The low-dimensional algebraic cohomology of the Witt and the Virasoro algebra

based on arXiv:1707.06106 and on-going work

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Abstract

The main aim of our work is to compute the third algebraic cohomology with values in the adjoint module of the Witt and the Virasoro algebra. More precisely, we show that the third algebraic cohomology of the Witt algebra is zero, whereas the third algebraic cohomology of the Virasoro algebra is one-dimensional. We also show that the third algebraic cohomology of the Witt and the Virasoro algebra with values in the trivial module is one-dimensional. We consider purely algebraic cohomology, i.e. our results are independent of any topology chosen.

1 Introduction

- The Witt and the Virasoro algebra are infinite-dimensional Lie algebras of outermost importance in physics, the Virasoro algebra being omnipresent in 2-dimensional conformal field theory and String Theory.
- Low-dimensional cohomology: interpretation in terms of invariants, outer derivations, extensions, deformations and obstructions, as well as crossed modules ↔ their knowledge allows a better understanding of the Lie algebra itself.
- *Algebraic* cohomology (arbitrary maps) vs. *continuous* cohomology (continuous maps): valid for any base field $\mathbb{K}$ with $char(\mathbb{K}) = 0$, independent of any topology chosen, independent of any concrete realization of the Lie algebra.

2 Main Objectives

- Aim: Compute the third algebraic cohomology with values in the adjoint module of the Witt and the Virasoro algebra.
- Byproduct: third algebraic cohomology with values in the trivial module of the Witt and the Virasoro algebra.
- Known: Second algebraic cohomology of the Witt algebra; see Schlichenmaier [8, 7] and also Fialowski [3, 2].
- Known: Second algebraic cohomology of the Virasoro algebra; see Schlichenmaier [8].
- Known: First algebraic cohomology of the Witt and the Virasoro algebra; see Ecker and Schlichenmaier [1].

3 The Witt and the Virasoro algebra

3.1 The Witt algebra

- The Witt algebra $\mathcal{W}$ is generated as vector space over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ by the basis elements $\{e_n \mid n \in \mathbb{Z}\}$ satisfying the following Lie structure:

$$[e_n, e_m] = (m-n)e_{n+m} \qquad n, m \in \mathbb{Z}$$

- $\mathbb{Z}$-graded Lie algebra: $deg(e_n) := n$
- Decomposition of $\mathcal{W}$: $\mathcal{W} = \bigoplus_{n\in\mathbb{Z}} \mathcal{W}_n$, with each $\mathcal{W}_n$ a 1-dimensional homogeneous subspace generated by e_n
- Internally graded: $[e_0, e_n] = ne_n = deg(e_n)e_n$, i.e. e_n is eigenvector of $ad_{e_0} := [e_0, \cdot]$ with eigenvalue n
- Algebraic realization: Lie algebra of derivations of Laurent polynomials $\mathbb{K}[Z^{-1}, Z]$
- Geometrical realization:
  - $\mathbb{K} = \mathbb{C}$, algebra of meromorphic vector fields on $\mathbb{CP}^1$ holomorphic outside of 0 and ∞, with $e_n = z^{n+1}\frac{d}{dz}$
  - Complexified Lie algebra of polynomial vector fields on S^1, with $e_n = e^{in\phi}\frac{d}{d\phi}$

3.2 The Virasoro algebra

- There is an unique, one-dimensional central extension of the Witt algebra, called Virasoro algebra $\mathcal{V}$, which is a universal central extension:

$$0 \longrightarrow \mathbb{K} \stackrel{i}{\longrightarrow} \mathcal{V} \stackrel{\pi}{\longrightarrow} \mathcal{W} \longrightarrow 0,$$

where $\mathbb{K}$ is in the center of $\mathcal{V}$.
- As a vector space, $\mathcal{V} = \mathbb{K} \oplus \mathcal{W}$ generated by $\hat{e}_n := (0, e_n)$ and $t := (1, 0)$
- Lie structure equation:

$$\begin{aligned} &[\hat{e}_n, \hat{e}_m] = (m-n)\hat{e}_{n+m} - \tfrac{1}{12}(n^3 - n)\delta_n^{-m} t, \\ &[\hat{e}_n, t] = [t, t] = 0 \end{aligned} \qquad n, m \in \mathbb{Z}$$

- $deg(\hat{e}_n) := deg(e_n) = n$ and $deg(t) = 0 \Rightarrow \mathcal{V}$ is $\mathbb{Z}$-graded

