Model and parameter identification through Bayesian inference in solid mechanics

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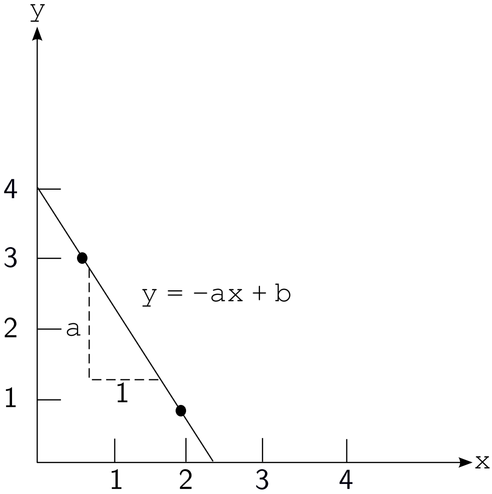
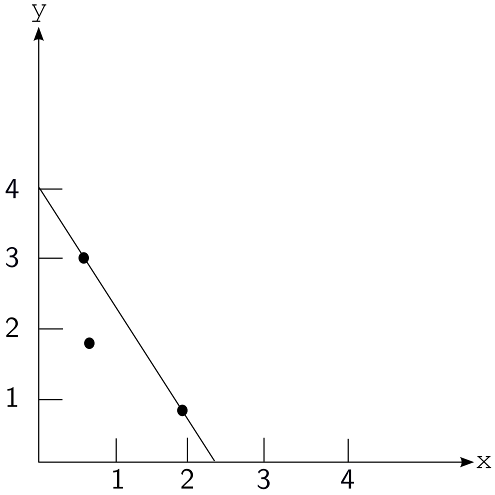
Introduction: Probabilistic modelling

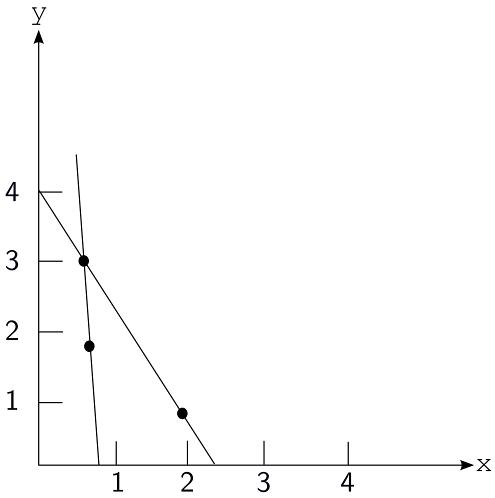


Introduction to Gaussian Processes, Neil Lawrence

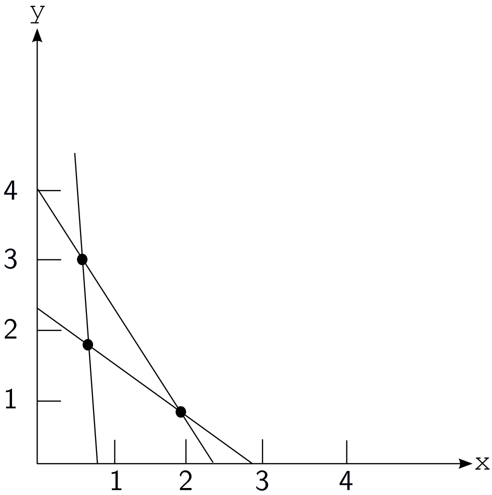
Introduction: Probabilistic modelling



Introduction: Probabilistic modelling



Introduction: Probabilistic modelling



Each point can be written as the model+ a corruption:

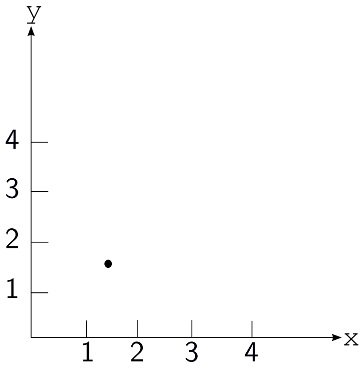
$$y_1 = ax + c + \omega_1$$
$$y_2 = ax + c + \omega_2$$
$$y_3 = ax + c + \omega_3$$

Introduction: Probabilistic modelling

Pierre-Simon Laplace
1749–1827
(source wikipedia)

The corruption term can be presented with a probability distribution

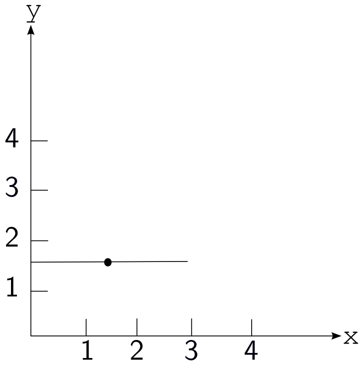
Introduction: Probabilistic modelling



How can we fit the $y = ax + b$ line, having only one point?

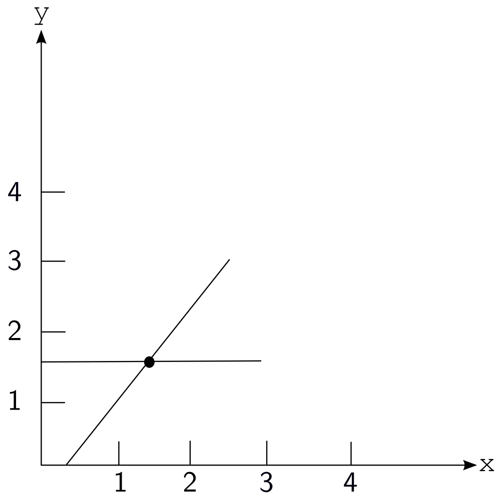
Introduction to Gaussian Processes, Neil Lawrence

