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Generalized Mittag-Leffler function method for solving Lorenz system

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ABSTRACT: In this paper, generalizations Mittag-Leffler function method is applied to solve approximate and analytical solutions of nonlinear fractional differential equation systems such as lorenz system of fractional oreder, and compared the results with the results of Homotopy perturbation method (HPM) and Variational iteration method (VIM) in the standard integer order form. The reason of using fractional order differential equations (FOD) is that fractional order differential equations are naturally related to systems with memory which exists in most systems. Also they are closely related to fractals which are abundant in systems. The results derived of the fractional system are of a more general nature. Respectively, solutions of fractional order differential equations spread at a faster rate than the classical differential equations, and may exhibit asymmetry. A few numerical methods for fractional differential equations models have been presented in the literature. However many of these methods are used for very specific types of differential equations, often just linear equations or even smaller classes put the results generalizations Mittag-Leffler function method show the high accuracy and efficiency of the approach. A new solution is constructed in power series. The fractional derivatives are described by Caputo's sense.

KEYWORDS: Lorenz system, Caputo fractional derivative, Mittag-Leffler function.

1 INTRODUCTION

Between a large number of chaotic systems obvious that the Lorenz model is the most classical and paradigmatic problem because it was the first model of chaotic behaviour. the Lorenz model is a simplification of a previous more complicated model of Saltzman to describe buoyancy driven convection patterns in the classical rectangular Rayleigh–Bénard problem applied to the thermal convection between two plates perpendicular to the direction of the Earth's gravitational force [1]-[5]. The components of the basic tree-component model are proportional to the convective velocity, the temperature difference between descending and ascending flows, and the mean convective heat flow is denoted respectively by $x(t)$, $y(t)$, $z(t)$. The famous Lorenz equations are [6]:

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$$\frac{dx}{dt} = s\,(y - x),$$
$$\frac{dy}{dt} = r\,x - y - xz, \qquad (1.1)$$
$$\frac{dz}{dt} = xy - bz.$$

with the initial conditions:

$$x(0) = M_1, \qquad y(0) = M_2, \qquad z(0) = M_3. \qquad (1.2)$$

s, b are real constants, and r so-called bifurcation parameter.

Now we introduce the generalized chaotic dynamical system (Lorenz system). The system is described by the following system of fractional differential equations:

$$D^{\alpha_1} x = s\,(y - x),$$
$$D^{\alpha_2} y = r\,x - y - xz, \qquad \text{where} \qquad 0 \prec \alpha_1, \alpha_2, \alpha_3 \le 1 \qquad (1.3)$$
$$D^{\alpha_3} z = xy - bz.$$

Where D^{α_i} , $i = 1,2,3$ is the derivative of order α_i in the sense of Caputo.

with the initial conditions

$$x(0) = M_1, \qquad y(0) = M_2, \qquad z(0) = M_3. \qquad (1.4)$$

The purpose of using fractional differential equations is that The fractional calculus approach provides a powerful tool for the description of memory and hereditary properties of various materials and processes [7]–[14]. It has been applied to many fields in science and engineering, such as viscoelasticity, anomalous diffusion, fluid mechanics, biology, chemistry, acoustics, control theory, etc.

The motivation of this paper is the application of the generalizations Mittag-Leffler function method for solving generalized lorenz system and compared the results with the results of Homotopy perturbation method (HPM) and Variational iteration method (VIM) in the standard integer order form [6].This method used by Rida and Arafa for solving Linear fractiona differential equations [15].

2 BASIC DEFINITIONS

In this section, we mention the basic definitions of the fractional calculus.

2.1 DEFINITION:

The fractional derivative of *f*(x) in the Caputo sense is defined as:

$$D^{\alpha} f(x) = J^{n-\alpha} D^{n} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} f^{(n)}(t)\,dt \qquad (2.1.1)$$

$for\ \ n-1 \prec \alpha \le n,\ n \in N,\ \ x \succ 0$, for the caputo derivative we have $D^{\alpha} c = 0$, c is constant

$$D^{\alpha} t^{m} == \begin{cases} 0 & m \le \alpha - 1 \\ \dfrac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} t^{m-\alpha}, & m > \alpha - 1 \end{cases} \qquad (2.1.2)$$

