

Alma Mater Studiorum – Università di Bologna

DOTTORATO DI RICERCA IN  
INGEGNERIA CIVILE, CHIMICA, AMBIENTALE E DEI  
MATERIALI

Ciclo XXX

**Settore Concorsuale: 08/B3**

**Settore Scientifico Disciplinare: ICAR/09**

**INNOVATIVE MODEL UPDATING PROCEDURE FOR DYNAMIC  
IDENTIFICATION AND DAMAGE ASSESSMENT OF STRUCTURES**

**Presentata da:** MICHELE TONDI

**Coordinatore Dottorato**

Prof. Luca Vittuari

**Supervisore**

Prof. Marco Savoia

**Co-Supervisore**

Dott. Marco Bovo

**Esame finale anno 2018**



# TABLE OF CONTENTS

TABLE OF CONTENTS.....	I
ACKNOWLEDGEMENTS.....	V
LIST OF TABLES.....	VII
LIST OF FIGURES.....	IX
LIST OF ABBREVIATIONS.....	XIII
ABSTRACT.....	XV
SOMMARIO.....	XVII
CHAPTER 1: INTRODUCTION.....	1
1.1 - Background about model updating and damage assessment.....	1
1.2 - Background about optimization algorithms.....	3
1.3 - Infills modeling criteria.....	11
1.4 - Organization of the thesis.....	12
Table of Chapter 1.....	15
Figures of Chapter 1.....	17
CHAPTER 2: DESCRIPTION OF THE INNOVATIVE PROCEDURE.....	19
2.1 - Introduction.....	19
2.2 - First step: system without eigenvalues equations.....	19
2.3 - Closed form solution of the first step.....	22
2.4 - Numerical example.....	26
2.5 - Second step: system with eigenvalues equations.....	30
2.6 - Description of the two steps algorithm.....	32
2.7 - Parameters uncertainties evaluation.....	33
2.8 - Two steps algorithm for the complete statistical analysis of the procedure.....	42
2.9 - Numerical example.....	45
2.10 - Generalized procedure with DE-Q algorithm.....	48
2.11 - Direct statistical distribution analysis.....	48
2.12 - Conclusions.....	52
Table of Chapter 2.....	53
Figures of Chapter 2.....	55
CHAPTER 3: THEORETICAL BASIS.....	63
3.1 - Introduction.....	63

3.2 - Maximum number of parameters.....	63
3.3 - Least squares solution of the problem.....	67
3.4 - Partial derivatives procedure for errors propagation.....	75
3.5 - Linear independence of parameters and uniqueness of the solution.....	80
3.6 - Maximum number of parameters in real structures.....	81
3.7 - Correctness of rigid diaphragm assumption.....	82
3.8 - Conclusions.....	84
Figures of Chapter 3.....	85
<b>CHAPTER 4: PRELIMINARY SENSITIVITY ANALYSIS ON SIMPLE STRUCTURES AND COMPARISON BETWEEN PROCEDURES.....</b>	<b>87</b>
4.1 - Introduction.....	87
4.2 - Procedure for the sensitivity analysis.....	87
4.3 - 2-D infilled frame sample.....	88
4.4 - 3-D infilled frame samples.....	89
4.5 - Comparison between sensitivity analysis using the two procedures.....	91
4.6 - Analysis of the dependence upon the number of modes and statistical distribution analysis.....	92
4.7 - Conclusions.....	93
Tables of Chapter 4.....	95
Figures of Chapter 4.....	101
<b>CHAPTER 5: DAMAGE IDENTIFICATION OF EXISTING INFILLED R.C. FRAMES.....</b>	<b>113</b>
5.1 - Introduction.....	113
5.2 - Damage parameter.....	113
5.3 - UCSD sample.....	113
5.4 - El-Centro building.....	117
5.5 - Conclusions.....	124
Tables of Chapter 5.....	125
Figures of Chapter 5.....	131
<b>CHAPTER 6: GENERALIZATION OF THE PROCEDURE TO NON-LINEAR PARAMETERS IN MODEL UPDATING.....</b>	<b>149</b>
6.1 - Introduction.....	149
6.2 - Unrelated non-linear parameters.....	149
6.3 - Related non-linear parameters.....	152



6.4 - System with damping.....	157
6.5 - Conclusions.....	157
Figures of Chapter 6.....	159
<b>CHAPTER 7: APPLICATION OF THE PROCEDURE FOR THE RETROFITTING OF</b>	
<b>EXISTING R.C. STRUCTURES.....</b>	
7.1 - Introduction.....	165
7.2 - Description of the procedure.....	165
7.3 - Case 1: Symmetric-Asymmetric structure.....	166
7.4 - Case 2: Totally asymmetric structure.....	167
7.5 - Non-linear parameters.....	168
7.6 - Conclusions.....	169
Tables of Chapter 7.....	171
Figures of Chapter 7.....	173
<b>CHAPTER 8: SUMMARY AND CONCLUSIONS.....</b>	<b>177</b>
<b>APPENDIX A: EXAMPLE OF DIRECT DISTRIBUTION ANALYSIS.....</b>	<b>181</b>
A.1 - Introduction.....	181
A.2 - Numerical example.....	181
A.3 - Conclusions.....	183
Table of Appendix A.....	185
Figures of Appendix A.....	187
<b>APPENDIX B: MAXIMUM NUMBER OF PARAMETERS.....</b>	<b>189</b>
B.1 - Introduction.....	189
B.2 - Numerical example.....	189
B.3 - Conclusions.....	190
Table of Appendix B.....	193
<b>APPENDIX C: RELATED NON-LINEAR PARAMETER CASE.....</b>	<b>195</b>
C.1 - Introduction.....	195
C.2 - Theoretical relations.....	195
C.3 - Numerical example.....	197
C.4 - Conclusions.....	198
Table of Appendix C.....	199
Figures of Appendix C.....	201
<b>REFERENCES.....</b>	<b>203</b>



## ACKNOWLEDGEMENTS

During my Ph.D., several people helped me and deserve my acknowledgement. First of all my parents Leonardo and Doriana, their sustain and kindness helped me to finish the Ph.D.. My grandparents Maddalena, Marisa and Silvano, they supported me for my whole life. I wish to thank Giulia for all the things who made during my Ph.D., especially when I was abroad. I very important acknowledge goes to Marco Bovo who helped me for the three years of Ph.D. with a helpful advices and ideas. Thanks to Professor Marco Savoia, one of the best professor and engineer I have never met, he gave me several and helpful advices. Thanks to Loris Vincenzi, he helped me in a very important way. Thanks also to Professor Claudio Mazzotti, his advices were very useful for my research activities. A very important appreciation goes to Sina, Hadi and Amir; without them my staying in Buffalo would have been horrible. Another very important appreciation go to Professor Stavridis, he helped me with my research more than he thinks. I wish to thank Professor Moaveni, I have never met him personally but his advices helped me in a very important way. I want to thank Michele Naldi, I appreciated our conversations and friendship. I want to thank all my friends of Castel d'Aiano and Montese, thank you for the fun, laughs and the wonderful evenings spent together.



## LIST OF TABLES

<b>Table 1.1:</b> Relations proposed in literature for the calculation of w/d ratio (Tarque et al., 2015). .....	15
<b>Table 2.1:</b> Standard deviations and CoVs of parameters from uncertainties evaluation and Monte Carlo realizations. ....	53
<b>Table 4.1:</b> Reference values of parameters. ....	95
<b>Table 4.2:</b> Mean values, errors and CoVs of parameters for 2-D sample. ....	95
<b>Table 4.3:</b> Reference values of the equivalent strut widths. ....	95
<b>Table 4.4:</b> Mean values, errors and CoVs for case AA-0-0, AS-0-0 and AAv-0-0. ....	95
<b>Table 4.5:</b> Mean values, errors and CoVs for case AA-1-I, AA-2-I and AA-3-I. ....	96
<b>Table 4.6:</b> Mean values, errors and CoVs for case AS-1-I, AS-2-I and AS-3-I. ....	96
<b>Table 4.7:</b> Mean values, errors and CoVs for case AA-2-I, AA-2-II and AA-2-III. ....	96
<b>Table 4.8:</b> Mean values, errors and CoVs for case AAv-2-I, AAv-2-II and AAv-2-III. ....	97
<b>Table 4.9:</b> Mean values, errors and CoVs for case AA-2-I with DE-Q and two steps procedures. ....	97
<b>Table 4.10:</b> Mean values, errors and CoVs for case AS-2-I with DE-Q and two steps procedures. ....	97
<b>Table 4.11:</b> Mean values, errors and CoVs for case AA-2-I with and without determinant equations using two steps procedure (for the system without determinant equations only the first step is needed).. ....	98
<b>Table 4.12:</b> Mean values, errors and CoVs for case AA-2-I with 6 and 9 modes. ....	98
<b>Table 4.13:</b> Chi-square tests for cases AA-2-I and AS-2-I. ....	99
<b>Table 4.14:</b> Statistical distributions of parameters for cases AA-2-I and AS-2-I. ....	99
<b>Table 5.1:</b> Dynamic tests performed in the specimen (from Moaveni et al,2013). ....	125
<b>Table 5.2:</b> Identified frequencies from white noise tests. ....	125
<b>Table 5.3:</b> Comparison between experimental and numerical frequencies and MAC values. ....	126
<b>Table 5.4:</b> Walls demolition sequence and resulted damage state. ....	126
<b>Table 5.5:</b> Summary of system identification results (Yousefianmoghadam et al., 2015). ....	126
<b>Table 5.6:</b> Comparison between modes components and re-built modes components. ....	126
<b>Table 5.7:</b> Values of parameters for 4 parameters case. ....	126
<b>Table 5.8:</b> Comparison between experimental and numerical frequencies and MAC values for 4 parameters case. ....	127

<b>Table 5.9:</b> Damage parameters for 4 parameters case. ....	127
<b>Table 5.10:</b> Mean values, standard deviations and CoVs for experimental frequencies. ....	127
<b>Table 5.11:</b> Mean values, standard deviations and CoV values of parameters. ....	128
<b>Table 5.12:</b> Mean values, standard deviations, CoVs and errors for frequencies and MAC values.....	128
<b>Table 5.13:</b> Values of parameters for 8 parameters case. ....	128
<b>Table 5.14:</b> Comparison between experimental and numerical frequencies and MAC values for 8 parameters case. ....	129
<b>Table 5.15:</b> Damage parameters for 8 parameters case. ....	129
<b>Table 5.16:</b> Comparison between experimental and numerical frequencies and MAC values for 8 parameters case without updating the ground storey parameters for DS2, DS3 and DS4. ....	129
<b>Table 7.1:</b> Modes components chosen for the procedure. ....	171
<b>Table 7.2:</b> w/d values for the 3 parameters - Case 1. ....	171
<b>Table 7.3:</b> Frequencies before and after retrofitting - Case 1. ....	171
<b>Table 7.4:</b> Modified modes components chosen for the procedure. ....	172
<b>Table 7.5:</b> w/d values for the 3 parameters - Case 2. ....	172
<b>Table 7.6:</b> Frequencies before and after retrofitting - Case 2. ....	172
<b>Table A.1:</b> Mean values, variances, standard deviations and CoVs of parameter from distribution propagation and Monte Carlo procedure. ....	185
<b>Table B.1:</b> CoVs of parameters changing the number of parameters updated. ....	193
<b>Table C.1:</b> Mean values, variances, standard deviations and CoVs of parameter from uncertainties evaluation and Monte Carlo procedure. ....	199

## LIST OF FIGURES

<b>Figure 1.1:</b> Algorithm scheme for DE method (Vincenzi and Savoia, 2010). .....	17
<b>Figure 1.2:</b> Mutation process by random combination (Vincenzi and Savoia, 2010). .....	17
<b>Figure 1.3:</b> Flowchart of the DE-Q algorithm (Vincenzi and Savoia, 2015). .....	18
<b>Figure 1.4:</b> Approximation of cost function by quadratic response surface. (Vincenzi and Savoia, 2010). .....	18
<b>Figure 2.1:</b> Two bays three storey infilled frame decomposition with two parameters. ....	55
<b>Figure 2.2:</b> Flowchart of the two steps algorithm. ....	55
<b>Figure 2.3:</b> Flowchart of the two steps algorithm with uncertainties evaluation. ....	56
<b>Figure 2.4:</b> Flowchart of the two steps algorithm for complete statistical analysis of the procedure. ....	57
<b>Figure 2.5:</b> Frequencies histograms of parameters $a_1, a_2$ from Monte Carlo analysis. ....	58
<b>Figure 2.6:</b> Flowchart of the algorithm with DE-Q procedure. ....	58
<b>Figure 2.7:</b> Flowchart of the algorithm with DE-Q procedure and uncertainties evaluation. ....	59
<b>Figure 2.8:</b> Flowchart of the algorithm with DE-Q procedure for complete statistical analysis. ....	60
<b>Figure 2.9:</b> Flowchart of the algorithm for the direct statistical distribution analysis. ....	61
<b>Figure 3.1:</b> General representation of rigid diaphragm components from $l+p$ displacements. ....	85
<b>Figure 3.2:</b> Flowchart of the procedure for obtaining of the maximum number of parameter in real structure. ....	85
<b>Figure 4.1:</b> View of the 2-D sample structure. ....	101
<b>Figure 4.2:</b> Parameters arrangement. ....	101
<b>Figure 4.3:</b> Frequencies histograms of parameters $a_1, a_2, a_3$ for 2-D sample - Case A. ....	102
<b>Figure 4.4:</b> Frequencies histograms of parameters $a_1, a_2, a_3$ for 2-D sample - Case B. ....	102
<b>Figure 4.5:</b> Frequencies histograms of parameters $a_1, a_2, a_3$ for 2-D sample - Case C. ....	103
<b>Figure 4.6:</b> Frequencies histograms of parameters $a_1, a_2, a_3$ for 2-D sample - Case D. ....	104
<b>Figure 4.7:</b> Plan view of 3-D samples. ....	104
<b>Figure 4.8:</b> Views of 3-D samples - Case AA: (a) North elevation, (b) West elevation, (c) South elevation, (d) East elevation. ....	104
<b>Figure 4.9:</b> Views of 3-D samples - Case AS: (a) North elevation, (b) West elevation, (c) South elevation, (d) East elevation. ....	105
<b>Figure 4.10:</b> Views of 3-D samples - Case AA $v$ : (a) North elevation, (b) West elevation, (c) South elevation, (d) East elevation - The blue infill is the model error. ....	105

<b>Figure 4.11:</b> Frequencies histograms of parameters $a_1, a_2, a_3, a_4$ for 3-D sample - Case AA-2-I. ....	106
<b>Figure 4.12:</b> Frequencies histograms of parameters $a_1, a_2, a_3, a_4$ for 3-D sample - Case AS-2-I. ....	106
<b>Figure 4.13:</b> Frequencies histograms of parameters $a_1, a_2, a_3, a_4$ for 3-D sample - Case AA <sub>v</sub> -2-I. ....	107
<b>Figure 4.14:</b> Frequencies histograms of parameters $a_1, a_2, a_3, a_4$ for 3-D sample - Case AA-2-I along with main fitted distributions: (a) parameter $a_1$ , (b) parameter $a_2$ , (c) parameter $a_3$ , (d) parameter $a_4$ . ....	109
<b>Figure 4.15:</b> Frequencies histograms of parameters $a_1, a_2, a_3, a_4$ for 3-D sample - Case AS-2-I along with main fitted distributions: (a) parameter $a_1$ , (b) parameter $a_2$ , (c) parameter $a_3$ , (d) parameter $a_4$ . ....	111
<b>Figure 5.1:</b> Plan view of the prototype structure (from Moaveni et al,2013). ....	131
<b>Figure 5.2:</b> Elevation view of the prototype structure (from Moaveni et al,2013). ....	131
<b>Figure 5.3:</b> Design of the tree storey specimen (dimensions in m) (from Stavridis et al, 2011). ...	131
<b>Figure 5.4:</b> Cross sections of R.C. members (dimensions in mm) (from Stavridis et al, 2011). ....	132
<b>Figure 5.5:</b> Front view of the specimen (from Moaveni et al, 2013). ....	132
<b>Figure 5.6:</b> Location of accelerometers at each floor level (from Moaveni et al, 2013). ....	132
<b>Figure 5.7:</b> Modes for damage state DS0: (a) Polar plot representation for complex mode shapes; (b) Vibration mode shapes (from Moaveni et al, 2013). ....	132
<b>Figure 5.8:</b> Stick model representation. ....	133
<b>Figure 5.9:</b> Parameters values for all damage states. ....	133
<b>Figure 5.10:</b> Story stiffness for all damage states. ....	133
<b>Figure 5.11:</b> Damage parameters for all damage states. ....	134
<b>Figure 5.12:</b> Comparison between experimental and numerical mode shapes (the red lines are the experimental mode shapes, the blue lines are the numerical ones): (a) DS0; (b) DS1; (c) DS2; (d) DS3; (e) DS4; (f) DS5; (g) DS6; (h) DS7. ....	135
<b>Figure 5.13:</b> Views of the structure under study: (a) north-west view; (b) north-east view (Song et al., 2017). ....	135
<b>Figure 5.14:</b> First floor plan view with joists details (Yousefianmoghadam et al., 2015). ....	136
<b>Figure 5.15:</b> Walls demolition sequence and resulted damage state (Yousefianmoghadam et al., 2015). ....	136
<b>Figure 5.16:</b> Structure instrumentations: (a) ground floor; (b) First floor; (c) Roof level. ....	137