Introduction: Probabilistic modelling

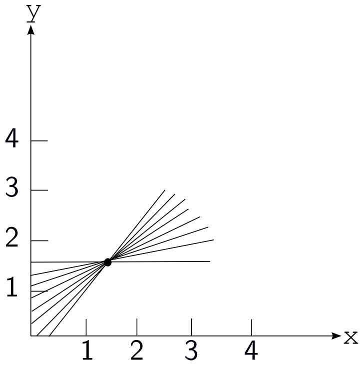


If b is fixed
$\Longrightarrow a = \frac{y-b}{x}$

Introduction: Probabilistic modelling



Introduction: Probabilistic modelling

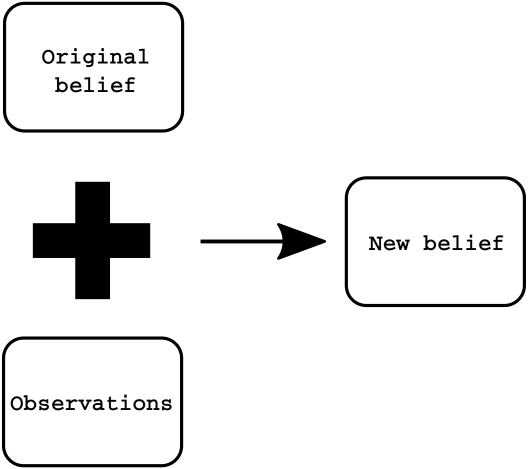


$$b \sim \pi_1 \implies a \sim \pi_2$$

Introduction: Probabilistic modelling

- This is called Bayesian treatment.
- The model parameters are treated as random variables.

Introduction: Bayesian perspective



Introduction: Bayesian formula (inverse probability)

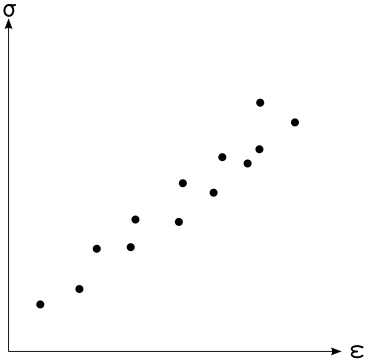
$$\overbrace{\pi(x|y)}^{\text{posterior}} = \frac{\overbrace{\pi(x)}^{\text{prior}} \times \overbrace{\pi(y|x)}^{\text{likelihood}}}{\underbrace{\pi(y)}_{\text{evidence}}}$$

y := observation
x := parameter
$\pi(x)$:= original belief
$\pi(y|x)$:= given by the mathematical model that relates y to x
$\pi(y)$:= is a constant number

Introduction: Bayesian formula (inverse probability)

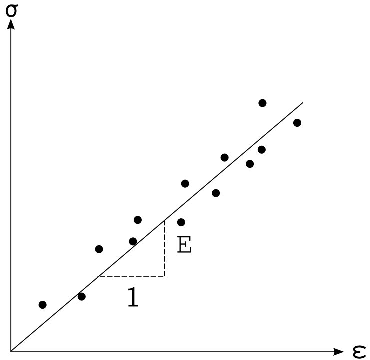
$$\pi(x|y) \propto \pi(x) \times \pi(y|x)$$

BI in computational mechanics



Linear elasticity

$$\sigma = E\varepsilon$$

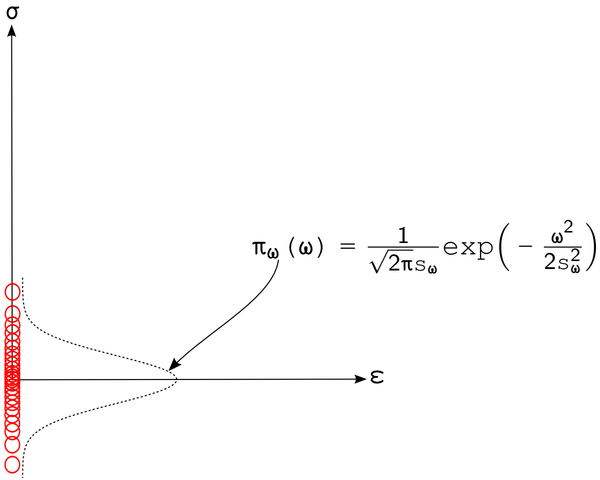


Linear elasticity

$$y = E\varepsilon + \omega$$
$$\Omega \sim \pi_\omega(\omega)$$

Capital letters denote a random variable

Linear elasticity



Noise PDF is modelled through calibration test.

Linear elasticity

Bayes' formula:

$$\pi(E|y) = \frac{\pi(E)\pi(y|E)}{\pi(y)} = \frac{\pi(E)\pi(y|E)}{k}$$

$$\pi(E|y) \propto \pi(E)\pi(y|E)$$

Linear elasticity

$$y = E\boldsymbol{\varepsilon} + \boldsymbol{\omega}$$
$$\Omega \sim N(0, s_{\omega}^{2})$$

Linear elasticity

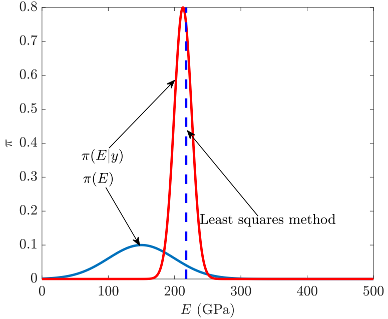
$$\pi(y|E) = \frac{1}{\sqrt{2\pi}s_{\omega}} \exp\left(- \frac{(y - E\varepsilon)^2}{2s_{\omega}^2} \right)$$