2.2 DEFINITION

For n to be the smallest integer that exceeds α, the Caputo fractional derivatives of order α > 0 is defined as

$$D^{\alpha}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha+1}\frac{\partial^n u(x,\tau)}{\partial \tau^n}d\tau, & for\ n-1<\alpha\le n,\\ \frac{\partial^n u(x,t)}{\partial t^n}, & for\ \alpha=n\in N.\end{cases} \quad (2.2.1)$$

3 ANALYSIS OF THE METHOD

The Mittag-Leffler (1902–1905) functions $E_{\alpha}\ and\ E_{\alpha,\beta}$ [16], defined by the power series

$$E_{\alpha}=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma[n\alpha+1]}\quad,\quad E_{\alpha,\beta}=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma[n\alpha+\beta]}\quad,\qquad \alpha,\beta\succ 0 \quad (3.1)$$

have already proved their efficiency as solutions of fractional order differential and integral equations and thus have become important elements of the fractional calculus theory and applications.

In this paper, we will show how to solve system of nonlinear fractional differential equations(lorenz system) through the imposition of the generalized Mittag-Leffler function $E_{\alpha}(z)$. The generalized Mittag-Leffler method suggests that $y_i(t),\quad i=1,2,3,...$ are decomposed by an infinite series of components [15]:

$$y_i(t)=E_{\alpha}(a_i\,t^{\alpha})=\sum_{n=0}^{\infty}a_i^n\frac{t^{n\alpha}}{\Gamma[n\alpha+1]}\quad,\qquad i=1,2,3,... \quad (3.2)$$

We will use the following definitions of fractional calculus:

$$D^{\alpha}y_i(t)=\sum_{n=1}^{\infty}a_i^n\frac{t^{(n-1)\alpha}}{\Gamma[(n-1)\alpha+1]}\,,\qquad i=1,2,3,... \quad (3.3)$$

This are based on the Caputo fractional is derivatives. The convergence of the Mittag Leffler function discussed in [16].

4 APPLICATIONS AND RESULTS

In this section, we applied the generalized Mittag-Leffler function method for solving system of fractional differential equations (lorenz system).and Compare the results with the results of other two methods. Considering the fractional lorenz system:

$$\begin{aligned} D^{\alpha_1}x&=s\,(y-x),\\ D^{\alpha_2}y&=r\,x-y-xz\,,\\ D^{\alpha_3}z&=xy-bz\,.\end{aligned}\qquad 0\prec\alpha_1,\,\alpha_2,\alpha_3\le 1 \quad (4.1)$$

with the initial conditions

$$x(0)=M_1\,,\qquad y(0)=M_2\,,\qquad z(0)=M_3\ . \quad (4.2)$$

By using generalized Mittag-Leffler function method we put

$$x(t) = E_\alpha(at^\alpha) = \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma[n\alpha+1]},$$
$$y(t) = E_\alpha(dt^\alpha) = \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma[n\alpha+1]}, \quad (4.3)$$
$$z(t) = E_\alpha(l\,t^\alpha) = \sum_{n=0}^{\infty} l^n \frac{t^{n\alpha}}{\Gamma[n\alpha+1]}$$

By using (4.3) and (3.3) into (4.1) when $\alpha_1, \alpha_2, \alpha_3 = \alpha$ we find this relation

$$\sum_{n=1}^{\infty} \frac{a^n\, t^{(n-1)\alpha}}{\Gamma[(n-1)\alpha+1]} - s\sum_{n=0}^{\infty} \frac{d^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} + s\sum_{n=0}^{\infty} \frac{a^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} = 0,$$
$$\sum_{n=1}^{\infty} \frac{d^n\, t^{(n-1)\alpha}}{\Gamma[(n-1)\alpha+1]} - r\sum_{n=0}^{\infty} \frac{a^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} + \sum_{n=0}^{\infty} \frac{d^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} + \sum_{n=0}^{\infty} c_1^n\, t^{n\alpha} = 0, \quad (4.4)$$
$$\sum_{n=1}^{\infty} \frac{l^n\, t^{(n-1)\alpha}}{\Gamma[(n-1)\alpha+1]} - \sum_{n=0}^{\infty} c_2^n\, t^{n\alpha} + b\sum_{n=0}^{\infty} \frac{l^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} = 0$$

