Rank One Preservers Between Spaces of Boolean Matrices

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ABSTRACT

In this paper, we characterize (i) linear transformations from one space of Boolean matrices to another that send pairs of distinct rank one elements to pairs of distinct rank one elements and (ii) surjective mappings from one space of Boolean matrices to another that send rank one matrices to rank one matrices and preserve order relation in both directions. Both results are proved in a more general setting of tensor products of two Boolean vector spaces of arbitrary dimension.

1. Introduction

Let $\mathscr{B}$ denote the Boolean algebra with two elements 0 and 1 with addition and multiplication defined as if 0 and 1 were real, except that $1 + 1 = 1$. A matrix with entries from $\mathscr{B}$ is called a Boolean matrix. Let $M_{m,n}(\mathscr{B})$ be the space of all $m \times n$ Boolean matrices. If A is an $m \times n$ non-zero Boolean matrix, its Boolean rank, $b(A)$, is the least integer k for which there exist $m \times k$ and $k \times n$ Boolean matrices B and C with $A = BC$. The Boolean rank of the zero matrix is 0. It is known that $b(A)$ is the least k such that A is the sum of k Boolean matrices of rank one (see [3]). An operator T from a space of Boolean matrices to another is called linear if T preserves sums and sends the zero matrix to the zero matrix.

In [1], Beasley and Pullman proved the following result.

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If T is a linear operator on $M_{m,n}(\mathscr{B})$, and $\min(m, n) \geqslant 2$, then the following statements are equivalent.

(i) T preserves Boolean ranks 1 and 2.
(ii) T is invertible and preserves Boolean rank 1.
(iii) There exist permutation matrices P and Q such that $T(A) = PAQ$ for all $A \in M_{m,n}(\mathscr{B})$ or $m = n$ and $T(A) = PA^tQ$ for all $A \in M_{m,n}(\mathscr{B})$.

In [4], Pullman gave a graph-theoretic interpretation of the above result.

A subset V of $M_{m,1}(\mathscr{B})$ is called a Boolean vector space if V contains 0 and is closed under addition. In this paper, we first introduce the concept of tensor products of two Boolean vector spaces and study some of their basic properties. We next characterize (i) linear transformations from one tensor product of two Boolean vector spaces to another that send pairs of distinct rank one elements to pairs of distinct rank one elements and (ii) surjective mappings from one tensor product of two Boolean vector spaces to another that send rank one elements to rank one elements and preserve order relation in both directions. We obtain from the above characterization theorems the corresponding results concerning rank one preservers between spaces of Boolean matrices as a special case.

2. Tensor products of Boolean vector spaces

Let X be a non-empty set. Let $\mathscr{B}_X$ denote the set of all functions f from X to $\mathscr{B}$ such that $\{x \in X : f(x) \neq 0\}$, the support of f, is a finite set. Let $|f|$ denote the cardinality of the support of f. For any $f, g \in \mathscr{B}_X$, let $f + g$ be the function from X to $\mathscr{B}$ such that $(f + g)(x) = f(x) + g(x)$ for any $x \in X$. Clearly $f + g \in \mathscr{B}_X$. For our purpose, we define a Boolean vector space to be any subset of $\mathscr{B}_X$ containing the zero function which is closed under addition.

If f and g are in $\mathscr{B}_X$, we write $f \geqslant g$ if $f(x) + g(x) = f(x)$ for any $x \in X$. Clearly $\mathscr{B}_X$ is a partially ordered set under this order relation. We write $f > g$ when $f \geqslant g$ and $f \neq g$.

Let U and V be Boolean vector spaces. If $U \subseteq V$, then U is called a subspace of V. Let S be a non-empty subset of U. Let $\langle S \rangle$ denote the intersection of all subspaces of U that contain S. Then $\langle S \rangle$ is a subspace of U called the subspace spanned by S. Note that $f \in \langle S \rangle$ if and only if f is a linear combination of a finite number of elements in S, i.e., $f = \sum_{i=1}^{k} \lambda_i s_i$ for some $s_1, \ldots, s_k$ in S and some $\lambda_i \in \mathscr{B}$, $i = 1, \ldots, k$. The set S is called independent if every element f in S is not the sum of any finite number of elements in $S \backslash \{f\}$. We regard the empty sum as the zero vector. A subset E of U is called a basis of U if E is independent and $\langle E \rangle = U$. We regard the empty set as the basis of the zero Boolean vector space.

The following result is known for the case where U is finite dimensional (see [2]).

Proposition 2.1. *Every Boolean vector space U has a unique basis.*

Proof. We may assume that $U \neq \{0\}$. Let $K = \{|f| : f \in U \backslash \{0\}\}$. We can write K as $\{k_i : i \in I\}$ where $I = \{1, 2, \ldots, n\}$ for some integer n or I is the set of all positive integers and $k_i < k_j$ if $i < j$. Let $J_i = \{f \in U : |f| = k_i\}$, $i \in I$. Let $H_1 = J_1$. If $j + 1 \in I$, we define H_{j+1} be the set of all elements f in J_{j+1} such that $\left(\bigcup_{i=1}^{j} H_i\right) \cup \{f\}$ is independent. Let $H = \bigcup_{i \in I} H_i$. It is clear that H forms a basis of U.

Suppose that M is a basis of U. We shall show that $M \supseteq H$. Suppose the contrary. Then there exists $h \in H$ such that $h \notin M$. Since M spans U, it follows that $h = g_1 + \cdots + g_m$ for some $g_1, \ldots, g_m$ in M. Since $h \notin M$, we have $h > g_i$ for $i = 1, \ldots, m$. Since H spans U and $h > g_i$ for $i = 1, \ldots, m$, it follows that each g_i is the sum of a finite number of elements in $H \backslash \{h\}$. Hence h is the sum of a finite number of elements in $H \backslash \{h\}$, a contradiction to the fact that H is independent. This shows that $M \supseteq H$. Since every element of $U \backslash H$ is a linear combination of some elements of H, it follows that $M = H$. $\square$

The cardinality of the basis of a Boolean vector space is called its dimension. For convenience, we call each element of the basis of a Boolean vector space a cell.

A non-empty subset H of a Boolean vector space U is called non-dominating if for any non-empty finite subset S of H and $u \in H \backslash S$, we have $\sum_{v \in S} v \ngeq u$.

Lemma 2.2. *Let $U \neq \{0\}$ be a Boolean subspace of $\mathscr{B}_Y$. Then the basis $\{f_i : i \in \Delta\}$ of U is non-dominating if and only if there exists an injective mapping $\sigma : \Delta \to Y$ such that for every $i \in \Delta$, $f_i(\sigma(i)) = 1$ and $f_j(\sigma(i)) = 0$ for all $j \neq i$.*

Proof. The sufficiency part is clear. We prove the necessity. Let $i \in \Delta$ and $Y_i = \{y \in Y : f_i(y) = 1\}$. For each $t_i \in Y_i$, let Z_{t_i} be the subset of all $f_j, j \neq i$, such that $f_j(t_i) = 1$. Suppose that $Z_{t_i} \neq \phi$ for all $t_i \in Y_i$. Let $h_{t_i} \in Z_{t_i}$. Then $\sum_{t_i \in Y_i} h_{t_i} \geqslant f_i$, a contradiction since $\{f_i : i \in \Delta\}$ is a non-dominating basis. Hence $Z_{s_i} = \phi$ for some $s_i \in Y_i$. This shows that $f_i(s_i) = 1, f_j(s_i) = 0$ for all $j \neq i$. Clearly, $s_i \neq s_j$ for all $i \neq j$. Hence the mapping $\sigma : \Delta \to Y$ defined by $\sigma(i) = s_i$ is injective. This proves the necessity. □

Let U be a subspace of $\mathscr{B}_Y$. It is possible that $\dim U > \dim \mathscr{B}_Y$ (see [1]). For example, if $\{f_1, \ldots, f_n\}$ is the basis of $\mathscr{B}_Y$ and $n > 2$, then the subspace $\langle f_1, f_1 + f_2, f_2 + f_3, f_3, \ldots, f_n\rangle$ is of dimension $n + 1$. However, the following is true:

Proposition 2.3. *If U is a subspace of $\mathscr{B}_Y$ with a non-dominating basis, then $\dim U \leqslant \dim \mathscr{B}_Y$.*

Proof. This follows from Lemma 2.2. □

Let U and V be Boolean vector spaces. Then a mapping $T : U \to V$ which preserves sums and 0 is said to be a (Boolean) linear transformation. A linear transformation T is called singular if $T(u) = 0$ for some non-zero vector u. We say that U is isomorphic to V if there exists a bijective linear transformation from U to V.