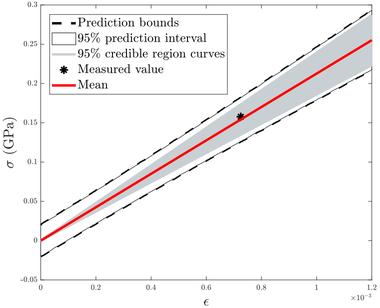
Linear elasticity

Posterior:

$$\pi(E|y) \propto \exp\left(-\frac{(E-\overline{E})^2}{2s_E^2}\right)\exp\left(-\frac{(y-E\boldsymbol{\varepsilon})^2}{2s_{\boldsymbol{\omega}}^2}\right)$$

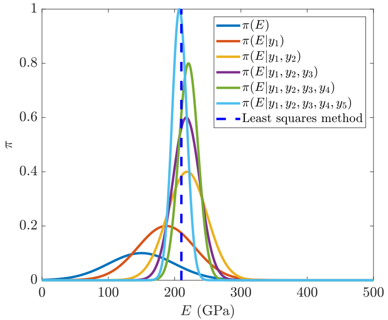
Linear elasticity

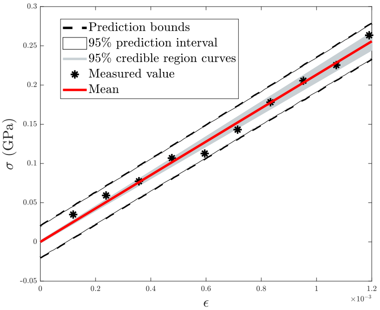




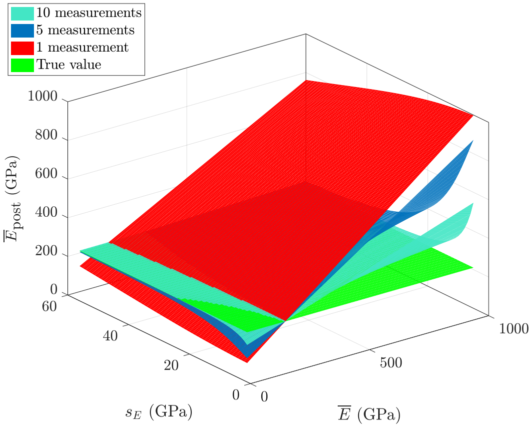
- Prediction interval: An estimate of an interval in which an observation will fall, with a certain probability.
- Credible region: A region of a distribution in which it is believed that a random variable lie with a certain probability.

Linear elasticity





Prior effect

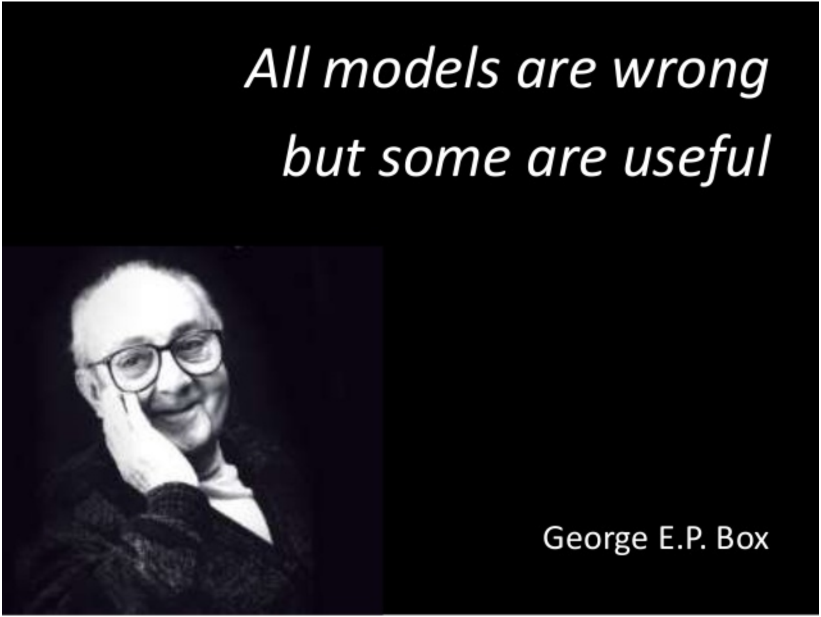


Model uncertainty and input error

$$y = f(\mathbf{x}, \varepsilon) + \omega$$
$$\Omega \sim \pi_\omega(\omega)$$

$\boldsymbol{\varepsilon}$ is the input variable and $\mathbf{x}$ is the parameter vector

Model uncertainty and input error



Model uncertainty and input error

Kennedy-O'Hagan (KOH) framework:

$$y = \underbrace{f(\mathbf{x}, \varepsilon) + d(\mathbf{x}_d, \varepsilon)}_{y_{true}} + \boldsymbol{\omega}$$

$\boldsymbol{\varepsilon}$ is the input variable and $\mathbf{x}$ is the parameter vector

Model uncertainty and input error

- Constant number d_0
- Deterministic function $\sum_{i=0}^{l} a_i \boldsymbol{\varepsilon}^i$
- Random variable from normal distribution $d \sim N(m, s_d^2)$
- Random variable from a normal distribution with input dependent mean and variance $d \sim N(m(\boldsymbol{\varepsilon}), s_d^2(\boldsymbol{\varepsilon}))$
- Gaussian process (GP)

...

Model uncertainty and input error

Bayes' formula:

$$\pi(\mathbf{x}, \mathbf{x}_d | y) \propto \pi(\mathbf{x})\pi(\mathbf{x}_d)\pi(y | \mathbf{x}, \mathbf{x}_d)$$

- Both material and model error parameters must be inferred.