<b>Figure 5.17:</b> Mode shapes for roof level: (a) Mode 1; (b) Mode 2 (Yousefianmoghadam et al., 2015). .....	137
<b>Figure 5.18:</b> Parameters definition for 4 parameters case: (a) Plan view of the first storey; (b) North elevation, 1 <sup>st</sup> parameter (in red); (c) East elevation, 2 <sup>nd</sup> parameter (in orange); (d) South elevation, 3 <sup>rd</sup> parameter (in green); (e) West elevation, 4 <sup>th</sup> parameter (in blue). .....	138
<b>Figure 5.19:</b> Parameters values for all the damage states for 4 parameters case. ....	139
<b>Figure 5.20:</b> Damage parameters for all the damage states for 4 parameters case. ....	139
<b>Figure 5.21:</b> Comparison between parameters for undamaged structure and DS1 for 4 parameters case. ....	140
<b>Figure 5.22:</b> Comparison between experimental and numerical mode shapes (the red lines are the experimental mode shapes, the blue lines are the numerical ones): (a) DS1 - Mode 1; (b) DS1 - Mode 2; (c) DS2 - Mode 1; (d) DS2 - Mode 2; (e) DS3 - Mode 1; (f) DS3 - Mode 2; (g) DS4 - Mode 1; (h) DS4 - Mode 2. ....	144
<b>Figure 5.23:</b> Frequencies histograms of parameters $a_1$ , $a_2$ , $a_3$ and $a_4$ for damage state DS1. ....	145
<b>Figure 5.24:</b> Mean values of parameters along with standard deviations for all the damage states. ....	145
<b>Figure 5.25:</b> Parameters definition for 8 parameters case: (a) Ground floor plan view; (b) First floor plan view; (c) North elevation, 1 <sup>st</sup> parameter in yellow, 5 <sup>th</sup> parameter in red; (d) East elevation, 2 <sup>nd</sup> parameter in black, 6 <sup>th</sup> parameter in orange; (e) South elevation, 3 <sup>rd</sup> parameter in violet, 7 <sup>th</sup> parameter in green (f) West elevation, 4 <sup>th</sup> parameter in dark blue, 8 <sup>th</sup> parameter in blue. ....	146
<b>Figure 5.26:</b> Parameters values for all the damage states for 8 parameters case. ....	147
<b>Figure 5.27:</b> Damage parameters for all the damage states for 8 parameters case. ....	147
<b>Figure 5.28:</b> Comparison between parameters for undamaged structure and DS1 for 8 parameters case. ....	148
<b>Figure 5.29:</b> Damage parameters between undamaged structure and DS1 for 8 parameters case. ....	148
<b>Figure 6.1:</b> Flowchart of the two steps algorithm for unrelated non-linear parameters. ....	159
<b>Figure 6.2:</b> Flowchart of the two steps algorithm with uncertainties evaluation for unrelated non-linear parameters. ....	160
<b>Figure 6.3:</b> Flowchart of the two steps algorithm for complete statistical analysis for unrelated non-linear parameters. ....	161

<b>Figure 6.4:</b> Flowchart of the algorithm with DE-Q procedure for related non-linear parameters. ....	162
<b>Figure 6.5:</b> Flowchart of the algorithm with DE-Q procedure and uncertainties evaluation for related non-linear parameters. ....	162
<b>Figure 6.6:</b> Flowchart of the algorithm with DE-Q procedure for complete statistical analysis for related non-linear parameters. ....	163
<b>Figure 7.1:</b> Flowchart of the procedure. ....	173
<b>Figure 7.2:</b> Plan view of Case 1 (dimensions in cm). ....	173
<b>Figure 7.3:</b> Elevation views of Case 1: (a) South elevation; (b) East elevation; (c) North elevation; (d) West elevation. ....	174
<b>Figure 7.4:</b> Elevation views of Case 2: (a) South elevation; (b) East elevation; (c) North elevation; (d) West elevation. ....	175
<b>Figure 7.5:</b> Flowchart of procedure for unrelated non-linear parameters. ....	175
<b>Figure 7.6:</b> Flowchart of procedure for related non-linear parameters. ....	176
<b>Figure A.1:</b> Analytical distribution along with calibrated Gaussian and Cauchy distributions. ....	187
<b>Figure A.2:</b> Analytical distribution along with normalized frequencies histogram from Monte Carlo analysis. ....	187
<b>Figure C.1:</b> View of the sample structure. ....	201
<b>Figure C.2:</b> Target function $H$ for not perturbed system. ....	201
<b>Figure C.3:</b> Frequencies histogram of parameter $a_1$ from Monte Carlo analysis. ....	201

## LIST OF ABBREVIATIONS

COMAC	Coordinate MAC
CoV	Coefficient of Variation
cov	Covariance
CR	Crossover Constant
DE	Differential Evolution
DMU	Direct Matrix Updating
DOF	Degree Of Freedom
DS	Damage State
ECM	Eigendynamic Constraint Method
EMM	Error Matrix Method
EQ	Earthquake base excitation
ERA	Eigensystem Realization Algorithm
FE	Finite Elements
FMAC	Frequency-scaled MAC
FRF	Frequency Response Function
GRSM	General Response Surface Method
IES	Inverse Eigensensitivity Method
LVDT	Linear Variable Displacement Transducer
IMAC	Improved MAC
MAC	Modal Assurance Criterion
MCE	Maximum Considered Earthquake

NEES	Network for Earthquake Engineering Simulation
NExT	Natural Excitation Technique
NCO	Normalized MAC
NMD	Normalized Modal Difference
pdf	Probability Density Function
R.C.	Reinforced Concrete
RFM	Response Function Method
RSM	Response Surface Methodology
UCSD	University of California San Diego
WN	White Noise base excitation

## ABSTRACT

The model updating technique allows the understanding of the dynamic behavior of a system and its damage state. In the last years, the structural monitoring has increased its applicability thanks to the decrease of the cost of sensors and improvements in the computational power. More and more structures are today instrumented in order to assess the intervention of progressive damages, understand their structural behavior and safety in almost real time. Nowadays, the real time identification of structural parameters and damage assessment is no longer unachievable. Moreover, the uncertainties evaluation is another important task required by the model updating procedures. Combining real time assessment and uncertainties evaluation, the algorithms can drive to a judgment about unsafety conditions in the buildings, with possible evacuation and securing of the structures, which is more and more required to structural health monitoring systems.

The algorithms developed in this work are focused on these topics, especially on very quick model updating procedure, with uncertainties evaluation, which allows to estimate the structural parameters along with an error assessment. The quickness of the algorithm enables for its use in real time monitoring of actual structures. The algorithm itself is based on an innovative two steps procedure, with uncertainties evaluation, solving the inverse eigenvalues problem. The first step is achieved with closed form solution (without considering the determinant equations). If the solution does not satisfy the fixed thresholds, the second iterative step should be performed in order to improve the agreement between experimental outcomes and numerical ones. This procedure allows us to write the partial derivatives of the problem itself, with respect to the experimental outcomes, in closed form. Therefore, the parameters uncertainties are computed using the errors propagation.

A second procedure is developed facing the complete problem entirely in iterative way, using a genetic algorithm with response surfaces (the so-called DE-Q algorithm). The uncertainties evaluation is done also for this procedure in closed form.

A sensitivity analysis has been performed on 2-D and 3-D infilled framed structures varying the perturbation values on frequencies and mode shapes and varying the parameters arrangement. Comparison between the two procedures has been done in terms of mean values and coefficients of variations of parameters. Another comparison has been performed in order to understand the influence of the determinant equations and the number of modes used.

Two real structures have been then analyzed with the algorithm, the first one is a three storey, two bays 2-D infilled frame tested with shake table in San Diego, CA, US, of which seven damage

states were achieved scaling accelerogram of actual earthquakes. The second one is a two storey, 3-D infilled framed structure located in El-Centro, CA, US, tested with ambient vibrations and forced vibrations through shaker. Four damage states were achieved artificially through infills removal at the first storey. For both the structures, a damage assessment has been achieved and, for the last one, a sensitivity analysis using several data windows for ambient vibrations is reported.

Then, a generalization of the procedure in order to take into account the possibility of non-linear parameters is studied.

In the last part of this thesis, the algorithm is used in order to find a first trial solution for the retrofitting problem of existing structures. Two sample structures are analyzed and the comparison between numerical and expected parameters is performed. A generalization for non-linear parameters is then developed.

**Keywords:** Model Updating, Damage Assessment, Dynamic identification, Inverse Eigenvalues Problem, Statistical Analysis, Error Propagation, Infilled Frames, Stick Model, Ambient Vibrations, Sensitivity Analysis, Monte Carlo Procedure, Genetic Algorithm, Response Surfaces Methodology, Seismic Retrofitting, Modal Assurance Criterion, Shake Table Tests, Shaker, Matlab, OpenSEES.

## SOMMARIO

La procedura di model updating è una tecnica alquanto datata che permette di comprendere il comportamento dinamico di un sistema e il suo stato di danno. Negli ultimi anni, il monitoraggio strutturale ha incrementato la sua applicabilità grazie al ridotto costo dei sensori e al miglioramento della potenza computazionale. Sempre più strutture sono oggi strumentate per valutare i loro danni e capire il comportamento dinamico stesso. La valutazione in tempo reale dei parametri strutturali e dello stato di danno è oggi non più irraggiungibile. La valutazione delle incertezze sui parametri (deviazioni standard) è, inoltre, richiesta ai moderni algoritmi di model updating. La combinazione della valutazione in tempo reale e dell'incertezza possono portare a un giudizio di situazioni potenzialmente pericolose in strutture esistenti con possibile evacuazione e messa in sicurezza della struttura stessa. Questa valutazione è sempre più richiesta ai sistemi di monitoraggio strutturale.

L'algoritmo sviluppato in questo lavoro è incentrato su questi aspetti, in particolare sulla rapida valutazione dei parametri strutturali (usando il model updating) e delle relative incertezze. La velocità dell'algoritmo permette l'uso dello stesso per il monitoraggio in tempo reale delle strutture. L'algoritmo è basato su una procedura innovativa a due fasi, con valutazione dell'incertezza, risolvendo un problema inverso agli autovalori. La prima fase è risolta con formulazione chiusa del problema (senza considerare le equazioni ai determinanti). Se la soluzione non soddisfa delle soglie prefissate per i parametri di controllo, la seconda fase, iterativa, deve essere eseguita in modo da migliorare la corrispondenza tra risultati sperimentali e numerici. La procedura permette, inoltre, di scrivere le derivate parziali del problema stesso, rispetto ai risultati sperimentali, in formulazione chiusa; pertanto le incertezze sui parametri sono calcolate mediante la teoria della propagazione degli errori.

Una seconda procedura è sviluppata affrontando direttamente il problema completamente in forma iterativa, usando un algoritmo genetico con superfici di risposta (chiamato algoritmo DE-Q). Le incertezze sono calcolate in formulazione chiusa anche per questo caso.

L'analisi di sensitività è stata eseguita su telai tamponati 2-D e 3-D variando il valore della perturbazione su frequenze e modi e, inoltre, variando la disposizione dei parametri. Il confronto tra le due procedure è stato fatto in termini di valori medi e coefficienti di variazione sui parametri. Inoltre, un confronto per capire l'influenza delle equazioni con i determinanti ed il numero di modi usati è stato analizzato.

Due strutture reali sono state successivamente analizzate con l'algoritmo, la prima è un telaio tamponato bidimensionale a tre piani e due campate, sottoposto a prove con tavola vibrante in San Diego, CA, US, raggiungendo sette stati di danno scalando accelerogrammi derivanti da sismi reali. La seconda è una struttura tridimensionale con telai tamponati a due piani in El-Centro, CA, sottoposta a prove con vibrazioni ambientali e mediante vibrodina. Quattro stati di danno sono stati artificialmente prodotti rimuovendo alcune tamponature al primo piano. Per entrambe le strutture, la valutazione del danno è stata eseguita; inoltre, per la seconda, un'analisi di sensitività è stata svolta sulla base dei dati derivanti da diverse finestre temporali di acquisizione per le vibrazioni ambientali.

Successivamente, la generalizzazione dell'algoritmo per tenere in conto di parametri non lineari è sviluppata.

Nell'ultima parte della tesi, l'algoritmo è usato per trovare una prima soluzione di tentativo per il problema del miglioramento/adequamento sismico di strutture esistenti. Due strutture campione sono state analizzate ed il confronto tra i parametri ottenuti e quelli attesi è riportato. La generalizzazione a parametri non lineari è successivamente studiata.

**Parole Chiave:** Aggiornamento del Modello, Valutazione del Danno, Identificazione Dinamica, Problema Inverso agli Autovalori, Analisi Statistica, Propagazione degli Errori, Telai Tamponati, Modello Stick, Vibrazioni Ambientali, Analisi di Sensitività, Procedura Monte Carlo, Algoritmo Genetico, Superfici di Risposta, Adeguamento sismico, Modal Assurance Criterion, Tavola Vibrante, Vibrodina, Matlab, OpenSEES.



# CHAPTER 1

## INTRODUCTION

The model updating is a dated technique which allows to understand the dynamic behavior of a structure or, more generally, a system whose dynamic behavior must be analyzed. In the last years, the structural monitoring increased its applicability thanks to the decrease of the cost of sensors and improvements in the computational power. More and more structures are today instrumented in order to monitor their dynamic behavior, to assess their damages and safety in almost real time. In this context, more powerful and less computationally demanding model updating procedures are needed in order to achieve a real time evaluation of structural parameters and real time damage assessment of the structure itself. Moreover, the uncertainties evaluation is another important task required by model updating algorithms. The real time assessment combined with uncertainties evaluation can drive to a judgment about dangerous situations in actual buildings, which is more and more required to structural health monitoring systems.

The algorithms developed in this work are focused on these topics, especially on very quick model updating procedure, with uncertainties evaluation, which allows to estimate the structural parameters along with an error assessment. The quickness of the algorithm enables to its use for real time monitoring of actual structure.

### 1.1 Background about model updating and damage assessment

Several literature proposals are available for the solution of model updating problems. Two main types of model updating methods are described in literature: (a) methods relying on comparison between experimental and numerical outcomes (frequencies, mode shapes, FRFs) using comparison coefficients; (b) methods which solve directly the system of eigenvalues equations.

A general review of those methods can be found in Ewins, 2000, Friswell and Mottershead, 1995, Mottershead and Friswell, 1993 and Imregun and Visser, 1991.

Different proposals for the first method has been found in literature. First of all a direct comparison between experimental frequencies and numerical ones (obtained using modeling of the structure) is described in the literature (Ewins, 2000). In order to compare the mode shapes, different coefficients were defined, starting with the most common one: the Modal Assurance Criterion (MAC coefficient) (Allemang, 1984; Allemang, 2003).

$$MAC(\underline{\varphi}_i, \overline{\varphi}_j^*) = \frac{\left| \langle \underline{\varphi}_i; \overline{\varphi}_j^* \rangle \right|^2}{\|\underline{\varphi}_i\|^2 \cdot \|\overline{\varphi}_j^*\|^2} \quad (1.1)$$

in which  $\overline{\varphi}_j$  is the j-th experimental modes and  $\underline{\varphi}_i$  is the i-th numerical one. The vectors with the star as apex are the complex conjugated of the corresponding without star as apex;  $\|\cdot\|$  and  $\langle \cdot \rangle$  mean the norm of the vector and the dot product between vectors, respectively. In the case of real eigenmodes, the formula must be simplified in this way:

$$MAC(\underline{\varphi}_i, \overline{\varphi}_j) = \frac{\left| \langle \underline{\varphi}_i; \overline{\varphi}_j \rangle \right|^2}{\|\underline{\varphi}_i\|^2 \cdot \|\overline{\varphi}_j\|^2} \quad (1.2)$$

this coefficient can assume values between 0 (no matching between experimental and numerical modes) and 1 (complete matching between the abovementioned modes).

In Savoia and Vincenzi, 2008; Vincenzi and Savoia, 2010; Vincenzi et al., 2013, there is the definition of the subsequent function to minimize as target function:

$$H = \sum_{i=1}^n [w_1 \cdot \left( \frac{f_i - \tilde{f}_i}{\tilde{f}_i} \right)^2 + w_2 \cdot NMD_i^2] \quad (1.3)$$

in which

$$NMD_i = \sqrt{\frac{1 - MAC(\underline{\varphi}_i, \overline{\varphi}_i)}{MAC(\underline{\varphi}_i, \overline{\varphi}_i)}} \quad (1.4)$$

$\overline{\varphi}_i, \underline{\varphi}_i$  is the i-th pair of corresponding modes and  $f_i, \tilde{f}_i$  is the i-th pair of corresponding frequencies.  $w_1$  e  $w_2$  are weighting functions,  $n$  is the number of experimentally identified modes.

Improvements about MAC coefficient have been performed in order to overcome some limits of the parameter itself. The Normalized MAC (NCO) (Ewins, 2000) takes into account also a weighting matrix in order to take into account also the mass matrix in the procedure and the Improved MAC (IMAC) is less sensitive to the DOFs chosen. In order to compare, in the same plot, frequencies and mode shapes, the Frequency-scaled MAC (FMAC) has been created (Ewins, 2000; Friswell and Mottershead, 1995). If the comparison is made for the i-th component of the mode, the COMAC has to be used (Ewins, 2000). In order to melt together frequencies and mode shapes comparisons, not only the FMAC was introduced in Literature. More recently target functions to minimize, which have one portion related to frequencies comparison and another one related to mode shapes comparison, have been defined. In Savoia and Vincenzi, 2008; Ewins, 2000 and Peeters and De

Roeck, 1999, a comparison in frequencies and NMD coefficient ("Normalized Modal Difference", related to MAC coefficient) are used to define a target function. In Teughels, 2003; Teughels and De Roeck, 2005 a sensitivity-based FE model updating strategy is defined, in which the structure is divided into substructure having the same value of parameters (or damages). Indications in order to reduce the estimation uncertainties were given in Mares et al., 2002. The last procedure was applied, for instance, in Moaveni et al., 2008; Moaveni et al., 2010; Moaveni et al., 2013 and Song et al., 2017. Other type of model updating uses the comparison between individual response functions or the correlation between the complete set of FRFs (Ewins, 2000, Friswell and Mottershead, 1995).

The second type of model updating relies on the solution of the dynamic eigenvalues problem. A review of the various proposals can be found in Mottershead and Friswell, 1993 and He, 1987. The first attempt was made in the 70s with the Direct Matrix Updating (DMU) in which the mass and stiffness matrices were updated directly (He, 1987). An enhanced procedure, with respect to the previous one, was developed by Lin, 1991 called Error Matrix Method (EMM) in which the error mass and stiffness matrices are computed. Another family of methods is the so-called indirect updating methods starting from the simplest and earliest case called Eigendynamic Constraint Method (ECM) in which the eigenvalues problem is solved iteratively (Ewins, 2000). The one with greatest application in practice, for the second type of model updating, is the Inverse Eigensensitivity Methods (IES) based on an equation of exactly the same general form as the ECM methods with the difference that the system matrix and vector are composed of properties which derive from the analytical model sensitivities and the discrepancies between predicted and measured modal properties. Also for this type, a method which uses the FRFs, the Response Function Methods (RFM), was developed (Visser, 1992 and Ewins et al., 1980). A probabilistic analysis of the problem has been done in Beck and Katafygiotis, 1998; Ching et al., 2006 and Muto and Beck, 2008.