Lemma 2.4. *Let U be a Boolean vector space with a non-dominating basis. Then for any non-zero vector u in U, there exists a unique set of cells $\{c_1, \ldots, c_k\}$ of U such that $u = c_1 + \cdots + c_k$.*

Proof. Suppose that $u = \sum_{i=1}^{k} c_i = \sum_{i=1}^{m} e_i$ where both $c_1, \ldots, c_k$ and $e_1, \ldots, e_m$ are distinct cells. Since $\sum_{i=1}^{m} e_i \geqslant c_j$ for each j, it follows that $c_j = e_{\sigma(j)}$ for some $\sigma(j) \leqslant m$. Hence $m \geqslant k$. Since $\sum_{j=1}^{k} c_j \geqslant e_i$ for any i, we see that $e_i = c_{\tau(i)}$ for some $\tau(i) \leqslant k$. Hence $k \geqslant m$. Therefore $k = m$ and the proof is complete. □

Proposition 2.5. *Let U be a Boolean vector space with a non-dominating basis $\{e_i : i \in I\}$. Then U is isomorphic to $\mathscr{B}_I$.*

Proof. For each non-empty finite subset J of I, let $A_J = \sum_{i \in J} e_i$ and let $f_J \in \mathscr{B}_I$ be such that $f_J(i) = 1$ if $i \in J$ and $f_J(i) = 0$ if $i \notin J$. By Lemma 2.4, we see that the mapping sending zero to zero and A_J to f_J is a well-defined bijective linear transformation from U to $\mathscr{B}_I$. □

Proposition 2.6. *Let U and V be Boolean vector spaces and $T : U \to V$ be a linear transformation. Then the following two conditions are equivalent:*

(i) *T is injective.*
(ii) *For all $u, v \in U$, $T(u) \geqslant T(v) \Rightarrow u \geqslant v$.*

If U has a non-dominating basis $\{e_i : i \in I\}$, then condition (ii) is equivalent to the following condition
(iii) *$\{T(e_i) : i \in I\}$ is a non-dominating basis for $\mathrm{Im}(T)$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $T(u) \geqslant T(v)$. Then $T(u) + T(v) = T(u + v) = T(u)$ and hence $u + v = u$. This shows that $u \geqslant v$.

(ii) $\Rightarrow$ (i): Suppose that $T(u) = T(v)$. Then the result follows from the hypothesis since $T(u) \geqslant T(v)$ and $T(v) \geqslant T(u)$.

Now we assume that U has a non-dominating basis $\{e_i : i \in I\}$.

(ii) $\Rightarrow$ (iii): Since for any non-empty finite subset H of I and any $j \notin H$, we have $\sum_{i\in H} e_i \ngeqslant e_j$, it follows from (ii) that

$$T\left(\sum_{i\in H} e_i\right) = \sum_{i\in H} T(e_i) \ngeqslant T(e_j).$$

This shows that $\{T(e_i) : i \in I\}$ is a non-dominating basis for $\mathrm{Im}(T)$.

(iii) $\Rightarrow$ (i): Suppose that $T(u) = T(v)$, but $u \neq v$. We may assume that $u \ngeqslant v$. Suppose that $u = 0$. Then $T(v) = 0$. Since $T(e_i) \neq 0$ for any $i \in I$, it follows that $v = 0$, a contradiction. Hence $u \neq 0$ and we have $u = \sum_{i\in H} e_i$ for some non-empty finite subset H of I. Clearly there exists $j \in I\backslash H$ such that $e_j \leqslant v$. Since T is linear, we have $T(v) \geqslant T(e_j)$. Since

$$\sum_{i\in H} T(e_i) \ngeqslant T(e_j),$$

it follows that

$$T(u) = \sum_{i\in H} T(e_i) \ngeqslant T(v),$$

a contradiction. This proves that T is injective. □

For any $f \in \mathscr{B}_X$ and $g \in \mathscr{B}_Y$, let $f \otimes g$ denote the function from $X \times Y$ to $\mathscr{B}$ such that $(f \otimes g)(x, y) = f(x)g(y)$ for any $x \in X$ and $y \in Y$. The map $f \otimes g$ is called a decomposable element. Clearly $f \otimes g \in \mathscr{B}_{X\times Y}$ and $f \otimes g = 0$ if and only if $f = 0$ or $g = 0$. For any $h \in \mathscr{B}_X$ and $k \in \mathscr{B}_Y$, we have

$$(f + h) \otimes g = f \otimes g + h \otimes g,$$
$$f \otimes (g + k) = f \otimes g + f \otimes k.$$

Let U and V be subspaces of $\mathscr{B}_X$ and $\mathscr{B}_Y$ respectively. Let $U \otimes V$ denote the subspace of $\mathscr{B}_{X\times Y}$ spanned by all the decomposable elements $f \otimes g$ with $f \in U$ and $g \in V$. We call $U \otimes V$ the tensor product of U and V. Clearly $\mathscr{B}_X \otimes \mathscr{B}_Y = \mathscr{B}_{X\times Y}$. If $X = \{1, 2, \ldots, m\}$ and $Y = \{1, 2, \ldots, n\}$, then $\mathscr{B}_{X\times Y}$ can be identified naturally with the space of all $m \times n$ Boolean matrices. Let A be a non-zero element in $U \otimes V$. Then A is said to have rank s if A is the sum of s, but not less than s, non-zero decomposable elements in $U \otimes V$. The rank of the zero element in $U \otimes V$ is 0.

For each non-zero vector u in U and each non-zero subspace K of V, $u \otimes K := \{u \otimes v : v \in K\}$ is called a left factor subspace of $U \otimes V$. Similarly, for each non-zero vector v in V and each non-zero subspace H of U, $H \otimes v := \{u \otimes v : u \in H\}$ is called a right factor subspace of $U \otimes V$.

Let T be a linear transformation from $U \otimes V$ to $W \otimes Z$ where W and Z are Boolean vector spaces. Then T is said to be induced by two linear transformations if one of the following conditions holds:

(i) there exist linear transformations $\theta : U \to W$ and $\varphi : V \to Z$ such that $T(u \otimes v) = \theta(u) \otimes \varphi(v)$ for any $u \in U$ and $v \in V$;
(ii) there exist linear transformations $\theta : U \to Z$ and $\varphi : V \to W$ such that $T(u \otimes v) = \varphi(v) \otimes \theta(u)$ for any $u \in U$ and $v \in V$.

For the first case, we write $T = \theta \otimes \varphi$, while for the second case, we write $T = \theta \breve{\otimes} \varphi$.

Let $U = \mathscr{B}_X, V = \mathscr{B}_Y, W = \mathscr{B}_I, Z = \mathscr{B}_J$, where $X = \{1, 2, \ldots, m\}, Y = \{1, 2, \ldots, n\}, I = \{1, 2, \ldots, p\}$, $J = \{1, 2, \ldots, q\}$. Then $\mathscr{B}_X \otimes \mathscr{B}_Y$ and $\mathscr{B}_I \otimes \mathscr{B}_J$ can be identified naturally with $M_{m,n}(\mathscr{B})$ and $M_{p,q}(\mathscr{B})$ respectively. If $T : U \otimes V \to W \otimes Z$ is a linear transformation satisfying condition (i), then $T(A) = PAQ$ for some $p \times m$ Boolean matrix P and some $n \times q$ Boolean matrix Q. If $T : U \otimes V \to W \otimes Z$ is a linear transformation satisfying condition (ii), then $T(A) = PA^tQ$ for some $p \times n$ Boolean matrix P and some $m \times q$ Boolean matrix Q.

