Spectral properties of localized continuum random Schrödinger operators

Dissertation
an der Fakultät für Mathematik, Informatik und Statistik
der Ludwig-Maximilians-Universität
München

eingereicht von

Adrian Dietlein

München, den 17. Mai 2018

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Zusammenfassung

Die Resultate, die ich im Rahmen meiner Dissertation vorstelle, sind hauptsächlich motiviert durch das Bestreben, das mathematische Verständnis des lokalisierten Spektralbereichs zufälliger quantenmechanischer Systeme zu verbessern. In der theoretischen (und experimentellen) Physik gelten verschiedene Spektraleigenschaften als charakteristische Indikatoren für das Vorliegen einer lokalisierten spektralen Phase. Das mathematische Bestätigen solcher Charakteristika in möglichst großer Allgemeinheit ist eines der Kernthemen der Theorie zufälliger Schrödingeroperatoren.

Im ersten Projekt dieser Dissertation, welches auf einer Zusammenarbeit mit Martin Gebert und Peter Müller basiert [37], wird die sogenannte *Anderson Orthogonalität* untersucht: Gegeben seien zwei nicht wechselwirkende Elektronensysteme, deren Einteilchenoperatoren sich nur um eine lokale Störung unterscheiden. Dann spricht man von Anderson Orthogonalität, falls der Überlapp der beiden Grundzustände der Elektronensysteme im makroskopischen Limes gegen null strebt. Wir zeigen, dass Anderson Othogonalität sowie deren Abwesenheit im lokalisierten Spektralbereich eines zufälligen Schrödingeroperators beide mit positiver Wahrscheinlichkeit auftreten. Folglich verschwindet der zufallsgemittelte Grundzustandsüberlapp nicht im makroskopischen Limes. In Kombination mit bereits bekannten Resultaten [51] zeigt dies, dass das Verhalten des Grundzustandüberlapps im makroskopischen Limes ein Indikator eines lokalisierten Spektralbereichs ist.

Ein weiterer Test für die Spektralstruktur eines zufälligen quantenmechanischen Systems ist dessen *lokale Eigenwertstatistik*. Es ist Teil der Folklore der Physik, dass eine poissonverteilte lokale Eigenwertstatistik ein universeller Indikator eines lokalisierten Systems ist. Andererseits funktionieren bekannte Beweise nur für das klassiche Andersonmodell und ähnliche Modelle auf dem Gitter. Unabhängig vom jeweiligen Modell ist eine notwendige Bedingung für eine poissonverteilte lokale Eigenwertstatistik bei der Referenzenergie E die strikte Positivität der Zustandsdichte an dieser Energie. Im zweiten Projekt, welches auf einer Zusammenarbeit mit Martin Gebert, Peter Hislop, Abel Klein und Peter Müller basiert [37], wird eine strikt positive untere Schranke an die Zustandsdichte von zufälligen Schrödingeroperatoren im Kontinuum etabliert. Danach präsentiere ich, basierend auf Resultaten die in Zusammenarbeit mit Alexander Elgart entstanden [36], einen neuen Beweis für die poissonsche lokale Eigenwertstatistik. Dieser ist deutlich flexibler als bekannte Beweise und ist zum Beispiel anwendbar auf zufällige Schrödingeroperatoren im Kontinuum.

Ein Phänomen, welches dem oben beschriebenen asymptotischen Verschwinden des Grundzustandsüberlapps ähnlich ist, ist die logarithmische Verstärkung der führenden Ordnung sogenannter *asymptotischer Szegő Spurformeln*. Die Absenz solcher logarithmischer Verstärkungen für lokalisierte zufällige Schrödingeroperatoren ist bereits bekannt [100, 43]. Aufbauend auf diesen Arbeiten beweise ich [35] eine komplette asymptotische Entwicklung für die Spur des Operators $h(g(H_\omega)_{[-L,L]^d})$ in der Längenskala L, wo h und g geeignete Funktionen sind und H_ω ein allgemeiner ergodischer Operator. Die Hauptannahme, unter der diese komplette asymptotische Entwicklung gültig ist, ist hinreichend schneller Abfall des Operatorkerns des Operators $g(H_\omega)$. Eine solche Annahme kann nachgewiesen werden unter entweder einer spektralen Lokalisierungsannahme für den Operator H_ω oder einer Regularitätsannahme für die Funktion g.

Abstract

The results presented in this thesis are mainly motivated by the attempt to improve the mathematical understanding of the localized spectral region of random quantum mechanical systems. It is common wisdom in theoretical (and experimental) physics that a variety of spectral properties are characteristic indicators for the presence of spectral localization. The mathematical verification of such characteristic properties at large is one of the key concerns of the theory of random Schrödinger operators.

The first topic we address, based on joint work with Martin Gebert and Peter Müller [37], is a phenomenon dubbed *Anderson orthogonality*: Given two non-interacting, quasi-free electron systems which only differ by a local perturbation, Anderson orthogonality refers to the vanishing of their ground-state overlap in the macroscopic limit. We prove that in the localized spectral region Anderson orthogonality and absence of Anderson orthogonality both typically appear with positive probability. As a consequence, the disorder-averaged ground-state overlap does not vanish in the macroscopic limit. Combined with the mathematical results from [51], this shows that the absence of Anderson orthogonality can indeed be viewed as a characteristic property of the localized spectral region.

Another test for the spectral structure of a random quantum mechanical system is its *local eigenvalue statistics*. On the one hand, it is common sense in physics that the eigenvalue statistics for a generic localized system are poissonian. But, on the other hand, previously known proofs only applied for the lattice Anderson model and similar lattice models. Irrespective of the concrete model, a mandatory requirement to obtain Poisson statistics of the local eigenvalue process around a reference energy E is a positive density of states at that point. As a first step towards Poisson statistics we prove, based on joint work with Martin Gebert, Peter Hislop, Abel Klein and Peter Müller [37], a strictly positive lower bound on the density of states for continuum random Schrödinger operators. Then, based on joint work with Alexander Elgart [36], we present a new proof for poissonian local eigenvalue statistics. It is more flexible than known methods and, for instance, applicable to continuum random Schrödinger operators.

A phenomenon reminiscent of the vanishing of the ground-state overlap described above is the logarithmic enhancement of asymptotic *Szegő-type trace formulas*. The absence of a logarithmic enhancement for the localized lattice Anderson model is already known [100, 43]. But motivated by those works, we prove [35] a full asymptotic expansion for the trace of $h(g(H_\omega)_{[-L,L]^d})$ in terms of the length-scale L, where h and g are suitable functions and H_ω is a general ergodic operator. Our key assumption here is that the operator kernel of $g(H_\omega)$ exhibits sufficient spatial decay, which can be verified either under a spectral localization assumption on H_ω or a regularity assumption on g.

Preface

The thesis consists of two introductory chapters followed by four chapters with a detailed description of the results, including proofs. The first chapter is a general introduction to random Schrödinger operators and provides an overview over the four topics of this thesis. The second chapter consists of a review of continuum random Schrödinger operators and a short description of what I consider the main results. The Chapters three to six then contain a detailed description of the results, including proofs.

Most of the results presented here were obtained in scientific collaboration, which resulted in the publications listed below. The relation to published material is highlighted at the beginning of each of the chapters three to six. Moreover, parts of the introduction coincide both in content and writing with material from the publications (i)-(iv) below.

Published content

(i) A. Dietlein, M. Gebert, P. Hislop, A. Klein and P. Müller, A bound on the averaged spectral shift function and a lower bound on the density of states for random Schrödinger operators on $\mathbb{R}^d$, *Int. Math. Res. Not.,* rnx092 (2017).
(ii) A. Dietlein, M. Gebert and P. Müller, Bounds on the effect of perturbations of continuum random Schrödinger operators and applications, Accepted for publication in: *J. Spectr. Theory*, arXiv:1701.02956.
(iii) A. Dietlein, Full Szegő-type trace asymptotics for ergodic operators on large boxes, Accepted for publication in: *Comm. Math. Phys.*, arXiv:1710.00201.
(iv) A. Dietlein and A. Elgart, Level spacing for continuum random Schrödinger operators with applications, arXiv:1712.03925, *submitted.*

We do not refer to the publications below by the numbers (i)–(iv) but by their respective numbers in the bibliography at the end of this thesis.

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CHAPTER 1

Introduction

Almost 60 years ago, P.W. Anderson proposed Schrödinger operators with a potential whose spatial profile depends on a random variable ω to study metal-insulator transitions in condensed matter physics. He considered the prototypical lattice tight binding model $H_\omega^A := -\Delta + \lambda V_\omega$, acting on the square-summable functions on $\mathbb{Z}^d$. Here $-\Delta$ is the graph Laplacian and the random potential V_ω acts as $V_\omega(k) = \omega_k$ $(k \in \mathbb{Z}^d)$ for a family $(\omega_k)_{k\in\mathbb{Z}^d}$ of independent random variables that are distributed according to the uniform distribution on $[-1, 1]$. Anderson argued that in $d \geqslant 3$ dimensions this model exhibits a phase transition in terms of the spectral structure as the disorder parameter λ is tuned [10]: For sufficiently small disorder strength the bulk of the system roughly behaves like the corresponding non-random system: The majority of the operator's spectrum consists of extended *delocalized* (generalized) eigenfunctions which spread spatially under time evolution. In contrast, for sufficiently large disorder strength the spectrum only consists of exponentially decaying *localized* eigenfunctions which do not spread spatially under time evolution. On an ad hoc level, this phase transition can be perceived by comparing the two components of H_ω^A: For small disorder, the graph Laplacian $-\Delta$ is the dominant part. Its spectrum is absolutely continuous, with (generalized) eigenfunctions that spread spatially under time evolution. Conversely, for large disorder, the random potential V_ω is the dominant part. Its spectrum is pure point with eigenfunctions that are localized on a single point of $\mathbb{Z}^d$. This conjectured phase transition kickstarted intense research in both physics and mathematics. The first key question in the context of random Schrödinger operators therefore is:

Does a given random Schrödinger operator exhibit
a localization-delocalization phase transition?

From a theoretical physics point of view, it is nowadays common sense that generic random Schrödinger operators exhibit a localization-delocalization phase transition in $d \geqslant 3$ dimensions, while in $d = 1, 2$ dimensions the random Schrödinger operator is spectrally localized as soon as any disorder is present $(\lambda > 0)$. Despite considerable effort, so far only partial results have been obtained mathematically. The first mathematical works in this direction proved spectral localization for one-dimensional random Schrödinger operators [56]. Besides that, there are two well-developed and quite flexible tools to prove the existence of a localized energy region in $d \geqslant 2$ dimensions. They both establish spectral localization in an energy region at the edges of the spectrum that grows with λ. For the lattice Anderson model H_ω^A, those methods ultimately (i.e. for sufficiently large λ) yield spectral localization for all of the spectrum. The first method that provided a mathematical proof of localization is the *multiscale* *analysis* developed by Fröhlich and Spencer [48]. It relies on the observation that eigenfunctions decay exponentially in energetically unfavorable spatial regions. For instance, for an eigenfunction associated to an eigenvalue at the lower spectral edge, most of the space looks unfavorable due to a non-vanishing random potential. This is combined with an inductive procedure which successively rules out resonances between spatial regions that are energetically favorable for the eigenfunction. The multiscale analysis proved to be a quite flexible method, capable to deal with technically involved models such as the continuum random Schrödinger operators with discrete disorder [23]. The second approach towards spectral localization is the *fractional moment method* developed by Aizenman and Molchanov [5]. It is tailored towards an application to random Schrödinger operators with a sufficiently regular random potential, and in many cases yields stronger results with seemingly less effort compared to the multiscale analysis. The price of those benefits however is less flexibility with respect to model parameters. Both methods, the multiscale analysis and the fractional moment method, are nowadays well developed. Only a mere fraction of generic random Schrödinger operators, most prominently probably the lattice Anderson model with Bernoulli-distributed single-site random variables, is currently out of reach via both methods. This cumulates in a mathematical understanding of the physically well-accepted statement that localization due to the introduction of a random potential is a universal feature of generic Schrödinger operators. On the other hand, barely anything is known about the existence of a delocalized spectral region in the presence of disorder. Besides results for random Schrödinger operators on tree(-type) graphs [7, 47, 79, 109], no mathematical proofs for the existence of a delocalized spectral region are presently available for generic random Schrödinger operators. Finding such a proof can be considered a cornerstone problem in spectral theory of Schrödinger operators [114]. The second key question in the context of random Schrödinger operators is:

What are characteristic properties of the spectrum in the localized (or delocalized) spectral region?

This is interesting in at least two regards. Firstly, there is a variety of properties of a random quantum mechanical system which are physically believed to be (almost) equivalent to localization or delocalization, respectively. It is, for instance, a commonplace in physics that the faith of a generic random quantum mechanical system is encoded in its local eigenvalue statistics. This topic is addressed in more detail below. A typical procedure in physics is to analyze, say, the local eigenvalue statistics of a system and solely on this basis conclude whether it is localized or delocalized. It is hence a natural question whether one can put this course of action on solid footing by means of mathematical proofs. Since no proof of a delocalized spectral region is currently available, we focus here on the localized spectral region. Another reason to investigate such questions from a mathematical point of view is that a better understanding of the localized spectral region often yields soft criteria for the presence of a delocalized spectral region. Let's for instance assume that spectral localization together with a bunch of generic general properties of random Schrödinger operators implies property (P). If we could prove that (P) is not true for a generic random Schrödinger operator, then we could at least confirm the existence of a non-localized spectral region (even though this would, of course, be a much weaker statement compared to the existence of a delocalized spectral region).

The questions above have been the original motivation of my research and the work on all four projects presented in this thesis was initially triggered by them. This is reflected by a natural intertwining between the projects. However, only two of the projects are directly related to those questions whereas the other two deal with mathematically interesting problems which have only a loose connection to those questions. In order to give each project sufficient space to breathe they are presented separately. Even though we considered the lattice Anderson model for the sake of illustration above, our results are formulated for operators on continuum space $\mathbb{R}^d$. To the best of our knowledge, the results presented below would also yield new results on the lattice for three of the four projects. The reasoning behind presenting them in the continuum setup is that the corresponding proofs in the lattice case can more often than not be retrieved from the continuum proofs. Moreover, covering a wide range of models is particularly interesting if phenomena are investigated which are physically believed to be universal (i.e. model independent to a high degree).

One way to obtain information on a quantum mechanical system is to study its reaction to an external perturbation. The ground-state overlap of the unperturbed and the perturbed system constitutes one among the several meaningful methods to quantify the effect of the perturbation. In order to rule out finite-volume fluctuations, it is most convenient to consider the ground-state overlap in the macroscopic limit. This setup was first examined by Anderson. He discovered that for a system of free fermions even a small local perturbation typically causes an asymptotic vanishing of the ground-state overlap [12, 11]. This phenomenon has subsequently been studied thoroughly in solid state physics and is nowadays known as *Anderson orthogonality* [98, 89]. Since the asymptotic behavior of the ground-state overlap crucially depends on the realization of the macroscopic limit, we need to be a bit more precise. Let $\mathbf{\Phi}_N^L$ be the ground state of a N-particle system of non-interacting fermions with single-particle operator H_{Λ_L}, the finite-volume restriction of the single-particle operator H onto $\Lambda_L := (-L/2, L/2)^d$. If H' is a local perturbation of H, then we denote by $\mathbf{\Psi}_N^L$ the corresponding ground state of a N-particle system of non-interacting fermions with single-particle operator H'_{Λ_L}. Then the modulus of their ground-state overlap is given by

$$S_{N,L} := \big|\big\langle \mathbf{\Phi}_N^L, \mathbf{\Psi}_N^L \big\rangle\big|, \tag{1.1}$$

where the scalar product is on the fermionic N-particle Hilbert space. The macroscopic limit realizing a Fermi energy $E \in \mathbb{R}$ is then given by the joint limit $N, L \to \infty$ subject to $N/L^d \to \mathcal{N}(E) > 0$. Here $\mathcal{N}(E)$ denotes the integrated density of states of the unperturbed operator H at energy E. Physically it is expected that an asymptotic vanishing of the ground-state overlap is not restricted to free fermions but an intrinsic property of delocalized Schrödinger operators. This has rather recently been verified mathematically in [85, 51, 52]: The ground-state overlap decays algebraically in the macroscopic limit as long as the Fermi energy E belongs to the absolutely continuous spectrum of the operator H. Moreover, in more special one-dimensional situations exact asymptotics [83, 50] for the ground-state overlap have been obtained. In contrast to this, no attention has been paid to the asymptotic behaviour of the ground-state overlap for localized random Schrödinger operators in both, mathematics and physics, for a long time. From a physics point of view it was established wisdom that the effect would not occur in this case [53]. Altogether, the common belief for a random Schrödinger operator H_ω (such as, say, the lattice Anderson model H_ω^A) was

$$\lim_{N/L^d\to\mathcal{N}(E)} S_{\omega,N,L} \quad \begin{cases} = 0 & \text{if } E \text{ is within the delocalized spectral region} \\ \neq 0 & \text{if } E \text{ is within the localized spectral region} \end{cases} \tag{1.2}$$

where the ω-subscript indicates that the ground-state overlap here is a random variable. The mathematical works [51, 52] have been sufficiently general to cover random Schrödinger operators via a pointwise (in the realization ω) application. If we are brave enough to expect absolutely continuous spectrum in the delocalized spectral region of a random Schrödinger operator, then the first point in (1.2) has already been established mathematically in those works. Concerning the second point in (1.2) it was only the recent studies [73, 34] which revealed that this picture has to be refined. Their non-rigorous analytical arguments and numerical evidence suggest that Anderson orthogonality does occur for random Schrödinger operators and a Fermi energy E that is within the localized spectral region, but with a probability strictly between 0 and 1. Hence the behavior (1.2) can only be expected to be true for the disorder-averaged ground-state overlap. We clarify the picture by providing a mathematical analysis of the asymptotic behavior of the ground-state overlap in the macroscopic limit for localized continuum random Schrödinger operators. The main result is that the finite-volume ground-state overlap almost surely converges towards what we dub the infinite-volume ground-state overlap $S_\omega(E)$. This random variable can be expressed in terms of spectral projections up to energy E of the single-particle operators H_ω and H'_ω. Subsequently we identify the effect which can cause an asymptotic vanishing of the ground-state overlap (i.e. $S_\omega(E) = 0$) in this situation: A non-vanishing spectral shift function. Finally, we prove that in a neighborhood at the bottom of the spectrum of the random Schrödinger operator H_ω both absence and presence of Anderson orthogonality occur with positive probability. This particularly proves that the picture from (1.2) only applies if the disorder average of $S_{\omega,N,L}$ is considered.

The next topic is not directly related to the spectral structure of localized random Schrödinger operators but is of independent mathematical interest. Moreover, it has two direct applications which we depict below. For the lattice Anderson model H_ω^A Wegner [127] proved that the integrated density of states $\mathcal{N}$ is Lipschitz continuous. Hence its derivative, the density of states $\mathcal{N}'$, is well defined and bounded almost everywhere. The argument he provided entails the slightly stronger statement that the expected number of eigenvalues of H_{ω,Λ_L} in the energy interval $J \subset \mathbb{R}$ is bounded above by $\sim |J|L^d$:

$$\mathbb{E}\big[\operatorname{tr} \mathbb{1}_J(H_{\Lambda_L})\big] \leqslant C|J|L^d, \tag{1.3}$$

where $\mathbb{E}\,[\,\cdot\,]$ denotes the disorder average and tr is the trace. This bound, nowadays dubbed *Wegner estimate*, has since been vastly generalized due to its fundamental role as a resonance killer within the multiscale analysis depicted above. In the same work [127] Wegner also provided a semi-rigorous argument that the expected number of eigenvalues of H_{ω,Λ_L} in the interval $J \subset \mathbb{R}$ is likewise bounded below by $\sim |J|L^d$:

$$\mathbb{E}\big[\operatorname{tr} \mathbb{1}_J(H_{\Lambda_L})\big] \geqslant C|J|L^d \tag{1.4}$$

for a constant $C > 0$. We refer to such a bound as a *reverse Wegner estimate*. For the lattice Anderson model the reverse Wegner estimate has been verified rigorously in [62, 71]. We extend this result to continuum random Schrödinger operators. Due to technical reasons we are unfortunately only able to prove a reverse Wegner estimate within the energy region of localization. In a similar vein as the Wegner estimate implies existence and boundedness of the density of states $\mathcal{N}'$ the reverse Wegner estimate implies a strictly positive lower bound on $\mathcal{N}'$. Hence $\mathcal{N}' > 0$ almost everywhere in the energy region for which the reverse Wegner estimate holds. We note that ergodicity of the random Schrödinger operator H_ω yields that its spectrum is almost surely non-random and agrees with the support of $\mathcal{N}'$. But this alone does not yet imply that $\mathcal{N}' > 0$ almost everywhere within the almost surely non-random spectrum. A counterexample for this implication is provided by the indicator function of the complement of a "fat" Cantor set (that is, of a nowhere dense set with positive Lebesgue measure). We apply the strict positivity of the density of states at two places. First, it is central for the proof that Anderson orthogonality does occur in the localized spectral region. Secondly, strict positivity $\mathcal{N}'(E)$ at energy E is going to be a prerequisite for our result on the local eigenvalue point process around energy E below.

A topic of similar flavor to the Anderson orthogonality is the leading-order asymptotic term of the (von Neumann) entanglement entropy for a system of non-interacting fermions with single-particle Schrödinger operator H. For fixed $E \in \mathbb{R}$ it is given by

$$S_L(E) = \operatorname{tr} h(\mathbb{1}_{(-\infty,E]}(H)_{\Lambda_L}). \tag{1.5}$$

Here $h(x) := -x \log x - (1-x)\log(1-x)$ for $x \in (0,1)$ and $h(x) = 0$ for $x \in \{0,1\}$, and $\mathbb{1}_{(-\infty,E]}(H)_{\Lambda_L}$ denotes the restriction of the spectral projection up to energy E onto the box Λ_L. The leading order asymptotic term of $S_L(E)$ in L is then expected to be sensitive to the spectral structure of H at energy E. More precisely, an area law (i.e. a leading order scaling $\sim L^{d-1}$) is expected if E is within a localized energy region while a logarithmically enhanced area law is expected if E is within the delocalized spectrum. For a random Schrödinger operator H_ω this can be summarized as

$$S_{\omega,L}(E) \sim \begin{cases} L^{d-1}\log L & \text{if } E \text{ is within the delocalized spectral region} \\ L^{d-1} & \text{if } E \text{ is within the localized spectral region} \end{cases} \tag{1.6}$$

where $\sim$ refers to leading order scaling (neglecting constants). The ω-subscript here again indicates that the entanglement entropy is a random variable. From an abstract point of view, $S_{\omega,L}(E)$ from (1.5) is a special case of the expression $\operatorname{tr} h(g(H_\omega)_{\Lambda_L})$ for suitable functions g and h, where h is referred to as a test function. In contrast to (1.6) the leading order term for this trace typically is L^d, the so-called density of states term. Its absence in (1.6) is a consequence of $h \circ \mathbb{1}_{(-\infty,E]} = 0$. In this more general context the general belief is that for a logarithmically enhanced subleading term to pop up, a function g with a discontinuity within a delocalized spectral region of the Hamiltonian H_ω is needed.

Such trace asymptotics have been studied extensively in the context of translation-invariant systems such as Toeplitz matrices and Wiener-Hopf operators and were pioneered by Szegő in 1915 [122]. In the context of Toeplitz matrices he proved a leading order asymptotic formula and, a couple of years later, established a two-term asymptotic formula under certain smoothness assumptions [123]. This was later complemented in [45, 15] where it was shown that the order of the subleading term crucially depends on those smoothness assumptions. More precisely, let $a : \mathbb{T} \to \mathbb{R}$ be a strictly positive function on the torus with Fourier coefficients $(a_k)_{k\in\mathbb{Z}}$. By T_L we denote the finite-volume truncation of the Toeplitz matrix $T := (a_{j-k})_{j,k\in\mathbb{N}}$. Moreover, let h be a sufficiently regular (say, analytic) test function.

In the context of trace asymptotics, and by omitting details, the above references can be summarized as

$$\operatorname{tr} h(T_L) = \begin{cases} L(h \circ a)_0 + B_1 + \mathrm{o}(1) & \text{if, say, } a \in \mathcal{C}^2 \\ L(h \circ a)_0 + \widetilde{B}_1 \log L + \mathrm{o}(\log L) & \text{if } a \text{ possesses a jump discontinuity} \end{cases} \tag{1.7}$$

with coefficients B_1 and $\widetilde{B}_1$ that depend on a and h. For further discussion of asymptotic expansions for determinants and traces of Toeplitz matrices, we refer to [20, 84, 33]. Our focus is on the multi-dimensional continuum version of the problem and full asymptotic expansions in case the subleading term is not logarithmically enhanced. For a symbol $a : \mathbb{R}^d \to \mathbb{C}$ and a domain $\Omega \subset \mathbb{R}^d$, with $\Omega_L := L\Omega$, the truncated Wiener-Hopf operator $W_L(a) := \mathcal{X}_{\Omega_L}\mathcal{F}^* a \mathcal{F}\mathcal{X}_{\Omega_L}$ is the multi-dimensional continuum analog of the truncated Toeplitz matrix A_L from above. Here $\mathcal{F}$ denotes the Fourier transform and $\mathcal{X}_{\Omega_L}$ is the spatial projection onto Ω_L. If we assume, for example, that the domain Ω is piecewise smooth and the symbol a is smooth and decaying sufficiently fast at infinity, then a natural analog of the asymptotic formula (1.7) holds for $W_L(a)$ and sufficiently smooth test functions h with $h(0) = 0$. Now the leading term is of order L^d and the subleading term is of order L^{d-1} with an error term of order $\mathrm{o}(L^{d-1})$. As in the one-dimensional Toeplitz case, the subleading term depends on the smoothness of the symbol a. Again, an additional term of order $L^{d-1}\log L$ emerges if the symbol possesses jump-type discontinuities [86, 129, 59, 116, 117]. Motivated by its connection with the bipartite entanglement entropy, those results have recently been extended to non-smooth test functions h, see [87, 88] and references therein. If the symbol a is smooth and the domain Ω is not only piecewise smooth but smooth, then one can go beyond the subleading term [107, 130]. For $M \in \mathbb{N}$

$$\operatorname{tr} h(W_L(a)) = \sum_{m=0}^{M} L^{d-m} B'_m + \mathrm{o}(L^{d-M}) \tag{1.8}$$

holds for recursively defined coefficients $B'_m = B'_m(a, h, \Omega)$. A similar asymptotic expansion for an N-dimensional analog of Toeplitz matrices on rectangular domains was established in [125].

Recently, subleading order trace asymptotics as in (1.7)–(1.8) have been studied for Schrödinger operators with non-trivial potential [100, 78, 43, 103], which all fit in the larger framework of ergodic operators. For a, say, $\mathbb{Z}^d$-ergodic and self-adjoint operator H_ω on $L^2(\mathbb{R}^d)$, the natural object to consider is the trace of the operator $h(g(H_\omega)_{\Lambda_L})$ for a suitable function g. In [78] the asymptotic behavior of such traces was studied for one-dimensional random and quasiperiodic Schrödinger operators on the lattice. For the random Anderson model and concrete choices of functions g and h the authors showed that the leading order term, which is of order L, obeys a central limit theorem. Hence, an additional Gaussian fluctuation of order $\sqrt{L}$ can contribute to the asymptotic expansion. Moreover, it was exemplified in [78] that spectral localization can suppress the logarithmic enhancement of the subleading term. The latter point was generalized in [100, 43] to the lattice Anderson model in any dimension and larger classes of functions g and h. On the other hand, in [103] it was proved that the logarithmic enhancement of the subleading term does occur for one-dimensional periodic continuum Schrödinger operators. Those mathematical findings are in line with the above described heuristic picture that a logarithmically enhanced subleading term is a consequence of a discontinuity of the function g within the delocalized energy region of H_ω.

We prove a full trace asymptotics as in (1.8) for a general class of self-adjoint $\mathbb{Z}^d$-ergodic operators H_ω on $L^2(\mathbb{R}^d)$. Besides mild general requirements we only impose sufficiently fast decay of the operator kernel of $g(H_\omega)$, which can be checked directly in many situations. Typically, it either stems from spectral properties of the operator H_ω, such as spectral localization, or smoothness properties of the function g. We confine ourselves to boxes as scaling domains, which is the generic setup for a $\mathbb{Z}^d$-ergodic model. Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded and compactly supported function such that the operator kernel of $g(H_\omega)$ decays sufficiently fast. Then, for sufficiently smooth functions $h : \mathbb{R} \to \mathbb{C}$ with $h(0) = 0$

$$\mathbb{E}\big[\operatorname{tr} h(g(H)_{\Lambda_L})\big] = \sum_{m=0}^{d} A_m (2L)^{d-m} + \mathcal{O}(L^{-\tau}). \tag{1.9}$$

Moreover, $\tau > 0$ depends on the rate of decay of the operator kernel of $g(H)$ and the regularity of h. In contrast to the expansion (1.8) for smooth domains, this expansion terminates at constant order. The coefficients A_m can be represented as ω-averaged traces of differences of operators of the form $h(g(H_\omega)_G)$, $G \subseteq \mathbb{R}^d$. For more explicit formulas of the coefficients one would have to confine to more concrete models. This can already be seen at the leading-order coefficient A_0, which can be interpreted as a density of states term.

We already highlighted above that local eigenvalue statistics are a frequently applied litmus test for the spectral structure. H_{Λ_L} again denotes the finite-volume restriction of a Schrödinger operator to the box Λ_L and E is a fixed reference energy. The point process of the centered and appropriately rescaled eigenvalues is

$$\xi_E^L = \sum_n \delta_{L^d(\lambda_n^L - E)}, \tag{1.10}$$

where λ_n^L are the eigenvalues of H_{Λ_L}, counted according to their multiplicity. If the energy E is within the localized spectral region, then ξ_E^L is expected to converge to a Poisson point process (Poi) in the macroscopic limit. On the other hand, in the presence of extended states it converges towards completely different processes such as the Gaussian orthogonal ensemble (GOE). This duality is known as the *spectral statistics conjecture*. For a random Schrödinger operator this summarizes to the conjectured behavior

$$\lim_{L\to\infty} \xi_{E,\omega}^L = \begin{cases} \text{(Poi)} & \text{if } E \text{ is within the localized spectral region} \\ \text{(GOE)} & \text{if } E \text{ is within the delocalized spectral region} \end{cases} \tag{1.11}$$

where the above convergence of random processes is in distribution. The ω-subscript here again reflects the randomness induced by the random Schrödinger operator. Establishing mathematical proofs of Poisson statistics in the localized spectral region has been a topic of interest in the theory of random Schrödinger operators since its beginning. They were first proven rigorously for a one-dimensional model by Molchanov [94]. Due to the works [82, 22] the one-dimensional situation is nowadays rather well understood for both continuum and lattice random Schrödinger operators. For the higher-dimensional case $d \geqslant 2$ Minami [93] first proved Poisson statistics in the localized spectral region for the lattice Anderson model. The key technical ingredient of Minami's proof besides localization is a probabilistic estimate on the event that two or more eigenvalues of H^A_{ω,Λ_L} are located in an interval $J \subset \mathbb{R}$:

$$\mathbb{P}\big(\operatorname{tr} \mathbb{1}_J(H^A_{\Lambda_L}) \geqslant 2\big) \leqslant CL^{2d}|J|^2, \tag{1.12}$$

where $\mathbb{P}(\cdot)$ denotes the probability measure associated to the random Schrödinger operator (i.e. for the lattice Anderson model the joint probability measure of the random couplings $(\omega_k)_{k\in\mathbb{Z}^d}$). Such estimates are referred to as *Minami estimates*. Minami's original proof for (1.12) has subsequently been generalized [17, 58, 27, 124, 61] with the aim to incorporate more general models than the lattice Anderson model. But Minami's strategy of proof for (1.12) heavily relies on the fact that the random potential itself already satisfies the same bound: If we set $-\Delta = 0$ in H^A_ω, then the bound trivially holds. As a consequence, generalizations were limited to the neighborhood of the lattice Anderson model. We present a new approach towards Minami's estimate, and hence Poisson statistics, which is much more flexible than Minami's original method. For instance, it is applicable to the continuum Anderson model. We note however that this flexibility comes at a price: Its applicability is (in contrast to (1.12), which is valid for all intervals $J \subset \mathbb{R}$) limited to an interval at the bottom of the spectrum, which does in general not cover the complete energy region of spectral localization. The starting point of our argument is an ever so slight switch of perspective: Instead of directly proving a Minami estimate, we focus on the following seemingly weaker consequence:

$$\mathbb{P}\big(\min_{i<j} |\lambda^L_i - \lambda^L_j| < \delta\big) \leqslant CL^{2d}\delta \tag{1.13}$$

for $\delta > 0$ and $L > 0$, where $(\lambda^L_{\omega,i})_i$ denote the random eigenvalues of H^A_{ω,Λ_L}. We refer to such an estimate as *level-spacing estimate* in the following. The estimate (1.13) follows from (1.12) by covering the spectrum of H^A_{ω,Λ_L} with intervals J of length $\sim \delta$. We argue in the opposite direction. First, we establish a weak version of (1.13) and then argue that under certain (not too wild) regularity assumptions on the random Schrödinger operator one can also deduce a weak version of (1.12) from (1.13). The rider weak here refers (besides the restriction to an energy interval at the bottom of the spectrum) to a much worse decay in $\delta \ll 1$. More precisely, the right hand side of (1.13) is bounded by $\leqslant CL^{2d}|\log\delta|^{-K}$ where K is arbitrarily large, but the constant C depends on K. That such an estimate is still strong enough to yield information on the eigenvalue distribution in the macroscopic limit can be seen as follows: Bulk eigenfunctions of H_{ω,Λ_L} in the energy region of spectral localization converge exponentially fast to the corresponding infinite-volume eigenfunctions of H_ω. The same is true for the respective eigenvalues. Hence, considering scales $\delta_L := e^{-\sqrt{L}}$ in the above estimates yield meaningful information on the macroscopic limit. But for this choice of δ_L we still have rapid decay of the probability above. Apart from its application to local eigenvalue statistics, a level-spacing estimate is also of independent interest. It implies, for instance, simplicity of point spectrum [81]. In a more general context level spacing is also expected to play an important role with regard to many-body localization studies for an interacting electron gas in a random environment. In this context, perturbative approaches [46, 9, 57, 14, 68] suggest the existence of a many-body localized phase for one-dimensional spin systems in the presence of weak interactions. Localization of many-body systems is a topic of current research in mathematical quantum mechanics. For recent developments we refer to [1, 110, 91, 16, 42, 41] and references therein.

Structure of the thesis. The remainder of the thesis is intended to be readable independently of the introduction. This inevitably induces some redundancy. I'm sorry for any inconveniences!

The next chapter starts with a short introduction to ergodic operators and continuum random Schrödinger operators. Afterward, a short description of what we consider our main results is given, formulated in the framework of a particularly simple concrete example of a continuum random Schrödinger operator. In the Chapters 3 - 6 we then present the four topics described in the introduction in full detail. Each of the chapters is arranged similarly. First, we state and discuss the results. An emphasis is put on technical aspects. Afterwards, we give a brief heuristic justification of the main results and describe the key steps leading to their proofs. In most cases this part also contains a discussion of related open problems. Finally, we provide proofs of the results.

Some methods are applied repeatedly throughout the thesis, but in slightly different contexts. One could prove them for a unified abstract setup. We did however not attempt to do so since on the one hand no new insights are provided and on the other hand the space saved this way would probably not compensate for the emerging (additional?) lack of readability. We did, however, allow ourselves to only execute repeatedly used arguments in full detail once. The instance where the argument is proved in detail is not necessarily the first instance where it appears but rather the place where it from my point of view most generically fits.

A word on notation. We try to follow mainstream notational conventions. Notation is briefly described whenever first introduced (at least if it is not mathematical basis notation). For the most part we try to maintain notation throughout the whole thesis, with some exceptions: The (in three out of four cases random) operator H_ω is specified at the beginning of each of the Chapters three to six. Moreover, slightly different additional assumptions (V_4) are introduced at the beginning of the Chapters 4 and 6. In Subsection 6.4 the notation resets almost completely as we consider a lattice model. Moreover, we occasionally recall some earlier introduced notation at the beginning of a new chapter. Given the introductory Chapter 2, each of the subsequent chapters is intended to be readable without having read the previous chapters. In addition, a bibliography is added at the end of the thesis. The value of constants is intended to reset in each theorem, remark and proof. Within a theorem or proof we typically number constants by $C_1, C_2, ...$ or $C, C', C'', ...$ and occasionally indicate their dependence on relevant parameters a, b, c by writing, say, $C_1 = C_{1,a,b,c}$. The usage of brackets follows a vague "whatever seems more readable" approach. If an expression is unambiguous without brackets then they are typically omitted. This also applies for a function and its argument. An example is the trace $\operatorname{tr} A$ of an operator A as in (1.12). Basically everything is assumed to be measurable. For instance, sets $G \subset \mathbb{R}^n$ are all required to be measurable and we apply the same convention for functions. Finally, operators are always assumed to be linear and closed.

CHAPTER 2

Model and main results

Context: The first section is a review of well-known properties of ergodic and random Schrödinger operators. Detailed references are included. The second section is an overview of what I consider the main results of this thesis. It contains material from the publications [38, 37, 36, 35] which were written in collaboration with (in alphabetic order): **Alexander Elgart, Martin Gebert, Peter Hislop, Abel Klein** and **Peter Müller**.

Content: We first review the models which we most frequently work with: Ergodic operators on continuum space $\mathbb{R}^d$ and in particular alloy-type continuum random Schrödinger operators. The first part on ergodic operators mainly follows [26, 99]. For the second part on continuum random Schrödinger operators we mostly directly cite the research literature but refer to [120, 26, 75, 60, 126] for monographs and reviews which cover certain aspects of continuum random Schrödinger operators. For a gentle introduction to random Schrödinger operators on the lattice we refer to the reviews [64, 76, 121], the paper [4] and the recent monograph [8] for further references.

Subsequently we present our main results in the context of a concrete and technically uncomplicated continuum random Schrödinger operator. The intend is to describe the essence of each topic without caring too much about details and elaborate formulations of the results for now. The discussion of the results is reduced to a minimum at this point. Heuristic discussions can be found in the introduction and the respective sections concerned with the proof's ideas. Technical remarks are likewise included in the Chapters 3 - 6. By discussing the results in this context we pay a price in terms of generality. For example, the asymptotic expansion presented in Section 2.2.3 can be generically formulated in the context of general $\mathbb{Z}^d$-translation invariant operators. For more elaborate formulations of the results we also refer to the Chapters 3 - 6.

2.1. Ergodic and continuum random Schrödinger operators in a nutshell

This thesis is for the most part concerned with $\mathbb{Z}^d$-ergodic operators on the Hilbert space $L^2(\mathbb{R}^d)$ of square-integrable functions $\psi : \mathbb{R}^d \to \mathbb{C}$. More precisely, let $(\Omega, \mathbb{P})$ be a probability space and

$$\begin{aligned} H : \Omega &\to \mathcal{L}_{\text{sa}}(L^2(\mathbb{R}^d)) \\ \omega &\mapsto H(\omega) := H_\omega \end{aligned} \tag{2.1}$$

a map into the self-adjoint operators on $L^2(\mathbb{R}^d)$. With some abuse of notation we refer to both, the map H and its realizations H_ω, as the random operator. H is said to be measurable if the following holds: For any measurable and bounded function $f \in \mathcal{M}_b(\mathbb{R})$ and any two vectors $\phi, \psi \in L^2(\mathbb{R}^d)$ the map

$$\Omega \ni \omega \mapsto \langle \phi, f(H_\omega)\psi \rangle \tag{2.2}$$

is measurable. We are only considering $\mathbb{Z}^d$ translation-invariant operators here. Let $\{U_j\}_{j\in\mathbb{Z}^d}$ be the unitary family of translation operators on $L^2(\mathbb{R}^d)$, that is $(U_j\psi)(x) := \psi(x-j)$ for $\psi \in L^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Then H is $\mathbb{Z}^d$-translation invariant if there exists a family $\{T_j\}_{j\in\mathbb{Z}^d}$ of measure preserving transformations $T_j : \Omega \to \Omega$ such that for all $f \in \mathcal{M}_b(\mathbb{R})$ and all $\omega \in \Omega$

$$U_j f(H_\omega) U_j^* = f(H_{T_j\omega}). \tag{2.3}$$

The statement that (2.3) holds for all measurable bounded functions is in abuse of notation typically abbreviated as $U_j H_\omega U_j^* = H_{T_j\omega}$. If the measure preserving family $\{T_j\}_{j\in\mathbb{Z}^d}$ is in addition an ergodic subgroup of the automorphisms of Ω as a measurable space (where we omitted the sigma-algebra induced by $\mathbb{P}$ in notation), then H is called an ergodic operator. A well-known consequence is that the spectrum of the H_ω is almost surely non random: There exists a closed set $\Sigma \subset \mathbb{R}^d$ such that $\sigma(H_\omega) = \Sigma$ almost surely, where the left hand side denotes the spectrum of H_ω. The same also holds true for the spectral subsets obtained from Lebesgue decomposition of the spectral measure. Let's denote by $\sigma_{ac}(H_\omega), \sigma_{sc}(H_\omega), \sigma_{pp}(H_\omega)$ the absolutely continuous, singular continuous and pure point spectrum of H_ω respectively. Then there exist non-random closed sets $\Sigma_{ac}, \Sigma_{sc}, \Sigma_{pp}$ such that $\sigma_\star(H_\omega) = \Sigma_\star$ holds almost surely for $\star \in \{ac, sc, pp\}$. By $E_0 := \inf \Sigma$ we denote the almost sure infimum of the spectrum of H_ω.

Beside the results on Szegő-type asymptotics in Chapter 5, which are formulated for general $\mathbb{Z}^d$-translation invariant operators, we mostly work with the random Schrödinger operators

$$H_\omega := \mu H_0 + V_\omega = \mu H_0 + \sum_{k\in\mathbb{Z}^d} \omega_k V_k. \tag{2.4}$$

Here $\mu > 0$ is the coupling parameter of the non-random operator H_0 and the random potential V_ω. The random potential is of alloy-type, with random coupling constants $\Omega \ni \omega = (\omega_k)_{k\in\mathbb{Z}^d}$ taken from a probability space $(\Omega, \mathbb{P})$ specified below. In addition, we impose the following technical assumptions:

- (K) The unperturbed operator is given by $H_0 := -\Delta + V_{\text{per}}$, where Δ is the Laplacian and $V_{\text{per}} \in L^\infty(\mathbb{R}^d)$ is a deterministic, $\mathbb{Z}^d$-periodic and bounded background potential.
- (V_1) The single-site bump functions V_k are translates of a function V_0, $V_k = V_0(\cdot - k)$ for $k \in \mathbb{Z}^d$. Moreover, there exist $v_-, v_+ \in (0,1]$ and $r, R \in (0,\infty)$ such that
$$v_- \mathcal{X}_{B_r(0)} \leqslant V_0 \leqslant v_+ \mathcal{X}_{B_R(0)}. \tag{2.5}$$
- (V_2) The random potential satisfies a covering condition: For constants $V_-, V_+ \in (0,1]$ we have
$$V_- \leqslant \sum_{k\in\mathbb{Z}^d} V_k \leqslant V_+. \tag{2.6}$$
- (V_3) The family of random couplings $\omega = (\omega_k)_{k\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ is distributed according to $\mathbb{P} := \bigotimes_{\mathbb{Z}^d} P_0$. Here, the single-site probability measure P_0 is absolutely continuous with

respect to the Lebesgue measure. Its density ρ satisfies $\rho \in L^\infty(\mathbb{R})$ and $\operatorname{supp}(\rho) \subseteq [0,1]$.

Those assumptions ensure that almost surely with respect to the probability measure $\mathbb{P}$ the operator H_ω is a self-adjoint operator with domain $\mathcal{D}(H_\omega) = \mathcal{D}(\Delta) = H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ and core $\mathcal{C}_c^\infty(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. It can be verified directly that the so-defined map $H : \Omega \to \mathcal{L}_{\mathrm{sa}}(L^2(\mathbb{R}^d))$ is measurable and ergodic with respect to the unitary group of $\mathbb{Z}^d$-translations. In addition to (V_1)–(V_3) we occasionally introduce additional assumptions on the single-site probability density ρ in later chapters (such as, e.g., an additional regularity assumption in Chapter 6). Not all of the assumptions above are essential. For example, the constraints $v_+, V_+ \leqslant 1$ and $\operatorname{supp}(\rho) \subset [0,1]$ are made for convenience. Moreover, in many cases (V_2) can be dropped completely since the assumption $V_- > 0$ can often be replaced by a unique continuation argument. The key technical tools for continuum random Schrödinger operators are also known for way less regular probability distributions than what we imposed in (V_3). For example, weaker versions of Wegner's estimate and spectral localization are also known if P_0 is a discrete probability measure. However, regularity of the single-site probability measure enters independently for most of our main results.

We frequently deal with finite-volume restrictions of H. For an open set $G \subset \mathbb{R}^d$ let $H_{\omega,G}$ denote the restriction of H_ω to G with Dirichlet boundary conditions. Similarly, for a box $\Lambda_L := (-L/2, L/2)^d$ of side length L we set $H_{\omega,L} := H_{\omega,\Lambda_L}$, which can be written as

$$H_{\omega,L} = -\mu\Delta_L + \mu V_{\mathrm{per}}^L + \sum_{k\in\Gamma_L} \omega_k V_k^L \tag{2.7}$$

for $\Gamma_L := \mathbb{Z}^d \cap \Lambda_{L+2R}$ and V_{per}^L,V_k^L the restrictions of V_{per} and V_k to $L^2(\Lambda_L)$. If G is bounded then the resolvent of $H_{\omega,G}$ is almost surely compact. Hence, the spectrum is discrete. Since the operators are moreover bounded from below by $-\|V_{\mathrm{per}}\|$ we can denote by $\lambda_{\omega,1}^G \leqslant \lambda_{\omega,2}^G \leqslant \ldots$ the ascendingly ordered eigenvalues of $H_{\omega,G}$, where the eigenvalues are counted according to their multiplicity. The above remarks on measurability also apply for the finite-volume restrictions $H_{\omega,G}$. Together with the min-max characterization of the eigenvalues $\lambda_{\omega,i}^G$ this implies that the functions $\Omega \ni \omega \to \lambda_{\omega,i}^G$ are measurable as well. By similar arguments pretty much all the ω-dependent quantities considered here are measurable.

We next turn to two slightly different manifestations of the idea that for local (in energy) spectral properties of H_ω, disorder averaging is akin to energy averaging: Wegner's estimate and fractional moment bounds for the resolvent. If we choose the number of eigenvalues in an energy interval $[E-\varepsilon, E+\varepsilon]$ as such a local spectral property, then this correspondence suggests that its disorder average should scale as εL^d. Such bounds are nowadays dubbed Wegner estimates and were first proved for the lattice Anderson model in [127]. Later they were generalized substantially due to the central role they play in the multiscale analysis. For further references and more recent developments we refer to [28, 23, 108, 80]. The following version of Wegner's estimate holds for our model.

(W) For fixed $E > 0$ there exists a constant C such that

$$\mathbb{E}\big[\operatorname{tr} \mathbb{1}_I(H_L)\big] \leqslant CL^d|I| \tag{2.8}$$

for all intervals $I \subset [0,E]$.

One of its implications is regularity of the integrated density of states. Due to ergodicity of H_ω, the integrated density of states of H_ω

$$\mathcal{N}(E) := \lim_{L\to\infty} \frac{1}{|\Lambda_L|} \operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{\omega,L}) \tag{2.9}$$

is non-random and almost surely well-defined for all $E \in \mathbb{R}$ [26, 99]. Wegner's estimate then ensures that $\mathcal{N}$ is locally Lipschitz continuous and possesses a Lebesgue density $\mathcal{N}'$, the density of states of H_ω. On the other hand, fractional moment bounds for the resolvent have been investigated in the context of the fractional moment method, where they take over the role Wegner's estimate plays in the multiscale analysis [5, 3].

(FM) For every $0 < s < 1$ and every $E > 0$ there exists a constant C such that

$$\sup_{\substack{E' \leqslant E \\ \eta \neq 0}} \sup_{\substack{G \subseteq \mathbb{R}^d \\ \text{open}}} \sup_{a,b\in\mathbb{R}^d} \mathbb{E}\left[\|\mathcal{X}_a R_{E'+i\eta}(H_G)\mathcal{X}_b\|^s\right] \leqslant C. \tag{2.10}$$

The most prominent feature of random Schrödinger operators is the existence of an energy region which exhibits the expected behavior of an insulator. For instance, the spectrum within this energy region consists of point spectrum with exponentially decaying eigenfunctions. Such a spectral structure is commonly referred to as spectral localization. Both known methods to prove spectral localization for random Schrödinger operators in $d > 1$ dimensions, the multiscale analysis and the fractional moment method, have first been developed for the lattice Anderson model and were subsequently extended to continuum random Schrödinger operators. This generalization was first done in [29] and [3], respectively. For more recent developments we refer, e.g., to [23, 69, 40, 54, 67] (where I have to note that I did not yet fully understand [67]). As a matter of convenience we work with the technically slightly stronger output generated by the fractional moment method. Both approaches in their original formulation have in common that they perform most of the analysis for the resolvent, and subsequently prove spectral localization as a consequence of suitable decay bounds of the averaged and spatially localized resolvent. In most cases, localization proofs for continuum random Schrödinger operators are either only performed in a neighborhood of the bottom of the spectrum or written in terms of finite-volume criteria. To avoid an ungeneric restriction of our results to the bottom of the spectrum we next define the energy region of spectral localization Σ_{FMB} as the set of energies which satisfies the conclusion of the fractional moment method. For a, say, closed operator A let $R_z(A) := (A - z)^{-1}$ denote the resolvent at $z \in \mathbb{C} \setminus \sigma(A)$. By $\|\cdot\|$ we denote the operator norm and for $a \in \mathbb{R}^d$ we denote by $\mathcal{X}_a := \mathcal{X}_{Q_a}$ the $L^2(\mathbb{R}^d)$-projection onto the set $Q_a := a + \Lambda_1$.

(Loc) $E \in \Sigma_{\text{FMB}}$ if there exists a neighborhood $U_E \subset \mathbb{R}$ of E such that the following holds: For every $0 < s < 1$ there exist constants $C, \mu > 0$ such that for all $a, b \in \mathbb{R}^n$

$$\sup_{\substack{E' \in U_E \\ \eta \neq 0}} \sup_{\substack{G \subseteq \mathbb{R}^d \\ \text{open}}} \mathbb{E}\left[\|\mathcal{X}_a R_{E'+i\eta}(H_G)\mathcal{X}_b\|^s\right] \leqslant C\, \mathrm{e}^{-\mu|a-b|}. \tag{2.11}$$

In [3] the bound (2.11) is proven at the bottom of the spectrum: $[E_0, E_0 + \varepsilon] \subset \Sigma_{\text{FMB}}$ for a sufficiently small $\varepsilon > 0$. The authors however consider a boundary-adapted distance function in the exponent to take into account the possible occurrence of extended boundary states for an operator which is localized in the bulk. But for random Schrödinger operators without magnetic potentials the bound (2.11) also holds with the usual distance $|\cdot|$ in a neighborhood of the bottom of the spectrum [24]. We remark that, strictly speaking, [3, 24] both prove (2.11) for $0 < s < 1/3$. By means of the estimate (2.10) and an interpolation argument this yields (2.11) for any $0 < s < 1$ and s-dependent constants $C, \mu > 0$, see [6]. Even though we for the most part only work with the resolvent bound (2.11), we remark that it yields spectral localization in its strongest form [3]: If $I \subset \Sigma_{\mathrm{FMB}}$ is a compact interval, then there exist constants $C, \mu > 0$ such that

$$\sup_{\substack{G\subseteq\mathbb{R}^d\\ \text{open}}} \mathbb{E}\big[\sup_f \|\mathcal{X}_a f(H_G)\mathcal{X}_b\|\big] \leqslant C\,\mathrm{e}^{-\mu|a-b|}, \tag{2.12}$$

where the supremum in f is over the measurable functions $f : \mathbb{R} \to \mathbb{C}$ with $\|f\|_\infty \leqslant 1$ and $\operatorname{supp}(f) \subset I$. Moreover $I \cap \sigma(H_\omega) \subset \sigma_{pp}(H_\omega)$ holds almost surely and the eigenfunctions of H_ω corresponding to eigenvalues in I are exponentially decaying.

We close this section with the Combes-Thomas estimate. In contrast to the properties described above, which crucially rely on the randomness induced by the random potential V_ω, it is a statement about deterministic Schrödinger operators. But for convenience we only state it in the context of the random Schrödinger operator H_ω. Among the various formulations we stick to the one stated in [55, Cor. 1]. Their result is formulated for Schrödinger operators on $L^2(\mathbb{R}^d)$ but the argument extends to Schrödinger operators on $L^2(G)$ for arbitrary open sets $G \subseteq \mathbb{R}^d$ – see also [111]. By $\operatorname{dist}(\cdot,\cdot)$ we denote the distance between two points, a point and a set or two sets in $\mathbb{R}^n$ or $\mathbb{C}^n$ with respect to the maximum norm $|\cdot|$ (typically, $n \in \{1, d\}$).

(CT) For every compact set $K \subset \mathbb{C}$ there exist constants $C, \mu > 0$, which are independent of $G \subseteq \mathbb{R}^d$ and $\omega \in \Omega$, such that

$$\|\mathcal{X}_a R_z(H_{\omega,G})\mathcal{X}_b\| \leqslant \frac{C}{\operatorname{dist}(z, \sigma(H_\omega, G))}\,\mathrm{e}^{-\mu \operatorname{dist}(z,\sigma(H_\omega,G))|a-b|} \tag{2.13}$$

holds for all $z \in K \setminus \sigma(H_{\omega,G})$.

The Combes-Thomas estimate quantifies the common sense that eigenfunctions of Schrödinger operators decay exponentially in energetically unfavorable spatial regions. From this point of view, and in the context of random Schrödinger operators, the Combes-Thomas estimate (2.13) is a precursor of spectral localization. This relation is made precise by the multiscale analysis, which in one of its various versions states that (CT) $\wedge$ (W) $\Rightarrow$ (Loc).

2.2. Main results in a simplified setup

In this section we solely consider the random Schrödinger operator

$$H_\omega := -\Delta + V_\omega = -\Delta + \sum_{k\in\mathbb{Z}^d} \omega_k \mathcal{X}_k, \tag{2.14}$$

i.e. $V_{\mathrm{per}} = 0$, $\mu = 1$ and $V_k = \mathcal{X}_k$ is the multiplication operator of the box $\Lambda_1(k)$. In addition to (V_3), which ensures independence of the random couplings, we assume that their joint probability measure P_0 is distributed according to the uniform distribution on the interval $[0, 1]$. This in particular implies $\Sigma = [0, \infty)$ for the almost surely non-random spectrum and $E_0 = 0$.

2.2.1. Absence of Anderson orthogonality.

Based on joint work with Martin Gebert and Peter Müller.

For a fixed bounded and compactly supported function $W \in L_c^\infty(\mathbb{R}^d)$ with $W \geqslant V_0$ let $H'_\omega := H_\omega + W$. The assumption $W \geqslant V_0$ on the one hand ensures that the perturbation is sign-definite and non-trivial and on the other hand enters as a technical necessity in the third point of the theorem below. We denote the random eigenvalues of $H_{\omega,L}$ and $H'_{\omega,L}$ by

$$\lambda_{\omega,1}^L \leqslant \lambda_{\omega,2}^L \leqslant \cdots \qquad \text{and} \qquad \mu_{\omega,1}^L \leqslant \mu_{\omega,2}^L \leqslant \cdots, \tag{2.15}$$

repeated according to their respective multiplicities. The corresponding eigenfunctions are denoted by $(\varphi_{\omega,k}^L)_{k\in\mathbb{N}}$ and $(\psi_{\omega,k}^L)_{k\in\mathbb{N}}$. For fixed $E \in \mathbb{R}$, referred to as *Fermi energy* in this context, we define the finite-volume particle number

$$N_{\omega,L}(E) := \operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{\omega,L}). \tag{2.16}$$

This yields a particular realization of the macroscopic limit with particle density given by the integrated density of states of the unperturbed operator, $\lim_{L\to\infty} N_{\omega,L}(E)/|\Lambda_L| = \mathcal{N}(E)$. The respective ground states of the two non-interacting $N_{\omega,L}(E)$-particle fermionic systems with single-particle Schrödinger operators $H_{\omega,L}$ and $H'_{\omega,L}$ are given by the totally antisymmetrized and normalized tensor products

$$\boldsymbol{\Phi}_\omega^L := \varphi_{\omega,1}^L \wedge \cdots \wedge \varphi_{\omega,N_{\omega,L}(E)}^L \qquad \text{and} \qquad \boldsymbol{\Psi}_\omega^L := \psi_{\omega,1}^L \wedge \cdots \wedge \psi_{\omega,N_{\omega,L}(E)}^L. \tag{2.17}$$

Those vectors are elements of the totally antisymmetrized tensor-product Hilbert spaces $\bigwedge_{j=1}^{N_{\omega,L}(E)} L^2(\Lambda_L)$, a subspace of the $N_{\omega,L}(E)$-fold tensor product of the Hilbert space $L^2(\Lambda_L)$. The modulus of the *finite-volume ground-state overlap* is then defined as

$$S_{\omega,L}(E) := \big|\langle \boldsymbol{\Phi}_\omega^L, \boldsymbol{\Psi}_\omega^L\rangle\big|. \tag{2.18}$$

Anderson orthogonality (at the Fermi energy E) refers to a vanishing of the ground-state overlap $S_{\omega,L}(E)$ as $L \to \infty$.

Theorem 2.1. *For $E \in \Sigma_{\text{FMB}} \cap (0,\infty)$ there exists a random variable $S_\omega(E)$, specified in Chapter 3, such that the following holds.*

(i) *$L^1(\mathbb{P})$ convergence in the macroscopic limit:*

$$\lim_{L\to\infty} \mathbb{E}\big[|S_L(E) - S(E)|\big] = 0. \tag{2.19}$$

(ii) *The following equivalence holds almost surely:*

$$S_\omega(E) = 0 \iff \operatorname{tr}\big(\mathbb{1}_{(-\infty,E]}(H_\omega) - \mathbb{1}_{(-\infty,E]}(H'_\omega)\big) \neq 0. \tag{2.20}$$

In particular, the trace on the right hand side is almost surely well defined.

(iii) *There exists $E_1 > 0$ such that for almost every $E \in (0, E_1]$*

$$1 > \mathbb{P}\big(S(E) = 0\big) > 0. \tag{2.21}$$

A more or less explicit expression for $S_\omega(E)$ is given in Chapter 3. The right hand side of (2.20) is reminiscent of the spectral shift function (where we neglect that the spectral shift function is strictly speaking only specified up to Lebesgue almost every $E \in \mathbb{R}$). We prove in Chapter 3 that they indeed agree in the localized spectral region. The second part of the theorem hence relates the occurrence of Anderson orthogonality to a non-vanishing spectral shift function.

2.2.2. A lower bound on the density of states.

Based on joint work with Martin Gebert, Peter Hislop, Abel Klein and Peter Müller.

Instead of directly stating a lower bound on the density of states we present slightly stronger result: A lower bound on the expected number of eigenvalues of the operator $H_{\omega,L}$ in a small energy region $J \subset \mathbb{R}$. In the following we refer to such a bound as a *reverse Wegner estimate.*

Theorem 2.2. *Let $I \subset \Sigma_{\text{FMB}} \cap (0,\infty)$ be a compact interval. Then there exist constants $C, \mathcal{L} > 0$ such that*

$$\mathbb{E}\big[\operatorname{tr} \mathbb{1}_J(H_L)\big] \geqslant C|J|L^d \tag{2.22}$$

holds for all intervals $J \subset I$ and all $L > \mathcal{L}$.

The reverse Wegner estimate from (2.22) directly implies the corresponding lower bound $\mathcal{N}'(E) \geqslant C$ for almost every $E \in I$ for the density of states, see also Corollary 4.2. Hence $\mathcal{N}'(E) > 0$ holds for almost every $E \in \Sigma_{\text{FMB}} \cap (0,\infty)$. One would probably expect the theorem to hold for any compact interval $I \subset (0,\infty)$. We comment on this in detail in Section 4.2. Luckily though, both our applications of the result are restricted to the energy region of localization anyways: First, the proof of Theorem 2.1 (iii) relies on strict positivity of the density of states. Secondly, strict positivity $\mathcal{N}'(E) > 0$ enters as an assumption in Theorem 2.5 below.