## **1.2 Background about the optimization algorithms**

### **1.2.1 Description of the Differential Evolution DE-Q Algorithm**

The DE-Q algorithm is a Differential Evolution algorithm that looks for the minimum value of a target function  $H$ . This type of algorithm combines the genetic algorithm with the response surface methodology. The main aspect of this algorithm are summarized in the following points,

taken from Vincenzi and Savoia, 2010, Vincenzi and Savoia, 2015 and Vincenzi and Gambarelli, 2017.

### **Genetic Evolution algorithm (DE algorithm)**

Differential Evolution (DE) is a heuristic direct search approach where NP vectors:

$$\underline{x}_{i,G} \text{ with } i = 1, 2, \dots, NP$$

are used (Storn and Price, 1997). Subscript G indicates the G-th generation of parameter vectors, called population. Vectors  $\underline{x}_{i,G}$  contain a number D of optimization parameters. The number NP of vectors of the population is kept constant during the minimization process.

In order to minimize the objective function, a direct search method is a strategy that generates variations of parameter vectors. Once a variation is generated, a decision must be made whether or not to accept the new parameters. A new vector of parameters is accepted only if it reduces the value of the objective function. A robust algorithm requires that the solution does not converge to a local minimum. Techniques like genetic and evolution algorithms are based on a calculation involving several vectors simultaneously (Goldberg, 1989; Vanderplaats, 1984). Hence, if some vectors reach local minima, they can be excluded because they are associated with higher values of the cost function.

The algorithmic scheme of the DE approach is shown in Figure 1.1. First of all, the initial population is chosen randomly. Then, DE generates a new parameter vector by adding the weighted difference vector between two vectors of the population, so generating a third vector (the mutant vector). This operation is called Mutation. Then, in the Crossover operation, a new trial vector is generated by selecting some components of the mutant vector and some of the original vector. If the trial vector gives a lower value of objective function than that of the old population, the new generated vector replaces the old vector (Selection operation).

#### – Mutation

For each vector of the G-th population:

$$\underline{x}_{i,G} \text{ with } i = 1, 2, \dots, NP$$

a trial vector  $\underline{v}_{i,G}$  is generated by adding to  $\underline{x}_{i,G}$  a contribution obtained as the difference between two other vectors of the same population.

Three different combination strategies can be used during the mutation process: the “random” combination, the “best” combination and an intermediate combination called “best-to-rand”.

According to Storn and Price, 1997, in the random combination, the mutant vector is generated according to the expression:

$$\underline{v}_{i,G+1} = \underline{x}_{r_1,G} + F \cdot (\underline{x}_{r_2,G} - \underline{x}_{r_3,G}) \quad (1.5)$$

where  $r_1, r_2, r_3 \in \{1, 2, \dots, NP\}$  are mutually different integer numbers. Moreover,  $F$  is a positive constant (scale parameter) controlling the amplitude of the mutation. The scale parameter  $F$  is taken smaller than 2. Figure 1.2 shows the *mutation* process according to “random” combination. “Best” combination is similar to the random combination, but the *mutant vector* is defined from the equation:

$$\underline{v}_{i,G+1} = \underline{x}_{best,G} + F \cdot (\underline{x}_{r_1,G} - \underline{x}_{r_2,G}) \quad (1.6)$$

where  $\underline{x}_{best,G}$  is the vector giving the minimum value of the object function of the  $G$ -th population. Finally, in the “best-to-rand” combination, mutant vector is generated according to the expression:

$$\underline{v}_{i,G+1} = \underline{x}_{i,G} + F \cdot (\underline{x}_{best,G} - \underline{x}_{i,G}) + F \cdot (\underline{x}_{r_1,G} - \underline{x}_{r_2,G}) \quad (1.7)$$

The effectiveness of one method depends on the regularity of the objective function. For regular functions with only one (global) minimum, “best” combination converges more rapidly since the best vector obtained from the previous generation is taken as the basic vector. In the presence of more minima, “random” or “best-to-rand” combinations are the best choices, since convergence to local minima can be avoided.

#### – Crossover

In order to increase the diversity of the vectors, crossover process is introduced in the DE algorithm. The *trial vector*  $\underline{u}_{i,G+1}$  is obtained by randomly exchanging the values of optimization parameters between the original vectors of the population  $\underline{x}_{i,G}$  and those of mutant population  $\underline{v}_{i,G+1}$ , i.e.:

$$\underline{u}_{i,G+1} = (u_{1i,G+1}, u_{2i,G+1}, \dots, u_{Di,G+1}) \quad (1.8)$$

where:

$$u_{ji,G+1} = \begin{cases} v_{ji,G+1} & \text{if } rand(j) \leq CR \\ x_{ji,G} & \text{if } rand(j) > CR \end{cases} \quad (1.9)$$

In Eq. 1.9,  $j = 1, 2, \dots, D$ , where  $D$  is the number of optimization parameters, and  $u_{ji}$  is the  $j$ -th component of vector  $\underline{u}_i$ . Moreover,  $rand(j)$  is the  $j$ -th value of a vector of uniformly distributed random numbers, and  $CR$  is the crossover constant, with  $0 < CR < 1$ . Constant  $CR$  indicates the percentage of mutations considered in the trial vector.

#### – Selection

In order to decide if a vector  $\underline{u}_i$  may be element of new population of generation  $G+1$ , each vector  $\underline{u}_{i,G+1}$  is compared with the previous vector  $\underline{x}_{i,G}$ . If vector  $\underline{u}_{i,G+1}$  gives a smaller value of objective

function  $H$  than  $\underline{x}_{i,G}$ ,  $\underline{u}_{i,G+1}$  is selected as the new vector of population  $G+1$ ; otherwise, the old vector  $\underline{x}_{i,G}$  is retained:

$$\underline{x}_{i,G+1} = \begin{cases} \underline{u}_{i,G+1} & \text{if } H(\underline{u}_{i,G+1}) < H(\underline{x}_{i,G}) \\ \underline{x}_{i,G} & \text{if } H(\underline{u}_{i,G+1}) \geq H(\underline{x}_{i,G}) \end{cases}$$

with  $i = 1, 2, \dots, NP$ .

– Convergence

In the convergence rule, values of the objective function obtained from the population  $G+1$  are compared. Vectors are ordered depending on values of objective function as:

$$\tilde{\underline{x}}_{1,G+1} \prec \tilde{\underline{x}}_{2,G+1} \prec \dots \prec \tilde{\underline{x}}_{NP,G+1}$$

such that:

$$H(\tilde{\underline{x}}_{1,G+1}) < H(\tilde{\underline{x}}_{2,G+1}) < \dots < H(\tilde{\underline{x}}_{NP,G+1})$$

Convergence rule is then based on the difference of values  $H$  of the objective function of the first  $NC$  vectors and the distances between the same vectors,  $NC$  being the number of controlled vectors. The first, convergence test can be expressed as:

$$\Delta_i^H = \frac{|H(\tilde{\underline{x}}_{i,G+1}) - H(\tilde{\underline{x}}_{i+1,G+1})|}{H(\tilde{\underline{x}}_{i,G+1})} < VTR_1 \quad (1.10)$$

where  $i = 1, 2, \dots, NC$  and  $VTR_1$  is the prescribed precision.

Control of values of objective function  $H$  only can be not sufficient when the object function has a low gradient close to the minimum solution. For this reason, convergence requires also that the relative distance between the components of the first  $NC$  vectors is small, i.e.:

$$\Delta_{ij}^x = \frac{|\tilde{x}_{ji,G+1} - \tilde{x}_{ji+1,G+1}|}{\tilde{x}_{ji,G+1}} < VTR_2 \quad (1.11)$$

### The response surface methodology

The basic concept of the response surface method is to approximate the original complex or implicit target function using a simple and explicit interpolation function.

The response surface method was originally proposed by Khuri and Cornell (1996) as a statistical tool, to find the operating conditions of a chemical process at which some response was optimized.

Subsequently, the use of RSMs has been extended to other fields, especially to engineering problems involving the execution of complex computer analysis codes. In this case, in fact, RS methods can be used to alleviate the computational effort. Khuri and Cornell (1996) provided

modern perspectives of RS method applied to structural reliability analyses. The base idea of the surface response method is that a cost function can be defined, such as:

$$\underline{H} = \underline{g}(\underline{x}) \quad (1.12)$$

where  $\underline{x}$  denotes the  $D$ -dimensional vector of design parameters and  $\underline{g}(\underline{x})$  is called response function.

If  $\underline{g}(\underline{x})$  is a continuous and differentiable function, it can be locally represented with a Taylor series expansion from an arbitrary point  $\underline{x}_k$  :

$$H = \underline{g}(\underline{x}_k) + \nabla \underline{g}(\underline{x}_k)^T \cdot \underline{p} + \frac{1}{2} \cdot \underline{p}^T \cdot \nabla^2 \underline{g}(\underline{x}_k) \cdot \underline{p} \quad (1.13)$$

where  $\nabla \underline{g}(\underline{x}_k)$  and  $\nabla^2 \underline{g}(\underline{x}_k)$  are, respectively, the gradient vector which contains the first-order partial derivatives of function  $\underline{g}$  and the Hessian matrix (second-order partial derivatives) evaluated at  $\underline{x}_k$ . Many practical evaluation techniques are available to define  $\underline{g}(\underline{x})$ . Among those methods, reduction of Eq. 1.13 to a polynomial expression is the idea of RSM.

In classical RSM, the response surface is obtained by combining first or second order polynomials fitting the objective function defined in a set of sampling points. Second order approximations are commonly used in structural problems due to the computational efficiency with acceptable accuracy. Higher order polynomials are rarely used because the number of coefficients to be determined strongly increases with the order. Furthermore, some authors used quadratic polynomials without the cross terms, originating incomplete polynomials.

Adopting a second-order approximation function, Eq. 1.13 can be written as follows:

$$H = \frac{1}{2} \cdot \underline{x}^T \cdot \underline{Q} \cdot \underline{x} + \underline{L}^T \cdot \underline{x} + \beta_0 \quad (1.14)$$

where  $\underline{Q}$  is a  $D \times D$  coefficient matrix collecting the quadratic terms,  $\underline{L}$  is a  $D$ -dimension vector of linear terms and  $\beta_0$  is a constant.

Following the procedure proposed by Khuri and Cornell (1996), a limited number of selected numerical simulations (called experiments) is used in order to obtain an analytical relation between the mean values of identification parameters  $x_1, x_2$  and the target function  $H$ . Without loss of generality and for the sake of simplicity, in the following only 2 parameters ( $x_1, x_2$ ) will be considered. Therefore, Eq. 1.14 can be written as follows:

$$H = \beta_0 + \beta_1 \cdot x_1 + \beta_2 \cdot x_2 + \beta_3 \cdot x_1^2 + \beta_4 \cdot x_2^2 + \beta_5 \cdot x_1 \cdot x_2 \quad (1.15)$$

where coefficients  $\beta$  are unknown. In this method, response surface function includes the first and second order terms.

If  $NS$  observations are available, Eq. 1.14 can be expressed in a linear matrix notation as:

$$\underline{H} = \underline{Z} \cdot \underline{\beta} \quad (1.16)$$

where vector  $\underline{\beta}$  collects the unknown parameters of the response surface and:

$$\underline{H} = \begin{bmatrix} H_1(x_{1,1}, x_{2,1}) \\ H_2(x_{1,2}, x_{2,2}) \\ \vdots \\ H_{NS}(x_{1,NS}, x_{2,NS}) \end{bmatrix} \quad (1.17)$$

$$\underline{Z}(x_1, x_2) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_{1,1} & x_{1,2} & \cdots & x_{1,NS} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,NS} \\ x_{1,1}^2 & x_{1,2}^2 & \cdots & x_{1,NS}^2 \\ x_{2,1}^2 & x_{2,2}^2 & \cdots & x_{2,NS}^2 \\ x_{1,1} \cdot x_{2,1} & x_{1,2} \cdot x_{2,2} & \cdots & x_{1,NS} \cdot x_{2,NS} \end{bmatrix} \quad (1.18)$$

Constants  $\underline{\beta}$  are determined by applying the least square estimates method, so obtaining:

$$\underline{\beta} = (\underline{Z}^T \cdot \underline{Z})^{-1} \cdot \underline{Z}^T \cdot \underline{H} \quad (1.19)$$

In Eq. 1.19, all coefficients  $\underline{\beta}$  have equal weight. However, a good RSM must be generated such that it describes the target function well close to the solution point. The following weighted regression method is proposed in Myers and Montgomery (1995) and Kaymaz and McMahon (2005) to determine the coefficients of the RSM:

$$\underline{\beta} = (\underline{Z}^T \cdot \underline{W} \cdot \underline{Z})^{-1} \cdot \underline{Z}^T \cdot \underline{W} \cdot \underline{H} \quad (1.20)$$

where  $\underline{W}$  is an  $NS \times NS$  diagonal matrix of weight coefficients. For them, the following expression can be used:

$$w_i = \exp\left(-\frac{g(\underline{x}_i) - y_{best}}{y_{best}}\right) \quad (1.21)$$

where:

$$y_{best} = \min(g(\underline{x}_i)) \quad (1.22)$$

Many algorithms have been proposed to select appropriate set of sampling points  $\underline{x}_k$ , in order to obtain better fitting of response function. Detailed description of RSM methods including implementation and sampling strategies can be obtained from Khuri and Cornell (1996). The main



disadvantage of the use of RSM is that a local minimum can be reached if the target function presents more than one local minima. The use of the so-called general response surface method (GRSM) (Alotto et al., 1997) can partially resolve this problem, but this approach is applicable to low dimensional problems only, since its practical efficiency deteriorates with a high number of design variables.

### Implementation

The RSM methodology is introduced in Differential Evolution algorithm to improve performance in term of speed rate and to obtain higher precision of results. The algorithmic scheme of the modified DE algorithm by the use of a quadratic response function (called in the following *DE-Q*) is shown in Figure 1.3.

First, the initial population is selected randomly. At each iteration,  $NP$  sets containing  $NS$  vectors are chosen (with  $NS < NP$ ). Starting from the  $NS$  sampling points, a *RS* is calibrated to fit the cost function  $H$ . Solving the linear system of Eq. 1.20, coefficients  $\underline{\beta}$  can be obtained and, from them, it can be checked if the *RS* function has a convex shape. If it is the case (Figure 1.4(a)), the new parameter vector is defined as the minimum of a second-order polynomial approximation, i.e.:

$$\underline{v}_{i,G+1} = \underline{x}^* \mid H(\underline{x}^*) = \min(g(\underline{x})) \quad (1.23)$$

Otherwise (Figure 1.4(b)), classical Mutation operation based on linear combination is performed to obtain the trial vector  $\underline{v}_{i,G+1}$ , Crossover and Selection operations are then defined as in the original DE algorithm.

It is worth noting that the shape of objective function is usually unknown. If it presents only one (global) minimum, second-order approximation provides for the solution in a very low number of iterations. On the other hand, even if local minima are present, global minimum is expected to be reached since multiple search points are used simultaneously. Moreover, if the minimum of second order approximation gives a higher values of target function (see Figure 1.4(c)), it can be rejected in the Selection operation (the old vector  $\underline{x}_{i,G}$  is retained). Finally, in order to detect the global minimum, several evaluations must be performed by using Genetic and Evolutionary algorithms, in order to obtain the prescribed precision. Close to the solution (Figure 1.4(d)), the second-order approximation gives very good performance in term of speed rate and higher accuracy with respect to original algorithm.

For these reasons, global performance in term of speed rate is strongly improved by introducing the second order approximation by RSM and high precision of results of the original DE algorithm is preserved. This procedure appears to be more efficient with respect to GRSM proposed in (Alotto et

al., 1997), where DE and RSM method are used alternatively, because the latter is characterized by sensitivity of performance with respect to rules governing the switch between the algorithms.

### 1.2.2 Trust-Region Algorithm

Starting from a function to minimize,  $f(x)$ , where the function takes vector arguments and returns scalars. Considering the unconstrained minimization problem, if you are at a point  $x$  in  $n$ -space and you want to move toward point with a lower function value, the basic idea is to approximate  $f$  with a simpler function  $q$ , which reasonably reflects the behavior of the original function  $f$  in a neighborhood  $N$  around point  $x$ . A trial step  $s$  is computed by minimizing over  $N$  (which is called Trust-Region). The sub-problem can be written as follows:

$$\min_s \{q(\underline{s}), \underline{s} \in N\} \quad (1.24)$$

The current point is updated to be  $x + s$  if  $f(x + s) < f(x)$ , otherwise the current point remains unchanged and  $N$  is shrunk and the trial step computation is repeated.

In the standard Trust-Region method (Moré and Sorensen, 1983), the quadratic approximation  $q$  is defined by the first two terms of the Taylor approximation of  $f$  at point  $x$ .  $N$  is usually spherical or ellipsoidal in shape. The Trust-Region sub-problem is stated as follows:

$$\min_s \left\{ \frac{1}{2} \cdot \underline{s}^T \cdot \underline{\tilde{H}} \cdot \underline{s} + \underline{s}^T \cdot \underline{g} \text{ such that } \|\underline{D} \cdot \underline{s}\| \leq \Delta \right\} \quad (1.25)$$

where  $\underline{g}$  is the gradient of  $f$  at current point  $x$ ,  $\underline{\tilde{H}}$  is the Hessian matrix,  $\underline{D}$  is a diagonal scaling matrix,  $\Delta$  is a positive scalar and  $\|\cdot\|$  is the 2-norm. The algorithm used for solving Eq. 1.25 involves the computation of a full eigensystem and a Newton process applied to the secular equation:

$$\frac{1}{\Delta} - \frac{1}{\|\underline{s}\|} = 0 \quad (1.26)$$

Such algorithm provide an accurate solution to Eq. 1.25 but requires time proportional to several factorizations of  $\underline{\tilde{H}}$ . Therefore, for Trust-Region problems a different approach is needed. Several approximation and heuristic strategies, based on Eq. 1.25, have been proposed in literature (Byrd et al., 1988; Steihaug, 1983). The approximation approach followed, for instance, in MATLAB (MathWorks, 2005, MathWorks, 2017) is to restrict the Trust-Region sub-problem to a two-dimensional subspace  $S$  (Branch et al., 1999; Byrd et al., 1988). Once the subspace  $S$  has been computed, the work to solve Eq. 1.25 is trivial even if full eigenvalue/eigenvector information is

needed (since in the subspace, the problem is only two-dimensional). The predominant work has now shifted to the determination of the subspace.