If $d(\mathbf{x}_d, \varepsilon)$ is deterministic (for simplicity):

$$\pi(\mathbf{x}, \mathbf{x}_d | y) \propto \pi(\mathbf{x})\pi(\mathbf{x}_d)\pi_\omega(y - f(\mathbf{x}, \varepsilon) - d(\mathbf{x}_d, \varepsilon))$$

Model uncertainty and input error

Bayes' formula:

$$\pi(\mathbf{x}, \mathbf{x}_d | y) \propto \pi(\mathbf{x})\pi(\mathbf{x}_d)\pi(y | \mathbf{x}, \mathbf{x}_d)$$

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$$\pi(\mathbf{x}, \mathbf{x}_d | y) \propto \pi(\mathbf{x})\pi(\mathbf{x}_d)\pi_{\boldsymbol{\omega}}(y - f(\mathbf{x}, \boldsymbol{\varepsilon}) - d(\mathbf{x}_d, \boldsymbol{\varepsilon}))$$

Model uncertainty and input error

But what about the input error?

Model uncertainty and input error

$$y = f(\mathbf{x}, \boldsymbol{\varepsilon}) + d(\mathbf{x}_d, \boldsymbol{\varepsilon}) + \boldsymbol{\omega}$$
$$\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon} + \boldsymbol{\omega}_{\boldsymbol{\varepsilon}}$$

$$\Omega \sim \pi(\boldsymbol{\omega})$$
$$\Omega_{\boldsymbol{\varepsilon}} \sim \pi(\boldsymbol{\omega}_{\boldsymbol{\varepsilon}})$$

Model uncertainty and input error

Bayes' formula:

$$\pi(\mathbf{x}, \mathbf{x}_d, \varepsilon | y, \varepsilon^*) \propto \pi(y | \mathbf{x}, \mathbf{x}_d, \varepsilon) \pi(\varepsilon | \varepsilon^*) \pi(\mathbf{x}) \pi(\mathbf{x}_d)$$

$$\pi(\mathbf{x}, \mathbf{x}_d | y, \varepsilon^*) \propto \int_0^b \pi(y | \mathbf{x}, \mathbf{x}_d, \varepsilon) \pi(\varepsilon | \varepsilon^*) \, d\varepsilon \, \pi(\mathbf{x}) \pi(\mathbf{x}_d)$$

$$\pi(y | \mathbf{x}, \mathbf{x}_d, \varepsilon) = \pi_\omega(y - f(\mathbf{x}, \varepsilon) - d(\mathbf{x}_d, \varepsilon))$$
$$\pi(\varepsilon | \varepsilon^*) = \pi_{\omega_\varepsilon}(\varepsilon^* - \varepsilon)$$

Model uncertainty and input error

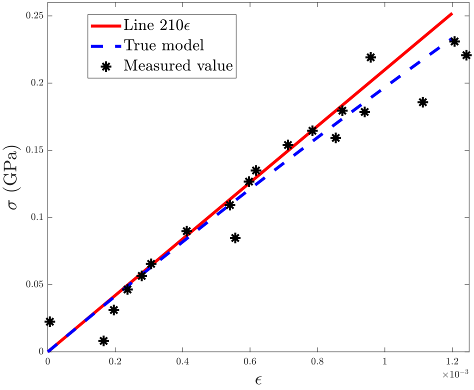
Bayes' formula:

$$\pi(\mathbf{x}, \mathbf{x}_d, \varepsilon | y, \varepsilon^*) \propto \pi(y|\mathbf{x}, \mathbf{x}_d, \varepsilon)\pi(\varepsilon|\varepsilon^*)\pi(\mathbf{x})\pi(\mathbf{x}_d)$$

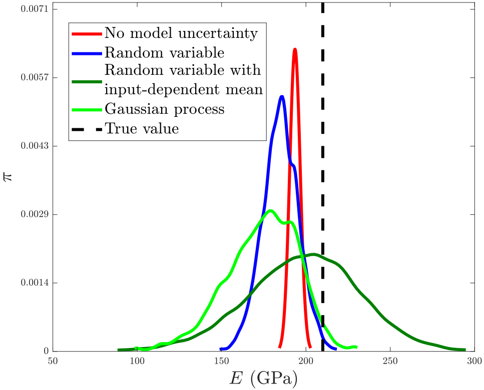
$$\pi(\mathbf{x}, \mathbf{x}_d | y, \varepsilon^*) \propto \int_0^b \pi(y|\mathbf{x}, \mathbf{x}_d, \varepsilon)\pi(\varepsilon|\varepsilon^*)\,d\varepsilon\, \pi(\mathbf{x})\pi(\mathbf{x}_d)$$

$$\pi(y|\mathbf{x}, \mathbf{x}_d, \varepsilon) = \pi_\omega(y - f(\mathbf{x}, \varepsilon) - d(\mathbf{x}_d, \varepsilon))$$
$$\pi(\varepsilon|\varepsilon^*) = \pi_{\omega_\varepsilon}(\varepsilon^* - \varepsilon)$$

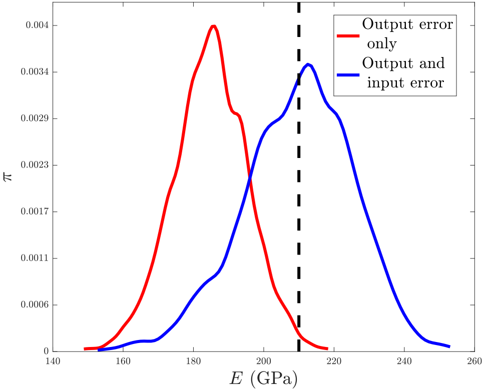
Model uncertainty and input error: Linear elasticity



Model uncertainty and input error: Effect of model uncertainty

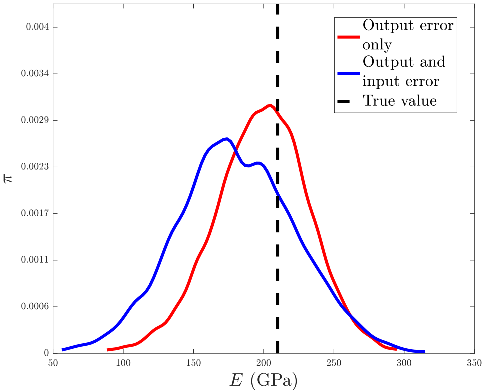


Model uncertainty and input error: Effect of model uncertainty as well as input error