For the following three results, we assume that U and V are subspaces of $\mathscr{B}_X$ and $\mathscr{B}_Y$ respectively.

Lemma 2.7. *If $\sum_{i=1}^m f_i \otimes g_i \geqslant \sum_{j=1}^n u_j \otimes v_j$ where $f_i \otimes g_i$, $u_j \otimes v_j$ are non-zero decomposable elements in $U \otimes V$, then $\sum_{i=1}^m f_i \geqslant \sum_{j=1}^n u_j$ and $\sum_{i=1}^m g_i \geqslant \sum_{j=1}^n v_j$.*

Proof. Suppose that $\sum_{i=1}^m f_i \ngeqslant \sum_{j=1}^n u_j$. Then there exists $x \in X$ such that $(\sum_{i=1}^m f_i)(x) = 0$ and $\left(\sum_{j=1}^n u_j\right)(x) = 1$. Hence there exists $1 \leqslant s \leqslant n$ such that $u_s(x) = 1$. Choose $y \in Y$ such that $v_s(y) = 1$. Clearly,

$$\left(\sum_{i=1}^m f_i \otimes g_i\right)(x, y) = 0 \quad \text{and} \quad \left(\sum_{j=1}^n u_j \otimes v_j\right)(x, y) = 1,$$

a contradiction. This shows that $\sum_{i=1}^m f_i \geqslant \sum_{j=1}^n u_j$. Similarly, we have $\sum_{i=1}^m g_i \geqslant \sum_{j=1}^n v_j$. $\square$

Corollary 2.8. *If $\sum_{i=1}^m f_i \otimes g_i = \sum_{j=1}^n u_j \otimes v_j$ where $f_i \otimes g_j$, $u_j \otimes v_j$ are non-zero decomposable elements in $U \otimes V$, then*

$$\sum_{i=1}^m f_i = \sum_{j=1}^n u_j \quad \text{and} \quad \sum_{i=1}^m g_i = \sum_{j=1}^n v_j.$$

From Corollary 2.8, we see that every non-zero decomposable element A of $U \otimes V$ has a unique representation $f \otimes g$ where $f \in U$ and $g \in V$. We call f the left factor of A and g the right factor of A.

Theorem 2.9. *Let C and D be bases of Boolean vector spaces U and V respectively. Let $E = \{u \otimes v : u \in C, v \in D\}$. Then*

(i) *E is the basis of $U \otimes V$;*
(ii) *C and D are non-dominating if and only if E is non-dominating.*

Proof. (i) It is clear that $U \otimes V$ is spanned by E. Suppose that E is not independent. Then there exists $u \otimes v \in E$ such that $u \otimes v$ is the sum of finite number of elements from $E \backslash \{u \otimes v\}$. We see that

$$u \otimes v = u_1 \otimes v_1 + \cdots + u_k \otimes v_k$$

for some distinct elements $u_1, \ldots, u_k \in C$ and some non-zero vectors $v_1, \ldots, v_k \in V$. By Corollary 2.8, we have $u = \sum_{i=1}^k u_i$. Since $\{u, u_1, \ldots, u_k\} \subseteq C$ and C is independent, it follows that $u = u_i$ for some i. Without loss of generality, we may assume that $u = u_1$. We have the following two cases:

Case 1. $k = 1$. We have $v = v_1$, a contradiction to $u_1 \otimes v_1 \in E \backslash \{u \otimes v\}$.

Case 2. $k \geqslant 2$. Since

$$u = u_1 + \cdots + u_k \quad \text{and} \quad u \neq u_2 + \cdots + u_k,$$

there exists $x \in X$ such that $u(x) = 1$ and $u_i(x) = 0$ for each $i \geqslant 2$. Note that $v \neq v_1$ and $v \geqslant v_1$. Hence there exists $y \in Y$ such that $v(y) = 1$ and $v_1(y) = 0$. This implies that $(u \otimes v)(x, y) = 1$. However,

$$(u_1 \otimes v_1)(x, y) = 0$$

since $v_1(y) = 0$, and

$$(u_i \otimes v_i)(x, y) = 0$$

for $i \geqslant 2$ since $u_i(x) = 0$ for $i \geqslant 2$. Hence,

$$u \otimes v \neq \sum_{i=1}^k u_i \otimes v_i,$$

a contradiction. This proves that E is independent.

(ii) ($\Rightarrow$) Suppose that E is dominating. Then there exist $u \otimes v \in E$ and $A_1, \ldots, A_m \in E \backslash \{u \otimes v\}$ such that

$$A_1 + \cdots + A_m \geqslant u \otimes v.$$

Let $\{u_1, \ldots, u_k\}$ be the subset of C consisting of the left factors of $A_1, \ldots, A_m$. Then

$$A_1 + \cdots + A_m = u_1 \otimes v_1 + \cdots + u_k \otimes v_k,$$

where for each $i = 1, \ldots, k$, v_i is the sum of the right factors of those A_j with u_i as their left factors. By Lemma 2.7, $\sum_{i=1}^{k} u_i \geqslant u$. Since C is non-dominating, it follows that $u = u_i$ for some i. Without loss of generality, we may assume that $u = u_1$. Since $A_j \neq u \otimes v$ for any j, it follows that $v_1 = w_1 + \cdots + w_s$ for some $w_i \in D \backslash \{v\}$. Since D is non-dominating, we have $v_1 \ngeqslant v$. If $k = 1$, then $u_1 \otimes v_1 \geqslant u \otimes v$ and hence by Lemma 2.7, $v_1 \geqslant v$, a contradiction. Now, suppose that $k \geqslant 2$. Since $v_1 \ngeqslant v$, there exists $y \in Y$ such that $v_1(y) = 0$ and $v(y) = 1$. Since C is non-dominating, it follows that $u_2 + \cdots + u_k \ngeqslant u$ and hence $u_i(x) = 0$ for $i \geqslant 2$ and $u(x) = 1$ for some $x \in X$. This shows that

$$\left(\sum_{i=1}^{k} u_i \otimes v_i \right) (x, y) = 0.$$

However, $(u \otimes v)(x, y) = 1$, a contradiction. This proves that E is non-dominating.

($\Leftarrow$) Suppose that C is dominating. Then $\sum_{i=1}^{k} u_i \geqslant u$ for some $u \in C$ and some $u_1, \ldots, u_k \in C \backslash \{u\}$. For any $v \in D$, we have

$$u_1 \otimes v + \cdots + u_k \otimes v \geqslant u \otimes v,$$

a contradiction since E is non-dominating. Hence C must be non-dominating. Similarly, we can show that D is non-dominating. $\square$

Proposition 2.10. *Suppose that $T : U \otimes V \to W \otimes Z$ is a linear transformation induced by two linear transformations θ and φ where $U \neq \{0\}$ and $V \neq \{0\}$. If T is injective, then both θ and φ are injective. If U or V has a non-dominating basis, then the converse is also true.*

Proof. Suppose that T is injective. Consider the case where $T = \theta \otimes \varphi$. Suppose that $\theta(f) = \theta(g)$ for some $f, g \in U$. Let $h \in V \backslash \{0\}$. Then $T(f \otimes h) = T(g \otimes h)$. Hence $f \otimes h = g \otimes h$. This shows that $f = g$. Hence θ is injective. Similarly, we can show that φ is injective. For the case where $T = \theta \breve{\otimes} \varphi$, the result can be proved similarly.