2.2.3. Full Szegő-type trace asymptotics.

The generic context of this section's result is more general than what we specified above: First, it can be formulated for general ergodic operators, including, for example, Wiener-Hopf operators. Moreover, it only relies on sufficiently fast decay of the (averaged) operator kernel of $g(H_\omega)$ for a function $g : \mathbb{R} \to \mathbb{C}$ as in the theorem below. Such decay can stem from spectral localization of the operator H_ω or from smoothness properties of the function g. For the sake of illustration we stick to the setup specified above. Moreover, we only cover the case in which the operator kernel of $g(H_\omega)$ decays as a consequence of spectral localization. A more generic presentation of the following result can be found in Chapter 5.

Theorem 2.3. *Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{C}$ be two compactly supported and bounded functions subject to $\operatorname{supp}(g) \subset \Sigma_{\text{FMB}}$ and $h \in \mathcal{C}^{\lfloor 2r+2 \rfloor}(\mathbb{R})$ for $r > 2d$ with $h(0) = 0$. Then, as $2\mathbb{N} \ni L \to \infty$, the asymptotic expansion*

$$\mathbb{E}\big[\operatorname{tr} h\big(g(H)_{\Lambda_L}\big)\big] = \sum_{m=0}^{d} A_m (2L)^{d-m} + \mathcal{O}(L^{2d-r}) \tag{2.23}$$

holds for suitable coefficients A_m, $m = 0, ..., d$.

The coefficients A_m can be expressed more or less explicitly in terms of traces over model operators, see Chapter 5. Moreover, the restriction to length-scales $L \in 2\mathbb{N}$ is due to the $\mathbb{Z}^d$-ergodicity of our model.

2.2.4. Level spacing and Poisson statistics.

Based on joint work with Alexander Elgart.

As above let $\lambda^L_{\omega,1} \leqslant \lambda^L_{\omega,2} \leqslant \cdots$ denote the random eigenvalues of $H_{\omega,L}$, repeated according to their multiplicities. We quantify the minimal eigenvalue spacing, also referred to as the *level spacing*, of the operator $H_{\omega,L}$ in the energy interval $[0,E]$ by

$$\operatorname{spac}_E(H_{\omega,L}) := \inf\left\{|\lambda^L_{\omega,i} - \lambda^L_{\omega,j}| : i \neq j,\, \lambda^L_{\omega,i}, \lambda^L_{\omega,j} \leqslant E\right\} \tag{2.24}$$

for $E \in \mathbb{R}$. The threshold energy from Chapter 6 for the concrete model considered here is

$$E_{\mathrm{sp}} := \frac{\pi^2}{6}. \tag{2.25}$$

We suspect that this could be improved slightly, see Section 6.4. But our method is fundamentally committed to energies below $\pi^2/2$. This limitation is almost certainly suboptimal and we comment on it in more detail in Chapter 6. Our level-spacing estimate reads as follows.

Theorem 2.4. *For fixed $E \in (0, E_{\mathrm{sp}})$ and $K > 0$ there exist constants $C, \mathcal{L} > 0$ such that*

$$\mathbb{P}\left(\operatorname{spac}_E(H_L) < \delta\right) \leqslant CL^{2d}|\log\delta|^{-K} \tag{2.26}$$

holds for $L \geqslant \mathcal{L}$ and $0 < \delta < 1$.

The above probabilistic estimate is way weaker than what is known for the classical lattice Anderson model: In this case, the left hand side of (2.26) is bounded by $CL^{2d}\delta$. But the above estimates are still sufficient for the applications that we have in mind. That (2.26) is indeed strong enough to yield information on the eigenvalue distribution in the macroscopic limit can be seen as follows: Bulk eigenfunctions of $H_{\omega,L}$ in the energy region of spectral localization converge exponentially fast to the corresponding infinite-volume eigenfunctions of H_ω. The same is true for the respective eigenvalues. Hence, considering scales $\delta_L := e^{-\sqrt{L}}$ in the above estimates yields meaningful information on the macroscopic limit. But for this choice of δ_L we still have rapid decay of the probability above.

The main conclusion we draw from Theorem 2.4 are Poissonian local eigenvalue statistics in the energy region of localization. The point process of the rescaled eigenvalues of $H_{\omega,L}$ around a fixed reference energy $E \in \mathbb{R}$ is given by

$$\xi^L_{\omega,E}(B) := \operatorname{tr} \mathbb{1}_{E+L^{-d}B}(H_{\omega,L}) \tag{2.27}$$

for bounded, Borel-measurable sets $B \subset \mathbb{R}$.

Theorem 2.5. *Let $E \in (0, E_{\mathrm{sp}}) \cap \Sigma_{\mathrm{FMB}}$ such that the integrated density of states $\mathcal{N}$ is differentiable at E, with $\mathcal{N}'(E) > 0$. Then the point process $\xi^L_{\omega,E}$ converges weakly to the Poisson point process on $\mathbb{R}$ with intensity measure $\mathcal{N}'(E)dx$ as $L \to \infty$.*

CHAPTER 3

Absence of Anderson orthogonality

Context: The main results presented in this chapter coincide with the main results from [38], which was written in collaboration with **Martin Gebert** and **Peter Müller**. Most of the proofs presented in Sections 3.3-3.5 coincide to a large extent with the respective proofs from [38]. The presentation of both, results and proofs, has been streamlined and rearranged in order to fit in the overall picture of this thesis.

Content: This chapter deals with the phenomenon of Anderson orthogonality within the localized spectrum of a (continuum) random Schrödinger operator. Most interestingly, we relate the appearance of Anderson orthogonality to the spectral shift function and prove that (at least at the bottom of the spectrum) both Anderson orthogonality and its absence do occur with non-zero probability. This partially confirms the recent non-rigorous findings [73, 34]. The chapter also contains two additional results. One of them is a collection of detailed bounds on the effect of a perturbation on the localized spectral region, Theorem 3.5. Even though those bounds do not yield new conceptual insights we view them as a useful technical tool. Apart from their direct applications in the present context they also yield a streamlined proof of Lemma 6.24 which was originally proved in [27] for continuum random Schrödinger operators. Moreover, a variant of such bounds enters the proof of the reverse Wegner estimate in Chapter 4. The other additional result is a description of the behavior of the spectral shift function in the localized energy region.

3.1. Discussion of results

We work with the random Schrödinger operator

$$H_\omega = H_0 + V_\omega = -\Delta + V_{\mathrm{per}} + V_\omega \tag{3.1}$$

from Section 2.1, subject to the assumptions (V_1) - (V_3) and with the choice $\mu = 1$. The choice $\mu = 1$ is solely for convenience. Most of the results in this chapter however only require the assumptions (V_1) and (V_2). For a compactly supported and bounded potential $W \in L^\infty_c(\mathbb{R}^d)$ we define the perturbed random Schrödinger operator

$$H'_\omega := H_\omega + W. \tag{3.2}$$

In Section 2.1 we reviewed standard properties of the random Schrödinger operator H_ω. Here we only remark that analogous results hold for the perturbed operator. Either the results are already formulated for non-ergodic random Schrödinger operators in the references given in Section 2.1 or the proofs directly extend to the present case of a local perturbation. More precisely, the Combes-Thomas estimate (CT), the Wegner estimate (W) and the finiteness of fractional moments (FM) hold for the operator H'_ω with constants that depend on W through $\operatorname{supp}(W)$ and $\|W\|_\infty$.

The notation for eigenvalues and eigenfunctions is as in Chapter 2: The eigenvalues of $H_{\omega,L}$ and $H'_{\omega,L}$ are denoted by

$$\lambda^L_{\omega,1} \leqslant \lambda^L_{\omega,2} \leqslant \cdots \qquad \text{and} \qquad \mu^L_{\omega,1} \leqslant \mu^L_{\omega,2} \leqslant \cdots , \tag{3.3}$$

with corresponding eigenfunctions $(\varphi^L_{\omega,k})_{k\in\mathbb{N}}$ and $(\psi^L_{\omega,k})_{k\in\mathbb{N}}$. They are as usual repeated according to their respective multiplicities. For $N \in \mathbb{N}$ we consider two non-interacting N-particle fermionic systems with single-particle Schrödinger operators $H_{\omega,L}$ and $H'_{\omega,L}$. The corresponding respective Schrödinger operators are

$$\boldsymbol{H}^{(\prime)}_{\omega,N,L} := \sum_{j=1}^{N} \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{(j-1)\text{times}} \otimes H^{(\prime)}_{\omega,L} \otimes \underbrace{\mathbb{1}\otimes\cdots\otimes\mathbb{1}}_{(N-j)\text{times}}, \tag{3.4}$$

acting on the totally antisymmetrized tensor-product space $\mathcal{H}_{N,L} := \bigwedge_{j=1}^N L^2(\Lambda_L)$, which is a Hilbert space with respect to the canonically induced scalar product. The respective ground states of $\boldsymbol{H}_{\omega,N,L}$ and $\boldsymbol{H}'_{\omega,N,L}$ are then given by the totally antisymmetrized and normalized tensor products

$$\boldsymbol{\Phi}^L_{\omega,N} := \varphi^L_{\omega,1}\wedge\cdots\wedge\varphi^L_{\omega,N} \qquad \text{and} \qquad \boldsymbol{\Psi}^L_{\omega,N} := \psi^L_{\omega,1}\wedge\cdots\wedge\psi^L_{\omega,N}. \tag{3.5}$$

The *modulus* of their scalar product is given by

$$S_{\omega,N,L} := \left|\left\langle \boldsymbol{\Phi}^L_{\omega,N}, \boldsymbol{\Psi}^L_{\omega,N}\right\rangle\right| = \left|\det\begin{pmatrix} \langle\varphi^L_{\omega,1},\psi^L_{\omega,1}\rangle & \cdots & \langle\varphi^L_{\omega,1},\psi^L_{\omega,N}\rangle \\ \vdots & & \vdots \\ \langle\varphi^L_{\omega,N},\psi^L_{\omega,1}\rangle & \cdots & \langle\varphi^L_{\omega,N},\psi^L_{\omega,N}\rangle \end{pmatrix}\right|, \tag{3.6}$$

where the equality follows from the Leibniz formula for determinants. The subsequent analysis crucially relies on the algebraic identity

$$S_{\omega,N,L} = \det\big(\mathbb{1} - (P_{\omega,N,L} - Q_{\omega,N,L})^2\big)^{1/4} \tag{3.7}$$

for the ground-state overlap, where we abbreviated (employing Dirac notation)

$$P_{\omega,N,L} := \sum_{j=1}^{N} |\varphi^L_{\omega,j}\rangle\langle\varphi^L_{\omega,j}| \quad \text{and} \quad Q_{\omega,N,L} := \sum_{k=1}^{N} |\psi^L_{\omega,k}\rangle\langle\psi^L_{\omega,k}|. \tag{3.8}$$

Its proof is contained in Lemma 3.16 below. The determinant in (3.7) is interpreted as a Fredholm determinant, which is well-defined since $(P_{\omega,N,L} - Q_{\omega,N,L})$ is a finite rank operator, hence in particular Hilbert-Schmidt [105, Sect. XIII.17]. We realize the macroscopic limit by fixing the particle number

$$N_{\omega,L}(E) := \operatorname{tr}\mathbb{1}_{(-\infty,E]}(H_{\omega,L}) \tag{3.9}$$

in the volume Λ_L, where $E \in \mathbb{R}$ is referred to as the *Fermi energy*. The limit $L \to \infty$ is then a particular realization of the macroscopic limit, with particle density given by the integrated density of states of H_ω. For this choice of particle number sequence the *finite-volume ground-state overlap* is given by

$$S_{\omega,L}(E) := \begin{cases} S_{\omega,N_{\omega,L}(E),L}, & \text{if} \quad N_{\omega,L}(E) \in \mathbb{N}, \\ 1, & \text{if} \quad N_{\omega,L}(E) = 0. \end{cases} \tag{3.10}$$

Motivated by identity (3.7) we define the *infinite-volume ground-state overlap*

$$S_\omega(E) := \det\big(\mathbb{1} - \big(\mathbb{1}_{(-\infty,E]}(H_\omega) - \mathbb{1}_{(-\infty,E]}(H'_\omega)\big)^2\big)^{1/4}, \tag{3.11}$$

which is interpreted as $S_\omega(E) = 0$ in case $\mathbb{1}_{(-\infty,E]}(H_{\omega,L}) - \mathbb{1}_{(-\infty,E]}(H'_{\omega,L})$ is not a Hilbert-Schmidt operator. We are now ready to state this chapter's main result, where we again denote by $\mathcal{N}'$ the density of states of H_ω.

Theorem 3.1 (Anderson orthogonality). (i) *For all* $E \in \Sigma_{\text{FMB}} \cap \operatorname{int}(\Sigma)$ *we have*

$$\lim_{L\to\infty} \mathbb{E}\big[|S_L(E) - S(E)|\big] = 0. \tag{3.12}$$

(ii) *Let* $E \in \Sigma_{\text{FMB}}$. *Then almost surely*

$$S_\omega(E) = 0 \iff 1 \in \sigma\Big(\big(\mathbb{1}_{(-\infty,E]}(H_{\omega,L}) - \mathbb{1}_{(-\infty,E]}(H'_{\omega,L})\big)^2\Big). \tag{3.13}$$

(iii) *If the perturbation is sufficiently large,* $W \geqslant cV_0$ *for some* $c > 0$, *then*

$$\mathbb{P}\big(S(E) = 0\big) > 0 \qquad \text{for almost every } E \in \Sigma_{\text{FMB}} \cap \{E' \in \mathbb{R} : \mathcal{N}'(E') > 0\}. \tag{3.14}$$

(iv) *Assume that the perturbation is sufficiently large,* $W \geqslant cV_0$ *for some* $c > 0$, *and that assumption* (V_4) *from Chapter 4 holds. Then there exists* $E_1 > E_0$ *such that*

$$1 > \mathbb{P}\big(S(E) = 0\big) > 0 \qquad \text{for almost every } E \in (E_0, E_1]. \tag{3.15}$$

Remarks 3.2. (i) Corollary 4.2 shows that the energy region on the right hand side of (3.14) is not too small and almost agrees with Σ_{FMB}.

(ii) The assumption $W \geqslant cV_0$ is slightly annoying. Clearly W is not allowed to vanish identically for the points three and four. But that W has to be lower bounded by a single-site potential seems to be an artifact of the technical (and not very smooth) proof of the third point of the theorem.

(iii) In order to prove the first statement of the theorem we verify the following slightly stronger pointwise (in ω) statement: For $E \in \Sigma_{\text{FMB}} \cap \operatorname{int}(\Sigma)$ and a sequence $(L_n)_{n\in\mathbb{N}}$ of length scales with $L_n/\log n \to \infty$ we almost surely have

$$\lim_{n\to\infty} S_{\omega,L_n}(E) = S_\omega(E). \tag{3.16}$$

(iv) For sign-definite perturbations W, the condition (3.13) for Anderson orthogonality to occur in the region of complete localization is almost surely equivalent to

$$S_\omega(E) = 0 \iff \operatorname{tr}\big(\mathbb{1}_{(-\infty,E]}(H_{\omega,L}) - \mathbb{1}_{(-\infty,E]}(H'_{\omega,L})\big) \neq 0. \tag{3.17}$$

Theorem 3.5 shows that the right hand side is almost surely well-defined for $E \in \Sigma_{\text{FMB}}$ and Theorem 3.3 relates it to the spectral shift function. The equivalence (3.17) follows from Theorem 3.3 and the analysis in [13], which is reviewed in Lemma 3.15 below.

Our result can be compared to the behavior of the ground-state overlap in a (hypothetical) delocalized spectral region. For simplicity, let $W \geqslant 0$. Suppose there is a spectral interval $J \subset \Sigma$ such that H_ω almost surely has, say, absolutely continuous spectrum in J. Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of lengths such that $L_n/e^{n^\alpha} \to \infty$ as $n \to \infty$ for some $\alpha > 1$. Then the results from [51, 52] apply pointwise (in ω) to the operators H_ω and H'_ω. Moreover, the choice of rapidly growing length scales L_n avoids the necessity of passing to a subsequence in their results. We infer that almost surely and for almost every $E \in J$

$$S_{\omega,L_n}(E) \leqslant L_n^{-0.5\gamma_\omega(E)+o(L_n^0)} \qquad \text{as } n \to \infty. \tag{3.18}$$

The decay exponent

$$\gamma_\omega(E) := \frac{1}{2\pi^2} \Big\| \arcsin \Big| \frac{\mathbb{1} - \mathbb{S}_{\omega,E}}{2} \Big| \Big\|_2^2 \tag{3.19}$$

relates to the energy-dependent scattering matrix $\mathbb{S}_{\omega,E}$ and is strictly positive if the perturbation W causes non-trivial scattering at energy E. This demonstrates on a mathematical level that the limiting behaviour of the averaged ground-state overlap is indeed a reasonable soft criterion for the spectral structure of H_ω at energy E.

The parts (ii)-(iv) from Theorem 3.1 above partially rely on an analysis of the spectral shift function in the localized energy region. Let's first introduce the relevant quantities. For self-adjoint operators A and B on a Hilbert space $\mathcal{H}$ and $E \in \mathbb{R}$, we call

$$T(E, A, B) := \mathbb{1}_{(-\infty,E]}(A) - \mathbb{1}_{(-\infty,E]}(B) \tag{3.20}$$

the *spectral shift operator* (for the operators A and B at energy E). Assume further that A and B are bounded from below and that $\mathrm{e}^{-A} - \mathrm{e}^{-B}$ is a trace class operator. Then there exists a unique function $\xi := \xi(\,\cdot\,, A, B) \in L^1_{\mathrm{loc}}(\mathbb{R})$, the space of locally integrable functions on $\mathbb{R}$, such that

$$\mathrm{tr}\,\big(f(A) - f(B)\big) = -\int_{\mathbb{R}} d\lambda\, f'(\lambda)\, \xi(\lambda, A, B) \tag{3.21}$$

holds for all test functions $f \in \mathcal{C}^\infty(\mathbb{R})$ with $\lim_{\lambda\to\infty} f(\lambda) = 0$ and $\mathrm{supp}(f')$ compact. The function $\xi(\cdot, A, B)$ is called the *spectral shift function* (for the operators A and B). Details and further properties can be found, e.g., in [131]. Finally, given two self-adjoint projections P and Q such that $\pm 1 \notin \sigma_{\mathrm{ess}}(P - Q)$, the essential spectrum of $P - Q$, we define their *Fredholm index* as

$$\mathrm{index}(P, Q) := \dim\ker(P - Q - 1) - \dim\ker(P - Q + 1). \tag{3.22}$$

This index was, to my best knowledge, first introduced in [13], where also further details and properties can be found. In the particular case of spectral projections $P = \mathbb{1}_{(-\infty,E]}(A)$ and $Q = \mathbb{1}_{(-\infty,E]}(B)$, we introduce the short-hand notation

$$\begin{aligned} \theta(E, A, B) &:= \mathrm{index}\,\big\{\mathbb{1}_{(-\infty,E]}(A), \mathbb{1}_{(-\infty,E]}(B)\big\} \\ &= \dim\ker\big(T(E, A, B) - \mathbb{1}\big) - \dim\ker\big(T(E, A, B) + \mathbb{1}\big). \end{aligned} \tag{3.23}$$

We return to our original setup of a random Schrödinger operator H_ω and its locally perturbed version $H'_\omega = H_\omega + W$. It is well-known, e.g. [65, Thm. 1], that $\mathrm{e}^{-H_{\omega(,L)}} - \mathrm{e}^{-H'_{\omega(,L)}}$ almost surely is trace class for all $L > 0$. The spectral shift function $\xi(\,\cdot\,, H_{\omega(,L)}, H'_{\omega(,L)})$ is therefore well-defined almost surely as a function in $L^1_{\mathrm{loc}}(\mathbb{R})$.

For energies outside of the almost surely non-random essential spectrum of H_ω it is known that the spectral shift function, the Fredholm index and the trace of the spectral shift operator are all well-defined and coincide, see e.g. Prop. 2.1 and its proof in [104]. Amongst others, we show in the next theorem that this equality extends to the essential spectrum in the localized spectral region. By $\|A\|_1$ we denote the trace norm of an operator A and by $\mathcal{S}^1$ the space of operators with finite trace norm.

Theorem 3.3 (Spectral shift function). (i) *The spectral shift function, the trace of the shift operator and the Fredholm index coincide. I.e. almost surely*

$$\xi(E, H_\omega, H'_\omega) = \operatorname{tr} T(E, H_\omega, H'_\omega) = \theta(E, H_\omega, H'_\omega) \qquad \text{for almost every } E \in \Sigma_{\mathrm{FMB}}. \tag{3.24}$$

In particular, all three quantities are almost surely well-defined for almost every $E \in \Sigma_{\mathrm{FMB}}$.

(ii) *Given a compact interval* $I \subset \Sigma_{\mathrm{FMB}}$*, there exist constants* $C, \mu > 0$ *such that for all* $E \in I$ *and all* $L > 0$

$$\mathbb{E}\left[\|T(E, H_L, H'_L) - T(E, H, H')\|_1\right] \leqslant C\,\mathrm{e}^{-\mu L}. \tag{3.25}$$

(iii) *Let* $I \subset \Sigma_{\mathrm{FMB}}$ *be a compact interval and* $\alpha \in (0,1)$*. Then there exists a constant* C *such that for all* $E, E' \in I$

$$\mathbb{E}\left[\big\|T(E, H, H') - T(E', H, H')\big\|_1\right] \leqslant C|E - E'|^\alpha. \tag{3.26}$$

(iv) *Assume that* $W \geqslant cV_0$ *for some* $c > 0$ *holds. Then*

$$\mathbb{E}\left[\xi(E, H, H')\right] > 0 \qquad \text{for almost every } E \in \Sigma_{\mathrm{FMB}} \cap \{E' \in \mathbb{R} : \mathcal{N}'(E') > 0\}. \tag{3.27}$$

Remarks 3.4. (i) It is a direct consequence of [13, Thm. 4.1] that the second equality in (3.24) follows from

$$T(E, H_\omega, H'_\omega) \in \mathcal{S}^1 \qquad \text{almost surely for every } E \in \Sigma_{\mathrm{FMB}}. \tag{3.28}$$

(ii) The statement also demonstrates that (since the Fredholm index is integer-valued by definition) the infinite-volume spectral shift function is integer valued in the localized spectral region.

(iii) For a particular choice of random Schrödinger operators and for a particular choice of the perturbation, [30] shows that the disorder-averaged finite- and infinite-volume spectral shift functions are locally bounded uniformly in the system size. See also the related discussion in Section 4.2.

(iv) Part two of the theorem in particular implies

$$\mathbb{E}\left[\big|\operatorname{tr} T(E, H_{\omega,L}, H'_{\omega,L}) - \operatorname{tr} T(E, H_\omega, H'_\omega)\big|\right] \leqslant C\,\mathrm{e}^{-\mu L}, \tag{3.29}$$

and the corresponding statements also hold for the index and the spectral shift function.

(v) The first part of the theorem establishes that the trace of the (infinite-volume) spectral shift operator is a representant of the (infinite-volume) spectral shift function. From this point of view, Hölder continuity of the averaged spectral shift function is established in part three of the theorem.

One can't expect the first and the second part of Theorem 3.3 to hold at large. For example, if E lies within the absolutely continuous spectrum of two operators H, H', both the index and the spectral shift function may be well defined. But in this situation the spectral shift function is typically not integer-valued (while the index by definition is). The same reasoning shows that the convergence (3.29) doesn't hold in general.

For a general Schrödinger operator $-\Delta + V$ and a perturbation by a bounded, compactly supported potential, only vague convergence of the finite-volume spectral shift function is known [63]. Moreover, for a sequence $(L_n)_{n\in\mathbb{N}}$ of length-scales with $L_n/\ln n \to \infty$ as $n \to \infty$ the first part of the theorem yields that

$$\lim_{n\to\infty} \xi(E, H_\omega, H'_\omega) = \xi(E, H_\omega, H'_\omega) \qquad \text{for a.e. } E \in \Sigma_{\mathrm{FMB}} \tag{3.30}$$

holds almost surely. For the pair $-\Delta$ and $-\Delta+W$, with $0 \leqslant W \in L^\infty_c(\mathbb{R}^d)$ non-zero, a partial converse is shown in [74]: For any diverging sequence $(L_n)_{n\in\mathbb{N}}$ there exists a dense subset $\mathcal{E} \subseteq (0,\infty)$ such that for all $E \in \mathcal{E}$

$$\sup_{n\in\mathbb{N}} \xi(E, -\Delta_{L_n}, -\Delta_{L_n} + W) = \infty. \tag{3.31}$$

This issue is revisited in the discussion of the proof of Chapter 4. For comparative purposes we note that for general deterministic continuum Schrödinger operators L^p-bounds for finite- or infinite-volume spectral shift functions are known, e.g. [65, 31, 66]. We apply such bounds in a different context in Section 6.5.2.

The proofs of both theorems above rely on trace-norm bounds on the effect of the local perturbation W. In all our applications, including the applications in other chapters of this thesis, it would be sufficient to prove the bounds below for Fermi projections and trace norms instead of general functions of bounded variation and arbitrary Schatten classes. Due to their multiple applications in our context we nevertheless state them in a fairly general context.

For $p > 0$ we denote by $\|A\|_p := (\operatorname{tr}|A|^p)^{1/p}$ the Schatten-p (quasi-)norm of an operator A on a Hilbert space $\mathcal{H}$ and by $\mathcal{S}^p$ we denote the Schatten-p ideal, i.e. the (quasi-)normed vector space of operators with finite Schatten-p (quasi-)norm. By $\mathrm{BV}(\mathbb{R})$ we denote the space of functions of bounded variation,

$$\mathrm{BV}(\mathbb{R}) := \big\{ f : \mathbb{R} \to \mathbb{R} \text{ measurable} :\ \mathrm{TV}(f) < \infty \big\}, \tag{3.32}$$

equipped with the total variation $\mathrm{TV}(f) := \sup_{(x_p)_p \in \mathcal{P}} \sum_p |f(x_{p+1}) - f(x_p)|$ (where the supremum is taken over the set $\mathcal{P}$ of all finite partitions of $\mathbb{R}$). For a given bounded interval $I = [I_-, I_+] \subset \mathbb{R}$ let

$$\mathcal{F}_I := \big\{ f \in \mathrm{BV}(\mathbb{R}) :\ f\big|_{(-\infty, I_-]} \equiv \text{ const.}, f\big|_{[I_+,\infty)} \equiv 0 \,\big\} \subset L^\infty(\mathbb{R}). \tag{3.33}$$

We also recall the short-hand notation $Q_a := \Lambda_1(a)$ for the cube of side-length 1 centered at $a \in \mathbb{R}^d$.

Theorem 3.5 (Effect of a perturbation)**.** *Fix $p > 0$ and let $I \subset \Sigma_{\mathrm{FMB}}$ be a compact interval. Then there exist constants $C, \mu > 0$ such that the following holds for all $f \in \mathcal{F}_I$:*

(i) *For all open sets $G \subseteq \mathbb{R}^d$ and all $a, b \in \mathbb{R}^d$ we have*

$$\mathbb{E}\big[\big\|\mathcal{X}_a\big(f(H_G) - f(H'_G)\big)\mathcal{X}_b\big\|_p\big] \leqslant C\,\mathrm{TV}(f)\,\mathrm{e}^{-\mu(|a|+|b|)}\,. \tag{3.34}$$

(ii) *For all open sets $G \subset \widetilde{G} \subseteq \mathbb{R}^d$ with $\operatorname{dist}(\partial\widetilde{G}, \partial G) \geqslant 1$ and all $a, b \in \mathbb{R}^d$ such that $Q_a \cap G \neq \varnothing$ or $Q_b \cap G \neq \varnothing$ we have*

$$\mathbb{E}\big[\big\|\mathcal{X}_a\big(f(H_G^{(\prime)}) - f(H_{\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\|_p\big] \leqslant C\ \mathrm{TV}(f)\,\mathrm{e}^{-\mu[\operatorname{dist}(a,\partial G)+\operatorname{dist}(b,\partial G)]}\,. \tag{3.35}$$

(iii) *For all open $G \subseteq \mathbb{R}^d$ we have*

$$\mathbb{E}\left[\|f(H_G) - f(H'_G)\|_p\right] \leqslant C\ \mathrm{TV}(f). \tag{3.36}$$

(iv) *For all $L > 0$ we have*

$$\mathbb{E}\big[\,\big\|\big(f(H_L) - f(H'_L)\big) - \big(f(H) - f(H')\big)\big\|_p\big] \leqslant C\,\mathrm{TV}(f)\,\mathrm{e}^{-\mu L}\,. \tag{3.37}$$

Remarks 3.6. (i) The separate exponential decay of (3.34) in a and b reflects that the operator $f(H_{\omega,G}) - f(H'_{\omega,G})$ is typically (in ω) exponentially small away from the support of $W = H'_{\omega,G} - H_{\omega,G}$.

(ii) Parts one and two of the theorem are basically the same: In the latter part, the role of the local perturbation is played by the boundary of G.

(iii) The assumption $I \subset \Sigma_{\mathrm{FMB}}$ initially only ensures spectral localization of the unperturbed operator H_ω in the energy region I, but (3.35) for H'_ω certainly requires the perturbed operator to be localized as well. The intuitively obvious statement that local perturbations do not alter the energy region of localization is proved in Lemma 3.10.

(iv) A more sophisticated version of the above estimates could be obtained by altering the function class $\mathcal{F}_I$ as follows: Instead of demanding f to be constant on $\mathbb{R} \setminus I$ we could require f to be smooth on $\mathbb{R} \setminus I$ and sufficiently fast decaying at ∞. This would yield algebraic decay (instead of exponential decay) in the theorem above, which would depend on the amount of smoothness of f on $\mathbb{R} \setminus I$. For two related results we refer to Lemma 4.5 and Theorem 5.5.

3.2. Proof's idea & more

The proof of Theorem 3.5 is mainly concerned with technical aspects, but we briefly outline the essence of the argument. Moreover, the first and the second part of the Theorem only differ on a technical level: In the second part the boundary plays the same role as the potential W in the first part. Hence we focus on Theorem 3.5 (i) and set $G = \mathbb{R}^d$ and $W = \mathcal{X}_0$ in the theorem's statement for convenience. Via a suitable functional calculus, in our case a version of the Helffer-Sjöstrand formula that is applicable to functions of bounded variation, we rewrite $f(H^{(\prime)}_\omega)$ in terms of their respective resolvents. This yields

$$\mathcal{X}_a\big(f(H_\omega) - f(H'_\omega)\big)\mathcal{X}_b \sim \int_{\mathbb{R}^2} d\zeta_f(x,y)\, \mathcal{X}_a\big(R_{x+iy}(H_\omega) - R_{x+iy}(H'_\omega)\big)\mathcal{X}_b \tag{3.38}$$

for a compactly supported and bounded measure ζ_f. The spatial projections $\mathcal{X}_a, \mathcal{X}_b$ here are no gimmick but essential: In case of a function of bounded variation the measure ζ_f in general does not decay in a vicinity of the real axis and hence can't salvage the $1/|y|$ divergence of the resolvents near the real axis. The spatial projections together with the finiteness of fractional moments of the resolvent ensure that the right hand side is well defined. The resolvent equation yields for $z = x + iy$

$$\mathcal{X}_a\big(R_z(H_\omega) - R_z(H'_\omega)\big)\mathcal{X}_b = \big(\mathcal{X}_a R_z(H_\omega)\mathcal{X}_0\big)\big(\mathcal{X}_0 R_z(H'_\omega)\mathcal{X}_b\big). \tag{3.39}$$

A local perturbation should not alter the energy region of localization. Hence the first factor on the right hand side of (3.39) decays as $e^{-\mu|a|}$ and the second factor decays as $e^{-\mu|b|}$. This is strictly speaking only true for the respective disorder averages and additional technical steps are in place since the product on the right hand side of (3.39) has to share one disorder average.

Let's now concentrate on Theorem 3.1. First of all we choose a sign-definite perturbation $W \geqslant 0$, which we think of as being localized at the origin. The rough picture for Theorem 3.1 is the following. Eigenfunctions attached to eigenvalues in the localized spectral region can be thought of as being centered at a fixed point in space and exponentially decaying away from this point. For the eigenfunctions of the finite-volume operator $H_{\omega,L}$ there are two possibilities:

If the eigenfunction is localized within the bulk (i.e. rather far apart from the boundary $\partial\Lambda_L$) then the eigenfunction also solves the corresponding eigenvalue equation for H'_ω up to an error $e^{-\mu L}$. Hence it also is (almost) an eigenfunction of the infinite-volume operator. The second possibility is that the eigenfunction is localized at the boundary. In this case it solves the corresponding eigenvalue equation for $H'_{\omega,L}$ up to an error $e^{-\mu L}$, hence (almost) is an eigenfunction of the finite-volume perturbed operator. This means that eventually, for sufficiently large $L \gg 1$, all eigenfunctions of the infinite-volume operator $H_\omega^{(\prime)}$ that are localized in the vicinity of the origin are also (almost) eigenfunctions of the finite-volume operator $H_{\omega,L}^{(\prime)}$. Moreover, if we further increase L then the additional eigenfunctions are localized at a distance $\sim L$ and hence yield no noteworthy contribution to the determinant (3.6). This shows that the ground-state overlap coverges as $L \to \infty$ and that effectively only the eigenfunctions of H_ω and H'_ω localized around the origin contribute to the ground-state overlap. But there is nevertheless an effect which can cause a vainishing of the overlap: Assume that an eigenpair (λ, φ) of $H_{\omega,L}$ with $\lambda < E$ gets affected by the perturbation sufficiently much such that it gets pushed above the Fermi energy. This in particular means that φ is localized in the bulk (in the sense specified above). If we denote the 'corresponding' (in quotation marks, as labelling in this context is touchy business) eigenpair of $H'_{\omega,L}$ by (λ', φ'), then $\lambda' > E$. For sufficiently large L an eigenfunction of $H_{\omega,L}$ – and hence $H'_{\omega,L}$ – corresponding to an eigenvalue between E and λ' enters the box Λ_L. This new eigenfunction now contributes to the perturbed gound state $\mathbf{\Phi}^L_{\omega,N}$ but not to the unperturbed ground state $\mathbf{\Psi}^L_{\omega,N}$. But since it is an eigenfunction of both $H_{\omega,L}$ and $H'_{\omega,L}$, there is a row in the matrix (3.6) that is ≈ 0.

We have to convert those heuristics into a technically more feasible approach. The starting point is formula (3.7), namely

$$S_{\omega,L}(E) = S_{\omega,N_{\omega,L}(E),L} = \det\big(\mathbb{1} - (P_{\omega,N_{\omega,L}(E),L} - Q_{\omega,N_{\omega,L}(E),L})^2\big)^{1/4}, \tag{3.40}$$

where $P_{\omega,N_{\omega,L}(E),L}$ and $Q_{\omega,N_{\omega,L}(E),L}$ are the projections onto the first $N_{\omega,L}(E)$ eigenvectors of $H_{\omega,L}$ and $H'_{\omega,L}$, respectively. By definition of the particle-number sequence $N_{\omega,L}(E)$ we have

$$P_{\omega,N_{\omega,L}(E),L} = \mathbb{1}_{(-\infty,E]}(H_{\omega,L}) =: P_1, \tag{3.41}$$

$$Q_{\omega,N_{\omega,L}(E),L} = \mathbb{1}_{(-\infty,E]}(H'_{\omega,L}) + \mathbb{1}_{(E,\mu^L_{\omega,N_{\omega,L}(E)}]}(H'_{\omega,L}) =: Q_1 + Q_2 \tag{3.42}$$

almost surely, where we neglect the randomness in the notation for P_1, Q_1 and Q_2. Here and in the following we typically denote projection operators by P and Q even though Q_a also denotes the cube in $\mathbb{R}^d$ around a. Strictly speaking, we used here that the eigenvalues of the finite-volume random Schrödinger operator $H'_{\omega,L}$ are almost surely simple. This statement is not used in the proof below, but at least at the bottom of the spectrum it follows from Theorem 6.1. Due to the above heuristics the eigenfunctions of $H'_{\omega,L}$ with eigenvalues in $(E, \mu^L_{N_{\omega,L}(E)}]$ are also almost eigenfunctions of $H_{\omega,L}$. Hence we have

$$Q_2 Q_1 = Q_1 Q_2 = 0 \approx Q_2 P_1 \approx P_1 Q_2, \tag{3.43}$$

which in turn yields

$$S_{\omega,L}(E)^4 \approx \det\big(\mathbb{1} - (P_1 - Q_1)^2 - Q_2\big) \approx \det\big(\mathbb{1} - (P_1 - Q_1)^2\big) \det\big(\mathbb{1} - Q_2\big). \tag{3.44}$$

But Q_2 is a projection, i.e. as long as Q_2 is not identically zero the right hand side of (3.44) is ≈ 0. We note that, even though the right hand side here is exactly zero if $Q_2 \neq 0$ the finite-volume ground-state overlap typically is non-zero. But we did a couple of errors of size $\approx e^{-L}$ in the above calculation. Hence the ground-state overlap $S_{\omega,L}(E)^4$ should be exponentially small if $Q_2 \neq 0$.

An interesting question which has not been addressed in Theorem 3.1, and which would need a refined method to prove convergence of the finite-volume ground state overlap towards the infinite-volume ground state overlap, is the speed of convergence. Our proof only yields information on the speed of convergence in case $S_\omega(E) \neq 0$. But in the physics literature it is in particular claimed that (at least with positive probability) the convergence $S_{\omega,L}(E) \to 0$ is exponentially fast if $S_\omega(E) = 0$. More precisely, it is claimed in [34] that

$$\mathbb{E}[\log S_{\omega,L}(E)] \sim -L. \tag{3.45}$$

This behavior of the ground-state overlap is dubbed *statistical Anderson orthogonality* to distinguish it from the usual algebraic decay for energies in the scattering regime.

3.3. Proof of the decay estimates

As outlined in Section 3.2 the proof relies on a suitable functional calculus to rewrite $f(H^{(\prime)}_{\omega,G})$ in terms of its resolvent. If the theorem was only formulated for the Fermi projection $f = 1_{(-\infty,E]}$, then the natural functional calculus would be provided by contour integration. To tackle the larger class of functions $\mathcal{F}_I$ we apply a suitably adapted version of the Helffer-Sjöstrand formula. This is the content of Subsection 3.3.1 below. The subsequent Subsection 3.3.2 contains two additional auxiliary results: Stability of the energy region Σ_{FMB} under local perturbations and a priori Schatten-p class bounds. Both statements are probably well-known and included for convenience. After those two preparatory subsections we prove Theorem 3.5 in Subsection 3.3.3.

3.3.1. Helffer-Sjöstrand formula. Let $\mathcal{H}$ be a Hilbert space, K a self-adjoint operator on $\mathcal{H}$ and $f : \mathbb{R} \to \mathbb{C}$ a sufficiently regular function. According to the Helffer-Sjöstrand formula, see e.g. [32], there exists a complex Borel measure ζ_f on $\mathbb{R}^2$ such that

$$f(K) = \frac{1}{2\pi} \int_{\mathbb{R}^2} d\zeta_f(x,y)\, R_{x+\mathrm{i}y}(K). \tag{3.46}$$

The smoothness of f determines the vanishing order of the measure ζ_f in the vicinity of the horizontal axis $y = 0$. This compensates for the potential divergence of the resolvent as $|y| \to 0$. For instance, if $f \in \mathcal{C}^2_c(\mathbb{R})$, then ζ_f can be chosen as $d\zeta_f(x,y) = dx\, dy\, \tilde{f}(x,y)$ with

$$\tilde{f}(x,y) := (\partial_x + \mathrm{i}\partial_y)\left((f(x) + \mathrm{i}yf'(x))\Xi(x,y)\right). \tag{3.47}$$

Here, $\Xi \in \mathcal{C}^\infty_c(\mathbb{R}^2)$ is a cutoff function with $\Xi \equiv 1$ in a neighbourhood N_f of $\operatorname{supp}(f) \times \{0\} \subset \mathbb{R}^2$. In particular, the integral in (3.46) is well defined in this case. For an application of the Helffer-Sjöstrand formula for smooth functions f we refer to Section 5.3.3. Here our aim is to formulate a version of the formula that allows for more singular functions such as the Fermi function $f = 1\!\!1_{(-\infty,E]}$. We first note that (3.46) implies

$$Af(K)B = \frac{1}{2\pi}\int_{\mathbb{R}^2} d\zeta_f(x,y)\, AR_{x+\mathrm{i}y}(K)B \tag{3.48}$$

for all bounded operators A, B on $\mathcal{H}$. If the product $AR_{x+iy}(K)B$ has a less severe divergence as $|y| \to 0$ than the resolvent alone the right-hand side of (3.48) is well defined for functions f which are less regular than what is needed for (3.46). This is the context of the next lemma.

For its formulation we consider functions in $\mathrm{BV}_c(\mathbb{R})$, the compactly supported functions of bounded variation. For $f \in \mathrm{BV}_c(\mathbb{R})$, we choose a fixed cutoff function $\Xi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ with $\Xi \equiv 1$ in a neighbourhood N_f of $\mathrm{supp}(f) \times \{0\} \subset \mathbb{R}^2$ and define the complex Borel measure ζ_f on $\mathbb{R}^2$ by

$$d\zeta_f(x,y) := df(x)\,dy\,\Xi(x,y) + dx\,dy\,f(x)\,(\partial_x + \mathrm{i}\partial_y)\,\Xi(x,y). \tag{3.49}$$

Here, df denotes Lebesgue-Stieltjes integration with respect to f. We write $|\zeta_f|$ for the total variation measure of ζ_f.

Lemma 3.7 (Helffer-Sjöstrand formula)**.** *Let $f \in \mathrm{BV}_c(\mathbb{R})$ and ζ_f as in* (3.49)*. Let K be a self-adjoint operator and let A, B be bounded operators on the Hilbert space $\mathcal{H}$. If*

$$\int_{\mathbb{R}^2} d|\zeta_f|(x,y)\,\|A\,R_{x+\mathrm{i}y}(K)\,B\| < \infty, \tag{3.50}$$

then

$$Af(K)B = \frac{1}{2\pi}\int_{\mathbb{R}^2} d\zeta_f(x,y)\,A\,R_{x+\mathrm{i}y}(K)\,B \tag{3.51}$$

holds, where the right-hand side is a Bochner integral with respect to operator norm.

Remark 3.8. Lemma 3.7 can be extended to appropriate Besov spaces $B^s_{p,q}$ ($1 \leqslant p,q \leqslant \infty$ and $0 < s < 1$) by using Dynkin's characterization of Besov spaces [39].

For the random Schrödinger operators $H^{(\prime)}_{\omega,G}$, the priori estimate (2.10) ensures that the lemma is applicable.

Corollary 3.9. *Let $G \subseteq \mathbb{R}^d$ be open and $a, b \in G$. For $f \in \mathrm{BV}_c(\mathbb{R})$ the equality*

$$\mathcal{X}_a f(H^{(\prime)}_{\omega,G})\mathcal{X}_b = \frac{1}{2\pi}\int_{\mathbb{R}^2} d\zeta_f(x,y)\,\mathcal{X}_a\,R_{x+\mathrm{i}y}(H^{(\prime)}_{\omega,G})\,\mathcal{X}_b \tag{3.52}$$

holds almost surely.

Proof. Let Ξ be a cutoff function as specified above and $\delta > 0$ such that $\mathrm{supp}(\Xi) \subset \mathbb{R} \times [-\delta,\delta]$. For fixed $0 < s < 1$, we use the norm bound $\|R_z(H^{(\prime)}_{\omega,G})\| \leqslant |\mathrm{Im} z|^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$, and estimate

$$\mathbb{E}\left[\int_{\mathbb{R}^2} d|\zeta_f|(x,y)\,\|\mathcal{X}_a\,R_{x+\mathrm{i}y}(H^{(\prime)}_G)\,\mathcal{X}_b\|\right] \leqslant C_s \int_{\mathbb{R}^2} \frac{d|\zeta_f|(x,y)}{|y|^{1-s}}, \tag{3.53}$$

where

$$C_s := \sup_{\substack{(E,\eta)\in\mathrm{supp}(\zeta_f)\\ \eta\neq 0}} \mathbb{E}\left[\|\mathcal{X}_a R_{E+\mathrm{i}\eta}(H^{(\prime)}_G)\mathcal{X}_b\|^s\right] \tag{3.54}$$

is finite. The assumption (3.50) of Lemma 3.7 now follows from

$$\int_{\mathbb{R}^2} \frac{d|\zeta_f|(x,y)}{|y|^{1-s}} \leqslant \frac{2}{s}\,\delta^s\big(\mathrm{TV}(f)\|\Xi\|_\infty + 2\|f\|_1\||\nabla\Xi|\|_\infty\big). \tag{3.55}$$

□

Proof of Lemma 3.7. The first part of the proof closely follows the standard proof of the Helffer-Sjöstrand formula. We note that assumption (3.50) implies that the right-hand side of (3.51) is well defined as a Bochner integral with respect to the operator norm. To show (3.51) we let $\varepsilon > 0$ and introduce the closed horizontal strip $\mathcal{C}_\varepsilon := \mathbb{R} \times [-\varepsilon, \varepsilon]$ in $\mathbb{R}^2$. We split the integral according to

$$\begin{aligned}\int_{\mathbb{R}^2} d\zeta_f(x,y)\, A\, R_{x+\mathrm{i}y}(K)\, B &= \int_{\mathcal{C}_\varepsilon} d\zeta_f(x,y)\, A\, R_{x+\mathrm{i}y}(K)\, B \\ &\quad + \int_{\mathbb{R}^2\setminus\mathcal{C}_\varepsilon} d\zeta_f(x,y)\, A\, R_{x+\mathrm{i}y}(K)\, B \\ &=: I_1^\varepsilon + I_2^\varepsilon. \end{aligned} \tag{3.56}$$

Because of (3.50), dominated convergence yields $\lim_{\varepsilon\downarrow 0}\|I_1^\varepsilon\| = 0$. As for the second integral, we note that $C \mapsto ACB$ is a norm-continuous linear map on the Banach space of bounded linear operators. The properties of the Bochner integral then imply

$$I_2^\varepsilon = A\bigg(\int_{\mathbb{R}^2\setminus\mathcal{C}_\varepsilon} d\zeta_f(x,y)\, R_{x+\mathrm{i}y}(K)\bigg)B. \tag{3.57}$$

Here, the right-hand side is well defined because $\mathrm{supp}(\zeta_f)$ is compact and the norm of the resolvent is uniformly bounded on $\mathbb{R}^2 \setminus \mathcal{C}_\varepsilon$. Now we choose $\varepsilon > 0$ so small that $\Xi \equiv 1$ on $\mathrm{supp}(f) \times [-\varepsilon, \varepsilon]$. It follows from Fubini's theorem that $I_2^\varepsilon = 2\pi A f_\varepsilon(K) B$ with

$$\begin{aligned} f_\varepsilon(\lambda) &:= \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus\mathcal{C}_\varepsilon} \frac{d\zeta_f(x,y)}{\lambda - x - \mathrm{i}y} = \frac{\mathrm{i}}{2\pi}\int_{\mathbb{R}} dx\,\bigg(-\frac{f(x)\Xi(x,\varepsilon)}{\lambda - x - \mathrm{i}\varepsilon} + \frac{f(x)\Xi(x,-\varepsilon)}{\lambda - x + \mathrm{i}\varepsilon}\bigg) \\ &= \int_{\mathbb{R}} dx\, f(x)\,\frac{\varepsilon/\pi}{(\lambda - x)^2 + \varepsilon^2} = \int_{\mathbb{R}} dx\,\frac{1/\pi}{x^2+1}\, f(\lambda + \varepsilon x) \end{aligned} \tag{3.58}$$

for $\lambda \in \mathbb{R}$, where the first equality relies on integration by parts and holomorphy $(\partial_x + \mathrm{i}\partial_y)(\lambda - x - \mathrm{i}y)^{-1} = 0$ on $\mathbb{R}^2 \setminus \mathcal{C}_\varepsilon$. The second equality follows from $\Xi(x,\varepsilon) = \Xi(x,-\varepsilon) = 1$ for all $x \in \mathrm{supp}(f)$.

The second part of the proof deals with the problem that discontinuity points of f challenge the convergence of I_2^ε as $\varepsilon \downarrow 0$. However, the regularity condition (3.50) ensures that they form only a null set of the relevant spectral measures and, thus, weak convergence still holds. To see this, let $\varphi, \psi \in \mathcal{H}$ and define the complex spectral measure $\mu_{\varphi,\psi} := \langle\varphi, A1\!\!1_\bullet(K)B\psi\rangle$ of K. The functional calculus, (3.58) and dominated convergence imply

$$\lim_{\varepsilon\downarrow 0}\langle\varphi, A f_\varepsilon(K) B\psi\rangle = \int_{\mathbb{R}} d\mu_{\varphi,\psi}(\lambda)\, \lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}} dx\,\frac{1/\pi}{x^2+1}\, f(\lambda + \varepsilon x). \tag{3.59}$$

We prove below that the set of discontinuity points of f is a $\mu_{\varphi,\psi}$-null set. Using this, another application of dominated convergence in (3.59) yields $\lim_{\varepsilon\downarrow 0} I_2^\varepsilon = 2\pi A f(K) B$ weakly, and the lemma follows.

It remains to prove that f is continuous $\mu_{\varphi,\psi}$-almost everywhere. Without loss of generality, we assume $\|\varphi\| = \|\psi\| = 1$. Since $f \in \mathrm{BV}_c(\mathbb{R})$, it has left and right limits at all points. Hence, the set $\mathcal{U} := \{\lambda \in \mathbb{R} : f \text{ not continuous in } \lambda\}$ consists of jump discontinuities only. Moreover, $\mathcal{U}$ is countable so that

$$\mu_{\varphi,\psi}(\mathcal{U}) = \sum_{\lambda\in\mathcal{U}} \mu_{\varphi,\psi}(\{\lambda\}). \tag{3.60}$$

We fix an arbitrary $\lambda \in \mathcal{U}$ and set $\delta f_\lambda := \lim_{\varepsilon\downarrow 0}[f(\lambda+\varepsilon) - f(\lambda-\varepsilon)] \neq 0$. We choose $y_0 > 0$ small enough such that $\Xi(\lambda, y) = 1$ whenever $|y| \leqslant y_0$. Assumption (3.50) implies

$$\begin{aligned}\infty > \int_{\mathbb{R}^2} d|\zeta_f|(x,y)\, \|A\, R_{x+\mathrm{i}y}(K)\, B\| &\geqslant |\delta f_\lambda| \int_{-y_0}^{y_0} dy\, \|A\, R_{\lambda+\mathrm{i}y}(K)\, B\| \\ &\geqslant |\delta f_\lambda| \int_{-y_0}^{y_0} \frac{dy}{|y|}\, |h(y)|,\end{aligned} \tag{3.61}$$

where $h(y) := \int_{\mathbb{R}} d\mu_{\varphi,\psi}(\lambda')\, y/(\lambda' - \lambda - \mathrm{i}y)$. Dominated convergence implies that $\mathbb{R} \ni y \mapsto h(y)$ is continuous and that $\lim_{y\to 0} h(y) = \mathrm{i}\, \mu_{\varphi,\psi}(\{\lambda\})$. Now, we assume that $\mu_{\varphi,\psi}(\{\lambda\}) \neq 0$. Then there exists $0 < y_1 \leqslant y_0$ such that $|h(y)| \geqslant |\mu_{\varphi,\psi}(\{\lambda\})|/2$ whenever $|y| \leqslant y_1$, and we conclude that (3.61) yields a contradiction. Therefore we must have $\mu_{\varphi,\psi}(\{\lambda\}) = 0$, and (3.60) implies the desired continuity of f. □

3.3.2. Further auxiliary results. The results presented in this subsection are probably well-known and only included for convenience. For example, estimates as in Lemma 3.11 below appeared, e.g., in [112, Sect. B.9], [3, App. A] and [24, App. A]. We first prove that the fractional moment bounds for the resolvent are stable under local perturbations.

Lemma 3.10 (Persistence of localization). $\Sigma_{\mathrm{FMB}}(H_\omega) = \Sigma_{\mathrm{FMB}}(H'_\omega)$. *More precisely: Let* $I \subset \Sigma_{\mathrm{FMB}}$ *be a compact interval. Then for any fixed* $0 < s < 1$ *there exist constants* $C, \mu > 0$ *such that for all open* $G \subseteq \mathbb{R}^d$ *and* $a, b \in \mathbb{R}^d$

$$\sup_{E\in I, \eta\neq 0} \mathbb{E}\left[\|\mathcal{X}_a R_{E+i\eta}(H'_G)\mathcal{X}_b\|^s\right] \leqslant C\, \mathrm{e}^{-\mu|a-b|}. \tag{3.62}$$

The second auxiliary statement establishes local Schatten-p estimates. It can be formulated more conveniently for the non-random Schrödinger operators

(D) $H := -\Delta + U$ with a bounded potential $U \in L^\infty(\mathbb{R}^d)$.

The restriction to bounded potentials is made for convenience and could be relaxed.

Lemma 3.11 (Local Schatten-class bounds). *Assume* (D) *and set* $E_{00} := \operatorname{ess\,inf}_{x\in\mathbb{R}^d} U(x)$. *Let* $p > 0$. *Then there exists a constant* C *such that for all open* $G \subseteq \mathbb{R}^d$*, all* $a \in \mathbb{R}^d$ *and all* $g \in L^\infty_c(\mathbb{R})$ *we have*

$$\|\mathcal{X}_a g(H_G)\|_p \leqslant C\|g\|_\infty\, \mathrm{e}^{0.5(\sup\operatorname{supp}(g) - E_{00})}. \tag{3.63}$$

Proof of Lemma 3.10. Let $z := E + i\eta$ with $E \in I$ and $\eta \neq 0$. For the moment, we also fix $0 < s < 1/2$. The resolvent equation

$$R_z(H'_{\omega,G}) = R_z(H_{\omega,G}) - R_z(H_{\omega,G}) W R_z(H'_{\omega,G}) \tag{3.64}$$

yields the upper bound

$$\begin{aligned}\mathbb{E}\left[\|\mathcal{X}_a R_z(H'_G)\mathcal{X}_b\|^s\right] &\leqslant \mathbb{E}\left[\|\mathcal{X}_a R_z(H_G)\mathcal{X}_b\|^s\right] + \mathbb{E}\left[\|\mathcal{X}_a R_z(H_G) W R_z(H'_G)\mathcal{X}_b\|^s\right] \\ &=: I_1 + I_2. \end{aligned} \tag{3.65}$$

The term I_1 can directly be bounded by the fractional moment bound (2.11) for the unperturbed operator $H_{\omega,G}$. For I_2 we estimate

$$\begin{aligned} I_2 &\leqslant \|W\|_\infty^s \sum_{c\in\Gamma_W} \mathbb{E}\left[\|\mathcal{X}_a R_z(H_G)\mathcal{X}_c\|^s \|\mathcal{X}_c R_z(H'_G)\mathcal{X}_b\|^s\right] \\ &\leqslant \|W\|_\infty^s \sum_{c\in\Gamma_W} \mathbb{E}\left[\|\mathcal{X}_a R_z(H_G)\mathcal{X}_c\|^{2s}\right]^{1/2} \mathbb{E}\left[\|\mathcal{X}_c R_z(H'_G)\mathcal{X}_b\|^{2s}\right]^{1/2}, \end{aligned} \tag{3.66}$$

where we abbreviated $\Gamma_W := \{n \in \mathbb{Z}^d : \Lambda_1(n) \cap \operatorname{supp}(W) \neq \varnothing)\}$. Since $2s < 1$, the first expectation can again be estimated via the fractional moment bounds. The second expectation can in turn be estimated via (2.10) for the perturbed operator. This yields

$$I_2 \leqslant C_1 \sum_{c\in\Gamma_W} \mathrm{e}^{-\mu|c-a|} \leqslant C_2\, \mathrm{e}^{-\mu|a|} \tag{3.67}$$

with constants C_1, C_2 that are independent of $G \subseteq \mathbb{R}^d$ and $a \in \mathbb{R}^d$. Since $R_z(H_{\omega,G}) W R_z(H'_{\omega,G}) = R_z(H'_{\omega,G}) W R_z(H_{\omega,G})$ we obtain along the same lines that $I_2 \leqslant C_2\, \mathrm{e}^{-\mu|b|}$. By multiplying this inequality with (3.67) we infer

$$I_2 \leqslant C_2\, \mathrm{e}^{-0.5\mu(|a|+|b|)} \leqslant C_2\, \mathrm{e}^{-0.5\mu(|a-b|)}. \tag{3.68}$$

This yields the assertion for $s < 1/2$. In order to conclude that the assertion holds for all $s \in (0,1)$ we apply an argument which was used, e.g., in [6]. So far, we in particular proved the assertion for $s_0 := 1/3$. We fix $q \in (1, \frac{1-s_0}{s-s_0})$ (which is possible due to $\frac{1-s_0}{s-s_0} > 1$) and let $p > 1$ be its Hölder conjugate. Via Hölder's inequality we then estimate

$$\begin{aligned}\mathbb{E}\left[\|\mathcal{X}_a R_z(H'_G)\mathcal{X}_b\|^s\right] &= \mathbb{E}\left[\|\mathcal{X}_a R_z(H'_G)\mathcal{X}_b\|^{s_0/p} \|\mathcal{X}_a R_z(H'_G)\mathcal{X}_b\|^{s-s_0/p}\right] \\ &\leqslant \mathbb{E}\left[\|\mathcal{X}_a R_z(H'_G)\mathcal{X}_b\|^{s_0}\right]^{1/p} \mathbb{E}\left[\|\mathcal{X}_a R_z(H'_G)\mathcal{X}_b\|^{qs-s_0q/p}\right]^{1/q}. \end{aligned} \tag{3.69}$$

Our choice of q above ensures that $qs - s_0q/p = q(s-s_0) + s_0 < 1$. Hence the finiteness of fractional moments of the resolvent can be applied to the second expectation in the last line of (3.69). □

Proof of Lemma 3.11. We first note that for $p' > 0$

$$\begin{aligned}\|\mathcal{X}_a g(H_G)\|_{p'} &= \|\mathcal{X}_a |g|^2(H_G)\mathcal{X}_a\|_{p'/2}^{1/2} \\ &\leqslant \|g\|_\infty\, \mathrm{e}^{0.5(\sup\operatorname{supp}(g)-E_{00})}\, \|\mathcal{X}_a e^{-(H_G-E_{00})}\mathcal{X}_a\|_{p'/2}^{1/2}. \end{aligned} \tag{3.70}$$

Hence it is sufficient to prove that for $p > 0$

$$\|\mathcal{X}_a e^{-(H_G-E_{00})}\mathcal{X}_a\|_p \leqslant C \tag{3.71}$$

for a constant C which only depends on p. For $p \geqslant 1$ this directly follows from

$$\begin{aligned}\|\mathcal{X}_a e^{-(H_G-E_{00})}\mathcal{X}_a\|_1 &= \|\mathcal{X}_a e^{-0.5(H_G-E_{00})}\|_2^2\\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d} d(x,y)\mathcal{X}_a(x)\big|e^{-0.5(H_G-E_{00})}(x,y)\big|^2\\ &\leqslant \int_{\mathbb{R}^d\times\mathbb{R}^d} d(x,y)\mathcal{X}_a(x)\big|e^{-0.5(-\Delta)}(x,y)\big|^2 < \infty,\end{aligned} \tag{3.72}$$

where we used the Feynman-Kac representation for Dirichlet restrictions, see e.g. [25, Sect. 6]: Since $U - E_{00} \geqslant 0$ it implies the estimate

$$0 \leqslant e^{-0.5(H_G-E_{00})}(x,y) \leqslant e^{-0.5(-\Delta)}(x,y) \tag{3.73}$$

for all $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$. For $0 < p < 1$ the adapted triangle inequality

$$\|A+B\|_p^p \leqslant \|A\|_p^p + \|B\|_p^p, \tag{3.74}$$

holds for, say, compact operators A and B [92]. Now, let $m \in \mathbb{N}$ be sufficiently large such that $pm \geqslant 2$. We set $a_0 := a$ and estimate with the adapted triangle inequality

$$\begin{aligned}\|\mathcal{X}_a e^{-(H_G-E_0)}\|_p^p &= \Big\|\mathcal{X}_{a_0} e^{-m^{-1}(H_G-E_{00})}\Big(\sum_{a_1\in\mathbb{Z}^d}\mathcal{X}_{a_1}\Big)e^{-m^{-1}(H_G-E_0)}\\ &\qquad\times\cdots\Big(\sum_{a_{m-1}\in\mathbb{Z}^d}\mathcal{X}_{a_{m-1}}\Big)e^{-m^{-1}(H_G-E_{00})}\Big(\sum_{a_m\in\mathbb{Z}^d}\mathcal{X}_{a_m}\Big)\Big\|_p^p\\ &\leqslant \sum_{a_1,\dots,a_m\in\mathbb{Z}^d}\Big\|\prod_{l=1}^m \mathcal{X}_{a_{l-1}}e^{-m^{-1}(H_G-E_{00})}\mathcal{X}_{a_l}\Big\|_p^p.\end{aligned} \tag{3.75}$$

Hölder's inequality for Schatten classes yields

$$\Big\|\prod_{l=1}^m \mathcal{X}_{a_{l-1}}e^{-m^{-1}(H_G-E_{00})}\mathcal{X}_{a_l}\Big\|_p^p \leqslant \prod_{l=1}^m \|\mathcal{X}_{a_{l-1}}e^{-m^{-1}(H_G-E_{00})}\mathcal{X}_{a_l}\|_{pm}^p. \tag{3.76}$$

With $pm \geqslant 2$ and an argument which is analogous to the one in (3.72) we find that there exist constant $C_1, \mu_1 > 0$ such that

$$\big\|\mathcal{X}_a e^{-m^{-1}(H_G-E_{00})}\mathcal{X}_b\big\|_{pm} \leqslant C_1 e^{-\mu_1|a-b|} \tag{3.77}$$

for all $a, b \in \mathbb{R}^d$. The statement follows if we first estimate the right hand side of (3.76) via (3.77) and subsequently apply the resulting bound to the right hand side of (3.75). $\square$

3.3.3. Proof of Theorem 3.5. In analogy to the constant defined in Section 3.3.2 we introduce a lower bound of the spectra of all considered random Schrödinger operators,

$$\begin{aligned}E_{00} &:= \min\big\{\operatorname*{ess\,inf}_{x\in\mathbb{R}^d} V_{\mathrm{per}}(x), \operatorname*{ess\,inf}_{x\in\mathbb{R}^d}\big(V_{\mathrm{per}}(x)+W(x)\big)\big\}\\ &\leqslant \min\big\{\min\sigma(H_{\omega,G}), \min\sigma(H'_{\omega,G})\big\},\end{aligned} \tag{3.78}$$

where the inequality holds almost surely for all open $G \subseteq \mathbb{R}^d$. For functions $f \in \mathrm{BV}(\mathbb{R})$, we denote by $\mathrm{supp}(f')$ the support of the (complex) measure defined by Lebesgue-Stieltjes integration with respect to f. We start with the proof of Theorem 3.5 (i), which we first prove for the operator norm instead of the Schatten norm in (3.34).