The two-dimensional subspace  $S$  is determined with the aid of the preconditioned conjugate gradient (PCG) process. The solver defines  $S$  as a linear space spanned by  $\underline{s}_1$  and  $\underline{s}_2$ , where  $\underline{s}_1$  is in the direction of the gradient  $\underline{g}$  and  $\underline{s}_2$  is either an approximate Newton direction, achieved solving:

$$\underline{\tilde{H}} \cdot \underline{s}_2 = -\underline{g} \quad (1.27)$$

or a direction of negative curvature:

$$\underline{s}_2^T \cdot \underline{\tilde{H}} \cdot \underline{s}_2 < 0 \quad (1.28)$$

A sketch of unconstrained minimization process using Trust-Region algorithm is now easy to give:

1. Formulate the two-dimensional Trust-Region sub-problem.
2. Solve Eq. 1.25 to determine the trial step  $s$ .
3. If  $f(x+s) < f(x)$ , then  $x' = x + s$ .
4. Adjust  $\Delta$  of Eq. 1.25.

These four steps have to be repeated until convergence. The Trust-Region dimension  $\Delta$  is adjusted according to standard rules. In particular, it is decreased if the trial step is not accepted, i.e.  $f(x+s) \geq f(x)$  (Coleman and Verma, 2001; Sorensen, 1994).

An overview of the entire method, also considering the case of constrained minimization process, can be found in Coleman and Li, 1996; Conn et al., 2000 and MathWorks, 2005.

### 1.3 Infills modeling criteria

A lot of researchers, in the early 90s, paid attention in defining a numerical models which simulate the behavior of R.C. infilled frames. These models, suggested by the different authors, for the study of interaction between infill panels and frames could be classified, through consolidated approach, in micro, meso and macro-models. Micro and meso-modeling are currently used to analyze portion of a buildings or single infill panels, which results too onerous (in terms of calculation time) for the study of whole buildings. Among the macro-models proposed to reproduce the interaction between frames and infills, almost all of them are based on the concept of equivalent strut (Crisafulli et al., 2000; Asteris et al., 2011; Tarque et al., 2015). Further classification among these models can be made on the basis of the number of equivalent struts considered to model the presence of the infill panel. In Crisafulli et al., 2000 it is shown how the single equivalent strut is not suitable to represent the distribution of stresses in the infilled frame and, when these stresses are required or necessary, modeling with multi-struts has to be made. Proposals of models with multi-

struts can be found in Thiruvengadam, 1985; Chrysostomou et al., 2002; El-Dakhkhni et al., 2003; Crisafulli and Carr, 2007. In Crisafulli et al., 2000 and Asteris et al., 2011 it is shown how these methodologies of modeling with single or multiple struts provides almost equal results when the knowledge of the global behavior of the structure is required (e.g. determination of mode shapes and frequencies of the structure). Polyakov, 1960 was the first who model the masonry infill, inserted in a frame, as diagonal strut having only axial stiffness. Then different authors have analyzed that issue using analog approach. The various models herein considered give different methodologies and expression to define the width ( $w$ ) of the equivalent strut that, if multiplied for the thickness of the panel ( $t$ ), gives the cross-sectional area to be assigned to diagonal strut. Regarding the definition of the elastic properties of the material constituting the strut is used to adopt the elastic modulus  $E$  of the masonry panel. In Table 1.1 are listed the proposals of various authors for the definition of the ratio  $w/d$  where, as said before,  $w$  is the width of the equivalent strut while  $d$  is the length of the strut. For further details about the different definitions of  $w/d$  from different authors, one can see the references. The different author's proposals have been tested on a real structure, a seven storey, R.C. framed building infilled with hollow clay blocks. The different literature proposals gave very different results in terms of frequencies and mode shapes (Tondi et al., 2018)

In this work the procedure outlined by Stavridis (Stavridis, 2009) is used for fixing the stiffness of infills which are not subject of updating. This procedure is used for intact infill panels but also for infill panels with opening(s) and infill panels already damaged. The definition of the reduction parameters for taking into account the openings and the damages can be found, again, in Stavridis, 2009.

## 1.4 Organization of the thesis

The method proposed in this work is based on the inverse eigenvalues problem in the second family of the abovementioned methods of model updating. Starting from the experimental frequencies and mode shape vectors, the eigenvalues/eigenvectors problem can be written in matrix form. For the purposes of this work, two procedures will be analyzed. The first one relies on a Two Step algorithm in which the first one is in closed form, facing the eigenvalues/eigenvectors problem without considering the determinant equations. Comparison parameters must be computed before running the second step because, if these parameters satisfy fixed thresholds, the second one can be avoided. If the thresholds are not satisfy, the first trial solution will be the starting point for the subsequent iterative procedure (with Trust-Region algorithm) in which also the determinant

equations have been taken into account. With this procedure also the uncertainties in the parameters can be computed, starting from known values of uncertainties in frequencies and modes components. Closed relations can be achieved for partial derivatives and the parameters standard deviations can be computed using the theory of error propagation. This algorithm is very quick and the time saving, with respect other algorithm, can be up to 98%, as will be pointed out in Chapter 4. The quickness of the algorithm, along with the possibility to evaluate the uncertainties in the parameters, allows us to use it also for real-time assessment using real-time structural monitoring.

The second procedure uses a genetic algorithm with response surfaces (the so-called DE-Q algorithm, Savoia and Vincenzi, 2008) in which the entire minimization problem is faced by the genetic algorithm and the solution is completely iterative. It will be found that the two steps procedure with Trust-Region optimization is more computationally efficient with respect to the DE-Q one. All these procedures will be outlined in Chapter 2. In the same Chapter will be also treated the statistical analysis of parameters starting from generally distributed frequencies and mode shape vectors. A general algorithm will be given for the two abovementioned procedures in order to achieve the statistical analysis of the problem. In the last part of Chapter 2, the distribution analysis of parameters will also be introduced.

In Chapter 3 will be reported some theoretical results for the procedure in order to compute the maximum number of parameters achievable by the algorithm for fixed number of frequencies and mode shapes, all the calculations for the closed form solution and uncertainties evaluation given in Chapter 2, the goodness-of-definition (uniqueness of the solution) of parameters themselves and a procedure to find the maximum number of parameters for a non-ideal case.

These procedures were originally conceived for the model updating of infilled framed structures and therefore the sensitivity analysis of Chapter 4 were done for infilled framed buildings. The sensitivity analysis were performed on 2-D and 3-D infilled frames with different values of perturbations in frequencies and mode shapes and different parameters arrangements. In Chapter 4 will be also reported a comparison between the two procedures (in term of mean values and coefficients of variation of parameters) and also the comparison between systems with or without determinant equations. Moreover, the influence of the mode shapes number at disposal in the procedure was also studied.

In Chapter 5 two real cases, in which frequencies and mode shape vectors were available, will be analyzed in order to figure out the parameters values and to detect damages in the structures for different damage states. The first structure is a three storey, two bays 2-D infilled frames, tested through shake table in San Diego, CA, US, in which seven damage states were created scaling the

accelerograms of real earthquakes. The second one was a two storey 3-D infilled framed structure (located in El-Centro, CA, US), tested with ambient vibrations and forced vibrations with shaker, in which the damages were created artificially through walls removal. Four damage states were induced for this specimen. Two models with different number of parameters were analyzed. For this structure a lot of data windows for ambient vibrations were at disposal and therefore a statistical analysis was also done.

In Chapter 6 will be introduced the generalization of the procedure in order to take into account the presence of non-linear parameters and the case of viscous damping for classically damped structures.

In the last part of this work, Chapter 7, the algorithm will be used for achieving a first trial retrofitting for existing structures using the eigenvalues/eigenvectors equations without determinant ones. Two sample cases will be analyzed and the trial solutions for the retrofitting compared with respect to expected ones. The algorithm will be then generalized in order to consider the case of non-linear parameters.

Three Appendix will be reported in which simple cases will be analyzed. The first one will treat the distribution analysis for a simple case with matrices of grade 3 and only one parameter. The second Appendix will treat the analysis of the maximum number of parameters for a sample case. The last Appendix will be focused on the statistical analysis of a case with related non-linear parameter.

## Table of Chapter 1

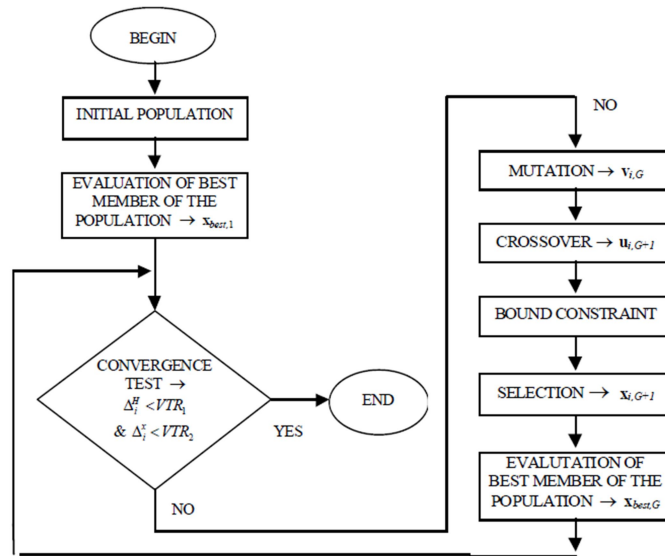
Author (year)	Expression	Notes
Holmes (1961)	$w/d = 1/3$	$\lambda_h < 2$
Stafford Smith (1967)	$0.10 < w/d < 0.25$	The value graphically depends on $\lambda_h$
Mainstone (1971)	$w/d = 0.16\lambda_h^{-0.3}$	For $\lambda_h$ see Ref.
Mainstone (1974)	$w/d = 0.175\lambda_h^{-0.4}$	Adopted by FEMA-274 (1997) and FEMA-306 (1998)
Bazan & Meli (1980)	$w = (0.35 + 0.022\beta)h_m$	$0.9 \leq \beta \leq 11$ ; for $\beta$ see Ref.
Hendry (1981)	$w = \frac{1}{2}\sqrt{z_b^2 + z_c^2}$	For $z_b$ e $z_c$ see Ref.
Tassios (1984)	$w/d = 0.20\beta\sin\theta$	$1 \leq \beta \leq 5$
Liau & Kwan (1984)	$w/d = \frac{0.95\sin(2\theta)}{2\sqrt{\lambda_h}}$	$25^\circ \leq \theta \leq 50^\circ$
Decanini & Fantin (1987) For uncracked panels	$\frac{w}{d} = 0.085 + \frac{0.748}{\lambda_h}$ $\frac{w}{d} = 0.130 + \frac{0.393}{\lambda_h}$	For $\lambda_h \leq 7.85$ For $\lambda_h > 7.85$
Decanini & Fantin (1987) For cracked panels	$\frac{w}{d} = 0.010 + \frac{0.707}{\lambda_h}$ $\frac{w}{d} = 0.040 + \frac{0.470}{\lambda_h}$	For $\lambda_h \leq 7.85$ For $\lambda_h > 7.85$
Paulay & Priestley (1992)	$w/d = 0.25$	For $\lambda_h < 4.00$
Durrani & Luo (1994)	$w/d = \gamma \cdot \sin(2\theta)$	For $\gamma$ see Ref.
Cavaleri et al. (2005) Amato et al. (2008) Campione et al. (2014)	$\frac{w}{d} = \frac{k}{z} \cdot \frac{c}{(\lambda^*)^\beta}$	$c$ e $\beta$ take account of the Poisson's ratio, $k$ takes into account the vertical load and $z$ is a geometrical parameter.

**Table 1.1:** Relations proposed in literature for the calculation of  $w/d$  ratio (Tarque et al., 2015).

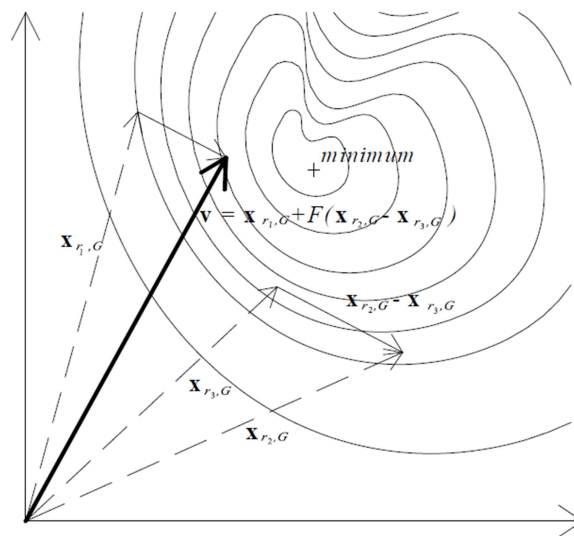




# Figures of Chapter 1



**Figure 1.1:** Algorithm scheme for DE method (Vincenzi and Savoia, 2010).



**Figure 1.2:** Mutation process by random combination (Vincenzi and Savoia, 2010).

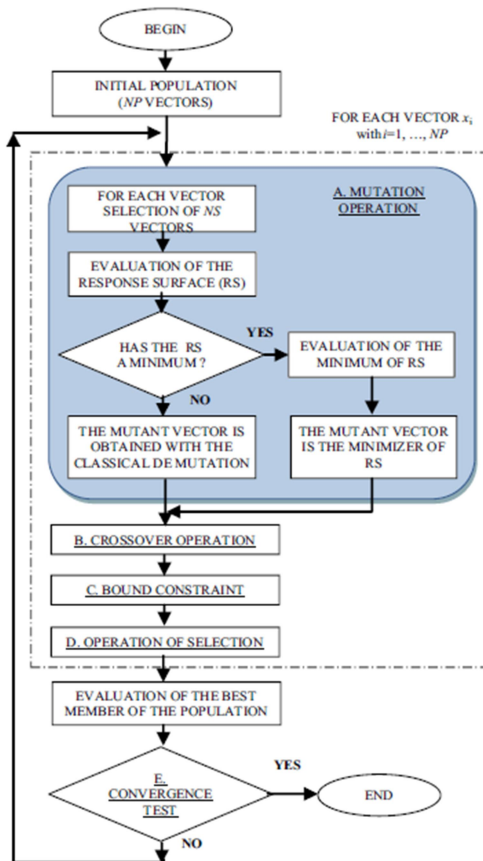


Figure 1.3: Flowchart of the DE-Q algorithm (Vincenzi and Savoia, 2015).

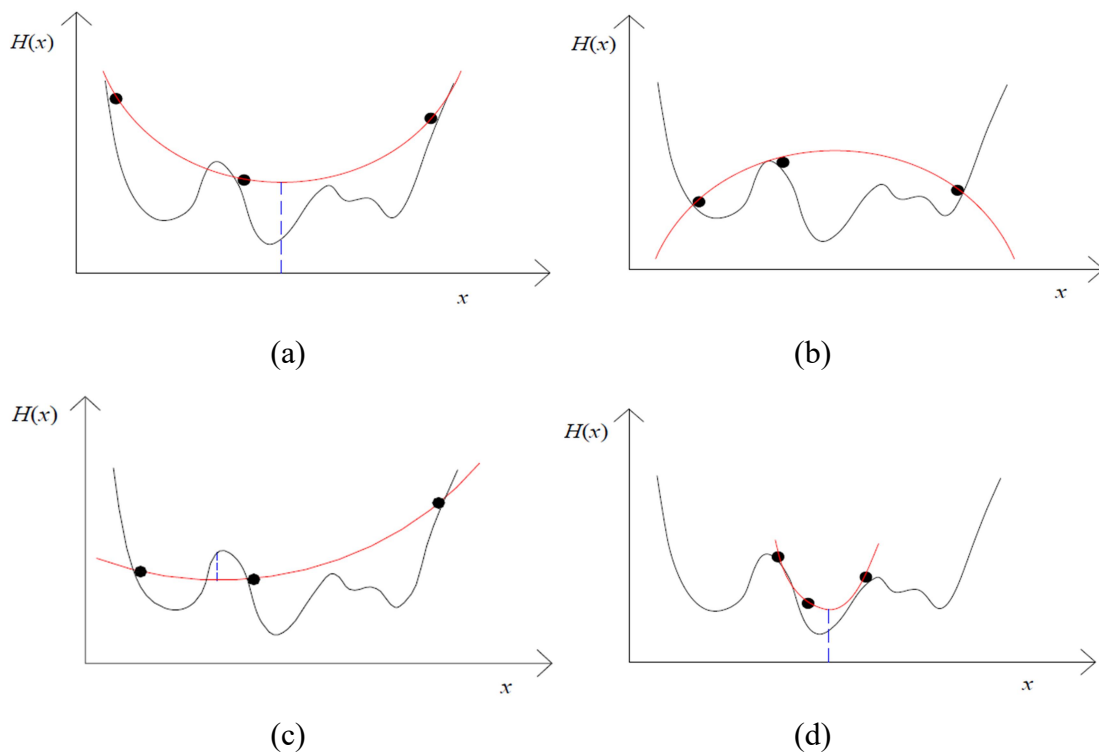


Figure 1.4: Approximation of cost function by quadratic response surface. (Vincenzi and Savoia, 2010).

## CHAPTER 2

### DESCRIPTION OF THE INNOVATIVE PROCEDURE

#### 2.1 Introduction

In this chapter, the definition of a Two Step algorithm, with uncertainties evaluation, will be developed starting from the definition of the system of equations to be solved and the new target function. Even the weighting functions will have to be defined and introduced in the procedure.

Firstly, the first step, in which system without determinant equations is used, will be studied and a closed form solution will be achieved.

Secondly, an improvement of the solution itself, with the second step, will be done using the Trust-Region algorithm with starting point being the previously computed solution (from direct formulation).

The system of equations allows us to write the partial derivatives in closed form, with respect to the frequencies and modes components. The standard deviations of parameters can therefore be achieved with direct formulation (using the theory of errors propagation). In this way the uncertainties in the parameters, starting from standard deviations in frequencies and modes components, can be achieved.

The statistical analysis utilizing the Monte Carlo procedure will then be introduced. For all these procedures, a numerical example will be studied.

After that, procedure with DE-Q algorithm will be introduced.

Eventually, a statistical distribution analysis of parameters, starting from known distributions for the frequencies and modes components, will be performed.