Suppose that θ and φ are injective. We have the following two cases:

Case 1. $T = \theta \otimes \varphi$. Suppose that U has a non-dominating basis C. Since θ is injective, it follows from Proposition 2.6 that $\theta(C)$ is a non-dominating basis of $\operatorname{Im} \theta$. Suppose that $T(A) = T(B)$ for some vectors A, B in $U \otimes V$. Either (i) $A = B = 0$ or (ii) not both A and B are zero. Consider case (ii). Without loss of generality, we may assume that $A \neq 0$. Note that

$$A = u_1 \otimes v_1 + \cdots + u_m \otimes v_m$$

for some distinct cells $u_1, \ldots, u_m \in C$ and some non-zero vectors $v_1, \ldots, v_m \in V$. Let $\theta(u_i) = w_i$, $\varphi(v_i) = z_i$, $i = 1, \ldots, m$. Then

$$T(A) = w_1 \otimes z_1 + \cdots + w_m \otimes z_m.$$

Since θ and φ are injective, we have $w_1 \neq 0$, $z_1 \neq 0$ and hence $w_1 \otimes z_1 \neq 0$. This shows that $T(A) \neq 0$ and hence $B \neq 0$. Thus

$$B = f_1 \otimes g_1 + \cdots + f_n \otimes g_n$$

for some distinct cells $f_1, \ldots, f_n \in C$ and some non-zero vectors $g_1, \ldots, g_n \in V$. Let $\theta(f_i) = h_i$, $\varphi(g_i) = k_i$, $i = 1, \ldots, n$. Then

$$\begin{aligned} T(A) &= w_1 \otimes z_1 + \cdots + w_m \otimes z_m \\ &= T(B) \\ &= h_1 \otimes k_1 + \cdots + h_n \otimes k_n. \end{aligned}$$

In view of Corollary 2.8,

$$w_1 + \cdots + w_m = h_1 + \cdots + h_n.$$

Since $\theta(C)$ is a non-dominating basis of $\operatorname{Im}\theta$, it follows from Lemma 2.4 that

$$m = n \quad \text{and} \quad \{w_1, \ldots, w_m\} = \{h_1, \ldots, h_m\}.$$

Without loss of generality, we may assume that $h_i = w_i$, $i = 1, \ldots, m$. Since θ is injective, it follows that $u_i = f_i$, $i = 1, \ldots, m$. Suppose that $m = 1$. Then $w_1 \otimes z_1 = h_1 \otimes k_1$ and hence by Corollary 2.8, $z_1 = k_1$. Since φ is injective, it follows that $v_1 = g_1$ and hence $A = B$. Now, suppose that $m > 1$. Since $\theta(C)$ is a non-dominating basis of $\operatorname{Im}\theta$, we have $\sum_{i=2}^{m} w_i \not\geqslant w_1$. Hence there exists an element p such that $w_1(p) = 1$ and $w_i(p) = 0$ for $i \geqslant 2$. For any element q in the domain of z_1, we have

$$\begin{aligned} &\left(\sum_{i=1}^{m} w_i \otimes z_i\right)(p, q) = w_1(p) z_1(q) = z_1(q) \\ &= \left(\sum_{i=1}^{m} w_i \otimes k_i\right)(p, q) = w_1(p) k_1(q) = k_1(q). \end{aligned}$$

Hence $z_1 = k_1$. Similarly we can show that $z_i = k_i$, $i \geqslant 2$. Since φ is injective, it follows that $v_i = g_i$, $i = 1, \ldots, m$. Hence $A = B$. This shows that T is injective. Similarly, if V has a non-dominating basis, we can show that T is injective.

Case 2. $T = \theta \breve{\otimes} \varphi$. The proof is similar to that of Case 1. $\square$

3. Rank one preservers between tensor products of Boolean vector spaces

Throughout this section, U, V, W and Z are Boolean vector spaces each of dimension at least two. We denote the set of all rank one elements in $U \otimes V$ by $\mathscr{D}(U, V)$.

Two elements u_1, u_2 of a Boolean vector space are said to be comparable if $u_1 > u_2$ or $u_2 > u_1$.

The following result was proved in [1, Lemma 2.6.2] for the space $M_{m,n}(\mathscr{B})$. It can be proved by using the same argument as in [1, Lemma 2.6.2].

Lemma 3.1. *Let A and B be two rank one elements in $U \otimes V$ such that $A + B$ is of rank one. If A, B are incomparable, then A and B have a common factor.*

Theorem 3.2. *Let U and V be two Boolean vector spaces both without comparable cells. Let $T : U \otimes V \to W \otimes Z$ be a linear transformation. Then T sends distinct rank 1 elements to distinct rank 1 elements if and only if one of the following is true:*

(i) *there exist a fixed non-zero element $w \in W$ and a linear transformation φ from $U \otimes V$ to Z such that*

$$T(A) = w \otimes \varphi(A)$$

for any A in $U \otimes V$ where $\varphi|_{\mathscr{D}(U,V)}$ is injective,

(ii) *there exist a fixed non-zero element $z \in Z$ and a linear transformation θ from $U \otimes V$ to W such that*

$$T(A) = \theta(A) \otimes z$$

for any A in $U \otimes V$ where $\theta|_{\mathscr{D}(U,V)}$ is injective,

(iii) *T is induced by two injective linear transformations.*

Proof. The sufficiency part is clear. We now prove the necessity. We first show that for any non-zero vector $u \in U$, $T(u \otimes V)$ is a factor subspace of $W \otimes Z$. Let v_1 and v_2 be two distinct cells in V. Then

$$T(u \otimes v_1) = w_1 \otimes z_1,$$
$$T(u \otimes v_2) = w_2 \otimes z_2$$

for some non-zero vectors w_1, w_2 in W and non-zero vectors z_1, z_2 in Z. If $w_1 \otimes z_1 \geqslant w_2 \otimes z_2$, then

$$T(u \otimes (v_1 + v_2)) = T(u \otimes v_1)$$

and hence by hypothesis, $u \otimes (v_1 + v_2) = u \otimes v_1$. This implies that $v_1 + v_2 = v_1$ and hence $v_1 > v_2$, a contradiction. Hence $w_1 \otimes z_1 \not\geqslant w_2 \otimes z_2$. Similarly we can show that $w_2 \otimes z_2 \not\geqslant w_1 \otimes z_1$. Hence by Lemma 3.1, either $w_1 = w_2$ or $z_1 = z_2$ since $w_1 \otimes z_1 + w_2 \otimes z_2$ is of rank 1. Suppose that $w_1 = w_2$. Then $z_1 \neq z_2$. Now for any cell v in V such that $v \notin \{v_1, v_2\}$, we have $T(u \otimes v) = w \otimes z$ for some non-zero vector w in W and non-zero vector z in Z. By the previous argument, we see that $w \otimes z$ and $w_i \otimes z_i$ have a common factor for $i = 1, 2$. Hence $w = w_1 = w_2$ since $z_1 \neq z_2$. This shows that $T(u \otimes V) \subseteq w \otimes Z$. Similarly, if $z_1 = z_2$, we have $T(u \otimes V) \subseteq W \otimes z_1$.

Using the same argument as above, one can show that for any non-zero vector $v \in V$, $T(U \otimes v)$ is a factor subspace of $W \otimes Z$.

Claim. *For any two distinct non-zero vectors u_1, u_2 in U, $T(u_1 \otimes V)$, $T(u_2 \otimes V)$ are either left factor subspaces or right factor subspaces. Suppose the contrary. Then there exist distinct non-zero vectors x, y in U such that*

$$T(x \otimes V) = x' \otimes Z_1,$$
$$T(y \otimes V) = W_1 \otimes y'$$

for some non-zero $x' \in W$, $y' \in Z$, some subspace Z_1 of Z, and some subspace W_1 of W. Choose a non-zero vector $g \in Z_1$ such that $g \neq y'$. Let $c \in V$ such that $T(x \otimes c) = x' \otimes g$. Since $T(x \otimes c)$ and $T(y \otimes c)$ have a common factor, it follows that $T(y \otimes c) = x' \otimes y'$. Hence $x' \in W_1$. Similarly, we can show that $y' \in Z_1$. Hence $T(x \otimes V) \cap T(y \otimes V)$ contains $x' \otimes y'$, a contradiction to the hypothesis. This proves the Claim.