Lemma 3.12. *For a fixed compact set $S \subset \Sigma_{\mathrm{FMB}}$ there exist constants $C, \mu > 0$ such that for all functions $f \in \mathrm{BV}_c(\mathbb{R})$ with $\mathrm{supp}(f') \subseteq S$, open $G \subseteq \mathbb{R}^d$ and $a, b \in \mathbb{R}^d$ we have*

$$\mathbb{E}\left[\left\|\mathcal{X}_a\big(f(H_G) - f(H'_G)\big)\mathcal{X}_b\right\|\right] \leqslant C\left(\|f\|_1 + \mathrm{TV}(f)\right) \mathrm{e}^{-\mu(|a|+|b|)}. \tag{3.79}$$

Proof. Let $S_- := \inf S$ and $S_+ := \sup S$. Independently of the function f we choose a cutoff function Ξ, which only depends on the set S, subject to the following properties:

(P1) $\Xi \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ with $0 \leqslant \Xi \leqslant 1$ and $\|\partial_x \Xi\|_\infty, \|\partial_y \Xi\|_\infty \leqslant 3$,

(P2) $\mathrm{supp}(\Xi) \subseteq [S_- - 1, S_+ + 1] \times [-1, 1]$,

(P3) $\Xi \equiv 1$ on $[S_- - 1/2, S_+ + 1/2] \times [-1/2, 1/2]$.

Let's now fix an open $G \subseteq \mathbb{R}^d$, $a, b \in \mathbb{R}^d$ and $f \in \mathrm{BV}_c(\mathbb{R})$ such that $\mathrm{supp}(f') \subseteq S$, i.e. in particular $\mathrm{supp}(f) \subseteq [S_-, S_+]$. By ζ_h we denote the complex Borel measure defined in (3.49). An application of Corollary 3.9 to the operators $f(H_{\omega,G})$ and $f(H'_{\omega,G})$ gives

$$\begin{aligned}
&\mathbb{E}\left[\left\|\mathcal{X}_a\big(f(H_G) - f(H'_G)\big)\mathcal{X}_b\right\|\right] \\
&\leqslant \frac{1}{2\pi}\mathbb{E}\left[\int_{\mathbb{R}^2} d|\zeta_f|(x,y)\, \left\|\mathcal{X}_a\big(R_{x+\mathrm{i}y}(H_G) - R_{x+\mathrm{i}y}(H'_G)\big)\mathcal{X}_b\right\|\right] \\
&\leqslant \frac{2^{1-s/2}}{2\pi}\mathbb{E}\left[\int_{\mathbb{R}^2} \frac{d|\zeta_f|(x,y)}{|y|^{1-s/2}} \left\|\mathcal{X}_a\big(R_{x+\mathrm{i}y}(H_G) - R_{x+\mathrm{i}y}(H'_G)\big)\mathcal{X}_b\right\|^{s/2}\right],
\end{aligned} \tag{3.80}$$

where, in the last step, we applied the norm bound $\|R_z(H^{(\prime)}_{\omega,G})\|^{1-s/2} \leqslant 1/|\mathrm{Im} z|^{1-s/2}$ for some $0 < s < 1$. We recall the notation $\Gamma_W := \{n \in \mathbb{Z}^d : \Lambda_1(n) \cap \mathrm{supp}(W) \neq \varnothing)\}$. From the resolvent equation and the Cauchy-Schwarz inequality we then obtain

$$\begin{aligned}
&\mathbb{E}\bigg[\int_{\mathbb{R}^2} \frac{d|\zeta_f|(x,y)}{|y|^{1-s/2}} \left\|\mathcal{X}_a\big(R_{x+\mathrm{i}y}(H_G) - R_{x+\mathrm{i}y}(H'_G)\big)\mathcal{X}_b\right\|^{s/2}\bigg] \\
&\leqslant \|W\|_\infty^{s/2} \sum_{c\in\Gamma_W} \int_{\mathbb{R}^2} \frac{d|\zeta_f|(x,y)}{|y|^{1-s/2}} \\
&\qquad \times \mathbb{E}\left[\|\mathcal{X}_a R_{x+\mathrm{i}y}(H_G)\mathcal{X}_c\|^{s/2} \|\mathcal{X}_c R_{x+\mathrm{i}y}(H'_G)\mathcal{X}_b\|^{s/2}\right] \\
&\leqslant \|W\|_\infty^{s/2} \sum_{c\in\Gamma_W} \int_{\mathbb{R}^2} \frac{d|\zeta_f|(x,y)}{|y|^{1-s/2}} \\
&\qquad \times \mathbb{E}\big[\|\mathcal{X}_a R_{x+\mathrm{i}y}(H_G)\mathcal{X}_c\|^s\big]^{1/2} \mathbb{E}\big[\|\mathcal{X}_c R_{x+\mathrm{i}y}(H'_G)\mathcal{X}_b\|^s\big]^{1/2}.
\end{aligned} \tag{3.81}$$

From (3.49), (P2) and (P3) we infer that

$$\begin{aligned}
\mathrm{supp}(\zeta_f) &\subseteq \Big(\mathrm{supp}(f') \times [-1,1]\Big) \cup \Big([S_-, S_+] \times \big([-1,-1/2] \cup [1/2,1]\big)\Big) \\
&=: Z_1 \cup Z_2.
\end{aligned} \tag{3.82}$$

On the set Z_2 we estimate the right hand side of (3.81) by the Combes-Thomas estimate stated Section 2.1. For $(x, y) \in Z_2$ we have

$$|x + \mathrm{i}y| \leqslant |x| + |y| \leqslant \max\{|S_-|, |S_+|\} + 1 =: C_S. \tag{3.83}$$

This implies that for a constant C_2 and $\mu_2 := 2s\mu_1/C_S > 0$

$$\sup_{(x,y)\in Z_2} \Big\{\mathbb{E}\big[\|\mathcal{X}_a R_{x+\mathrm{i}y}(H_G)\mathcal{X}_c\|^s\big]\,\mathbb{E}\big[\|\mathcal{X}_c R_{x+\mathrm{i}y}(H'_G)\mathcal{X}_b\|^s\big]\Big\} \leqslant C_2\,\mathrm{e}^{-\mu_2(|a-c|+|c-b|)}\,. \tag{3.84}$$

On the set Z_1 we use the fractional-moment bounds (2.11) for H_ω and H'_ω, which can be applied because of $\operatorname{supp}(f') \subseteq \Sigma_{\mathrm{FMB}}$ and Lemma 3.10. Hence, there exist constants $C_3, \mu_3 > 0$ such that

$$\sup_{(x,y)\in Z_1} \Big\{\mathbb{E}\big[\|\mathcal{X}_a R_{x+\mathrm{i}y}(H_G)\mathcal{X}_c\|^s\big]\,\mathbb{E}\big[\|\mathcal{X}_c R_{x+\mathrm{i}y}(H'_G)\mathcal{X}_b\|^s\big]\Big\} \leqslant C_3\,\mathrm{e}^{-\mu_3(|a-c|+|c-b|)}\,. \tag{3.85}$$

Collecting the estimates in (3.80), (3.81), (3.84) and (3.85), we obtain constants C_4, C_5 and $\mu_4 := \min\{\mu_2,\mu_3\}/2 > 0$, which depend on s and S but are independent of G, such that

$$\begin{aligned}\mathbb{E}\left[\big\|\mathcal{X}_a\big(f(H_G)-f(H'_G)\big)\mathcal{X}_b\big\|\right] &\leqslant C_4\|W\|_\infty^{s/2}\sum_{c\in\Gamma_W}\mathrm{e}^{-\mu_4(|a-c|+|c-b|)}\int_{\mathbb{R}^2}\frac{d|\zeta_f|(x,y)}{|y|^{1-s/2}}\\ &\leqslant C_5\big(\|f\|_1+\mathrm{TV}(f)\big)\,\mathrm{e}^{-\mu_4(|a|+|b|)}\,.\end{aligned} \tag{3.86}$$

□

PROOF OF THEOREM 3.5(i). We set $I_- := \min I$, $I_+ := \max I$ and assume without loss of generality that $I_+ \geqslant E_{00}$ (since $E_{00} \leqslant \inf\sigma(H^{(\prime)}_{\omega,G})$). Moreover, we can without loss of generality restrict ourselves to $0 < p < 1$. Let's fix an open $G \subseteq \mathbb{R}^d$, $a, b \in \mathbb{R}^d$ and $f \in \mathcal{F}_I$. The plan is to combine Theorem 3.12 with the a prioi bounds from Lemma 3.11. In order to apply the former Theorem we switch from the function f to the function $h := f\mathbb{1}_{[E_0-1,\infty)}$. This function satisfies $h \in \mathrm{BV}_c(\mathbb{R})$, $\mathrm{TV}(h) \leqslant 2\,\mathrm{TV}(f)$, $\operatorname{supp}(h') \subseteq I \cup \{E_0 - 1\} \subset \Sigma_{\mathrm{FMB}}$ (by the Combes-Thomas estimate) and

$$f(H^{(\prime)}_{\omega,G}) = h(H^{(\prime)}_{\omega,G}). \tag{3.87}$$

Because of (3.87) we obtain for any $0 < r < p$ that

$$\begin{aligned}\big\|\mathcal{X}_a\left(f(H_{\omega,G})-f(H'_{\omega,G})\right)\mathcal{X}_b\big\|_p &= \big\|\mathcal{X}_a\left(h(H_{\omega,G})-h(H'_{\omega,G})\right)\mathcal{X}_b\big\|_p\\ &\leqslant \big\|\mathcal{X}_a\left(h(H_{\omega,G})-h(H'_{\omega,G})\right)\mathcal{X}_b\big\|^{r/p}\big\|\mathcal{X}_a\left(h(H_{\omega,G})-h(H'_{\omega,G})\right)\mathcal{X}_b\big\|_{p-r}^{1-r/p},\end{aligned} \tag{3.88}$$

where we also used that $\|A\|_p^p \leqslant \|A\|^\varepsilon\|A\|_{p-\varepsilon}^{p-\varepsilon}$ for, say, a compact operator A. The adapted triangle inequality (3.74) and Lemma 3.11 yield

$$\begin{aligned}\big\|\mathcal{X}_a\left(h(H_G)-h(H'_{\omega,G})\right)\mathcal{X}_b\big\|_{p'}^{p'} &\leqslant \|\mathcal{X}_a h(H_{\omega,G})\mathcal{X}_b\|_{p'}^{p'} + \|\mathcal{X}_a h(H'_{\omega,G})\mathcal{X}_b\|_{p'}^{p'}\\ &\leqslant C_{p'}\|h\|_\infty^{p'}\end{aligned} \tag{3.89}$$

for every $0 < p' \leqslant 1$, where $C_{p'}$ depends on I_+, but is independent of h and uniform in the disorder and open sets $G \subseteq \mathbb{R}^d$. We apply (3.89) with $p' = p - r$ to estimate the expectation of (3.88) by

$$\mathbb{E}\Big[\big\|\mathcal{X}_a\left(f(H_G)-f(H'_G)\right)\mathcal{X}_b\big\|_p\Big] \leqslant C_{p-r}^{1/p}\,\|h\|_\infty^{1-s_1}\,\mathbb{E}\Big[\big\|\mathcal{X}_a\big(h(H_G)-h(H'_G)\big)\mathcal{X}_b\big\|\Big]^{s_1}, \tag{3.90}$$

where we introduced $s_1 := r/p < 1$ and applied Jensen's inequality. Now, we choose $0 < s_2 < 1$ and apply Theorem 3.12 with $S = I \cup \{E_{00} - 1\}$ to the expectation on the right-hand side of (3.90). This yields constants $C_1, \mu_1 > 0$ (which depend on s_2 and I) such that

$$\mathbb{E}\Big[\big\|\mathcal{X}_a\big(h(H_G)-h(H'_G)\big)\mathcal{X}_b\big\|\Big] \leqslant C_1(I_+ - E_{00} + 2)\,\mathrm{TV}(f)\;\mathrm{e}^{-\mu_1(|a|+|b|)}, \tag{3.91}$$

where we also used that

$$\|h\|_1 + \mathrm{TV}(h) \leqslant 2(I_+ - E_0 + 2)\,\mathrm{TV}(f). \tag{3.92}$$

Inserting (3.91) into (3.90) and observing

$$\|h\|_\infty \leqslant \mathrm{TV}(h) \leqslant 2\,\mathrm{TV}(f), \tag{3.93}$$

we obtain

$$\mathbb{E}\left[\big\|\mathcal{X}_a\left(f(H_G)-f(H'_G)\right)\mathcal{X}_b\big\|_p\right] \leqslant C\,\mathrm{TV}(f)\,\mathrm{e}^{-s_1\mu_1(|a|+|b|)} \tag{3.94}$$

with suitable constants $C, s_1, \mu_1 > 0$. □

Proof of Theorem 3.5 (iii). Without loss of generality we take $0 < p < 1$ such that $p^{-1} \in \mathbb{N}$. For $f \in \mathcal{F}_I$ and open $G \subseteq \mathbb{R}^d$ we abbreviate

$$T_{\omega,f} := f(H_{\omega,G}) - f(H'_{\omega,G}). \tag{3.95}$$

In virtue of the adapted triangle inequality (3.74) we have

$$\|T_{\omega,f}\|_p^p \leqslant \sum_{a,b\in\mathbb{Z}^d} \|\mathcal{X}_a T_{\omega,f}\mathcal{X}_b\|_p^p. \tag{3.96}$$

A $k := p^{-1}$-fold application of (3.96) yields

$$\mathbb{E}\left[\|T_f\|_p\right] = \mathbb{E}\big[\big(\|T_f\|_p^p\big)^k\big] \leqslant \sum_{\substack{a_1,\dots,a_k\in\mathbb{Z}^d\\ b_1,\dots,b_k\in\mathbb{Z}^d}} \mathbb{E}\Big[\prod_{l=1}^k \|\mathcal{X}_{a_l}T_f\mathcal{X}_{b_l}\|_p^p\Big]. \tag{3.97}$$

Next, we apply Hölder's inequality to the expectation on the right hand side of (3.97) to obtain

$$\mathbb{E}\left[\|T_f\|_p\right] \leqslant \sum_{\substack{a_1,\dots,a_k\in\mathbb{Z}^d\\ b_1,\dots,b_k\in\mathbb{Z}^d}} \prod_{l=1}^k \mathbb{E}\big[\|\mathcal{X}_{a_l}T_f\mathcal{X}_{b_l}\|_p\big]^p. \tag{3.98}$$

Theorem 3.5 (i) implies the existence of finite constants $C, \mu > 0$ (which only depend on p and I) such that

$$\mathbb{E}\big[\|\mathcal{X}_a T_f \mathcal{X}_b\|_p\big] \leqslant C\,\mathrm{TV}(f)\,\mathrm{e}^{-\mu(|a|+|b|)} \tag{3.99}$$

for all $a, b \in \mathbb{Z}^d$. By estimating the right hand side of (3.98) by (3.99) we arrive at (recall that $pk = 1$)

$$\mathbb{E}\left[\|T_f\|_p\right] \leqslant C_2\,\mathrm{TV}(f), \tag{3.100}$$

for a constant $C_2 = C_{2,p,I}$. □

Proof of Theorem 3.5 (ii). We will follow the strategy in the proofs of Lemma 3.12 and Theorem 3.5 (i). We also apply the notation introduced in those proofs. We again assume without loss of generality that $I_+ \geqslant E_{00}$ and $0 < p \leqslant 1$.

Let $a, b \in \mathbb{R}^d$ and $f \in \mathcal{F}_I$, where we assume that $Q_b \cap G \neq \varnothing$. We again consider the truncation $h := f \mathbb{1}_{[E_{00}-1,\infty)}$ and write ζ_h for the complex measure defined as in (3.49), with a cutoff function $\Xi = \Xi_h$ that satisfies (P1) – (P3), where f is replaced by h. Proceeding along the lines of (3.90) and (3.80), we obtain for any $s, s' \in (0,1)$

$$\begin{aligned}\mathbb{E}\Big[\big\|\mathcal{X}_a\big(f(H_G^{(\prime)}) - f(H_{\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\|_p\Big] &= \mathbb{E}\left[\big\|\mathcal{X}_a\big(h(H_G^{(\prime)}) - h(H_{\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\|_p\right] \\ &\leqslant C_1 \,\|h\|_\infty^{1-s'}\, \mathbb{E}\left[\big\|\mathcal{X}_a\big(h(H_G^{(\prime)}) - h(H_{\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\|\right]^{s'} \\ &\leqslant C_2 \,\|h\|_\infty^{1-s'}\, \mathbb{E}\left[\int_{\mathbb{R}^2} \frac{d|\zeta_h|(x,y)}{|y|^{1-s/2}} \big\|\mathcal{X}_a\big(R_{x+\mathrm{i}y}(H_G^{(\prime)}) - R_{x+\mathrm{i}y}(H_{\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\|^{s/2}\right]^{s'}\end{aligned} \tag{3.101}$$

for constants $C_1 = C_{1,p,s',I_+}$ $C_2 = C_{2,p,s,s',I_+}$. We are going to apply via a geometric resolvent equality in order to localize the difference of resolvents on the right hand side to the boundary ∂G. We distinguish between two cases.

Case 1: $\operatorname{dist}(a, \partial G) > 1$ and $\operatorname{dist}(b, \partial G) > 1$. Since $Q_b \cap G \neq \varnothing$ we have $Q_b \subset G$ in this case. Hence the geometric resolvent inequality, see e.g. [120, Lemma 2.5.2], can be applied to the operator norm in the last line of (3.101). Even though it is only stated for boxes there, the key estimate, [120, Lemma 2.5.3], covers our setup (this is where the assumption $\operatorname{dist}(\partial G, \partial \widetilde{G}) > 1$ enters). Hence we obtain

$$\big\|\mathcal{X}_a\big(R_z(H_{\omega,G}^{(\prime)}) - R_z(H_{\omega,\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\| \leqslant C_3 \sum_{c \in (\delta G)^\#} \|\mathcal{X}_a R_z(H_{\omega,\widetilde{G}}^{(\prime)})\mathcal{X}_c\| \|\mathcal{X}_{\Lambda_2(c)} R_z(H_{\omega,G}^{(\prime)})\mathcal{X}_b\|, \tag{3.102}$$

where $\Lambda_2(c) := c + \Lambda_2$ and $\delta G := \big\{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial G) \leqslant 1\big\}$. The constant C_3 is uniform in $z \in \mathbb{C}$ on each compact subset of $\mathbb{C}$. Via (3.102) we can estimate the right hand side of (3.101) as

$$\begin{aligned}\mathbb{E}\Big[\int_{\mathbb{R}^2} \frac{d|\zeta_h|(x,y)}{|y|^{1-s/2}} &\big\|\mathcal{X}_a\big(R_{x+\mathrm{i}y}(H_G^{(\prime)}) - R_{x+\mathrm{i}y}(H_{\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\|^{s/2}\Big] \\ &\leqslant C_4 \sum_{c \in (\delta G)^\#} \int_{\mathbb{R}^2} \frac{d|\zeta_h|(x,y)}{|y|^{1-s/2}}\, \mathbb{E}\Big[\big\|\mathcal{X}_a R_{x+\mathrm{i}y}(H_{\widetilde{G}}^{(\prime)})\mathcal{X}_c\big\|^s\Big]^{1/2} \\ &\qquad \times \mathbb{E}\Big[\big\|\mathcal{X}_{\Lambda_2(c)} R_{x+\mathrm{i}y}(H_G^{(\prime)})\mathcal{X}_b\big\|^s\Big]^{1/2}\end{aligned} \tag{3.103}$$

for a constant $C_4 = C_{4,s,I_+}$. We again decompose the support of ζ_h as in (3.82) and treat the product of the expectations on Z_1 via the fractional moment estimate as in (3.85) and on Z_2 with the Combes-Thomas estimate as in (3.84). The remaining integral is then estimated as in (3.55). Overall this yields constants $C_5, C_6, \mu > 0$, which only depend on s and I, such that

$$\mathbb{E}\Big[\int_{\mathbb{R}^2}\frac{d|\zeta_h|(x,y)}{|y|^{1-s/2}}\,\big\|\mathcal{X}_a\big(R_{x+iy}(H_G^{(\prime)})-R_{x+iy}(H_{\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\|^{s/2}\Big]$$
$$\leqslant C_5\sum_{c\in(\delta G)^{\#}}\mathrm{e}^{-\mu(|a-c|+|c-b|)}\big(\|h\|_1+\mathrm{TV}(h)\big)$$
$$\leqslant C_6\,\mathrm{TV}(f)\,\mathrm{e}^{-(\mu/2)[\mathrm{dist}(a,\partial G)+\mathrm{dist}(b,\partial G)]}, \tag{3.104}$$

where in the last step we also applied (3.92). Now, the claim follows upon inserting (3.104) into (3.101) and observing (3.93). This finishes the first case.

Case 2: $\mathrm{dist}(a,\partial G)\leqslant 1$ or $\mathrm{dist}(b,\partial G)\leqslant 1$. This in particular means that

$$|a-b|\geqslant\max\big\{\mathrm{dist}(a,\partial G),\mathrm{dist}(b,\partial G)\big\}-1. \tag{3.105}$$

Hence we can estimate the operator norm on the right hand side of (3.101) by the triangle inequality. Each of the resulting two terms then decays exponentially in $|a-b|$ by the fractional moment estimate (2.11) and Lemma 3.10. The remaining integral is again estimated by (3.55) and (3.92). Overall, we obtain

$$\mathbb{E}\Big[\int_{\mathbb{R}^2}\frac{d|\zeta_h|(x,y)}{|y|^{1-s/2}}\,\big\|\mathcal{X}_a\big(R_{x+iy}(H_G^{(\prime)})-R_{x+iy}(H_{\widetilde{G}}^{(\prime)})\big)\mathcal{X}_b\big\|^{s/2}\Big]\leqslant C_7\,\mathrm{e}^{-\mu|a-b|}\,\mathrm{TV}(f) \tag{3.106}$$

for constants $C_7,\mu>0$ which only depend on on s and I. The claim follows upon inserting (3.106) into (3.101) together with the observations (3.93) and (3.105). □

Proof of Theorem 3.5 (iv). As in the proof of Theorem 3.5 (iii) can without loss of generality confine to $0<p<1$ with $k:=p^{-1}\in\mathbb{N}$. For fixed $f\in\mathcal{F}_I$ and $L>0$ we define

$$T_{\omega,f,L}:=\big(f(H_{\omega,L})-f(H'_{\omega,L})\big)-\big(f(H_\omega)-f(H'_\omega)\big). \tag{3.107}$$

Applied to the present context, the argument leading to (3.98) in the proof of Theorem 3.5 (iii) gives

$$\mathbb{E}\left[\|T_{f,L}\|_p\right]\leqslant\sum_{\substack{a_1,\dots,a_k\in\mathbb{Z}^d\\ b_1,\dots,b_k\in\mathbb{Z}^d}}\prod_{l=1}^{k}\mathbb{E}\big[\|\mathcal{X}_{a_l}T_{f,L}\mathcal{X}_{b_l}\|_p\big]^p. \tag{3.108}$$

Below, we will decompose $T_{\omega,f,L}=A_{\omega,f,L}-B_{\omega,f,L}$ in two different ways. The first decomposition is with respect to the components

$$A_{\omega,f,L}:=f(H_{\omega,L})-f(H_\omega)\quad\text{and}\quad B_{\omega,f,L}:=f(H'_{\omega,L})-f(H'_\omega), \tag{3.109}$$

and the second is with respect ot the components

$$A_{\omega,f,L}:=f(H_{\omega,L})-f(H'_{\omega,L})\quad\text{and}\quad B_{\omega,f,L}:=f(H_\omega)-f(H'_\omega). \tag{3.110}$$

In both casees, the adapted triangle inequality (3.74) and Minkowski's inequality on $L^k(\Omega,\mathbb{P})$ imply (recall that $kp=1$) for all $a,b\in\mathbb{R}^d$

$$\mathbb{E}\big[\|\mathcal{X}_aT_{f,L}\mathcal{X}_b\|_p\big]^p\leqslant\sum_{j=1}^{2}\mathbb{E}\big[\|\mathcal{X}_aT_{f,L}^{(j)}\mathcal{X}_b\|_p\big]^p. \tag{3.111}$$

This yields the bound

$$\mathbb{E}\,[\|T_{f,L}\|_p] \leqslant C_1 \sum_{\substack{a_1,\dots,a_k\in\mathbb{Z}^d\\ b_1,\dots,b_k\in\mathbb{Z}^d}} \prod_{l=1}^{k} \Big(\sum_{j=1}^{2} \mathbb{E}\big[\|\mathcal{X}_{a_l} T_{f,L}^{(j)} \mathcal{X}_{b_l}\|_p\big]^p \Big). \tag{3.112}$$

Next we split the summation over each pair $(a_l, b_l) \in \mathbb{Z}^d \times \mathbb{Z}^d$ into two parts:

$$\Lambda_{L/2}^{2,\#} := (\Lambda_{L/2} \times \Lambda_{L/2}) \cap (\mathbb{Z}^d \times \mathbb{Z}^d) \quad \text{and} \quad \Lambda_{L/2}^{2,\#,c} := (\mathbb{Z}^d \times \mathbb{Z}^d) \setminus \Lambda_{L/2}^2. \tag{3.113}$$

For $(a_l, b_l) \in \Lambda_{L/2}^2$ we use the decomposition (3.109). Theorem 3.5 (ii) then provides constants $C_2 = C_{2,p,I}$ and $\mu_2 = \mu_{2,p,I} > 0$ such that

$$\begin{aligned} \sum_{(a,b)\in\Lambda_{L/2}^{2,\#}} \sum_{j=1}^{2} \mathbb{E}\big[\|\mathcal{X}_{a_l} T_{f,L}^{(j)} \mathcal{X}_{b_l}\|_p\big]^p & \\ &\leqslant C_2\, \mathrm{TV}(f)^p \sum_{(a,b)\in\Lambda_{L/2}^{2,\#}} \mathrm{e}^{-\mu_2 p[(\mathrm{dist}(a,\partial\Lambda_L)+\mathrm{dist}(b,\partial\Lambda_L))]} \\ &\leqslant C_2\, \mathrm{TV}(f)^p\, (L+1)^{2d}\, \mathrm{e}^{-\mu_2 L p/2}\,. \end{aligned} \tag{3.114}$$

For $(a_l, b_l) \in \Lambda_{L/2}^{2,\#,c}$ we compare unperturbed and perturbed operators, that is, we choose the decomposition (3.110). Theorem 3.5 (i) now provides constants C_3, C_4 and $\mu_3 > 0$ which depend on p and I such that

$$\begin{aligned} \sum_{(a,b)\in\Lambda_{L/2}^{2,\#,c}} \sum_{j=1}^{2} \mathbb{E}\big[\|\mathcal{X}_{a_l} T_{f,L}^{(j)} \mathcal{X}_{b_l}\|_p\big]^p &\leqslant C_3\, \mathrm{TV}(f)^p \sum_{(a,b)\in\Lambda_{L/2}^{2,\#,c}} \mathrm{e}^{-\mu_3 p(|a|+|b|)} \\ &\leqslant 2C_3\, \mathrm{TV}(f)^p \sum_{\substack{a\in\mathbb{Z}^d\\ b\in\mathbb{Z}^d\setminus\Lambda_{L/2}}} \mathrm{e}^{-\mu_3 p(|a|+|b|)} \\ &\leqslant C_4\, \mathrm{TV}(f)^p (L+1)^{d-1}\, \mathrm{e}^{-\mu_3 L p/4}\,. \end{aligned} \tag{3.115}$$

We conclude from (3.112), (3.114) and (3.115) that

$$\mathbb{E}\,[\|T_{f,L}\|_p] \leqslant C_5\, \mathrm{TV}(f)^{pk}\, \mathrm{e}^{-\mu_5 L} \tag{3.116}$$

with constants $C_5 = C_{5,p,q,I}$ and $\mu_5 = \mu_{5,p,q,I} > 0$. ☐

3.4. Proof of results on the spectral shift function

The proofs of the statements (i)–(iv) from Theorem 3.3 are more or less independent and we prove them one after another. The first part follows from a corresponding deterministic statement, Lemma 3.13 below. The second and third part are almost immediate consequences of the technical estimates from Theorem 3.5. The proof of part (iv) is a bit more elaborate. For a very specific situation the statement was in principle proven in [30] and the appearing complications for general random Schrödinger operators and generic potentials have a technical flavor.

Lemma 3.13. *Let A and B be two self-adjoint and lower bounded operators on a Hilbert space $\mathcal{H}$. Assume that $\mathrm{e}^{-A} - \mathrm{e}^{-B} \in \mathcal{S}^1$ and that, for some open interval $I \subset \mathbb{R}$, the mapping*

$$I \ni E \mapsto \|T(E,A,B)\|_1 \tag{3.117}$$

is an $L^1(I)$-function. Then the spectral shift function and the trace of the shift operator coincide, i.e.

$$\xi(E,A,B) = \operatorname{tr} T(E,A,B) \qquad \text{for almost every } E \in I. \tag{3.118}$$

Proof of Lemma 3.13. We show that the function $E \mapsto \operatorname{tr} T(E,A,B)$ satisfies (3.21) for every $f \in C^\infty(\mathbb{R})$ with $\operatorname{supp}(f') \subseteq I$ and $\lim_{\lambda\to\infty} f(\lambda) = 0$. Given such a function, assumption (3.117) implies that

$$\mathbb{R} \ni E \mapsto |f'(E)| \|T(E,A,B)\|_1 \in L^1(\mathbb{R}). \tag{3.119}$$

Hence $\int_{\mathbb{R}} dE\, f'(E)\, T(E,A,B)$ is well-defined as a trace-norm Bochner integral. Moreover, the identity

$$f(B) - f(A) = \int_{\mathbb{R}} dE\, f'(E)\, T(E,A,B) \tag{3.120}$$

holds, as we argue below. Since the mapping $\mathcal{S}^1 \ni B \mapsto \operatorname{tr}(B)$ is a bounded linear functional on $\mathcal{S}^1$, it interchanges with the Bochner integral. Hence (3.120) implies

$$\operatorname{tr}\big(f(B) - f(A)\big) = \int_{\mathbb{R}} dE\, f'(E) \operatorname{tr} T(E,A,B). \tag{3.121}$$

It remains to prove (3.120). We compute, using Fubini's theorem,

$$f(A) = -\int_{\mathbb{R}} dE_A(\lambda) \int_\lambda^\infty d\mu\, f'(\mu) = -\int_{\mathbb{R}} d\mu\, f'(\mu) 1\!\!1_{(-\infty,\mu]}(A), \tag{3.122}$$

where we denoted the spectral measure of A by E_A The analogous computation for B gives

$$\begin{aligned} f(B) - f(A) &= \int_{\mathbb{R}} d\mu\, f'(\mu)\, T(\mu,A,B) \\ &= \int_{\mathbb{R}} d\mu\, f'(\mu)\, T(\mu,A,B). \end{aligned} \tag{3.123}$$

□

Proof of Theorem 3.3 (i). We only have to prove the left equality, cf. Remark 3.4 (i). For a compact interval $I \subset \Sigma_{\mathrm{FMB}}$, Theorem 3.5 (iii) (for $f = 1\!\!1_{(-\infty,E]}$) and Fubini's Theorem yield

$$\mathbb{E}\Big[\int_I dE\, \big\|T(E,H,H')\big\|_1\Big] < \infty. \tag{3.124}$$

This implies that almost surely $\|T(\,\cdot\,,H_\omega,H'_\omega)\|_1 \in L^1(I)$ holds, and Lemma 3.13 is applicable. □

Proof of Theorem 3.3 (ii). The statement directly follows from Theorem 3.5 (iv) (for $f = 1\!\!1_{(-\infty,E]}$). □

Proof of Theorem 3.3 (iii). Let $E, E' \in I$ with $E < E'$ and denote $J := [E, E']$. Lemma 3.11 implies

$$\begin{aligned}\mathbb{E}\big[\big\|T(E,H,H') - T(E',H,H')\big\|_1\big] \\ &= \mathbb{E}\big[\|\mathbb{1}_J(H) - \mathbb{1}_J(H')\|_1\big] \\ &\leqslant C_1 \sum_{a,b\in\mathbb{Z}^d} \mathbb{E}\big[\big\|\mathcal{X}_a\big(\mathbb{1}_J(H) - \mathbb{1}_J(H')\big)\mathcal{X}_b\big\|\big]^{\theta}, \end{aligned} \tag{3.125}$$

for $\theta \in (0,1)$ and a constant $C_1 = C_{1,\theta,I}$. Hölder's inequality, Lemma 3.12 and the bound $\|\mathbb{1}_J(H) - \mathbb{1}_J(H')\| \leqslant 1$ yield for $p > 1$

$$\begin{aligned}\mathbb{E}\big[\big\|\mathcal{X}_a\big(\mathbb{1}_J(H) - \mathbb{1}_J(H')\big)\mathcal{X}_b\big\|\big] \\ \leqslant C_2 e^{-\mu_1|a-b|}\Big(\mathbb{E}\big[\big\|\mathcal{X}_a\mathbb{1}_J(H)\mathcal{X}_b\big\|\big]^{1/p} + \mathbb{E}\big[\big\|\mathcal{X}_a\mathbb{1}_J(H')\big)\mathcal{X}_b\big\|\big]^{1/p}\Big)\end{aligned} \tag{3.126}$$

for constants $C_2 = C_{2,p}$ and $\mu_1 = \mu_{1,p}$. That the remaining expectation scales with (a fraction of) the interval length $|J|$ follows, e.g., from the local Wegner estimate [28]. For convenience we present a short alternative derivation which exploits boundedness of the resolvent's fractional moments. Let γ_J be the contour parametrized via

$$[0, 2\pi] \ni t \mapsto \gamma_J(t) := \frac{E' + E}{2} + \frac{E' - E}{2} e^{it}. \tag{3.127}$$

Then (FM) yields for $s \in (0,1)$

$$\begin{aligned}\mathbb{E}\big[\big\|\mathcal{X}_a\mathbb{1}_J(H^{(\prime)})\mathcal{X}_b\big\|\big] &\leqslant C_3 \int_0^{2\pi} \frac{dt}{\mathrm{Im}(\gamma_J(t))^{1-s}}\, \mathbb{E}\big[\big\|\mathcal{X}_a R_{\gamma_J(t)}(H^{(\prime)})\mathcal{X}_b\big\|^s\big] \\ &\leqslant C_4|J|^s.\end{aligned} \tag{3.128}$$

The statement then follows from combining (3.125), (3.126) and (3.128), together with suitable choices of θ, p and s (subject to $s\theta/p = \alpha$ with α as in the theorem's statement). □

For the proof of Theorem 3.3 (iv) we apply the following auxiliary statement.

Lemma 3.14. *Let $E \in \mathbb{R}$ and let $\Upsilon \subset \mathbb{R}^d$ be a Borel set with* $\mathrm{int}(\Upsilon) \neq \varnothing$. *Then there exists $\gamma > 0$ such that for every non-negative, measurable function $f : \mathbb{R} \to [0, \infty)$ with support* $\mathrm{supp}(f) \subseteq (-\infty, E]$ *the lower bound*

$$\mathbb{E}\big[\,\mathrm{tr}\,\big(\mathcal{X}_\Upsilon f(H)\big)\big] \geqslant \gamma \int_{\mathbb{R}} dE'\, \mathcal{N}'(E') f(E') \tag{3.129}$$

holds. If the operator $\mathcal{X}_\Upsilon f(H)$ is not trace class, the left hand side is interpreted as $+\infty$.

Proof. By standard reasoning in integration theory, the lemma follows from the corresponding statement for indicator functions of Borel sets $B \subseteq (-\infty, E)$. By the comparison theorem for measures, see e.g. [44, Thm. II.5.8], it is moreover enough to prove it for semi-open intervals $J \subset (-\infty, E]$. For the rest of the argument we fix such an interval I. The left-hand side of (3.129) is monotone in, and invariant under $\mathbb{Z}^d$-translations of, the set Υ. We hence can without loss of generality assume that $\Upsilon = B_r(x_0) \subset \Lambda_1$. Below we apply an infinite-volume unique continuation principle for spectral projections. Such a unique continuation principle has recently been established in [97]. For convenience we derive of such an infinite-volume unique continuation principle from the more classical finite-volume version from [96, Cor. 2.3] and [80, Thm. 1.1]. In our context it reads

$$\mathbb{1}_I(H_{\omega,L}) \leqslant \frac{1}{\gamma}\mathbb{1}_I(H_{\omega,L})\, \mathcal{X}_{\Upsilon_L}\mathbb{1}_I(H_{\omega,L}) \tag{3.130}$$

for intervals $I \subset (-\infty, E]$ and length scales $L \in \mathbb{N}$, where $\Upsilon_L := \Lambda_L \cap \big(\bigcup_{k\in\mathbb{Z}^d}(k+\Upsilon)\big)$. The finite non-random constant $\gamma > 0$ only depends on d, E, V_{per} and V_0. Let's fix a typical $\omega \in \Omega$. For a dense set of energies $E' \subset (-\infty, E]$ (that depends on the realization ω) we know that E' is not an eigenvalue of H_ω. Fix such E' and let $I = (-\infty, E']$, then

$$\operatorname*{s-lim}_{L\to\infty} \mathbb{1}_{I_{E'}}(H_{\omega,L}) = \mathbb{1}_{I_{E'}}(H_\omega) \quad \text{and} \quad \operatorname*{s-lim}_{L\to\infty} \mathcal{X}_{\Upsilon_L} = \mathcal{X}_{\Upsilon_\infty}, \tag{3.131}$$

where s-lim refers to convergence in the strong operator topology and $\mathcal{X}_{\Upsilon_\infty} := \bigcup_{k\in\mathbb{Z}^d}(k+\Upsilon)$. Since moreover $\|\mathcal{X}_{\Upsilon_L}\| = \|\mathbb{1}_{I_{E'}}(H_{\omega,L})\| = 1$ for all $L \geqslant 2$, we obtain

$$\mathbb{1}_{I_{E'}}(H_\omega) \leqslant \frac{1}{\gamma}\mathbb{1}_{I_{E'}}(H_\omega)\, \mathcal{X}_{\Upsilon_\infty}\mathbb{1}_{I_{E'}}(H_\omega). \tag{3.132}$$

This form inequality holds for all $I = (-\infty, E']$ with E' as specified above. Let $B \subset (-\infty, E)$ be a Borel-measurable set. If we choose E' sufficiently large such that $B \subset (-\infty, E']$ then evaluation of the form inequality (3.132) for functions $\mathbb{1}_B(H_\omega)\phi \in L^2(\mathbb{R}^d)$ with $\phi \in L^2(\mathbb{R}^d)$ yields

$$\mathbb{1}_B(H_\omega) \leqslant \frac{1}{\gamma}\mathbb{1}_B(H_\omega)\, \mathcal{X}_{\Upsilon_\infty}\mathbb{1}_B(H_\omega), \tag{3.133}$$

which hence holds almost surely for all Borel-measurable sets $B \subset (-\infty, E')$. We now apply this for the interval $J \subset (-\infty, E)$ as specified at the proof's beginning. Together with the $\mathbb{Z}^d$-ergodicity of H_ω this yields

$$\begin{aligned}\gamma\,\mathbb{E}\left[\operatorname{tr}(\mathcal{X}_0\mathbb{1}_J(H))\right] &\leqslant \mathbb{E}\left[\operatorname{tr}(\mathcal{X}_0\mathbb{1}_J(H)\mathcal{X}_{\Upsilon_\infty}\mathbb{1}_J(H)\mathcal{X}_0)\right]\\ &= \sum_{k\in\mathbb{Z}^d}\mathbb{E}\left[\operatorname{tr}(\mathcal{X}_{k+\Upsilon}\mathbb{1}_J(H)\mathcal{X}_0\mathbb{1}_J(H)\mathcal{X}_{k+\Upsilon})\right]\\ &= \sum_{k\in\mathbb{Z}^d}\mathbb{E}\left[\operatorname{tr}(\mathcal{X}_{\Upsilon}\mathbb{1}_J(H)\mathcal{X}_k\mathbb{1}_J(H)\mathcal{X}_{\Upsilon})\right]\\ &= \mathbb{E}\left[\operatorname{tr}(\mathcal{X}_{\Upsilon}\mathbb{1}_J(H))\right].\end{aligned} \tag{3.134}$$

□

Proof of Theorem 3.3 (iv). Throughout the proof we abbreviate the spectral shift functions by $\xi_{\omega(,L)} := \xi(\,\cdot\,, H_{\omega(,L)}, H'_{\omega(,L)})$. Since $\xi_{\omega(,L)}$ depends monotonously on the perturbation and $W \geqslant cV_0$, we assume without loss of generality that $W = cV_0$.

By Theorem 3.3 (iii) and the subsequent Remark 3.29 the function $E \mapsto \mathbb{E}[\xi(E)]$ is for any $\alpha \in (0,1)$ Hölder continuous with exponent α on compact intervals in Σ_{FMB}. Let $E \in \Sigma_{\text{FMB}}$ and $\varepsilon_0 > 0$ such that $I_{\varepsilon_0} := [E-\varepsilon_0, E+\varepsilon_0] \subset \Sigma_{\text{FMB}}$. Consider any $\varepsilon \in (0, \varepsilon_0]$. Then

$$\mathbb{E}\left[\xi(E)\right] \geqslant \frac{1}{2\varepsilon}\int_{I_\varepsilon} dE'\,\mathbb{E}\left[\xi(E')\right] - C_1\varepsilon^\alpha, \tag{3.135}$$

where C_1 depends on α, E and ε_0, but not on ε. We denote by $\mathbb{E}_{\neq 0}[\,\cdot\,]$ the averaging with respect to all random variables but ω_0 and infer from the Birman-Solomyak formula [19]

$$\begin{aligned}\int_{I_\varepsilon} dE'\,\mathbb{E}\left[\xi(E')\right] &= \int_0^1 ds\,\mathbb{E}\left[\operatorname{tr}\left(cV_0\,\mathbb{1}_{I_\varepsilon}(H+scV_0)\right)\right]\\ &= \int_0^c ds \int_0^1 d\omega_0\,\rho(\omega_0)\,\mathbb{E}_{\neq 0}\Big[\operatorname{tr}\big(V_0\mathbb{1}_{I_\varepsilon}(H+sV_0)\big)\Big].\end{aligned} \tag{3.136}$$

Let's fix a parameter $s_0 \in (0, \min\{1, c\}]$ to be determined later and abbreviate

$$J := \int_0^{s_0} d\omega_0\,\mathbb{E}_{\neq 0}\left[\operatorname{tr}\left(V_0\mathbb{1}_{I_\varepsilon}(H)\right)\right]. \tag{3.137}$$

In (3.136) we perform the change of variables $\omega_0 \mapsto \omega_0 + s$, and subsequently restrict the s-integration to $[0, s_0]$ and the ω_0-integration to $[s_0, 1]$. This way we obtain

$$\begin{aligned}\int_{I_\varepsilon} dE'\,\mathbb{E}\left[\xi(E')\right] &\geqslant s_0\rho_- \int_{s_0}^1 d\omega_0\,\mathbb{E}_{\neq 0}\Big[\operatorname{tr}\big(V_0\mathbb{1}_{I_\varepsilon}(H)\big)\Big]\\ &\geqslant s_0\rho_-\left(\frac{1}{\rho_+}\mathbb{E}\left[\operatorname{tr}\left(V_0\mathbb{1}_{I_\varepsilon}(H)\right)\right] - \int_0^{s_0} d\omega_0\,\mathbb{E}_{\neq 0}\left[\operatorname{tr}\left(V_0\mathbb{1}_{I_\varepsilon}(H)\right)\right]\right)\\ &\geqslant s_0\rho_-\left(\frac{\gamma\mathcal{N}(I_\varepsilon)}{\rho_+} - J\right),\end{aligned} \tag{3.138}$$

where the last inequality follows from $V_0 \geqslant v_-\mathcal{X}_{B_r(0)}$ together with Lemma 3.14. To estimate J we first exchange the operator H_ω by its finite-volume restriction $H_{\omega,L}$. The error arising from this modification of the operator can be bounded by Theorem 3.5 (ii), which yields constants $C_2, \mu_1 > 0$ such that

$$\mathbb{E}\left[\left|\operatorname{tr}\left(V_0\mathbb{1}_{I_\varepsilon}(H)\right) - \operatorname{tr}\left(V_0\mathbb{1}_{I_\varepsilon}(H_L)\right|\right] \leqslant C_2\,\mathrm{e}^{-\mu_1 L} \tag{3.139}$$

for all $L > 0$. This implies the bound

$$J \leqslant s_0 \sup_{\omega_0\in[0,s_0]} \mathbb{E}_{\neq 0}\left[\operatorname{tr}\left(V_0\mathbb{1}_{I_\varepsilon}(H_L)\right)\right] + \frac{C_2}{\rho_-}\,\mathrm{e}^{-\mu_1 L} \tag{3.140}$$

for every $L > 0$. Let $e_1 := (1, 0, ..., 0) \in \mathbb{R}^d$ be the unit vector along the first coordinate axis. For the subset of lattice points $A_L := \big(e_1 + (3\mathbb{Z})^d\big) \cap \Lambda_L$ we denote by $\omega_{A_L} = (\omega_k)_{k\in A_L}$ the family of random couplings supported in A_L and by $\omega_{A_L^c}$ the family of the remaining random variables. Moreover we introduce the random background operator

$$\widetilde{H}_{\omega_{A_L^c},L} := H_{0,L} + \sum_{k\in\mathbb{Z}^d\setminus A_L} \omega_k V_k, \tag{3.141}$$

where $H_{0,L}$ is the Dirichlet restriction of H_0 to Λ_L. However, for any fixed realization of coupling constants $\omega_{A_L^c}$, we view $\widetilde{H}_L := \widetilde{H}_{\omega_{A_L^c},L}$ as a non-random operator with a non-periodic background potential that is bounded uniformly in $\omega_{A_L^c}$. Now, the Dirichlet restriction of $H_{\omega,L}$ takes the form

$$H_{\omega,L} = \widetilde{H}_{\omega_{A_L^c},L} + \sum_{k\in A_L} \omega_k V_k. \tag{3.142}$$

After scaling by a factor 1/3 and (if required) introducing a energy shift, it constitutes a *crooked Anderson Hamiltonian* in the sense of [80]. We apply the Wegner estimate [80, Thm. 1.4] and obtain a finite constant $C_3 > 0$ such that

$$\begin{aligned}\sup_{\omega_k\in[0,1],\, k\in\mathbb{Z}^d\setminus A_L} \mathbb{E}_{A_L}\left[\operatorname{tr}\left(V_0 \mathbb{1}_{I_\varepsilon}(H_L)\right]\right. &\leqslant \|V_0\|_\infty \sup_{\omega_k\in[0,1],\, k\in\mathbb{Z}^d\setminus A_L} \mathbb{E}_{A_L}\left[\operatorname{tr}\left(\mathbb{1}_{I_\varepsilon}(H_L)\right]\right. \\ &\leqslant C_3 2\varepsilon L^d, \end{aligned} \tag{3.143}$$

where $\mathbb{E}_{A_L}$ denotes the average over the couplings ω_{A_L}. The estimate (3.143) holds for all $\varepsilon \in (0, \varepsilon_1]$ and all length-scales $L \geqslant L_1$ such that $L/3 \in \mathbb{N}$ is odd, and ε_1 and L_1 depend only on model parameters. We insert (3.143) into (3.140) and obtain

$$J \leqslant C_3 2\varepsilon s_0 L^d + \frac{C_2}{\rho_-}\, \mathrm{e}^{-\mu L}. \tag{3.144}$$

Combining (3.135), (3.138) and (3.144), we conclude

$$\mathbb{E}\left[\xi(E)\right] \geqslant \rho_- s_0 \left(\frac{\gamma}{\rho_+} \frac{\mathcal{N}(I_\varepsilon)}{2\varepsilon} - C_3 s_0 L^d - \frac{C_2}{\rho_-} \frac{\mathrm{e}^{-\mu L}}{2\varepsilon} \right) - C_1 \varepsilon^\alpha \tag{3.145}$$

for all $\varepsilon \in (0, \min\{\varepsilon_0, \varepsilon_1\}]$, all $s_0 \in (0, \min\{1, C\}]$ and all $L \geqslant L_1$ such that $L/3 \in \mathbb{N}$ is odd.

Now, suppose that E is a Lebesgue point of the integrated density of states $\mathcal{N}$, which is the case Lebesgue-almost everywhere. Then we have $\mathcal{N}(I_\varepsilon)/2\varepsilon \to \mathcal{N}'(E)$ as $\varepsilon \downarrow 0$. Suppose also that $\mathcal{N}'(E) > 0$. Then, there exists $\varepsilon_2 \in (0, \min\{\varepsilon_0, \varepsilon_1\}]$ such that $\mathcal{N}(I_\varepsilon)/2\varepsilon \geqslant \mathcal{N}'(E)/2$ for every $\varepsilon \in (0, \varepsilon_2]$. Finally, we choose $L \geqslant L_1$ with $L/3 \in \mathbb{N}$ odd so large that $\varepsilon := L^{-4d/\alpha} \leqslant \varepsilon_2$ and $s_0 := L^{-2d} \leqslant \min\{1, C\}$. In this case (3.145) yields

$$\mathbb{E}\left[\xi(E)\right] \geqslant \frac{\gamma\rho_-}{2\rho_+} L^{-2d}\Big(\mathcal{N}'(E) - O(L^{-d})\Big). \tag{3.146}$$

The right hand side of (3.146) is strictly positive by possibly enlarging L even further. ☐

We close this section with a lemma that is essentially contained in [13] and which yields a proof of Remark 3.2 (iv).The lemma is reviewed here for convenience and is stated and proved in a more convenient general framework.

Lemma 3.15. *Let A, B be self-adjoint operators in a Hilbert space, which are both bounded from below. Assume that the perturbation $B - A$ is sign-definite (that is, there exists $\alpha \in \{\pm 1\}$ such that $\alpha(B - A) \geqslant 0$) and that $T(E, A, B) \in \mathcal{S}^1$ for some $E \in \mathbb{R}$. Then, we have*

$$\dim\ker\big(T(E, A, B) + \alpha\mathbb{1}\big) = 0 \tag{3.147}$$

and

$$\theta(E, A, B) = \alpha \dim\ker\big(T(E, A, B) - \alpha\mathbb{1}\big). \tag{3.148}$$

In particular, we have

$$1 \in \sigma\big(T(E, A, B)^2\big) \iff \theta(E, A, B) \neq 0. \tag{3.149}$$

Proof. The implication "⇐" in (3.149) follows immediately from (3.148); for "⇒", use (3.148) and (3.147). The equality (3.148) follows from the definition of the index and (3.147). Thus, it remains to prove (3.147). We restrict ourselves to the case $B - A \geqslant 0$ and define the linear subspace

$$\mathcal{V}_- := \ker\big(\mathbb{1}_{(-\infty,E]}(A)\big) \cap \operatorname{ran}\big(\mathbb{1}_{(-\infty,E]}(B)\big). \tag{3.150}$$

Clearly, $\mathcal{V}_- \subseteq \ker\big(T(E,A,B)+\mathbb{1}\big)$. Conversely, let $\psi \in \ker\big(T(E,A,B)+\mathbb{1}\big)$, then

$$\langle \psi, \mathbb{1}_{(-\infty,E]}(A)\psi\rangle + \langle \psi, (1-\mathbb{1}_{(-\infty,E]}(B))\psi\rangle = 0, \tag{3.151}$$

which implies $\psi \in \mathcal{V}_-$. This proves $\mathcal{V}_- = \ker\big(T(E,A,B)+\mathbb{1}\big)$. It remains to show that $\mathcal{V}_- = \{0\}$. Pick $E_0 \in \mathbb{R}$ such that $A, B > E_0$. If $E \leqslant E_0$ then $\operatorname{ran}\big(\mathbb{1}_{(-\infty,E]}(B)\big) = \{0\}$ and so $\mathcal{V}_- = \{0\}$. Now, consider the case $E > E_0$. We assume that there exists $\psi \in \mathcal{V}_-$ with $\|\psi\| = 1$. Thus we have both $\psi \in \ker\big(\mathbb{1}_{(-\infty,E]}(A)\big) = \operatorname{ran}\big(\mathbb{1}_{(E,\infty)}(A)\big)$ and $\psi \in \operatorname{ran}\big(\mathbb{1}_{(-\infty,E]}(B)\big)$. Using operator-monotonicity of the resolvent below the spectrum, we conclude

$$(E-E_0)^{-1} \leqslant \langle \psi, R_{E_0}(B)\psi\rangle \leqslant \langle \psi, R_{E_0}(A)\psi\rangle < (E-E_0)^{-1}, \tag{3.152}$$

a contradiction. □

3.5. Proof of results on Anderson orthogonality

We start with a proof of the algebraic identity (3.7). This identity is the starting point for the proof of the first part of the theorem, which constitutes the bulk part of this section. The remaining parts (ii) and (iii) follow more or less directly from Theorem 3.3.

Lemma 3.16. *Let $P_{\omega,N,L}$ and $Q_{\omega,N,L}$ denote the projections defined in* (3.8) *and $S_{\omega,N,L}$ the ground-state overlap defined in* (3.6)*. Then the ground-state overlap can be written as*

$$\begin{aligned} S_{\omega,N,L} &= \det\big(\mathbb{1} - (P_{\omega,N,L} - Q_{\omega,N,L})^2\big)^{1/4} \\ &= \det\big(\mathbb{1} - P_{\omega,N,L}(\mathbb{1} - Q_{\omega,N,L})P_{\omega,N,L}\big)^{1/2} && (3.153) \\ &= \det\big(\mathbb{1} - (\mathbb{1} - P_{\omega,N,L})Q_{\omega,N,L}(\mathbb{1} - P_{\omega,N,L})\big)^{1/2}. && (3.154) \end{aligned}$$

Proof. For the proof we abbreviate $P := P_{\omega,N,L}$, $Q := Q_{\omega,N,L}$ and $S := S_{\omega,N,L}$. For the $N \times N$-matrix $M := \big(\langle \varphi^L_{\omega,j}, \psi^L_{\omega,k}\rangle\big)_{1\leqslant j,k\leqslant N}$, the matrix entries of MM^* and M^*M are given by

$$\begin{aligned} (MM^*)_{jl} &= \sum_{k=1}^N \langle \varphi^L_{\omega,j}, \psi^L_k\rangle\langle \psi^L_{\omega,k}, \varphi^L_{\omega,l}\rangle = \langle \varphi^L_{\omega,j}, PQP\varphi^L_{\omega,l}\rangle, \\ (M^*M)_{jl} &= \sum_{k=1}^N \langle \psi^L_{\omega,j}, \varphi^L_{\omega,k}\rangle\langle \varphi^L_{\omega,k}, \psi^L_{\omega,l}\rangle = \langle \psi^L_{\omega,j}, QPQ\psi^L_{\omega,l}\rangle \end{aligned} \tag{3.155}$$

for $1 \leqslant j,l \leqslant N$. Thus, S can be written as

$$\begin{aligned} S^2 &= \det(MM^*) = \det\big(PQP\big|_{\operatorname{ran} P}\big) = \det\big(\mathbb{1} - P(\mathbb{1}-Q)P\big) && (3.156) \\ &= \det(M^*M) = \det\big(QPQ\big|_{\operatorname{ran} Q}\big) = \det\big(\mathbb{1} - Q(\mathbb{1}-P)Q\big). && (3.157) \end{aligned}$$

The above determinants are well-defined Fredholm determinants as the operators $P(\mathbb{1}-Q)P$ and $Q(\mathbb{1}-P)Q$ have finite rank. Because the non-zero singular values of the operator $Q(\mathbb{1}-P)$ coincide with the non-zero singular values of its adjoint $(\mathbb{1}-P)Q$ we moreover have

$$\det\big(\mathbb{1} - Q(\mathbb{1}-P)Q\big) = \det\big(\mathbb{1} - (\mathbb{1}-P)Q(\mathbb{1}-P)\big). \tag{3.158}$$

Combining (3.157) with (3.158) and multiplying the result with (3.156) gives

$$S^4 = \det\big(\mathbb{1} - P(\mathbb{1}-Q)P\big)\det\big(\mathbb{1} - (\mathbb{1}-P)Q(\mathbb{1}-P)\big)$$
$$= \det\big(\mathbb{1} - P(\mathbb{1}-Q)P - (\mathbb{1}-P)Q(\mathbb{1}-P)\big). \tag{3.159}$$

where the last line follows from Lemma 3.17 below. □

The following Lemma is essentiall contained in [13], its short proof is included for convenience (and because I like it).

Lemma 3.17. *Let P and Q be two orthogonal projections on a joint Hilbert space and let $\overline{P} := \mathbb{1} - P$, $\overline{Q} := \mathbb{1} - Q$. Then, the formulas*

$$(P-Q)^{2n-1} = (P\overline{Q})^n - (\overline{P}Q)^n \tag{3.160}$$

and

$$(P-Q)^{2n} = (P\overline{Q}P)^n + (\overline{P}Q\overline{P})^n \tag{3.161}$$

hold for each $n \in \mathbb{N}$.

Proof. We compute

$$P - Q = P(\overline{Q} + Q) - (P + \overline{P})Q = P\overline{Q} - \overline{P}Q \tag{3.162}$$

and

$$(P-Q)^2 = (P-Q)(P-Q)^* = P\overline{Q}P + \overline{P}Q\overline{P}. \tag{3.163}$$

Formula (3.161) follows from iterated multiplications of (3.163) with itself. Formula (3.160) follows from multiplying the $(n-1)^{\text{st}}$ power of (3.163) with (3.162). □

For the remainder of this section we fix some short-hand notation. For $L > 0$ and $E \in \mathbb{R}$ the index and the shift operator of $H_{\omega(,L)}$ and $H'_{\omega(,L)}$ are abbreviated by

$$\theta_{\omega(,L)}(E) := \theta(E, H_{\omega(,L)}, H'_{\omega(,L)}), \tag{3.164}$$
$$T_{\omega(,L)}(E) := T(E, H_{\omega(,L)}, H'_{\omega(,L)}), \tag{3.165}$$

respectively. The limiting behavior of the finite-volume ground-state overlap is closely related to the spectral shift function respectively index respectively shift operator. In the following proofs we work with the Fredholm index, but this choice is rather arbitrary. Note, however, that $\theta_\omega(E)$ is shorter than $\operatorname{tr} T_\omega(E)$.