Combining the alike terms and replacing (n) by (n +1) in the first sum, we assume the form

$$\sum_{n=0}^{\infty} \frac{a^{n+1}\, t^{n\alpha}}{\Gamma[n\alpha+1]} - s\sum_{n=0}^{\infty} \frac{d^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} + s\sum_{n=0}^{\infty} \frac{a^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} = 0,$$
$$\sum_{n=0}^{\infty} \frac{d^{n+1}\, t^{n\alpha}}{\Gamma[n\alpha+1]} - r\sum_{n=0}^{\infty} \frac{a^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} + \sum_{n=0}^{\infty} \frac{d^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} + \sum_{n=0}^{\infty} c_1^n\, t^{n\alpha} = 0, \quad (4.5)$$
$$\sum_{n=0}^{\infty} \frac{l^{n+1}\, t^{n\alpha}}{\Gamma[n\alpha+1]} - \sum_{n=0}^{\infty} c_2^n\, t^{n\alpha} + b\sum_{n=0}^{\infty} \frac{l^n\, t^{n\alpha}}{\Gamma[n\alpha+1]} = 0$$

Where

$$c_1^n = \sum_{k=0}^{n} \frac{a^k l^{n-k}}{\Gamma[k\alpha+1]\Gamma[(n-k)\alpha+1]},$$
$$c_2^n = \sum_{k=0}^{n} \frac{a^k d^{n-k}}{\Gamma[k\alpha+1]\Gamma[(n-k)\alpha+1]} \quad (4.6)$$

We have

$$\sum_{n=0}^{\infty} (a^{n+1} - sd^n + sa^n) \frac{t^{n\alpha}}{\Gamma[n\alpha+1]} = 0,$$
$$\sum_{n=0}^{\infty} (d^{n+1} - r\,a^n + d^n + c_1^n\, \Gamma[n\alpha+1]) \frac{t^{n\alpha}}{\Gamma[n\alpha+1]} = 0, \quad (4.7)$$
$$\sum_{n=0}^{\infty} (l^{n+1} - c_2^n\, \Gamma[n\alpha+1] + b\,l^n) \frac{t^{n\alpha}}{\Gamma[n\alpha+1]} = 0$$

With the coefficient of $t^{n\alpha}$ equal to zero and identifying the coefficients, we obtain recurrence relations:

$$a^{n+1} - sd^n + sa^n = 0 ,$$
$$d^{n+1} - r\,a^n + d^n + c_1^n\,\Gamma[n\alpha+1] = 0, \qquad (4.8)$$
$$l^{n+1} - c_2^n\,\Gamma[n\alpha+1] + b\,l^n = 0$$

we can obtain a few first terms being calculated:

$$a^0 = M_1 , \quad d^0 = M_2 , \quad l^0 = M_3 .$$

When n=0 we have

$$a^1 = s(M_2 - M_1),$$
$$d^1 = rM_1 - M_2 - M_1M_3 ,$$
$$l^1 = M_1M_2 - bM_3 .$$

When n=1 we have

$$a^2 = s(rM_1 - M_2 - M_1M_3 - s\,M_2 + sM_1),$$
$$d^2 = rs(M_2 - M_1) - rM_1 + M_2 + M_1M_3 - M_1{}^2M_2 + bM_3M_1 - sM_2M_3 + sM_3M_1 ,$$
$$l^2 = rM_1{}^2 - M_1M_2 - M_1{}^2M_3 + sM_2{}^2 - sM_1M_2 - bM_1M_2 + b^2M_3 .$$

When n=2 we have

$$a^3 = rs^2(M_2 - M_1) - rsM_1 + sM_2 + sM_1M_3 - sM_1{}^2M_2 + sbM_3M_1 - s^2M_2M_3$$
$$+ s^2M_3M_1 - s^2rM_1 + s^2M_2 + s^2M_1M_3 + s^3M_2 - s^3M_1 ,$$
$$d^3 = rs(rM_1 - M_2 - M_1M_3 - sM_2 + sM_1) - rs(M_2 - M_1) + r\,M_1 - M_2 - M_1M_3$$
$$+ M_1{}^2M_2 - bM_3M_1 + sM_2M_3 - sM_3M_1 - rM_1{}^3 + M_1{}^2M_2 + M_1{}^3M_3 - sM_1M_2{}^2$$
$$+ sM_1{}^2M_2 + bM_1{}^2M_2 - b^2M_1M_3 - sM_3(rM_1 - M_2 - M_1M_3 - sM_2 + sM_1)$$
$$-(sM_2 - sM_1)(M_1M_2 - bM_3)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2} ,$$
$$l^3 = rsM_1(M_2 - M_1) - rM_1{}^2 + M_1M_2 + M_1{}^2M_3 - M_1(M_1{}^2M_2 - bM_1M_3) - sM_1M_2M_3$$
$$+ sM_1{}^2M_3 + sM_2(rM_1 - M_2 - M_1M_3 - sM_2 + sM_1) + s(M_2 - M_1)(rM_1 - M_2 - M_1M_3)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}$$
$$- brM_1{}^2 + bM_1M_2 + bM_1{}^2M_3 - sbM_2{}^2 + sbM_1M_2 + b^2M_1M_2 - b^3M_3 .$$