We have the following two cases:

Case (i). For any non-zero vector e in U, $T(e \otimes V)$ is a left factor subspace of $W \otimes Z$.

We have $T(e \otimes V) = e' \otimes Z_e$ for some non-zero vector $e' \in W$ and some subspace Z_e of Z.

Suppose there exists a non-zero vector $f \in V$ such that

$$T(U \otimes f) = f' \otimes K_f$$

for some non-zero vector $f' \in W$ and some subspace K_f in Z. Since

$$e \otimes f \in (e \otimes V) \cap (U \otimes f),$$

it follows that $f' = e'$. In this case we have $\mathrm{Im}(T) \subseteq f' \otimes Z$. Hence there exist a linear transformation φ from $U \otimes V$ to Z such that

$$T(A) = f' \otimes \varphi(A)$$

for any A in $U \otimes V$ where $\varphi|_{\mathscr{D}(U,V)}$ is injective.

Suppose now that for each non-zero vector $f \in V$,

$$T(U \otimes f) = W_f \otimes \bar{f}$$

for some subspace W_f of W and some non-zero vector $\bar{f} \in Z$. This implies that

$$T(e \otimes f) = e' \otimes \bar{f}$$

for any non-zero vector e in U and any non-zero vector $f \in V$. Let $\theta : U \to W$ be the mapping such that $\theta(e) = e'$ and $\varphi : V \to Z$ be the mapping such that $\varphi(f) = \bar{f}$. Since T is a linear transformation, it follows that both θ and φ are linear transformations. Hence $T = \theta \otimes \varphi$. Clearly both θ and φ are injective.

Case (ii). For any non-zero vector e in U, $T(e \otimes V)$ is a right factor subspace. By using a similar argument as in Case (i), we can show that either there exist a fixed non-zero element $z \in Z$ and a linear transformation θ from $U \otimes V$ to W such that $T(A) = \theta(A) \otimes z$ for any A in $U \otimes V$ where $\theta|_{\mathscr{D}(U,V)}$ is injective, or T is induced by some injective linear transformations $\eta : U \to Z$ and $\xi : V \to W$. $\square$

The following example shows that Theorem 3.2 is not true if one of the Boolean vector spaces U and V has comparable cells.

Example 3.3. Let U be a Boolean vector space consisting of three elements 0, e_1, e_2 where $e_1 < e_2$. Let V and W be Boolean vector spaces with non-dominating bases $\{f_1, f_2\}$ and $\{g_1, g_2, g_3\}$ respectively. Then there exists a linear transformation T from $U \otimes V$ to $W \otimes W$ such that

$$T(e_1 \otimes f_1) = g_1 \otimes g_1, \quad T(e_1 \otimes f_2) = g_1 \otimes g_2,$$
$$T(e_2 \otimes f_1) = (g_1 + g_3) \otimes (g_1 + g_3),$$
$$T(e_2 \otimes f_2) = (g_1 + g_3) \otimes (g_2 + g_3).$$

Note that $U \otimes V$ has six rank one elements and

$$T(e_1 \otimes (f_1 + f_2)) = g_1 \otimes (g_1 + g_2),$$
$$T(e_2 \otimes (f_1 + f_2)) = (g_1 + g_3) \otimes (g_1 + g_2 + g_3).$$

Hence T sends distinct rank 1 elements to distinct rank 1 elements. However, $\mathrm{Im}(T)$ is not a factor subspace of $W \otimes W$ and also T is not induced by two injective linear transformations. We note that T sends rank 2 elements to rank 2 elements.

Remark 3.4. A linear transformation $U \otimes V$ to $W \otimes Z$ sending pairs of distinct rank one elements to pairs of distinct rank one elements is not necessarily injective. For example, the linear transformation $T : M_2(\mathscr{B})$ to $M_{2,4}(\mathscr{B})$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b & c & a+d \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

has this property.

Remark 3.5. Theorem 3.2 is analogous to the following result of Westwick [6]: If T is a linear transformation from one tensor product of two vector spaces over a field to another that sends non-zero decomposable elements to non-zero decomposable elements, then either the image of T consists of decomposable elements or T is induced by two injective linear transformations.

Lemma 3.6. *Let $P \in M_{m,n}(\mathscr{B})$. Then the linear transformation $\theta : M_{n,1}(\mathscr{B}) \to M_{m,1}(\mathscr{B})$ defined by $\theta u = Pu$, is injective if and only if P contains an $n \times n$ permutation submatrix.*

Proof. Let $\{e_i : i = 1, \ldots, n\}$ be the standard basis of $M_{n,1}(\mathscr{B})$. Using Proposition 2.6 and Lemma 2.2, we see that

θ is injective
$\Leftrightarrow$ $\{Pe_1, \ldots, Pe_n\}$ is a non-dominating basis of $\mathrm{Im}\,\theta$
$\Leftrightarrow$ There exists an injective mapping $\sigma : \{1, \ldots, n\} \to \{1, \ldots, m\}$ such that the $\sigma(i)$th coordinate of Pe_i is 1 and the $\sigma(i)$th coordinate of Pe_j is 0 for any $j \neq i$
$\Leftrightarrow$ P contains an $n \times n$ permutation submatrix. $\square$

The following result follows from Theorem 3.2 and Lemma 3.6.

Corollary 3.7. *Let $T : M_{m,n}(\mathscr{B}) \to M_{k,\ell}(\mathscr{B})$ be a linear transformation where $\min\{m, n, k, l\} \geqslant 2$. Then T sends distinct rank one matrices to distinct rank one matrices if and only if one of the following is true:*

(i) *there exist a fixed non-zero vector w in $M_{k,1}(\mathscr{B})$ and a linear transformation φ from $M_{m,n}(\mathscr{B})$ to $M_{1,l}(\mathscr{B})$ such that*
$$T(A) = w\varphi(A)$$
for any A in $M_{m,n}(\mathscr{B})$ where the restriction of φ to the set of all rank one matrices is injective,

(ii) *there exist a fixed non-zero element z in $M_{1,l}(\mathscr{B})$ and a linear transformation θ from $M_{m,n}(\mathscr{B})$ to $M_{k,1}(\mathscr{B})$ such that*
$$T(A) = \theta(A)z$$
for any A in $M_{m,n}(\mathscr{B})$ where the restriction of θ to the set of all rank one matrices is injective,

(iii) *$T(A) = PAQ$ for some $P \in M_{k,m}(\mathscr{B})$ and some $Q \in M_{n,\ell}(\mathscr{B})$ where P contains an $m \times m$ permutation submatrix and Q contains an $n \times n$ permutation submatrix,*

(iv) *$T(A) = PA^tQ$ for some $P \in M_{k,n}(\mathscr{B})$ and some $Q \in M_{m,\ell}(\mathscr{B})$ where P contains an $n \times n$ permutation submatrix and Q contains an $m \times m$ permutation submatrix.*

Example 3.8. Let T_1 and T_2 be any two linear rank one preservers on $M_2(\mathscr{B})$. Let $T : M_4(\mathscr{B}) \to M_4(\mathscr{B})$ be defined by
$$T\left(\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]\right) = \left[\begin{array}{c|c} T_1(A) & 0 \\ \hline 0 & T_2(\mathscr{B}) \end{array}\right],$$
$$T\left(\left[\begin{array}{c|c} A & C \\ \hline D & B \end{array}\right]\right) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$
if $C \neq 0$ or $D \neq 0$, where $A, B \in M_2(\mathscr{B})$. Then T is a linear rank one preserver which is not of the form (i) or (ii) mentioned in Corollary 3.7. Note that $T(E_{11})$ and $T(E_{13})$ do not have a common factor, T cannot be of the form (iii) or (iv) mentioned in Corollary 3.7. Here E_{ij} denotes the matrix with 1 in position i, j and 0 elsewhere.