Proof of Theorem 3.1 (i). For the proof we assume that Theorem 3.1 (ii), which is proven below, holds. Let $E \in \Sigma_{\text{FMB}} \cap \operatorname{int}(\Sigma)$ be fixed and let $(L_n)_{n\in\mathbb{N}}$ be a sequence of length scales with $L_n/\log(n) \to \infty$ as $n \to \infty$. Our aim is to prove that $S_{\omega,L}(E) \to S_\omega(E)$ as $n \to \infty$ almost surely. This implies (3.12) by virtue of a subsubsequence argument, applied to the real-valued sequence $(\mathbb{E}[|S_L(E) - S(E)|])_{L>0}$: Once we arrived at a subsubsequence (which for simplicity is again denoted by $(L_n)_n$) with $L_n/\log(n) \to \infty$, the above pointwise convergence holds. Since moreover $|S_{\omega,L}(E) - S_\omega(E)| \leqslant 1$ the claim follows from dominated convergence.

For the sequence $(L_n)_{n\in\mathbb{N}}$ of length scales chosen above, an application of Theorem 3.3 (see also Remark 3.4 (iv)) yields

$$\sum_{n\in\mathbb{N}} \mathbb{E}\left[|\theta_{L_n}(E)-\theta(E)|\right] \leqslant \sum_{n\in\mathbb{N}} \mathbb{E}\left[\|T_{L_n}(E)-T(E)\|_1\right] \leqslant C\sum_{n\in\mathbb{N}} \mathrm{e}^{-\mu L_n} < \infty, \tag{3.166}$$

which in turn almost surely yields the pointwise convergence $\theta_{\omega,L_n}(E)\to\theta_\omega(E)$ as $n\to\infty$. We included the corresponding statement for the shift operators here since it is employed below. Because the index is integer valued, we can define a (E-dependent) random variable $n_0 := \Omega\to\mathbb{N}$ such that almost surely

$$\theta_{\omega,L_n}(E)=\theta_\omega(E) \qquad \text{for all } n\geqslant n_0(\omega). \tag{3.167}$$

For technical reasons we split the proof into two cases: The case $\theta_\omega(E)=0$, in which case $S_\omega(E)\neq 0$, and the case $\theta_\omega(E)\neq 0$, in which case $S_\omega(E)=0$. For the remainder of this proof we denote

$$P_{\omega,n} := P_{\omega,N_{\omega,L_n}(E),L_n} \quad \text{and} \quad Q_{\omega,n} := Q_{\omega,N_{\omega,L_n}(E),L_n}. \tag{3.168}$$

Case $\theta_\omega(E)=0$*:* Let $n\geqslant n_0(\omega)$. In this situation we have

$$P_{\omega,n}=\mathbb{1}_{(-\infty,E]}(H_{\omega,L_n}) \quad \text{and} \quad Q_{\omega,n}=\mathbb{1}_{(-\infty,E]}(H'_{\omega,L_n}). \tag{3.169}$$

The first relation follows from the definition of the particle-number sequence $N_{\omega,L_n}(E)$ and is always true. The second relation follows from $n\geqslant n_0(\omega)$ and $\theta_\omega(E)=0$. By virtue of the representation from Lemma 3.16 those relations yield

$$S_{\omega,L_n}(E)^4=\det\big(\mathbb{1}-T_{\omega,L_n}(E)^2\big) \tag{3.170}$$

for the finite-volume ground-state overlap. Continuity of the Fredholm determinant with respect to the trace norm [115, Thm. 3.4] implies

$$\begin{aligned}\big|S_{\omega,L_n}(E)^4-S_\omega(E)^4\big| &= \big|\det\big(\mathbb{1}-T_{\omega,L_n}(E)^2\big)-\det\big(\mathbb{1}-T_\omega(E)^2\big)\big| \\ &\leqslant 2\|T_{\omega,L_n}(E)-T_\omega(E)\|_1 \exp\big(\|T_{\omega,L_n}(E)\|_2^2+\|T_\omega(E)\|_2^2+1\big).\end{aligned} \tag{3.171}$$

The estimate (3.166) above now yields the almost-sure convergence

$$\lim_{n\to\infty}\|T_{\omega,L_n}(E)-T_\omega(E)\|_1 \exp\big(\|T_{\omega,L_n}(E)\|_2^2+\|T_\omega(E)\|_2^2+1\big)=0. \tag{3.172}$$

This implies the desired pointwise convergence in case that $\theta_\omega(E)=0$.

Case $\theta_\omega(E)\neq 0$. By definition of the index we have $1\in\sigma\big((P_{\omega,n}-Q_{\omega,n})^2\big)$ in this case. Let us assume the strong convergence

$$\operatorname*{s-lim}_{n\to\infty}(P_{\omega,n}-Q_{\omega,n})^2=T_\omega(E)^2 \qquad \text{almost surely} \tag{3.173}$$

for the time being. Note that, since $Q_{\omega,n}\neq\mathbb{1}_{(-\infty,E]}(H'_{\omega,L_n})$, this is not a completely immediate consequence of Theorem 3.3. Since $1\in\sigma\big(T_\omega(E)^2\big)$ we almost surely can find a sequence $(\alpha_{\omega,n})_{n\in\mathbb{N}}$ with $\alpha_{\omega,n}\in\sigma\big((P_{\omega,n}-Q_{\omega,n})^2\big)$ and $\alpha_{\omega,n}\to 1$ as $n\to\infty$ [106, Thm. VIII.24(a)]. Moreover, $0\leqslant(P_{\omega,n}-Q_{\omega,n})^2\leqslant 1$, and we conclude

$$S_{\omega,L_n}(E)^4=\det\big(\mathbb{1}-(P_{\omega,n}-Q_{\omega,n})^2\big)\leqslant 1-\alpha_{\omega,n}\to 0 \tag{3.174}$$

as $n\to\infty$ almost surely. This is the assertion in the case $\theta_\omega(E)\neq 0$.

Thus, it remains to prove (3.173), which we prove for $\theta_\omega(E) > 0$. The other case follows along the same lines. Moreover, it suffices to prove the strong convergence $P_{\omega,n} - Q_{\omega,n} \to T_\omega(E)$ as $n \to \infty$. Let $\eta \in L^2(\mathbb{R}^d)$. We add and subtract the term $T_{L_n}(E)$ and estimate

$$\begin{aligned} \big\|\big((P_{\omega,n} - Q_{\omega,n}) - T_\omega(E)\big)\eta\big\| \leqslant \big\|\big(\mathbb{1}_{(-\infty,E]}(H'_{\omega,L_n}) - Q_{\omega,n}\big)\eta\big\| \\ + \big\|\big(T_{\omega,L_n}(E) - T_\omega(E)\big)\eta\big\|. \end{aligned} \tag{3.175}$$

The second term on the right hand side of (3.175) converges to 0 as $n \to \infty$ almost surely, again by virtue of (3.166). For the first term on the right hand side we infer from (3.167) and Theorem 3.3 (i) that that since $n \geqslant n_0(\omega)$

$$\theta_\omega(E) = \theta_{\omega,L_n}(E) = \operatorname{tr} T_{\omega,L_n}(E) = \operatorname{tr}\big(Q_{\omega,n} - \mathbb{1}_{(-\infty,E]}(H'_{\omega,L_n})\big). \tag{3.176}$$

We recall that the eigenvalues of $H'_{\omega,L}$ are denoted by $\mu^L_{\omega,1} \leqslant \mu^L_{\omega,2} \leqslant ...$, with corresponding orthonormal basis of eigenfunctions $(\psi^L_{\omega,k})_{k\in\mathbb{N}}$. Since we assumed that $\theta_\omega(E) > 0$ and $Q_{\omega,n}$ is the orthogonal projection on the eigenspaces of all eigenvalues up to the $N_{\omega,L_n}(E)$th eigenvalue of H'_{ω,L_n}, we obtain that

$$\mu^{L_n}_{\omega,N_{\omega,L_n}(E)-\theta_\omega(E)} \leqslant E \leqslant \mu^{L_n}_{\omega,N_{\omega,L_n}(E)-\theta_\omega(E)+1} \leqslant \mu^L_{\omega,N_{\omega,L_n}(E)}. \tag{3.177}$$

At this point, I'm kind of sorry for the overloaded notation. Anyway, from (3.177) we defer

$$Q_{\omega,n} - \mathbb{1}_{(-\infty,E]}(H'_{\omega,L_n}) = \sum_{k=0}^{\theta_\omega(E)-1} |\psi^{L_n}_{\omega,N_{\omega,L_n}(E)-k}\rangle\langle\psi^{L_n}_{\omega,N_{\omega,L_n}(E)-k}| \tag{3.178}$$

hold almost surely for $n \geqslant n_0(\omega)$. Note that $E \in \operatorname{int}(\Sigma)$ and $\Sigma = \sigma_{\mathrm{ess}}(H'_\omega)$. Via the strong resolvent convergence of H'_{ω,L_n} to H'_ω and Fatou's lemma we obtain for any $\varepsilon > 0$ that almost surely

$$\lim_{n\to\infty} \operatorname{tr}\big(\mathbb{1}_{[E,E+\varepsilon)}(H'_{\omega,L_n})\big) \geqslant \operatorname{tr}\big(\mathbb{1}_{[E,E+\varepsilon)}(H'_\omega)\big) = \infty. \tag{3.179}$$

This implies that for any $\varepsilon > 0$ there exists a random variable $0 < n_1 : \Omega \to \mathbb{N}$ such that almost surely

$$\sum_{k=0}^{\theta_\omega(E)-1} |\psi^{L_n}_{\omega,N_{\omega,L_n}(E)-k}\rangle\langle\psi^{L_n}_{\omega,N_{\omega,L_n}(E)-k}| \leqslant \mathbb{1}_{[E,E+\varepsilon)}(H'_\omega) \tag{3.180}$$

holds for all $n \geqslant n_1(\omega)$. This yields for any fixed $\varepsilon > 0$ the almost sure bound

$$\limsup_{n\to\infty} \big\|\big(Q_{\omega,n} - \mathbb{1}_{(-\infty,E]}(H'_{\omega,L_n})\big)\eta\big\| \leqslant \limsup_{n\to\infty} \big\|\mathbb{1}_{[E,E+\varepsilon)}(H'_{\omega,L_n})\eta\big\|. \tag{3.181}$$

Let's fix sequence $(\varepsilon_\ell)_{\ell\in\mathbb{N}} \subset (0,\infty)$ with $\varepsilon_\ell \to 0$ as $\ell \to \infty$. The points $E, E+\varepsilon_\ell$ almost surely are no eigenvalue of H'_ω. A proof of this standard consequence of spectral averaging is, for convenience, included at the end of this proof in Lemma 3.18. As a conesquence [106, Thm. VIII.24(b)] we almost surely have the strong convergence

$$\operatorname*{s-lim}_{L\to\infty} \mathbb{1}_{[E,E+\varepsilon_\ell)}(H'_{\omega,L}) = \mathbb{1}_{[E,E+\varepsilon_l)}(H'_\omega). \tag{3.182}$$

Together with (3.181) this yields

$$\begin{aligned} \limsup_{n\to\infty} \big\|\big(Q_{\omega,n} - \mathbb{1}_{(-\infty,E]}(H'_{\omega,L_n})\big)\eta\big\| &\leqslant \limsup_{\ell\to\infty} \big\|\mathbb{1}_{[E,E+\varepsilon_\ell)}(H'_\omega)\eta\big\| \\ &= \big\|\mathbb{1}_{\{E\}}(H'_\omega)\eta\big\| = 0 \end{aligned} \tag{3.183}$$

almost surely, where the last equality follows again from E being almost surely not an eigenvalue of H'_ω. □

Lemma 3.18. *Let $E \in \mathbb{R}$. Then E is not an eigenvalue of H'_ω almost surely.*

Proof. The strong convergence $-i\varepsilon R_{E+i\varepsilon}(H'_\omega) \to \mathbb{1}_{\{E\}}(H'_\omega)$ as $\varepsilon \to 0$ follows from the spectral theorem and the convergence $-i\varepsilon(x - E - i\varepsilon)^{-1} \to \mathbb{1}_{\{E\}}(x)$ as $\varepsilon \to 0$. From this, Fatou's lemma and the finiteness of fractional moments of the resolvent we infer that

$$\mathbb{E}\big[\|\mathcal{X}_a \mathbb{1}_{\{E\}}(H')\mathcal{X}_b\|^s\big] \leqslant \liminf_{\varepsilon\to 0} \varepsilon^s\, \mathbb{E}\big[\|\mathcal{X}_a R_{E+i\varepsilon}(H')\mathcal{X}_b\|^s\big] = 0 \tag{3.184}$$

for some $0 < s < 1$ and any $a, b \in \mathbb{Z}^d$. Since $\mathbb{Z}^d$ is countable, we obtain that almost surely $\mathbb{1}_{\{E\}}(H'_\omega) = 0$. □

Proof of Theorem 3.1 (ii). Let $E \in \Sigma_{\mathrm{FMB}}$. Then Theorem 3.5 (iii) implies that $T_\omega(E) \in \mathcal{S}^2$ almost surely and consequently $S_\omega(E) = \det\big(\mathbb{1} - T_\omega(E)^2\big)^{1/4}$ almost surely. By definition $S_\omega(E) = 0$ in case 1 is an eigenvalue of $T_\omega(E)^2$. Suppose, on the other hand, that 1 is not an eigenvalue of $T_\omega(E)^2$. This in particular implies that $\|T_\omega(E)\| < 1$. If we denote the non-increasingly ordered sequence of eigenvalues of $T_\omega(E)^2$ by $1 > b_{\omega,1} \geqslant b_{\omega,2} \geqslant \ldots \geqslant 0$, then the identity $\log\det(A) = \operatorname{tr}\exp(A)$ yields

$$\begin{aligned} S_\omega(E)^4 &= \exp\Big\{\sum_{n\in\mathbb{N}} \log(1 - b_{\omega,n})\Big\} = \exp\Big\{-\sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \frac{b^k_{\omega,n}}{k}\Big\} \\ &\geqslant \exp\Big\{-\sum_{n\in\mathbb{N}} b_n \sum_{k\in\mathbb{N}} b^{k-1}_{\omega,1}\Big\} = \exp\Big\{-\frac{\|T_\omega(E)\|_2^2}{1 - \|T_\omega(E)\|^2}\Big\} > 0. \end{aligned} \tag{3.185}$$

□

Proof of Theorem 3.1 (iii). By Theorem 3.3 (i) the identity $\theta(E, H_\omega, H'_\omega) = \xi(E, H_\omega, H'_\omega)$ holds almost surely for almost every $E \in \Sigma_{\mathrm{FMB}}$. The equivalence (3.17) thus implies

$$\mathbb{P}\big(S(E) = 0\big) = \mathbb{P}\big(\theta(E, H, H') > 0\big) = \mathbb{P}\big(\xi(E, H, H') > 0\big) \tag{3.186}$$

for almost every $E \in \Sigma_{\mathrm{FMB}}$. The statement now follows from Theorem 3.3 (iv). □

Proof of Theorem 3.1 (iv). We employ a Lifschitz-tail argument. From Theorem 3.1 (iii) and Corollary 4.2 we already known that there exists $E_2 > E_0$ such that $\mathbb{P}(S(E) = 0) > 0$ for almost every $E \in (E_0, E_2)$. Here E_2 is chosen so that $[E_0, E_2] \subset \Sigma_{\mathrm{FMB}}$. In virtue of Theorem 3.1 (ii) it suffices to prove for some $E_3 > E_0$ that

$$\mathbb{E}\big[\xi(E, H, H')\big] < 1 \tag{3.187}$$

for almost every $E \in [E_0, E_3]$. Together with the almost sure statement $\xi(E, H_\omega, H'_\omega) \in \mathbb{N}_0$ already implies $\mathbb{P}(\xi(E, H, H') = 0) > 0$. Our starting point is yet again the Birman-Solomyak formula, which yields

$$\begin{aligned} \mathbb{E}\big[\xi(E, H, H')\big] &= \lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon} \int_{E-\varepsilon}^{E+\varepsilon} dE'\, \mathbb{E}\big[\xi(E', H, H')\big] \\ &= \lim_{\varepsilon\searrow 0} \frac{1}{2\varepsilon} \int_0^1 ds\, \mathbb{E}\big[\operatorname{tr}\big(W \mathbb{1}_{[E-\varepsilon, E+\varepsilon]}(H + sW)\big)\big] \end{aligned} \tag{3.188}$$

We now apply the local Wegner estimate with Lifschitz tail constant [28, Thm. 2.4(ii)]. The results are formulated for finite-volume operators with periodic boundary conditions but extend to the infinite-volume operator. This yields

$$\mathbb{E}\big[\operatorname{tr}\big(W\mathbb{1}_{[E-\varepsilon,E+\varepsilon]}(H+sW)\big)\big] \leqslant \frac{2\varepsilon}{2} = \varepsilon \tag{3.189}$$

for all $E > E_0$ and $\varepsilon > 0$ such that $E+\varepsilon \leqslant E_2$ for some fixed energy $E_2 > E_0$. Together with (3.188) this yields $\mathbb{E}\big[\xi(E,H,H')\big] \leqslant 1/2$ for almost every $E < [E_0, E_2]$. The statement follows with $E_1 := \min\{E_2, E_3\}$. □

CHAPTER 4

A lower bound on the density of states

Context: The main result presented in this chapter coincide with the main results from [37], which was written in collaboration with **Martin Gebert, Peter Hislop, Abel Klein** and **Peter Müller**. The proof of the main result presented below follows a slightly different strategy and has, in the present form, not been published previously.

Content: This chapter is devoted to a proof of the reverse Wegner estimate and its consequence, a strictly positive lower bound on the density of states. Two auxiliary results are contained which may be of independent interest, Lemmas 4.4 and 4.5. But since similar results are contained in Chapters 3 and 5, respectively, the auxiliary lemmas are only stated in the very specific setup needed below.

4.1. Discussion of results

We again work with the random Schrödinger operator

$$H_\omega = H_0 + V_\omega = -\Delta + V_{\mathrm{per}} + V_\omega \tag{4.1}$$

from Section 2.1, subject to the assumptions (V_1) - (V_3) and again with the choice $\mu = 1$. Moreover, we require the following additional assumption.

(V_4) The single-site probability density ρ is bounded from below on its support,

$$\rho_- := \operatorname*{ess\,inf}_{x\in[0,1]} \rho(x) > 0. \tag{4.2}$$

As usual Σ_0 denotes the spectrum of the non-random operator $H_0 = -\Delta + V_{\mathrm{per}}$ and Σ denotes the almost surely non-random spectrum of H_ω. Its infimum is denoted by $E_0 = \inf \Sigma$. The set $\Gamma_L = \mathbb{Z}^d \cap \Lambda_{L+2R} \subset \mathbb{Z}^d$ is the index set of random couplings on which the finite-volume restriction $H_{\omega,L}$ depends. This chapter's results, the lower Wegner estimate and the lower bound on the density of states, read as follows.

Theorem 4.1 (Reverse Wegner estimate). *Let $I \subset \Sigma_{\mathrm{FMB}} \cap \operatorname{int}\big(\Sigma_0 + [0, V_-]\big)$ be a compact interval. Then there exists a constant $C > 0$ and an initial length scale $\mathcal{L} > 0$ such that*

$$\mathbb{E}\left[\operatorname{tr} \mathbb{1}_J(H_L)\right] \geqslant C|J||\Lambda_L| \tag{4.3}$$

holds for all intervals $J \subset I$ and all $L > \mathcal{L}$.

Corollary 4.2 (Strict positivity of $\mathcal{N}'$). *Let $I \subset \Sigma_{\mathrm{FMB}} \cap \operatorname{int}\big(\Sigma_0 + [0, V_-]\big)$ be a compact interval and $C > 0$ the constant from Theorem 4.1. Then*

$$\operatorname*{ess\,inf}_{E\in I} \mathcal{N}'(E) \geqslant C. \tag{4.4}$$

Remarks 4.3. (i) The assumption $I \subset \Sigma_{\mathrm{FMB}}$ is unnatural here and was imposed to salvage technical issues of the proof, see Section 4.2 for details. That this localization assumption can indeed be dropped has recently been proved in [49] after this thesis was finished.

(ii) Let's assume that no spectral gaps are present in Σ_0. Then $\Sigma_0 = [E_0, \infty)$ and consequently $\Sigma_0 + [0, V_-] = \Sigma_0 = \Sigma$ due to $V_\omega \geqslant 0$. But in general only

$$\Sigma_0 + [0, V_-] \subseteq \Sigma \subseteq \Sigma_0 + [0, V_+] \tag{4.5}$$

holds. Hence, if Σ_0 has a spectral gap of size $> V_-$ then the random potential in general can't close the spectral gap. In this situation our method does not directly yield a lower bound at the right edge of the corresponding spectral gap of Σ.

(iii) We pick up on a remark already made in the introduction. It is well-known [26, Prop. VI.1.3] that $\Sigma = \operatorname{supp}(\mathcal{N}')$. But this alone does not yet imply $\mathcal{N}' > 0$ Lebesgue-almost everywhere on Σ. An example is provided by the indicator function of the complement of a "fat" Cantor set: A nowhere dense set with positive Lebesgue measure.

4.2. Proof's idea & more

Locally uniform strict positivity of the density of states has been proved for the classical Anderson model on the lattice $\mathbb{Z}^d$ [71, 62], following an argument given in [128]. The key observation that was made in the last mentioned publication in this context is the following: The change of variables for the random couplings

$$(\omega_k)_{k\in\Gamma_L} \to (\eta_k)_{k\in\Gamma_L} \quad \text{with} \quad \eta_k := \begin{cases} \omega_k - \omega_0 & \text{for } k \in \Gamma_L \setminus \{0\}, \\ \omega_0 & \text{for } k = 0, \end{cases} \tag{4.6}$$

yields a coupling η_0 which roughly acts as an energy shift on the finite-volume operator $H_{\omega,L}$. To emphasize this we set $V_- = V_+ = 1$ for the remainder of this subsection, since in this case the coupling η_0 indeed acts as an energy shift. Downsizing the support of all the other random couplings $(\eta_k)_{k\in\Gamma_L\setminus\{0\}}$ results in a lower bounds for the expected number of eigenvalues of $H_{\omega,L}$ in some small energy interval $[E, E+\varepsilon]$. Hence we first downsize the support of all the random couplings $\eta_k, k \neq 0$, until they are small compared to the energy shift performed by η_0. This roughly yields (neglecting any sort of details)

$$\mathbb{E}\big[\operatorname{tr} \mathbb{1}_{[E,E+\varepsilon]}(H_L)\big] \geqslant C^{|\Gamma_L|-1} \int_0^1 d\eta_0\, \mathbb{E}\big[\operatorname{tr} \mathbb{1}_{[E,E+\varepsilon]}(H_{0,L} + \eta_0)\big], \tag{4.7}$$

where the constant $C < 1$ was created by the downsizing of one of the random couplings. The Birman-Solomyak formula [19] (or a direct computation) yields

$$\begin{aligned} \frac{(4.7)}{C^{|\Gamma_L|-1}} &= \int_E^{E+\varepsilon} d\lambda\, \mathbb{E}\big[\operatorname{tr} \mathbb{1}_{(-\infty,\lambda]}(H_{0,L}) - \operatorname{tr} \mathbb{1}_{(-\infty,\lambda]}(H_{0,L} + 1)\big] \\ &\approx \varepsilon\big(\operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{0,L}) - \operatorname{tr} \mathbb{1}_{(-\infty,E-1]}(H_{0,L})\big). \end{aligned} \tag{4.8}$$

The difference of traces on the right hand side is $\approx L^d(\mathcal{N}_{H_0}(E) - \mathcal{N}_{H_0}(E-1)) =: L^d C_1$ for large L, where $\mathcal{N}_{H_0}$ is the integrated density of states of the non-random periodic operator H_0. Moreover $C_1 > 0$ for typical energies $E \in \Sigma_0$. This yields a lower bound of the form

$C_1 L^d C^{|\Gamma_L|-1}\varepsilon$ on (4.7). The ε-scaling of this bound is sound, but unfortunately it is super-exponentially decaying in L because of $C < 1$ (instead of being of order L^d as needed in order to retrieve information on the infinite-volume density of states).

This issue was salvaged for the lattice Anderson model in [71, 62]. Instead of performing the above mentioned change of variables on all of Λ_L at once one can instead first partition the box Λ_L into subboxes $\Lambda_\ell(j)$ with $L \gg \ell \gg 0$. Subsequently, the change of variables (4.6) is performed on each box $\Lambda_\ell(j)$ separately. The key point then is to choose ℓ sufficiently large but fixed such that (4.8) for ℓ instead of L is bounded below by εC for a constant $C > 0$. Let's go into some more details. First, the change of variables yields the lower bound

$$\begin{aligned}\mathbb{E}\big[\operatorname{tr} \mathbb{1}_{[E,E+\varepsilon]}(H_L)\big] &\geqslant C^{|\Gamma_\ell|-1} \sum_{j\in(\ell\mathbb{Z})^d\cap\Lambda_L} \int_0^1 d\eta_0\, \mathbb{E}\big[\operatorname{tr}\big(\mathcal{X}_{\Lambda_\ell(j)} \mathbb{1}_{[E,E+\varepsilon]}(\widetilde{H}_{L,\ell,j} + \eta_0 \mathcal{X}_{\Lambda_\ell(j)})\big)\big] \\ &\approx C^{\ell^d} \frac{L^d}{\ell^d} \varepsilon\, \mathbb{E}\big[\operatorname{tr} \mathbb{1}_{(-\infty,E]}(\widetilde{H}_{L,\ell,0}) - \operatorname{tr} \mathbb{1}_{(-\infty,E]}(\widetilde{H}_{L,\ell,0} + \mathcal{X}_{\Lambda_\ell})\big], \end{aligned} \tag{4.9}$$

where $C < 1$ is the same constant as above and where we abbreviated

$$\widetilde{H}_{\omega,L,\ell,j} := H_{0,L} + \sum_{\substack{k\in\Gamma_L\\ |k-j|>\ell}} \omega_k V_k^L. \tag{4.10}$$

Here V_k^L is the restriction of V_k to $L^2(\Lambda_L)$. For the second line in (4.9) we used the arguments from (4.7) and (4.8) and that the terms in the j-sum are approximately equal. The right hand side of (4.9) looks pretty good. First, the disorder-averaged difference of traces should, for sufficiently large but fixed ℓ, at the very least be strictly positive uniformly in $L \gg \ell$. If we fix such ℓ then the right hand side of (4.9) reads $C_2 L^d \varepsilon$ for $C_2 = C_{2,\ell} > 0$. It remains to prove that the disorder-averaged difference of traces is indeed strictly positive for large ℓ uniformly in $L \gg \ell$. The proof of this point, which we present in Section 4.3 below, deviates from the proof in [62] and its adaption to continuum random Schrödinger operators in [37]. The underlying ideas are however similar and the method from [62, 37] is a bit more convenient for our present discussion. The remainder of this outline hence deals with the related argument from [37]. The disjoint union $\Lambda_\ell \cup \operatorname{int}(\Lambda_L \setminus \Lambda_\ell)$ equals Λ_L up to a set of Lebesgue-measure zero. Hence Dirichlet-Neumann decoupling along the surface $\partial\Lambda_\ell$ yields

$$\begin{aligned}\mathbb{E}\big[\operatorname{tr} \mathbb{1}_{(-\infty,E]}(\widetilde{H}_{L,\ell,0}) &- \operatorname{tr} \mathbb{1}_{(-\infty,E]}(\widetilde{H}_{L,\ell,0} + \mathcal{X}_{\Lambda_\ell})\big] \\ &\geqslant \operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{0,\ell}^D) - \operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{0,\ell}^N + 1) \\ &\quad + \mathbb{E}\big[\operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{\Lambda_L\setminus\Lambda_\ell}^D) - \operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{\Lambda_L\setminus\Lambda_\ell}^N)\big] \end{aligned} \tag{4.11}$$

where the D/N-superscript refers to either Dirichlet or Neumann boundary conditions along the newly added boundary $\partial\Lambda_\ell$. Here we ignored that the random potential of $\widetilde{H}_{\omega,L,\ell,0}$ possibly is non-zero in a strip of width R along $\partial\Lambda_\ell$. The Dirichlet boundary conditions along $\partial\Lambda_L$ remain unchanged. With the same argument as after (4.8) we conclude that the first term is $\approx \ell^d C_3$ for a constant $C_3 > 0$. It remains to argue that the second summand is negligible, i.e. of order $o(\ell^d)$ uniformly in $L \gg \ell$. Such a bound is immediate on the lattice: In this case the difference of traces (which coincides with the corresponding spectral shift function for the finite-volume operators considered here, see Section 3.1) can be estimated by the rank of the difference of the operators $H_{\omega,\Lambda_L\setminus\Lambda_\ell}^D$ and $H_{\omega,\Lambda_L\setminus\Lambda_\ell}^N$. In general such uniform bounds on the spectral shift function are touchy business for continuum Schrödinger operators. For instance at least for a dense set of energies $\mathcal{E} \subset [0,\infty)$ it was proven in [74] that for $E \in \mathcal{E}$

$$\limsup_{\mathbb{N}\ni L\to\infty} \Big(\operatorname{tr} \mathbb{1}_{(-\infty,E]}((-\Delta)^D_{\Lambda_L\setminus\Lambda_\ell}) - \operatorname{tr} \mathbb{1}_{(-\infty,E]}((-\Delta)^N_{\Lambda_L\setminus\Lambda_\ell}) \Big) = \infty. \tag{4.12}$$

We also refer to the related discussion in Section 3.1. In [37] we proved a sufficiently detailed bound on the second summand in (4.11) via the methods developed in Chapter 3. This is where the localization assumption enters. That spectral localization implies such a bound can be guessed from the related Theorem 3.5. If a version of Theorem 3.5 (ii) was also established for Neumann boundary conditions then the second summand in (4.11) could already be estimated by $\lesssim \ell^{d-1}$. We note at this point that operator-norm versions of the estimates from Theorem 3.5 directly extend to Neumann boundary conditions. But the trace bounds from Lemma 3.11 do not in full generality (a restriction of the domains under consideration is required). Even though (4.12) indicates that such bounds do not hold in arbitrary generality one can still argue that a localization assumption here is too harsh and spectral averaging should be sufficient. See also Remark 4.3 (i). To provide evidence in this direction let us consider the original finite-volume operator $H_{\omega,L}$ and its perturbation $H_{\omega,L} + \mathcal{X}_{\partial\Lambda_\ell^+}$ with $\partial\Lambda_\ell^+ := \{x \in \Lambda_L : \operatorname{dist}(x, \partial\Lambda_\ell) \leqslant 1\}$ for $L \gg \ell$. Then the Birman-Solomyak formula yields

$$\begin{aligned}\mathbb{E}\big[\xi(E, H_L, H_L + \mathcal{X}_{\partial\Lambda_\ell^+})\big] &= \lim_{\varepsilon\searrow 0} \frac{1}{\varepsilon} \int_0^1 ds\, \mathbb{E}\big[\operatorname{tr}\big(\mathcal{X}_{\partial\Lambda_\ell^+} \mathbb{1}_{[E,E+\varepsilon]}(H_L + s\mathcal{X}_{\partial\Lambda_\ell^+})\big)\big] \\ &\leqslant C\ell^{d-1},\end{aligned} \tag{4.13}$$

where the estimate follows from the local Wegner estimate from [28].

4.3. Proof of the reverse Wegner estimate

This whole section is dedicated to the proof of this chapter's main result. For the sake of lucidity the argument is subdivided into three subsections. Let

$$N_{\omega,L}(E) := \operatorname{tr} \mathbb{1}_{(-\infty,E]}(H_{\omega,L}) \tag{4.14}$$

denote the non-normalized non-averaged finite-volume integrated density of states. By virtue of Wegner's estimate the normalized disorder average $\mathcal{N}_L(E) := L^{-d}\,\mathbb{E}[N_L(E)]$ is a locally Lipschitz-continuous function in E. We denote its Lebesgue density by $\mathcal{N}_L'$.

4.3.1. Disorder averaging results in energy averaging. Let $E \in \mathbb{R}$ and $\varepsilon > 0$ be fixed. Our starting point is

$$\begin{aligned}\frac{1}{\varepsilon}\big(\mathcal{N}_L(E+\varepsilon) - \mathcal{N}_L(E)\big) &= \frac{1}{L^d\varepsilon}\,\mathbb{E}\big[\operatorname{tr} \mathbb{1}_{(E,E+\varepsilon]}(H_L)\big] \\ &\geqslant \frac{1}{L^d\varepsilon V_+} \sum_{k\in\Gamma_L} \mathbb{E}\big[\operatorname{tr}\big(V_k \mathbb{1}_{(E,E+\varepsilon]}(H_L)\big)\big],\end{aligned} \tag{4.15}$$

where $\Gamma_L = \Lambda_{L+2R} \cap \mathbb{Z}^d$. Apart from a boundary layer, we partition the cube Λ_L into smaller cubes $\Lambda_{\ell,j} := \Lambda_\ell(j)$ of side length $\ell \in \mathbb{N}$, $\ell \leqslant L$, and centered at the points

$$j \in \Gamma_L^\ell := \big\{k \in (\ell\mathbb{Z})^d : |k| \leqslant (L-\ell)/2 - R - \mathcal{R}/2 - 1\big\}, \tag{4.16}$$

where $\mathbb{R}$ is specified above of (4.30) below. As always, $|\cdot|$ denotes the maximum norm on $\mathbb{R}^d$ and $B^\# := B \cap \mathbb{Z}^d$ for a set $B \subset \mathbb{R}^d$. For $j \in \Gamma_L^\ell$, we abbreviate

$$F_{\Lambda_{\ell,j}}(\omega) := \sum_{k \in \Lambda_{\ell,j}^\#} \operatorname{tr}\big(V_k \mathbb{1}_{(E,E+\varepsilon]}(H_{\omega,L})\big) \tag{4.17}$$

and infer from (4.15) that

$$\frac{1}{\varepsilon}\big(\mathcal{N}_L(E+\epsilon) - \mathcal{N}_L(E)\big) \geqslant \frac{1}{L^d \varepsilon V_+} \sum_{j \in \Gamma_L^\ell} \mathbb{E}\left[F_{\Lambda_{\ell,j}}\right]. \tag{4.18}$$

Let's denote by $\omega_{\Lambda_{\ell,j}}$ and $\omega_{\Lambda_{\ell,j}^c}$ the collection of random variables corresponding to single-site potentials centered inside (i.e. at the lattice points $\Lambda_{\ell,j}^\#$) respectively outside (i.e. at the lattice points $\Gamma_L \setminus \Lambda_{\ell,j}^\#$) the cube $\Lambda_{\ell,j}$. In virtue of assumption (V_4) we can estimate

$$\mathbb{E}[F_{\Lambda_{\ell,j}}] \geqslant \rho_-^{\theta(\ell)}\, \mathbb{E}_{\Lambda_{\ell,j}^c}\left[\int_{[0,1]^{\theta(\ell)}} d\omega_{\Lambda_{\ell,j}}\, F_{\Lambda_{\ell,j}}\big((\omega_{\Lambda_{\ell,j}}, \omega_{\Lambda_{\ell,j}^c})\big)\right], \tag{4.19}$$

where we denoted the expectation with respect to the random variables $\omega_{\Lambda_{\ell,j}^c}$ by $\mathbb{E}_{\Lambda_{\ell,j}^c}[\,\cdot\,]$ and $\theta(\ell) := |\Lambda_{\ell,j}^\#|$ denotes the cardinality of the sets $\Lambda_{\ell,j}^\#$ (which is independent of j). For fixed $j \in \Gamma_L^\ell$ we perform the same change of variables as in [127, 62], namely

$$\omega_{\Lambda_{\ell,j}} = (\omega_k)_{k \in \Lambda_{\ell,j}^\#} \mapsto \eta := (\eta_k)_{k \in \Lambda_{\ell,j}^\#} \text{ with } \eta_k := \begin{cases} \omega_k - \omega_j & \text{for } k \in \Lambda_{\ell,j}^\# \setminus \{j\}, \\ \omega_j & \text{for } k = j. \end{cases} \tag{4.20}$$

This change of variables yields

$$\begin{aligned} &\int_{[0,1]^{\theta(\ell)}} d\omega_{\Lambda_{\ell,j}}\, F_{\Lambda_{\ell,j}}\big((\omega_{\Lambda_{\ell,j}}, \omega_{\Lambda_{\ell,j}^c})\big) \\ &\qquad = \int_0^1 d\eta_j \int_{[-\eta_j, 1-\eta_j]^{\theta(\ell)-1}} \Bigg(\prod_{k \in \Lambda_{\ell,j}^\# \setminus \{j\}} d\eta_k\Bigg) F_{\Lambda_{\ell,j}}\big((\omega_{\Lambda_{\ell,j}}(\eta), \omega_{\Lambda_{\ell,j}^c})\big) \\ &\qquad \geqslant \int_\delta^{1-\delta} d\eta_j \int_{[-\delta,\delta]^{\theta(\ell)-1}} \Bigg(\prod_{k \in \Lambda_{\ell,j}^\# \setminus \{j\}} d\eta_k\Bigg) F_{\Lambda_{\ell,j}}\big((\omega_{\Lambda_{\ell,j}}(\eta), \omega_{\Lambda_{\ell,j}^c})\big) \end{aligned} \tag{4.21}$$

for any fixed $0 < \delta < 1/4$. The reason for this constraint on δ will become clear in (4.25) and (4.26) below, where it ensures non-negativity of the right hand side of (4.26). The η_j-integral on the right hand side of (4.21) will be evaluated by the Birman-Solomyak formula. Let's therefore emphasize the η_j-dependence of $H_{\omega,L}$. We recall that V_k^L is the restriction of the single-site potential V_k to the box Λ_L. With the potentials

$$U_j := \sum_{k\in\Lambda^{\#}_{\ell,j}} V_k, \qquad W_{j,\eta} := \sum_{\substack{k\in\Lambda^{\#}_{\ell,j}\\ k\neq j}} \eta_k V_k, \qquad W^c_{j,\omega} := \sum_{k\in\Gamma_L\setminus\Lambda^{\#}_{\ell,j}} \omega_k V^L_k \tag{4.22}$$

at hand, we can view the operator as a one-parameter operator family with respect to the parameter η_j:

$$H_{\omega,L} = H_{0,L} + W^c_{j,\omega} + W_{j,\eta} + \eta_j U_j =: \widetilde{H}_{\omega,\eta,L,\ell,j} + \eta_j U_j. \tag{4.23}$$

Also note that the operator $\widetilde{H}_{\omega,\eta,L,\ell,j}$ in fact only depends on the families $\omega_{\Lambda^c_{\ell,j}}$ and $(\eta_k)_{k\in\Lambda^{\#}_{\ell,j}\setminus\{j\}}$ of parameters. The Birman-Solomyak formula [19] yields

$$\begin{aligned}\int_\delta^{1-\delta} d\eta_j\, F_{\Lambda_{\ell,j}}\big((\omega_{\Lambda_{\ell,j}}(\eta), \omega_{\Lambda^c_{\ell,j}})\big)&\\ &= \int_\delta^{1-\delta} d\eta_j\ \mathrm{tr}\,\big(U_j \mathbb{1}_{(E,E+\varepsilon]}(\widetilde{H}_{\omega,\eta,L,\ell,j} + \eta_j U_j)\big)\\ &= \int_E^{E+\varepsilon} d\lambda\, \xi\big(\lambda, \widetilde{H}_{\omega,\eta,L,\ell,j} + \delta U_j, \widetilde{H}_{\omega,\eta,L,\ell,j} + (1-\delta)U_j\big),\end{aligned} \tag{4.24}$$

where $\xi(\cdot, A, B)$ denotes the spectral shift function of the operators A, B. For its definition and more we refer to Section 3.1. For the values of the parameters $(\eta_k)_{k\in\Lambda^{\#}_{\ell,j}\setminus\{j\}}$ in the integration in (4.21), we have the estimate $-\delta U_j \leqslant W_{j,\eta} \leqslant \delta U_j$. Moreover, due to the covering condition (V_2) and the maximal range R of the support of V_0, we also have $V_- \mathcal{X}_{\Lambda_{\ell-2R}(j)} \leqslant U_j \leqslant \mathcal{X}_{\Lambda_{\ell+2R}(j)} V_+$. Those two estimates together yield

$$\begin{aligned}\widetilde{H}_{\omega,\eta,L,\ell,j} + (1-\delta)U_j \geqslant H_{\omega,L,\ell,j,+} &:= H_{0,L} + W^c_{j,\omega} + (1-2\delta)V_- \mathcal{X}_{\Lambda_{\ell-2R}(j)},\\ \widetilde{H}_{\omega,\eta,L,\ell,j} + \delta U_j \leqslant H_{\omega,L,\ell,j,-} &:= H_{0,L} + W^c_{j,\omega} + 2\delta V_+ \mathcal{X}_{\Lambda_{\ell+2R}(j)}\end{aligned} \tag{4.25}$$

in the form sense. If we plug those estimates into (4.24) we end up with

$$\int_\delta^{1-\delta} d\eta_j\, F_{\Lambda_{\ell,j}}\big((\omega_{\Lambda_{\ell,j}}(\eta), \omega_{\Lambda^c_{\ell,j}})\big) \geqslant \int_E^{E+\varepsilon} d\lambda\, \xi\big(\lambda, H_{\omega,L,\ell,j,-}, H_{\omega,L,\ell,j,+}\big), \tag{4.26}$$

which is uniform in the parameters $(\eta_k)_{k\in\Lambda^{\#}_{\ell,j}\setminus\{j\}}$ for values as appearing in the integration in (4.21). Note that the spectral shift function on the right hand side may for unfortunate choices of ℓ and δ not be $\geqslant 0$ anymore. We are going to choose suitable values for ℓ and δ below. Combining (4.26), (4.21) and (4.19), we find

$$\mathbb{E}[F_{\Lambda_{\ell,j}}] \geqslant \frac{(2\delta\rho_-)^{\theta(\ell)}}{2\delta} \int_E^{E+\varepsilon} d\lambda\ \mathbb{E}\big[\xi\big(\lambda, H_{L,\ell,j,-}, H_{L,\ell,j,+}\big)\big]. \tag{4.27}$$

The expectation on the right hand side is effectively only the partial expectation $\mathbb{E}_{\Lambda^c_{\ell,j}}$ because the two operators $H_{\omega,L,\ell,j,\pm}$ no longer depend on the random variables ω_k for $k \in \Lambda^{\#}_{\ell,j}$. By substituting this lower bound into (4.18) and subsequently taking the limit $\varepsilon \searrow 0$ in (4.15), we obtain that for Lebesgue almost every $E \in \mathbb{R}$

$$\mathcal{N}_L'(E) \geqslant \frac{(2\delta\rho_-)^{\theta(\ell)}}{2V_+\delta}\frac{1}{L^d}\sum_{j\in\Gamma_L^\ell}\mathbb{E}\big[\xi\big(E, H_{L,\ell,j,-}, H_{L,\ell,j,+}\big)\big]. \tag{4.28}$$

So far we accomplished the local energy shift. In the next step this energy shift has to be isolated from the rest of the operator.

4.3.2. Separation of the local energy shift. We for now choose a fixed $\mathcal{E} \in I$ and $0 < \varepsilon \leqslant V_-/4$ sufficiently small such that $[\mathcal{E}-\varepsilon, \mathcal{E}+\varepsilon] \subset I$. The plan is now to find a uniform lower bound on (4.28) for all $E \in [\mathcal{E}-\varepsilon, \mathcal{E}+\varepsilon]$ and sufficiently large $L \gg \ell \gg 0$ (in the sense specified below). Moreover, we choose a fixed

$$\delta = \delta_\varepsilon < \frac{V_- - 2\varepsilon}{2(V_+ + V_-)}. \tag{4.29}$$

For the time being we also fix an arbitrary $j \in \Gamma_L^\ell$ and an energy $E \in [\mathcal{E}-\varepsilon, \mathcal{E}+\varepsilon]$. The potentials of the Schrödinger operators $H_{\omega,L,\ell,j,\pm}$ in the box $\Lambda_{\ell-2R,j}$ constitute an energy shift of magnitude $(1-2\delta)V_-$ and $2\delta V_+$, respectively. In the buffer zone $\Lambda_{\ell+2R,j} \setminus \Lambda_{\ell-2R,j}$ the potential is either given by $W_{j,\omega}^c$ in case of the operator $H_{\omega,L,\ell,j,+}$ or a combination of the energy shift $2\delta V_+$ and $W_{\omega,j}^c$ in case of the operator $H_{\omega,L,\ell,j,-}$. Finally, potentials of the operators agree with each other and with the original potential of V_ω on $\Lambda_L \setminus \Lambda_{\ell+2R,j}$. Let $\mathcal{R} > 0$ be the constant such that [3, Lemma 3.3] (respectively the subsequent remark there) yields the following: For fixed $0 < s < 1$ there exists a constant $C_1 = C_{1,s}$ such that for all $a, b \in \Lambda_L$

$$\mathbb{E}_{\Lambda_\mathcal{R}(a)\cup\Lambda_\mathcal{R}(b)}\big[\|\mathcal{X}_a R_z(H_L)\mathcal{X}_b\|^s\big] \leqslant C_1, \tag{4.30}$$

where $\mathbb{E}_{\Lambda_\mathcal{R}(a)\cup\Lambda_\mathcal{R}(b)}$ is the expectation over the random variables indexed by $\Lambda_\mathcal{R}(a)^\# \cup \Lambda_\mathcal{R}(b)^\#$. This is nothing but a slightly refined version of the boundedness of fractional moments (FM) reviewed in Chapter 2.1. Hence, if we define the box $\Lambda_{\ell,j}^+ := \Lambda_{\ell+2R+\mathcal{R}}(j) \cap \Lambda_L$, then (4.30) also holds for the operators $H_{\omega,L,\ell,j,\pm}$ and all $a, b \in \Lambda_{\ell,j}^{+,c}$. This is due to the observation that the potentials of $H_{\omega,L,\ell,j,\pm}$ on $\Lambda_L \setminus \Lambda_{\ell+2R}(j)$ agree with $W_{\omega,j}^c$, which in turn agrees with V_ω in this spatial region.

Since the operators $H_{\omega,L,\ell,j,\pm}$ are finite-volume operators, the spectral shift function in (4.28) is given in terms of the trace over the corresponding spectral shift operator, see Section 3.1. With the further abbreviation $\Lambda_{\ell,j}^- := \Lambda_{\ell-2R,j}$ we obtain

$$\begin{aligned}\xi\big(E, H_{\omega,L,\ell,j,-}, H_{\omega,L,\ell,j,+}\big) &= \operatorname{tr}\big(\mathcal{X}_{\Lambda_{\ell,j}^-}\big(\mathbb{1}_{(-\infty,E]}(H_{\omega,L,\ell,j,-}) - \mathbb{1}_{(-\infty,E]}(H_{\omega,L,\ell,j,+})\big)\big)\\ &\quad + \operatorname{tr}\big(\mathcal{X}_{\Lambda_{\ell,j}^+\setminus\Lambda_{\ell,j}^-}\big(\mathbb{1}_{(-\infty,E]}(H_{\omega,L,\ell,j,-}) - \mathbb{1}_{(-\infty,E]}(H_{\omega,L,\ell,j,+})\big)\big)\\ &\quad + \operatorname{tr}\big(\mathcal{X}_{\Lambda_L\setminus\Lambda_{\ell,j}^+}\big(\mathbb{1}_{(-\infty,E]}(H_{\omega,L,\ell,j,-}) - \mathbb{1}_{(-\infty,E]}(H_{\omega,L,\ell,j,+})\big)\big)\\ &=: \text{(I)} + \text{(II)} + \text{(III)}.\end{aligned} \tag{4.31}$$

The next step in the proof is to show that (I) is bounded below by a term of order ℓ^d, while the disorder average of (III) is of lower order in ℓ. The term (II) can be estimated directly (and uniformly in pretty much everything): Lemma 3.11 yields a constant $C_2 = C_{2,I,R,\mathcal{R}}$ such that

$$\text{(II)} \leqslant C_2 \ell^{d-1} \tag{4.32}$$

for, say, $\ell \geqslant 1$. Let's continue with the term (I). For given $\eta > 0$ specified below we choose two smooth switch functions $f_{\pm\eta} \in \mathcal{C}^\infty(\mathbb{R})$ subject to

$$\mathbb{1}_{(-\infty,-\eta]} \leqslant f_{-\eta} \leqslant \mathbb{1}_{(-\infty,0]} \quad \text{and} \quad \mathbb{1}_{(-\infty,0]} \leqslant f_{\eta} \leqslant \mathbb{1}_{(-\infty,\eta]}. \tag{4.33}$$

The functions $f_{E,\pm\eta} := f_{\pm\eta}(\cdot - E) \in \mathcal{C}^\infty(\mathbb{R})$ then satisfy

$$\mathbb{1}_{(-\infty,E-\eta]} \leqslant f_{E,-\eta} \leqslant \mathbb{1}_{(-\infty,E]} \quad \text{and} \quad \mathbb{1}_{(-\infty,E]} \leqslant f_{E,+\eta} \leqslant \mathbb{1}_{(-\infty,E+\eta]}, \tag{4.34}$$

respectively. This yields

$$\text{(I)} \geqslant \operatorname{tr}\big(\mathcal{X}_{\Lambda^-_{\ell,j}}\big(f_{E,-\eta}(H_{\omega,L,\ell,j,-}) - f_{E,+\eta}(H_{\omega,L,\ell,j,+})\big)\big). \tag{4.35}$$

In Lemma 4.5 below we prove that there exists a constant $C_3 = C_{3,\eta}$ (which is in particular independent of the realization ω), such that

$$\big|\operatorname{tr}\big(\mathcal{X}_{\Lambda^-_{\ell,j}} f_{E,-\eta}(H_{\omega,L,\ell,j,-})\big) - \operatorname{tr}\big(f_{E,-\eta}(H_{0,\Lambda^-_{\ell,j}} + 2\delta V_+)\big)\big| \leqslant C_3 \ell^{d-1}, \tag{4.36}$$

$$\big|\operatorname{tr}\big(\mathcal{X}_{\Lambda^-_{\ell,j}} f_{E,\eta}(H_{\omega,L,\ell,j,+})\big) - \operatorname{tr}\big(f_{E,\eta}(H_{0,\Lambda^-_{\ell,j}} + (1-2\delta)V_-)\big)\big| \leqslant C_3 \ell^{d-1}. \tag{4.37}$$

Together with (4.35) this yields

$$\begin{aligned}\text{(I)} &\geqslant \operatorname{tr}\big(f_{E,-\eta}(H_{0,\Lambda^-_{\ell,j}} + 2\delta V_+) - f_{E,\eta}(H_{0,\Lambda^-_{\ell,j}} + (1-2\delta)V_-)\big) - 2C_3\ell^{d-1} \\ &\geqslant \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}-\varepsilon-\eta-2\delta V_+]}(H_{0,\Lambda^-_{\ell,j}}) - \mathbb{1}_{(-\infty,\mathcal{E}+\varepsilon+\eta-(1-2\delta)V_-]}(H_{0,\Lambda^-_{\ell,j}})\big) - 2C_3\ell^{d-1}\end{aligned} \tag{4.38}$$

for all $E \in [\mathcal{E}-\varepsilon, \mathcal{E}+\varepsilon]$. If we now choose δ, η sufficiently small such that

$$E_+ := \mathcal{E} - \varepsilon - \eta - 2\delta V_+ > \mathcal{E} + \varepsilon + \eta - (1-2\delta)V_- =: E_-, \tag{4.39}$$

then we have that

$$\text{(I)} \geqslant \operatorname{tr} \mathbb{1}_{[E_-,E_+]}(H_{0,\Lambda^-_{\ell,j}}) - 2C_3\ell^{d-1}. \tag{4.40}$$

For the term (III) we prove in Lemma 4.4 below that

$$\mathbb{E}\big[|\text{(III)}|\big] \leqslant C_4 \ell^{d-1}. \tag{4.41}$$

By combining (4.40) and (4.41) we find that there exists a constant C_5 (uniform in $1 \leqslant \ell \leqslant L$ and $E \in [\mathcal{E}-\varepsilon, \mathcal{E}+\varepsilon]$) such that

$$\mathcal{N}'_L(E) \geqslant \frac{(2\delta\rho_-)^{\theta(\ell)} b_{L,\ell}}{2V_+\delta} \left\{ \frac{1}{|\Lambda^-_\ell|} \operatorname{tr} \mathbb{1}_{[E_-,E_+]}(H_{0,\Lambda^-_\ell}) - \frac{C_5}{\ell} \right\} \tag{4.42}$$

for all length scales $1 \leqslant \ell \leqslant L$. Here, we introduced the notations $\Lambda^-_\ell := \Lambda^-_{\ell,0}$, $b_{L,\ell} := |\Gamma^\ell_L||\Lambda^-_\ell|/L^d$ and made use of the $\mathbb{Z}^d$-ergodicity of H_0. For later use we observe that, given any length $\ell \geqslant 1$, there exists an initial length $\mathcal{L}_\ell$ such that

$$b_{L,\ell} \geqslant \frac{1}{2} \qquad \text{for every } L \geqslant \mathcal{L}_\ell. \tag{4.43}$$

We have now completely isolated the energy shift that we previously obtained by means of the random potential. In the last part of the proof we are going to argue that this energy shift yields to a strictly positive lower bound on the density of states.

4.3.3. Strictly positive lower bound. Since H_0 is a periodic operator, we have the pointwise convergence

$$\lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell^-|} \mathbb{E}\big[\operatorname{tr} \mathbb{1}_{(-\infty,\tilde{E}]}\big(H_{0,\Lambda_\ell^-}\big)\big] = \mathcal{N}_{H_0}(\tilde{E}) \tag{4.44}$$

for all $\tilde{E} \in \mathbb{R}$, where $\mathcal{N}_{H_0}$ stands for the integrated density of states of the non-random periodic operator H_0. In particular we have that

$$\lim_{\ell\to\infty} \frac{1}{|\Lambda_\ell^-|} \mathbb{E}\big[\operatorname{tr} \mathbb{1}_{[E_-,E_+]}(H_{0,\Lambda_\ell^-})\big] = \mathcal{N}_{H_0}(E_+) - \mathcal{N}_{H_0}(E_-). \tag{4.45}$$

The right hand side of (4.45) is obviously non-negative. In the following we are going to use $[\mathcal{E} - \varepsilon, \mathcal{E} + \varepsilon] \subset \operatorname{int}(\Sigma_0 + [0, V_-])$ to show strict positivity. First of all, there exists $\mathcal{E}_0 \in \Sigma_0$ and $\lambda \in (0, 1)$ such that $\mathcal{E} = \mathcal{E}_0 + \lambda V_-$ and. From (4.39) we also infer that

$$\mathcal{E}_0 - (1 - \lambda)V_- < E_- < E_+ < \mathcal{E}_0 + \lambda V_-. \tag{4.46}$$

We distinguish three cases to finish the argument for strict positivity of (4.45).
First case: $\mathcal{E}_0 \in (E_-, E_+)$. In this case, the claim follows directly because Σ_0 is the set of growth points of the integrated density of states $\mathcal{N}_{H_0}$.
Second case: $\mathcal{E}_0 \in [E_+, \mathcal{E}_0 + \lambda V_-)$. In this case, we decrease the values of ε and δ_ε and obtain again $\mathcal{E}_0 \in (E_-, E_+)$ as in the first case.
Third case: $\mathcal{E}_0 \in (\mathcal{E}_0 - (1 - \lambda)V_-, E_-]$. Again, by making ε and δ_ε smaller, we obtain $\mathcal{E}_0 \in (E_-, E_+)$, and the argument is complete. By taking (4.43) into account we infer that there exists a constant $C_6 = C_{6,E_0,\varepsilon} > 0$ and an initial length $\mathcal{L}_0$ such that

$$(4.42) \geqslant \frac{(2\delta\rho_-)^{\theta(\mathcal{L}_0)}}{4V_+\delta} C_6 \tag{4.47}$$

for all $L \geqslant \mathcal{L}_0$ and all $E \in [\mathcal{E} - \varepsilon, \mathcal{E} + \varepsilon]$. By compactness, we cover I with finitely many intervals of the form $(\mathcal{E} - \varepsilon, \mathcal{E} + \varepsilon) \subset \operatorname{int}(\Sigma_0 + [0, V_-])$. We finally arrive at the reverse Wegner estimate by integrating the so-obtained lower bound on $\mathcal{N}_L'$ over energies E from E_1 to E_2.

□

In the above proof we applied the following two lemmas. Similar results are contained at other places of this thesis: The first lemma almost follows from Theorem 3.5. However, the operators $H_{\omega,L,\ell,j,\pm}$ from the proof above are strictly speaking not localized for large ℓ. Some smaller modifications are necessary to salvage this issue. The second lemma is almost a consequence of Theorem 5.5. But this theorem is formulated for general bounded operators instead of the specific unbounded operators considered here. Due to those similar results we state both lemmas in the exact way they were applied above, not striving for any generality. Since the proofs follow along the lines of the above mentioned similar results only an outline of the respective arguments is included. Moreover, we stick to the notation introduced in the proof of Theorem 4.1 above.

Lemma 4.4. *Let $H_{\omega,L,\ell,j,\pm}$, $j \in \Gamma_L^\ell$ and $R, \mathcal{R} > 0$ be as in the proof of Theorem 4.1. For compact intervals $I \subset \Sigma_{\mathrm{FMB}}$ there exists a constant C such that*

$$\mathbb{E}\big[|(\mathrm{III})|\big] = \mathbb{E}\big[\big|\operatorname{tr}\big(\mathcal{X}_{\Lambda_L \setminus \Lambda_{\ell,j}^+}\big(\mathbb{1}_{(-\infty,E]}(H_{L,\ell,j,-}) - \mathbb{1}_{(-\infty,E]}(H_{L,\ell,j,+})\big)\big)\big|\big] \leqslant C\ell^{d-1} \tag{4.48}$$

for all $1 \leqslant \ell \leqslant L$ and all $E \in I$.

Lemma 4.5. *Let $H_{\omega,L,\ell,j,\pm}$, $j \in \Gamma_L^\ell$, $f_{E,\pm\eta}$ be as in the proof of Theorem* 4.1 *and $I \subset \mathbb{R}$ a compact interval. There exists a constant C such that* (4.36) *and* (4.37) *hold for all $1 \leqslant \ell \leqslant L$ and $E \in I$.*

PROOF OF LEMMA 4.4. By cyclicity of the trace and a triangle inequality the lemma follows from

$$\mathbb{E}\big[\big|\operatorname{tr}\big(\mathcal{X}_a\big(\mathbb{1}_{(-\infty,E]}(H_{L,\ell,j,\pm}) - \mathbb{1}_{(-\infty,E]}(H_{\Lambda_L\setminus\Lambda_{\ell,j}^+})\big)\mathcal{X}_a\big)\big|\big] \leqslant Ce^{-\mu d(a,\Lambda_{\ell,j}^+)} \tag{4.49}$$

for some constants $C, \mu > 0$ and all $a \in \Lambda_L \setminus \Lambda_{\ell+1,j}^+$ and $E \in I$. We from now on fix such a and E but note that the constants appearing below are uniform in both of them. By virtue of Lemma 3.11, and with the same line of arguments as in the proof of Theorem 3.5 (ii), it is sufficient to prove

$$\mathbb{E}\big[\big\|\mathcal{X}_a\big(\mathbb{1}_{(-\infty,E]}(H_{L,\ell,j,\pm}) - \mathbb{1}_{(-\infty,E]}(H_{\Lambda_L\setminus\Lambda_{\ell,j}^+})\big)\mathcal{X}_a\big\|\big] \leqslant Ce^{-\mu\,\mathrm{dist}(a,\Lambda_{\ell,j}^+)}. \tag{4.50}$$

Due to (4.30) the assumption of Lemma 3.7 is satisfied for the operators $H_{\omega,L,\ell,j,\pm}$ and a as above. With estimates as in the proof of Lemma 3.12 we obtain

$$(4.50) \leqslant C_1 \sup_{(x,y)\in Z} \mathbb{E}\big[\big\|\mathcal{X}_a\big(R_{x+iy}(H_{L,\ell,j,\pm}) - R_{x+iy}(H_{\Lambda_L\setminus\Lambda_{\ell,j}^+})\big)\mathcal{X}_a\big\|^{1/4}\big], \tag{4.51}$$

where the set Z is defined similarly as in (3.82), namely:

$$\Big(I\times[-1,1]\Big) \cup \Big([E_0-1, I_+ + 1]\times\big([-1,-1/2]\cup[1/2,1]\big)\Big) =: Z_1 \cup Z_2 =: Z, \tag{4.52}$$

where $I_+ = \sup I$ and $E_0 = \inf\Sigma$. We now apply the geometric resolvent inequality as in the proof of Theorem 3.5 (ii). This yields

$$(4.51) \leqslant C_2 \sup_{(x,y)\in Z} \sum_{b\in(\delta\Lambda_{\ell,j}^+)^\#} \mathbb{E}\big[\big\|\mathcal{X}_a\big(R_{x+iy}(H_{L,\ell,j,\pm})\mathcal{X}_b\big\|^{1/4}\big\|\mathcal{X}_{\Lambda_2(b)}R_{x+iy}(H_{\Lambda_L\setminus\Lambda_{\ell,j}^+})\big)\mathcal{X}_a\big\|^{1/4}\big], \tag{4.53}$$

where $\delta\Lambda_{\ell,j}^+ := \big\{x\in\mathbb{R}^d : \mathrm{dist}(x,\partial\Lambda_{\ell,j}^+) \leqslant 1\big\}$. We can now decouple the two operator norms in the expectation by means of Hölder's inequality. The expectation involving the resolvent of $H_{L,\ell,j,\pm}$ can then be estimated by means of (4.30) by a constant while the expectation involving the resolvent of $H_{\Lambda_L\setminus\Lambda_{\ell,j}^+}$ can be bounded by $C_3e^{-\mu_1|a-b|}$ due to $E\in I\subset\Sigma_{\mathrm{FMB}}$. □

PROOF OF LEMMA 4.5. The two bounds are analogous and we focus on (4.36), that is

$$\big|\operatorname{tr}\big(\mathcal{X}_{\Lambda_{\ell,j}^-} f_{E,-\eta}(H_{\omega,L,\ell,j,-})\big) - \operatorname{tr}\big(f_{E,-\eta}(H_{0,\Lambda_{\ell,j}^-} + 2\delta V_+)\big)\big| \leqslant C\ell^{d-1} \tag{4.54}$$

for a constant C that depends on η through the choice of $f_{\pm\eta}\in\mathcal{C}^\infty$ as specified in the theorem above but is independent of L,ℓ and $E\in I$. With the usual arguments it suffices to prove

$$\big\|\mathcal{X}_a\big(f_{E,-\eta}(H_{\omega,L,\ell,j,-}) - f_{E,-\eta}(H_{0,\Lambda_{\ell,j}^-} + 2\delta V_+)\big)\mathcal{X}_a\big\| \leqslant \frac{C}{\mathrm{dist}(a,\partial\Lambda_{\ell,j}^-)^m} \tag{4.55}$$

for all $a\in\Lambda_{\ell-1,j}^-$ and a sufficiently large m (for instance $m\geqslant 3$ would be sufficient). For this we apply the classical Helffer-Sjöstrand formula, which we comment on in more detail in the

Sections 3.3.1 and 5.3.3. With it we can write

$$\begin{aligned} & f_{E,-\eta}(H_{\omega,L,\ell,j,-}) - f_{E,-\eta}(H_{0,\Lambda^-_{\ell,j}} + 2\delta V_+) \\ & \quad = \frac{1}{2\pi}\int_{\mathbb{R}^2} dxdy\, \zeta_f(x,y)\left(R_{x+iy}(H_{\omega,L,\ell,j,-}) - R_{x+iy}(H_{0,\Lambda^-_{\ell,j}} + 2\delta V_+)\right) \end{aligned} \tag{4.56}$$

for a function ζ_f subject to

- $\operatorname{supp}(\zeta_f) \subset \left(\operatorname{supp}(f) + [-1,1]\right) \times [-1,1] =: Z$,
- $|\zeta_f(x,y)| \leqslant C_1 |y|^{m+d+2}$ for all $x, y \in \mathbb{R}$.