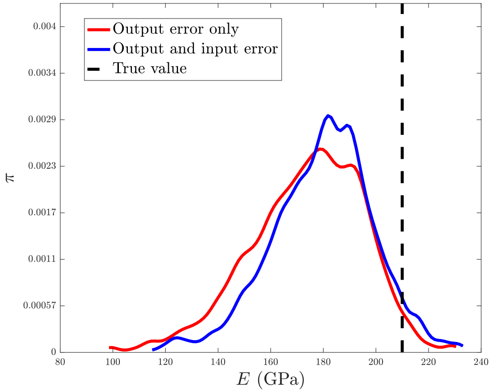
Normal distribution with constant parameters

Model uncertainty and input error: Effect of model uncertainty as well as input error



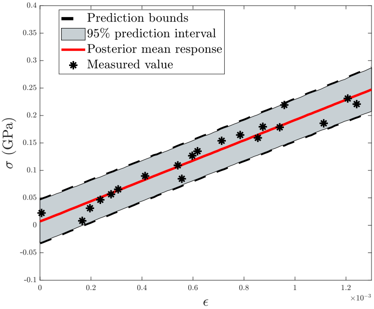
Normal distribution with an input-dependent mean

Model uncertainty and input error: Effect of model uncertainty as well as input error

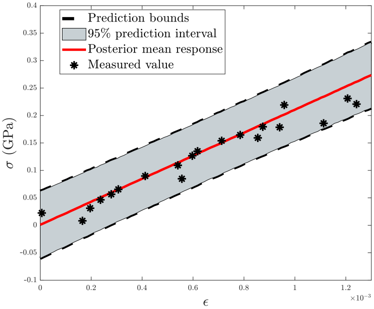


Gaussian process

Model uncertainty and input error: Effect of model uncertainty as well as input error



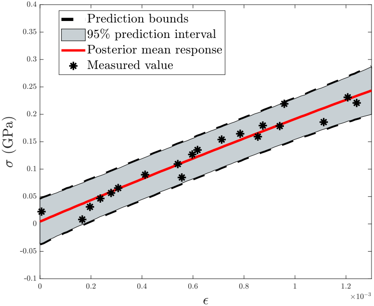
without input error



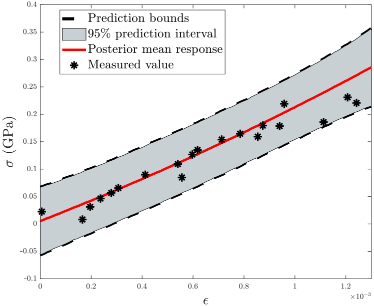
with input error

Normal distribution with constant parameters

Model uncertainty and input error: Effect of model uncertainty as well as input error



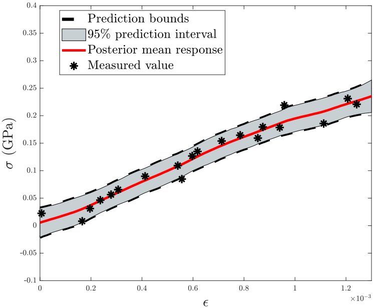
without input error



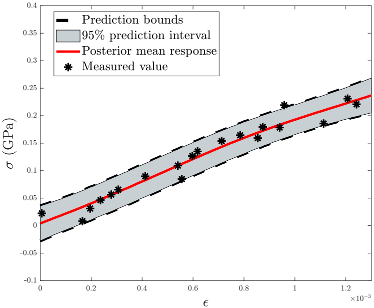
with input error

Normal distribution with an input-dependent mean

Model uncertainty and input error: Effect of model uncertainty as well as input error



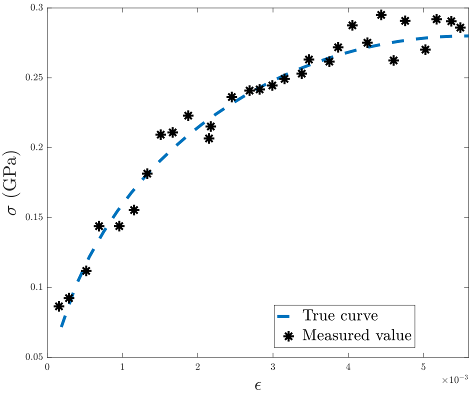
without input error



with input error

Gaussian process

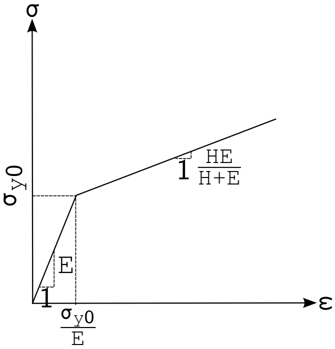
Model uncertainty and input error: Linear elastic-linear hardening



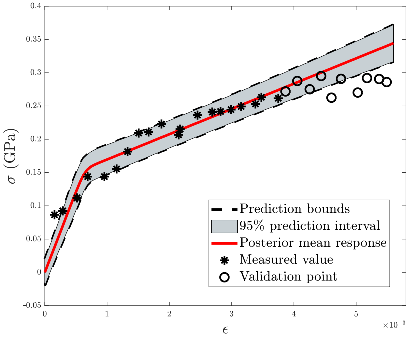
Model uncertainty and input error: Linear elastic-linear hardening

$$\sigma(\varepsilon, \mathbf{x}) = \begin{cases} E\varepsilon & \text{if } \varepsilon \leq \frac{\sigma_{y0}}{E} \\ \sigma_{y0} + \frac{HE}{H+E}\left(\varepsilon - \frac{\sigma_{y0}}{E}\right) & \text{if } \varepsilon > \frac{\sigma_{y0}}{E} \end{cases}$$