The following result was proved in [1,5] for the space $M_{m,n}(\mathscr{B})$. Our proof here is very short.

Lemma 3.9. *Let A and B be distinct rank one elements in $U \otimes V$ where both U and V have no comparable cells. Then there exists a rank one element C in $U \otimes V$ such that $\{\text{rank}(A + C), \text{rank}(B + C)\} = \{1, 2\}$.*

Proof. Let $A = u \otimes v$ and $B = x \otimes y$. Since $A \neq B$, we may assume that $y \neq v$. Either $y \ngeqslant v$ or $v \ngeqslant y$. We consider only the first case as the second case can be proved similarly. Let w be a cell of U such that $x \geqslant w$ and let z be another cell of U. Since U has no comparable cells, it follows that $z \ngeqslant w$. Hence $z \ngeqslant x$. Let $C = z \otimes v$. Then $A + C$ is of rank one and by Lemma 3.1, $B + C$ is of rank 2. □

Remark 3.10. It can be shown that Lemma 3.9 holds true under the weaker hypothesis that either $U\backslash\{0\}$ or $V\backslash\{0\}$ has no least element. However we do not need it for the following corollary.

Corollary 3.11. *Let $T : U \otimes V \to W \otimes Z$ be a linear transformation where both U and V have no comparable cells. Then the following two conditions are equivalent:*

(i) *T sends rank k elements to rank k elements when $k = 1, 2$.*
(ii) *T is induced by two injective linear transformations.*

Proof. (i) $\Rightarrow$ (ii): Suppose that A and B are two distinct rank one elements in $U \otimes V$ such that $T(A) = T(B)$. By Lemma 3.9, there exists a rank one element C in $U \otimes V$ such that

$$\{\operatorname{rank}(A + C), \operatorname{rank}(B + C)\} = \{1, 2\}.$$

Hence

$$\{\operatorname{rank} T(A + C), \operatorname{rank} T(B + C)\} = \{1, 2\},$$

a contradiction since $T(A + C) = T(B + C)$. This proves that T sends distinct rank one elements to distinct rank one elements and hence the result follows from Theorem 3.2.

(ii) $\Rightarrow$ (i): Suppose that T is induced by two injective linear transformations θ and φ.We consider only the case $T = \theta \otimes \varphi$ as the proof for the other case is similar. Clearly T sends rank 1 elements to rank 1 elements. Suppose that A is of rank 2. Then $A = u_1 \otimes v_1 + u_2 \otimes v_2$ for some $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Hence $T(A) = B_1 + B_2$, where $B_i = \theta(u_i) \otimes \varphi(v_i)$, $i = 1, 2$. If B_1, B_2 have a common factor, say $\theta(u_1) = \theta(u_2)$, then $u_1 = u_2$ and hence A is of rank $\leqslant 1$, a contradiction. If $B_1 \geqslant B_2$, then by Lemma 2.7, $\theta(u_1) \geqslant \theta(u_2)$ and $\varphi(v_1) \geqslant \varphi(v_2)$. By Proposition 2.6, we have $u_1 \geqslant u_2$ and $v_1 \geqslant v_2$. This implies that $A = u_1 \otimes v_1$, a contradiction. Similarly it is not possible that $B_2 \geqslant B_1$. By Lemma 3.1, $T(A)$ is of rank 2. This completes the proof. $\square$

Remark 3.12. Example 3.3 shows that Corollary 3.11 is not true if one of the Boolean vector spaces U and V has comparable cells.

Theorem 3.13. *Let T be a linear transformation on $U \otimes U$ where U is finite dimensional and $U\backslash\{0\}$ has no least element. Then T sends maximal left factor subspaces to maximal factor subspaces if and only if $T = \theta \otimes \varphi$ or $T = \theta \breve{\otimes} \varphi$ for some non-singular linear transformation θ on U and some bijective linear transformation φ on U.*

Proof. The sufficiency part is clear. We now prove the necessity. Let E be the basis of U and n be its cardinality. Since E is a finite partially ordered set, it follows that E has a minimal element e_1. Similarly $E\backslash\{e_1\}$ has a minimal element e_2. Continue the process, we can choose a minimal element e_s from $E\backslash\{e_1, \ldots, e_{s-1}\}$ if $n \geqslant s > 2$. Hence $E = \{e_1, e_2, \ldots, e_n\}$ where e_s is a minimal element of $\{e_s, e_{s+1}, \ldots, e_n\}$, $s = 1, \ldots, n$.

Suppose that

$$T(u_1 \otimes U) = f \otimes U \quad \text{and} \quad T(u_2 \otimes U) = U \otimes g$$

for some distinct $u_1, u_2 \in U\backslash\{0\}$ and for some $f, g \in U\backslash\{0\}$. Since $T((u_1 + u_2) \otimes U)$ is a maximal factor subspace, it follows that

$$T((u_1 + u_2) \otimes U) = f' \otimes U$$

for some $f' \in U\backslash\{0\}$ or

$$T((u_1 + u_2) \otimes U) = U \otimes g'$$

for some $g' \in U\backslash\{0\}$. Consider the first case. There exists $v_k \in U$ such that

$$T((u_1 + u_2) \otimes v_k) = f' \otimes e_k, \quad k = 1, \ldots, n.$$

Since

$$T((u_1 + u_2) \otimes v_k) \geqslant T(u_2 \otimes v_k),$$

it follows that $e_k \geqslant g$ for any k. Since $U\backslash\{0\}$ has no least element, it follows that $g = 0$, a contradiction. Similarly, the second case leads to a contradiction. Hence $\{T(u \otimes U) : u \in U\}$ consists of maximal left factor subspaces or consists of maximal right factor subspaces. Consider the first case. We have

$$T(e_i \otimes U) = f_i \otimes U$$

for some $f_i \in U\backslash\{0\}$, $i = 1, \ldots, n$. For each $i = 1, \ldots, n$, there exists a bijective linear transformation φ_i on U such that

$$T(e_i \otimes e) = f_i \otimes \varphi_i(e)$$

for any cell e. Note that $E = \{\varphi_i(e_1), \ldots, \varphi_i(e_n)\}$.

Suppose that $f_1 = f_2 = \cdots = f_n$. Let $f := f_1$. Note that for any distinct i and j,

$$T((e_i + e_j) \otimes U) = f \otimes U$$

and hence for each $s = 1, \ldots, n$, there exists $c_s \in E$ such that

$$T((e_i + e_j) \otimes c_s) = f \otimes e_s.$$

Hence

$$f \otimes e_s = f \otimes \varphi_i(c_s) + f \otimes \varphi_j(c_s).$$

This implies that

$$e_s \geqslant \varphi_i(c_s) \quad \text{and} \quad e_s \geqslant \varphi_j(c_s).$$

Since e_1 is a minimal element of E, it follows that

$$e_1 = \varphi_i(c_1) = \varphi_j(c_1).$$

Suppose that

$$e_s = \varphi_i(c_s) = \varphi_j(c_s), \quad s = 1, \ldots, k-1$$

where k is a fixed positive integer such that $1 < k < n$. Then

$$\begin{aligned}\{e_k, \ldots, e_n\} &= \{\varphi_i(c_k), \ldots, \varphi_i(c_n)\} \\ &= \{\varphi_j(c_k), \ldots, \varphi_j(c_n)\}.\end{aligned}$$

Since e_k is a minimal element $\{e_k, e_{k+1}, \ldots, e_n\}$, it follows that

$$e_k = \varphi_i(c_k) = \varphi_j(c_k).$$

By induction, we see that

$$e_s = \varphi_i(c_s) = \varphi_j(c_s)$$

for any $s = 1, \ldots, n$. Hence $\varphi_i = \varphi_j$ for any i and j. Let θ be the linear transformation on U such that $\theta(u) = f$ for any non-zero vector $u \in U$. Clearly θ is non-singular and $T = \theta \otimes \varphi_1$.