We again apply the geometric resolvent inequality in a similar fashion as in the proof of Theorem 3.5 (ii). This yields for $z = x + iy$

$$\begin{aligned} & \left\|\mathcal{X}_a\left(R_z(H_{\omega,L,\ell,j,-}) - R_z(H_{0,\Lambda^-_{\ell,j}} + 2\delta V_+)\right)\mathcal{X}_a\right\| \\ & \qquad \leqslant C_1 \sum_{b \in \left(\delta\Lambda^-_{\ell,j}\right)^{\#}} \|\mathcal{X}_a R_z(H_{\omega,L,\ell,j,-})\mathcal{X}_b\| \|\mathcal{X}_{\Lambda_2(b)} R_z(H_{0,\Lambda^-_{\ell,j}} + 2\delta V_+)\mathcal{X}_a\|, \end{aligned} \tag{4.57}$$

where $\delta\Lambda^-_{\ell,j} := \left\{x \in \mathbb{R}^d : \operatorname{dist}(x, \partial\Lambda^-_{\ell,j}) \leqslant 1\right\}$. Together with the Combes-Thomas estimate we obtain

$$\begin{aligned} & \left\|\mathcal{X}_a\left(f_{E,-\eta}(H_{\omega,L,\ell,j,-}) - f_{E,-\eta}(H_{0,\Lambda^-_{\ell,j}} + 2\delta V_+)\right)\mathcal{X}_a\right\| \\ & \qquad \leqslant C_1 \sum_{c \in \left(\delta\Lambda^-_{\ell,j}\right)^{\#}} \int_{\mathbb{R}^2} dxdy\, \frac{|\zeta_f(x,y)|}{|y|^2}\, e^{-2\mu|y||a-c|} \\ & \qquad \leqslant C_2 \sum_{c \in \left(\delta\Lambda^-_{\ell,j}\right)^{\#}} |a-c|^{-m-d-1} \\ & \qquad \leqslant \frac{C_3}{\operatorname{dist}(a, \partial\Lambda^-_{\ell,j})^m} \end{aligned} \tag{4.58}$$

with arguments as in Section 5.3.3. $\square$

CHAPTER 5

Full Szegő-type trace asymptotics

Context: The majority of this chapter coincides both in content and writing with [35].

Acknowledgements: I am very grateful to (alphabetically ordered) Peter Müller, Bernhard Pfirsch and Alexander Sobolev for many illuminating discussions on this topic.

Content: This chapter deals with Szegő-type asymptotic trace expansions for ergodic self-adjoint operators on $L^2(\mathbb{R}^d)$. Let g be a bounded, compactly supported and real-valued function such that the (averaged) operator kernel of $g(H_\omega)$ decays sufficiently fast, and let h be a sufficiently smooth compactly supported function. We then prove a full asymptotic expansion of the averaged trace of the operator $h(g(H_\omega)_{\Lambda_L})$ in terms of the length scale L. The result consists of two parts, an algebraic part and an analytic part. The algebraic part is a scheme of iterated regularizations that allows us to elaborate the contribution of a face of the cube to the different asymptotic orders. This part may also be of interest with regard to an asymptotic analysis of the corresponding non-averaged traces for more concrete models such as random Schrödinger operators. In the analytic part we argue that this algebraic scheme indeed yields a full asymptotic expansion.

5.1. Discussion of results

In contrast to the other chapters we are for now working with a general $\mathbb{Z}^d$-translation invariant operator. That is, for a probability space $(\Omega, \mathbb{P})$ we consider a measurable map

$$\Omega \ni \omega \mapsto H_\omega \in \mathcal{L}_{\mathrm{sa}}(L^2(\mathbb{R}^d)) \tag{5.1}$$

into the self-adjoint operators on $L^2(\mathbb{R}^d)$ that is $\mathbb{Z}^d$-translation invariant in the sense specified in Section 2.1. We keep the notation introduced there, i.e. $(U_j)_{j\in\mathbb{Z}^d}$ denotes the group of unitary translation operators and $(T_j)_{j\in\mathbb{Z}^d}$ is the associated family of measure preserving transformations. We also impose the following additional core requirement. As usual we write $\|A\|_p := (\operatorname{tr}|A|^p)^{1/p}$ for the Schatten-p (quasi-)norm ($p > 0$) of a, say, compact operator A.

($\mathcal{A}_1$) For any $p > 0$ and every (measurable) bounded and compactly supported function $g : \mathbb{R} \to \mathbb{R}$,

$$C := \sup_{a,b\in\mathbb{R}^d} \operatorname*{ess\,sup}_{\omega\in\Omega} \|\mathcal{X}_a g(H_\omega)\mathcal{X}_b\|_p < \infty. \tag{5.2}$$

Remark 5.1. The bound (5.2) for instance holds for Schrödinger operators $H = -\Delta + V$ with, for simplicity, bounded potential V. See Lemma 3.11 and the references listed in Section 3.3.2.

The $\mathbb{Z}^d$-ergodicity and $(\mathcal{A}_1)$ are our core assumptions. For such operators we prove this chapter's main result under the assumption that the operator kernel of $g(H_\omega)$ has sufficient spatial decay. A precise notion of this is given below. Besides those essential requirements we facilitate life by introducing additional symmetry. For a measure preserving transformation $T : \Omega \to \Omega$ we introduce the short-hand notation $H_{T\omega} =: H_\omega^T$.

$(\mathcal{A}_2)$ Symmetry of spatial directions: For $\pi \in \mathcal{S}^d$, the group of permutations on $\{1, ...d\}$, we define the unitary operator U_π on $L^2(\mathbb{R}^d)$ acting as $(U_\pi\psi)(x) := \psi(x_\pi)$, where $x_\pi := (x_{\pi(1)}, ..., x_{\pi(d)})$. Then for any $\pi \in \mathcal{S}^d$ there exists a measure preserving transformation $P_\pi : \Omega \to \Omega$ such that

$$U_\pi H_\omega U_\pi^* = H_\omega^{P_\pi}. \tag{5.3}$$

$(\mathcal{A}_3)$ Reflection symmetry: For $\sigma = (\sigma_i)_{i=1}^d \in \{0, 1\}^d =: \mathcal{R}^d$ we define the unitary operator U_σ on $L^2(\mathbb{R}^d)$ acting as $(U_\sigma\psi)(x) := \psi(x_\sigma)$, where $x_\sigma := ((-1)^{\sigma_1}x_1, ..., (-1)^{\sigma_d}x_d)$. Then for any $\sigma \in \mathcal{R}^d$ there exists a measure preserving transformation $R_\sigma : \Omega \to \Omega$ such that

$$U_\sigma H_\omega U_\sigma = H_\omega^{R_\sigma}. \tag{5.4}$$

In (5.4) we used that $U_\sigma = U_\sigma^*$ for $\sigma \in \mathcal{R}^d$. Those two additional assumptions are made for convenience and could be dropped. We included them because they make statement and proof of our results less notationally involved; for instance $(\mathcal{A}_2)$ allows to reduce up to $|S_d| = d!$ terms to only one. Our guiding example are Schrödinger operators $H_\omega = -\Delta + V_\omega$, where V_ω is an ω-dependent and real-valued potential that satisfies $U_j V_\omega U_j^* = V_{T_j\omega}$. Concrete examples are periodic Schrödinger operators and the random Schrödinger operators from Section 2.1. Those two are also the most extreme examples in terms of the amount of randomnes present in the system. Here we only consider an asymptotic expansion of the disorder-averaged trace of $h(g(H)_{\Lambda_L})$, which allows us to treat those substantially different cases at once. In case of non-deterministic operators, i.e. for $(\Omega, \mathbb{P})$ sufficiently rich, averaged asymptotics are also a starting point for a pointwise analysis. For this purpose, however, it seems more realistic to obtain results for one specific type of models at a time. This is also discussed at the end of Section 5.2.

Apart from the technical properties $(\mathcal{A}_1)$–$(\mathcal{A}_3)$ introduced above, our main assumption is sufficiently fast decay of the operator kernel of $g(H_\omega)$, where $g : \mathbb{R} \to \mathbb{R}$ is a compactly supported and bounded function. The two guiding examples for which decay of the operator kernel is known are spectral localization of the operator H_ω and a sufficiently smooth function g. To cover both cases in a convenient way we assume that one of the following two conditions holds.

$(\mathcal{L}_{1,q})$ For fixed $q > 0$ there exists a constant $C_{1,q}$ such that for all $a, b \in \mathbb{R}^d$

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \|\mathcal{X}_a g(H_\omega)\mathcal{X}_b\| \leqslant \frac{C_{1,q}}{(1 + |a - b|)^q}. \tag{5.5}$$

($\mathcal{L}_2$) There exist constants $C_2, \mu > 0$ such that for all $a, b \in \mathbb{R}^d$

$$\mathbb{E}\left[\|\mathcal{X}_a g(H)\mathcal{X}_b\|\right] \leqslant C_2 e^{-\mu|a-b|}. \tag{5.6}$$

The first condition holds for a large class of operators which obey a Combes-Thomas estimate and with a power of q that depends on the regularity of g. More concretely, if $H_\omega = -\Delta + V_\omega$ is a Schrödinger operator with, for simplicity, uniformly (in $x \in \mathbb{R}^d$ and in $\omega \in \Omega$) bounded potential V_ω, then $g \in \mathcal{C}_c^{q+2}(\mathbb{R})$ implies that ($\mathcal{L}_{1,q}$) holds [55, Thm. 2]. The second bound for example holds if H_ω is the random Schrödinger operator from Section 2.1 and g is a bounded function such that $\operatorname{supp}(g) \subset \Sigma_{\mathrm{FMB}}$.

In order to state the main result, and to define the asymptotic coefficients, we introduce some more notation. If g is bounded then $g(H_\omega)$ is a bounded operator. For $G \subset \mathbb{R}^d$ we denote its restriction to the space $L^2(G)$ by $g(H_\omega)_G$. With some abuse of notation this operator coincides with the operator $\mathcal{X}_G g(H_\omega)\mathcal{X}_G$ on $L^2(\mathbb{R}^d)$. For the sake of consistency (and even though the latter is prettier) we abbreviate $g(H_\omega)_L := g(H_\omega)_{\Lambda_L}$ for $L > 0$. For $n = 0, ..., d$ we define the model operators

$$f_{\omega,n} := h(g(H_\omega)_{\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n}}), \tag{5.7}$$

which approximate $h(g(H_\omega)_L)$ in respective areas of the cube Λ_L. Here, and in the following, $\mathbb{R}^n_{\geqslant 0} := (\mathbb{R}_{\geqslant 0})^n = (0,\infty)^n$ for $n = 1, .., d$. For instance, $f_{\omega,0}$ is an approximation of the operator in the bulk of Λ_L and $f_{\omega,1}$ is an approximation of the operator along a face of Λ_L (taking the symmetries ($\mathcal{A}_2$) and ($\mathcal{A}_3$) into account). Moreover, for $1 \leqslant n \leqslant m \leqslant d$ we set

$$c_{m,n} := \frac{(-1)^{m-n}2^m d!}{(m-n)!(d-m)!}, \tag{5.8}$$

$$\widehat{\mathcal{X}}_{m,n} := \mathcal{X}_{\mathbb{R}^d_{\geqslant 0}}\mathcal{X}_{\{x_1\leqslant\ldots\leqslant x_n\}}\mathcal{X}_{\{x_n\geqslant x_{n+1},\ldots,x_m\}}\mathcal{X}_{\{x_{m+1},\ldots,x_d\in[0,1]\}}. \tag{5.9}$$

The constant $c_{m,n}$ is a combinatorial factor which stems from collecting terms via the symmetry assumptions ($\mathcal{A}_2$) and ($\mathcal{A}_3$). The projection operator $\widehat{\mathcal{X}}_{m,n}$ ensures that the first n coordinates are ordered increasingly and, in addition, that the n-th coordinate is not smaller than the first m coordinates. If $n = m$, we interpret $\mathcal{X}_{\{x_n\geqslant x_{n+1},\ldots,x_m\}} = \mathcal{X}_{\mathbb{R}^d} = \mathrm{id}_{L^2(\mathbb{R}^d)}$ in (5.9), and, in the same vein, $\mathcal{X}_{\{x_{d+1},\ldots,x_d\in[0,1]\}} = \mathcal{X}_{\mathbb{R}^d} = \mathrm{id}_{L^2(\mathbb{R}^d)}$. Finally, for a fixed bounded function $g : \mathbb{R} \to \mathbb{R}$ we set

$$\widehat{\Sigma}_g := [\inf \Sigma_g, \sup \Sigma_g], \tag{5.10}$$

where Σ_g denotes the (almost surely non-random) spectrum of $g(H_\omega)$.

Theorem 5.2 (The asymptotic expansion). *Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{C}$ be two compactly supported and bounded functions with $h(0) = 0$. If one of the following two conditions is satisfied for $\widetilde{q} > 2d$*

(i) ($\mathcal{L}_2$) *holds and $h \in \mathcal{C}^{\lfloor 2\widetilde{q}+2\rfloor}(\mathbb{R})$,*

(ii) ($\mathcal{L}_{1,q}$) *holds for $q > 2d + \widetilde{q}$ and h can be continued analytically to $\{z \in \mathbb{C} : \operatorname{dist}(z, \widehat{\Sigma}_g) < C_{g,\widetilde{q}}\}$ for the constant $C_{g,\widetilde{q}}$ specified in* (5.34) *below,*

then, as $\mathbb{N} \ni L \to \infty$*, the asymptotic expansion*

$$\mathbb{E}\big[\operatorname{tr} h(g(H)_{2L})\big] = \sum_{m=0}^{d} A_m (2L)^{d-m} + \mathcal{O}(L^{2d-\widetilde{q}}) \tag{5.11}$$

holds. For $m \neq 0$ *the coefficients* A_m *are defined as*

$$A_m := \sum_{n=1}^{m} c_{m,n} \operatorname{tr}\Big(\mathbb{E}\Big[\widehat{\mathcal{X}}_{m,n}\{f_n - f_{n-1}\}\widehat{\mathcal{X}}_{m,n}\Big]\Big), \tag{5.12}$$

and for $m = 0$ *as* $A_0 := \mathbb{E}\big[\operatorname{tr}\big(\mathcal{X}_{[0,1]^d} f_0\big)\big]$.

Remark 5.3. The representation (5.12) of the coefficients is not unique and depends on the partition of corners for the cube $(-L/2, L/2)^d$, $L \in 2\mathbb{N}$, which we choose in the proof. At the end of Section 5.4 we show that the coefficients also have a partition-free representation:

$$A_m = \lim_{L\to\infty} \sum_{n=0}^{m} \widetilde{c}_{m,n} \mathbb{E}\Big[\operatorname{tr}\Big(f_n \mathcal{X}_{[0,L]^d} \mathcal{X}_{\{x_{m+1},..,x_d \in [0,1]\}}\Big)\Big] \tag{5.13}$$

for constants $\widetilde{c}_{m,n}$ defined in Section 5.4.3 below (and which are of alternating sign in n for fixed m). The operator $f_{\omega,n}\mathcal{X}_{[0,\infty)^d}\mathcal{X}_{\{x_{m+1},\dots,x_d\in[0,1]\}}$ is trace class only if $m = 0$, which corresponds to the coefficient A_0. The L-limit can therefore not be interchanged with the sum appearing in (5.13).

Remarks 5.4. (i) The validity of the asymptotic expansion (5.11) is not restricted to assumptions (*i*) or (*ii*), which serve as two relevant examples. A rather different setup is described in Remark 5.7 (v) below.

(ii) The uncommon ordering of expectation and trace norm in (5.12) stems from Lemma 5.10 and is only necessary under assumption (*i*).

(iii) Under reasonable assumptions, the theorem can be extended to length-scales $L \in \mathbb{R}_{>0}$. For $\mathbb{Z}^d$-ergodic operators the coefficients A_m then become functions of the fractional part of L. This dependence in turn does not show up if the operator is invariant under $\mathbb{R}^d$-translations.

Theorem 5.2 can be split into two parts, Theorem 5.5 and Theorem 5.6 below. The aim of this subdivision is to split the result into an analytic part, Theorem 5.5, and an algebraic part, Theorem 5.6. We recall that $Q_a = \Lambda_1(a)$ is the cube of side length 1 centered at $a \in \mathbb{R}^d$.

Theorem 5.5 (Asymptotic expansion - analytic part)**.** *Let* $g : \mathbb{R} \to \mathbb{R}$ *and* $h : \mathbb{R} \to \mathbb{C}$ *be two compactly supported, bounded functions with* $h(0) = 0$*. If, additionally, one of the following two conditions is satisfied for fixed* $\widetilde{q} > 0$

(i) $(\mathcal{L}_2)$ *holds and* $h \in \mathcal{C}^{\lfloor 2\widetilde{q}+2\rfloor}(\mathbb{R})$,

(ii) $(\mathcal{L}_{1,q})$ *holds for* $q > 2d + \widetilde{q}$ *and* h *can be continued analytically to* $\{z \in \mathbb{C} : \operatorname{dist}(z, \widehat{\Sigma}_g) < C_{g,\widetilde{q}}\}$ *for the constant* $C_{g,\widetilde{q}}$ *specified in* (5.34) *below,*

then the following holds: There exists a constant $C = C_{g,h,\widetilde{q}}$ such that for all $G \subset G' \subseteq \mathbb{R}^d$ and all $a, b \in G'$ with $Q_a \subset G$ or $Q_b \subset G$

$$\|\mathbb{E}\left[\mathcal{X}_a\{h(g(H)_G) - h(g(H)_{G'})\}\mathcal{X}_b\right]\|_1 \leqslant \frac{C}{\operatorname{dist}(a, G' \setminus G)^{\widetilde{q}} + \operatorname{dist}(b, G' \setminus G)^{\widetilde{q}}}. \tag{5.14}$$

Theorem 5.6 (Asymptotic expansion - algebraic part)**.** *Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{C}$ be two compactly supported and bounded functions such that there exist constants $C_h, \gamma_h > 0$ with $|h(x)| \leqslant C_h |x|^{\gamma_h}$ holds for all $x \in \mathbb{R}$. If (5.14) holds for $\widetilde{q} > 2d$, then, as $\mathbb{N} \ni L \to \infty$*

$$\mathbb{E}\big[\operatorname{tr} h(g(H)_{2L})\big] = \sum_{m=0}^{d} A_m (2L)^{d-m} + \mathcal{O}(L^{2d-\widetilde{q}}). \tag{5.15}$$

Remarks 5.7. (i) If $(\mathcal{L}_2)$ holds for a deterministic model, i.e. $\Omega = \{0\}$, then the proof of Theorem 5.5 and Remark 5.11 show that $h \in \mathcal{C}^{\widetilde{q}+1}(\mathbb{R})$ implies (5.14). This is probably also true for the general case but would require a refined version of the Combes-Thomas estimate from Lemma 5.10.

(ii) For fixed pairs of functions g and h in Theorems 5.5 and 5.6 it is sufficient to assume (5.2) for the function g and a sufficiently small h-dependent value of $0 < p < 1$.

(iii) Under assumption (*ii*), the expectation in (5.14) is obsolete and the bound holds almost surely.

(iv) For the special case of a random Schrödinger operator as in Section 2.1 and a function g such that $\operatorname{supp}(g) \subset \Sigma_{\mathrm{FMB}}$, the bound (5.14) seems to be a weak conclusion from $(\mathcal{L}_2)$: It is for instance known that in this case $\mathbb{E}\left[\|\mathcal{X}_a (h \circ g)(H) \mathcal{X}_b\|\right]$ is exponentially decaying in $|a - b|$ for any bounded function h. But in order to conclude (5.14) without any smoothness assumption on h we would have to rule out extended boundary states for the random operator $g(H_\omega)$. To my knowledge this is not known in such generality for $d > 1$.

(v) The bound (5.14) is not restricted to the assumptions (*i*) and (*ii*) in Theorem 5.5. A special yet very different situation is the following. If H_ω is a random Schrödinger operator as in Section 2.1 and we take $g = \mathrm{id}_{\mathbb{R}}$ and $h = \mathbb{1}_{(-\infty, E]}$ for an energy $E \in \Sigma_{\mathrm{FMB}}$, then (5.14) holds with exponential decay in $\operatorname{dist}(a, G' \setminus G)$ and $\operatorname{dist}(b, G' \setminus G)$. This follows from Theorem 3.5. From the perspective of Remark 5.7 (iv) above, this is the trivial case in which extended boundary states for $g(H_\omega)$ can be ruled out in the relevant spectral region.

5.2. Proof's idea & more

Let's first discuss the technical Theorem 5.5. Its statement is reminiscent of Theorem 3.5 and most of the methods leading to its proof are well known from different contexts. For each of the assumptions (*i*) and (*ii*) we first prove an operator-norm version of the estimate (5.14) and in both cases we employ a suitable functional calculus to rewrite $h(g(H_\omega)_{G^{(\prime)}})$ in terms of the resolvent of $g(H_\omega)_{G^{(\prime)}}$. Via a Combes-Thomas estimate and the geometric resolvent equation we then localize the operator $h(g(H_\omega)_G) - h(g(H_\omega)_{G'})$ to $\partial G' \cap G$. Finally, the corresponding trace-norm estimate follows from interpolation with Schatten-p bounds ($p < 1$) for the difference of operators on the left-hand side of (5.14). Such bounds are a consequence of $(\mathcal{A}_1)$, see Lemma 5.8 below. We quickly comment on both cases separately. Under assumption $(\mathcal{L}_{1,q})$ a polynomial Combes-Thomas estimate for the resolvent of $g(H_\omega)_G$,

$G \subseteq \mathbb{R}^d$, is known to hold [2, App. II]. The holomorphic functional calculus then lifts this mild decay of the resolvent to decay of the operator $h(g(H_\omega)_G) - h(g(H_\omega)_{G'})$. In case of assumption $(\mathcal{L}_2)$ only decay of the averaged operator kernel is known. While this seems to shut down the standard approach for the Combes-Thomas estimate we show below that an alternative approach - power series expansion of the resolvent far apart from the spectrum and subsequent interpolation in the complex energy parameter - is flexible enough. To the best of our knowledge this approach is not covered in the literature. This is why we included a detailed proof in the next section. Once the Combes-Thomas estimate is established we apply the Helffer-Sjöstrand formula to rewrite $h(g(H_\omega)_G)$ in terms of the resolvent of $g(H_\omega)_G$. This final step is essentially contained in [55].

We now turn the asymptotic expansion. For the sake of discussion, we argue by means of a $\mathbb{Z}^d$-translation invariant operator H for this part. Hence disorder averages can be omitted. The guiding example we have in mind is a $\mathbb{Z}^d$-periodic Schrödinger operator $H = -\Delta + V_{\text{per}}$. Moreover, we for simplicity confine to $d = 2$ dimensions and assume that the operator kernel of $g(H)$ is exponentially decaying. Strictly speaking, exponential decay in this situation is a bit too harsh as it basically forces the function g to have an entire extension. This is in conflict with our assumption of compact support. We ignore this technical issue for the present discussion. Finally we choose $L \in \mathbb{N}$ in the following and note that $g(H)_{2L} = g(H)_{(-L,L)^2} = g(H)_{[-L,L]^2}$ as operators on $L^2(\mathbb{R}^2)$.

Due to the decay assumption on the operator kernel of $g(H)$ the finite-volume restriction $g(H)_{2L}$ is well-approximated by the respective model operators which approximate $g(H)_{2L}$ in the different regions of the cube Λ_{2L}. To illustrate this we first discuss the well-known first-order asymptotics, known as the density of states term, and the second-order asymptotics which for instance was recently proven for the localized lattice Anderson model [43] but was established earlier in the context of Toeplitz matrices and Wiener-Hopf operators. For instance, it implicitly already appeared in the seminal work [123] from Szegő. In order to stick closer to the actual proof below (and to simplify notation) we first employ the $\mathbb{Z}^d$-translation invariance and the symmetries $\mathcal{A}_2, \mathcal{A}_3$ to rewrite

$$\begin{aligned} \operatorname{tr} h(g(H)_{2L}) &= 4 \operatorname{tr}\big(\mathcal{X}_{[0,L]^2} h(g(H)_{[0,2L]^2})\big) \\ &= 4 \operatorname{tr}\big(\mathcal{X}_{[0,L]^2} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big) + \mathcal{O}(L^2 e^{-\mu L}). \end{aligned} \tag{5.16}$$

For the second identity we used that the trace in the middle expression is evaluated in the spatial region $[0, L]^2$, which has distance L from the spatial area where we erased the boundary. That this results in an error of order $L^2 e^{-\mu L}$ is a consequence of the exponentially decaying operator kernel $\mathcal{X}_a g(H) \mathcal{X}_b$ and the discussion of Theorem 5.5 above. Similar arguments are employed repeatedly below. For example, similar reasoning yields that

$$\operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\{h(g(H)_{\mathbb{R}^2_{\geqslant 0}}) - (h \circ g)(H)\}\big) = \mathcal{O}(L). \tag{5.17}$$

Here we decomposed $[0, L]^2$ into strips $\{x \in [0, L]^2 : k \leqslant \operatorname{dist}(x, \partial\mathbb{R}^2_{\geqslant 0}) \leqslant k+1\}$ and estimated the trace on each such strip by $\lesssim L e^{-\mu k}$. If we plug this into (5.16) and use the $\mathbb{Z}^2$-translation invariance of H we obtain

$$\operatorname{tr} h(g(H)_{2L}) = (2L)^2 \operatorname{tr}\big(\mathcal{X}_0 (h \circ g)(H))\big) + \mathcal{O}(L). \tag{5.18}$$

This can be restated as the convergence of the finite-volume density of states and which holds in much broader generality than the scenario considered here. Arguing via the right hand side of (5.16) is quite artificial in this case but for the second and third order asymptotics it will simplify notation significantly. To get beyond the leading order term we have to take into account that the operator $h(g(H)_{\mathbb{R}^2_{\geqslant 0}})$ along the boundary of $\mathbb{R}^2_{\geqslant 0}$ can be approximated more adequately by the half-space operators $h(g(H)_{\mathbb{R}^2_{\geqslant 0}})$. To this end, we choose an L-dependent distance $\ell := (\log L)^2$ and decompose

$$\operatorname{tr}\big(\mathcal{X}_{[0,L]^2} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big) = \operatorname{tr}\big(\mathcal{X}_{[\ell,L]^2} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big) + \operatorname{tr}\big(\mathcal{X}_{[0,\ell]^2} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big) + \operatorname{tr}\big(\mathcal{X}_{[\ell,L]\times[0,\ell]} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big) + \operatorname{tr}\big(\mathcal{X}_{[0,\ell]\times[\ell,L]} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big). \tag{5.19}$$

The first term on the right hand side is the bulk term. Substituting $h(g(H)_{\mathbb{R}^2_{\geqslant 0}})$ by $(h \circ g)(H)$ on $[\ell, L]^2$ results in an error of order $O(L^2 e^{-\mu L})$. The second term is of order $O(\ell^2)$ and can be neglected. This leaves us with the last two terms, which due to the symmetry assumptions $\mathcal{A}_2$, $\mathcal{A}_3$ are equal. Overall, this yields

$$\operatorname{tr}\big(\mathcal{X}_{[0,L]^2} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big) = (L-\ell)^2 \operatorname{tr}\big(\mathcal{X}_0 (h\circ g)(H))\big) + 2\operatorname{tr}\big(\mathcal{X}_{[0,\ell]\times[\ell,L]} h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\big) + \mathcal{O}(L^2 e^{-\mu L}) + \mathcal{O}((\log L)^2). \tag{5.20}$$

The second summand already contains the contribution of the face of the cube but likewise still contain a bulk contribution. To extract the bulk contribution we write

$$\operatorname{tr}\big(\mathcal{X}_{[0,\ell]\times[\ell,L]} h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\big) = (L-\ell)\ \operatorname{tr}\big(\mathcal{X}_{[0,\ell]\times[0,1]}\{h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}}) - (h\circ g)(H)\}\big) + (L-\ell)\ell\ \operatorname{tr}\big(\mathcal{X}_0 (h\circ g)(H)\big), \tag{5.21}$$

where we also used that the operator $h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})$ is invariant under $\mathbb{Z}$-translations in the x_2-direction. From (5.14) we can grasp that the first trace on the right hand side converges (fast enough) as $\ell \to \infty$. Hence we overall found that

$$\operatorname{tr} h(g(H)_{\Lambda_{2L}}) = (2L)^2 \operatorname{tr}\big(\mathcal{X}_0 (h\circ g)(H)\big) + 4(2L)\operatorname{tr}\big(\mathcal{X}_{\mathbb{R}_{\geqslant 0}\times[0,1]}\{h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}}) - (h\circ g)(H)\}\big) + \mathcal{O}((\log L)^2). \tag{5.22}$$

The error term here is suboptimal and due to our choice of ℓ above. The basic steps in order to go beyond the second order are similar as going from first to second order: We have to take the corner term into account which was neglected as a lower order term so far. Similarly as we regularized the face term by subtracting its bulk contribution in (5.21) we have to extract both, the bulk and the face contribution from the corner term. The above procedure relied on separating the leading and subleading order via introducing a second length scale $\ell \ll L$. From my point of view a similar approach to lower order asymptotics in general is inconvenient at best and impossible at worst. We follow a slightly different approach, based on an algebraic identity that separates the contributions to the respective orders L^2, L^1 and L^0 of the asymptotic formula directly. To this end we again employ $\mathcal{A}_1, \mathcal{A}_2$ to rewrite

$$\operatorname{tr}\big(\mathcal{X}_{[0,L]^2} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big) = 2\operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\leqslant x_2\}} h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big). \tag{5.23}$$

Here $\mathcal{X}_{\{x_1\leqslant x_2\}}$ stands for the $L^2(\mathbb{R}^2)$-projection onto the set $\{(x_1,x_2)\in\mathbb{R}^2 : x_1 \leqslant x_2\}$. The advantage of the right hand side is that $\mathcal{X}_{\{x_1\leqslant x_2\}}$ makes the expression robust under manipulation of the boundary along the $x_2 = 0$ face. This is

$$\begin{aligned} \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\leqslant x_2\}}h(g(H)_{\mathbb{R}^2_{\geqslant 0}})\big) &= \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\leqslant x_2\}}\{h(g(H)_{\mathbb{R}^2_{\geqslant 0}}) - h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\}\big) \\ &\quad + \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\leqslant x_2\}}h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\big). \end{aligned} \tag{5.24}$$

With similar arguments as above the first trace on the right hand side converges (fast enough) as $L \to \infty$. This is going to be the first contribution to the third order of the asymptotics. The second summand can be rewritten as

$$\begin{aligned} \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\leqslant x_2\}}h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\big) &= \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\big) \\ &\quad - \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\geqslant x_2\}}h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\big). \end{aligned} \tag{5.25}$$

The first trace on the right hand side can be handled similarly as the corresponding term for the second order asymptotics. Namely,

$$\begin{aligned} \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\big) &= L\ \operatorname{tr}\big(\mathcal{X}_{[0,L]\times[0,1]}\{h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}}) - (h\circ g)(H)\}\big) \\ &\quad + L^2 \operatorname{tr}\big(\mathcal{X}_0(h\circ g)(H)\big). \end{aligned} \tag{5.26}$$

Here, the first trace on the right hand side again converges (sufficiently fast) as $L \to \infty$. Finally we regularize the second term on the right hand side of (5.25) by

$$\begin{aligned} \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\geqslant x_2\}}h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\big) &= \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\geqslant x_2\}}\{h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}}) - (h\circ g)(H)\}\big) \\ &\quad + \operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\geqslant x_2\}}(h\circ g)(H)\big). \end{aligned} \tag{5.27}$$

The second trace on the right hand side is due to $\mathcal{A}_2, \mathcal{A}_3$ equal to $(L^2/2)\operatorname{tr}\big(\mathcal{X}_0(h\circ g)(H)\big)$ and the first trace is again convergent (sufficiently fast) as $L \to \infty$. Overall we proved the algebraic identity

$$\operatorname{tr} h(g(H)_{2L}) = (2L)^2 A_0 + (2L)A_1^{(L)} + A_2^{(L)} \tag{5.28}$$

with coefficients

$$A_0 := \operatorname{tr}\big(\mathcal{X}_0(h\circ g)(H))\big), \tag{5.29}$$

$$A_1^{(L)} := 4\operatorname{tr}\big(\mathcal{X}_{[0,L]\times[0,1]}\{h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}}) - (h\circ g)(H)\}\big), \tag{5.30}$$

$$\begin{aligned} A_2^{(L)} &:= 8\operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\leqslant x_2\}}\{h(g(H)_{\mathbb{R}^2_{\geqslant 0}}) - h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}})\}\big) \\ &\quad - 8\operatorname{tr}\big(\mathcal{X}_{[0,L]^2}\mathcal{X}_{\{x_1\geqslant x_2\}}\{h(g(H)_{\mathbb{R}_{\geqslant 0}\times\mathbb{R}}) - (h\circ g)(H)\}\big). \end{aligned} \tag{5.31}$$

As we have seen along the lines of the above argument the finite-volume coefficients $A_i^{(L)}$, $i = 1, 2$, converge (sufficiently fast) towards their respective infinite-volume counterparts.

The asymptotic expansion (5.15) could be extended and improved in many directions. First of all, our smoothness constraints on the function h do exclude the relevant case of the entanglement entropy. Due to the works [87, 88, 118, 119] it is known that a second-order asymptotic expansion of $\operatorname{tr} h(g(H)_{2L})$ for smooth test functions $h \in \mathcal{C}^\infty$ can be extended to functions as singular as $[0,1] \ni x \to h(x) = x\log x$ by means of rather general functional analytic arguments. Whether or not this is possible in a similar fashion for lower order terms of the asymptotic expansion is unclear to me. In principle, this problem should also be much easier for the special case of a localized random Schrödinger operator H_ω (or, more precisely, $\operatorname{supp}(g) \subset \Sigma_{\mathrm{FMB}}$) because of Remark 5.7(iv): In this situation one would expect a much stronger bound than (5.14).

Another interesting question is whether the above procedure is substantially restricted to the special (and highly symmetric) case of cubes or if comparable results hold for more general scaling domains $\Omega_L := L\Omega$. A full asymptotic expansion has recently been proved [102, 101] for general curvilinear polygons in $d = 2$ dimensions.

Finally, in the context of random Schrödinger operators (or, more generally, ergodic operators with a sufficiently rich space $(\Omega, \mathbb{P})$) the averaged asymptotic expansion (5.6) and the applied methods could serve as a starting point for a non-averaged asymptotic expansion. Let's for the sake of simplicity consider the random Schrödinger operator H_ω from Section 2.1. The results and heuristics from [78] suggest that if $\operatorname{supp}(g) \subset \Sigma_{\mathrm{FMB}}$ and h is a sufficiently smooth function, then

$$\operatorname{tr} h(g(H)_{2L}) = \sum_{m=0}^{\lfloor d/2 \rfloor} A_m (2L)^{d-m} + L^{d/2} \mathcal{N}(0, \sigma^2), \tag{5.32}$$

where the above equality refers to convergence in distribution and $\mathcal{N}(0, \sigma^2)$ is a normal distribution with non-trivial variance $\sigma > 0$.

5.3. Proof of the decay estimate

Most of the methods from this section are well-known from different contexts. For instance, the passage from a Combes-Thomas estimate for $g(H_\omega)$ to decay of the operator kernel of $h(g(H_\omega)_G)$ via a Helffer-Sjöstrand formula is essentially contained in [55]. Our proof of Theorem 5.5 under assumption (i) follows along the lines of their proof.

Let $g : \mathbb{R} \to \mathbb{R}$ be bounded and compactly supported. For the whole section we abbreviate $A_\omega := g(H_\omega)$ and $\widehat{\Sigma} = \widehat{\Sigma}_g$. The restriction of the (uniformly in ω) bounded operator A_ω to a subset $G \subset \mathbb{R}^d$ is denoted by $A_{\omega,G}$. In the following we stick to our original setup but one can think of A_ω as an arbitrary bounded (ω-dependent) operator satisfying $\mathcal{A}_1$.

5.3.1. Proof of Theorem 5.5 under assumption (ii)**.** For operators on $\mathbb{Z}^d$ a polynomial Combes-Thomas estimate is proved in [2, App. II] and reviewed in [8, Ch. 10.3]. Their proof carries over to our setup. For the next few lines the notation closely sticks to [8]. If matrix elements are substituted by operator kernels $\mathcal{X}_a (A_{\omega,G} - z)^{-1} \mathcal{X}_b$ for $a, b \in \mathbb{Z}^d \cap G$, then the proof works if we choose a distance function which is constant on unit cubes Q_a, $a \in \mathbb{Z}^d$. The transition to arbitrary $a, b \in G$ then induces a slightly enlarged constant in (5.33) below. The term $(|a - b| + 2)^{q'}$ in (5.34) below instead of $(|a - b| + 1)^{q'}$ in [8] is due to the transition from the $\mathbb{Z}^d$-adapted distance to the original distance. Let $\varepsilon > 0$ such that $q = \widetilde{q} + 2d + \varepsilon$ and define $q' = q - d - \varepsilon/2 = \widetilde{q} + d + \varepsilon/2$. Then, via the polynomial Combes-Thomas estimate,

$$\|\mathcal{X}_a R_z(A_{\omega,G}) \mathcal{X}_b\| \leqslant \frac{C_1}{(|a - b| + 1)^{q'}} \tag{5.33}$$

holds for all $z \in \mathbb{C}$ that satisfy

$$\operatorname{dist}(z, \widehat{\Sigma}) \geqslant 1 + \sup_{a \in \mathbb{Z}^d} \sum_{b \in \mathbb{Z}^d} \|\mathcal{X}_a A_{\omega,G} \mathcal{X}_b\| \Big((|a - b| + 2)^{q'} - 1 \Big) =: C_{g,\widetilde{q}} - 1. \tag{5.34}$$

Fix $a, b \in G'$ such that $Q_a \subset G$. By assumption the function h can be continued analytically onto $\{z \in \mathbb{C} : \operatorname{dist}(z, \widehat{\Sigma}) < C_{g,\widetilde{q}}\}$. Let Γ be a smooth oriented curve, with winding number $= 1$ for the set $\widehat{\Sigma}$, such that

$$\operatorname{ran}(\Gamma) \subset \{z \in \mathbb{C} : C_{g,\widetilde{q}} - 1 < \operatorname{dist}(z, \widehat{\Sigma}) < C_{g,\widetilde{q}}\} \tag{5.35}$$

holds for the range of Γ. The holomorphic functional calculus then yields

$$\mathcal{X}_a \left(h(A_{\omega,G}) - h(A_{\omega,G'})\right) \mathcal{X}_b = \frac{1}{2\pi i} \int_\Gamma \mathrm{d}z\, h(z) \mathcal{X}_a R_z(A_{\omega,G}) \left(\mathcal{X}_G A_\omega \mathcal{X}_{G' \setminus G}\right) R_z(A_{\omega,G'}) \mathcal{X}_b, \tag{5.36}$$

where we also applied the geometric resolvent equation and used $Q_a \subset G$. For a set $U \subseteq \mathbb{R}^d$ we define $U_+^\# := \{n \in (\mathbb{Z}+1/2)^d : Q_n \cap U \neq \varnothing\}$, where the $+$ subscript refers to the different choice of cube centers compared to our usual definition of the set $U^\#$. The operator norm of (5.36) can then be estimated as

$$\begin{aligned} \|(5.36)\| &\leqslant C_2 \sum_{\substack{l \in G_+^\# \\ k \in (G' \setminus G)_+^\#}} \frac{1}{(|a-l|+1)^{q'}} \frac{1}{(|l-k|+1)^{q}} \frac{1}{(|k-b|+1)^{q'}} \\ &\leqslant \frac{C_3}{\operatorname{dist}(a, G' \setminus G)^{q'}}, \end{aligned} \tag{5.37}$$

where we used the inequality $xy \geqslant x/2 + y/2$ for $x, y \geqslant 1$ and the relations $q' > d$, $q > q' + d$. Because the same bound holds with b instead of a on the right-hand side of (5.37) we obtain

$$\|(5.36)\| \leqslant \frac{C_4}{\operatorname{dist}(a, G' \setminus G)^{q'} + \operatorname{dist}(b, G' \setminus G)^{q'}}. \tag{5.38}$$

Finally we interpolate (5.38) with Schatten-class bounds for the operator kernel of $A_{\omega,G}$. Such bounds follow from $(\mathcal{A}_1)$ and the next lemma.

Lemma 5.8 (Local Schatten-class bounds)**.** *Let $p > 0$ and let B be a self-adjoint bounded operator on $L^2(\mathbb{R}^d)$ such that*

$$C := \sup_{a \in \mathbb{R}^d} \|\mathcal{X}_a B^2 \mathcal{X}_a\|_p < \infty \tag{5.39}$$

holds. Then, for functions $h : \mathbb{R} \to \mathbb{C}$ such that $|h(x)| \leqslant C_h |x|^{\gamma_h}$ holds for all $x \in \mathbb{R}$ and constants C_h and $0 < \gamma_h \leqslant 1$,

$$\sup_{G \subseteq \mathbb{R}^d} \sup_{a \in \mathbb{R}^d} \|\mathcal{X}_a h(B_G)\|_{\frac{2p}{\gamma_h}} \leqslant C^p C_h^{\frac{2p}{\gamma_h}}. \tag{5.40}$$

The proof of the Lemma is given below. For $p > \delta > 0$ and an operator K the bound

$$\|K\|_p^p \leqslant \|K\|^\delta \|K\|_{p-\delta}^{p-\delta} \tag{5.41}$$

holds. With $p = 1$ and $\delta = \widetilde{q}/q' \in (0,1)$ this yields

$$\begin{aligned} \left\|\mathcal{X}_a\{h(A_{\omega,G}) - h(A_{\omega,G'})\}\mathcal{X}_b\right\|_1 &\leqslant \left\|\mathcal{X}_a\{h(A_{\omega,G}) - h(A_{\omega,G'})\}\mathcal{X}_b\right\|^{\widetilde{q}/q'} \\ &\quad \times \left\|\mathcal{X}_a\{h(A_{\omega,G}) - h(A_{\omega,G'})\}\mathcal{X}_b\right\|_{1-\widetilde{q}/q'}^{1-\widetilde{q}/q'}. \end{aligned} \tag{5.42}$$

The first term on the right hand side of (5.42) can be estimated via (5.38). For the second term we apply Lemma 5.8 to the operator A_ω^2. Assumption $(\mathcal{A}_1)$ ensures that (5.39) holds and the bound on h follows from smoothness and $h(0) = 0$. The lemma yields

$$\begin{aligned}\big\|\mathcal{X}_a\{h(A_{\omega,G}) - h(A_{\omega,G'})\}\mathcal{X}_b\big\|_{1-\widetilde{q}/q'}^{1-\widetilde{q}/q'} &\leqslant \|\mathcal{X}_a h(A_{\omega,G})\mathcal{X}_b\|_{1-\widetilde{q}/q'}^{1-\widetilde{q}/q'} + \big\|\mathcal{X}_a h(A_{\omega,G'})\mathcal{X}_b\big\|_{1-\widetilde{q}/q'}^{1-\widetilde{q}/q'}\\ &\leqslant C_5 \end{aligned} \tag{5.43}$$

and overall we found that

$$\big\|\mathcal{X}_a\{h(A_{\omega,G}) - h(A_{\omega,G'})\}\mathcal{X}_b\big\|_1 \leqslant \frac{C_6}{\operatorname{dist}(a, G' \setminus G)^{\widetilde{q}} + \operatorname{dist}(b, G' \setminus G)^{\widetilde{q}}}. \tag{5.44}$$

for $a, b \in G'$ such that $Q_a \subset G$. The proof for $Q_b \subset G$ follows along the same lines. ☐

Proof of Lemma 5.8. The singular values of $\mathcal{X}_a h(B_G)$ are

$$\mu_n(\mathcal{X}_a h(B_G)) = \sqrt{\lambda_n(\mathcal{X}_a |h|^2(B_G)\mathcal{X}_a)}, \tag{5.45}$$

where $\lambda_n(\cdot)$ denotes the n-th eigenvalue. Moreover, because $|h(x)| = |x|^{\gamma_h}\widetilde{h}(x)$, $x \in \mathbb{R}$, for some non-negative function $\widetilde{h}$ which is bounded by C_h, the form inequality

$$\begin{aligned}\mathcal{X}_a|h|^2(B_G)\mathcal{X}_a &= \mathcal{X}_a|B_G|^{\gamma_h}\widetilde{h}^2(B_G)|B_G|^{\gamma_h}\mathcal{X}_a\\ &\leqslant C_h^2\mathcal{X}_a|B_G|^{2\gamma_h}\mathcal{X}_a\end{aligned} \tag{5.46}$$

holds. The function $x \to x^{\gamma_h}$ is operator monotone because $0 < \gamma_h < 1$. Hence the form inequality

$$|B_G|^{2\gamma_h} = (\mathcal{X}_G B \mathcal{X}_G B \mathcal{X}_G)^{\gamma_h} \leqslant (\mathcal{X}_G B^2 \mathcal{X}_G)^{\gamma_h} \tag{5.47}$$

holds. By Jensen's inequality for concave functions

$$\big\langle \psi, \mathcal{X}_a(\mathcal{X}_G B^2 \mathcal{X}_G)^{\gamma_h}\mathcal{X}_a\psi\big\rangle \leqslant \big\langle \psi, \mathcal{X}_a\mathcal{X}_G B^2 \mathcal{X}_G\mathcal{X}_a\psi\big\rangle^{\gamma_h} \tag{5.48}$$

holds for normalized $\psi \in L^2(\mathbb{R}^d)$. (5.46) – (5.48) together with the min-max principle yield

$$\lambda_n(\mathcal{X}_a|h|^2(B_G)\mathcal{X}_a) \leqslant C_h^2\big(\lambda_n(\mathcal{X}_a\mathcal{X}_G B^2\mathcal{X}_G\mathcal{X}_a)\big)^{\gamma_h}. \tag{5.49}$$

Let $p' = 2p/\gamma_h$. Together with (5.45) this yields

$$\begin{aligned}\|\mathcal{X}_a h(B_G)\|_{p'}^{p'} &= \sum_{n\in\mathbb{N}} \mu_n(\mathcal{X}_a h(B_G))^{p'} \leqslant C_h^{p'} \sum_{n\in\mathbb{N}} \lambda_n(\mathcal{X}_a\mathcal{X}_G B^2\mathcal{X}_G\mathcal{X}_a)^{p'\gamma_h/2}\\ &\leqslant C_h^{2p/\gamma_h}\|\mathcal{X}_a B^2\mathcal{X}_a\|_p^p.\end{aligned} \tag{5.50}$$

☐

5.3.2. Combes-Thomas estimate under assumption $(\mathcal{L}_2)$. In this section we prove that averaged decay of the operator kernel of A_ω is sufficient to deduce averaged decay for the operator kernel of the resolvent at complex energies away from the spectrum. We state two different versions of this result, Lemma 5.9 and Lemma 5.10. The first Lemma is not needed for the proof of Theorem 5.5 but serves to illustrate the method and can be directly compared to the classical Combes-Thomas estimate.

Lemma 5.9 (Combes-Thomas estimate, version 1)**.** *Assume that* $(\mathcal{L}_2)$ *holds and let* $0 < \theta < 1/2$ *be fixed. Then there exist constants* $C, \mu > 0$ *such that for* $z \in \mathbb{C} \setminus \widehat{\Sigma}$

$$\|\mathbb{E}\left[\mathcal{X}_a R_z(A_G)\mathcal{X}_b\right]\| \leqslant \frac{C}{\operatorname{dist}(z,\widehat{\Sigma})} e^{-\mu \operatorname{dist}(z,\widehat{\Sigma})|a-b|^\theta}. \tag{5.51}$$

Lemma 5.10 (Combes-Thomas estimate, version 2)**.** *Assume that* $(\mathcal{L}_2)$ *holds and let* $0 < \theta < 1/2$ *be fixed. Then there exist constants* $C_\theta, \mu_\theta > 0$ *such that for* $G \subset G' \subseteq \mathbb{R}^d$ *and* $a, b \in G'$ *with* $Q_a \subset G$ *or* $Q_b \subset G$ *the bound*

$$\|\mathbb{E}\left[\mathcal{X}_a \left(R_z(A_G) - R_z(A_{G'})\right) \mathcal{X}_b\right]\| \leqslant \frac{C}{\operatorname{dist}(z,\widehat{\Sigma})} e^{-\mu \operatorname{dist}(z,\widehat{\Sigma})\left(\operatorname{dist}(a,G'\setminus G)^\theta + \operatorname{dist}(b,G'\setminus G)^\theta\right)} \tag{5.52}$$

holds for all $z \in \mathbb{C} \setminus \widehat{\Sigma}$.

Remark 5.11. The reason why only fractional exponential decay is established stems from the rather bold application of Hölder's inequality in (5.58) below. For a deterministic model, i.e. $\Omega = \{0\}$, the proof yields exponential decay ($\theta = 1$ in (5.51) and (5.52)).

Proof of Lemma 5.9. For convenience we fix $\theta = 1/4$ for the proof. Note that $\|A_\omega\| \leqslant \|g\|_\infty$ almost surely. Let $a, b \in \mathbb{R}^d$ be fixed. Then, for fixed m with $0 < m < M := 2(\|g\|_\infty + 1)$,

$$\{z \in \mathbb{C} : \; m < \operatorname{Im}(z) < M\} =: S_{m,M} \ni z \mapsto f(z) := \mathbb{E}\left[\mathcal{X}_a R_z(A)\mathcal{X}_b\right], \tag{5.53}$$

is an operator-valued analytic map which is continuous on $\overline{S_{m,M}}$ and bounded by $1/m$. For $m \leqslant t \leqslant M$ we define

$$F_t := \sup_{x\in\mathbb{R}} \|f(x+it)\| \leqslant \frac{1}{t}. \tag{5.54}$$

Then the Stein interpolation theorem [18] states that for $m \leqslant t \leqslant M$ the bound

$$F_t \leqslant F_m^{\frac{M-t}{M-m}} F_M^{\frac{t-m}{M-m}} \tag{5.55}$$

holds, where F_m can be estimated by $1/m$. In order to estimate F_M we expand the resolvent $R_z(A_\omega)$ as a Neumann series. This yields

$$\begin{aligned} F_M = \sup_{x\in\mathbb{R}} |f(x+iM)| &\leqslant \sum_{l=0}^{N} \frac{\mathbb{E}\left[\|\mathcal{X}_a A^l \mathcal{X}_b\|\right]}{2^{l+1}(\|g\|_\infty + 1)^{l+1}} + \sum_{l=N+1}^{\infty} \frac{\mathbb{E}\left[\|\mathcal{X}_a A^l \mathcal{X}_b\|\right]}{2^{l+1}(\|g\|_\infty + 1)^{l+1}} \\ &=: I_1 + I_2 \end{aligned} \tag{5.56}$$

for some $N > 0$ which is specified below. We estimate I_2 as

$$I_2 \leqslant \sum_{l=N+1}^{\infty} \frac{\|g\|_\infty^l}{2^{l+1}(\|g\|_\infty + 1)^{l+1}} \leqslant \frac{1}{2^N}. \tag{5.57}$$

To estimate I_1 we set, for fixed $l > 0$, $k_0 := a$ and $k_l := b$. An application of Hölder's inequality then yields

$$\mathbb{E}\left[\|\mathcal{X}_a A^l \mathcal{X}_b\|\right] \leqslant \sum_{k_1,\dots,k_{l-1}\in\mathbb{Z}^d} \mathbb{E}\Big[\prod_{j=1}^{l} \|\mathcal{X}_{k_{j-1}} A \mathcal{X}_{k_j}\|\Big]$$
$$\leqslant \sum_{k_1,\dots,k_{l-1}\in\mathbb{Z}^d} \prod_{j=1}^{l} \mathbb{E}\left[\|\mathcal{X}_{k_{j-1}} A \mathcal{X}_{k_j}\|^l\right]^{1/l}$$
$$\leqslant C_2^l \|g\|_\infty^{l-1} \sum_{k_1,\dots,k_{l-1}\in\mathbb{Z}^d} \prod_{j=1}^{l} e^{-\frac{\mu}{l}|k_{j-1}-k_j|}, \tag{5.58}$$

where, for the last inequality, we used ($\mathcal{L}_2$) and C_2 is the same constant as there. The product in (5.58) can be estimated as

$$\prod_{j=1}^{l} e^{-\frac{\mu}{l}|k_{j-1}-k_j|} \leqslant e^{-\frac{\mu}{2l}|a-b|} \prod_{j=1}^{l} e^{-\frac{\mu}{2l}|k_{j-1}-k_j|}. \tag{5.59}$$

Let's assume that $|a-b| > 1$. Then the sum defining I_1 starts at $l = 1$ and we obtain the upper bound

$$I_1 \leqslant \sum_{l=1}^{N} \frac{\|g\|_\infty^{l-1}}{2^{l+1}(\|g\|_\infty + 1)^{l+1}} e^{-\frac{\mu}{2l}|a-b|} \sum_{k_1,\dots,k_{l-1}\in\mathbb{Z}^d} \prod_{j=1}^{l} e^{-\frac{\mu}{2l}|k_{j-1}-k_j|}$$
$$\leqslant \sum_{l=1}^{N} 2^{-(l+1)} e^{-\frac{\mu}{2l}|a-b|} \left(\sum_{k\in\mathbb{Z}^d} e^{-\frac{\mu}{2l}|k|}\right)^{l-1}. \tag{5.60}$$

The k-sum on the right-hand side of (5.60) can be estimated from above by Bl^d for an l-independent constant B. Hence F_M can be estimated as

$$F_M \leqslant e^{-N\log(2)} + e^{-\frac{\mu}{2N}|a-b|} \sum_{l=1}^{N} \frac{(Bl^d)^{l-1}}{2^{l+1}}$$
$$\leqslant e^{-N\log(2)} + e^{-\frac{\mu}{2N}|a-b|} N B^N N^{dN}. \tag{5.61}$$

For the choice $N = |a-b|^{1/4}$, this yields

$$F_M \leqslant e^{-\log(2)|a-b|^{1/4}} + e^{-\frac{\mu}{2}|a-b|^{3/4}} |a-b|^{1/4} e^{|a-b|^{1/4}\left(\log(B)+\frac{d}{4}\log|a-b|\right)}$$
$$\leqslant C_3 e^{-\mu_2|a-b|^{1/4}} \tag{5.62}$$

for $\mu_2 := \min\{\mu/2, \log(2)\}$. With (5.55) for $t = 2m$ we arrive at

$$F_{2m} \leqslant F_m^{\frac{M-2m}{M-m}} F_M^{\frac{m}{M-m}} \leqslant \left(\frac{1}{m}\right)^{\frac{M-2m}{M-m}} C_4^{\frac{m}{M-m}} e^{-\frac{\mu_3 m}{M-m}|a-b|^{1/4}}. \tag{5.63}$$

For $\eta > 0$ this can be written as

$$\sup_{E\in\mathbb{R}} \left\|\mathbb{E}\left[\mathcal{X}_a R_{E+i\eta}(A)\mathcal{X}_b\right]\right\| \leqslant \left(\frac{2}{\eta}\right)^{\frac{M-\eta}{M-\eta/2}} C_5^{\frac{\eta}{2M-\eta}} e^{-\frac{\mu_4\eta}{2M-\eta}|a-b|^{1/4}}. \tag{5.64}$$

Because $M \geqslant 2$ we get for $\eta \in (0,1)$ the more appealing bound

$$\sup_{E\in\mathbb{R}} \left\|\mathbb{E}\left[\mathcal{X}_a, R_{E+i\eta}(A)\mathcal{X}_b\right]\right\| \leqslant \frac{C_6}{\eta} e^{-\mu_5\eta|a-b|^{1/4}} \tag{5.65}$$

for constants $C_6, \mu_5 > 0$ that are independent of $\eta \in (0,1)$ and $a, b \in \mathbb{R}^d$. For $\eta < 0$ the same interpolation argument can be performed below the real axis. This yields (5.51) in case $z = E + i\eta \in \mathbb{C} \setminus \widehat{\Sigma}$ is such that $\operatorname{dist}(E, \widehat{\Sigma}) \leqslant |\eta|$. If $\operatorname{dist}(E, \widehat{\Sigma}) \geqslant |\eta|$, then (5.51) would follow from interpolation on a vertical strip. But in this case interpolation is not even needed since the resolvent can directly be expanded. □

Proof of Lemma 5.10. We again choose $\theta = 1/4$ for notational convenience and do the proof for $z = E + i\eta$ with $\eta > 0$ and $E \in \widehat{\Sigma}$. Let $G \subset G' \subseteq \mathbb{R}^d$ and choose $a \in G$ with $Q_a \subset G$ and $b \in G'$. Fix $0 < m < M$ with $M := 2(\|g\|_\infty + 1)$. Except of the bound for F_M the proof is then the same as the proof of Lemma 5.9. We start by rewriting the difference $R_z(A_{\omega,G}) - R_z(A_{\omega,G'})$ via the resolvent equation:

$$\mathcal{X}_G \left(R_z(A_{\omega,G}) - R_z(A_{\omega,G'})\right) \mathcal{X}_{G'} = R_z(A_{\omega,G}) \left(\mathcal{X}_G A_\omega \mathcal{X}_{G'\setminus G}\right) R_z(A_{\omega,G'}). \tag{5.66}$$

Hölder's inequality then yields for a, b as chosen above and $z \in \mathbb{C}$ with $\operatorname{Im}(z) = M$

$$\begin{aligned}
&\left\|\mathbb{E}\left[\mathcal{X}_a \left(R_z(A_G) - R_z(A_{G'})\right) \mathcal{X}_b\right]\right\| \\
&\quad\leqslant \sum_{\substack{k\in(G'\setminus G)_+^{\#} \\ l\in G_+^{\#}}} \mathbb{E}\left[\|\mathcal{X}_a R_z(A_{G'})\mathcal{X}_k\|^3\right]^{1/3} \mathbb{E}\left[\|\mathcal{X}_k A \mathcal{X}_l\|^3\right]^{1/3} \mathbb{E}\left[\|\mathcal{X}_l R_z(A_G)\mathcal{X}_b\|^3\right]^{1/3} \\
&\quad\leqslant \left(\frac{C_1\|g\|_\infty^2}{M^4}\right)^{1/3} \sum_{\substack{k\in(G'\setminus G)_+^{\#} \\ l\in G_+^{\#}}} e^{-\mu|k-l|/3} \mathbb{E}\left[\|\mathcal{X}_a R_z(A_{G'})\mathcal{X}_k\|\right]^{1/3} \mathbb{E}\left[\|\mathcal{X}_l R_z(A_G)\mathcal{X}_b\|\right]^{1/3},
\end{aligned} \tag{5.67}$$

where for the last inequality we used $(\mathcal{L}_2)$ and estimated $\|\mathcal{X}_a R_z(A_{\omega,G})\mathcal{X}_k\|$ respectively $\|\mathcal{X}_k R_z(A_{\omega,G})\mathcal{X}_b\|$ by $1/|\operatorname{Im}(z)| = 1/M$. Here we again employed the notation $U_+^{\#} := \{n \in (\mathbb{Z}+1/2)^d : Q_n \cap U \neq \varnothing\}$ for $U \subset \mathbb{R}^d$. The two remaining expectations can now be estimated as in the proof of Lemma 5.9. Because the operator kernel of $A_{\omega,G^{(\prime)}}$ can be estimated by the operator kernel of A_ω, there exist constants $C_2, \mu_1 > 0$, which are independent of G and G', such that

$$\mathbb{E}\left[\|\mathcal{X}_a R_{E+iM}(A_G)\mathcal{X}_k\|\right] \leqslant C_2 e^{-\mu_1|a-k|^{1/4}}, \tag{5.68}$$

$$\mathbb{E}\left[\|\mathcal{X}_l R_{E+iM}(A_{G'})\mathcal{X}_b\|\right] \leqslant C_2 e^{-\mu_1|l-b|^{1/4}} \tag{5.69}$$

for $a, b \in \mathbb{R}^d$. Estimating (5.67) via (5.69) then implies

$$\begin{aligned} F_M &\leqslant C_3 \sum_{\substack{k \in (G' \setminus G)_+^{\#} \\ l \in G_+^{\#}}} e^{-\mu_1 |a-k|^{1/4}/3} e^{-\mu_1 |k-l|/3} e^{-\mu_1 |l-b|^{1/4}/3} \\ &\leqslant C_4 e^{-\mu_2 (\mathrm{dist}(a, G' \setminus G)^{1/4} + \mathrm{dist}(b, G' \setminus G)^{1/4})} \end{aligned} \tag{5.70}$$

for constants $C_3, C_4, \mu_2 > 0$. If $b \in G$ with $Q_b \subset G$ and $a \in G'$ the proof follows along the same lines. ☐

5.3.3. Proof of Theorem 5.5 under assumption (i). The following argument is essentially contained in [55].

Via the Helffer-Sjöstrand formula [32], see also Section 3.3.1, we first rewrite the left hand side of (5.14) in terms of the resolvents of $A_{\omega,G}$ and $A_{\omega,G'}$. In one of its standard formulations the Helffer-Sjöstrand formula states that for a self-adjoint operator K and a compactly supported function $f \in \mathcal{C}_c^n(\mathbb{R})$, $n \geqslant 2$, the operator $f(A)$ can be written as

$$f(K) = \frac{1}{2\pi} \int_{\mathbb{R}^2} dxdy\, \zeta_f(x, y) R_{x+iy}(K), \tag{5.71}$$

where $\zeta_f := (\partial_x + i\partial_y)\widetilde{f}$ and $\widetilde{f}$ is a quasi-analytic continuation of f, see e.g. [32]. Moreover, $\widetilde{f}$ can be chosen such that

$$|\zeta_f(x, y)| \leqslant C_1 |y|^{n-1}, \tag{5.72}$$

$$\mathrm{supp}(\zeta_f) \subseteq \big(\mathrm{supp}(f) + [-1, 1]\big) \times [-1, 1], \tag{5.73}$$

where the constant C_1 only depends on f and n. Let h be as in Theorem 5.5 and let $n := \lfloor 2\widetilde{q} + 2 \rfloor$. Because $h \in \mathcal{C}_c^n(\mathbb{R})$ we can choose a quasi-analytic continuation $\widetilde{h}_n$ such that $\zeta_{h,n} := (\partial_x + i\partial_y)\widetilde{h}_n$ meets (5.72) and (5.73). For open subsets $G \subset G' \subseteq \mathbb{R}^d$ and $a, b \in G'$ such that $Q_a \subset G$ the Helffer Sjöstrand formula gives

$$\begin{aligned} &\mathcal{X}_a \{h(A_{\omega,G}) - h(A_{\omega,G'})\} \mathcal{X}_b \\ &= \frac{1}{2\pi} \int_{\mathbb{R}^2} dxdy\, \zeta_{h,n}(x, y) \mathcal{X}_a \{R_{x+iy}(A_{\omega,G}) - R_{x+iy}(A_{\omega,G'})\} \mathcal{X}_b \\ &=: \frac{1}{2\pi} \int_{\mathbb{R}^2} dxdy\, \zeta_{h,n}(x, y) T^{a,b}_{\omega,x+iy}(G, G'), \end{aligned} \tag{5.74}$$

where we have abbreviated

$$T^{a,b}_{\omega,x+iy}(G, G') := \mathcal{X}_a \{R_{x+iy}(A_{\omega,G}) - R_{x+iy}(A_{\omega,G'})\} \mathcal{X}_b. \tag{5.75}$$

Upon averaging both sides of (5.74) we obtain the bound

$$\|\mathbb{E}\left[\mathcal{X}_a \{h(A_G) - h(A_{G'})\} \mathcal{X}_b\right]\| \leqslant \frac{1}{2\pi} \int_{\mathbb{R}^2} dxdy\, |\zeta_{h,n}(x, y)| \big\|\mathbb{E}\big[T^{a,b}_{x+iy}(G, G')\big]\big\|. \tag{5.76}$$

Lemma 5.10 implies that for $0 < \theta < 1/2$ there exist constants $C_2, \mu_1 > 0$ such that for $z = x + iy$ and $z \in \mathbb{C} \setminus \widehat{\Sigma}$

$$\big\|\mathbb{E}\big[T^{a,b}_z(G, G')\big]\big\| \leqslant \frac{C_2}{\mathrm{dist}(z, \widehat{\Sigma})} e^{-\mu\, \mathrm{dist}(z, \widehat{\Sigma}) \left(\mathrm{dist}(a, G' \setminus G)^{\theta} + \mathrm{dist}(b, G' \setminus G)^{\theta}\right)}. \tag{5.77}$$

Estimating the right hand side of (5.76) by (5.77) yields

$$\begin{aligned} &\|\mathbb{E}\,[\mathcal{X}_a\{h(A_G)-h(A_{G'})\}\mathcal{X}_b]\| \\ &\quad\leqslant C_3\int_{-1}^{1}dy\,|y|^{n-2}e^{-\mu|y|\left(\operatorname{dist}(a,G'\setminus G)^\theta+\operatorname{dist}(b,G'\setminus G)^\theta\right)}, \end{aligned}\tag{5.78}$$

where we also used (5.72) and (5.73). A change of variables shows that

$$(5.78)\leqslant\frac{C_4}{\operatorname{dist}(a,G'\setminus G)^{(n-1)\theta}+\operatorname{dist}(b,G'\setminus G)^{(n-1)\theta}},\tag{5.79}$$

where the constant C_4 depends on θ. Because $n-1=\lfloor 2\widetilde{q}+2\rfloor-1>2\widetilde{q}$ we can choose $\theta<1/2$ such that $(n-1)\theta>\widetilde{q}$.