Compensation from the previous recurrence relations in this series

$$x(t) = \sum_{n=0}^{\infty} a^n \frac{t^{n\alpha}}{\Gamma[n\alpha+1]} = a^0 + a^1\frac{t^{\alpha}}{\Gamma[\alpha+1]} + a^2\frac{t^{2\alpha}}{\Gamma[2\alpha+1]} + a^3\frac{t^{3\alpha}}{\Gamma[3\alpha+1]} + a^4\frac{t^{4\alpha}}{\Gamma[4\alpha+1]} + \ldots\ldots.$$
$$y(t) = \sum_{n=0}^{\infty} d^n \frac{t^{n\alpha}}{\Gamma[n\alpha+1]} = d^0 + d^1\frac{t^{\alpha}}{\Gamma[\alpha+1]} + d^2\frac{t^{2\alpha}}{\Gamma[2\alpha+1]} + d^3\frac{t^{3\alpha}}{\Gamma[3\alpha+1]} + d^4\frac{t^{4\alpha}}{\Gamma[4\alpha+1]} + \ldots\ldots.$$
$$z(t) = \sum_{n=0}^{\infty} l^n \frac{t^{n\alpha}}{\Gamma[n\alpha+1]} = l^0 + l^1\frac{t^{\alpha}}{\Gamma[\alpha+1]} + l^2\frac{t^{2\alpha}}{\Gamma[2\alpha+1]} + l^3\frac{t^{3\alpha}}{\Gamma[3\alpha+1]} + l^4\frac{t^{4\alpha}}{\Gamma[4\alpha+1]} + \ldots\ldots.$$

Therefore:

$$x(t)=M_1+s(M_2-M_1)\frac{t^{\alpha}}{\Gamma[\alpha+1]}+s(r\,M_1-M_2-M_1M_3-s\;M_2+s\,M_1)\frac{t^{2\alpha}}{\Gamma[2\alpha+1]}$$
$$+\{r\,s^2\,(M_2-M_1)-r\,s\,M_1+s\,M_2+s\,M_1M_3-s\,M_1{}^2M_2+s\,b\,M_3M_1-s^2\;M_2M_3$$
$$+s\;^2M_3M_1-s^2r\,M_1+s^2\;M_2+s^2\;M_1M_3+s^3\;M_2-s^3M_1\}\frac{t^{3\alpha}}{\Gamma[3\alpha+1]}+....$$

$$y(t)=M_2+\{rM_1-M_2-M_1M_3\}\frac{t^{\alpha}}{\Gamma[\alpha+1]}+\{r\,s\,(M_2-M_1)-r\,M_1+M_2+M_1M_3-M_1{}^2M_2$$
$$+b\,M_3M_1-s\,M_2M_3+s\,M_3M\}\frac{t^{2\alpha}}{\Gamma[2\alpha+1]}+\{r\,s\,(r\,M_1-M_2-M_1M_3-s\,M_2+s\,M_1)-r\,s\,(M_2-M_1)$$
$$+r\;M_1-M_2-M_1M_3+M_1{}^2M_2-b\,M_3M_1+s\,M_2M_3-s\,M_3M_1-r\,M_1{}^3+M_1{}^2M_2+M_1{}^3M_3-s\,M_1M_2{}^2$$
$$+s\,M_1{}^2M_2+b\,M_1{}^2M_2-b^2M_1M_3-s\,M_3\,(r\,M_1-M_2-M_1M_3-s\,M_2+s\,M_1)$$
$$-(s\,M_2-s\,M_1)\,(M_1M_2-b\,M_3)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}\}\frac{t^{3\alpha}}{\Gamma[3\alpha+1]}+.....$$