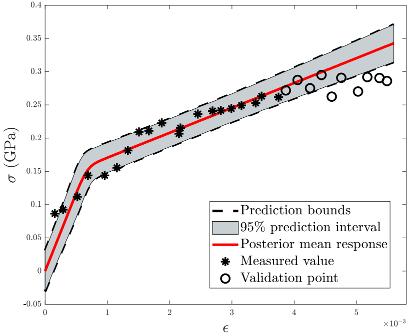
$$\mathbf{x} = [E, \sigma_{y0}, H]$$



Model uncertainty and input error: Linear elastic-linear hardening, extrapolation

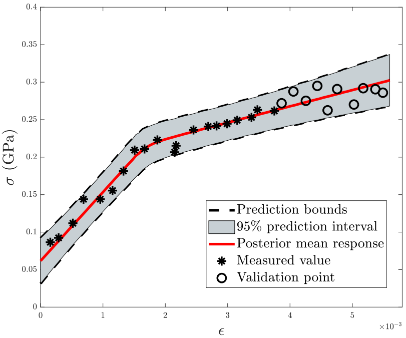


Error in the stress measurements only

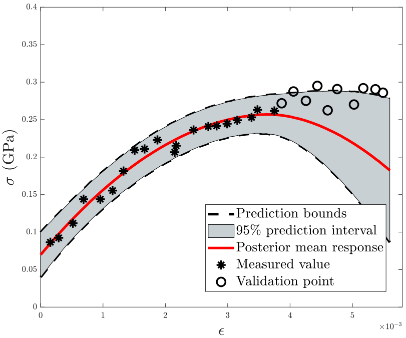


Error in both the stress and the strain measurements

Model uncertainty and input error: Linear elastic-linear hardening, extrapolation



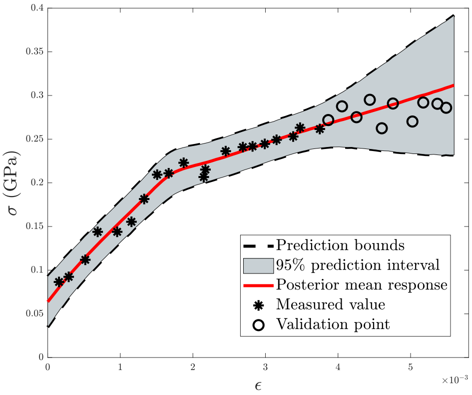
Normal distribution with constant parameters



Normal distribution with an input-dependent mean

Input error is considered for both cases.

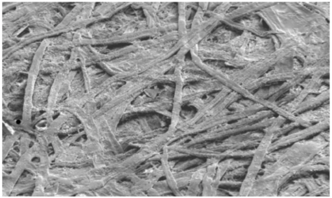
Model uncertainty and input error: Linear elastic-linear hardening, extrapolation

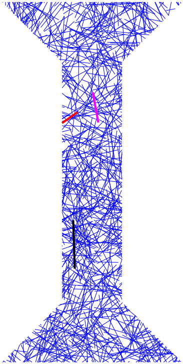


Gaussian process

Input error is also considered.

Material parameter distribution



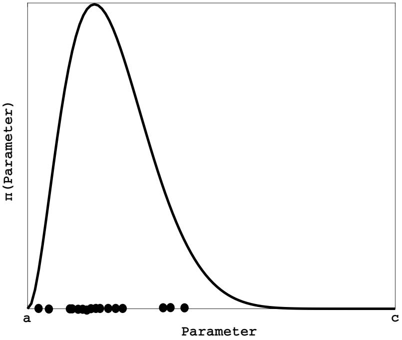


- Geometrical randomness
- Material parameter randomness

Material parameter distribution

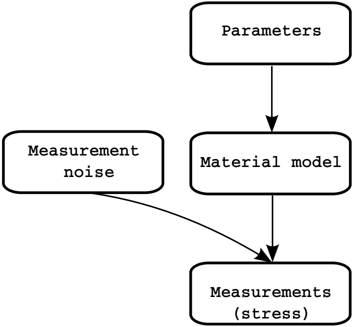
Objective

Find the distribution from which the parameters of the specimens are coming with **limited** number of specimens, N=20.



Identification scheme

$$\pi(x|y) \propto \pi(x)\pi(y|x)$$



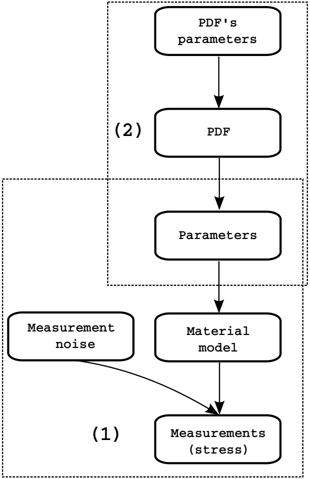
Identification scheme

$y \sim \pi(y|x_m) \longrightarrow$ (1)

$x_m \sim \pi(x_m|x_{PDF}) \longrightarrow$ (2)

$x_{PDF} \sim \pi(x_{PDF}) \longrightarrow$ prior

$$\pi(x_m, x_{PDF}|y) \propto \pi(y|x_m)\pi(x_m|x_{PDF})\pi(x_{PDF})$$



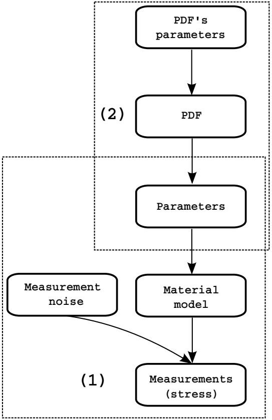
Hard to incorporate when the parameter distribution or model is not standard!