□

5.4. Proof of the asymptotic expansion

The whole section deals with the proof of Theorem 5.6, which consists of two parts. In the main part, which is purely algebraic, we rewrite $\mathbb{E}\,[\operatorname{tr}h(g(H)_{2L})]$ via the transformations $\{T_j\}_{j\in\mathbb{Z}^d}$, $\{P_\pi\}_{\pi\in\mathcal{S}^d}$ and $\{R_\sigma\}_{\sigma\in\mathcal{R}^d}$ as

$$\mathbb{E}\big[\operatorname{tr}\big(h(g(H)_{2L})\big)\big]=\sum_{m=0}^{d}(2L)^{d-m}A_m^{(L)}+\mathcal{E}^{(L)},\tag{5.80}$$

where the $A_m^{(L)}$ are finite-volume versions of the coefficients A_m from (5.12) and $\mathcal{E}^{(L)}$ is an error term. In this part of the proof we work with the non-averaged quantities $\operatorname{tr}h(g(H_\omega)_G)$ as long as possible. For a concrete model such as the random Anderson model, and additional (model-specific) assumptions, the pointwise formula (5.106) would be the starting point for an almost sure pointwise or stochastic asymptotic analysis beyond the results from [78]. In the second part we then apply Theorem 5.5 to show that the coefficients A_m defined in (5.12) are well-defined for $\widetilde{q}>2d$ and that there exist constants C,C' such that

$$|A_m^{(L)}-A_m|\leqslant CL^{2m-\widetilde{q}},\tag{5.81}$$

$$|\mathcal{E}^{(L)}|\leqslant C'L^{d-\widetilde{q}}.\tag{5.82}$$

A short calculation at the end of the section verifies the alternative representation (5.13) of the coefficients A_m. To keep the formulas in this section admissibly short we we from now on drop all ω-subscripts and simply write H for H_ω or f_n for $f_{\omega,n}$ (where the latter was defined in (5.7)).

5.4.1. First part of the proof. The definitions of the measure preserving transformations $\{T_j\}_{j\in\mathbb{Z}^d}$, $\{P_\pi\}_{\pi\in\mathcal{S}^d}$ and $\{R_\sigma\}_{\sigma\in\mathcal{R}^d}$ can be found in Sections 2.1 and 5.1. For the whole first part we choose a fixed length scale $L\in\mathbb{N}$. In order to shorten notation we abbreviate

$$f_n^T:=h\big(g(H^T)_{\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n}}\big)\tag{5.83}$$

for a transformation $T : \Omega \to \Omega$, where H^T is the random operator defined by $H^T_\omega = H_{T\omega}$, $\omega \in \Omega$. We first decompose the cube Λ_{2L} of side length $2L$ into 2^d subcubes

$$\Lambda_L(\sigma) := \{x \in \mathbb{R}^d : x_\sigma \in [-L, 0]^d\}, \qquad \sigma \in \mathcal{R}^d, \tag{5.84}$$

of side length L, which are disjoint up to sets of Lebesgue measure zero. Here we denoted $x_\sigma := ((-1)^{\sigma_1} x_1, ..., (-1)^{\sigma_d} x_d)$. Under the measure preserving transformation R_σ from assumption $(\mathcal{A}_3)$ the difference of the operators $h(g(H)_{2L})$ and $f_0 = (h \circ g)(H)$ transforms as

$$\begin{aligned} U_\sigma \mathcal{X}_{\Lambda_L(\sigma)}\{h(g(H)_{2L}) - f_0\} U_\sigma &= \mathcal{X}_{[-L,0]^d} U_\sigma \{h(g(H)_{2L}) - f_0\} U_\sigma \\ &= \mathcal{X}_{[-L,0]^d}\{h(g(H^{R_\sigma})_{2L}) - f_0^{R_\sigma}\}. \end{aligned} \tag{5.85}$$

Via the unitary group $\{U_j\}_{j\in\mathbb{Z}^d}$ of translations defined in Section 2.1 we can further rewrite the right hand side of (5.85) as

$$\begin{aligned} &\mathcal{X}_{[-L,0]^d}\{h(g(H^{R_\sigma})_{2L}) - f_0^{R_\sigma}\} \\ &\quad = U_L^* \mathcal{X}_{[0,L]^d}\{h(g(H^{T_L R_\sigma})_{[0,2L]^d}) - f_0^{T_L R_\sigma}\} U_L, \end{aligned} \tag{5.86}$$

where U_L and T_L are a short-cut for the unitary operator $U_{(L,...,L)}$ and the measure preserving transformation $T_{(L,...,L)}$, respectively. After combining (5.85) and (5.86) we take the trace and sum over $\sigma \in \mathcal{R}^d$ to arrive at

$$\begin{aligned} &\operatorname{tr}\left(\mathcal{X}_{\Lambda_{2L}}\{h(g(H)_{2L}) - f_0\}\right) \\ &\quad = \sum_{\sigma\in\mathcal{R}^d} \operatorname{tr}\left(\mathcal{X}_{[0,L]^d}\{h(g(H^{T_L R_\sigma})_{[0,2L]^d}) - f_0^{T_L R_\sigma}\}\right). \end{aligned} \tag{5.87}$$

With the error term

$$\mathcal{E}^{(L)} := \sum_{\sigma\in\mathcal{R}^d} \operatorname{tr}\left(\mathcal{X}_{[0,L]^d}\{h(g(H^{T_L R_\sigma})_{[0,2L]^d}) - f_d^{T_L R_\sigma}\}\right) \tag{5.88}$$

the formula (5.87) reads

$$(5.87) = \sum_{\sigma\in\mathcal{R}^d} \operatorname{tr}\left(\mathcal{X}_{[0,L]^d}\{f_d^{T_L R_\sigma} - f_0^{T_L R_\sigma}\}\right) + \mathcal{E}^{(L)}. \tag{5.89}$$

So far we reduced the problem to a corner of the cube of linear size L and absorbed the effect of those boundary parts of Λ_{2L} into an error term that are far apart from the corner under consideration. Let's continue by decomposing the box $[0, L]^d$ as

$$[0, L]^d = \bigcup_{\pi\in\mathcal{S}^d} \{x \in [0, L]^d : x_{\pi(1)} \leqslant ... \leqslant x_{\pi(d)}\}, \tag{5.90}$$

where the union is disjoint up to a set of Lebesgue-measure zero. The single sets on the right-hand side of (5.90) can be transformed into each other via relabeling coordinates: If we set

$$\mathcal{X}_{L,\pi} := \mathcal{X}_{[0,L]^d} \mathcal{X}_{\{x_{\pi(1)} \leqslant ... \leqslant x_{\pi(d)}\}} \tag{5.91}$$

for $\pi \in \mathcal{S}^d$, then

$$\mathcal{X}_{L,\pi} = U_\pi \mathcal{X}_{L,\mathrm{id}} U_\pi^* \tag{5.92}$$

(where 'id' here stands for the neutral element in $\mathcal{S}^d$). We extend the shortcut (5.83) as follows. For a transformation $T:\Omega\to\Omega$ we set

$$f_{n,\pi}^T := h\big(U_\pi \mathcal{X}_{(\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n})} U_\pi^* g(H^T) U_\pi \mathcal{X}_{(\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n})} U_\pi^*\big), \tag{5.93}$$

i.e. $f_{n,\mathrm{id}}^T = f_n^T$ as operators on $L^2(\mathbb{R}^d)$. Moreover, $f_{d,\pi}^T = f_d^T$ and $f_{0,\pi}^T = f_0^T$ hold for any $\pi\in\mathcal{S}^d$. Via a telescopic expansion we arrive at

$$\begin{aligned}&\mathrm{tr}\left(\mathcal{X}_{[0,L]^d}\{f_d^{T_L R_\sigma} - f_0^{T_L R_\sigma}\}\right)\\ &\qquad = \sum_{\pi\in\mathcal{S}^d}\sum_{n=1}^{d}\mathrm{tr}\left(\mathcal{X}_{L,\pi}\{f_{n,\pi}^{T_L R_\sigma} - f_{n-1,\pi}^{T_L R_\sigma}\}\right).\end{aligned} \tag{5.94}$$

For $n,l = 1,...,d$ we define

$$\vec{K}_{n,l}^d := \big\{\vec{k} = (k_1,...,k_{n-1}) :\ k_i\in\{1,...,d\}\setminus\{l\},\ k_i\neq k_j (i\neq j)\big\}, \tag{5.95}$$

and for $\vec{k} = (k_1,...,k_{n-1})\in\vec{K}_{n,l}^d$

$$\mathcal{S}_n^d(\vec{k},l) := \{\pi\in\mathcal{S}^d :\ (\pi(1),...,\pi(n)) = (k_1,...,k_{n-1},l)\}\subseteq\mathcal{S}^d. \tag{5.96}$$

For fixed $n=1,...,d$ the sets $\mathcal{S}_n^d(\vec{k},l)$, $l\in\{1,,,.d\}$ and $\vec{k}\in\vec{K}_{n,l}^d$, form a disjoint partition of $\mathcal{S}^d$. Hence (5.94) can be written as

$$(5.94) = \sum_{n=1}^{d}\sum_{l=1}^{d}\sum_{\vec{k}\in\vec{K}_{n,l}^d}\ \sum_{\pi\in\mathcal{S}_n^d(\vec{k},l)}\mathrm{tr}\left(\mathcal{X}_{L,\pi}\{f_{n,\pi}^{T_L R_\sigma} - f_{n-1,\pi}^{T_L R_\sigma}\}\right). \tag{5.97}$$

For fixed $\vec{k},l$ we choose an arbitrary but fixed $\pi_0 = \pi_0(\vec{k},l)\in\mathcal{S}^d$ such that $\pi_0^{-1}\in\mathcal{S}_n^d(\vec{k},l)$ and calculate for $\pi\in\mathcal{S}_n^d(\vec{k},l)$

$$\begin{aligned}f_{n,\pi}^{T_L R_\sigma} &= h\big(U_\pi\mathcal{X}_{(\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n})}U_\pi^* g(U_L U_{\pi_0}^* H^{P_{\pi_0}R_\sigma}U_{\pi_0}U_L^*)U_\pi\mathcal{X}_{(\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n})}U_\pi^*\big)\\ &= U_{\pi_0}^* h\big(\mathcal{X}_{(\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n})} g(U_L H^{P_{\pi_0}R_\sigma}U_L^*)\mathcal{X}_{(\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n})}\big)U_{\pi_0}\\ &= U_{\pi_0}^* f_{n,\mathrm{id}}^{T_L P_{\pi_0}R_\sigma}U_{\pi_0}.\end{aligned} \tag{5.98}$$

Here we used that U_L commutes with U_π, $\pi\in\mathcal{S}^d$, and

$$U_{\pi_0\circ\pi}\mathcal{X}_{(\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n})}U_{\pi_0\circ\pi}^* = \mathcal{X}_{(\mathbb{R}^n_{\geqslant 0}\times\mathbb{R}^{d-n})}. \tag{5.99}$$

Since $(\pi_0\circ\pi)(n) = n$ a similar calculation can be performed for $f_{n-1,\pi}^{T_L R_\sigma}$. Combining them yields

$$\mathrm{tr}\left(\mathcal{X}_{L,\pi}\{f_{n,\pi}^{T_L R_\sigma} - f_{n-1,\pi}^{T_L R_\sigma}\}\right) = \mathrm{tr}\left(U_{\pi_0}\mathcal{X}_{L,\pi}U_{\pi_0}^*\{f_n^{T_L P_{\pi_0}R_\sigma} - f_{n-1}^{T_L P_{\pi_0}R_\sigma}\}\right). \tag{5.100}$$

Via the inclusion-exclusion principle we rewrite the sum over $\pi \in \mathcal{S}_n^d(\vec{k}, l)$ of the operators $U_{\pi_0}\mathcal{X}_{L,\pi}U_{\pi_0}^*$ as

$$\begin{aligned}\sum_{\pi\in\mathcal{S}_n^d(\vec{k},l)} U_{\pi_0}\mathcal{X}_{L,\pi}U_{\pi_0}^* &= \mathcal{X}_{[0,L]^d}\mathcal{X}_{\{x_1\leqslant\ldots\leqslant x_n\}}\prod_{i=n+1}^{d}\mathcal{X}_{\{x_n\leqslant x_i\}} \\ &= \mathcal{X}_{[0,L]^d}\mathcal{X}_{\{x_1\leqslant\ldots\leqslant x_n\}}\prod_{i=n+1}^{d}\left(\mathcal{X}_{\mathbb{R}^d} - \mathcal{X}_{\{x_n\geqslant x_i\}}\right) \\ &= \mathcal{X}_{[0,L]^d}\mathcal{X}_{\{x_1\leqslant\ldots\leqslant x_n\}}\sum_{j=0}^{d-n}(-1)^j \sum_{\substack{\mathcal{M}\subseteq\{n+1,\ldots,d\}:\\ |\mathcal{M}|=j}} \mathcal{X}_{\{\forall t\in\mathcal{M}:\, x_n\geqslant x_t\}}.\end{aligned} \tag{5.101}$$

For the $j = 0$ summand the second sum is interpreted as $\mathcal{X}_{\mathbb{R}^d}$. By summing (5.100) over $\pi \in \mathcal{S}_n^d(\vec{k}, l)$ we obtain

$$\begin{aligned}&\sum_{\pi\in\mathcal{S}_n^d(\vec{k},l)} \operatorname{tr}\left(\mathcal{X}_{L,\pi}\{f_{n,\pi}^{T_L R_\sigma} - f_{n-1,\pi}^{T_L R_\sigma}\}\right) \\ &\quad= \sum_{j=0}^{d-n}(-1)^j \sum_{\substack{\mathcal{M}\subseteq\{n+1,\ldots,d\}:\\ |\mathcal{M}|=j}} \operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}}\{f_n^{T_L P_{\pi_0} R_\sigma} - f_{n-1}^{T_L P_{\pi_0} R_\sigma}\}\right),\end{aligned} \tag{5.102}$$

where we abbreviated

$$\mathcal{X}_{L,n,\mathcal{M}} := \mathcal{X}_{[0,L]^d}\mathcal{X}_{\{x_1\leqslant\ldots\leqslant x_n\}}\mathcal{X}_{\{\forall t\in\mathcal{M}:\, x_n\geqslant x_t\}}. \tag{5.103}$$

Let's summarize the above calculation. For $n, m = 1, ..., d$ and $j = 0, ..., d - n$ we define

$$b_{n,j}^{(L)} := (-1)^j \sum_{\sigma\in\mathcal{R}^d}\sum_{l=1}^{d}\sum_{\vec{k}\in\vec{K}_{n,l}^d}\sum_{\substack{\mathcal{M}\subseteq\{n+1,\ldots,d\}:\\ |\mathcal{M}|=j}} \operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}}\{f_n^{T_L P_{\pi_0} R_\sigma} - f_{n-1}^{T_L P_{\pi_0} R_\sigma}\}\right), \tag{5.104}$$

$$b_m^{(L)} := \sum_{n=1}^{m} b_{n,m-n}^{(L)}. \tag{5.105}$$

Our above calculation then shows that

$$\operatorname{tr}\left(\mathcal{X}_{2L}\{h(g(H)_{\Lambda_L}) - f_0\}\right) = \sum_{m=1}^{d} b_m^{(L)} + \mathcal{E}^{(L)}. \tag{5.106}$$

Now we take expectations and exploit that P_{π_0} and R_σ are measure preserving transformations. For $m = 1, ..., d$, $n = 1, ..., m$ and $\mathcal{M} \subset \{n + 1, ..., d\}$ a set of size $|\mathcal{M}| = m - n$ as appearing in the coefficients $b_{n,m-n}^{(L)}$ this yields

$$\begin{aligned}\mathbb{E}\left[\operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}}\{f_n^{T_L P_{\pi_0} R_\sigma} - f_{n-1}^{T_L P_{\pi_0} R_\sigma}\}\right)\right] &= \mathbb{E}\left[\operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}}\{f_n - f_{n-1}\}\right)\right] \\ &= \mathbb{E}\left[\operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}_{m,n}}\{f_n - f_{n-1}\}\right)\right],\end{aligned} \tag{5.107}$$

where in the second step we substituted the set $\mathcal{M}$ by $\mathcal{M}_{m,n} := \{n+1, ..., m\}$ (with $\mathcal{M}_{m,m} = \varnothing$). This is possible because the measure preserving transformation $T_{\mathcal{M}}$ associated to the unitary operator $U_{\mathcal{M}}$, which acts via relabeling the coordinates indexed by $\mathcal{M}$ into those indexed by $\mathcal{M}_{m,n}$, satisfies

$$U_{\mathcal{M}}\{f_n - f_{n-1}\}U_{\mathcal{M}}^* = f_n^{T_{\mathcal{M}}} - f_{n-1}^{T_{\mathcal{M}}}. \tag{5.108}$$

The right-hand side of (5.107) now is independent of $\sigma \in \mathcal{R}^d$, $l = 1, ..., d$ and $\vec{k} \in \vec{K}_{n,l}^d$ and therefore

$$\mathbb{E}\big[b_{n,m-n}^{(L)}\big] = \frac{(-1)^{m-n}2^d d!}{(m-n)!(d-m)!}\mathbb{E}\left[\operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}_{m,n}}\{f_n - f_{n-1}\}\right)\right]. \tag{5.109}$$

Hence, if we set

$$c_{m,n} := \frac{(-1)^{m-n}2^m d!}{(m-n)!(d-m)!}, \tag{5.110}$$

we arrive at

$$\mathbb{E}\big[b_m^{(L)}\big] = 2^{d-m}\sum_{n=1}^{m} c_{m,n}\mathbb{E}\big[\operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}_{m,n}}\{f_n - f_{n-1}\}\right)\big]. \tag{5.111}$$

For $1 \leqslant n \leqslant m \leqslant d$ the operator on the right-hand side of (5.109) is invariant under translations in the last $d-m$ coordinates. For a cube $Q_a \subset [0, L]^{d-m}$ of side-length 1 and with center $a \in (\mathbb{Z} + 1/2)^{d-m}$ this gives

$$\begin{aligned}
&\mathbb{E}\left[\operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}_{m,n}}\mathcal{X}_{\{(x_{m+1},...,x_d)\in Q_a\}}\{f_n - f_{n-1}\}\right)\right] \\
&\quad = \mathbb{E}\left[\operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}_{m,n}}\mathcal{X}_{\{(x_{m+1},...,x_d)\in[0,1]^{d-m}\}}\{f_n - f_{n-1}\}\right)\right].
\end{aligned} \tag{5.112}$$

For $m = 1, ..., d$ we obtain

$$\begin{aligned}
\mathbb{E}\big[b_m^{(L)}\big] =&(2L)^{d-m}\sum_{n=1}^{m} c_{m,n}\mathbb{E}\left[\operatorname{tr}\left(\mathcal{X}_{L,n,\mathcal{M}_{m,n}}\mathcal{X}_{\{x_{m+1},...,x_d\in[0,1]\}}\{f_n - f_{n-1}\}\right)\right] \\
=:&(2L)^{d-m}A_m^{(L)}
\end{aligned} \tag{5.113}$$

and for $m = 0$ we set

$$A_0^{(L)} := \mathbb{E}\big[\operatorname{tr}\left(\mathcal{X}_{[0,1]^d} f_0\right)\big], \tag{5.114}$$

which already is L-independent. This finishes the first part of the proof, which can be summarized as

$$\mathbb{E}\left[\operatorname{tr}\left(h(g(H)_{2L})\right)\right] = \sum_{m=0}^{d}(2L)^{d-m}A_m^{(L)} + \mathbb{E}\big[\mathcal{E}^{(L)}\big]. \tag{5.115}$$

5.4.2. Second part of the proof. We start by proving that $\mathcal{E}^{(L)}$ defined in (5.88) is indeed a negligible error term. For a set $U \subseteq \mathbb{R}^d$ we recall the notation $U_+^{\#} := \{n \in (\mathbb{Z} + 1/2)^d : Q_n \cap A \neq \varnothing\}$. Because of $(\mathcal{A}_1)$ and Lemma 5.8 we may interchange trace and expectation in

(5.88) to obtain

$$\begin{aligned} \big|\mathbb{E}\big[\mathcal{E}^{(L)}\big]\big| &= 2^d \,\big|\operatorname{tr}\big(\mathbb{E}\big[\mathcal{X}_{[0,L]^d}\{h(g(H)_{[0,2L]^d}) - f_d\}\big]\big)\big| \\ &\leqslant 2^d \sum_{a\in([0,L]^d)^{\#}_{+}} \big\|\mathbb{E}\big[\mathcal{X}_a\{h(g(H)_{[0,2L]^d}) - f_d\}\mathcal{X}_a\big]\big\|_1. \end{aligned} \tag{5.116}$$

Next, we apply estimate (5.14) (which by assumption holds for $\widetilde{q} > 2d$) with $G = [0,2L]^d$ and $G' = \mathbb{R}^d_{\geqslant 0}$, in which case $\operatorname{dist}([0,L]^d, \mathbb{R}^d_{\geqslant 0} \setminus [0,2L]^d) = L$. This implies that

$$\big\|\mathbb{E}\big[\mathcal{X}_a\{h(g(H)_{[0,2L]^d}) - f_d\}\mathcal{X}_a\big]\big\|_1 \leqslant C_1 L^{-\widetilde{q}} \tag{5.117}$$

holds for $a \in ([0,L]^d)^{\#}_{+}$, and consequently

$$|\mathbb{E}\big[\mathcal{E}^{(L)}\big]| \leqslant C_2 L^{d-\widetilde{q}}. \tag{5.118}$$

Now let us turn to (5.81). We first introduce the abbreviation

$$\widehat{\mathcal{X}}_{L,m,n} := \mathcal{X}_{L,n,\mathcal{M}_{m,n}} \mathcal{X}_{\{x_{m+1},\dots,x_d\in[0,1]\}} \tag{5.119}$$

for $L \in \mathbb{N} \cup \{\infty\}$, and $1 \leqslant n \leqslant m \leqslant d$. We note that $\widehat{\mathcal{X}}_{\infty,m,n} = \widehat{\mathcal{X}}_{m,n}$, where the latter operators were defined in (5.9). The natural limiting candidates for the coefficients $A^{(L)}_m$ defined in (5.113) are

$$A_m := \sum_{n=1}^{m} c_{m,n} \operatorname{tr}\Big(\mathbb{E}\Big[\widehat{\mathcal{X}}_{\infty,m,n}\{f_n - f_{n-1}\}\widehat{\mathcal{X}}_{\infty,m,n}\Big]\Big). \tag{5.120}$$

Here we exchanged the order of trace and expectation to ensure that the coefficients A_m are well-defined via the bound (5.14) and the calculation below. To prove convergence of $A^{(L)}_m$ towards A_m we prove that the single summands which contribute to $A^{(L)}_m$ converge towards their respective infinite-volume counterparts. For brevity we abbreviate for $1 \leqslant n \leqslant m \leqslant d$

$$A^{(L)}_{m,n} := \operatorname{tr}\Big(\mathbb{E}\Big[\widehat{\mathcal{X}}_{L,m,n}\{f_n - f_{n-1}\}\widehat{\mathcal{X}}_{L,m,n}\Big]\Big), \tag{5.121}$$

$$A_{m,n} := \operatorname{tr}\Big(\mathbb{E}\Big[\widehat{\mathcal{X}}_{\infty,m,n}\{f_n - f_{n-1}\}\widehat{\mathcal{X}}_{\infty,m,n}\Big]\Big). \tag{5.122}$$

We first prove that that the operator $\mathbb{E}\big[\widehat{\mathcal{X}}_{\infty,m,n}\{f_n - f_{n-1}\}\widehat{\mathcal{X}}_{\infty,m,n}\big]$ is trace class. The trace norm of this operator can be estimated via the operator kernel of $f_n - f_{n-1}$ as

$$\begin{aligned} &\Big\|\mathbb{E}\Big[\widehat{\mathcal{X}}_{\infty,m,n}\{f_n - f_{n-1}\}\widehat{\mathcal{X}}_{\infty,m,n}\Big]\Big\|_1 \\ &\quad\leqslant \sum_{\substack{a\in(\mathbb{R}^d_{\geqslant 0})^{\#}_{+}:\\ \widehat{\mathcal{X}}_{\infty,m,n}\mathcal{X}_a\neq 0}} \; \sum_{\substack{b\in(\mathbb{R}^d_{\geqslant 0})^{\#}_{+}:\\ \widehat{\mathcal{X}}_{\infty,m,n}\mathcal{X}_b\neq 0}} \|\mathbb{E}\left[\mathcal{X}_a\{f_n - f_{n-1}\}\mathcal{X}_b\right]\|_1 \\ &\quad\leqslant C_2\Big(\sum_{\substack{a\in(\mathbb{R}^d_{\geqslant 0})^{\#}_{+}:\\ \widehat{\mathcal{X}}_{\infty,m,n}\mathcal{X}_a\neq 0}} \frac{1}{(|a_n|+1)^{\widetilde{q}/2}}\Big)^2, \end{aligned} \tag{5.123}$$

where we used the assumption that (5.14) holds for $\widetilde{q}$ and that

$$\mathbb{R}^{n-1}_{\geqslant 0} \times \mathbb{R}^{d-(n-1)} \cap \operatorname{int}\big(\mathbb{R}^n_{\geqslant 0} \times \mathbb{R}^{d-n}\big) = \mathbb{R}^{n-1}_{>0} \times \{0\} \times \mathbb{R}^{d-n}, \tag{5.124}$$

where $\operatorname{int}(A)$ as usual denotes the topological interior of a set $A \subset \mathbb{R}^d$. By definition of the operator $\widehat{\mathcal{X}}_{\infty,m,n}$ the right-hand side of (5.123) can be estimated by

$$\sqrt{(5.123)} \leqslant C_2^{1/2} \sum_{a_n\in\mathbb{N}} \sum_{\substack{a_1,\dots,a_{n-1}\in\mathbb{N}_0:\\ a_i\leqslant a_n+1}} \sum_{\substack{a_{n+1},\dots,a_m\in\mathbb{N}_0:\\ a_i\leqslant a_n+1}} a_n^{-\widetilde{q}/2} \leqslant C_3 \sum_{a_n\in\mathbb{N}} a_n^{m-1-\widetilde{q}/2}, \tag{5.125}$$

which is finite for $\widetilde{q} > 2m$. Finally we prove that $|A_m^{(L)} - A_m| = \mathcal{O}(L^{2m-\widetilde{q}})$. We proved above that the operator $T_{m,n} := \mathbb{E}\left[\widehat{\mathcal{X}}_{\infty,m,n}\{f_n - f_{n-1}\}\widehat{\mathcal{X}}_{\infty,m,n}\right]$ is trace class. Cyclicity of the trace then yields

$$\begin{aligned}|A_{m,n}^{(L)} - A_{m,n}| &= \left|\operatorname{tr}\left(\mathcal{X}_{[0,L]^d}T_{m,n}\right) - \operatorname{tr} T_{m,n}\right| \\ &= \left|\operatorname{tr}\left((\mathcal{X}_{\mathbb{R}^d_{\geqslant 0}} - \mathcal{X}_{[0,L]^d})T_{m,n}(\mathcal{X}_{\mathbb{R}^d_{\geqslant 0}} - \mathcal{X}_{[0,L]^d})\right)\right| \\ &\leqslant \left\|\mathbb{E}\left[\widehat{\mathcal{X}}_{\infty,m,n}(\mathcal{X}_{\mathbb{R}^d_{\geqslant 0}} - \mathcal{X}_{[0,L]^d})\{f_n - f_{n-1}\}(\mathcal{X}_{\mathbb{R}^d_{\geqslant 0}} - \mathcal{X}_{[0,L]^d})\widehat{\mathcal{X}}_{\infty,m,n}\right]\right\|_1.\end{aligned} \tag{5.126}$$

With estimates as for (5.123) and (5.125) we arrive at

$$(5.126) \leqslant C_4 \left(\sum_{a_n=L-1}^{\infty} a_n^{m-1-\widetilde{q}/2}\right)^2 \leqslant C_5 L^{2m-\widetilde{q}}. \tag{5.127}$$

□

5.4.3. Proof of Remark 5.3. Because of the analysis in the second part of the proof

$$A_m = \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} A_m^{(L)} = \lim_{\substack{L\to\infty\\ L\in\mathbb{N}}} \sum_{n=1}^{m} c_{m,n}\mathbb{E}\left[\operatorname{tr}\left(\widehat{\mathcal{X}}_{L,m,n}\{f_n - f_{n-1}\}\right)\right] \tag{5.128}$$

with constants $c_{m,n}$ defined in (5.110). The presence of the finite-volume projection now allows to rearrange terms in the above sum. This leads to

$$\begin{aligned}A_m^{(L)} =& c_{m,m}\mathbb{E}\left[\operatorname{tr}\left(\widehat{\mathcal{X}}_{L,m,m}f_m\right)\right] - c_{m,1}\mathbb{E}\left[\operatorname{tr}\left(\widehat{\mathcal{X}}_{L,m,1}f_0\right)\right] \\ &+ \sum_{n=1}^{m-1}\left(c_{m,n}\mathbb{E}\left[\operatorname{tr}\left(\widehat{\mathcal{X}}_{L,m,n}f_n\right)\right] - c_{m,n+1}\mathbb{E}\left[\operatorname{tr}\left(\widehat{\mathcal{X}}_{L,m,n+1}f_n\right)\right]\right).\end{aligned} \tag{5.129}$$

We next use that f_n is invariant under permutation of the first n and last $d-n$ coordinates to find that for $1 \leqslant n \leqslant m-1$

$$\begin{aligned}&\mathbb{E}\left[\operatorname{tr}\left(\widehat{\mathcal{X}}_{L,m,n}f_n\right)\right] \\ &\quad= \frac{1}{n!}\sum_{k=1}^{n} \mathbb{E}\left[\operatorname{tr}\left(f_n\mathcal{X}_L\mathcal{X}_{\{x_{m+1},\dots,x_d\in[0,1]\}}\mathcal{X}_{\{x_k\geqslant x_1,\dots,x_m\}}\right)\right],\end{aligned} \tag{5.130}$$

$$\begin{aligned}&\mathbb{E}\left[\operatorname{tr}\left(\mathcal{X}_L\mathcal{X}_{L,m,n+1}f_n\right)\right] \\ &\quad= \frac{1}{n!(m-n)}\sum_{k=n+1}^{m} \mathbb{E}\left[\operatorname{tr}\left(f_n\mathcal{X}_L\mathcal{X}_{\{x_{m+1},\dots,x_d\in[0,1]\}}\mathcal{X}_{\{x_k\geqslant x_1,\dots,x_m\}}\right)\right].\end{aligned} \tag{5.131}$$

For the constants $c_{m,n}$ and $c_{m,n+1}$ the relation

$$\frac{1}{n!}c_{m,n} = \frac{-1}{n!(m-n)}c_{m,n+1} = \frac{d!}{(-1)^{m-n}2^m n!(m-n)!(d-m)!} =: \widetilde{c}_{m,n} \tag{5.132}$$

holds, which yields

$$\begin{aligned} &c_{m,n}\mathbb{E}\left[\operatorname{tr}\left(\widehat{\mathcal{X}}_{L,m,n}f_n\right)\right] - c_{m,n+1}\mathbb{E}\left[\operatorname{tr}\left(\widehat{\mathcal{X}}_{L,m,n+1}f_n\right)\right] \\ &\quad = \widetilde{c}_{m,n}\mathbb{E}\left[\operatorname{tr}\left(f_n\mathcal{X}_L\mathcal{X}_{\{x_{m+1},\dots,x_d\in[0,1]\}}\right)\right]. \end{aligned} \tag{5.133}$$

After performing similar calculations for the $n=0$ and the $n=d$ term appearing on the right-hand side of (5.129), we arrive at

$$A_m^{(L)} = \lim_{L\to\infty}\sum_{n=0}^{m}\widetilde{c}_{m,n}\mathbb{E}\left[\operatorname{tr}\left(f_n\mathcal{X}_L\mathcal{X}_{\{x_{m+1},\dots,x_d\in[0,1]\}}\right)\right] \tag{5.134}$$

□

CHAPTER 6

Level spacing and Poisson statistics

Context: The main results presented in this chapter coincide with the main results from [36], which was written in collaboration with **Alexander Elgart**. Most of the proofs presented in Sections 6.3 and 6.5-6.7 coincide to a large extent with the respective proofs from [36], but have been streamlined occasionally. Moreover, the results contained in Section 6.4 are novel and have not been published previously.

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Content: We prove a probabilistic level-spacing estimate and Poissonian local eigenvalue statistics for the continuum random Schrödinger operator from Section 2.1 in the localized spectral region. Even though the latter has been our motivation we consider the level-spacing estimate as this chapter's main technical result. Instead of presenting two slightly different versions of the level-spacing estimate as in [36] we focus on one version. To showcase the proof's main ideas we exemplify it for the technically more convenient lattice dimer (or polymer) model in Section 6.4. At the cost of some redundancy this section can be read independently of the rest of the chapter. Two additional results are contained in this chapter: First, we do not immediately deduce Poisson statistics from the level-spacing estimate. As an intermediate step we first prove a Minami-type estimate, Theorem 6.2 below. Another consequence of a level-spacing estimate is simplicity of the pure point spectrum within the localized spectrum. The respective steps from a Minami-type estimate to Poisson statistics and from a level-spacing estimate to simplicity of point spectrum in the localized spectrum have been established in [81, 93, 94, 27]. For convenience proofs are contained in Section 6.7.

6.1. Discussion of results

We work with the random Schrödinger operator

$$H_\omega = -\mu\Delta + V_\omega \tag{6.1}$$

defined in Chapter 2.1, subject to assumptions (V_1)-(V_3) and without periodic potential ($V_{\text{per}} = 0$) for convenience. Even though the method still works for generic periodic potentials it would occasionally mess up the theorem's statements (however, a fake periodic potential is introduced for some of the intermediate steps in the proofs below for technical reasons).The coupling μ is not needed directly for our proofs but needed to discuss the energy range that our method covers. Besides that, we also assume the following.

(V_4) The single-site probability density ρ is Lipschitz continuous and bounded below,

$$\mathcal{K} := \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|\rho(x)-\rho(y)|}{|x-y|} < \infty \quad \text{and} \quad \rho_- := \operatorname*{ess\,inf}_{x\in[0,1]} \rho(x) > 0. \tag{6.2}$$

The assumption (V_4) can be avoided for the level-spacing estimate via the slightly altered second approach presented in [36]. However, our transition from the level-spacing estimate to a Minami-type estimate crucially relies on this assumption anyway.

Let's first recall and introduce some notation. For the eigenvalues $(\lambda^L_{\omega,i})_{i\in\mathbb{N}}$ of $H_{\omega,L}$ (which are, as usual, ascendingly ordered and repeated according to their multiplicity) the minimal eigenvalue spacing on in interval $I \subset \mathbb{R}$ is given by

$$\operatorname{spac}_I(H_{\omega,L}) = \inf\left\{|\lambda^L_{\omega,i} - \lambda^L_{\omega,j}| : i \neq j,\, \lambda^L_{\omega,i}, \lambda^L_{\omega,j} \in I\right\}, \tag{6.3}$$

with $\operatorname{spac}_E(H_{\omega,L}) = \operatorname{spac}_{(-\infty,E]}(H_{\omega,L})$ for $E \in \mathbb{R}$. Moreover, we define the two threshold constants

$$E_{\text{sp}} := \frac{\mu\pi^2 V_-}{(2R)^2(2R+1)^d v_+} \quad \text{and} \quad E_{\text{M}} := \frac{E_{\text{sp}}}{V_+}. \tag{6.4}$$

They are the energies up to which we can establish a level-spacing estimate and a Minami-type estimate, respectively. We note that the constants E_{sp} and E_{M} are linearly growing in the coupling constant μ. Putting aside that a restriction of the level-spacing or the Minami-type estimate to the bottom of the spectrum is probably not natural in general, this is not an ungeneric behavior in the coupling: A stronger kinetic energy leads to stronger repulsion which breaks down the local symmetries of the random potential more effectively. See also the discussion in Section 6.2. But for applications we need the level-spacing estimate and the Minami-type estimate in the localized energy region Σ_{FMB} which at least on a heuristic level is shrinking as μ is growing. For small $\mu \ll 1$ we hence only establish the two estimates in a small region at the bottom of the localized energy region.

The main technical results of this chapter then read as follows.

Theorem 6.1 (Level-spacing estimate)**.** *For fixed $E < E_{\text{sp}}$ and $K > 0$ there exist constants $\mathcal{L}, C$ such that*

$$\mathbb{P}\left(\operatorname{spac}_E(H_L) < \delta\right) \leqslant CL^{2d}|\log\delta|^{-K} \tag{6.5}$$

holds for $L \geqslant \mathcal{L}$ and $0 < \delta < 1$.

Theorem 6.2 (Minami-type estimate)**.** *For fixed $E' < E_{\text{M}}$ and $K > 0$ there exist constants $\mathcal{L}, C$ such that*

$$\mathbb{P}\left(\operatorname{tr} \mathbb{1}_J(H_L) \geqslant 2\right) \leqslant CL^{4d}\delta|\log\delta|^{-K} \tag{6.6}$$

holds for all intervals $J \subset [0, E']$ with $|J| \leqslant \delta$ and $L \geqslant \mathcal{L}$.

Remarks 6.3. (i) The dependence of the threshold energies $E_{\text{M}}, E_{\text{sp}}$ on V_ω (through the constants V_-, v_+ and R) is certainly sub-optimal. But, regardless of the choice of random potential, our method is limited to $E_{\text{sp}} \leqslant \mu\lambda_2^{(N)}/2$, where $\lambda_2^{(N)}$ is the second eigenvalue of the Neumann Laplacian on $\operatorname{supp}(V_0)$ (provided that the boundary of this set is sufficiently regular). The limiting factor here is Lemma 6.21 below.

(ii) A slight adaption of the proofs likely yield the following: For every $E_1 > 0$ and $K > 0$ there exist constants $C, \mathcal{L}, R > 0$ such that

$$\mathbb{P}\left(\operatorname{tr} \mathbb{1}_J(H_L) \geqslant R\right) \leqslant C L^{4d} \delta |\log \delta|^{-K} \tag{6.7}$$

for intervals $J \subset [0, E_1]$ and $L \geqslant \mathcal{L}$.

(iii) From a mathematical point of view it is an interesting technical problem to prove a version of Minami's original estimate (i.e. with an optimal factor of δ^2 instead of $\delta|\log \delta|^{-K}$ on the right hand side of (6.6)) for the continuum Anderson model. But, at least for the applications we have in mind, the above estimates are sufficient.

Degenerate eigenvalues of Schrödinger operators are typically caused by symmetry. It is common sense that randomness tends to break symmetries. The first result on spectral simplicity of random Schrödinger operators goes back to Simon [113], who considered the classical lattice Anderson model. Those ideas were subsequently generalized in [70, 95], but for instance continuum random Schrödinger operators can presently not be handled through this approach. We instead work with an approach due to Klein and Molchanov [81] which however relies on spectral localization. However, as the above cited works suggest, simplicity of the pure point (or singular) spectrum should in general not be restricted to the localized spectral region.

Corollary 6.4 (Eigenvalue simplicity). *The spectrum in* $[0, E_{\mathrm{sp}}] \cap \Sigma_{\mathrm{FMB}} \cap \sigma(H_\omega)$ *almost surely only consists of simple eigenvalues.*

Poissonian local eigenvalue statistics follow from (6.2) via the method developed in [93, 94, 27]. We recall from Section 2.1 that the local point process of the rescaled eigenvalues of $H_{\omega,L}$ around a fixed reference energy $E \in \mathbb{R}$ is given by

$$\xi^L_{\omega,E}(B) := \operatorname{tr} \mathbb{1}_{E+L^{-d}B}(H_{\omega,L}) \tag{6.8}$$

for bounded, Borel-measurable sets $B \subset \mathbb{R}$.

Corollary 6.5 (Poisson statistics). *Let* $E < E_{\mathrm{M}}$ *with* $E \in \Sigma_{\mathrm{FMB}}$*, and such that the integrated density of states* $\mathcal{N}$ *is differentiable at* E *with derivative* $\mathcal{N}'(E) > 0$*. Then, as* $L \to \infty$*, the point process* $\xi^L_{E,\omega}$ *converges weakly to the Poisson point process on* $\mathbb{R}$ *with intensity measure* $\mathcal{N}'(E)dx$*.*

Remark 6.6. Due to $V_{\mathrm{per}} = 0$ and (V_4) we have $\Sigma = \Sigma_0 = [0, \infty)$. Hence Corollary 4.2 shows that $\mathcal{N}'(E) > 0$ for almost every $E \in \mathbb{R}$.

6.2. Proof's idea & more

As mentioned in the introduction, a Minami estimate is the central estimate needed to establish Poissonian local eigenvalue statistics. Moreover, the step from a Minami estimate to the eigenvalue statistics can be performed by a well-known procedure which moreover is rather model independent. This leaves us with proving a Minami-type estimate for continuum random Schrödinger operators.

For $d \geqslant 2$ dimensions the known strategies to obtain a Minami estimate heavily rely on the fact that the random potential itself, i.e. the operator V_ω, already satisfies this bound. In fact, the standard proofs all crucially rely on the rank-1 structure of the single-site potentials of the lattice Anderson model H_ω^A, and extensions are limited to slight modifications of the model. For example, the method already breaks down for the dimer potential, where the single-site potentials are translates of $u = \mathcal{X}_{\{0,1\}}$, a rank-2 operator. This illustrates that the effect of the kinetic energy term H_0 has to be taken into account in order to prove a Minami-type estimate for more general random Schrödinger operators.

Typically, degenerate eigenvalues are a manifestation of symmetry within the system. Due to translation invariance a 'typical' kinetic energy term on a generic domain, say the Laplace operator on a box, only possesses – if any – global symmetries. In contrast, independence at distance of the random potential ensures that the symmetries of the random potential – if any – are local. The guiding idea of what we describe below is to harness the random potential to destroy global symmetries of the kinetic energy and, in turn, to use the repulsion of the kinetic energy to destroy local symmetries. A qualitative implementation of this observation was employed in the works [113, 95] and [70] to prove simplicity of point spectrum and singular spectrum, respectively.

Let's now slightly switch perspective. At first glance, a level-spacing estimate seems to be a weaker result than a Minami-type estimate. This can already be guessed from the observation that a level-spacing estimate can be obtained from a Minami estimate by simply summing over energy intervals. Below we argue that if the probability space is sufficiently regular then the converse is also true: A Minami-type estimate can be recovered from a level-spacing estimate. This is the key point where the additional regularity assumption (V_4) enters. For now we focus on the level-spacing estimate.

In a nutshell, our procedure can be described as follows. Let $L > 0$ be fixed. By making the above heuristics concerning local vs. global symmetries quantitative we obtain a subset of $\Omega_L = [0,1]^{\Gamma_L}$ of configurations for which the eigenvalue spacing is relatively large. Subsequently, we apply an analytic argument to conclude that the eigenvalue spacing is not too small for most configurations $\omega \in \Omega_L$.

Let's discuss this in some more detail. For the sake of illustration, we assume that the continuum random Schrödinger operator $H_{\omega,L}$ on the box Λ_L is bounded, with roughly L^d eigenvalues (in case you feel uncomfortable with this, think of the dimer model from Section 6.4 below). For the moment we also assume that we had a configuration $\omega_0 \in \Omega_L$ such that the spectrum is perfectly spaced, $\lambda^L_{\omega_0,i+1} - \lambda^L_{\omega_0,i} \sim L^{-d}$ for all i. The analytic part of the argument is based on the following observation: Let $f : [0,1] \to \mathbb{C}$ with $a \in [0,1]$ such that $|f(a)| = \varepsilon > 0$. If f can be extended to a holomorphic function in a sufficiently large complex neighborhood of $[0,1]$, and if the absolute value of this extension is uniformly bounded by one, then for all $0 < \delta < 1$

$$|\{x \in [0,1] : |f(x)| < \delta\}| \lesssim \exp\left(-c\left|\frac{\log \delta}{\log \varepsilon}\right|\right). \tag{6.9}$$

Such an estimate, going by the name Cartan's estimate, has for instance been applied by Bourgain in the related context of a Wegner estimate [21]. It can be interpreted as a quantitative version of the identity theorem from complex analysis. The discriminant of $H_{\omega,L}$ is an analytic function in each of the couplings ω_k, $k \in \Gamma_L$, and yields an upper and lower bound on the minimal level spacing via

$$\Big(\prod_{i<j} \big(\lambda^L_{\omega,i} - \lambda^L_{\omega,j}\big)^2\Big)^{L^{-2d}} \geqslant \min\{|\lambda^L_{\omega,i} - \lambda^L_{\omega,j}| : i \neq j\} \geqslant \prod_{i<j} \big(\lambda^L_{\omega,i} - \lambda^L_{\omega,j}\big)^2 . \tag{6.10}$$

Strictly speaking, the inequalities only hold if $|\lambda^L_{\omega,i} - \lambda^L_{\omega,j}| \leqslant 1$ for all i, j. An extension of the Cartan estimate (6.9) to multiple variables and an adaption to the present situation together with (6.10) yields

$$\mathbb{P}\left(\mathrm{spac}(H_L) < \delta\right) \lesssim \exp\left(-\frac{c}{L^{2d}} \frac{|\log \delta|}{|\log \mathrm{spac}(H_{\omega_0,L})|}\right) \sim \exp\left(-\frac{c'|\log \delta|}{L^{2d}\log L}\right) \tag{6.11}$$

where the absolute continuity of the random single-site couplings crucially enters. The details of this step are contained in Lemma 6.10 below. This estimate already would yield almost sure simplicity of the eigenvalues of $H_{\omega,L}$. Unfortunately (6.11) is next to useless if considered in the macroscopic limit. In order to obtain valuable information on the eigenvalue behavior in the macroscopic limit we have to choose $\delta_L \ll e^{-L}$. But in this case the right hand side of (6.11) is ~ 1. The main problem here is the L^{2d} in the exponent on the right hand side of (6.11), which in turn originates from the left estimate in (6.10).

We salvage this by breaking the analysis of the minimal eigenvalue spacing in a macroscopically large energy region down to the analysis of small clusters $\mathcal{C}$ of eigenvalues that are separated from the remainder of the spectrum. Assume that for a given configuration $\omega_0 \in \Omega_L$ we had a cluster of eigenvalues $\mathcal{C} := \lambda^L_{\omega_0,m+1}, ..., \lambda^L_{\omega_0,m+n}$ that has distance $\varepsilon > 0$ from the remainder of the spectrum. Then (6.10) reads

$$\Big(\prod_{m<i<j\leqslant m+n} \big(\lambda^L_{\omega_0,i} - \lambda^L_{\omega_0,j}\big)^2\Big)^{n^{-2}} \geqslant \min_{m<i<m+n} \big(\lambda^L_{\omega_0,i+1} - \lambda^L_{\omega_0,i}\big) \geqslant \prod_{m<i<j\leqslant m+n} \big(\lambda^L_{\omega_0,i} - \lambda^L_{\omega_0,j}\big)^2 . \tag{6.12}$$

The right hand side of (6.12) can be interpreted as a local version of the discriminant for the cluster of eigenvalues. By virtue of our assumption that the cluster is separated from the remainder of the spectrum by a spectral gap of size ε the local discriminant can be extended to an analytic function in the vicinity of the configuration ω_0. This allows us to deduce a rescaled version of (6.11) for the local discriminant in a vicinity of ω_0. If we further assume that the cluster is perfectly spaced at the realization ω_0 (i.e. $\lambda^L_{\omega_0,m+j+1} - \lambda^L_{\omega_0,m+j} \sim L^{-d}$ for $j = 1, ..., n-1$) then this yields

$$\mathbb{P}\left(\mathrm{spac}(H_L) < \delta,\, \omega_0 + [-\varepsilon, \varepsilon]^{\Gamma_L}\right) \lesssim \exp\left(-\frac{c''|\log \delta|}{n^2 \log L}\right). \tag{6.13}$$

In view of the above considerations this constitutes a good bound for our purposes in case the cluster size n is $\ll \sqrt{L}$. We prove in Lemma 6.19 below that the typical cluster size is indeed not too big for continuum random Schrödinger operators. Two key assumptions were made in the above toy calculation: That typical clusters of eigenvalues are separated from the remainder of the spectrum and that for the realization ω_0 above the cluster of eigenvalues is indeed well spaced. The first point is a consequence of Wegner's estimate, hence we can focus on the second point. Note that the procedure outlined above only allows us to consider cubes of side-length ε in the configuration space Ω_L. Hence, we first have to decompose the configuration space into $\sim \varepsilon^{-|\Gamma_L|}$ such cubes Q. Then we separately for each of those cubes Q decompose the spectrum into clusters $\mathcal{C}$ of eigenvalues. To conclude (6.13) we therefore need a good configuration for each cluster $\mathcal{C}$ and each cube Q: Good configurations have to be rather dense in Ω_L.

For a cluster $\mathcal{C}$ of eigenvalues that is separated from the rest of the spectrum our starting point is a Hellman-Feynman type estimate, Lemma 6.7. The Hellman-Feynman theorem states that for self-adjoint operators A, B and the one-parameter operator family $s \to A + sB$ we have $\operatorname{tr}(P_s B) = (\partial_s \bar{E}^s) \operatorname{tr} P_s$, where P_s denotes the projection onto a cluster of eigenvalues and $\bar{E}^s$ denotes the central energy, i.e. the arithmetic mean of the eigenvalues of the cluster. In Lemma 6.7 we show that a refined statement holds under the assumption that the cluster is tightly concentrated around its central energy $\bar{E}^s$:

$$P_s B P_s \approx \frac{\operatorname{tr}(P_s B)}{\operatorname{tr} P_s} P_s. \tag{6.14}$$

We next argue why this implies that low lying eigenvalues can't remain clustered even in a small neighborhood of the configuration ω_0. Let's assume we have bad luck and the cluster is tightly concentrated around its central energy for configurations in a small neighborhood of ω_0. We then apply (6.14) for every $k \in \Gamma_L$ to the spectral family $s \to H_{\omega_0,L} + sV_k$. This shows that the tight concentration of the cluster originates from high amount of local symmetry. More precisely, for every $k \in \Gamma_L$ one of the following two scenarios applies: Either all eigenfunctions of the cluster have almost no mass on $\operatorname{supp}(V_k)$ or they form an almost orthogonal family with respect to V_k:

$$\langle \varphi^L_{\omega_0,m+i}, V_k \varphi^L_{\omega_0,m+j} \rangle \approx \frac{\operatorname{tr}(P_{\omega_0} V_k)}{\operatorname{tr} P_{\omega_0}} \delta_{i,j}, \tag{6.15}$$

where $(\varphi^L_{\omega_0,m+i})_{i=1}^n$ are the eigenfunctions associated to the eigenvalues of the cluster $\mathcal{C}$. We utilize this orthogonality relation to conclude via a bracketing argument that the central energy $\bar{E}^{\omega_0}$ of the cluster has to be $\gtrsim \lambda_2^{(N)}$, the second eigenvalue of the Laplacian $-\Delta$ restricted to $\operatorname{supp}(V_k)$ with Neumann boundary conditions. In conclusion we obtain a quite rich set of configurations for which the eigenvalues of the cluster are rather far apart from each other. This finishes the proof of the level-spacing estimate.

For the proof of the Minami-type estimate let's for the moment assume that $\sum_{k \in \mathbb{Z}^d} V_k = 1$. The main idea leading from the level-spacing estimate – which is semi-global in energy – to the Minami-type estimate – which is local in energy – is to clone the interval $J := J_0 := [E - \delta, E + \delta]$ for which we want to prove a Minami-type bound. Let $\{J_k\}_{k=1}^K$ be K disjoint intervals of length 2δ and such that $\operatorname{dist}(J_k, J_0) \lesssim K\delta \ll 1$. We now utilize that (in view of $\sum_k V_k = 1$) a shift $(\omega_k)_{k \in \Gamma_L} \to (\omega_k + \varepsilon)_{k \in \Gamma_L}$ in the configuration space results in an energy shift by ε. Together with the homogeneity of the single-site probability measures (here the additional assumption (V_4) is crucial) it implies that

$$\mathbb{P}\left(\operatorname{spac}_{J_0}(H_L) < \delta\right) \sim \mathbb{P}\left(\operatorname{spac}_{J_k}(H_L) < \delta\right). \tag{6.16}$$

Summing both sides over $1 \leqslant k \leqslant K$ then yields

$$\mathbb{P}\left(\operatorname{spac}_{J_0}(H_L) < \delta\right) \lesssim K^{-1} \mathbb{P}\left(\operatorname{spac}_{E_{\mathrm{sp}}}(H_L) < \delta\right), \tag{6.17}$$

by arguing that the events on the right hand side of (6.16) are more or less disjoint. With the choice $K = (L^d\delta)^{-1}$ we can ensure that $\operatorname{dist}(J_k, J_0) \lesssim L^{-d}$. The level-spacing estimate can now be applied to the right hand side of (6.17) to finish the argument. In order to remove the constraint $V := \sum_{k\in\mathbb{Z}^d} V_k = 1$ we consider the auxiliary operator $\widetilde{H}^E_\omega := V^{-1/2}(H_\omega - E)\,V^{-1/2}$. This motivates the introduction of the larger class of deformed random Schrödinger operators in Section 6.5 for which Theorem 6.1 is proven, see Theorem 6.23. The line of arguments above shows that (6.6) also holds for the operator $\widetilde{H}^E_\omega$ at energy zero. By exploiting that the spectrum of H_ω around energy E and the spectrum of $\widetilde{H}^E_\omega$ around energy zero are in good agreement, see Lemma 6.16 for details, we finally obtain the Minami-type estimate for H_ω.