Suppose now that $f_i \neq f_j$ for some distinct i and j. We have

$$T((e_i + e_j) \otimes U) = u \otimes U$$

for some non-zero vector u in U. For each $s = 1, \ldots, n$, there exists $w_s \in E$ such that

$$T((e_i + e_j) \otimes w_s) = u \otimes e_s.$$

Hence

$$u \otimes e_s = f_i \otimes \varphi_i(w_s) + f_j \otimes \varphi_j(w_s).$$

This implies that

$$e_s \geqslant \varphi_i(w_s) \quad \text{and} \quad e_s \geqslant \varphi_j(w_s).$$

By the same argument as in the last paragraph, we have

$$e_s = \varphi_i(w_s) = \varphi_j(w_s)$$

for $s = 1, \ldots, n$. Hence $\varphi_i = \varphi_j$ since $\{w_1, \ldots, w_n\}$ is the basis of U. For any positive integer $k \leqslant n$, we have either $f_k \neq f_i$ or $f_k \neq f_j$. Hence $\varphi_k = \varphi_i$. This shows that

$$T(e_s \otimes v) = f_s \otimes \varphi_1(v)$$

for $s = 1, \ldots, n$ and any $v \in U$. Since T is a linear transformation, it follows that there exists a linear transformation θ on U such that $\theta(e_s) = f_s$, $s = 1, \ldots, n$. Clearly θ is non-singular and $T = \theta \otimes \varphi_1$.

For the case where $\{T(u \otimes U) : u \in U\}$ consists of maximal right factor subspaces, it can be proved similarly that $T = \alpha \breve{\otimes} \beta$ for some non-singular linear transformation α on U and some bijective linear transformation β on U. $\square$

The following example shows that the condition that $U\backslash\{0\}$ has no least element is necessary for Theorem 3.13.

Example 3.14. Let U be the Boolean vector space consisting of three elements 0, e_1, e_2 where $e_1 < e_2$. Then there exists a linear transformation T on $U \otimes U$ such that

$$T(e_i \otimes e_i) = e_i \otimes e_i, \quad i = 1, 2,$$
$$T(e_1 \otimes e_2) = T(e_2 \otimes e_1) = e_1 \otimes e_2.$$

We have $T(e_1 \otimes U) = e_1 \otimes U$ and $T(e_2 \otimes U) = U \otimes e_2$. Clearly T cannot be induced by any two linear transformations on U.

Theorem 3.13 is not true if U is infinite dimensional as shown by the following example.

Example 3.15. Let N be the set of all positive integer. Let $\{e_i : i \in N\}$ be the basis of $\mathscr{B}_N$. Let T be the linear transformation on $\mathscr{B}_N \otimes \mathscr{B}_N$ such that

$$T(e_1 \otimes e_1) = e_1 \otimes (e_1 + e_2),$$
$$T(e_n \otimes e_1) = e_1 \otimes e_1 \quad \text{for any } n \geqslant 2,$$
$$T(e_n \otimes e_i) = e_1 \otimes e_{i-1} \quad \text{for any } n \in N \text{ and } i \geqslant 2.$$

Then T sends every maximal left factor subspaces to $e_1 \otimes \mathscr{B}_N$. However, T is clearly not induced by any two linear transformations on $\mathscr{B}_N$.

The following example shows that there exist surjective linear rank one preservers from $U \otimes U$ to $V \otimes V$ that send maximal left factor subspaces to maximal factor subspaces which are not induced by any two non-singular linear transformations.

Example 3.16. Let $T : M_3(\mathscr{B}) \to M_2(\mathscr{B})$ be the linear transformation defined by

$$T\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \begin{bmatrix} a+c+g+i & b+h+i \\ d+f+g+i & e+h+i \end{bmatrix}.$$

We check that T is a rank one preserver. Let $U := M_{3,1}(\mathscr{B})$ and $V := M_{2,1}(\mathscr{B})$. Let $\{e_1, e_2, e_3\}$ be the standard basis of U. Then $T(e_1 \otimes U) = \binom{1}{0} \otimes V$, $T(e_2 \otimes U) = \binom{0}{1} \otimes V$ and $T(u \otimes U) = \binom{1}{1} \otimes V$ for any non-zero vector $u \notin \{e_1, e_2\}$. Hence T is surjective and it sends maximal left factor subspaces to maximal factor subspaces. Since

$$T(E_{13}) = E_{11} \quad \text{and} \quad T(E_{33}) = E_{11} + E_{12} + E_{21} + E_{22},$$

it is easy to see that there do not exist matrices P and Q such that $T(A) = PAQ$ for all A in $M_3(\mathscr{B})$ or $T(A) = PA^tQ$ for all A in $M_3(\mathscr{B})$.

The following result is a characterization of surjective mappings from a tensor product of two Boolean vector spaces without comparable cells to another that send rank one elements to rank one elements and preserve order relation in both directions.

Theorem 3.17. *Let U, V, W, and Z be Boolean vector spaces where both U and V have no comparable cells. If $T : U \otimes V \to W \otimes Z$ is a surjective mapping sending rank one elements to rank one elements and*

$$T(A) > T(B) \Leftrightarrow A > B \quad \text{for any } A, B \in U \otimes V$$

then T is linear and induced by two bijective linear transformations.

Proof. We first show that T is injective. Suppose that $T(A) = T(B)$, but $A \neq B$. Since $T(A) \not> T(B)$ and $T(B) \not> T(A)$, it follows from the hypothesis that $A \not> B$ and $B \not> A$. We have the following cases:

Case 1. One of A and B, say B, is not a cell in $U \otimes V$. Since $A \not> B$, there is a cell C in $U \otimes V$ such that $B > C$ but $A \not\geqslant C$. This implies that $T(B) > T(C)$ and $T(A) \not> T(C)$ and, a contradiction since $T(A) = T(B)$.

Case 2. Both A and B are cells of $U \otimes V$. Then by Theorem 2.9,

$$A = c_1 \otimes d_1, \quad B = c_2 \otimes d_2$$

for some cells c_1, c_2 in U and some cells d_1, d_2 in V. Suppose that A and B have a common factor, say $c_1 = c_2$. Since $A \not> B$, we have $d_1 \not> d_2$. Let e be a cell in U distinct from c_1. Let $D = (e + c_1) \otimes d_1$. Then $D > A$ and $D \not> B$. Hence

$$T(D) > T(A) \quad \text{and} \quad T(D) \not> T(B),$$

a contradiction. Suppose now that A and B have no common factors. Let $K = (c_1 + c_2) \otimes d_1$. Then $K > A$ and $K \not> B$, since U and V have no comparable cells. Hence

$$T(K) > T(A) \quad \text{and} \quad T(K) \not> T(B),$$

a contradiction.

Since both cases lead to a contradiction, we have $A = B$ and hence T is injective.

We shall show that T is linear. Let $\{E_i : i \in I\}$ be the basis of $U \otimes V$. Let A be a non-zero element in $U \otimes V$ which is not a cell. Then $A = \sum_{j \in J} E_j$ for some finite subset J of I where $|J| \geqslant 2$. Since $T(A) \geqslant T(E_j)$ for any j in J, it follows that

$$T(A) \geqslant \sum_{j \in J} T(E_j).$$

Since T is surjective, we have

$$T(B) = \sum_{j \in J} T(E_j)$$

for some B in $U \otimes V$. Hence $A \geqslant B$. Since $T(B) \geqslant T(E_j)$ for any j in J, it follows that $B \geqslant E_j$ for any j in J. Hence

$$B \geqslant \sum_{j \in J} E_j = A.$$

This shows that $A = B$. Hence $T(A) = \sum_{j \in J} T(E_j)$.