Identification scheme: Bayesian updating-Least squares

(1) $\longrightarrow$ Least squares

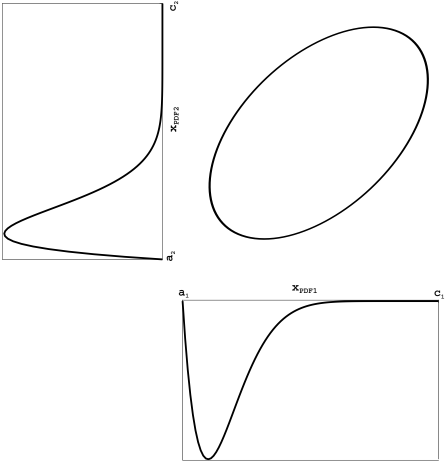
(2) $\longrightarrow$
Bayesian updating

$\pi(x_{PDF}|\overline{x}_m) \propto \pi(x_{PDF}|\overline{x}_m)\pi(x_{PDF})$

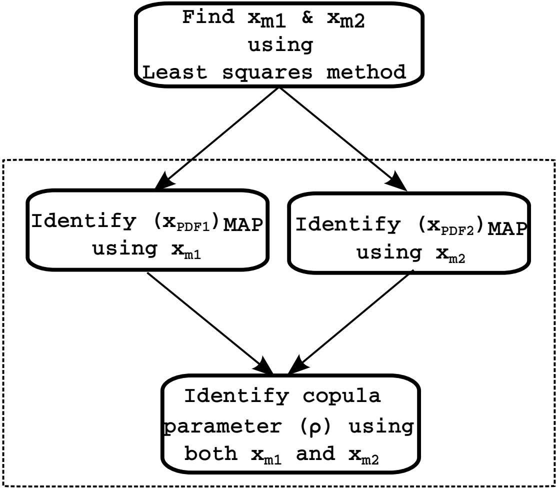


Copula

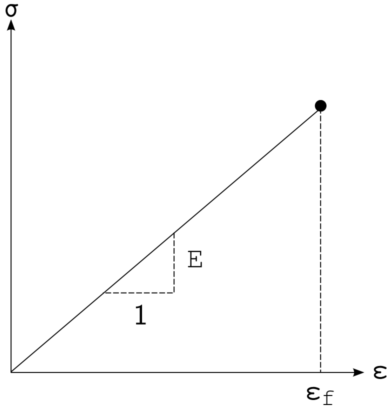
Copulas are tools enable us to model dependence of several random variables in terms of their marginal distribution.



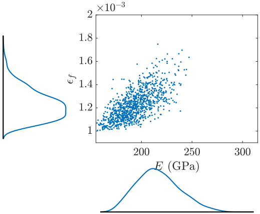
Proposed framework



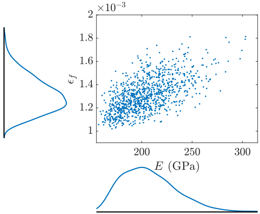
Identification results: Brittle damage



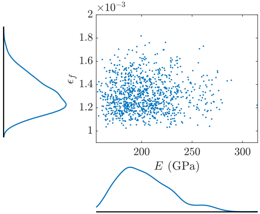
Identification results: Brittle damage



True

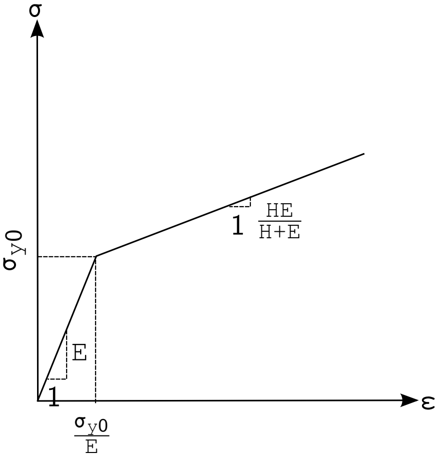


Identified

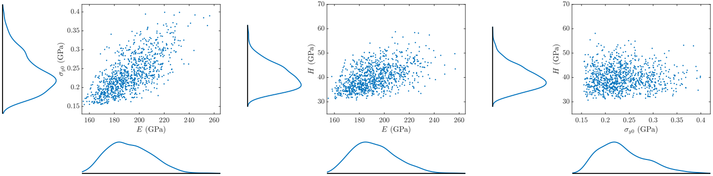


Identified without correlation

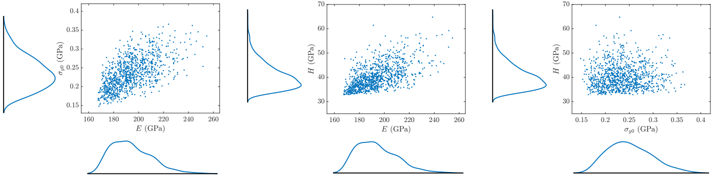
Identification results: Liner elastic-linear hardening