$$z(t)=M_3+\{M_1M_2-b\,M_3\}\frac{t^{\alpha}}{\Gamma[\alpha+1]}+\{r\,M_1{}^2-M_1M_2-M_1{}^2M_3+s\,M_2{}^2-s\,M_1M_2-b\,M_1M_2+b^2M_3\}\frac{t^{2\alpha}}{\Gamma[2\alpha+1]}$$
$$+\{r\,s\,M_1\,(M_2-M_1)-r\,M_1{}^2+M_1M_2+M_1{}^2M_3-M_1(M_1{}^2M_2-b\,M_1M_3)-s\,M_1M_2M_3$$
$$+s\,M_1{}^2M_3+s\,M_2\,(r\,M_1-M_2-M_1M_3-s\,M_2+s\,M_1)+s\,(M_2-M_1)\,(r\,M_1-M_2-M_1\,M_3)\frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2}$$
$$-b\,r\,M_1{}^2+b\,M_1M_2+b\,M_1{}^2M_3-s\,b\,M_2{}^2+s\,b\,M_1M_2+b^2\;M_1M_2-b^3M_3\}\frac{t^{3\alpha}}{\Gamma[3\alpha+1]}+......$$

When $\alpha=1$, then we have the same results with the results of Homotopy perturbation method and Variational iteration method to solve Lorenz system in [6].

5 CONCLUSION

In this paper, The Mittag-Leffler function and its generalizations used to obtain a new method for solving system of nonlinear fractional differential equations (Lorenz system). And compared the results with the results of Homotopy perturbation method (HPM) and Variational iteration method (VIM), the results of HPM and results of VIM in the standard integer order form When $\alpha=1$ in Eq(1.3). The new generalization is based on the Caputo fractional derivative. From the results we seen that this method is a very powerful and efficient technique in finding approximate solutions for wide classes of fractional differential equations.

REFERENCES

[1] R. Barrio, S. Serrano, "A three-parametric study of the Lorenz model," *Physica D* 229, 43–51, 2007.
[2] E. Lorenz, J. Atmos. Sci. 20 (1963) 130.
[3] M. Viana, Math. Intelligencer 22 (2000) 6.
[4] J. Argyris, G. Faust, M. Haase, "An exploration of chaos, in:Texts on Computational Mechanics," VII, North-Holland Publishing Co, Amsterdam, 1994.
[5] C. Sparrow, "The Lorenz equations: Bifurcations, chaos, and strange attractors," in: *Applied Mathematical Sciences*, vol. 41, Springer-Verlag, New York, 1982.
[6] M. MERDAN, "Variational iteration and homotopy perturbation method for solving Lorenz system," BAÜ FBE Dergisi 1, 3-14, 2009.
[7] K.B. Oldham, J. Spanier, The Fractional Calculus, Academic, New York, 1974.

[8] K.S. Miller, B. Ross, "An Introduction to the Fractional Calculus and Fractional Differential Equations," Wiley, New York, 1993.

[9] V. D. Gejji, H. Jafari, "Analysis of a system of nonautonomous fractional differential equations involving Caputo derivatives," *J. Math. Anal. Appl*. 328 (2) 1026–1033, 2007.

[10] A.M.A. El-Sayed, S. Z. Rida, A.A.M. Arafa, "On the Solutions of the generalized reaction-diffusion model for bacteria growth," *Acta Appl. Math*., 110, 1501–1511, 2010.

[11] S. Z. Rida, A.M.A. El-Sayed, A.A.M. Arafa, "On the solutions of time-fractional reaction–diffusion equations," *Commun Nonlinear Sci Numer Simulat* 15, 3847–3854, 2010.

[12] A.M.A. El-Sayed ,S. Z. Rida, A.A.M. Arafa, "Exact Solutions of Some Time- Fractional Biological Population Models by Using Generalized Differential Transform Method," *Int. J. of Math. modeling, simulation and Appl*. Vol. 3, No. 3, 231-239, 2010.

[13] A.M.A. El-Sayed ,S. Z. Rida, A.A.M. Arafa, "On the Solutions of Time-fractional Bacterial Chemotaxis in a Diffusion Gradient Chamber," *Int. J. of Nonlinear Science*, Vol.7, No.4, pp.485-492, 2009.

[14] E. Hairer, G. Wanner, "Solving ordinary differential equations II: stiff and differential–algebraic problems," Springer-Verlag; 1991.

[15] S. Z. Rida, A.A.M. Arafa, "New Method for Solving Linear Fractional Differential Equations," *International Journal of Differential Equations* 2011.

[16] I. Podlubny, Fractional differential equations. New York: Academic Press; 1999.