#### 2.2 First step: system without eigenvalues equations

The new target function relies on the superimposition of the stiffness and mass matrices of the system with the parameters chosen for the model updating. This procedure has got a general applicability, in all the civil structures. In the following paragraphs, such procedure is presented with respect to the problem of model updating of infilled frames (because of was originally conceived for that type of structure) but its applicability, once again, could be generalized.

### 2.2.1 Parameters definition and stiffness matrix decomposition

Starting from a structure with  $N$  different parameters, it is called  $K_T$  the total stiffness matrix of the model representing the actual structure, with all parameters having the proper value.  $K_0$  is, instead, the stiffness matrix with all the parameters set to null value. Then the  $K_j$  is the stiffness matrix with the  $j$ -th parameter having unit value and the other ones having null values. Then, through subtraction, one can obtain:

$$\begin{aligned}
 \underline{\underline{K}}_{r,1} &= \underline{\underline{K}}_1 - \underline{\underline{K}}_0 \\
 \underline{\underline{K}}_{r,2} &= \underline{\underline{K}}_2 - \underline{\underline{K}}_0 \\
 &\cdot \\
 &\cdot \\
 &\cdot \\
 \underline{\underline{K}}_{r,N} &= \underline{\underline{K}}_N - \underline{\underline{K}}_0
 \end{aligned} \tag{2.1}$$

therefore, the total stiffness matrix can be reconstructed in the following way:

$$\underline{\underline{K}}_T = \underline{\underline{K}}_0 + a_1 \cdot \underline{\underline{K}}_{r,1} + a_2 \cdot \underline{\underline{K}}_{r,2} + \dots + a_N \cdot \underline{\underline{K}}_{r,N} \tag{2.2}$$

in which  $a_1, a_2, \dots, a_N$  are the unknown parameters.

This procedure is illustrated graphically in an example of two bays three storey infilled frame, with two parameters, in Figure 2.1.

### 2.2.2 Eigenvalues/Eigenvectors problem

In order to define the new target function, the global eigenvalues/eigenvectors problem has to be analyzed. The problem can be set in the following way:

$$\begin{cases}
 p_1 \cdot \left[ \underline{\underline{K}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{K}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{M}} \right] \cdot \underline{\underline{\varphi}}_1 = \underline{\underline{v}}_1 \\
 \vdots \\
 p_i \cdot \left[ \underline{\underline{K}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{K}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{M}} \right] \cdot \underline{\underline{\varphi}}_i = \underline{\underline{v}}_i \\
 \vdots \\
 p_n \cdot \left[ \underline{\underline{K}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{K}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{M}} \right] \cdot \underline{\underline{\varphi}}_n = \underline{\underline{v}}_n
 \end{cases} \tag{2.3}$$

in which  $\underline{\underline{K}}_0, \underline{\underline{K}}_{r,s}$  are matrices previously defined,  $\omega_i$  is the  $i$ -th experimentally identified frequency,  $\underline{\underline{\varphi}}_i$  is the corresponding experimentally identified mode shape,  $\underline{\underline{v}}_i$  is the vector of residue deriving from the equations of  $i$ -th frequency and mode shape. If the exact value of

frequencies, mode shapes and parameters were used, the system would be simplified with all the residue equal to 0.

Each of the  $n$  matrix equations previously described contains  $m$  scalar equations, in which  $m$  is the size of the problem (the size of the stiffness and mass matrices). These scalar equations are not all linearly independent but there is a precise relation which connects the number of linearly independent equations (and therefore the maximum number of parameters obtainable). In Chapter 3 the relationship will be analyzed.

In order to include the possibility that also the mass matrix presents some unknown values (and therefore it needs the definition of some parameters), the system of Eq. 2.3 can be modified in the following way:

$$\begin{cases} p_1 \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_1 = \underline{\underline{\mathbf{v}}}_1 \\ \vdots \\ p_i \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_i = \underline{\underline{\mathbf{v}}}_i \\ \vdots \\ p_n \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_n = \underline{\underline{\mathbf{v}}}_n \end{cases} \quad (2.4)$$

in which  $q$  is the number of parameters in the stiffness matrix and  $N-q$  being the number of parameters in the mass matrix.  $\underline{\underline{\mathbf{M}}}_s$  is the mass matrix associated with the  $s$ -th coefficient.

Using the residue obtained in this way, the target function (to minimize) can be defined as the 2-norm of the residual vector:

$$H = \sqrt{\sum_{i=1}^n \sum_{j=1}^m v_{ji}^2} \quad (2.5)$$

### 2.2.3 Definition of weight functions

The introduction of weight functions has been made in order to drive the solution to a better understanding of dynamic behavior of those modes which have more influence in the global behavior of the structure. For this purpose the  $p_i$  coefficients have been defined as follows:

- For 2D structures:

$$p_i = \frac{\tilde{M}_i \cdot \sum_{k=1}^n \omega_k}{\sum_{k=1}^n \tilde{M}_k \cdot \omega_i} \quad (2.6)$$

in which:

$$\tilde{M}_i = \frac{(\underline{\varphi}_i^T \cdot \underline{M}_0 \cdot \underline{1})^2}{\underline{\varphi}_i^T \cdot \underline{M}_0 \cdot \underline{\varphi}_i} \quad (2.7)$$

is the effective modal mass excited from the  $i$ -th mode of vibration and  $\omega_i$  is the  $i$ -th circular frequency.

– For 3D structures:

$$p_i = \frac{\tilde{M}_{x,i} + \tilde{M}_{y,i} + \tilde{M}_{\theta,i}}{\sum_{k=1}^n (\tilde{M}_{x,k} + \tilde{M}_{y,k} + \tilde{M}_{\theta,k})} \cdot \frac{\sum_{k=1}^n \omega_k}{\omega_i} \quad (2.8)$$

in which  $\tilde{M}_{x,i}, \tilde{M}_{y,i}, \tilde{M}_{\theta,i}$  are the effective modal masses, computed with respect to  $\underline{M}_0$ , for the  $i$ -th mode of vibration in the  $x$ ,  $y$  direction and rotation about the vertical axes, respectively.  $\omega_i$  is the  $i$ -th circular frequency.

Another weighting parameters, without physical meaning, have been included in the equations in order to take into account the different unit and order of magnitude of rotational DOFs with respect to the translational ones, only in 3-D problems (Ewins, 2000). These coefficients are not shown in the equations already written because of they have to be inserted in the stiffness and mass matrices. These parameters have to be calibrated from time to time in order to obtain the same order of magnitude of residue values. For 3-D infilled frames has been seen that if all the rotational rows of the stiffness and mass matrices are divided by the coefficient presented in Eq. 2.9, the results are satisfactory.

$$p_{\theta,l} = \frac{M_{\theta,l}}{M_{t,l}} \quad (2.9)$$

in which  $M_{\theta,l}$  is the rotational mass and  $M_{t,l}$  is the translational mass ( $M_x$  or  $M_y$ ),  $l=1,2,\dots,t$  in which  $t$  is the number of the storey. Once again, this coefficients are computed with respect to  $\underline{M}_0$ .

## 2.3 Closed form solution of the first step

The direct non-iterative formulation is presented for the system of Eq. 2.4. Two formulations have been achieved, the first one uses the pseudo-inverse of the coefficient matrix. Rewriting the system, a second formulation has been reached through partial derivatives of the problem.

### 2.3.1 Direct formulation

System of Eq. 2.4 can be rewritten in order to find a closed formulation of the problem:

$$\underline{\underline{B}} \cdot \underline{a} = \underline{c} \quad (2.10)$$

in which  $\underline{\underline{B}}$  is the coefficients matrix of dimensions  $n \times N$ ,  $\underline{a}$  is the parameters vector of dimensions  $N \times 1$  and  $\underline{c}$  is the know-values vector of dimensions  $n \times 1$ .

The abovementioned matrix and vectors have the following definition (rewriting system of Eq. 2.4):

$$\begin{cases} \underline{Q}_{1:m,s} = p_1 \cdot \underline{K}_{r,s} \cdot \underline{\varphi}_1 \\ \underline{Q}_{m+1:2m,s} = p_2 \cdot \underline{K}_{r,s} \cdot \underline{\varphi}_2 \\ \vdots \\ \underline{Q}_{(n-1)m+1:nm,s} = p_n \cdot \underline{K}_{r,s} \cdot \underline{\varphi}_n \end{cases} \quad (2.11)$$

in which  $s = 1, 2, \dots, q$  and  $\underline{Q}$  is a  $n \cdot m \times q$  matrix.

$$\begin{cases} \underline{R}_{1:m,s} = -p_1 \cdot \omega_1^2 \cdot \underline{M}_s \cdot \underline{\varphi}_1 \\ \underline{R}_{m+1:2m,s} = -p_2 \cdot \omega_2^2 \cdot \underline{M}_s \cdot \underline{\varphi}_2 \\ \vdots \\ \underline{R}_{(n-1)m+1:nm,s} = -p_n \cdot \omega_n^2 \cdot \underline{M}_s \cdot \underline{\varphi}_n \end{cases} \quad (2.12)$$

in which  $s = q + 1, q + 2, \dots, N$  and  $\underline{R}$  is a  $n \cdot m \times N - q$  matrix,  $\underline{Q}_{1:m,s}$  and  $\underline{R}_{1:m,s}$  indicating the rows from 1 to m and s-th column of matrix  $\underline{Q}$  and  $\underline{R}$  respectively.

The complete coefficients matrix is as follows:

$$\underline{\underline{B}} = \left[ \underline{Q} \quad \underline{R} \right] \quad (2.13)$$

which is a  $n \cdot m \times N$  matrix.

$$\underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_q \\ a_{q+1} \\ a_{q+2} \\ \vdots \\ a_N \end{bmatrix} \quad (2.14)$$

is the parameters row.

$$\begin{cases} \underline{c}_{1:m} = p_1 \cdot (\omega_1^2 \cdot \underline{M}_0 - \underline{K}_0) \cdot \underline{\varphi}_1 \\ \underline{c}_{m+1:2m} = p_2 \cdot (\omega_2^2 \cdot \underline{M}_0 - \underline{K}_0) \cdot \underline{\varphi}_2 \\ \vdots \\ \underline{c}_{(n-1)m+1:nm} = p_n \cdot (\omega_n^2 \cdot \underline{M}_0 - \underline{K}_0) \cdot \underline{\varphi}_n \end{cases} \quad (2.15)$$

is the known-values row.

The weighting functions defined in Eq. 2.9 have to be applied also in this case in order to meet the requirement of same order of magnitude between equations in the system of Eq. 2.4.

The system of Eq. 2.4, rewritten in the form of Eq. 2.10, is usually a over-determined system and can be solved with pseudo-inverse procedure:

$$\underline{a} = \underline{B}^+ \cdot \underline{c} \quad (2.16)$$

or

$$\underline{a} = (\underline{B}^T \cdot \underline{B})^{-1} \cdot \underline{B}^T \cdot \underline{c} \quad (2.17)$$

in which  $\underline{B}^+$  is the pseudo-inverse of matrix  $\underline{B}$ ; or through matrix decomposition and SVD (Ewins, 2000). Both look for a least-squares solution of the system.

The maximum number of parameters achievable through this procedure, the same of the procedure described above, in function of the number of modes available, will be treated in Chapter 3.

### 2.3.2 Alternative solution of the system

Using the procedure with partial derivatives (all the mathematical computations are given in Chapter 3) another closed-form relation can be achieved in which the system is determined (the solution is the same as in the previous case):

$$\underline{\tilde{A}} \cdot \underline{a} = \underline{\tilde{b}} \quad (2.18)$$

in which:

$$\underline{\tilde{A}} = \begin{bmatrix} \|\underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}\|^2 & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \|\underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}\|^2 & \cdots & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \|\underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}\|^2 & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \|\underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}\|^2 & \cdots & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \|\underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}\|^2 \end{bmatrix} \quad (2.19)$$



$$\underline{\tilde{b}} = \begin{bmatrix} \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \vdots \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \vdots \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \end{bmatrix} \quad (2.20)$$

in which:

$$\underline{\tilde{K}}_{r,s} = \begin{bmatrix} p_1 \cdot \underline{K}_{r,s} & \underline{0} & \cdots & \underline{0} \\ \underline{0} & p_2 \cdot \underline{K}_{r,s} & \cdots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \cdots & p_n \cdot \underline{K}_{r,s} \end{bmatrix} \quad (2.21)$$

for  $1 \leq s \leq q$ ;

$$\underline{\tilde{M}}_s = \begin{bmatrix} -p_1 \cdot \omega_1^2 \cdot \underline{M}_s & \underline{0} & \cdots & \underline{0} \\ \underline{0} & -p_2 \cdot \omega_2^2 \cdot \underline{M}_s & \cdots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \cdots & -p_n \cdot \omega_n^2 \cdot \underline{M}_s \end{bmatrix} \quad (2.22)$$

for  $q+1 \leq s \leq N$ ;

$$\underline{\tilde{\varphi}} = \begin{bmatrix} \underline{\varphi}_1 \\ \underline{\varphi}_2 \\ \vdots \\ \underline{\varphi}_n \end{bmatrix} \quad (2.23)$$

$$\underline{\tilde{\psi}} = \underline{\tilde{M}}_0 \cdot \underline{\tilde{\varphi}} - \underline{\tilde{K}}_0 \cdot \underline{\tilde{\varphi}} \quad (2.24)$$

$$\underline{\tilde{K}}_0 = \begin{bmatrix} p_1 \cdot \underline{K}_0 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & p_2 \cdot \underline{K}_0 & \cdots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \cdots & p_n \cdot \underline{K}_0 \end{bmatrix} \quad (2.25)$$

$$\underline{\tilde{M}}_0 = \begin{bmatrix} p_1 \cdot \omega_1^2 \cdot \underline{M}_0 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & p_2 \cdot \omega_2^2 \cdot \underline{M}_0 & \cdots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \cdots & p_n \cdot \omega_n^2 \cdot \underline{M}_0 \end{bmatrix} \quad (2.26)$$

$\| \cdot \|$  and  $\langle \cdot \rangle$  mean the norm of the vector and the dot product between vectors, respectively.

$\underline{\underline{\tilde{A}}}$  matrix is symmetric, because of the dot products satisfy the commutative property, is semi-positive defined and, for the uniqueness condition, is positive defined (the demonstrations are reported in Chapter 3).

One parameter can be written in the following closed relation, from the Cramer formulation (Casali et al., 2016):

$$a_s = \frac{\det(\underline{\underline{\tilde{A}}}_s)}{\det(\underline{\underline{\tilde{A}}})} \quad (2.27)$$

for  $s = 1, 2, \dots, N$ , in which  $\underline{\underline{\tilde{A}}}_s$  is the  $\underline{\underline{\tilde{A}}}$  matrix with the s-th column replaced by the  $\underline{\underline{\tilde{b}}}$  vector (vector of known terms).

Otherwise solving the determined system through matrix inversion:

$$\underline{\underline{a}} = \underline{\underline{\tilde{A}}}^{-1} \cdot \underline{\underline{\tilde{b}}} \quad (2.28)$$

### 2.3.3 Comparison parameters

After obtaining the complete set of parameters, the stiffness matrix can be computed through Eq. 2.2. After that, a posteriori eigenvalues analysis can be performed, frequencies and mode shapes ( $\bar{f}_i, \bar{\varphi}_i$ ) can be computed. With those parameters and the experimental ones, one can perform a comparison in terms of MAC and frequencies error:

$$MAC(\underline{\underline{\varphi}}_i, \bar{\underline{\underline{\varphi}}}_i) = \frac{|\langle \underline{\underline{\varphi}}_i; \bar{\underline{\underline{\varphi}}}_i \rangle|^2}{\|\underline{\underline{\varphi}}_i\|^2 \cdot \|\bar{\underline{\underline{\varphi}}}_i\|^2} \quad (2.29)$$

$$\Delta f_i [\%] = \frac{\bar{f}_i - f_i}{f_i} * 100 \quad (2.30)$$

in which  $i = 1, 2, \dots, n$ ;  $\underline{\underline{\varphi}}_i$  and  $f_i$  being the experimental modes and frequencies. If the values of comparison parameters don't satisfy fixed thresholds, the second step must be performed; otherwise the analysis can stop after the first one.

## 2.4 Numerical example

A structure with matrix of grade 3 (for example three storey, 2-D framed structure) is going to be studied considering 2 parameters updated (only in the stiffness matrix) and only 1 frequency and 1 mode shape included in the procedure. First of all the general equations is going to be rewritten for this particular case, after that the numerical values will be included in the example.

### 2.4.1 Direct formulation of the problem

For a structure with matrix of grade 3, 2 parameters updated (only in the stiffness matrix) and only 1 frequency and 1 mode shape included in the procedure, Eq. 2.18 can be written in extensive way as follows:

$$\begin{bmatrix} \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle & \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{\psi} \rangle \\ \langle \underline{K}_{r,2} \cdot \underline{\varphi}; \underline{\psi} \rangle \end{bmatrix} \quad (2.31)$$

in which, having only one mode:

$$\underline{\varphi} = \begin{bmatrix} \varphi_{11} \\ \varphi_{21} \\ \varphi_{31} \end{bmatrix} \quad (2.32)$$

$$\underline{\psi} = \omega_1^2 \underline{M}_0 \cdot \underline{\varphi} - \underline{K}_0 \cdot \underline{\varphi} \quad (2.33)$$

Matrix  $\tilde{\underline{A}}$  is semi positive defined:

$$\|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 \geq 0 \quad (2.34)$$

$$\det(\tilde{\underline{A}}) = \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 \cdot \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 - \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle^2 \geq 0 \quad (2.35)$$

Eq. 2.35 is the inequality of Cauchy-Schwartz (particular case of Eq. 3.63 of Chapter 3).

The uniqueness of the solution is ensured if:

$$\|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 \cdot \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 - \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle^2 \neq 0 \quad (2.36)$$

$$\|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 \cdot \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \neq \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle^2 \quad (2.37)$$

therefore, if:

$$\underline{K}_{r,1} \cdot \underline{\varphi} \neq l \cdot \underline{K}_{r,2} \cdot \underline{\varphi} \quad (2.38)$$

$\forall l \in \mathfrak{R}$ , which is a particular case of Eqs. 3.51 and 3.52 of Chapter 3.