Identification results: Liner elastic-linear hardening

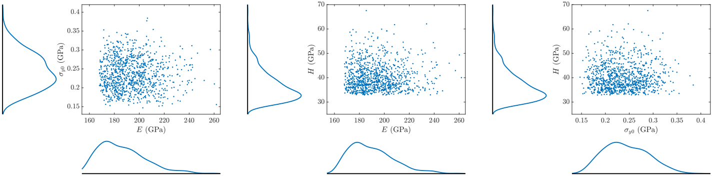


True



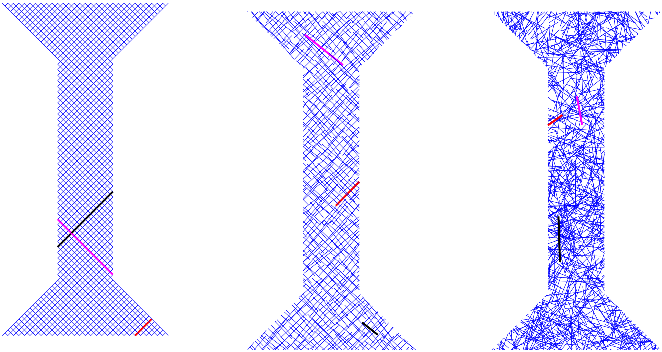
Identified

Identification results: Liner elastic-linear hardening



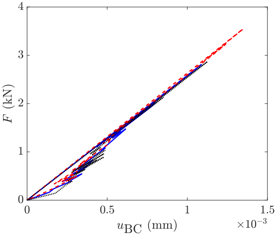
Identified without correlation

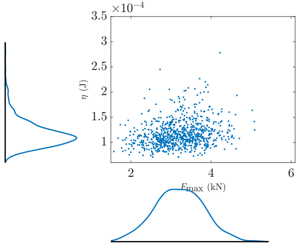
Influence of geometrical randomness

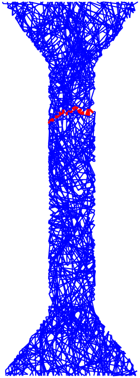


A: Regular B: Partially random C: Random

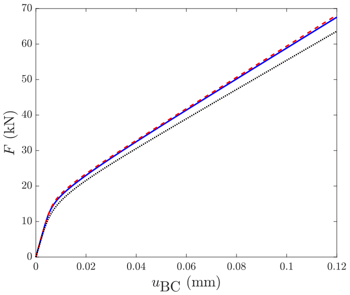
Influence of geometrical randomness: Brittle damage

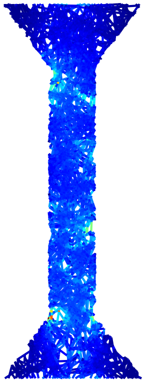






Influence of geometrical randomness: Linear elastic-linear hardenig





Equivalent parameters scatter plots are given.

Influence of geometrical randomness

- Including correlation has no significant influence in damage case.
- The effect of correlation becomes significant for the elastoplastic case.
- Increasing geometry randomness decreases the influence of correlation.
- Correlation is more important for long fibres and large fibre densities.

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Summary and conclusion

- Probability is the natural way of modelling lack of our knowledge (what Laplace calls it our ignorance).
- From Bayesian perspective (inverse probability) the parameters are treated as random variables.
- In addition to number of measurements the influence of the prior is dependent to the model (e.g. for viscoelastcity the prior has more significant effect than elastoplasticity).
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- Bayesian updating-Least squares framework proposed for material distribution identification.
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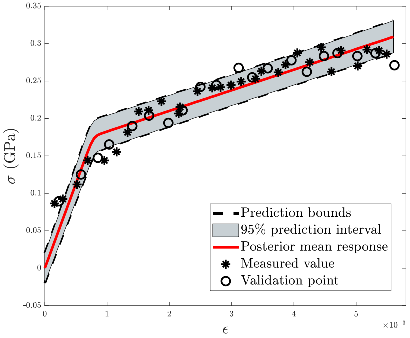
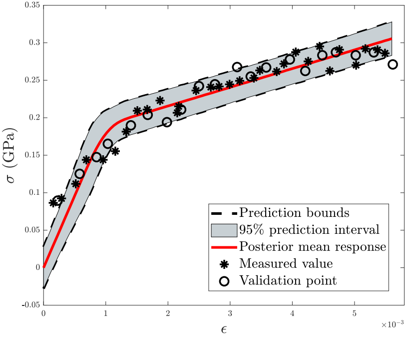
The End

Model uncertainty and input error: Linear elastic-linear hardening, interpolation

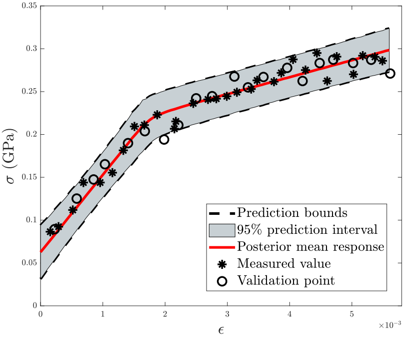


Error in the stress measurements only

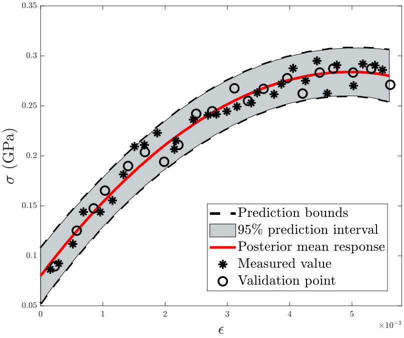


Error in both the stress and the strain measurements

Model uncertainty and input error: Linear elastic-linear hardening, interpolation

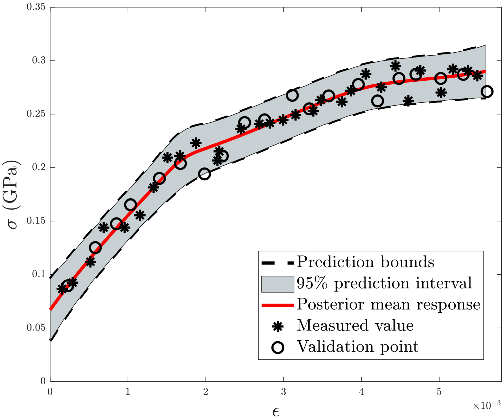


Normal distribution with constant parameters



Normal distribution with an input-dependent mean

Model uncertainty and input error: Linear elastic-linear hardening, interpolation



Gaussian process