result in a higher peak velocity.

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