The solution can be written as follows (particular case of Eq. 2.27):

$$a_1 = \frac{\det \begin{bmatrix} \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{\psi} \rangle & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,2} \cdot \underline{\varphi}; \underline{\psi} \rangle & \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \end{bmatrix}}{\det \begin{bmatrix} \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle & \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \end{bmatrix}} \quad (2.39)$$

$$a_2 = \frac{\det \begin{bmatrix} \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{\psi} \rangle \\ \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle & \langle \underline{K}_{r,2} \cdot \underline{\varphi}; \underline{\psi} \rangle \end{bmatrix}}{\det \begin{bmatrix} \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle & \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \end{bmatrix}} \quad (2.40)$$

or, developing the calculations:

$$a_1 = \frac{\langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{\psi} \rangle \cdot \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 - \langle \underline{K}_{r,2} \cdot \underline{\varphi}; \underline{\psi} \rangle \cdot \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle}{\|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 \cdot \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 - \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle^2} \quad (2.41)$$

$$a_2 = \frac{\langle \underline{K}_{r,2} \cdot \underline{\varphi}; \underline{\psi} \rangle \cdot \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 - \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{\psi} \rangle \cdot \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle}{\|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 \cdot \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 - \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle^2} \quad (2.42)$$

## 2.4.2 Numerical example

A numerical example is performed using the following matrices as known ones (only 4 decimal digits are going to be used):

$$\underline{K}_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad (2.43)$$

$$\underline{K}_{r,1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (2.44)$$

$$\underline{K}_{r,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad (2.45)$$

$$\underline{M}_0 = \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.02 \end{bmatrix} \quad (2.46)$$

$$\omega_{1,ref}^2 = 2.4031 \cdot 10^2 \text{ rad} / \text{s} \quad (2.47)$$

$$f_{1,ref} = 2.4672 \text{ Hz} \quad (2.48)$$

$$\underline{\varphi}_{ref} = \begin{bmatrix} 0.3012 \\ 0.5466 \\ 0.7813 \end{bmatrix} \quad (2.49)$$

All the relations introduced in paragraph 2.4.1 take the subsequent values:

$$\underline{\psi} = \omega_1^2 \underline{M}_0 \cdot \underline{\varphi} - \underline{K}_0 \cdot \underline{\varphi} = 2.4031 \cdot 10^2 \cdot \begin{bmatrix} 0.02 & 0 & 0 \\ 0 & 0.02 & 0 \\ 0 & 0 & 0.02 \end{bmatrix} \cdot \begin{bmatrix} 0.3012 \\ 0.5466 \\ 0.7813 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.3012 \\ 0.5466 \\ 0.7813 \end{bmatrix} =$$

$$\underline{\psi} = \begin{bmatrix} 1.3918 \\ 2.6164 \\ 3.5204 \end{bmatrix} \quad (2.50)$$

$$\underline{K}_{r,1} \cdot \underline{\varphi} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0.3012 \\ 0.5466 \\ 0.7813 \end{bmatrix} = \begin{bmatrix} 0.0558 \\ 0.2454 \\ 0 \end{bmatrix} \quad (2.51)$$

$$\underline{K}_{r,2} \cdot \underline{\varphi} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.3012 \\ 0.5466 \\ 0.7813 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2347 \\ 0.2347 \end{bmatrix} \quad (2.52)$$

$$\langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{\psi} \rangle = [0.0558 \quad 0.2454 \quad 0] \cdot \begin{bmatrix} 1.3918 \\ 2.6164 \\ 3.5204 \end{bmatrix} = 0.7197 \quad (2.53)$$

$$\langle \underline{K}_{r,2} \cdot \underline{\varphi}; \underline{\psi} \rangle = [0 \quad -0.2347 \quad 0.2347] \cdot \begin{bmatrix} 1.3918 \\ 2.6164 \\ 3.5204 \end{bmatrix} = 0.2122 \quad (2.54)$$

$$\| \underline{K}_{r,1} \cdot \underline{\varphi} \|^2 = [0.0558 \quad 0.2454 \quad 0] \cdot \begin{bmatrix} 0.0558 \\ 0.2454 \\ 0 \end{bmatrix} = 0.0633 \quad (2.55)$$

$$\| \underline{K}_{r,2} \cdot \underline{\varphi} \|^2 = [0 \quad -0.2347 \quad 0.2347] \cdot \begin{bmatrix} 0 \\ -0.2347 \\ 0.2347 \end{bmatrix} = 0.1102 \quad (2.56)$$

$$\langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle = [0.0558 \quad 0.2454 \quad 0] \cdot \begin{bmatrix} 0 \\ -0.2347 \\ 0.2347 \end{bmatrix} = -0.0576 \quad (2.57)$$

Using Eqs. 2.41 and 2.42, the solution becomes as follows:

$$a_1 = \frac{0.7197 \cdot 0.1102 + 0.2122 \cdot 0.0576}{0.0633 \cdot 0.1102 - 0.0033} = 24.90 \quad (2.58)$$

$$a_2 = \frac{0.2122 \cdot 0.0633 + 0.7197 \cdot 0.0576}{0.0633 \cdot 0.1102 - 0.0033} = 14.93 \quad (2.59)$$

The reference solution, used to compute the circular frequency and the mode shape of Eqs. 2.47 and 2.49 is as follows:

$$a_{1,ref} = 25, \quad a_{2,ref} = 15 \quad (2.60)$$

Comparing the reference values of Eq. 2.60 with the ones achieved using the optimization procedure (Eqs. 2.58 and 2.59), one can see the good agreement between the two pairs of values (the error in the first parameter is about -0.40%, in the second parameter about -0.47%).

Computing the frequency and first mode components using values of Eqs. 2.58 and 2.59, the results obtained are as follows:

$$\omega_1^2 = 2.3936 \cdot 10^2 \text{ rad} / \text{s} \quad (2.61)$$

$$f_1 = 2.4623 \text{ Hz} \quad (2.62)$$

$$\underline{\varphi} = \begin{bmatrix} 0.3011 \\ 0.5466 \\ 0.7814 \end{bmatrix} \quad (2.63)$$

The difference between the reference value and the computed one of the first frequency is very low, about -0.20%. The MAC value assumes value approximately equal to 1.00. The agreement is therefore very good for both frequency and mode components.

## 2.5 Second step: system with eigenvalues equations

Starting from the system defined in paragraph 2.2, an improvement can be done in order to take into account the eigenvalues equations. This improvement was done because the eigenvalues equations allow us to achieve better agreement between experimental and numerical frequencies. The mode shapes, conversely, are practically insensitive to the introduction of the new equations. Eq. 2.3 can be therefore modified as follows, adding the eigenvalues equations:

$$\left\{ \begin{array}{l}
p_1 \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}} \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_1 = \underline{\underline{\mathbf{v}}}_1 \\
\vdots \\
p_i \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}} \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_i = \underline{\underline{\mathbf{v}}}_i \\
\vdots \\
p_n \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}} \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_n = \underline{\underline{\mathbf{v}}}_n \\
\hline
p_1 \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}} \right] / w_1 = r_1 \\
\vdots \\
p_i \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}} \right] / w_i = r_i \\
\vdots \\
p_n \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^N (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}} \right] / w_n = r_n
\end{array} \right. \quad (2.64)$$

A generalization, in order to take into account the presence of parameters in the mass matrix, can be performed:

$$\left\{ \begin{array}{l}
p_1 \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_1 = \underline{\underline{\mathbf{v}}}_1 \\
\vdots \\
p_i \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_i = \underline{\underline{\mathbf{v}}}_i \\
\vdots \\
p_n \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_n = \underline{\underline{\mathbf{v}}}_n \\
\hline
p_1 \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_1 = r_1 \\
\vdots \\
p_i \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_i = r_i \\
\vdots \\
p_n \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_n = r_n
\end{array} \right. \quad (2.65)$$

the values of  $p_i$  and  $p_{\theta,l}$  have been already introduced. The residual vector presents  $(m+1) \cdot n$  different components.

The other set of parameters,  $w_i$ , have been used in order to make all the equations in the problem of the same order of magnitude. Because of the residue from the eigenvalues equations, for the same set of values  $a_s$ , are much higher with respect to the ones from the eigenvectors equations, these

additional parameters have been added to the system with eigenvalues equations. These ones have no physical meaning and they serve only from the mathematical point of view. These weights are defined in the following way:

- the system has to be run with  $w_i$  all equal to 1;
- the trial residue are obtained and the new weights are achieved with the following relation:

$$w_i = \frac{r_i}{\max_{j=1,2,\dots,m} v_{ji}} \quad (2.66)$$

in which  $v_{ji}$  is the  $j$ -th component of residual vector  $\underline{v}_i$ .

After this procedure, the residue have all the same order of magnitude inside the system.

The coefficients  $p_{\theta,l}$  must be applied only in the eigenvalues/eigenvectors equations in the system of Eq. 2.64 and not in the equations deriving from determinant of the system (last  $n$  equations).

Using the residue obtained in this way, the target function (to minimize) can be defined as the 2-norm of the residual vector:

$$H = \sqrt{\sum_{i=1}^n \sum_{j=1}^m v_{ji}^2 + \sum_{i=1}^n r_i^2} \quad (2.67)$$

The system of Eq. 2.65, unfortunately, cannot be written in a closed relation because of the nonlinearities of the eigenvalues equations. The closed form solution from Eq. 2.16 or Eq. 2.28 can be used as starting point for a iterative solution with Trust-Region algorithm, described in the next paragraph.

## 2.6 Description of the two steps algorithm

In Paragraphs 2.2 and 2.3 the target function definition and a direct formulation have been given. Starting from Eq. 2.4 and the knowledge of frequencies and modes, the definition of weights  $p_i$  and  $p_{\theta,l}$  has to be made. After that, solving Eq. 2.16 or Eq. 2.28 in closed form, a first trial solution has been achieved. The computation of comparison parameters from Eqs. 2.29 and 2.30 must be performed. If these values satisfy a fixed thresholds, the procedure stops; otherwise the trial solution will be the starting point for a gradient-type algorithm (the Trust-Region algorithm, Coleman and Li, 1996; Conn et al., 2000; MathWorks, 2005) used for the model updating of the system of Eq. 2.65. The definition of weights  $w_i$ , using Eq. 2.66 and the procedure of Paragraph 2.5, has to be done. The iterative solution of the complete system, eventually, leads to the final parameters values which satisfy the eigenvalues/eigenvectors equations; minimizing the target



function defined in Eq. 2.67. The comparison parameters can be eventually recomputed. In Figure 2.2 the flowchart for the complete process of the algorithm is depicted.

### 2.6.1 Numerical example

Analyzing the same example of paragraph 2.4.2 and starting from the results achieved there, the procedure outlined in paragraph 2.6 is used in order to improve the solution. The starting points for the iterative procedure are the parameters values of Eqs. 2.58 and 2.59. Running the algorithm, new parameters values are achieved:

$$a_1 = 25.00, \quad a_2 = 15.02 \quad (2.68)$$

Comparing with the reference values of Eq. 2.60, one can see that the solution is achieved in a almost perfect way (error of about 0% for the first parameter and about 0.13% for the second one).

Computing the frequency and first mode components, the values obtained are as follows:

$$f_1 = 2.4675\text{Hz} \quad (2.69)$$

$$\underline{\varphi} = \begin{bmatrix} 0.3013 \\ 0.5468 \\ 0.7812 \end{bmatrix} \quad (2.70)$$

The difference between the reference value and the computed one of the first frequency is about 0.01%. The MAC value assumes value approximately equal to 1.00.

An improvement in the solution is therefore achieved using the iterative procedure.

## 2.7 Parameters uncertainties evaluation

The procedure outlined in the previous paragraphs is exploitable if deterministic frequencies and mode shapes are available from experimental tests. In order to improve that procedure, a statistical development has been done to include the possibility of several experimental outcomes (different data windows of acquisition) or, anyway, to make a statistical analysis of parameters achieved with numerical perturbation of frequencies and mode shapes (procedure followed for sensitivity analysis and presented in Chapter 4) or, very common in practice, having at disposal from experimental tests the values of frequencies and modes components and their standard deviations.

### 2.7.1 Statistical analysis of experimental outcomes

Assuming of having several data windows of acquisition in which several values of frequencies and mode shapes, for the same structure and same damage state, have been obtained,

the most important statistical parameters are mean values, standard deviations and coefficients of variation (CoVs) for the frequencies and modes themselves.

The means have the following relations, assuming  $l$  data windows acquired:

$$f_i = \frac{\sum_{k=1}^l f_i^k}{l} \quad (2.71)$$

$$\underline{\varphi}_i = \frac{\sum_{k=1}^l \underline{\varphi}_i^k}{l} \quad (2.72)$$

in which  $i=1,2,\dots,n$ ,  $f_i^k$  and  $\underline{\varphi}_i^k$  are the frequency and mode acquired from the  $k$ -th data windows.

The standard deviations are as follows:

$$\sigma_{f,i} = \sqrt{\frac{\sum_{k=1}^l |f_i^k - f_i|^2}{l-1}} \quad (2.73)$$

$$\sigma_{\varphi,ji} = \sqrt{\frac{\sum_{k=1}^l |\varphi_{ji}^k - \varphi_{ji}|^2}{l-1}} \quad (2.74)$$

and

$$\underline{\sigma}_{\varphi,i} = \begin{bmatrix} \sigma_{\varphi,1i} \\ \sigma_{\varphi,2i} \\ \vdots \\ \sigma_{\varphi,ji} \\ \vdots \\ \sigma_{\varphi,mi} \end{bmatrix} \quad (2.75)$$

in which  $i=1,2,\dots,n$ . Eventually, the coefficients of variation are as follows:

$$CoV_{f,i} = \frac{\sigma_{f,i}}{f_i} \quad (2.76)$$

$$CoV_{\varphi,ji} = \frac{\sigma_{\varphi,ji}}{\varphi_{ji}} \quad (2.77)$$

$$\underline{CoV}_{\varphi,i} = \begin{bmatrix} CoV_{\varphi,1i} \\ CoV_{\varphi,2i} \\ \vdots \\ CoV_{\varphi,ji} \\ \vdots \\ CoV_{\varphi,mi} \end{bmatrix} \quad (2.78)$$

From  $f_i$  and  $CoV_{f,i}$  is possible to compute  $\omega_i^2$  and  $CoV_{\omega^2,i}$  which are needed for the solution of the eigenvalues problem:

$$\omega_i^2 = (2 \cdot \pi \cdot f_i)^2 \quad (2.79)$$

$$\omega_i^2 + \omega_i^2 \cdot CoV_{\omega^2,i} = [2 \cdot \pi \cdot (f_i + f_i \cdot CoV_{f,i})]^2 \quad (2.80)$$

and therefore:

$$\omega_i^2 \cdot CoV_{\omega^2,i} = 4 \cdot \pi^2 \cdot f_i^2 \cdot (CoV_{f,i}^2 + 2 \cdot CoV_{f,i}) \quad (2.81)$$

$$CoV_{\omega^2,i} = CoV_{f,i}^2 + 2 \cdot CoV_{f,i} \quad (2.82)$$

or, using the same procedure

$$\sigma_{\omega^2,i} = 4 \cdot \pi^2 \cdot (\sigma_{f,i}^2 + 2 \cdot f_i \cdot \sigma_{f,i}) \quad (2.83)$$

### 2.7.2 Closed form relations for partial derivatives

Starting from system of Eq. 2.4, the problem can be rewritten in the following way (all the computations are reported in Chapter 3):

$$\min \left\| \underline{\tilde{K}}_0 \cdot \underline{\tilde{\varphi}} + \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} - \underline{\tilde{M}}_0 \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} \right\| \quad (2.84)$$

with matrices and vectors defined in Eqs. 2.21 to 2.26. The function to minimize, therefore, is defined as follows:

$$f(a_1, a_2, \dots, a_N) = \left\| \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\| \quad (2.85)$$

deriving it with respect to the experimental outcomes ( $\varphi_{ji}$  and  $\omega_i^2$ ), all the partial derivatives are achieved (all the partial derivatives are computed with parameters values known, point of calculation, and achieved using procedure developed in paragraph 2.6, for sake of brevity the point in which the partial derivatives are computed will be omitted hereafter). The systems to be solved are of the following form (the complete computations are reported in Chapter 3):

$$\underline{\underline{\tilde{A}}} \cdot \frac{\partial \underline{a}}{\partial \varphi_{ji}} = \underline{\underline{\tilde{l}}}_{ji} \quad (2.86)$$

$$\underline{\underline{\tilde{A}}} \cdot \frac{\partial \underline{a}}{\partial \omega_i^2} = \underline{\underline{\tilde{r}}}_i \quad (2.87)$$

for  $i = 1, 2, \dots, n$ ;  $j = 1, 2, \dots, m$ ;  $\underline{\underline{\tilde{A}}}$  already defined in Eq. 2.19;  $\frac{\partial \underline{a}}{\partial \varphi_{ji}}$ ,  $\underline{\underline{\tilde{l}}}_{ji}$ ,  $\frac{\partial \underline{a}}{\partial \omega_i^2}$  and  $\underline{\underline{\tilde{r}}}_i$  defined as follows:

$$\frac{\partial \underline{a}}{\partial \varphi_{ji}} = \begin{bmatrix} \frac{\partial a_1}{\partial \varphi_{ji}} \\ \frac{\partial a_2}{\partial \varphi_{ji}} \\ \vdots \\ \frac{\partial a_q}{\partial \varphi_{ji}} \\ \frac{\partial a_{q+1}}{\partial \varphi_{ji}} \\ \vdots \\ \frac{\partial a_N}{\partial \varphi_{ji}} \end{bmatrix} \quad (2.88)$$

$$\underline{\underline{\tilde{l}}}_{ji} = \begin{bmatrix} \underline{\underline{\tilde{l}}}_{1,ji} \\ \underline{\underline{\tilde{l}}}_{2,ji} \\ \vdots \\ \underline{\underline{\tilde{l}}}_{q,ji} \\ \underline{\underline{\tilde{l}}}_{q+1,ji} \\ \vdots \\ \underline{\underline{\tilde{l}}}_{N,ji} \end{bmatrix} \quad (2.89)$$

with:

$$\begin{aligned} \underline{\underline{\tilde{l}}}_{s,ji} = & - \sum_{k=1}^q a_k \cdot \left[ \left\langle \underline{\underline{\tilde{K}}}_{r,s}(:,j); \underline{\underline{\tilde{K}}}_{r,k} \cdot \underline{\underline{\tilde{\varphi}}} \right\rangle + \left\langle \underline{\underline{\tilde{K}}}_{r,s} \cdot \underline{\underline{\tilde{\varphi}}}; \underline{\underline{\tilde{K}}}_{r,k}(:,j) \right\rangle \right] - \\ & \sum_{k=q+1}^N a_k \cdot \left[ \left\langle \underline{\underline{\tilde{K}}}_{r,s}(:,j); \underline{\underline{\tilde{M}}}_k \cdot \underline{\underline{\tilde{\varphi}}} \right\rangle + \left\langle \underline{\underline{\tilde{K}}}_{r,s} \cdot \underline{\underline{\tilde{\varphi}}}; \underline{\underline{\tilde{M}}}_k(:,j) \right\rangle \right] + \\ & \left\langle \underline{\underline{\tilde{K}}}_{r,s}(:,j); \underline{\underline{\tilde{\psi}}} \right\rangle + \left\langle \underline{\underline{\tilde{K}}}_{r,s} \cdot \underline{\underline{\tilde{\varphi}}}; \underline{\underline{\tilde{M}}}_0(:,j) - \underline{\underline{\tilde{K}}}_0(:,j) \right\rangle \end{aligned} \quad (2.90)$$

for  $s = 1, 2, \dots, q$  and

$$\underline{\underline{\tilde{l}}}_{s,ji} = - \sum_{k=1}^q a_k \cdot \left[ \left\langle \underline{\underline{\tilde{M}}}_s(:,j); \underline{\underline{\tilde{K}}}_{r,k} \cdot \underline{\underline{\tilde{\varphi}}} \right\rangle + \left\langle \underline{\underline{\tilde{M}}}_s \cdot \underline{\underline{\tilde{\varphi}}}; \underline{\underline{\tilde{K}}}_{r,k}(:,j) \right\rangle \right] -$$

$$\sum_{k=q+1}^N a_k \cdot \left[ \langle \underline{\tilde{M}}_s(:,j); \underline{\tilde{M}}_k \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_k(:,j) \rangle \right] + \langle \underline{\tilde{M}}_s(:,j); \underline{\tilde{\psi}} \rangle + \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_0(:,j) - \underline{\tilde{K}}_0(:,j) \rangle \quad (2.91)$$

for  $s = q+1, q+2, \dots, N$ ; in which  $(:, j)$  means the  $j$ -th column of the matrix. The matrices with tilde were defined in Eqs. 2.21 to 2.26.