6.3. Clusters of eigenvalues

For this section it is convenient to consider a more general framework. Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. We derive some preliminary properties of eigenvalue clusters that are separated from the rest of the spectrum by a spectral gap. More precisely, we consider an interval $I \subset \mathbb{R}$ (the energy region we are interested in) and $\varepsilon > 0$. Throughout the section we assume that

$$n := \operatorname{tr} \mathbb{1}_I(A) < \infty \qquad \text{and} \qquad \operatorname{dist}(I, \sigma(A)\setminus I) \geqslant 6\varepsilon \tag{6.18}$$

hold, where the latter is the spectral gap assumption. To simplify notation we also impose the constraints

$$|I| \leqslant \frac{1}{2} \qquad \text{and } 0 < \varepsilon < \frac{1}{12}. \tag{6.19}$$

The choice of numerical values in (6.18) and (6.19) is not important.

6.3.1. A Hellmann-Feynman type estimate. For a fixed self-adjoint and bounded operator B with $\|B\| \leqslant 1$ we consider the one-parameter operator family

$$(-\varepsilon, \varepsilon) \ni s \mapsto A_s := A + sB. \tag{6.20}$$

For the buffered interval $I_\varepsilon := I + (-\varepsilon, \varepsilon)$ the assumptions (6.18) yield

$$n = \operatorname{tr} \mathbb{1}_{I_\varepsilon}(A_s) \qquad \text{and} \qquad \operatorname{dist}(I_\varepsilon, \sigma(A_s)\setminus I_\varepsilon) \geqslant 4\varepsilon \tag{6.21}$$

for all $s \in (-\varepsilon, \varepsilon)$. By $\lambda_{s,1}, \ldots, \lambda_{s,n}$ we denote the eigenvalues of A_s in I_ε, with arithmetic mean $\bar\lambda_s := n^{-1}\sum_{i=1}^n \lambda_{s,i}$. In this context the classical Hellmann-Feynman formula gives $\operatorname{tr}(\mathbb{1}_{I_\varepsilon}(A_s)B) = n\partial_s\bar\lambda_s$. The next lemma provides additional information under the assumption that the n eigenvalues in I_ε are moving as a tightly concentrated cluster (in comparison to the gap size ε) as the coupling parameter s is tuned. We denote $P_s := \mathbb{1}_{I_\varepsilon}(A_s)$ for $s \in (-\varepsilon, \varepsilon)$.

Lemma 6.7. *Let $0 < \delta < \varepsilon$. If the bound*

$$\sup_{s\in(-\varepsilon,\varepsilon)} \sup_{i=1,\ldots,n} |\lambda_{s,i} - \bar\lambda_s| \leqslant \delta, \tag{6.22}$$

holds, then also

$$\sup_{s\in(-\varepsilon,\varepsilon)} \big\|P_s\big(B - \partial_s\bar\lambda_s\big)P_s\big\| \leqslant 9\sqrt{\frac{\delta}{\varepsilon}}. \tag{6.23}$$

In the proof of Lemma 6.7 we apply the following bounds. For convenience, a proof is provided at the end of this section.

Lemma 6.8. *For $s \in (-\varepsilon, \varepsilon)$ we have*

$$\|\partial_s P_s\| \leqslant \frac{1}{2\varepsilon} \quad \text{and} \quad \|\partial_s^2 P_s\| \leqslant \frac{1}{\pi\varepsilon^2}. \tag{6.24}$$

If moreover (6.22) *holds for a given $0 < \delta < \varepsilon$, then also*

$$\left\|\partial_s^2 \big(P_s(A_s - \bar{\lambda}_s)P_s\big)\right\| \leqslant \frac{7}{\varepsilon}. \tag{6.25}$$

Proof of Lemma 6.7. The assumption (6.22) yields

$$\|(A_s - \bar{\lambda}_s)P_s\| \leqslant \delta. \tag{6.26}$$

Differentiation of the operator $T_s := P_s(A_s - \bar{\lambda}_s)P_s$ together with (6.24) and (6.26) yields

$$\begin{aligned} \left\|P_s\big(B - \partial_s\bar{\lambda}_s\big)P_s\right\| &\leqslant 2\|\partial_s P_s\|\|(A_s - \bar{\lambda}_s)P_s\| + \|\partial_s T_s\| \\ &\leqslant \frac{\delta}{\varepsilon} + \|\partial_s T_s\|. \end{aligned} \tag{6.27}$$

The bound (6.23) follows once we establish

$$\sup_{s\in(-\varepsilon,\varepsilon)} \|\partial_s T_s\| = \sup_{s\in(-\varepsilon,\varepsilon)} \max_{\phi\in\mathcal{H}} |\,\langle\psi, (\partial_s T_s)(s_0)\psi\rangle\,| \leqslant 8\sqrt{\frac{\delta}{\varepsilon}}. \tag{6.28}$$

Set $T_{s,\psi} := \langle\psi, T_s\psi\rangle$ for $\psi \in \mathcal{H}$ and assume, for contradiction, that there exists $s_0 \in (-\varepsilon, \varepsilon)$ and a normalized $\psi \in \mathcal{H}$ such that

$$|(\partial_s T_{s,\psi})(s_0)| = |\,\langle\psi, (\partial_s T_s)(s_0)\psi\rangle\,| > 8\sqrt{\frac{\delta}{\varepsilon}}. \tag{6.29}$$

Then either $(\partial_s T_{s,\psi})(s_0) > 8\sqrt{\delta/\varepsilon}$ or $(\partial_s T_{s,\psi})(s_0) < -8\sqrt{\delta/\varepsilon}$. Without loss of generality we assume the former relation. Using the bound (6.25) we get that for $s_1 \in (-\varepsilon, \varepsilon)$

$$(\partial_s T_{s,\psi})(s_1) \geqslant (\partial_s T_{s,\psi})(s_0) - \frac{7}{\varepsilon}|s_1 - s_0| \geqslant 8\sqrt{\frac{\delta}{\varepsilon}} - \frac{7}{\varepsilon}|s_1 - s_0| \tag{6.30}$$

by the fundamental theorem of calculus. This implies that for any s in

$$\mathcal{S} := \left\{s \in (-\varepsilon, \varepsilon) : \ |s - s_0| \leqslant \frac{\sqrt{\delta\varepsilon}}{2}\right\}$$

we have $(\partial_s T_{s,\psi})(s) > 9\sqrt{\delta}/(2\sqrt{\varepsilon})$. Thus there exists $s_2 \in \mathcal{S}$ such that

$$\delta \geqslant |T_{s_2,\psi}| \geqslant \frac{\sqrt{\delta\varepsilon}}{2}\frac{9\sqrt{\delta}}{2\sqrt{\varepsilon}} - |T_{s_0,\psi}| \geqslant \frac{5}{4}\delta; \tag{6.31}$$

a contradiction. □

Proof of Lemma 6.8. For the proof we abbreviate $\dot{P}_s := \partial_s P_s$ and $\ddot{P}_s := \partial_s^2 P_s$. Let $I_+ = \sup I$ and $I_- = \inf I$. By $\gamma_{I,\varepsilon}$ we denote the contour consisting of the two counter-clockwise oriented line segments $[I_- - 3\varepsilon + i\infty, I_- - 3\varepsilon - i\infty]$ and $[I_+ + 3\varepsilon - i\infty, I_+ + 3\varepsilon + i\infty]$. For

$x + iy \in \operatorname{ran}(\gamma_{I,\varepsilon})$ the resolvent of A_s can be estimated as $\|R_{x+iy}(A_s)\| \leqslant ((2\varepsilon)^2 + y^2)^{-1/2}$. This yields the bounds

$$\|\dot{P}_s\| = \frac{1}{2\pi}\Big\| \int_{\gamma_{I,\varepsilon}} dz\, R_z(A_s) B R_z(A_s) \Big\| \leqslant \frac{1}{\pi}\int_{\mathbb{R}} dy\, \frac{1}{(2\varepsilon)^2 + y^2} = \frac{1}{2\varepsilon}, \tag{6.32}$$

$$\|\ddot{P}_s\| = \frac{1}{\pi}\Big\| \int_{\gamma_{I,\varepsilon}} dz\, R_z(A_s) B R_z(A_s) B R_z(A_s) \Big\| \leqslant \frac{2}{\pi}\int_{\mathbb{R}} dy\, \frac{1}{((2\varepsilon)^2 + y^2)^{3/2}} = \frac{1}{\pi\varepsilon^2}. \tag{6.33}$$

We turn to estimate (6.25). The first two derivatives of $P_s(A_s - \bar{\lambda}_s)P_s$ are

$$\partial_s\big(P_s(A_s - \bar{\lambda}_s)P_s\big) = \dot{P}_s P_s(A_s - \bar{\lambda}_s)P_s + P_s(A_s - \bar{\lambda}_s)P_s\dot{P}_s + P_s(B - \dot{\bar{\lambda}}_s)P_s, \tag{6.34}$$

$$\partial_s^2\left(P_s(A_s - \bar{\lambda}_s)P_s\right) = \Big\{\Big(\ddot{P}_s(A_s - \bar{\lambda}_s)P_s + \dot{P}_s^2(A_s - \bar{\lambda}_s)P_s + \dot{P}_s P(B - \dot{\bar{\lambda}}_s)P_s + \dot{P}_s P_s(A_s - \bar{\lambda}_s)\dot{P}_s\Big) + h.c.\Big\} + \Big\{\dot{P}_s(B - \dot{\bar{\lambda}}_s)P_s + h.c.\Big\} - P_s\ddot{\bar{\lambda}}_s P_s, \tag{6.35}$$

where $h.c.$ stands for the respective adjoint of the operator to its left side. The latter formula yields the bound

$$\left\|\partial_s^2\left(P_s(A_s - \bar{\lambda}_s)P_s\right)\right\| \leqslant 2\|\ddot{P}_s\|\|(A_s - \bar{\lambda}_s)P_s\| + 4\|\dot{P}_s\|^2\|\left(A_s - \bar{\lambda}_s\right)P_s\| + 8\|\dot{P}_s\| + |\ddot{\bar{\lambda}}_s|, \tag{6.36}$$

where we used $\|P_s\| = 1$, $\|B\| \leqslant 1$, and that the first derivative of $\bar{\lambda}_s$ satisfies

$$-1 \leqslant \dot{\bar{\lambda}}_s = \frac{1}{n}\Big(2\operatorname{tr}(P_s\dot{P}_s A_s) + \operatorname{tr}(P_s B)\Big) = \frac{1}{n}\operatorname{tr}(P_s B) \leqslant 1. \tag{6.37}$$

Via the bounds (6.24) and (6.26), and via the formula $\ddot{\bar{\lambda}}_s = n^{-1}\operatorname{tr}\big(\dot{P}_s B\big)$, we conclude that

$$\left\|\partial_s^2\left(P_s(A_s - \bar{\lambda}_s)P_s\right)\right\| \leqslant 2\frac{\delta}{\pi\varepsilon^2} + 4\frac{\delta}{4\varepsilon^2} + \frac{4}{\varepsilon} + \frac{1}{2\varepsilon} \leqslant \frac{2\delta}{\varepsilon^2} + \frac{5}{\varepsilon}. \tag{6.38}$$

□

6.3.2. The local discriminant and a Cartan estimate. If $n \geqslant 2$ for n from (6.18), i.e. if at least two eigenvalues of A are inside of I, then we define the local discriminant of A_s on I_ε as

$$\operatorname{disc}_{I_\varepsilon}(A_s) := \prod_{1\leqslant i<j\leqslant n} (\lambda_{s,i} - \lambda_{s,j})^2 \tag{6.39}$$

for $s \in (-\varepsilon, \varepsilon)$. Below we prove the following regularity result for the discriminant, which is a consequence of the spectral gap assumption (6.21).

Lemma 6.9. *The local discriminant, interpreted as a function $(-\varepsilon, \varepsilon) \ni s \mapsto \operatorname{disc}_{I_\varepsilon}(A_s)$, has an extension to a complex analytic function on $B_{3\varepsilon}^{\mathbb{C}} := \{z \in \mathbb{C} : |z| < 3\varepsilon\}$ that is bounded by 1.*

Let now $N \in \mathbb{N}$ and $0 \leqslant B_k \leqslant 1$ be self-adjoint operators for $k = 1, ..., N$ such that $\sum_k B_k \leqslant 1$. We consider the N-parameter operator family

$$(-\varepsilon, \varepsilon)^N \ni \mathbf{s} := (s_1, .., s_N) \mapsto A + \sum_{k=1}^{N} s_k B_k. \tag{6.40}$$

In this context a version of Cartan's lemma holds for the local discriminant and yields a probabilistic bound on the spacing of eigenvalues in the interval I.

Lemma 6.10. *If for fixed $0 < \delta_0 < \varepsilon$ there exists $\mathbf{s}_0 \in (-\varepsilon, \varepsilon)^N$ such that*

$$\operatorname{spac}_{I_\varepsilon}(A_{\mathbf{s}_0}) > \delta_0, \tag{6.41}$$

then there exist constants C_1, C_2 (which are independent of all the relevant parameters) such that

$$\left|\left\{\mathbf{s} \in (-\varepsilon, \varepsilon)^N : \operatorname{spac}_{I_\varepsilon}(A_{\mathbf{s}}) < \delta\right\}\right| \leqslant C_1 N (2\varepsilon)^N \exp\left(-\frac{C_2}{n^2}\left|\frac{\log \delta}{\log \delta_0}\right|\right) \tag{6.42}$$

holds for all $0 < \delta < 1$.

Proof of Lemma 6.9. Due to (6.18) we have $\mathbb{1}_{I_\varepsilon}(A_s) = \mathbb{1}_{I_{3\varepsilon}+i\mathbb{R}}(A_s)$ and $\mathbb{1}_{I_\varepsilon^c}(A_s) = \mathbb{1}_{I_{3\varepsilon}^c+i\mathbb{R}}(A_s)$ for $s \in (-\varepsilon, \varepsilon)$. Hence the two projections can be extended to the complex analytic operators

$$B_{3\varepsilon}^{\mathbb{C}} \ni s \mapsto \mathbb{1}_{I_{3\varepsilon}+i\mathbb{R}}(A_s), \tag{6.43}$$

$$B_{3\varepsilon}^{\mathbb{C}} \ni s \mapsto \mathbb{1}_{I_{3\varepsilon}^c+i\mathbb{R}}(A_s), \tag{6.44}$$

which are defined via the holomorphic functional calculus [72]. The function

$$z \mapsto p_s(z) = \det\left(\mathbb{1}_{I_{3\varepsilon}}(A_s)(A_s - z) + \mathbb{1}_{I_{3\varepsilon}^c}(A_s)\right) = \prod_{i=1}^{n}(\lambda_{s,i} - z), \tag{6.45}$$

is a polynomial of degree n in z. Here the $\lambda_{i,s}$, $i = 1, ..., n$, are the eigenvalues of A_s in $I_{3\varepsilon}$ for $s \in (-3\varepsilon, 3\varepsilon)$. For fixed $z \in \mathbb{C}$ the function $s \mapsto p_s(z)$ can be extended to a complex analytic function $\widetilde{p}_s(z)$ on $B_{3\varepsilon}^{\mathbb{C}}$, given by

$$B_{3\varepsilon}^{\mathbb{C}} \ni s \mapsto \widetilde{p}_s(z) = \det\left(\mathbb{1}_{I_{3\varepsilon}+i\mathbb{R}}(A_s)(A_s - z) + \mathbb{1}_{I_{3\varepsilon}^c+i\mathbb{R}}(A_s)\right). \tag{6.46}$$

If we write the polynomial as $\widetilde{p}_s(z) = \sum_{k=0}^{n} a_k(s) z^k$, then the coefficients $a_k(s)$ are also complex analytic on $B_{3\varepsilon}^{\mathbb{C}}$ since they can be expressed via evaluations of $\widetilde{p}_s(z)$ at different values of z, for instance via Lagrange polynomials. For $s \in B_{3\varepsilon}^{\mathbb{C}}$ the resultant of $\widetilde{p}_s$ and $\widetilde{p}'_s$, which is a polynomial of degree $n(n-1)$ in each of the coefficients $a_n(s)$, is then

$$\operatorname{res}(p_s, p'_s) = (-1)^{n(n-1)/2} \prod_{i<j} (\mu_i(s) - \mu_j(s))^2, \tag{6.47}$$

where the $\mu_i(s)$ are an arbitrary enumeration of the zero's of $\widetilde{p}_s$. For $s \in (-\varepsilon, \varepsilon)$ this agrees, up to the prefactor ± 1 in (6.47) with the local discriminant $\operatorname{disc}_{I_\varepsilon}(A_s)$ for A_s defined above. This proves the first part of the lemma. For the second part we note that the $\mu_i(s)$ in (6.47) are the eigenvalues of A_s in $B_{3\varepsilon}^{\mathbb{C}}$. Because $\sigma(A_s) \subset \sigma(A) + B_{3\varepsilon}^{\mathbb{C}}$ for $s \in B_{3\varepsilon}^{\mathbb{C}}$, and because $|I| \leqslant 1/2$ and $\varepsilon < 1/12$, this shows that $|\mu_i(s) - \mu_j(s)| \leqslant 1$ holds for $s \in B_{3\varepsilon}^{\mathbb{C}}$. □

Proof of Lemma 6.10. We define the map

$$(-\varepsilon,\varepsilon)^N \ni \mathbf{z} := (z_1,..,z_N) \mapsto F(z) := \operatorname{disc}_{I_\varepsilon}\left(A + \sum_{k=1}^{N} z_k B_k\right). \tag{6.48}$$

Lemma 6.9 implies that for $\xi = (\xi_i)_i \in [-1,1]^N$ the map

$$(-\varepsilon,\varepsilon) \ni s \mapsto F(s\xi_1,...,s\xi_N) \tag{6.49}$$

can be extended to a complex analytic map on $B^{\mathbb{C}}_{3\varepsilon}$. If we set $F_\varepsilon(z) := F(2\varepsilon z)$ for $z \in [-1/2,1/2]^N$ then $[-1/2,1/2] \ni s \mapsto F_\varepsilon(s\xi_1,...,s\xi_N)$ is real analytic and can be extended to a complex analytic map on $B^{\mathbb{C}}_{3/2}$ with $|F_\varepsilon| \leqslant 1$. Since by assumption there exists $z_0 \in [-1/2,1/2]^N$ such that $|F_\varepsilon(z_0)| > \delta_0^{n^2}$ Lemma 1 from [21] is applicable and yields

$$\left|\left\{z \in [-1/2,1/2]^N : |F_\varepsilon(z)| < \delta\right\}\right| \leqslant C_1 N \exp\left(-\frac{C_2}{n^2}\left|\frac{\log\delta}{\log\delta_0}\right|\right) \tag{6.50}$$

for $\delta \in (0,1)$ and constants C_1, C_2 that are uniform in all relevant parameters. Estimate (6.42) now follows from (6.50) and

$$\begin{aligned}\left|\left\{\mathbf{s} \in (-\varepsilon,\varepsilon)^N : \operatorname{spac}_{I_\varepsilon}(A_{\mathbf{s}}) < \delta\right\}\right| &\leqslant \left|\left\{\mathbf{s} \in (-\varepsilon,\varepsilon)^N : \operatorname{disc}_{I_\varepsilon}(A_{\mathbf{s}}) < \delta\right\}\right| \\ &= \left|\left\{\mathbf{s} \in (-\varepsilon,\varepsilon)^N : |F(\mathbf{s})| < \delta\right\}\right| \\ &= (2\varepsilon)^N \left|\left\{z \in [-1/2,1/2]^N : |F_\varepsilon(z)| < \delta\right\}\right|.\end{aligned} \tag{6.51}$$

□

6.4. Illustration of the method for the polymer model

In this section we illustrate this chapter's main results – and their proofs – in the context of a technically more convenient model: A version of the dimer (or polymer) model on the lattice, which is described in detail below. Known methods for proving either a level-spacing estimate or a Minami estimate already break down for the dimer model: It contains the key difficulties which have to be overcome for continuum random Schrödinger operators. But on the other hand the dimer model allows us to focus on conceptual difficulties for now and defer additional technical difficulties which arise when considering continuum random Schrödinger operators to a later point. In this spirit, we do not hesitate to impose further restrictions and artificially downgrade the results whenever it clarifies the presentation.

Let $R \in \mathbb{N}$ be fixed and let $\Gamma := (R\mathbb{Z})^d \subset \mathbb{Z}^d$ be the sublattice with mesh size R. Throughout this section we work with the random Schrödinger operator

$$H_\omega := -\Delta + V_\omega := -\Delta + \sum_{k\in\Gamma} \omega_k V_k, \tag{6.52}$$

acting on the Hilbert space $\ell^2(\mathbb{Z}^d)$ of square-summable sequences $(x_n)_{n\in\mathbb{Z}^d} \subset \mathbb{C}$. Here, $-\Delta$ is the graph Laplacian on the graph $\mathbb{Z}^d$, acting as

$$\big(-\Delta\psi\big)(n) = -\sum_{\substack{m\in\mathbb{Z}^d:\\ |n-m|_1=1}} \big(\psi(m) - \psi(n)\big) \tag{6.53}$$

for $\psi \in \ell^2(\mathbb{Z}^d)$ and $n \in \mathbb{Z}^d$, where $|\cdot|_1$ denotes the 1-norm on $\mathbb{Z}^d$. The graph Laplacian can be thought of as a finite difference analog of the continuum Laplacian. In contrast to his continuum brother in spirit, the graph Laplacian is a bounded operator with $\sigma(H) = \sigma_{ac}(H) = [0, 4d]$. This, together with the fact that the single-site potentials V_k are of finite rank, facilitates the analysis of random Schrödinger operators on the lattice to some extent. For the most part, and unless explicitly mentioned otherwise, we impose the following further assumptions.

($\widetilde{V}_1$) $V_k := V_0(\cdot - k)$ for $k \in \Gamma$, where V_0 is the $\ell^2(\mathbb{Z}^d)$-projection onto the cube $\mathcal{V}_0 := \{0, ..., R-1\}^d \subset \mathbb{Z}^d$ for the $R \in \mathbb{N}$ chosen above (i.e. $V_0 := \mathcal{X}_{\mathcal{V}_0}$ and $\mathcal{V}_0 = \operatorname{supp} V_0$).
($\widetilde{V}_2$) The random couplings $\omega = (\omega_k)_{k\in\Gamma} \in \mathbb{R}^\Gamma$ are independent and identically distributed according to the uniform distribution on the interval $[0, 1]$.

Those assumptions in particular imply that $\sum_{k\in\Gamma} V_k = \mathcal{X}_{\mathbb{Z}^d} = \mathrm{id}_{\ell^2(\mathbb{Z}^d)}$ holds. For $R = 1$ we simply reobtain the lattice Anderson model. But for $R > 1$ we arrive at a model for which the random potential V_ω alone certainly does not satisfy a level-spacing estimate: The random eigenvalues in this case (almost surely) have multiplicity R^d. The polymer model and similar models are an interesting playground to analyze the questions considered here [95, 61, 90, 77]. In [61] it was proved that for the polymer model in the localized spectrum the local eigenvalue process is compound Poisson. In [95] a procedure was presented that yields simplicity of the point spectrum of the infinite-volume operator H_ω. However, their statement was conditioned on the verification of a criterion which rules out local symmetries. Even though the criterion most likely can be verified for 'any' $R > 0$, computations quickly get involved and the criterion was only verified explicitly for $R = 2$. This idea was catched up and generalized in [90] to a larger class of polymer models. But the argument used there picked up ungeneric constraints on R.

Pretty much all the properties described in Section 2.1 in the context of continuum random Schrödinger operators likewise hold for the polymer model. This in particular applies for the Wegner estimate, which enters the proofs below. For $G \subset \mathbb{Z}^d$ we denote by $-\Delta_G$ the finite-volume restriction of the graph Laplacian onto G with graph boundary conditions. I.e. $-\Delta_G$ is acting as

$$(-\Delta_G \psi)(n) = - \sum_{\substack{m\in G:\\ |n-m|_1=1}} \big(\psi(m) - \psi(n)\big) \tag{6.54}$$

for $\psi \in \ell^2(G)$ and $n \in G$. We note that graph boundary conditions on the lattice are equivalent to Neumann boundary conditions in the sense that they allow for Neumann bracketing [76]. More precisely, if $G = G_1 \cup G_2$ is a disjoint union, then

$$-\Delta_G \geqslant -\Delta_{G_1} \oplus -\Delta_{G_2} \tag{6.55}$$

holds in the form sense. For fixed $L \in \mathbb{N}$ we denote by $\Lambda_L := \{0, ..., RL-1\}^d$ the box of linear size RL and set $\Gamma_L := \Lambda_L \cap \Gamma$. This specific choice of the finite-volume boxes makes the structure of the random potential on V_ω close to the boundary of the box slightly simpler compared to the general situation. The finite-volume restriction of H_ω onto $\ell^2(\Lambda_L)$ is given by

$$H_{\omega,L} = -\Delta_L + \sum_{k\in\Gamma_L} \omega_k V_k, \tag{6.56}$$

where we abbreviated $-\Delta_L := -\Delta_{\Lambda_L}$. The configuration space is given by $\Omega_L = [0,1]^{\Gamma_L}$. An adaption of the proof of Minami's estimate for the lattice Anderson model from [27] to the polymer model yields that

$$\mathbb{P}\big(\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) > R^d\big) \leqslant C|I|^2 L^{2d} \tag{6.57}$$

for all intervals $I \subset \mathbb{R}$ and a constant C that only depends on R [61]. This estimate serves as an upper bound for the typical size of clusters of tightly concentrated eigenvalues. In view of an extension to continuum random Schrödinger operators we note that the upper bound for the typical cluster size, R^d, agrees with the rank of the single-site potentials V_k. One of the points which have to be addressed separately for the continuum case is therefore a substitute for (6.57). As in the continuum case we denote by $\lambda^G_{\omega,1} \leqslant \lambda^G_{\omega,2} \leqslant \dots$ the non-decreasingly ordered eigenvalues of $H_{\omega,G}$ for $G \subset \mathbb{Z}^d$ (as usual counted according to their multiplicity). The critical energy up to which our approach works for the polymer model is

$$E_{\mathrm{sp}} := \frac{\lambda^{\mathcal{V}_0}_{0,2}}{2}. \tag{6.58}$$

Here,

$$0 = \lambda^{\mathcal{V}_0}_{0,1} < \lambda^{\mathcal{V}_0}_{0,2} \leqslant \lambda^{\mathcal{V}_0}_{0,3} \leqslant \dots \tag{6.59}$$

are the non-decreasingly ordered eigenvalues of $-\Delta_{\mathcal{V}_0}$, the restriction of the graph Laplacian to the support of the single-site potential V_0. Let's now have a look at the polymer model versions of Theorem 6.1 and Theorem 6.2.

Theorem 6.11 (Probabilistic level-spacing estimate). *For fixed $E < E_{\mathrm{sp}}$ there exist constants $C_1, C_2, \mathcal{L}$ such that*

$$\mathbb{P}\left(\operatorname{spac}_E(H_L) < \delta\right) \leqslant C_1 L^{2d} \exp\left(-C_2|\log\delta|^{1/4}\right) \tag{6.60}$$

holds for all $\mathbb{N} \ni L \geqslant \mathcal{L}$ and $0 < \delta \leqslant 1$.

Theorem 6.12 (Minami-type estimate). *For fixed $E < E_{\mathrm{sp}}$ there exist constants $C_1, C_2, \mathcal{L}$ such that*

$$\mathbb{P}\left(\operatorname{tr} \mathbb{1}_J(H_L) \geqslant 2\right) \leqslant C_1 L^{4d} \delta \exp\left(-C_2|\log\delta|^{1/4}\right) \tag{6.61}$$

holds for all intervals $J \subset (-\infty, E]$ with $|J| \leqslant \delta$ and all $\mathbb{N} \ni L \geqslant \mathcal{L}$.

Remarks 6.13. (i) The proof below shows that the first theorem holds for a quite general class of random Schrödinger operators on the lattice as long as the single-site probability distribution is absolutely continuous with a bounded density.

(ii) One can again deduce simplicity of the spectrum of H_ω and Poissonian local eigenvalue statistics in the localized spectrum from the above theorems.

(iii) The faster decay for small δ in the above theorems compared to the corresponding statements from Section 6.1 stems from the potent estimate (6.57) on the typical size of clusters of eigenvalues.

(iv) In view of the Remark in Section 2.1 and the discussion in Section 6.2 the energy E_{sp} from (6.58) is probably the sharp energy threshold for our method.

(v) Since $V_- = V_+ = 1$ (with V_-, V_+ as in Section 2.1) for the polymer model we have $E_{\mathrm{M}} := E_{\mathrm{sp}}/V_+ = E_{\mathrm{sp}}$. This is why we did not introduce E_{M} here and Theorem 6.12 holds up to energy E_{sp}.

The starting point for the proof of the level-spacing estimate is the following lemma. Frankly speaking, it ensures the existence of a sufficiently dense subset of good configurations in the configuration space $\Omega_L = [0,1]^{\Gamma_L}$ for which the eigenvalues of $H_{\omega,L}$ are well spaced. Let

$$\xi_{L,n} := \lambda_{0,2}^{\mathcal{V}_0} \left(\frac{1}{2} - \frac{23\sqrt{n}}{L} \right). \tag{6.62}$$

For convenience we also define $\lambda^L_{\omega,-1} := -\infty$ and $\lambda^L_{\omega,(RL)^d+1} := \infty$.

Lemma 6.14 (Good configurations)**.** *Let $0 < \varepsilon < 1/12$, $2 \leqslant n \in \mathbb{N}$, and let $L \in \mathbb{N}$ be sufficiently large such that $\xi_{L,n} > 0$. Let $\omega_0 \in \Omega_L$ such that $Q_\varepsilon(\omega_0) := \omega_0 + [-\varepsilon,\varepsilon]^{\Gamma_L} \subset \Omega_L$ and assume that there exists $m \in \mathbb{N}$ such that*

(i) $\lambda^L_{\omega_0,m+n} < \xi_{L,n}$

(ii) The cluster $\mathcal{C}^n_{\omega_0,m} := \lambda^L_{\omega_0,m+1}, \ldots, \lambda^L_{\omega_0,m+n}$ is isolated from the rest of the spectrum:

$$\min\left\{\lambda^L_{\omega_0,m+1} - \lambda^L_{\omega_0,m}, \lambda^L_{\omega_0,m+n+1} - \lambda^L_{\omega_0,m+n}\right\} \geqslant 6\varepsilon. \tag{6.63}$$

Then there exists $\widehat{\omega} \in Q_\varepsilon(\omega_0)$ such that

$$\min_{i=1,\ldots,n-1} |\lambda^L_{\widehat{\omega},m+i+1} - \lambda^L_{\widehat{\omega},m+i}| \geqslant 6\varepsilon L^{-(n-1)(2d+2)}. \tag{6.64}$$

Up to an iteration procedure the lemma follows from the following statement.

Lemma 6.15. *Assume that the assumptions of Lemma 6.14 are satisfied. Then there exists $\widehat{\omega} \in Q_{\varepsilon-\varepsilon L^{-(2d+2)}}(\omega_0)$ and $1 \leqslant k \leqslant n-1$ such that*

$$\lambda^L_{\widehat{\omega},m+k+1} - \lambda^L_{\widehat{\omega},m+k} > 6\varepsilon L^{-(2d+2)}. \tag{6.65}$$

Proof of Lemma 6.15. In order to slightly compactify formulas we abbreviate

$$E_i^\omega := \lambda^L_{\omega,m+i} \tag{6.66}$$

for $i = 1, \ldots, n$ and $\bar{E}^\omega := n^{-1}\sum_{i=1}^n E_i^\omega$. If we set $I := [E_1^{\omega_0}, E_n^{\omega_0}]$ then the min-max characterization of eigenvalues implies that

$$\operatorname{tr} \mathbb{1}_{I_\varepsilon}(H_{\omega,L}) = n \tag{6.67}$$

for all $\omega \in Q_\varepsilon(\omega_0)$, where $I_\varepsilon := I + [-\varepsilon,\varepsilon]$. Hence the eigenvalues $E_1^\omega, \ldots, E_n^\omega$ form a cluster of isolated eigenvalues as considered in Section 6.3. If we prove that

$$\max_{\omega \in Q_\varepsilon(\omega_0)} \max_{i=1,\ldots,n} |E_i^\omega - \bar{E}^\omega| > 8n\varepsilon L^{-(2d+2)},$$

then there exists $\omega' \in Q_\varepsilon(\omega_0)$ such that $E_n^{\omega'} - E_1^{\omega'} \geqslant 8n\varepsilon L^{-(2d+2)}$. Hence there exists $j \in \{1, \ldots, n-1\}$ such that $E_{j+1}^{\omega'} - E_j^{\omega'} > 8\varepsilon L^{-(2d+2)}$. If $\omega' \in Q_{\varepsilon-\varepsilon L^{-(2d+2)}}(\omega_0)$ then we can set $\widehat{\omega} := \omega'$ and the claim holds. If $\omega' \notin Q_{\varepsilon-\varepsilon L^{-(2d+2)}}(\omega_0)$ then we can choose $\widehat{\omega} \in Q_{\varepsilon-\varepsilon L^{-(2d+2)}}(\omega_0)$

such that $|\widehat{\omega} - \omega'| \leqslant \varepsilon L^{-(2d+2)}$ and the claim follows from Weyl's theorem on the movement of eigenvalues. Let us assume, for contradiction, that

$$\max_{\omega \in Q_\varepsilon(\omega_0)} \max_{i=1,\dots,n} |E_i^\omega - \bar{E}^\omega| \leqslant 8n\varepsilon L^{-(2d+2)}. \tag{6.68}$$

For fixed $k \in \Gamma_L$ Lemma 6.7 can be applied for ε and $\delta = 8n\varepsilon L^{-(2d+2)}$ to the one-parameter operator family

$$(-\varepsilon, \varepsilon) \ni s \mapsto H_{\omega_0,L} + sV_k. \tag{6.69}$$

With the interval I_ε defined above we denote $P_\omega := \mathbb{1}_{I_\varepsilon}(H_{\omega,L})$ for $\omega \in Q_\varepsilon(\omega_0)$. Let

$$\alpha_k := (\partial_s \bar{E}^s)(0) = (\partial_{\omega_k} \bar{E})(\omega_0) = \frac{1}{n} \operatorname{tr} P_{\omega_0} V_k \geqslant 0, \tag{6.70}$$

where we have used the Hellmann-Feynman theorem in the second step. Evaluation of (6.23) at $s = 0$ then yields the bound

$$\left\| P_{\omega_0} \left(V_k - (\partial_{\omega_k} \bar{E}^\omega)(\omega_0) \right) P_{\omega_0} \right\| \leqslant 23\sqrt{n} L^{-d-1} \tag{6.71}$$

for every $k \in \Gamma_L$. Neumann decoupling, see (6.55), yields

$$\operatorname{tr}\left(P_{\omega_0} H_{\omega_0,L} \right) \geqslant \sum_{k \in \Gamma_L} \operatorname{tr}\Big(P_{\omega_0} \big(-\Delta_{\mathcal{V}_k} \big) \Big), \tag{6.72}$$

where we also used that $V_k \geqslant 0$ and $\omega_{0,k} \geqslant 0$ for all $k \in \Gamma$. Since $0 = \lambda_{0,1}^{\mathcal{V}_0} < \lambda_{0,2}^{\mathcal{V}_0}$ holds for the first two eigenvalues of $-\Delta_{\mathcal{V}_k}$, we conclude that

$$\operatorname{tr}\left(P_{\omega_0} H_{\omega_0,L} \right) \geqslant \lambda_{0,2}^{\mathcal{V}_0} \sum_{k \in \Gamma_L} \operatorname{tr}\left(C_{\omega_0,k} R_k \right); \quad C_{\omega_0,k} := V_k P_{\omega_0} V_k, \tag{6.73}$$

where R_k is the projection onto $\operatorname{ran}(\Delta_{\mathcal{V}_k})$. Next, we bound the trace on the right hand side as

$$\operatorname{tr}\left(C_{\omega_0,k} R_k \right) = \operatorname{tr} C_{\omega_0,k} - \operatorname{tr}\left(C_{\omega_0,k}(V_k - R_k) \right) \geqslant \operatorname{tr} C_{\omega_0,k} - \|C_{\omega_0,k}\| = {\sum}' \nu_j, \tag{6.74}$$

where $(\nu_j)_j$ are the eigenvalues of $C_{\omega_0,k}$ counted with multiplicity and $\sum'$ stands for the sum of all but the largest eigenvalue of $C_{\omega_0,k}$. Here we also used that $V_k - R_k$ is the rank-1 projection onto the ground state of $-\Delta_{\mathcal{V}_k}$. Since $\sigma(C_{\omega_0,k}) \setminus \{0\} = \sigma(P_{\omega_0} V_k P_{\omega_0}) \setminus \{0\}$ and

$$P_{\omega_0} V_k P_{\omega_0} \geqslant \left(\alpha_k - 23\sqrt{n} L^{-d-1} \right) P_{\omega_0}, \tag{6.75}$$

we deduce by the min-max principle

$$\begin{aligned} \operatorname{tr}\left(C_{\omega_0,k} R_k \right) &\geqslant {\sum}' \nu_j \geqslant \left(\alpha_k - 23\sqrt{n} L^{-d-1} \right) (\operatorname{tr} P_{\omega_0} - 1) \\ &= (n-1) \left(\alpha_k - 23\sqrt{n} L^{-d-1} \right). \end{aligned} \tag{6.76}$$

Putting all bounds together, we obtain

$$\bar{E}^{\omega_0} = \frac{1}{n} \operatorname{tr}\left(P_{\omega_0} H_{\omega_0,L} \right) \geqslant \lambda_{0,2}^{\mathcal{V}_0} \left(\frac{1}{2} - \frac{23\sqrt{n}}{L} \right) = \xi_{L,n}. \tag{6.77}$$

This stands in conflict with our assumption $\lambda_{\omega_0,n+m} < \xi_{L,n}$. □

Proof of Lemma 6.14. We use the notation introduced in the proof of Lemma 6.15. First, we directly apply Lemma 6.15 to the cluster $E_1^{\omega_0}, ..., E_n^{\omega_0}$ and the set $Q_0 := Q_\varepsilon(\omega_0)$ in configuration space. Hence there exists $\omega_1 \in Q_1 := Q_{\varepsilon - \varepsilon L^{-(2d+2)}}(\omega_0)$ and $1 \leqslant k_1 \leqslant n-1$ such that

$$E_{k_1+1}^{\omega_1} - E_{k_1}^{\omega_1} > 6\varepsilon L^{-(2d+2)}. \tag{6.78}$$

If $k_1 = 1$ or $k_1 = n-1$ then we isolated one eigenvalue from the rest of the eigenvalues and only proceed with one cluster of eigenvalues. In the other cases we obtain two sets of eigenvalues $E_1^{\omega_1} \leqslant ... \leqslant E_{k_1}^{\omega_1}$ and $E_{k_1+1}^{\omega_1} \leqslant ... \leqslant E_n^{\omega_1}$ which both satisfy (6.63) for $\varepsilon_1 := \varepsilon L^{-(2d+2)}$. We apply Lemma 6.15 for $\varepsilon = \varepsilon_1$ to the cluster of eigenvalues $E_1^{\omega_1}, ..., E_{k_1}^{\omega_1}$. This yields $\omega_2 \in Q_2 := Q_{\varepsilon_1 - \varepsilon_1 L^{-(2d+2)}}(\omega_1)$ and $1 \leqslant k_2 \leqslant k_1 - 1$ such that

$$E_{k_2+1}^{\omega_2} - E_{k_2}^{\omega_2} > 6\varepsilon_1 L^{-(2d+2)}. \tag{6.79}$$

Set $\varepsilon_2 := \varepsilon_1 L^{-(2d+2)}$. Then, since $|\omega_2 - \omega_1| \leqslant \varepsilon_1 - \varepsilon_2$ we have

$$E_{k_1+1}^{\omega_2} - E_{k_1}^{\omega_2} > 6\varepsilon_1 - 2(\varepsilon_1 - \varepsilon_2) \geqslant 6\varepsilon_2 \tag{6.80}$$

by Weyl's theorem on the movement of eigenvalues and we can apply Lemma 6.15 to the set $E_{k_1+1}^{\omega_{0,2}}, ..., E_n^{\omega_{0,2}}$ of eigenvalues. Overall we found $\omega_3 \in Q_3 := Q_{\varepsilon_2 - \varepsilon_2 L^{-(2d+2)}}(\omega_2)$ and up to four clusters of eigenvalues which are separated from each other (and the rest of the spectrum of $H_{\omega,L}$) by $6\varepsilon_3 := 6\varepsilon_2 L^{-(2d+2)}$. We repeat this procedure at most $n-1$ times until each cluster consists of exactly one eigenvalue. □

We are now prepared to prove the level-spacing estimate.

Proof of Theorem 6.11. Let $E \in (0, E_{\mathrm{sp}})$ and fix a constant $0 < \kappa \leqslant E_{\mathrm{sp}}$ to be specified later. We first decompose the interval $[0, E]$ into a family $(K_i)_{i \in \mathcal{I}}$ of intervals of side length $|K_i| \leqslant \kappa$, with $|K_{i+1} \cap K_i| \geqslant \kappa/2$ and such that $|\mathcal{I}| \leqslant 4E_{\mathrm{sp}}\kappa^{-1}$. Let $i \in \mathcal{I}$ and define $K_{i,\eta} := K_i + [-\eta, \eta]$ for $\eta > 0$. Then the probability of the event

$$\Omega_{i,\varepsilon} := \left\{ \operatorname{tr} \mathbb{1}_{I_i}(H_{\omega,L}) \leqslant R^d \text{ and } \operatorname{tr} \mathbb{1}_{K_{i,8\varepsilon} \setminus K_i}(H_{\omega,L}) = 0 \right\} \tag{6.81}$$

can be estimated by Wegner's estimate and (6.57) as $\mathbb{P}(\Omega_{i,\varepsilon}) \geqslant 1 - C_1 L^d \varepsilon - C_2 L^{2d} \kappa^2$, where the constants C_1, C_2 only depend on the dimension and R. The same is true for all the constants below. This yields for $0 < \delta < \kappa/2$ that

$$\begin{aligned} &\mathbb{P}(\operatorname{spac}_E(H_L) < \delta) \\ &\leqslant \sum_{i \in \mathcal{I}} \mathbb{P}\left(\left\{ \operatorname{spac}_{K_i}(H_L) < \delta \right\} \cap \Omega_{i,\varepsilon}\right) + C_1 |\mathcal{I}| L^d \varepsilon + C_2 |\mathcal{I}| L^{2d} \kappa^2. \end{aligned} \tag{6.82}$$

We next partition the configuration space $\Omega_L = [0,1]^{\Gamma_L}$ into cubes Q_j, $j \in \mathcal{J}$, of side length 2ε. Hence $|Q_j| = (2\varepsilon)^{|\Gamma_L|}$ and $\mathcal{J} \leqslant 2(2\varepsilon)^{-|\Gamma_L|}$. Now, fix $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that $Q_j \cap \Omega_{i,\varepsilon} \neq \varnothing$, and let $\omega_{i,j} \in Q_j \cap \Omega_{i,\varepsilon}$. If we denote the center of Q_j by $\omega_{0,j}$ then

$$n_{i,j} := \operatorname{tr} \mathbb{1}_{K_{i,\varepsilon}}(H_{\omega_{0,j},L}) \leqslant R^d \quad \text{and} \quad \operatorname{dist}\left(K_{i,\varepsilon}, \sigma(H_{\omega_{0,j},L}) \setminus K_{i,\varepsilon}\right) \geqslant 6\varepsilon. \tag{6.83}$$

This follows from $|\omega_{0,j} - \omega_{i,j}| \leqslant \varepsilon$ and the min-max characterization of the eigenvalues of $H_{\omega,L}$. Hence Lemma 6.14 is applicable for $L \geqslant \mathcal{L}$, where $\mathcal{L}$ depends on the E chosen above. This yields $\widehat{\omega}_{i,j} \in Q_j$ such that

$$\operatorname{spac}_{K_{i,\varepsilon}}(H_{\widehat{\omega}_{i,j},L}) \geqslant 6\varepsilon L^{-(n_{i,j}-1)(2d+2)}. \tag{6.84}$$

This in turn can be used as an input for Lemma 6.10 with $\delta_0 := 6\varepsilon L^{-(n_{i,j}-1)(2d+2)}$. With $n_{i,j} \leqslant R^d$ and assumption $(\widetilde{V}_2)$ this yields

$$\begin{aligned}\mathbb{P}\big(Q_j \cap \big\{ \operatorname{spac}_{K_{i,2\varepsilon}}(H_{\omega,L}) < \delta\big\}\big) &= \big|\big\{\omega \in Q_j : \operatorname{spac}_{K_{i,2\varepsilon}}(H_{\omega,L}) < \delta\big\}\big| \\ &\leqslant C_3 L^d |Q_j| \exp\left(\frac{-C_4|\log\delta|}{|\log(6\varepsilon)| + R^d(2d+2)\log L}\right). \end{aligned} \tag{6.85}$$

If we set

$$\mathcal{J}_i := \{j \in \mathcal{J} : Q_j \cap \Omega_{i,\varepsilon} \neq \varnothing\} \tag{6.86}$$

for $i \in \mathcal{I}$, then

$$\begin{aligned}(6.82) &\leqslant \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i} \mathbb{P}\left(\big\{\operatorname{spac}_{K_i}(H_{\omega,L}) < \delta\big\} \cap Q_j\right) + C_1|\mathcal{I}|L^d\varepsilon + C_2|\mathcal{I}|L^{2d}\kappa^2 \\ &\leqslant 4E_{\mathrm{sp}}C_1\kappa^{-1}L^d\varepsilon + 4E_{\mathrm{sp}}C_2L^{2d}\kappa + 8E_{\mathrm{sp}}C_3L^d\kappa^{-1}\exp\left(\frac{-C_5|\log\delta|}{|\log\varepsilon| + \log L}\right). \end{aligned} \tag{6.87}$$

We now choose $\varepsilon := \exp\left(-|\log\delta|^{1/2}\right)$ and $\kappa := \exp\left(-|\log\delta|^{1/4}\right)$ and $\delta \leqslant \exp\left(-(\log L)^2\right)$. After possibly enlarging $\mathcal{L}$ this yields for $L \geqslant \mathcal{L}$ that $\delta \leqslant \kappa/2$. Hence we obtain

$$\mathbb{P}\left(\operatorname{spac}_E(H_L) < \delta\right) \leqslant C_6 L^{2d} \exp\left(-C_7|\log\delta|^{1/4}\right) \tag{6.88}$$

for constants C_6, C_7 which can be chosen to be independent of everything but the dimension and R. If $1 \geqslant \delta > \exp\left(-(\log L)^2\right)$ then the right hand side of (6.88) is $\geqslant 1$ anyways (at least after a suitable enlargement of C_6 if necessary). □

Before we start with the proof of the Minami-type estimate we make a preliminary remark. Let's consider for $a > 0$ the following slightly more general polymer model H^a_ω: Instead of $(\widetilde{V}_2)$ we assume that the random couplings are identically distributed according to the uniform distribution on the interval $[0, a]$. This means that for the original polymer model we have $H_\omega = H^1_\omega$. An inspection of the proof of Theorem 6.11 shows that the result still holds for the operators H^a_ω and that the constants $C_1, C_2, \mathcal{L} > 0$ are locally uniform in a: For fixed $E < E_{\mathrm{sp}}$ there exists an initial scale $\mathcal{L}$ and constants C_1, C_2 such that for all $a \in [1/2, 3/2]$, $\mathbb{N} \ni L \geqslant \mathcal{L}$ and $0 < \delta \leqslant 1$

$$\mathbb{P}_a\left(\operatorname{spac}_E(H_L) < \delta\right) \leqslant C_1 L^{2d} \exp\left(-C_2|\log\delta|^{1/4}\right). \tag{6.89}$$

Here we wrote $\mathbb{P}_a$ and considered the operator H_ω within the probability with some abuse of notation. This notation is also used in the following proof.

Proof of Theorem 6.12. For fixed $E_1 \in (0, E_{\mathrm{sp}})$ we denote by $C_1, C_2, \mathcal{L}$ the constants from the remark above. Let $E_2 < E_1$ be fixed and possibly enlarge $\mathcal{L}$ such that $\mathcal{L}^{-d} \leqslant E_1 - E_2$ holds. We now prove the Minami-type estimate for intervals $J \subset [0, E_2]$ with $J =: [E_J - \delta, E_J + \delta]$ for suitable $E_J < E_2$ and $\delta = \delta_J > 0$. We can without loss of generality assume that $C_2 < d$. This allows us to confine to the case $\delta < \exp(-|\log L|^4)$ since the statement follows from Wegner's estimate in the other case. Our starting point is

$$\mathbb{P}\left(\operatorname{tr}\mathbb{1}_J(H_L) \geqslant 2\right) \leqslant \sum_{j=1}^{(RL)^d} \mathbb{P}\left(\operatorname{spac}_{E_2}(H_L) < 2\delta,\ \lambda^L_j \in J\right). \tag{6.90}$$

In the sequel each summand on the right hand side is estimated separately. Let's first introduce some notation: After possibly another enlargement of $\mathcal{L}$ we have $\mathcal{L}^d \exp(-|\log \mathcal{L}|^4) \leqslant 1/4$. Together with the constraint on δ above this implies that there exists $N \in \mathbb{N}$ such that $(2L^d\delta)^{-1} - 1 < N \leqslant (2L^d\delta)^{-1}$. We define

$$J_i := J + (i-1)2\delta \quad \text{for } i \in \{1, ..., N\} \tag{6.91}$$

and for $j \in \mathbb{N}$, $i \in \{1, ..., N\}$ and $\mathcal{E} \in \mathbb{R}$

$$\Omega_{i,j}^{\mathcal{E}} := \big\{ \operatorname{spac}_{\mathcal{E}}(H_{\omega,L}) < 2\delta \big\} \cap \big\{ \lambda_{\omega,j}^L \in J_i \big\}. \tag{6.92}$$

Note that the event on the right hand side of (6.90) is $\Omega_{1,j}^{E_2}$. Due to the specific choice of the random potential a simultaneous shift of the random couplings $(\omega_j)_{j\in\Gamma_L}$ results in an energy shift. In general, if the random potential is not 'plain' in the sense that $V_- = V_+$ the following argument has to be performed for deformed auxiliary operators, see the proof of Theorem 6.2 below. In formulas this reads as follows: If we denote $\boldsymbol{\tau} = (\tau, ..., \tau) \in \Gamma_L$ for fixed $\tau \in \mathbb{R}$, then

$$H_{\omega+\boldsymbol{\tau},L} = H_{\omega,L} + \tau \tag{6.93}$$

as operators on $\ell^2(\Lambda_L)$. This implies that

$$\operatorname{spac}_K(H_{\omega,L}) = \operatorname{spac}_{K+\tau}(H_{\omega+\boldsymbol{\tau},L}) \tag{6.94}$$

for any interval $K \subset \mathbb{R}$. Let $\eta_i := (i-1)2\delta$ for $i = 1, ..., N$. The change of variables $(\omega_k)_{k\in\Gamma_L} \to (\omega_k + \eta_i)_{k\in\Gamma_L}$ and (6.94) give

$$\begin{aligned}
\mathbb{P}\left(\Omega_{1,j}^{E_2}\right) &= \int_{[\eta_i, 1+\eta_i]^{\Gamma_L}} \mathbb{1}_{\Omega_{i,j}^{E_2+\eta_i}}(\omega) \prod_{k\in\Gamma_L} d\omega_k \\
&\leqslant (1+L^{-d})^{|\Gamma_L|} \int_{[0,1+L^{-d}]^{\Gamma_L}} \mathbb{1}_{\Omega_{i,j}^{E_2+L^{-d}}}(\omega) \prod_{k\in\Gamma_L} \frac{d\omega_k}{1+L^{-d}} \\
&\leqslant (1+L^{-d})^{L^d}\, \mathbb{P}_{1+L^{-d}}\left(\Omega_{i,j}^{E_1}\right),
\end{aligned} \tag{6.95}$$

where we also used $\eta_i \leqslant 2N\delta \leqslant L^{-d} \leqslant \mathcal{L}^{-d} \leqslant E_1 - E_2$. Summation of (6.95) over $i = 1, ..., N$ yields

$$\mathbb{P}\left(\Omega_{1,j}^{E_2}\right) \leqslant C_3\, 4L^d\delta\, \mathbb{P}_{1+L^{-d}}\left(\operatorname{spac}_{E_1}(H_L) < 2\delta\right), \tag{6.96}$$

where we used that $N^{-1} \leqslant 4L^d\delta$ and that for $i_1 \neq i_2$

$$\mathbb{P}\big(\big\{\lambda_{\omega,j}^L \in J_{i_1}\big\} \cap \big\{\lambda_{\omega,j}^L \in J_{i_2}\big\}\big) = 0. \tag{6.97}$$

For the probability on the right hand side of (6.96) we utilize the remark right before the proof's beginning to estimate

$$\mathbb{P}_{1+L^{-d}}\left(\operatorname{spac}_{E_1}(H_L) < 2\delta\right) \leqslant C_1 L^{2d} \exp\left(-C_2|\log(2\delta)|^{1/4}\right). \tag{6.98}$$

Combining (6.96) and (6.98) yields constants C_4, C_5 such that

$$\mathbb{P}\left(\Omega_{1,j}^{E_2}\right) \leqslant C_4 L^{3d}\delta \exp\left(-C_5|\log\delta|^{1/4}\right) \tag{6.99}$$

Inserting (6.99) into (6.90) finally yields

$$\mathbb{P}\left(\operatorname{tr} \mathbb{1}_J(H_L) \geqslant 2\right) \leqslant C_5 L^{4d}\delta \exp\left(-C_5|\log\delta|^{1/4}\right). \tag{6.100}$$

□

6.5. Proof of the level-spacing estimate

The main objective of this section is to prove Theorem 6.1. But in the proof of Theorem 6.2 we have to apply the level-spacing estimate from Theorem 6.1 for the auxiliary operators $\widetilde{H}_\omega^E$ described in Section 6.2. In order to prove Theorem 6.1 and simultaneously establish the same level-spacing estimate for the auxiliary operators, we prove a variant of Theorem 6.1 for deformed random Schrödinger operators. More precisely, we work with the following model in this section: By H_ω we denote

$$H_\omega := -\mu G \Delta G + V_{\text{per}} + V_\omega, \tag{6.101}$$

were G, V_{per} are real-valued, bounded and $\mathbb{Z}^d$-periodic potentials and $V_\omega = \sum_{k\in\mathbb{Z}^d} \omega_k V_k$ is as introduced in Section 2.1. In particular, the properties (V_1)–(V_3) from Section 2.1 and (V_4) from Section 6.1 still hold. Moreover, we assume that G satisfies $G_- \leqslant G \leqslant G_+$ with constants $G_-, G_+ \in (0,\infty)$.

The section is arranged as follows: We start with some elementary properties of deformed random Schrödinger operators. Subsequently, we prove a probabilistic bound on the typical size of eigenvalue clusters and the existence of good configurations for such clusters. In the last subsection we then prove a version of Theorem 6.1 for deformed random Schrödinger operators. Parts of this section are very similar to the content from Section 6.4. In order to keep the present section self contained we nevertheless reintroduce notation and include proofs even if they are already contained in Section 6.4.

6.5.1. Properties of deformed Schrödinger operators. In this preliminary section we establish two basic properties of the deformed random Schrödinger operator H_ω defined above: An a priori trace bound and a Wegner estimate. Both of them are proven by relating the spectrum of the deformed random Schrödinger operator to the spectrum of an auxiliary standard random Schrödinger operator via the following lemma.

Lemma 6.16. *Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$, let S be an invertible contraction on $\mathcal{H}$ (i.e. $\|S\| \leqslant 1$), and let $C_\varepsilon(A) := \operatorname{tr} \mathbb{1}_{[-\varepsilon,\varepsilon]}(A)$. Then we have*

$$C_\varepsilon(A) \leqslant C_\varepsilon(SAS^*). \tag{6.102}$$

Proof. Consider $B := \mathbb{1}_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)}(A)A$. Then $C_0(B) = C_\varepsilon(A)$ and, by the invertibility of S, we have $C_0(SBS^*) = C_0(B)$. But

$$SAS^* = SBS^* + S\mathbb{1}_{[-\varepsilon,\varepsilon]}(A)AS^* \quad \text{and} \quad \left\|S\mathbb{1}_{[-\varepsilon,\varepsilon]}(A)AS^*\right\| \leqslant \varepsilon.$$

This yields that the eigenvalues of SAS^* and SBS^* differ by at most ε, which in turn implies that

$$C_0(SBS^*) \leqslant C_\varepsilon(SAS^*). \tag{6.103}$$

□

Lemma 6.17 (Trace bound)**.** *For every $E < \infty$ there exists a constant C such that we have for (almost) every ω and $L > 0$*

$$\operatorname{tr} \mathbb{1}_{(-\infty,E]} (H_{\omega,L}) \leqslant CL^d. \tag{6.104}$$

Proof. With the constant $c := \operatorname{ess\,inf}_{x\in\mathbb{R}^d} V_{\mathrm{per}}(x)$ we have

$$H_{\omega,L} \geqslant -\mu G\Delta_L G - c.$$

Hence by the min-max principle

$$\operatorname{tr} \mathbb{1}_{(-\infty,E]} (H_{\omega,L}) \leqslant \operatorname{tr} \mathbb{1}_{(-\infty,E+c]} (-\mu G\Delta_L G) = \operatorname{tr} \mathbb{1}_{[-\kappa,\kappa]} (-\mu U\Delta_L U^*)$$

for $E < \infty$, where $U = U^* := G_-^{-1}G$ and $\kappa := (E + c)\, G_-^{-2}$. Since $S := U^{-1}$ satisfies $\|S\| \leqslant 1$, we conclude from Lemma 6.16 that

$$\operatorname{tr} \mathbb{1}_{(-\infty,E]} (H_{\omega,L}) \leqslant \operatorname{tr} \mathbb{1}_{[-\kappa,\kappa]} (-\mu\Delta_L) \leqslant C_1 L^d$$

for a constant $C_1 = C_{1,\mu,E}$, where the latter bound is a special case of (3.11). □

Lemma 6.18 (Wegner estimate)**.** *For every $E > 0$ there exists C such that for all intervals $I \subset (-\infty, E]$*

$$\mathbb{P}\,(\operatorname{tr} \mathbb{1}_I(H_L) \geqslant 1) \leqslant CL^d\, |I|\,. \tag{6.105}$$

Proof. Let $I = \mathcal{E}+[-\delta, \delta]$ for suitable $\mathcal{E} < E$ and $\delta > 0$. With Lemma 6.16 and $S := G_- G^{-1}$ we have

$$\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) = \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(H_{\omega,L} - \mathcal{E}) \leqslant \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(S\,(H_{\omega,L} - \mathcal{E})\,S^*). \tag{6.106}$$

If we introduce the auxiliary periodic potential $\widetilde{V}_{\mathrm{per},\mathcal{E}} := G_-^2 G^{-2}(V_{\mathrm{per}} - \mathcal{E})$ and the random potential $\widetilde{V}_\omega := G_-^2 G^{-2} V_\omega$, then

$$\widetilde{H}_{\omega,L} := S\,(H_{\omega,L} - \mathcal{E})\,S^* = -\mu G_-^2 \Delta + \widetilde{V}_{\mathrm{per},\mathcal{E}} + \widetilde{V}_\omega$$

is a standard $\mathbb{Z}^d$-ergodic random Schrödinger operator for which the Wegner estimate is known. The statement follows since the constant for Wegner's estimate at energy zero can be chosen to be stable in the norm of the periodic background potential. This can for instance be seen from [28, Theorem 2.4]. As mentioned in the proof of Theorem 3.1 (iv) and also Theorem 2.5 below, the proof from [28] extends to Dirichlet boundary conditions. □

6.5.2. The size of typical eigenvalue clusters. For the polymer model, and lattice models in general, we've seen in Section 6.4 that probabilistic bounds on the typical size of eigenvalue clusters follow from a direct adaption of the proof of Minami's estimate for the lattice Anderson model. See also [27, 61]. The following Lemma extends this idea.

Lemma 6.19 (Typical cluster size)**.** *For fixed $E > 0$ and $\theta, \vartheta \in (0, 1)$ there exist constants $c = c_{\theta,E}, C = C_{\vartheta,E} > 0$ such that*

$$\mathbb{P}\big(\operatorname{tr} \mathbb{1}_I(H_L) > c|I|^{-\theta}\big) \leqslant CL^{2d}|I|^{2-\vartheta} \tag{6.107}$$

holds for all intervals $I \subset (-\infty, E]$.