Let A_1 and A_2 be two non-zero elements in $U \otimes V$. Then $A_i = \sum_{j \in J_i} E_j$ for some finite subsets J_i of I, $i = 1, 2$. Clearly $A_1 + A_2 = \sum_{j \in J_1 \cup J_2} E_j$. Hence

$$\begin{aligned} T(A_1 + A_2) &= \sum_{j \in J_1 \cup J_2} T(E_j) \\ &= \sum_{j \in J_1} T(E_j) + \sum_{j \in J_2} T(E_j) \\ &= T(A_1) + T(A_2). \end{aligned}$$

This shows that T is linear and hence the result follows from Theorem 3.2. □

If a non-zero vector u in a Boolean vector space with a non-dominating basis is the sum of k distinct cells, then k is called the height of u and is denoted by $\rho(u) = k$.

Lemma 3.18. *Let U be a Boolean vector space with a non-dominating basis. If $u \in U \backslash \{0\}$ has height k and $u \geqslant c_i$ for k distinct cells $c_1, \ldots, c_k$, then $u = \sum_{i=1}^{k} c_i$.*

Proof. By hypothesis, $u = \sum_{i=1}^{k} e_i$ for some cells $e_1, \ldots, e_k$. Since $u \geqslant c_i$, it follows that $c_i = e_{\sigma(i)}$ for some $\sigma(i) \leqslant k$. Hence $u = c_1 + \cdots + c_k$. □

The following result is analogous to Theorem 3.17.

Proposition 3.19. *Let U, V, W, and Z be Boolean vector spaces where each of them has a non-dominating basis. Then* $T : \mathscr{D}(U, V) \to \mathscr{D}(W, Z)$ *is a surjective mapping such that*

$$T(A) > T(B) \Leftrightarrow A > B \quad \text{for any } A, B \in \mathscr{D}(U, V)$$

if and only if T could be extended to a linear transformation from $U \otimes V$ *to* $W \otimes Z$ *which is induced by two bijective linear transformations.*

Proof. The sufficiency part of the result is clear. We now prove the necessity. From the first paragraph of the proof of Theorem 3.17, we see that T is injective.

Let $\{E_i : i \in I\}$ be the basis of $U \otimes V$. Then by Theorem 2.9, $E_i \in \mathscr{D}(U, V)$, $i \in I$, and $\{E_i : i \in I\}$ is non-dominating. Since T preserves order relation in both directions, it follows that $\{T(E_i) : i \in I\}$ is the set of all cells of $W \otimes Z$.

Let A be an element in $\mathscr{D}(U, V)$ which is not a cell. Then $A = \sum_{j \in J} E_j$ for some finite subset J of I where $|J| \geqslant 2$. Since $T(A) \geqslant T(E_j)$ for any j in J, it follows from Lemma 3.18 that $\rho(T(A)) \geqslant k$ where $k = |J|$. If $T(A) > T(E_s)$ for some $s \notin J$, then $A > E_s$, a contradiction. This shows that $\rho(T(A)) = k$ and hence by Lemma 3.18, we have

$$T(A) = \sum_{j \in J} T(E_j).$$

Now from the last paragraph of the proof of Theorem 3.17, we see that T can be extended to a bijective linear transformation from $U \otimes V$ to $W \otimes Z$. Hence the result follows from Theorem 3.2. □

Corollary 3.20. *Let U, V, W, and Z be finite dimensional Boolean vector spaces where each of them has a non-dominating basis. If* $T : U \otimes V \to W \otimes Z$ *is a bijective mapping sending rank one elements to rank one elements and* $A > B \Rightarrow T(A) > T(B)$ *for any* $A, B \in U \otimes V$, *then T is linear and induced by two bijective linear transformations.*

Proof. Let $\dim U = s$, $\dim V = t$, $\dim W = p$ and $\dim Z = q$. In view of Theorem 2.9, both $U \otimes V$ and $W \otimes Z$ have non-dominating bases. Since T is bijective, it follows from Lemma 2.4 that $U \otimes V$ and $W \otimes Z$ have the same number of cells. Hence $st = pq$ and the maximal height of all elements in $U \otimes V$ and in $W \otimes Z$ are the same. Let $\{E_i : i \in I\}$ be the basis of $U \otimes V$ where $I = \{1, \ldots, st\}$.

Let A be an element of $U \otimes V$ of height $k > 0$. Then $A = \sum_{j \in J} E_j$ for some non-empty finite subset J of I. Clearly there exist elements A_i of height i, $i = 1, \ldots, m$ where $m = st$ such that $A_k = A$ and $A_i < A_{i+1}$ for $i = 1, \ldots, m-1$. Since $T(A_i) < T(A_{i+1})$ for $i = 1, \ldots, m-1$, it follows that $T(A_k)$ is of height k. This shows that $\{T(E_i) : i \in I\}$ is the set of all cells of $W \otimes Z$ and T sends zero to zero. Since $T(A) \geqslant T(E_j)$ for any j in J and $\rho(T(A)) = k$, it follows from Lemma 3.18 that $T(A) = \sum_{j \in J} T(E_j)$.

Suppose that $T(B) > T(C) > 0$. Since $\{T(E_i) : i \in I\}$ is a non-dominating basis of $W \otimes Z$, we have

$$T(B) = \sum_{j \in K} T(E_j), \quad T(C) = \sum_{j \in H} T(E_j),$$

where H is a proper subset of K. Therefore $B = \sum_{j \in K} E_j$ and $C = \sum_{j \in H} E_j$. This shows that $B > C$. The corollary now follows from Theorem 3.17. □

Remark 3.21. From Corollary 3.20, we have the following corresponding result for spaces of Boolean matrices:

Let $T : M_{m,n}(\mathscr{B}) \to M_{k,\ell}(\mathscr{B})$ be a bijective mapping where $\min\{m, n, k, l\} \geqslant 2$. If T sends rank one matrices to rank one matrices and $A > B \Rightarrow T(A) > T(B)$ for any A, B in $M_{m,n}(\mathscr{B})$, then $\{m, n\} = \{k, l\}$ and there exist permutation matrices P and Q such that $T(A) = PAQ$ for all $A \in M_{m,n}(\mathscr{B})$ or $T(A) = PA^tQ$ for all $A \in M_{m,n}(\mathscr{B})$.

Let $B(m, n)$ denote the set of all bipartite graphs with bipartition (X, Y) where $|X| = m$, $|Y| = n$. Let $G \in B(m, n)$. Then it was shown in [3] that the biclique covering number of G, $bcc(G)$, is the same as the Boolean rank of the (0,1)-incidence matrix of G. Following [4], the above result can be translated into graph-theoretic terms as follows:

Let $T : B(m, n) \rightarrow B(k, l)$ be a bijective mapping where $\min\{m, n, k, l\} \geqslant 2$.

If $bcc(G) = 1 \Rightarrow bcc(T(G)) = 1$ for any G in $B(m, n)$ and H is a subgraph of K implies that $T(H)$ is a subgraph of $T(K)$ for every $H, K \in B(m, n)$, then $\{m, n\} = \{k, l\}$ and $T(G)$ is isomorphic to G for all G in $B(m, n)$.

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References

[1] L.B. Beasley, N.J. Pullman, Boolean-rank-preserving operators and Boolena-rank-1 spaces,Linear Algebra Appl. 59 (1984) 55–77.
[2] K.H. Kim, Boolean Matrix Theory and Applications, Marcel Dekker, New York, 1982.
[3] J. Orlin, Contentment in graph theory: covering graphs with cliques, Indag. Math. 39 (1977) 406–424.
[4] N.J. Pullman, Linear operators that preserve clique covering numbers, Ars Combin. 19A (1985) 27–36.
[5] S.-Z. Song, Separability of distinct Boolean rank-1 matrices, J. Appl. Math. Comput. 18 (2005) 197–204.
[6] R. Westwick, Transformations on tensor spaces, Pacific J. Math. 23 (1967) 613–620.