$$\frac{\partial \underline{a}}{\partial \omega_i^2} = \begin{bmatrix} \frac{\partial a_1}{\partial \omega_i^2} \\ \frac{\partial a_2}{\partial \omega_i^2} \\ \vdots \\ \frac{\partial a_q}{\partial \omega_i^2} \\ \frac{\partial a_{q+1}}{\partial \omega_i^2} \\ \vdots \\ \frac{\partial a_N}{\partial \omega_i^2} \end{bmatrix} \quad (2.92)$$

$$\underline{\tilde{r}}_i = \begin{bmatrix} \tilde{r}_{i,1} \\ \tilde{r}_{i,2} \\ \vdots \\ \tilde{r}_{i,q} \\ \tilde{r}_{i,q+1} \\ \vdots \\ \tilde{r}_{i,N} \end{bmatrix} \quad (2.93)$$

$$\tilde{r}_{s,i} = - \sum_{k=q+1}^N a_k \cdot \langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{k,i} \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{0,i} \cdot \underline{\tilde{\varphi}} \rangle \quad (2.94)$$

for  $s = 1, 2, \dots, q$  and

$$\tilde{r}_{s,i} = - \sum_{k=1}^q a_k \cdot \langle \underline{\tilde{K}}_{r,k} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{s,i} \cdot \underline{\tilde{\varphi}} \rangle - \sum_{k=q+1}^N a_k \cdot \left[ \langle \underline{\tilde{M}}_{k,i} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{M}}_k \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{s,i} \cdot \underline{\tilde{\varphi}} \rangle \right] + \langle \underline{\tilde{M}}_{s,i} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle + \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{0,i} \cdot \underline{\tilde{\varphi}} \rangle \quad (2.95)$$

for  $s = q+1, q+2, \dots, N$  with:

$$\underline{\underline{\tilde{M}}}_{s,i} = \begin{bmatrix} -p_1 \cdot \underline{\underline{0}} \cdot \underline{\underline{M}}_s & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & -p_2 \cdot \underline{\underline{0}} \cdot \underline{\underline{M}}_s & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & -p_i \underline{\underline{M}}_s & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & -p_n \cdot \underline{\underline{0}} \cdot \underline{\underline{M}}_s \end{bmatrix} \quad (2.96)$$

$$\underline{\underline{\tilde{M}}}_{0,i} = \begin{bmatrix} p_1 \cdot \underline{\underline{0}} \cdot \underline{\underline{M}}_0 & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & p_2 \cdot \underline{\underline{0}} \cdot \underline{\underline{M}}_0 & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & p_i \underline{\underline{M}}_0 & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & p_n \cdot \underline{\underline{0}} \cdot \underline{\underline{M}}_0 \end{bmatrix} \quad (2.97)$$

or, in the same way:

$$\underline{\underline{\tilde{M}}}_{s,i} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & -p_i \underline{\underline{M}}_s & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \end{bmatrix} \quad (2.96')$$

$$\underline{\underline{\tilde{M}}}_{0,i} = \begin{bmatrix} \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & p_i \underline{\underline{M}}_0 & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \end{bmatrix} \quad (2.97')$$

Solving systems of Eqs. 2.86 and 2.87, all the partial derivatives are achieved:

$$\frac{\partial a_s}{\partial \varphi_{ji}} = \frac{\det(\underline{\underline{\tilde{A}}}_{ji,s})}{\det(\underline{\underline{\tilde{A}}})} \quad (2.98)$$

for  $j=1,2,\dots,m$ ;  $i=1,2,\dots,n$ ; in which  $\underline{\underline{\tilde{A}}}_{ji,s}$  is the  $\underline{\underline{\tilde{A}}}$  matrix with the s-th column replaced by  $\underline{\underline{\tilde{l}}}_{ji}$

vector;

$$\frac{\partial a_s}{\partial \omega_i^2} = \frac{\det(\underline{\underline{\tilde{A}}}_{\omega i,s})}{\det(\underline{\underline{\tilde{A}}})} \quad (2.99)$$

for  $i=1,2,\dots,n$ ; in which  $\underline{\underline{\tilde{A}}}_{\omega i,s}$  is the  $\underline{\underline{\tilde{A}}}$  matrix with the s-th column replaced by  $\underline{\underline{\tilde{r}}}_i$  vector.

Otherwise solving the determined system through matrix inversion:

$$\frac{\partial \underline{a}}{\partial \varphi_{ji}} = \underline{\tilde{A}}^{-1} \cdot \underline{\tilde{l}}_{ji} \quad (2.100)$$

$$\frac{\partial \underline{a}}{\partial \omega_i^2} = \underline{\tilde{A}}^{-1} \cdot \underline{\tilde{r}}_i \quad (2.101)$$

### 2.7.3 Variance propagation

For the experimental outcomes, the covariance matrix can be defined in the following way:

$$\underline{\underline{C}} = \begin{bmatrix} \sigma_{\varphi,11}^2 & \text{cov}(\varphi_{11}; \varphi_{21}) & \cdots & \text{cov}(\varphi_{11}; \varphi_{mn}) & \text{cov}(\varphi_{11}; \omega_1^2) & \cdots & \text{cov}(\varphi_{11}; \omega_n^2) \\ \text{cov}(\varphi_{21}; \varphi_{11}) & \sigma_{\varphi,21}^2 & \cdots & \text{cov}(\varphi_{21}; \varphi_{mn}) & \text{cov}(\varphi_{21}; \omega_1^2) & \cdots & \text{cov}(\varphi_{21}; \omega_n^2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \text{cov}(\varphi_{mn}; \varphi_{11}) & \text{cov}(\varphi_{mn}; \varphi_{21}) & \cdots & \sigma_{\varphi,mn}^2 & \text{cov}(\varphi_{mn}; \omega_1^2) & \cdots & \text{cov}(\varphi_{mn}; \omega_n^2) \\ \text{cov}(\omega_1^2; \varphi_{11}) & \text{cov}(\omega_1^2; \varphi_{21}) & \cdots & \text{cov}(\omega_1^2; \varphi_{mn}) & \sigma_{\omega^2,1}^2 & \cdots & \text{cov}(\omega_1^2; \omega_n^2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{cov}(\omega_n^2; \varphi_{11}) & \text{cov}(\omega_n^2; \varphi_{21}) & \cdots & \text{cov}(\omega_n^2; \varphi_{mn}) & \text{cov}(\omega_n^2; \omega_1^2) & \cdots & \sigma_{\omega^2,n}^2 \end{bmatrix} \quad (2.102)$$

$\sigma_{\varphi,ji}^2$ ,  $\sigma_{\omega^2,i}^2$  are the variance (square of the standard deviation) for the j-th component of i-th mode shape vector and the variance of the square of the i-th circular frequency, respectively as defined in Eq . 2.71 to 2.83.  $\text{cov}(\varphi_{ji}; \varphi_{hk})$ ,  $\text{cov}(\varphi_{ji}; \omega_k^2)$ ,  $\text{cov}(\omega_i^2; \omega_k^2)$  are the covariance between modes components, between modes components and square of circular frequencies and between square of circular frequencies respectively.  $\underline{\underline{C}}$  matrix is symmetric, because of covariance definition satisfy the commutative property.

Knowing all the partial derivatives with respect to the experimental outcomes from Eqs. 2.100 and 2.101 (or through Eqs. 2.98 and 2.99) and using the theory of the error propagation (Taylor, 1997), also called theory of variance propagation, the standard deviations can be achieved using the following equations:

$$\begin{aligned} \sigma_{a,s} \cong & \left( \sum_{j=1}^m \sum_{i=1}^n \sum_{h=1}^m \sum_{k=1}^n \frac{\partial a_s}{\partial \varphi_{ji}} \cdot \frac{\partial a_s}{\partial \varphi_{hk}} \cdot \text{cov}(\varphi_{ji}; \varphi_{hk}) + \sum_{j=1}^m \sum_{i=1}^n \sum_{k=1}^n \frac{\partial a_s}{\partial \varphi_{ji}} \cdot \frac{\partial a_s}{\partial \omega_k^2} \cdot \text{cov}(\varphi_{ji}; \omega_k^2) + \right. \\ & \left. + \sum_{i=1}^n \sum_{k=1}^n \frac{\partial a_s}{\partial \omega_i^2} \cdot \frac{\partial a_s}{\partial \omega_k^2} \cdot \text{cov}(\omega_i^2; \omega_k^2) \right)^{0.5} \end{aligned} \quad (2.103)$$

for  $s = 1, 2, \dots, N$ , with  $\sigma_{\varphi,ji}^2 = \text{cov}(\varphi_{ji}; \varphi_{ji})$ ,  $\sigma_{\omega^2,i}^2 = \text{cov}(\omega_i^2; \omega_i^2)$ . Defining the gradient vector of parameter  $a_s$  (partial derivatives of parameter  $a_s$ ) as follows:

$$\nabla a_s = \begin{bmatrix} \frac{\partial a_s}{\partial \varphi_{11}} \\ \frac{\partial a_s}{\partial \varphi_{21}} \\ \vdots \\ \frac{\partial a_s}{\partial \varphi_{mn}} \\ \frac{\partial a_s}{\partial \omega_1^2} \\ \vdots \\ \frac{\partial a_s}{\partial \omega_n^2} \end{bmatrix} \quad (2.104)$$

the rule of error propagation can be rewritten in the following way:

$$\sigma_{a,s} \cong \sqrt{\nabla a_s^T \cdot \underline{\underline{C}} \cdot \nabla a_s} \quad (2.105)$$

for  $s = 1, 2, \dots, N$ .

The variance for parameter  $a_s$  can be computed as the square of the standard deviation obtained in Eq. 2.105. The CoV can be achieved in the following way:

$$CoV_{a,s} = \frac{\sigma_{a,s}}{a_s} \quad (2.106)$$

Then, the standard deviation vector of parameters can be defined as follows:

$$\underline{\underline{\sigma}}_a = \begin{bmatrix} \sigma_{a,1} \\ \sigma_{a,2} \\ \vdots \\ \sigma_{a,s} \\ \vdots \\ \sigma_{a,N} \end{bmatrix} \quad (2.107)$$

the variance and CoV vectors can be defined as follows:

$$\underline{\underline{\sigma}}_a^2 = \begin{bmatrix} \sigma_{a,1}^2 \\ \sigma_{a,2}^2 \\ \vdots \\ \sigma_{a,s}^2 \\ \vdots \\ \sigma_{a,N}^2 \end{bmatrix} \quad (2.108)$$



$$\underline{CoV}_a = \begin{bmatrix} CoV_{a,1} \\ CoV_{a,2} \\ \vdots \\ CoV_{a,s} \\ \vdots \\ CoV_{a,N} \end{bmatrix} \quad (2.109)$$

if all the covariance terms are null, Eq. 2.103 simplifies itself in:

$$\sigma_{a,s} \cong \sqrt{\sum_{j=1}^m \sum_{i=1}^n \left(\frac{\partial a_s}{\partial \varphi_{ji}}\right)^2 \cdot \sigma_{\varphi,ji}^2 + \sum_{i=1}^n \left(\frac{\partial a_s}{\partial \omega_i^2}\right)^2 \cdot \sigma_{\omega^2,i}^2} \quad (2.110)$$

$s = 1, 2, \dots, N$ . Otherwise, in matrix form:

$$\sigma_{a,s} \cong \sqrt{\nabla a_s^T \cdot \underline{\underline{V}} \cdot \nabla a_s} \quad (2.111)$$

with:

$$\underline{\underline{V}} = \begin{bmatrix} \sigma_{\varphi,11}^2 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \sigma_{\varphi,21}^2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \sigma_{\varphi,mn}^2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \sigma_{\omega^2,1}^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & \sigma_{\omega^2,n}^2 \end{bmatrix} \quad (2.112)$$

#### 2.7.4 Algorithm for obtaining of the mean values and standard deviations of parameters

Starting from the solution of the system of Eq. 2.10 or 2.18, using the mean values for modes components and circular frequencies (Eqs. 2.71, 2.72 and 2.79), the first trial mean values of parameters can be achieved. Then, the comparison parameters have to be computed and if they don't satisfy the thresholds, the second step of the procedure must be performed. Following the procedure outlined in paragraph 2.6 and the flowchart in Figure 2.2, the iterative solution is achieved and these values are labeled as the mean values of parameters.

After that, all the systems of Eqs. 2.86 and 2.87 must be computed for  $i = 1, 2, \dots, n$ ;  $j = 1, 2, \dots, m$ . In this way all the partial derivatives of parameters with respect to the experimental outcomes are computed. Then, using Eq. 2.103 or 2.105 (for  $s = 1, 2, \dots, N$ ), the standard deviations of parameters are obtained.

The solution can therefore be written as follows:

$$\tilde{a}_s = a_s \pm \sigma_{a,s} \quad (2.113)$$

for  $s = 1, 2, \dots, N$ .

If the identification procedure gave only mean values and standard deviations for frequencies and mode shapes, the procedure abovementioned works again but Eqs. 2.71 to 2.83 are no longer needed by the algorithm itself and directly the standard deviations of experimental outcomes have to be put in Eq. 2.103. The flowchart of the algorithm is depicted in Figure 2.3.

The main limit of this procedure is that I cannot find the distribution of parameters starting from fixed distributions of frequencies and modes components. Therefore, this procedure can be used in order to obtain an estimation of the  $\sigma_{a,s}$  if the complete statistical analysis is not run and in order to obtain the error propagation inside the algorithm itself. In Chapter 4 is presented the analysis of the distribution of parameters (with  $\chi^2$  test) starting from normally distributed frequencies and modes components. Unfortunately the parameters are not all normally distributed and therefore a general rule cannot be achieved.

Anyway, if one is not interested in the distribution of parameters but only to the standard deviation, the procedure outlined in this paragraph allows to find the mean values and errors of parameters themselves. Moreover is very quick and usable in real time evaluation problems.

## **2.8 Two steps algorithm for complete statistical analysis of the procedure**

In order to overcome the issues outlined in the previous paragraph, a complete statistical analysis will be treated in this paragraph. Unfortunately, the procedure outlined in this paragraph is quite computational demanded and therefore it cannot be used for real time evaluation of parameters.

### **2.8.1 Definition of data**

If different sets of data are available (with different sets of experimental frequencies and mode shapes), the systems of Eqs. 2.4 and 2.65 must be solved and the procedure outlined in Paragraph 2.6 must be followed for each of those experimental data.

Otherwise, if only one value for frequencies and modes is available from the tests, with the associated standard deviation, a set of modified experimental outcomes has to be created following the subsequent procedure:

- values of  $CoV_{f,i}$  and  $CoV_{\phi,ji}$  can be obtained with Eqs. 2.76 to 2.82 for all the experimental frequencies and modes components;

- the k-th realization for frequencies and mode shapes is as follows

$$f_i^k = f_i \cdot (1 + r_{k,i} \cdot CoV_{f,i}) \quad (2.114)$$

$$\varphi_{ji}^k = \varphi_{ji} \cdot (1 + s_{k,ji} \cdot CoV_{\varphi,ji}) \quad (2.115)$$

in which  $k = 1, 2, \dots, l$ .  $f_i^k$  and  $\varphi_{ji}^k$  are the k-th realization of the i-th frequency and i-th mode shape, j-th component,  $f_i$  and  $\varphi_{ji}$  are the experimental outcomes,  $r_{k,i}$  and  $s_{k,ji}$  are independent normally distributed random numbers with unitary  $CoV$ ,  $CoV_{f,i}$  and  $CoV_{\varphi,ji}$  are the coefficients of variation for the i-th frequency and the i-th mode shape, j-th component.

With these procedure, a set of  $l$  frequencies and mode shapes, for each experimental one, can be obtained, in which  $l$  is the number of realization performed.

## 2.8.2 Description of the procedure

Once the data are defined, the procedure of Paragraph 2.6 has to be applied for each of the  $l$  realizations of the experimental outcomes. In Figure 2.4 is presented the flowchart of the procedure. After all these run, a set of  $l$  different parameters are obtained,  $a_s^k$  with  $s = 1, 2, \dots, N$  and  $k = 1, 2, \dots, l$ . From these outcomes, a statistical analysis has to be carried out in order to obtain the mean values, standard deviations and coefficients of variation of parameters themselves:

$$a_s = \frac{\sum_{k=1}^l a_s^k}{l} \quad (2.116)$$

$$\sigma_{a,s} = \sqrt{\frac{\sum_{k=1}^l |a_s^k - a_s|^2}{l-1}} \quad (2.117)$$

and

$$CoV_{a,s} = \frac{\sigma_{a,s}}{a_s} \quad (2.118)$$

The solution can therefore be written in the same form as Eq. 2.113.