Remark 6.20. A slight adaption of the proof shows the following. For fixed $\vartheta \in (0,1)$, $\gamma \in [1,\infty)$ and $E > 0$ there exists a constant $C = C_{\vartheta,\gamma,E}$ such that for all $M \geqslant 1$

$$\mathbb{P}\left(\operatorname{tr} \mathbb{1}_I(H_L) > M\right) \leqslant CL^{2d}|I|^{1-\vartheta} \max\left\{|I|, M^{-\gamma}\right\} \tag{6.108}$$

holds for all intervals $I \subset (-\infty, E]$.

Proof. We again apply Lemma 6.16 to estimate for a fixed interval $I := \mathcal{E}_0 + [-\delta G_-^{-1}, \delta G_-^{-1}] \subset (-\infty, E]$

$$\operatorname{tr} \mathbb{1}_I(H_{\omega,L}) \leqslant \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}), \tag{6.109}$$

where $\widetilde{H}_\omega := -\mu\Delta + G^{-2}(V_{\text{per}} - \mathcal{E}_0) + G^{-2}V_\omega$. Then (6.109) implies

$$\mathbb{P}\big(\operatorname{tr} \mathbb{1}_I(H_L) > M\big) \leqslant \mathbb{P}\big(\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L) > M\big) \tag{6.110}$$

for any $M > 0$. By $\xi(\mathcal{E}, \widetilde{H}_{\omega,L}^{\omega_k=0}, \widetilde{H}_{\omega,L}^{\omega_k=1}) \geqslant 0$ we denote the the spectral shift function at energy $\mathcal{E}$ of the operators

$$\widetilde{H}_{\omega,L}^{\omega_k=0} := \widetilde{H}_{\omega,L} - \omega_k G^{-2} V_k \quad \text{and} \quad \widetilde{H}_{\omega,L}^{\omega_k=1} := \widetilde{H}_{\omega,L} + (1-\omega_k) G^{-2} V_k. \tag{6.111}$$

For a definition of the spectral shift function and more we refer to Section 3.1. We then define the random variable

$$X_\omega := \sup_{k \in \Gamma_L} \operatorname*{ess\,inf}_{\mathcal{E} \in [-\delta,\delta]} \xi(\mathcal{E}, \widetilde{H}_{\omega,L}^{\omega_k=0}, \widetilde{H}_{\omega,L}^{\omega_k=1}) \geqslant 0, \tag{6.112}$$

where $\Gamma_L = \Lambda_{L+2R} \cap \mathbb{Z}^d$ as usual. Because X_ω is (almost surely) integer valued, we have

$$\begin{aligned} &\mathbb{P}\big(\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L) > X\big) \\ &\qquad \leqslant \mathbb{E}\big[\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L)(\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L) - X) \mathbb{1}_{\{\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L}) > X\}}\big]. \end{aligned} \tag{6.113}$$

Omitting the ω, L-subscripts for the moment, we get for $\mathcal{E} \in [-\delta,\delta]$ and $k \in \Gamma_L$

$$\begin{aligned} \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}) &= \operatorname{tr}\big(\mathbb{1}_{(-\infty,\delta]}(\widetilde{H}) - \mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H})\big) + \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}) - \mathbb{1}_{(-\infty,-\delta]}(\widetilde{H})\big) \\ &\leqslant \operatorname{tr}\big(\mathbb{1}_{(-\infty,\delta]}(\widetilde{H}^{\omega_k=0}) - \mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}^{\omega_k=0})\big) \\ &\quad + \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}^{\omega_k=0}) - \mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H})\big) \\ &\quad + \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}^{\omega_k=1}) - \mathbb{1}_{(-\infty,-\delta]}(\widetilde{H}^{\omega_k=1})\big) \\ &\quad + \operatorname{tr}\big(\mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}) - \mathbb{1}_{(-\infty,\mathcal{E}]}(\widetilde{H}^{\omega_k=1})\big) \\ &\leqslant \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_k=0}) + \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_k=1}) \\ &\quad + \xi(\mathcal{E}, \widetilde{H}^{\omega_k=0}, \widetilde{H}^{\omega_k=1}). \end{aligned} \tag{6.114}$$

Since the inequality holds for all $\mathcal{E} \in [-\delta,\delta]$ we obtain

$$\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}) \leqslant \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_k=0}) + \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\omega_k=1}) + X. \tag{6.115}$$

Next we use (6.115) to estimate (6.113). We first note that for a constant C_1 the Wegner estimate

$$\mathbb{E}\big[\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L^{\omega_k=1})\big] \leqslant C_1 L^d \delta, \tag{6.116}$$

holds, for instance via [28] or [80]. We refer to the proof of Theorem 3.3 (iv) for details. With (6.116) at hand we obtain a constant $C = C_\vartheta$ such that

$$
\begin{aligned}
(6.113) &\leqslant V_- G_+^2 \sum_{k\in\Gamma_L} \mathbb{E}\Big[\operatorname{tr}\big(G^{-2} V_k \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L)\big) \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L^{\omega_k=0})\Big] \\
&\quad + V_- G_+^2 \sum_{k\in\Gamma_L} \mathbb{E}\Big[\operatorname{tr}\big(G^{-2} V_x \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L)\big) \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_L^{\omega_k=1})\Big] \\
&\leqslant C (2\delta)^{2-\vartheta} L^{2d}.
\end{aligned} \tag{6.117}
$$

The last inequality can be seen as follows. First, the Birman-Solomyak formula [19] yields

$$
\int_{[0,1]} d\omega_k \operatorname{tr}\big(G^{-2} V_k \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}_{\omega,L})\big) = \int_{[-\delta,\delta]} d\mathcal{E}\, \xi(\mathcal{E}, \widetilde{H}_{\omega,L}^{\omega_k=0}, \widetilde{H}_{\omega,L}^{\omega_k=1}). \tag{6.118}
$$

The estimate then follows from the local L^p-boundedness of the spectral shift function as a function in energy [31], applied for $p = \vartheta^{-1}$.

We finish the argument by proving the (almost sure) upper bound $X_\omega \leqslant c|I|^{-\theta}$, where $c = c_\theta$ does not depend on ω. After estimating X_ω via Hölder's inequality as

$$
\begin{aligned}
X_\omega &\leqslant \sup_{k\in\Gamma_L} \frac{1}{2\delta} \int_{[-\delta,\delta]} d\mathcal{E}\, \xi(\mathcal{E}, \widetilde{H}_{\omega,L}^{\omega_k=0}, \widetilde{H}_{\omega,L}^{\omega_k=1}) \\
&\leqslant \sup_{k\in\Gamma_L} (2\delta)^{-\theta} \Big(\int_{[-\delta,\delta]} d\mathcal{E}\, \xi(\mathcal{E}, \widetilde{H}_{\omega,L}^{\omega_k=0}, \widetilde{H}_{\omega,L}^{\omega_k=1})^{1/\theta} \Big)^{\theta}
\end{aligned} \tag{6.119}
$$

we can again apply the local L^p-boundedness of the spectral shift function, this time for $p = 1/\theta$, to obtain $X_\omega \leqslant c|I|^{-\theta}$ for a suitable constant $c = c_\theta$. ☐

6.5.3. Existence of good configurations. For $L > 0$ and $n \in \mathbb{N}$ we define

$$
\xi_{L,n} := \frac{\mu \pi^2 G_-^2}{(2R)^2 (2R+1)^d v_+} \Big(V_- - 23\sqrt{n} L^{-d-1} \Big) - \|V_{\mathrm{per}}\|. \tag{6.120}
$$

Moreover we again set $\lambda_{\omega,-1}^L := -\infty$ for notational convenience.

Lemma 6.21. *Let $0 < \varepsilon < 1/12$, $L \geqslant 1$, $2 \leqslant n \in \mathbb{N}$ and $\omega_0 \in \Omega_L = [0,1]^{\Gamma_L}$ such that $Q_\varepsilon(\omega_0) \subset \Omega_L$ and assume that there exists $m \in \mathbb{N}$ such that*

(i) $\lambda_{\omega_0,m+n}^L \leqslant \xi_{L,n}$.

(ii) The cluster $\mathcal{C}_{\omega_0,m}^n := \lambda_{\omega_0,m+1}^L, \ldots, \lambda_{\omega_0,m+n}^L$ is isolated from the rest of the spectrum:

$$
\min\big\{\lambda_{\omega_0,m+1}^L - \lambda_{\omega_0,m}^L, \lambda_{\omega_0,m+n+1}^L - \lambda_{\omega_0,m+n}^L\big\} \geqslant 6\varepsilon. \tag{6.121}
$$

Then there exists $\widehat{\omega} \in Q_\varepsilon(\omega_0)$ such that

$$
\min_{i=1,\ldots,n-1} |\lambda_{\widehat{\omega},m+i+1}^L - \lambda_{\widehat{\omega},m+i}^L| \geqslant 6\varepsilon L^{-(n-1)(4d+2)}. \tag{6.122}
$$

The lemma follows from the same iteration procedure as in Section 6.4.

Lemma 6.22. *Assume that the assumptions of Lemma* 6.21 *are satisfied. Then there exists* $\widehat{\omega} \in Q_{\varepsilon-\varepsilon L^{-(2d+2)}}(\omega_0)$ *and* $1 \leqslant k \leqslant n-1$ *such that*

$$\lambda^L_{\widehat{\omega},m+k+1} - \lambda^L_{\widehat{\omega},m+k} \geqslant 6\varepsilon L^{-(4d+2)}. \tag{6.123}$$

PROOF OF LEMMA 6.22. We closely stick to the proof of Lemma 6.15 in reasoning and notation. For $i = 1, ..., n$ we again abbreviate

$$E^\omega_i := \lambda^L_{\omega,m+i} \tag{6.124}$$

and $\bar{E}^\omega := n^{-1}\sum_{i=1}^n E^\omega_i$. The first part of the argument is verbatimly the same as in the proof of Lemma 6.15. It is included here to keep the proof for the continuum case self contained. If we set $I := [E^{\omega_0}_1, E^{\omega_0}_n]$ then the min-max characterization of eigenvalues implies that

$$\operatorname{tr} \mathbb{1}_{I_\varepsilon}(H_{\omega,L}) = n \tag{6.125}$$

for all $\omega \in Q_\varepsilon(\omega_0)$, where $I_\varepsilon := I + [-\varepsilon, \varepsilon]$. Hence the eigenvalues $E^\omega_1, ..., E^\omega_n$ form a cluster of isolated eigenvalues as considered in Section 6.3. If we prove that

$$\max_{\omega\in Q_\varepsilon(\omega_0)} \max_{i=1,...,n} |E^\omega_i - \bar{E}^\omega| > 8n\varepsilon L^{-(4d+2)},$$

then there exists $\omega' \in Q_\varepsilon(\omega_0)$ such that $E^{\omega'}_n - E^{\omega'}_1 \geqslant 8n\varepsilon L^{-(4d+2)}$. Hence there exists $j \in \{1, ..., n-1\}$ such that $E^{\omega'}_{j+1} - E^{\omega'}_j > 8\varepsilon L^{-(4d+2)}$. If $\omega' \in Q_{\varepsilon-\varepsilon L^{-(4d+2)}}(\omega_0)$ then we can set $\widehat{\omega} := \omega'$ and the claim holds. If $\omega' \notin Q_{\varepsilon-\varepsilon L^{-(4d+2)}}(\omega_0)$ then we can choose $\widehat{\omega} \in Q_{\varepsilon-\varepsilon L^{-(4d+2)}}(\omega_0)$ such that $|\widehat{\omega} - \omega'| \leqslant \varepsilon L^{-(4d+2)}$ and the claim follows from Weyl's theorem on the movement of eigenvalues. Let us assume, for contradiction, that

$$\max_{\omega\in Q_\varepsilon(\omega_0)} \max_{i=1,...,n} |E^\omega_i - \bar{E}^\omega| \leqslant 8n\varepsilon L^{-(4d+2)}. \tag{6.126}$$

For fixed $k \in \Gamma_L$ Lemma 6.7 can be applied for ε and $\delta = 8n\varepsilon L^{-(4d+2)}$ to the one-parameter operator family

$$(-\varepsilon, \varepsilon) \ni s \mapsto H_{\omega_0,L} + sV^L_k, \tag{6.127}$$

where as usual V^L_k is the restriction of V_k to $L^2(\Lambda_L)$. With the interval I_ε defined above we denote $P_\omega := \mathbb{1}_{I_\varepsilon}(H_{\omega,L})$ for $\omega \in Q_\varepsilon(\omega_0)$. Let

$$\alpha_k := (\partial_s \bar{E}^s)(0) = (\partial_{\omega_k}\bar{E})(\omega_0) = \frac{1}{n} \operatorname{tr} P_{\omega_0} V^L_k \geqslant 0, \tag{6.128}$$

where in the second step we have used the Hellmann-Feynman theorem. Evaluation of (6.23) at $s = 0$ then yields the bound

$$\left\| P_{\omega_0} \left(V^L_k - \alpha_k\right) P_{\omega_0} \right\| \leqslant 23\sqrt{n} L^{-2d-1} \tag{6.129}$$

for every $k \in \Gamma_L$. From now on the proof slightly deviates from the proof of Lemma 6.14. We next decompose Γ_L into disjoint subsets $(U_t)_{t\in\mathcal{T}}$ such that $|k - l| > 2R$ holds for $k, k' \in U_t$, $k \neq k'$, and such that $|\mathcal{T}| \leqslant (2R+1)^d$. For the sets $\Lambda^L_{2R}(k) := \Lambda_{2R}(k) \cap \Lambda_L$, $k \in \Gamma_L$, we then have $\Lambda^L_{2R}(k) \cap \Lambda^L_{2R}(k') = \varnothing$ for $k, k' \in U_t$ with $k \neq k'$. For fixed $t \in \mathcal{T}$ Neumann decoupling yields

$$\operatorname{tr}\left(P_{\omega_0} H_{\omega_0,L}\right) \geqslant \sum_{k\in U_t} \operatorname{tr}\Big(P_{\omega_0} G\big(-\mu\Delta^{(N)}_{\Lambda^L_{2R}(k)}\big)G\Big) - n\|V_{\mathrm{per}}\|, \tag{6.130}$$

where we also used that $\omega_{1,k}V_k \geqslant 0$ for all $k \in \Gamma_L$. After summing (6.130) over $t \in \mathcal{T}$, we obtain

$$\operatorname{tr}\big(P_{\omega_0} H_{\omega_0,L}\big) \geqslant (2R+1)^{-d} \sum_{k\in\Gamma_L} \operatorname{tr}\Big(P_{\omega_0} G\big(-\mu\Delta^{(N)}_{\Lambda^L_{2R}(k)}\big)G\Big) - n\|V_{\mathrm{per}}\|. \tag{6.131}$$

Since $\Lambda^L_{2R}(k)$ is a hyperrectangle with side-length bounded by $2R$, we have

$$-\Delta^{(N)}_{\Lambda^L_{2R}(k)} \geqslant \frac{\pi^2}{(2R)^2} R_k, \tag{6.132}$$

where R_k is the projection onto $\operatorname{ran}(\Delta^{(N)}_{\Lambda^L_{2R}(k)})$. With the shorthand notation

$$C_{\omega_0,k} := G\mathcal{X}_{\Lambda^L_{2R}(k)} P_{\omega_0} \mathcal{X}_{\Lambda^L_{2R}(k)} G$$

we conclude that

$$(6.131) \geqslant \frac{\mu\pi^2}{(2R)^2(2R+1)^d} \sum_{k\in\Gamma_L} \operatorname{tr}\big(C_{\omega_0,k}R_k\big) - n\|V_{\mathrm{per}}\|. \tag{6.133}$$

Next, we bound the trace on the right hand side as

$$\operatorname{tr}\big(C_{\omega_0,k}R_k\big) = \operatorname{tr} C_{\omega_0,k} - \operatorname{tr}\big(C_{\omega_0,k}(\mathcal{X}_{\Lambda^L_{2R}(k)} - R_k)\big) \geqslant \operatorname{tr} C_{\omega_0,k} - \|C_{\omega_0,k}\| = {\sum}' \nu_j, \tag{6.134}$$

where $(\nu_j)_j$ are the eigenvalues of $C_{\omega_0,k}$ counted with multiplicity and $\sum'$ stands for the sum of all but the largest eigenvalue of $C_{\omega_0,k}$. Here we also used that $\operatorname{rank}(\mathcal{X}_{\Lambda^L_{2R}(k)} - R_k) = 1$. Since $\sigma(C_{\omega_0,k}) \setminus \{0\} = \sigma(P_{\omega_0}\mathcal{X}_{\Lambda^L_{2R}(k)} G^2 \mathcal{X}_{\Lambda^L_{2R}(k)} P_{\omega_0}) \setminus \{0\}$ and since

$$P_{\omega_0}\mathcal{X}_{\Lambda^L_{2R}(k)} G^2 \mathcal{X}_{\Lambda^L_{2R}(k)} P_{\omega_0} \geqslant \frac{G_-^2}{v_+} P_{\omega_0} V_k^L P_{\omega_0} \geqslant \frac{G_-^2}{v_+}\Big(\alpha_k - 23\sqrt{n}L^{-2d-1}\Big) P_{\omega_0} \tag{6.135}$$

by (6.129), we deduce from the min-max principle that

$$\begin{aligned}\operatorname{tr}\big(C_{\omega_0,k}R_k\big) &\geqslant {\sum}'\nu_j \geqslant \frac{G_-^2}{v_+}\Big(\alpha_k - 23\sqrt{n}L^{-2d-1}\Big)(\operatorname{tr} P_{\omega_0} - 1)\\ &= \frac{(n-1)G_-^2}{v_+}\Big(\alpha_k - 23\sqrt{n}L^{-2d-1}\Big).\end{aligned} \tag{6.136}$$

After estimating (6.133) by (6.136) we arrive at

$$\operatorname{tr}\big(P_{\omega_0} H_{\omega_0,L}\big) \geqslant \frac{\mu\pi^2 G_-^2(n-1)}{(2R)^2(2R+1)^d v_+} \sum_{k\in\Gamma_L}\Big(\alpha_k - 23\sqrt{n}L^{-2d-1}\Big) - n\|V_{\mathrm{per}}\|. \tag{6.137}$$

Moreover (6.128) yields

$$\sum_{k\in\Gamma_L} \alpha_k = \frac{1}{n}\sum_{k\in\Gamma_L} \operatorname{tr}\big(P_{\omega_0} V_k^L\big) \geqslant V_-. \tag{6.138}$$

By putting all bounds together we obtain

$$\begin{aligned}\bar{E}^{\omega_0} = \frac{1}{n}\operatorname{tr}\big(P_{\omega_0} H_{\omega_0,L}\big) &\geqslant \frac{\mu\pi^2 G_-^2}{(2R)^2(2R+1)^d v_+}\Big(V_- - 23\sqrt{n}L^{-d-1}\Big) - \|V_{\mathrm{per}}\|\\ &= \xi_{L,n}.\end{aligned} \tag{6.139}$$

□

Proof of Lemma 6.21. The proof is literally the same as the proof of Lemma 6.14. The argument is included here for convenience. We use the notation introduced in the proof of Lemma 6.22. First, we directly apply Lemma 6.22 to the cluster $E_1^{\omega_0}, ..., E_n^{\omega_0}$ and the set $Q_0 := Q_\varepsilon(\omega_0)$ in configuration space. Hence there exists $\omega_1 \in Q_1 := Q_{\varepsilon - \varepsilon L^{-(4d+2)}}(\omega_0)$ and $1 \leqslant k_1 \leqslant n-1$ such that

$$E_{k_1+1}^{\omega_1} - E_{k_1}^{\omega_1} > 6\varepsilon L^{-(4d+2)}. \tag{6.140}$$

If $k_1 = 1$ or $k_1 = n-1$ then we isolated one eigenvalue from the rest of the eigenvalues and only proceed with one cluster of eigenvalues. In the other cases we obtain two sets of eigenvalues $E_1^{\omega_1} \leqslant ... \leqslant E_{k_1}^{\omega_1}$ and $E_{k_1+1}^{\omega_1} \leqslant ... \leqslant E_n^{\omega_1}$ which both satisfy (6.121) for $\varepsilon_1 := \varepsilon L^{-(4d+2)}$. We apply Lemma 6.22 for $\varepsilon = \varepsilon_1$ to the cluster of eigenvalues $E_1^{\omega_1}, ..., E_{k_1}^{\omega_1}$. This yields $\omega_2 \in Q_2 := Q_{\varepsilon_1 - \varepsilon_1 L^{-(4d+2)}}(\omega_1)$ and $1 \leqslant k_2 \leqslant k_1 - 1$ such that

$$E_{k_2+1}^{\omega_2} - E_{k_2}^{\omega_2} > 6\varepsilon_1 L^{-(4d+2)}. \tag{6.141}$$

Set $\varepsilon_2 := \varepsilon_1 L^{-(4d+2)}$. Then, since $|\omega_2 - \omega_1| \leqslant \varepsilon_1 - \varepsilon_2$ we have

$$E_{k_1+1}^{\omega_2} - E_{k_1}^{\omega_2} > 6\varepsilon_1 - 2(\varepsilon_1 - \varepsilon_2) \geqslant 6\varepsilon_2 \tag{6.142}$$

by Weyl's theorem on the movement of eigenvalues and we can apply Lemma 6.22 to the set $E_{k_1+1}^{\omega_{0,2}}, ..., E_n^{\omega_{0,2}}$ of eigenvalues. Overall we found $\omega_3 \in Q_3 := Q_{\varepsilon_2 - \varepsilon_2 L^{-(4d+2)}}(\omega_2)$ and up to four clusters of eigenvalues which are separated from each other (and the rest of the spectrum of $H_{\omega,L}$) by $6\varepsilon_3 := 6\varepsilon_2 L^{-(4d+2)}$. We repeat this procedure at most $n-1$ times until each cluster consists of exactly one eigenvalue. ☐

6.5.4. Proof of Theorem 2.4. Next is this section's main result, which for $G = \mathcal{X}_{\mathbb{R}^d} = \mathrm{id}_{L^2(\mathbb{R}^d)}$ gives Theorem 6.1. Let

$$E_{\mathrm{sp}} := E_{\mathrm{sp},G,V_{\mathrm{per}}} := \frac{\mu\pi^2 G_-^2 V_-}{(2R)^2(2R+1)^d v_+} - \|V_{\mathrm{per}}\|. \tag{6.143}$$

Theorem 6.23. *For fixed $E \in (0, E_{\mathrm{sp}})$ with E_{sp} as defined in* (6.143) *and $K > 0$ there exist constants $C, \mathcal{L}$ such that*

$$\mathbb{P}(\mathrm{spac}_E(H_L) < \delta) \leqslant C L^{2d} |\log \delta|^{-K} \tag{6.144}$$

holds for $L \geqslant \mathcal{L}$ and $0 < \delta < 1$.

In order to extract the Minami-type estimate, Theorem 6.2, at energy E from (6.144) we have to apply Theorem 6.23 multiple times for the E-dependent periodic potential $V_{\mathrm{per}} = EV^{-1}$, where $V := \sum_{k\in\mathbb{Z}^d} V_k$, and for a set of slightly varying L-dependent coupling constants μ_L. This is why we will occasionally comment in the sequel on the stability of constants in terms of V_{per} and μ variables.

Proof of Theorem 6.23. Let $E \in (0, E_{\mathrm{sp}})$ and $0 < \kappa < E_{\mathrm{sp}}$ be fixed. We first decompose the interval $[-\|V_{\mathrm{per}}\|, E]$ into a family $(K_i)_{i\in\mathcal{I}}$ of intervals of side length $|K_i| = \kappa < E_{\mathrm{sp}}$, with $|K_{i+1} \cap K_i| \geqslant \kappa/2$, and such that $|\mathcal{I}| \leqslant 4(E_{\mathrm{sp}} + \|V_{\mathrm{per}}\|)\kappa^{-1} + 1$. Let $i \in \mathcal{I}$ and define $K_{i,\eta} := K_i + [-\eta, \eta]$ for $\eta > 0$. Let $\theta \in (0,1)$. Then the probability of the event

$$\Omega_{i,\varepsilon} := \left\{ \mathrm{tr}\, \mathbb{1}_{K_i}(H_{\omega,L}) \leqslant c_\theta |K_i|^{-\theta} \text{ and } \mathrm{tr}\, \mathbb{1}_{K_{i,8\varepsilon}\setminus K_i}(H_{\omega,L}) = 0 \right\} \tag{6.145}$$

can be estimated by the Wegner estimate from Lemma 6.18 and the bound on typical clusters from Lemma 6.19 (with $\vartheta = 1/2$) as

$$\mathbb{P}\left(\Omega_{i,\varepsilon}\right) \geqslant 1 - C_1 L^d \varepsilon - C_2 L^{2d} \kappa^{3/2}. \tag{6.146}$$

Below we again decompose the configuration space Ω_L into small cubes, but have to be slightly more careful than in Lemma 6.11 due to the presence of a non-trivial single-site probability density ρ. Let $0 < \varepsilon < 1/12$ be fixed and $M \in \mathbb{N}$ be such that $2\varepsilon M \leqslant 1 < (M+1)2\varepsilon$. We define $\Omega'_L := [0, 2\varepsilon M]^{\Gamma_L}$ (interpreted as an event in Ω_L) and note that $\mathbb{P}(\Omega'_L) \geqslant 1 - C_3 \varepsilon L^d$ for a constant $C_3 = C_{3,\rho_+}$. For $0 < \delta < \kappa/2$ this yields

$$\begin{aligned}&\mathbb{P}\left(\operatorname{spac}_E(H_L) < \delta\right) \qquad (6.147)\\ &\leqslant \sum_{i\in\mathcal{I}} \mathbb{P}\left(\left\{\operatorname{spac}_{K_i}(H_{\omega,L}) < \delta\right\} \cap \Omega_{i,\varepsilon} \cap \Omega'_L\right) + C_1|\mathcal{I}|L^d\varepsilon + C_2|\mathcal{I}|L^{2d}\kappa^{3/2} + C_3 L^d \varepsilon.\end{aligned}$$

We next partition the event Ω'_L into disjoint cubes Q_j, $j \in \mathcal{J}$ of side length 2ε, $|Q_j| = (2\varepsilon)^{|\Gamma_L|}$, such that $|\mathcal{J}| = M^{|\Gamma_L|} \leqslant (2\varepsilon)^{-|\Gamma_L|}$. In contrast to the situation in Lemma 6.11 the cubes Q_j, $j \in \mathcal{J}$, here are disjoint and cover all of Ω'_L.

Now, fix $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that $Q_j \cap \Omega_{i,\varepsilon} \neq \varnothing$, and let $\omega_{i,j} \in Q_j \cap \Omega_{i,\varepsilon}$. If we denote the center of Q_j by $\omega_{0,j}$ then

$$n_{i,j} := \operatorname{tr} \mathbb{1}_{K_{i,\varepsilon}}(H_{\omega_{0,j},L}) \leqslant c_\theta \kappa^{-\theta} \quad \text{and} \quad \operatorname{dist}\left(K_{i,\varepsilon}, \sigma(H_{\omega_{0,j},L}) \setminus K_{i,\varepsilon}\right) \geqslant 6\varepsilon. \tag{6.148}$$

This follows from $|\omega_{0,j} - \omega_{i,j}| \leqslant \varepsilon$ and the min-max characterization of the eigenvalues of $H_{\omega,L}$. Let $\mathcal{L}$ sufficiently large such that for $L \geqslant \mathcal{L}$ we have $E < \xi_{L,L^d}$. Then Lemma 6.21 is applicable and yields $\widehat{\omega}_{i,j} \in Q_j$ such that

$$\operatorname{spac}_{K_{i,\varepsilon}}(H_{\widehat{\omega}_{i,j},L}) \geqslant 6\varepsilon L^{-(n_{i,j}-1)(3d+2)}. \tag{6.149}$$

This in turn can be used as an input for Lemma 6.10 with $\delta_0 := 6\varepsilon L^{-(n_{i,j}-1)(3d+2)}$. For $Q_j =: \times_{k\in\Gamma_L}[a_{j,k}, b_{j,k}]$ we obtain

$$\begin{aligned}&\mathbb{P}\left(Q_j \cap \left\{\operatorname{spac}_{K_{i,2\varepsilon}}(H_{\omega,L}) < \delta\right\}\right)\\ &\qquad\leqslant \Big(\prod_{k\in\Gamma_L} \sup_{x\in[a_{j,k},b_{j,k}]} \rho(x)\Big)\left|\left\{\omega \in Q_j : \operatorname{spac}_{K_{i,2\varepsilon}}(H_{\omega,L}) < \delta\right\}\right|\\ &\qquad\leqslant C_4 \left(1 + \frac{\mathcal{K}2\varepsilon}{\rho_-}\right)^{|\Gamma_L|} L^d\, \mathbb{P}\left(Q_j\right) \exp\left(\frac{-c'_\theta \kappa^{2\theta}|\log\delta|}{|\log(6\varepsilon)| + c''_\theta \kappa^{-\theta}\log L}\right). \qquad (6.150)\end{aligned}$$

Here we used that $n_{i,j} \leqslant c_\theta \kappa^{-\theta}$ and that ρ satisfies (V_4), which for $k \in \Gamma_L$ gives

$$\sup_{x\in[a_{j,k},b_{j,k}]} \rho(x) \leqslant \inf_{x\in[a_{j,k},b_{j,k}]} \rho(x) + \mathcal{K}2\varepsilon \leqslant \inf_{x\in[a_{j,k},b_{j,k}]} \rho(x)\left(1 + \frac{\mathcal{K}2\varepsilon}{\rho_-}\right). \tag{6.151}$$

The above estimate (6.150) holds for all pairs $i \in \mathcal{I}, j \in \mathcal{J}$ such that $Q_j \cap \Omega_{i,\varepsilon} \neq \varnothing$. So far we assumed that $0 < \varepsilon < 1/12$ and $0 < \delta < \kappa/2 < E_{\mathrm{sp}}/2$. If we set $\mathcal{J}_i := \{j \in \mathcal{J} : Q_j \cap \Omega_{i,\varepsilon} \neq \varnothing\}$

for $i \in \mathcal{I}$, then

$$
\begin{aligned}
(6.147) &\leqslant \sum_{i\in\mathcal{I}}\sum_{j\in\mathcal{J}_i}\mathbb{P}\left(\left\{\operatorname{spac}_{K_i}(H_{\omega,L})<\delta\right\}\cap Q_j\right)+C_1|\mathcal{I}|L^d\varepsilon+C_2|\mathcal{I}|L^{2d}\kappa^{3/2}+C_3L^d\varepsilon\\
&\leqslant C_5L^d\kappa^{-1}\varepsilon+C_6L^{2d}\kappa^{1/2}\\
&\quad+C_4L^d\left(1+\frac{\mathcal{K}2\varepsilon}{\rho_-}\right)^{|\Gamma_L|}(1+4\varepsilon|\Gamma_L|\rho_+)\kappa^{-1}\exp\left(\frac{-c'_\theta\kappa^{2\theta}|\log\delta|}{|\log(6\varepsilon)|+c''_\theta\kappa^{-\theta}\log L}\right).
\end{aligned}
\tag{6.152}
$$

For $0<\delta\leqslant\exp\left(-(\log L)^5\right)$ we now choose

$$
\kappa := |\log\delta|^{-1/(4\theta)} \quad\text{and}\quad \varepsilon:=\exp\left(-|\log\delta|^{1/4}\right). \tag{6.153}
$$

Those choices in particular imply $\delta<\kappa/2$ for sufficiently large L. Because $\varepsilon|\Gamma_L|\leqslant 1$ for sufficiently large L we end up with

$$
\begin{aligned}
\mathbb{P}\left(\operatorname{spac}_E(H_{\omega,L})<\delta\right)&\leqslant C_7L^{2d}|\log\delta|^{-1/(8\theta)}+C_8L^d|\log\delta|^{1/(4\theta)}\exp\left(-\tilde{c}_\theta|\log\delta|^{1/20}\right)\\
&\leqslant C_9L^{2d}|\log\delta|^{-1/(8\theta)}
\end{aligned}
\tag{6.154}
$$

for suitable constants C_7, C_8, C_9 and for $L\geqslant\mathcal{L}$, where $\mathcal{L}$ is sufficiently large. □

6.6. Proof of the Minami-type estimate

Before we start with the proof of Theorem 6.2 we make a preliminary remark. Let $H_\omega^\mu = -\mu\Delta + V_\omega$ be the standard random Schrödinger operator from Section 2.1. Here we stress the dependence on μ in notation because, as mentioned earlier, we'll have to work with L-dependent couplings μ_L in some small neighborhood of a fixed μ. We recall that $V=\sum_{k\in\mathbb{Z}^d}V_k$. The random operator

$$
\widetilde{H}_\omega^{\mu,E} := V^{-1/2}(H_\omega - E)V^{-1/2} = -\mu V^{-1/2}\Delta V^{-1/2}+\widetilde{V}_{\mathrm{per}}^E+\widetilde{V}_\omega \tag{6.155}
$$

is a deformed random Schrödinger operator with deformation $V^{-1/2}$, periodic potential $\widetilde{V}_{\mathrm{per}}^E := -EV^{-1}$ and random potential $\widetilde{V}_\omega := \sum_{k\in\mathbb{Z}^d}\omega_k\widetilde{V}_k$, where $\widetilde{V}_k := V^{-1}V_k$. Tracking constants in Section 6.5 shows the following: For fixed $E_1\in(0,E_{\mathrm{M}})$, with E_{M} as defined in (6.4), and $K>0$ there exists $\varepsilon>0$ and constants $C,\mathcal{L}>0$ such that for all $\mu'\in[\mu-\varepsilon,\mu+\varepsilon]$ and all $E\in[0,E_1]$

$$
\mathbb{P}\left(\operatorname{spac}_{[-\varepsilon,\varepsilon]}(\widetilde{H}_L^{\mu',E})<\delta\right)\leqslant CL^{2d}|\log\delta|^{-K} \tag{6.156}
$$

holds for all $L\geqslant\mathcal{L}$ and $0<\delta<1$.

Proof of Theorem 6.2. We fix $E_1\in(0,E_{\mathrm{M}})$ and $K>0$ and denote by $\varepsilon,\mathcal{L},C$ the constants from above. After possibly enlarging $\mathcal{L}$ we have $\delta\leqslant\mathcal{L}^{-d}\leqslant\varepsilon/2$ and $4\delta L^d\leqslant 1$ for L,δ that satisfy $L\geqslant\mathcal{L}$ and $\delta\leqslant\exp(-(\log L)^{5d})$. If $\delta\geqslant\exp(-(\log L)^{5d})$ then the bound (6.6) follows from Wegner's estimate (after possibly increasing $\mathcal{L}$ suitably).

We now consider a fixed interval $[E-V_-\delta, E+V_-\delta]$ with $E\leqslant E_1$. Moreover, let $L\geqslant\mathcal{L}$ and $0<\delta\leqslant\exp\left(-(\log L)^{5d}\right)$ be fixed. Our starting point is Lemma 6.16, which, applied for

$A = H^{\mu}_{\omega,L} - E$, $S = V_-^{1/2}V^{-1/2}$ and $\varepsilon = \delta V_-$ (for ε as in the lemma, which is not the same ε as in (6.156)), yields

$$\operatorname{tr} \mathbb{1}_{[E-\delta V_-, E+\delta V_-]}(H^{\mu}_{\omega,L}) = \operatorname{tr} \mathbb{1}_{[-\delta V_-, \delta V_-]}(H^{\mu}_{\omega,L} - E) \leqslant \operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\mu,E}_{\omega,L}). \tag{6.157}$$

By $\widetilde{E}^{\mu,E}_{\omega,j}$, $j \in \mathbb{N}$, we denote the eigenvalues of $\widetilde{H}^{\mu,E}_{\omega,L}$ in ascending order. If C_1 denotes the constant from Lemma 6.17, then

$$\mathbb{P}\left(\operatorname{tr} \mathbb{1}_{[-\delta,\delta]}(\widetilde{H}^{\mu,E}_{L}) \geqslant 2\right) \leqslant \sum_{j=1}^{C_1 L^d} \mathbb{P}\left(\operatorname{spac}_{[-\varepsilon/2,\varepsilon/2]}(\widetilde{H}^{\mu,E}_{L}) < 2\delta,\ \widetilde{E}^{\mu,E}_{j} \in [-\delta,\delta]\right), \tag{6.158}$$

where we used that $\delta \leqslant \varepsilon/2$. In the sequel each term on the right hand side is estimated separately. Let's first introduce some notation. Let $N \in \mathbb{N}$ such that $(2L^d\delta)^{-1} - 1 < N \leqslant (2L^d\delta)^{-1}$ and

$$J_i := [-\delta,\delta] + (i-1)2\delta \quad \text{for } i \in \{1,...,N\}. \tag{6.159}$$

Moreover, for $i \in \{1,...,N\}$, $j \in \mathbb{N}$ and $\theta > 0$ we define

$$\Omega^{\mathcal{E}}_{i,j} := \left\{\operatorname{spac}_{[-\mathcal{E},\mathcal{E}]}(\widetilde{H}^{\mu,E}_{\omega,L}) < 2\delta\right\} \cap \left\{\widetilde{E}^{\mu,E}_{\omega,j} \in J_i\right\}. \tag{6.160}$$

Let $\kappa := (1 + L^{-d})^{-1}$. We claim that for some constant C_2 we have

$$\mathbb{P}\left(\Omega^{\varepsilon/2}_{1,j}\right) \leqslant C_2\, \mathbb{P}\left(\operatorname{spac}_{[-\varepsilon,\varepsilon]}(\widetilde{H}^{\kappa\mu,\kappa E}_{L}) < 2\delta,\ \widetilde{E}^{\kappa\mu,\kappa E}_{j} \in \kappa J_i\right). \tag{6.161}$$

In this case, summation of (6.161) over $i \in \{1,...,N\}$ yields

$$\mathbb{P}\left(\Omega^{\varepsilon/2}_{1,j}\right) \leqslant 4C_2 L^d \delta\, \mathbb{P}\left(\operatorname{spac}_{[-\varepsilon,\varepsilon]}(\widetilde{H}^{\kappa\mu,\kappa E}_{L}) < 2\delta\right), \tag{6.162}$$

where we used that $N^{-1} \leqslant 4L^d\delta$ and that for $i_1 \neq i_2$

$$\left\{\widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa J_{i_1}\right\} \cap \left\{\widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa J_{i_2}\right\} = \varnothing. \tag{6.163}$$

The statement now follows from an application of (6.156) to the right hand side of (6.162). We are left with proving (6.161). For the operator $\widetilde{H}^{\mu,E}_{\omega,L}$ a shift of random couplings results in an energy shift. If we denote $\boldsymbol{\tau} = (\tau,...,\tau) \in \Gamma_L$ for fixed $\tau \in \mathbb{R}$, then

$$\widetilde{H}^{\mu,E}_{\omega+\boldsymbol{\tau},L} = \widetilde{H}^{\mu,E}_{\omega,L} + \tau \mathcal{X}_{\Lambda_L} V V^{-1} \mathcal{X}_{\Lambda_L} = \widetilde{H}^{\mu,E}_{\omega,L} + \tau \tag{6.164}$$

as operators on $L^2(\Lambda_L)$. This implies that

$$\operatorname{spac}_K(\widetilde{H}^{\mu,E}_{\omega,L}) = \operatorname{spac}_{K+\tau}(\widetilde{H}^{\mu,E}_{\omega+\boldsymbol{\tau},L}) \tag{6.165}$$

for any interval $K \subset \mathbb{R}$. Let $\eta_i := (i-1)2\delta$ denote the centers of the intervals J_i. The change of variables $\omega_k \to \omega_k + \eta_i$ and (6.165) give

$$\mathbb{P}\left(\Omega^{\varepsilon/2}_{1,j}\right) \leqslant \int_{[\eta_i, 1+\eta_i]^{\Gamma_L}} \mathbb{1}_{\Omega^{\varepsilon}_{i,j}}(\omega) \prod_{k\in\Gamma_L} \rho(\omega_k - \eta_i)\, d\omega_k, \tag{6.166}$$

where we also used $\eta_i \leqslant L^{-d} \leqslant \varepsilon/2$ and (6.165). Another change of variables $\omega_k \to \kappa\omega_k$ yields

$$(6.166) \leqslant \kappa^{-|\Gamma_L|} \int_{[a_i,b_i]^{\Gamma_L}} \mathbb{1}_{\Omega^{\varepsilon}_{i,j}}(\kappa^{-1}\omega) \prod_{k\in\Gamma_L} \rho(\kappa^{-1}\omega_k - \eta_i)\, d\omega_k, \tag{6.167}$$

where $a_i := \kappa\eta_i$ and $b_i := \kappa(1+\eta_i)$ (which both depend on L through κ). Note that we have

$$\widetilde{H}^{\mu,E}_{\kappa^{-1}\omega,L} = \kappa^{-1}\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}, \tag{6.168}$$

and hence by definition of the events $\Omega^{\varepsilon}_{i,j}$

$$\kappa^{-1}\omega \in \Omega^{\varepsilon}_{i,j} \iff \omega \in \kappa\Omega^{\varepsilon}_{i,j} \iff \begin{cases} \operatorname{spac}_{\kappa\varepsilon}(\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}) < \kappa 2\delta \\ \quad\text{and} \\ \widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa J_i. \end{cases} \tag{6.169}$$

Because $\kappa < 1$ the relation (6.169) yields

$$\kappa\Omega^{\varepsilon}_{i,j} \subset \left\{ \operatorname{spac}_{\varepsilon}(\widetilde{H}^{\kappa\mu,\kappa E}_{\omega,L}) < 2\delta,\ \widetilde{E}^{\kappa\mu,\kappa E}_{\omega,j} \in \kappa J_i \right\}. \tag{6.170}$$

Moreover, since ρ satisfies (V_4) we have for $x \in (a_i, b_i) \subset (0,1)$ that $\kappa^{-1}x - \eta_i \in (0,1)$ as well and

$$\rho(\kappa^{-1}x - \eta_i) \leqslant \rho(x) + 2\mathcal{K}L^{-d} \leqslant \rho(x)\left(1 + \frac{2\mathcal{K}}{L^d\rho_-}\right). \tag{6.171}$$

Estimating (6.167) via (6.170) and (6.171) yields

$$(6.167) \leqslant C_2\,\mathbb{P}\left(\operatorname{spac}_{\varepsilon}(\widetilde{H}^{\kappa\mu,\kappa E}_{L}) < 2\delta,\ \widetilde{E}^{\kappa\mu,\kappa E}_{j} \in \kappa J_i\right). \tag{6.172}$$

□

6.7. Proof of applications

As mentioned above, both corollaries follow from Theorem 6.1 respectively Theorem 6.2 and the techniques from [81, 27] respectively [93, 94, 27]. For convenience we recap the arguments here, closely sticking to the above references.

The proof of Corollary 6.4 exploits the following consequence of (Loc). For a compact interval $I \subset \Sigma_{\mathrm{FMB}}$ and $0 < \mu' < \mu$ the following holds with probability one: For all normalized eigenpairs (ψ, λ) of H_ω with $\lambda \in \Sigma_{\mathrm{FMB}}$ there exists a constant C_ψ such that for all $x \in \mathbb{R}^d$

$$\|\psi\|_x \leqslant C_\psi e^{-\mu'|x|}. \tag{6.173}$$

Here, the localization center has been absorbed into the (ω-dependent) constant C_ψ.

Proof of Corollary 6.4. Let $I \subset \Sigma_{\mathrm{FMB}}$ be a fixed compact interval. First we note that by Theorem 6.1 there exists $\mathcal{L}$ such that for $L \geqslant \mathcal{L}$

$$\mathbb{P}\left(\operatorname{spac}_E(H_L) < 3e^{-\sqrt{L}}\right) \leqslant L^{-2}. \tag{6.174}$$

Since the right hand side is summable over $L \in \mathbb{N}$ the Borel-Cantelli lemma yields that the set

$$\Omega_\infty := \left\{ \operatorname{spac}_E(H_{\omega,L}) < 3e^{-\sqrt{L}} \text{ for infinitely many } L \in \mathbb{N} \right\} \tag{6.175}$$

is of measure zero with respect to $\mathbb{P}$. Let $\Omega_{\mathrm{loc},I}$ be a set of measure one such that (6.173) holds for all eigenpairs (ψ, λ) with $\lambda \in I$ and for all $\omega \in \Omega_{\mathrm{loc},I}$. We now choose a fixed

$$\omega \in \Omega_{\mathrm{loc},I} \cap \left\{ \exists E \in I : \operatorname{tr} \mathbb{1}_{\{E\}}(H_\omega) \geqslant 2 \right\} =: \Omega_{\mathrm{loc},I} \cap \Omega_{\geqslant 2,I}; \tag{6.176}$$

i.e. for the configuration ω there exists $E \in I$ such that E is an eigenvalue of H_ω with two linearly independent, normalized and exponentially decaying eigenfunctions ϕ, ψ. We now apply [81, Lemma 1] with the slightly modified choice $\varepsilon_L = L^d e^{-\mu' L/2} \ll e^{-\sqrt{L}}$. The lemma is formulated for the lattice but generalizes to the continuum as has been remarked in [27]. This implies that for $J_{E,L} := [E - e^{-\sqrt{L}}, E + e^{-\sqrt{L}}]$ and all sufficiently large $L \in \mathbb{N}$

$$\operatorname{tr} \mathbb{1}_{J_{E,L}}(H_{\omega,L}) \geqslant 2 \tag{6.177}$$

holds, and consequently $\Omega_{\mathrm{loc},I} \cap \Omega_{\geqslant 2,I} \subset \Omega_\infty$. But the latter set is of $\mathbb{P}$-measure zero. □

Proof of Theorem 2.5. The proof closely follows [27, Sect. 6]. Let $E < E_{\mathrm{M}}$ with $E \in \Sigma_{\mathrm{FMB}}$ be a fixed energy with $\mathcal{N}'(E) > 0$. The starting point is to construct a triangular array of point processes which approximate $\xi_\omega^L := \xi_{E,\omega}^L$ sufficiently well. To this end, let L be fixed and $\ell := (\log L)^2$. Then we define point processes $\xi_\omega^{L,m}$ for $m \in \Upsilon_L := (\ell + 2\lceil R \rceil)\mathbb{Z}^d \cap \Lambda_{L-\ell}$ via $\xi_\omega^{L,m}(B) := \operatorname{tr} \mathbb{1}_{E+L^{-d}B}(H_{\omega,\Lambda_\ell(m)})$ ($B \subset \mathbb{R}$ Borel measurable). This definition ensures that for $m, n \in \Upsilon_L$, $m \neq n$, the processes $\xi_\omega^{L,m}$ and $\xi_\omega^{L,n}$ are independent.

The proof now consists of two parts. In the first part one shows that the superposition $\widetilde{\xi}_\omega^L := \sum_{m\in\Upsilon_L} \xi_\omega^{L,m}$ is a good approximation of the process ξ_ω^L in the sense that, if one of them converges weakly, then the other converges weakly as well and they share the same weak limit. This is a consequence of spectral localization, and the arguments are very similar to [27]. However, slight adaptions are in place since we work with different finite-volume restrictions of H_ω. We comment on this below. In the second part one then proves that the process $\widetilde{\xi}_\omega^L$ weakly converges towards the Poisson point process with intensity measure $\mathcal{N}'(E)dx$. This is the case if and only if for all bounded intervals $I \subset \mathbb{R}$ the three conditions

$$\lim_{L\to\infty} \max_{m\in\Upsilon_L} \mathbb{P}\left(\xi^{L,m}(I) \geqslant 1\right) = 0, \tag{6.178}$$

$$\lim_{L\to\infty} \sum_{m\in\Upsilon_L} \mathbb{P}\left(\xi^{L,m} \geqslant 1\right) = |I|\,\mathcal{N}'(E), \tag{6.179}$$

$$\lim_{L\to\infty} \sum_{m\in\Upsilon_L} \mathbb{P}\left(\xi^{L,m}(I) \geqslant 2\right) = 0 \tag{6.180}$$

hold. We assume for convenience that $|I| \leqslant 1$ and note that (6.178) follows from Wegner's estimate. Let L be sufficiently large such that $\ell \geqslant \mathcal{L}$, where $\mathcal{L}$ is the initial scale from Theorem 6.2. We apply the theorem for $K = 12d$ to estimate

$$\mathbb{P}\left(\xi^{L,m}(I) \geqslant 2\right) \leqslant C_1 \ell^{-2d} L^{-d} \tag{6.181}$$

for all $m \in \Upsilon_L$, which ensures (6.180). Moreover, for $n > C_2 \ell^d$ (with C_2 as in Lemma 6.17, see also Lemma 3.11) we have $\mathbb{P}\big(\xi_\omega^{L,m} \geqslant n\big) = 0$. The estimate

$$\begin{aligned}\sum_{m\in\Upsilon_L} \sum_{n=2}^{\infty} \mathbb{P}\left(\xi^{L,m}(I) \geqslant n\right) &\leqslant C_1 \ell^d |\Upsilon_L| \sup_{m\in\Upsilon_L} \mathbb{P}\left(\xi^{L,m}(I) \geqslant 2\right) \\ &\leqslant C_3 \ell^{-d}\end{aligned} \tag{6.182}$$

then already yields (6.180). Moreover, it also shows that (6.179) would follow from

$$\lim_{L\to\infty} \sum_{m\in\Upsilon_L} \mathbb{E}\left[\xi^{L,m}(I)\right] = \mathcal{N}'(E)|I|. \tag{6.183}$$

In order to verify (6.183) we use the following lemma, which is a slight variant of [27, Lem. 6.1].

Lemma 6.24. *Let $E \in \Sigma_{\mathrm{FMB}}$. For bounded intervals $J \subset \mathbb{R}$ we have*

$$\lim_{L\to\infty} \mathbb{E}\big[\big|\widetilde{\xi}^L(J) - \xi^L(J)\big|\big] = 0, \tag{6.184}$$

$$\lim_{L\to\infty} \mathbb{E}\big[\big|\Theta^L - \xi^L(J)\big|\big] = 0, \tag{6.185}$$

where $\Theta^L_\omega(J) := \mathrm{tr}\,\big(\mathcal{X}_{\Lambda_L} \mathbb{1}_{E+L^{-d}J}(H_\omega)\big)$.

A sketch of proof for the lemma is given below. By combining (6.184) and (6.185) we obtain

$$\lim_{L\to\infty} \sum_{m\in\Upsilon_L} \mathbb{E}\left[\xi^{L,m}(I)\right] = \lim_{L\to\infty} \mathbb{E}\left[\Theta^L\right] = \mathcal{N}'(E)|I| \tag{6.186}$$

for the interval I from above. Hence (6.181)–(6.183) hold and $\widetilde{\xi}^L_\omega$ converges weakly to the Poisson process with intensity measure $\mathcal{N}'(E)dx$. As argued in [27], the convergence (6.184) and the density of step functions in $L^1(\mathbb{R})$ are sufficient to prove that ξ^L_ω converges weakly, with the same weak limit as $\widetilde{\xi}^L_\omega$. □

Proof of Lemma 6.24. We first note that for our model a local Wegner estimate holds, i.e. there exists C_1 such that

$$\sup_{x\in\mathbb{R}^d\cap\Lambda_L} \mathbb{E}\left[\mathrm{tr}\,\big(\mathcal{X}_x \mathbb{1}_J(H_L)\big)\right] \leqslant C_1|J| \tag{6.187}$$

for all intervals $J \subset (-\infty, E_{\mathrm{M}}]$. This is proved in [28, Thm. 2.4] for periodic boundary conditions, but the argument also applies for Dirichlet boundary conditions. The second ingredient is Theorem 3.5 (ii). Applied to the Fermi-projection $f = \mathbb{1}_{(-\infty,E]}$ it reads as follows: There exist constants $C_2, \mu_1 > 0$ such that for open sets $G \subset G' \subset \mathbb{R}^d$ with $\mathrm{dist}(\partial G', \partial G) \geqslant 1$ and $a \in G$ we have

$$\mathbb{E}\left[\|\mathcal{X}_a \left(\mathbb{1}_J(H_G) - \mathbb{1}_J(H_{G'})\right) \mathcal{X}_a\|_1\right] \leqslant C_2 e^{-\mu_1 \,\mathrm{dist}(a,\partial G)} \tag{6.188}$$

for intervals $J \subset (-\infty, E_{\mathrm{M}}] \cap \Sigma_{\mathrm{FMB}}$. We now establish (6.184). The proof of (6.185) is similar. To this end, we split each $\Lambda_\ell(m)$, $m \in \Upsilon_L$, into a bulk part $\Lambda^{(i)}_\ell(m) := \Lambda_{\ell-\ell^{2/3}}(m)$ and a boundary part $\Lambda^{(o)}_\ell(m) := \Lambda_\ell(m) \setminus \Lambda^{(i)}_\ell(m)$. If we abbreviate $J_{E,L} := E + L^{-d}J$ then this splitting yields

$$\begin{aligned}
\mathbb{E}\Big[\big|\widetilde{\xi}^L(J) - \xi^{L,m}(J)\big|\Big] &= \sum_{m\in\Upsilon_L} \mathbb{E}\Big[\big|\,\mathrm{tr}\,\big(\mathcal{X}_{\Lambda^{(i)}_\ell(m)} \big(\mathbb{1}_{J_{E,L}}(H_{\Lambda_\ell(m)}) - \mathbb{1}_{J_{E,L}}(H_L)\big)\,\big)\big|\Big] \\
&\quad + \sum_{m\in\Upsilon_L} \mathbb{E}\Big[\big|\,\mathrm{tr}\,\big(\mathcal{X}_{\Lambda^{(o)}_\ell(m)} \big(\mathbb{1}_{J_{E,L}}(H_{\Lambda_\ell(m)}) - \mathbb{1}_{J_{E,L}}(H_L)\big)\,\big)\big|\Big] \\
&\quad + \mathbb{E}\Big[\mathrm{tr}\,\Big(\big(\mathcal{X}_{\Lambda_L} - \sum_{m\in\Upsilon_L} \mathcal{X}_{\Lambda_\ell(m)}\big)\mathbb{1}_{J_{E,L}}(H_L)\Big)\Big] \\
&=: (\text{bulk}) + (\text{boundary}) + (\text{rest}).
\end{aligned} \tag{6.189}$$

For the latter two terms we apply the local Wegner estimate from (6.187) to get

$$\text{(boundary)} \leqslant |\Upsilon_L| C_1 L^{-d} d\ell^{d-1}(\sqrt{\ell} + 2R) \leqslant C_1' \ell^{-1/2}, \tag{6.190}$$

$$\text{(rest)} \leqslant C_1 L^{-d} |\Upsilon_L| \ell^{d-1}(2R + 2) \leqslant C_1'' \ell^{-1}. \tag{6.191}$$

On the bulk contribution we in turn apply localization via (6.188) to get

$$\text{(bulk)} \leqslant |\Upsilon_L| C_3 \ell^d e^{-m' \ell^{2/3}} = C_3'' L^d e^{-\mu \ell^{3/2}}. \tag{6.192}$$

Because $L = e^{\sqrt{\ell}}$ all three terms (6.190)–(6.192) converge to zero as $L \to \infty$. □

Nomenclature

$-\Delta$	Non-negative Laplace operator on $L^2(\mathbb{R}^d)$ (or $\ell^2(\mathbb{Z}^d)$)
$\mathrm{BV}(\mathbb{R})$	Space of functions $f:\mathbb{R}\to\mathbb{C}$ with bounded total variation
$\mathcal{X}_a$	$L^2(\mathbb{R}^n)$-projection onto $Q_a=\Lambda_1(a)$
$\mathcal{X}_G$	$L^2(\mathbb{R}^n)$-projection onto $G\subset\mathbb{R}^n$
δG	$=\{x\in\mathbb{R}^n:\ \mathrm{dist}(x,\partial G)\leqslant 1\}$
$\det(A)$	Determinant or Fredholm determinant of a trace-class operator A
$\dim(X)$	Dimension of the linear space X in an ambient vector space
$\mathrm{dist}(x,A)$	Distance of a point $x\in\mathbb{C}^n$ to a set $A\subset\mathbb{C}^n$ w.r.t. $\lvert\cdot\rvert$
$\mathbb{E}$	Integral w.r.t. the measure $\mathbb{P}$
$\operatorname{ess\,inf}_{x\in X}$	Essential infimum; see essential supremum
$\operatorname{ess\,sup}_{x\in X}$	Essential supremum w.r.t. the canonical measure on X (typically either Lebesgue measure if $X\subset\mathbb{R}^n$ or $\mathbb{P}$ if $X=\Omega$)
Γ_L	$=\mathbb{Z}^d\cap\Lambda_{L+2R}$ with R as in (V_1)
$\mathbb{1}_A$	Indicator function of a set A
$\mathrm{int}(A)$	Topological interior of $A\subset\mathbb{R}^n$
$\ker(A)$	The kernel of an operator A
Λ_L	Open cube $(-L/2,L/2)^n\subset\mathbb{R}^n$ of side-length L and centered at the origin
$\Lambda_L(a)$	Open cube of side-length L and centered at $a\in\mathbb{R}^n$
$\langle\cdot,\cdot\rangle$	Scalar product on an ambient Hilbert space
$\log$	The natural logarithm
$\mathcal{C}^k(G)$	Space of k-fold differentiable functions $f:G\to\mathbb{C}$ for open $G\subset\mathbb{R}^n$
$\mathcal{C}^k_c(G)$	Space of functions $f\in\mathcal{C}^k(G)$ with compact support
$\mathcal{L}_{\mathrm{sa}}(\mathcal{H})$	Self-adjoint operators on the Hilbert space $\mathcal{H}$
$\mathcal{N}$	Integrated density of states
$\mathcal{N}'$	Density of states
$\mathcal{S}^p$	Schatten-p class of operators on an ambient Hilbert space $\mathcal{H}$ for $p\in(0,\infty)$
$\mathbb{N}$	Natural numbers $1,2,3,...$ without 0
$\mathbb{P}$	Probability measure
∂G	Topological boundary of a set $G\subset\mathbb{R}^n$
ρ	Lebesgue density of P_0
Σ	Almost surely non-random spectrum of an ergodic operator H_ω
$\sigma(A)$	Spectrum of a closed operator A
Σ_0	Spectrum of the operator H_0
Σ_{FMB}	Energy region of spectral localization
$\mathrm{supp}(g)$	support of a function g
$\mathrm{TV}(f)$	Total variation of a function $f:\mathbb{R}\to\mathbb{C}$

$\operatorname{tr} A$	Trace of an operator A (either trace-class or sign-definite)
$\|A\|$	Operator norm of an operator A
$\|A\|_p$	Schatten-p (semi-)norm of an operator A for $p \in (0, \infty)$
$\|f\|_p$	p-(semi-)norm of a function $f : \mathbb{R}^n \to \mathbb{C}$ for $p \in (0, \infty]$
$\lvert G\rvert$	Lebesgue measure of a set $G \subset \mathbb{R}^n$
$\lvert x\rvert$	Maximum norm of $x \in \mathbb{C}^n$, $\lvert x\rvert = \lvert x\rvert_\infty$
$\lvert x\rvert_p$	p-(semi-)norm of $x \in \mathbb{C}^n$ for $p \in (0, \infty]$
$\xi(\cdot, A, B)$	The spectral shift function of the operators A, B (if well defined)
A_G	Restriction of the operator A to $G \subset \mathbb{R}^n$ (typically subject to Dirichlet boundary conditions if the Laplacian is involved)
A_L	Restriction of the operator A to Λ_L, see also A_G
E_0	Minimum of Σ
$G^\#$	$= G \cap \mathbb{Z}^n$ for $G \subset \mathbb{R}^n$
$G^\#_+$	$= \{k \in (\mathbb{Z} + 1/2)^n : Q_k \cap G \neq \varnothing\}$ for $G \subset \mathbb{R}^n$
$L^p(\mathbb{R}^n)$	Space of p-integrable functions $\psi : \mathbb{R}^n \to \mathbb{C}$ for $p \in (0, \infty]$
$L^p_c(\mathbb{R}^n)$	Space of functions $f \in L^p(\mathbb{R}^n)$ with compact support for $p \in (0, \infty]$
P_0	Single-site probability distribution
Q_a	$= \Lambda_1(a)$, cube of side-length 1 centered at $a \in \mathbb{R}^n$
$R_z(A)$	Inverse of $A - z$ for $z \in \mathbb{C} \setminus \sigma(A)$ for a closed operator A
V_k	Single-site potential, $V_k = V_0(\cdot - k)$
V_k^L	Restriction of V_k to $L^2(\Lambda_L)$

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Eidesstattliche Versicherung

(Siehe Promotionsordnung vom 12.07.11, § 8, Abs. 2 Pkt. .5.)

Hiermit erkläre ich an Eidesstatt, dass die Dissertation von mir selbstständig, ohne unerlaubte Beihilfe angefertigt ist.

Dietlein, Adrian

Name, Vorname

München, 29. August 2018

Ort, Datum

Unterschrift Doktorand/in

Formular 3.2