4 The Chevalley-Eilenberg cohomology

- Let $\mathcal{L}$ be a Lie algebra, M a $\mathcal{L}$-module and $C^q(\mathcal{L}, M)$ the vector space of q-multilinear alternating maps with values in M, called q-cochains ($q \in \mathbb{N}$)
  Convention: $C^0(\mathcal{L}, M) := M$
- Coboundary operators δ_q defined by:

$$\forall q \in \mathbb{N}, \qquad \delta_q : C^q(\mathcal{L}, M) \to C^{q+1}(\mathcal{L}, M) : \psi \mapsto \delta_q\psi,$$

$$\begin{aligned} (\delta_q\psi)(x_1, \ldots x_{q+1}) : &= \textstyle\sum_{1\le i<j\le q+1} (-1)^{i+j+1}\, \psi([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}) \\ &+ \textstyle\sum_{i=1}^{q+1} (-1)^i\, x_i \cdot \psi(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1}), \end{aligned}$$

with $x_1, \ldots, x_{q+1} \in \mathcal{L}$

- Examples of modules M:
  - ⋆ Adjoint module $M = \mathcal{L}$ with product $x \cdot m = [x, m]$
  - ⋆ trivial module $M = \mathbb{K}$ with product $x \cdot m = 0$
- $\delta_{q+1} \circ \delta_q = 0 \; \forall\, q \in \mathbb{N} \to$ complex of vector spaces:

$$\{0\} \xrightarrow{\delta_{-1}} M \xrightarrow{\delta_0} C^1(\mathcal{L}, M) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{q-2}} C^{q-1}(\mathcal{L}, M) \xrightarrow{\delta_{q-1}} C^q(\mathcal{L}, M) \xrightarrow{\delta_{q+1}} C^{q+1}(\mathcal{L}, M) \xrightarrow{\delta_{q+1}} \cdots$$

where $\delta_{-1} := 0$
- q-cocycles: $Z^q(\mathcal{L}, M) := \ker \delta_q$
- q-coboundaries: $B^q(\mathcal{L}, M) := \operatorname{im} \delta_{q-1}$
- q^{th} cohomology group of $\mathcal{L}$ with values in M:

$$\mathrm{H}^q(\mathcal{L}, M) := Z^q(\mathcal{L}, M)/B^q(\mathcal{L}, M)$$

- Chevalley-Eilenberg cohomology: $\mathrm{H}^*(\mathcal{L}, M) := \bigoplus_{q=0}^{\infty} \mathrm{H}^q(\mathcal{L}, M)$

5 Results

Theorem 5.1: $\mathrm{H}^3(\mathcal{W}, \mathcal{W}) = \{0\}$

The third algebraic cohomology of the Witt algebra $\mathcal{W}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the adjoint module is zero, i.e.

$$\mathrm{H}^3(\mathcal{W}, \mathcal{W}) = \{0\}$$

Theorem 5.2: $\dim(\mathrm{H}^3(\mathcal{V}, \mathcal{V})) = 1$

The third algebraic cohomology of the Virasoro algebra $\mathcal{V}$ over a field $\mathbb{K}$ with $char(\mathbb{K}) = 0$ and values in the adjoint module is one-dimensional, i.e.

$$dim(\mathrm{H}^3(\mathcal{V}, \mathcal{V})) = 1$$

Theorem 5.3: $\mathrm{H}^k(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^k(\mathcal{W}, \mathcal{W})$

$$\begin{aligned} &\text{If} && \mathrm{H}^j(\mathcal{W}, \mathcal{W}) = 0 \quad \text{for} \quad k-2 \le j \le k-1, \\ &\text{then} && \mathrm{H}^k(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^k(\mathcal{W}, \mathcal{W}). \\ &\text{In particular,} && \mathrm{H}^3(\mathcal{V}, \mathcal{W}) \cong \mathrm{H}^3(\mathcal{W}, \mathcal{W}). \end{aligned}$$

Theorem 5.4: $\dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = \dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$

The third algebraic cohomology of the Witt and the Virasoro algebra with values in the trivial module $\mathbb{K}$ is one-dimensional, i.e.:

$$dim(\mathrm{H}^3(\mathcal{W}, \mathbb{K})) = dim(\mathrm{H}^3(\mathcal{V}, \mathbb{K})) = 1$$

6 Conclusion

- A geometrical realization of the Witt algebra is given by the complexified Lie algebra of polynomial vector fields on the circle ⇝ forms a dense subalgebra of the complexified Lie algebra of smooth vector fields on the circle, $Vect(S^1)$.
- The continuous cohomology of $Vect(S^1)$ with values in $\mathbb{R}$ is known, in particular we have $dim(\mathrm{H}^3_{\mathcal{C}}(Vect(S^1), \mathbb{R})) = 1$, see Fuks and Gelfand [5, 6].
- The continuous cohomology of $Vect(S^1)$ with values in the adjoint module is also known, in particular we have $\mathrm{H}^3_{\mathcal{C}}(Vect(S^1), Vect(S^1)) = \{0\}$, see Fialowski and Schlichenmaier [4].
- For the low-dimensional cohomology with values in the adjoint and in the trivial module of the Witt algebra, algebraic and continuous cohomology agree.

References

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