## 2.8.3 A posteriori statistical analysis of comparison parameters

After obtaining the complete set of parameters, the stiffness matrix can be computed through Eq. 2.2, for each of the  $l$  realizations. After that, a posteriori eigenvalues analysis can be performed and frequencies and mode shapes ( $\bar{f}_i^k, \bar{\varphi}_i^k$ ) can be computed. With these numerical outcomes and

the ones from the realizations, one can perform a comparison in terms of MAC parameters and frequencies error:

$$MAC^k(\underline{\varphi}_i, \overline{\varphi}_i) = \frac{\left| \left\langle \underline{\varphi}_i^k; \overline{\varphi}_i^{-k} \right\rangle \right|^2}{\left\| \underline{\varphi}_i^k \right\|^2 \cdot \left\| \overline{\varphi}_i^{-k} \right\|^2} \quad (2.119)$$

$$\Delta f_i^k [\%] = \frac{\overline{f}_i^k - f_i^k}{f_i^k} * 100 \quad (2.120)$$

in which  $k = 1, 2, \dots, l$  and  $i = 1, 2, \dots, n$ ;  $\underline{\varphi}_i^k$  and  $f_i^k$  being the k-th realization of experimental modes and frequencies.

With those outcomes, a statistical analysis of parameters  $MAC$  and  $\Delta f$  can be performed in order to obtain mean values, standard deviations and coefficients of variation:

$$MAC(\underline{\varphi}_i, \overline{\varphi}_i) = \frac{\sum_{k=1}^l MAC^k(\underline{\varphi}_i, \overline{\varphi}_i)}{l} \quad (2.121)$$

$$\Delta f_i [\%] = \frac{\sum_{k=1}^l \Delta f_i^k}{l} \quad (2.122)$$

$$\sigma_{MAC,i} = \sqrt{\frac{\sum_{k=1}^l \left| MAC^k(\underline{\varphi}_i, \overline{\varphi}_i) - MAC(\underline{\varphi}_i, \overline{\varphi}_i) \right|^2}{l-1}} \quad (2.123)$$

$$\sigma_{\Delta f,i} = \sqrt{\frac{\sum_{k=1}^l \left| \Delta f_i^k - \Delta f_i \right|^2}{l-1}} \quad (2.124)$$

and

$$CoV_{MAC,i} = \frac{\sigma_{MAC,i}}{MAC(\underline{\varphi}_i, \overline{\varphi}_i)} \quad (2.125)$$

$$CoV_{\Delta f,i} = \frac{\sigma_{\Delta f,i}}{\Delta f_i} \quad (2.126)$$

## 2.8.4 $\chi^2$ test of results

From the analysis results, one can perform  $\chi^2$  test on parameters in order to figure out in which way those quantities are distributed. Starting from  $a_s^k$  with  $s = 1, 2, \dots, N$  and  $k = 1, 2, \dots, l$ , the statistical absolute frequencies  $F_{a,s}^t$  can be computed, assigned some intervals  $t$  for the parameters itself.

An assigned distribution for the parameters can be reconstructed:

$$D(\hat{a}_s) = D(P_{a,i}^1, P_{a,i}^2; \dots) \quad (2.127)$$

in which  $P_{a,i}^1, P_{a,i}^2, \dots$  are known coefficients for the distribution itself. The new statistical absolute frequencies can be computed in the same way as for the ones of original parameters, using the same intervals used before:

$$F_{D(a),s}^t$$

after that, the chi-square test (Ross, 2014) can be performed:

$$\chi_s^2 = \sum_k \frac{(f_{a,s}^k - f_{D(a),s}^k)^2}{f_{D(a),s}^k} \quad (2.128)$$

from the values of chi-square, using the tables of the distribution with fixed number of degrees of freedom, one can evaluate if the assumption of assigned distribution matches the numerical one or not. Usually a 5% level of significance is chosen for the comparison:

$$\chi_k^2 \leq \chi_{k,5\%}^2 \quad (2.129)$$

if the inequality 2.129 is satisfied, the assigned distribution is acceptable.

## 2.9 Numerical example

In this paragraph is analyzed the same example of paragraph 2.4.2. In the first part, the equations introduced in paragraph 2.7.2 are particularized for the example studied. Then, the numerical values are going to be introduced, using the results already achieved in paragraph 2.4.2 and 2.6.1.

### 2.9.1 Direct formulation of partial derivatives

Starting from Eq. 2.31 and following the procedure outlined in paragraph 2.7.2 and 3.4, the partial derivatives with respect to  $\varphi_{11}$ ,  $\varphi_{21}$ ,  $\varphi_{31}$  and  $\omega_1^2$  can be obtained solving the subsequent systems of equations:

- Derivative with respect to  $\varphi_{11}$ :

$$\begin{bmatrix} \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle & \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial a_1}{\partial \varphi_{11}} \\ \frac{\partial a_2}{\partial \varphi_{11}} \end{bmatrix} = \begin{bmatrix} \tilde{l}_{1,11} \\ \tilde{l}_{2,11} \end{bmatrix} \quad (2.130)$$

in which:

$$\tilde{l}_{1,11} = -2 \cdot a_1 \cdot \langle \underline{K}_{r,1}(:,1); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle - a_2 \cdot \langle \underline{K}_{r,1}(:,1); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_2 \cdot \langle \underline{K}_{r,2}(:,1); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle + \left\langle \frac{\partial \underline{\psi}}{\partial \varphi_{11}}; \underline{K}_{r,1} \cdot \underline{\varphi} \right\rangle + \langle \underline{\psi}; \underline{K}_{r,1}(:,1) \rangle \quad (2.131)$$

$$\tilde{l}_{2,11} = -2 \cdot a_2 \cdot \langle \underline{K}_{r,2}(:,1); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_1 \cdot \langle \underline{K}_{r,1}(:,1); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_1 \cdot \langle \underline{K}_{r,2}(:,1); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle + \left\langle \frac{\partial \underline{\psi}}{\partial \varphi_{11}}; \underline{K}_{r,2} \cdot \underline{\varphi} \right\rangle + \langle \underline{\psi}; \underline{K}_{r,2}(:,1) \rangle \quad (2.132)$$

$$\frac{\partial \underline{\psi}}{\partial \varphi_{11}} = \omega_1^2 \cdot \underline{M}_0(:,1) - \underline{K}_0(:,1) \quad (2.133)$$

(:,1) means the 1<sup>st</sup> column of the matrix.

– Derivative with respect to  $\varphi_{21}$  :

$$\begin{bmatrix} \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle & \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial a_1}{\partial \varphi_{21}} \\ \frac{\partial a_2}{\partial \varphi_{21}} \end{bmatrix} = \begin{bmatrix} \tilde{l}_{1,21} \\ \tilde{l}_{2,21} \end{bmatrix} \quad (2.134)$$

in which:

$$\tilde{l}_{1,21} = -2 \cdot a_1 \cdot \langle \underline{K}_{r,1}(:,2); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle - a_2 \cdot \langle \underline{K}_{r,1}(:,2); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_2 \cdot \langle \underline{K}_{r,2}(:,2); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle + \left\langle \frac{\partial \underline{\psi}}{\partial \varphi_{21}}; \underline{K}_{r,1} \cdot \underline{\varphi} \right\rangle + \langle \underline{\psi}; \underline{K}_{r,1}(:,2) \rangle \quad (2.135)$$

$$\tilde{l}_{2,21} = -2 \cdot a_2 \cdot \langle \underline{K}_{r,2}(:,2); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_1 \cdot \langle \underline{K}_{r,1}(:,2); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_1 \cdot \langle \underline{K}_{r,2}(:,2); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle + \left\langle \frac{\partial \underline{\psi}}{\partial \varphi_{21}}; \underline{K}_{r,2} \cdot \underline{\varphi} \right\rangle + \langle \underline{\psi}; \underline{K}_{r,2}(:,2) \rangle \quad (2.136)$$

$$\frac{\partial \underline{\psi}}{\partial \varphi_{21}} = \omega_1^2 \cdot \underline{M}_0(:,2) - \underline{K}_0(:,2) \quad (2.137)$$

(:,2) means the 2<sup>nd</sup> column of the matrix.

– Derivative with respect to  $\varphi_{31}$  :

$$\begin{bmatrix} \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle & \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial a_1}{\partial \varphi_{31}} \\ \frac{\partial a_2}{\partial \varphi_{31}} \end{bmatrix} = \begin{bmatrix} \tilde{l}_{1,31} \\ \tilde{l}_{2,31} \end{bmatrix} \quad (2.138)$$

in which:

$$\tilde{l}_{1,31} = -2 \cdot a_1 \cdot \langle \underline{K}_{r,1}(:,3); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle - a_2 \cdot \langle \underline{K}_{r,1}(:,3); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_2 \cdot \langle \underline{K}_{r,2}(:,3); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle + \left\langle \frac{\partial \psi}{\partial \varphi_{31}}; \underline{K}_{r,1} \cdot \underline{\varphi} \right\rangle + \langle \psi; \underline{K}_{r,1}(:,3) \rangle \quad (2.139)$$

$$\tilde{l}_{2,31} = -2 \cdot a_2 \cdot \langle \underline{K}_{r,2}(:,3); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_1 \cdot \langle \underline{K}_{r,1}(:,3); \underline{K}_{r,2} \cdot \underline{\varphi} \rangle - a_1 \cdot \langle \underline{K}_{r,2}(:,3); \underline{K}_{r,1} \cdot \underline{\varphi} \rangle + \left\langle \frac{\partial \psi}{\partial \varphi_{31}}; \underline{K}_{r,2} \cdot \underline{\varphi} \right\rangle + \langle \psi; \underline{K}_{r,2}(:,3) \rangle \quad (2.140)$$

$$\frac{\partial \psi}{\partial \varphi_{31}} = \omega_1^2 \cdot \underline{M}_0(:,3) - \underline{K}_0(:,3) \quad (2.141)$$

(:,3) means the 3<sup>rd</sup> column of the matrix.

– Derivative with respect to  $\omega_1^2$  :

$$\begin{bmatrix} \|\underline{K}_{r,1} \cdot \underline{\varphi}\|^2 & \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{K}_{r,2} \cdot \underline{\varphi} \rangle & \|\underline{K}_{r,2} \cdot \underline{\varphi}\|^2 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial a_1}{\partial \omega_1^2} \\ \frac{\partial a_2}{\partial \omega_1^2} \end{bmatrix} = \begin{bmatrix} \langle \underline{K}_{r,1} \cdot \underline{\varphi}; \underline{M}_0 \cdot \underline{\varphi} \rangle \\ \langle \underline{K}_{r,2} \cdot \underline{\varphi}; \underline{M}_0 \cdot \underline{\varphi} \rangle \end{bmatrix} \quad (2.142)$$

## 2.9.2 Variance propagation

Once obtained all the partial derivatives, and being the standard deviations of the experimental outcomes known, Eq. 2.110 can be used (it is used the simplified relation because it is conjectured that all the experimental outcomes are independent each other). The relation can be written as follows:

$$\sigma_{a,1} \cong \sqrt{\left(\frac{\partial a_1}{\partial \varphi_{11}}\right)^2 \cdot \sigma_{\varphi,11}^2 + \left(\frac{\partial a_1}{\partial \varphi_{21}}\right)^2 \cdot \sigma_{\varphi,21}^2 + \left(\frac{\partial a_1}{\partial \varphi_{31}}\right)^2 \cdot \sigma_{\varphi,31}^2 + \left(\frac{\partial a_1}{\partial \omega_1^2}\right)^2 \cdot \sigma_{\omega^2,1}^2} \quad (2.143)$$

$$\sigma_{a,2} \cong \sqrt{\left(\frac{\partial a_2}{\partial \varphi_{11}}\right)^2 \cdot \sigma_{\varphi,11}^2 + \left(\frac{\partial a_2}{\partial \varphi_{21}}\right)^2 \cdot \sigma_{\varphi,21}^2 + \left(\frac{\partial a_2}{\partial \varphi_{31}}\right)^2 \cdot \sigma_{\varphi,31}^2 + \left(\frac{\partial a_2}{\partial \omega_1^2}\right)^2 \cdot \sigma_{\omega^2,1}^2} \quad (2.144)$$

### 2.9.3 Numerical example

If CoVs of 5% are used for the frequency  $f_1$  and for all the mode components, using the direct and inverse relations introduced in Eqs. from 2.76 to 2.83, the variances assume the following values:

$$\sigma_{\varphi,11}^2 = 2.2680 \cdot 10^{-4}, \sigma_{\varphi,21}^2 = 7.4693 \cdot 10^{-4}, \sigma_{\varphi,31}^2 = 1.5261 \cdot 10^{-3}, \sigma_{\omega^2,1}^2 = 606.7157 \quad (2.145)$$

Computing all the partial derivatives from Eqs. 2.130, 2.134, 2.138, 2.142 and applying the variance propagation of Eqs. 2.143 and 2.144, the results listed in Table A.1 are achieved. In Table A.1 are also listed the results from Monte Carlo analysis of the complete system of Eq. 2.64 or 2.65. The results are in good agreement with the standard deviation of the first parameter, practically they are coincident. More error is reached for the second parameter (parameter with less value and therefore less influence in the global behavior). For two parameters and only one mode shape considered the results are anyway very good.

The distributions of parameters from the Monte Carlo analysis are depicted in Figure 2.5.

## 2.10 Generalized procedure with DE-Q algorithm

Starting from the generalized version of the problem (Eq. 2.65), using the target function definition in Eq. 2.67 and the DE-Q algorithm defined in paragraph 1.2.1, it can be obtained the procedure and the statistical one using the genetic algorithm. These procedures are depicted in Figures 2.6, 2.7 and 2.8 as flowcharts. In this case the procedures don't require the initial evaluation of the trial solution because of the algorithm handles directly with the entire system of Eq. 2.65. Nonetheless, in Chapter 4 is shown how the two steps procedure reduces the computational effort.

## 2.11 Direct statistical distribution analysis

In order to overcome the problems of the direct statistical analysis (mentioned in paragraph 2.7.4) and avoiding the Monte Carlo analysis, a direct statistical distribution analysis can be performed.

Starting from Eq. 2.4, the experimental outcomes can be considered as random variables of known distribution:

$$\varphi_{ji} = X_{ji} = X_{ji}(P^1_{\varphi,ji}; P^2_{\varphi,ji}; \dots) \quad (2.146)$$

$$\omega_i^2 = Y_i = Y_i(P^1_{\omega^2,i}; P^2_{\omega^2,i}; \dots) \quad (2.147)$$



for  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, m$  and  $P^1_{\varphi, ji}; P^2_{\varphi, ji}; \dots$ ,  $P^1_{\omega^2, i}; P^2_{\omega^2, i}; \dots$  are known coefficients of the distributions.

Equation 2.4 can be rewritten as follows:

$$\begin{cases} p_1 \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \underline{\underline{\mathbf{K}}}_{r,s}) - Y_1 \cdot \underline{\underline{\mathbf{M}}}_0 - Y_1 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\mathbf{X}}}_1 = \underline{\underline{\mathbf{v}}}_1 \\ \vdots \\ p_i \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \underline{\underline{\mathbf{K}}}_{r,s}) - Y_i \cdot \underline{\underline{\mathbf{M}}}_0 - Y_i \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\mathbf{X}}}_i = \underline{\underline{\mathbf{v}}}_i \\ \vdots \\ p_n \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (a_s \underline{\underline{\mathbf{K}}}_{r,s}) - Y_n \cdot \underline{\underline{\mathbf{M}}}_0 - Y_n \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\mathbf{X}}}_n = \underline{\underline{\mathbf{v}}}_n \end{cases} \quad (2.148)$$

in which:

$$\underline{\underline{\mathbf{X}}}_i = \begin{bmatrix} X_{1i} \\ X_{2i} \\ \vdots \\ X_{ji} \\ \vdots \\ X_{mi} \end{bmatrix} \quad (2.149)$$

The system of Eq. 2.18 is still valid with:

$$\underline{\underline{\tilde{\mathbf{A}}}} = \begin{bmatrix} \|\underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}\|^2 & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \dots & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \dots & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle \\ \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \|\underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}}\|^2 & \dots & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \dots & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \dots & \|\underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}}\|^2 & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \dots & \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle \\ \langle \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \langle \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \dots & \langle \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \|\underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}\|^2 & \dots & \langle \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \langle \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \dots & \langle \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \langle \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}} \rangle & \dots & \|\underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}}\|^2 \end{bmatrix} \quad (2.150)$$

$$\underline{\underline{\tilde{\mathbf{b}}}} = \begin{bmatrix} \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\psi}}} \rangle \\ \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,2} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\psi}}} \rangle \\ \vdots \\ \langle \underline{\underline{\tilde{\mathbf{K}}}}_{r,q} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\psi}}} \rangle \\ \langle \underline{\underline{\tilde{\mathbf{M}}}}_{q+1} \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\psi}}} \rangle \\ \vdots \\ \langle \underline{\underline{\tilde{\mathbf{M}}}}_N \cdot \underline{\underline{\tilde{\mathbf{X}}}}; \underline{\underline{\tilde{\psi}}} \rangle \end{bmatrix} \quad (2.151)$$

in which  $\underline{\underline{\tilde{\mathbf{K}}}}_{r,s}$ ,  $\underline{\underline{\tilde{\mathbf{K}}}}_0$  are defined in Eqs. 2.21 and 2.25 respectively;

$$\underline{\underline{\tilde{M}}}_s = \begin{bmatrix} -p_1 \cdot Y_1 \cdot \underline{\underline{M}}_s & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & -p_2 \cdot Y_2 \cdot \underline{\underline{M}}_s & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & -p_n \cdot Y_n \cdot \underline{\underline{M}}_s \end{bmatrix} \quad (2.152)$$

for  $q+1 \leq s \leq N$ ;

$$\underline{\underline{\tilde{X}}} = \begin{bmatrix} \underline{\underline{X}}_1 \\ \underline{\underline{X}}_2 \\ \vdots \\ \underline{\underline{X}}_n \end{bmatrix} \quad (2.153)$$

$$\underline{\underline{\tilde{\psi}}} = \underline{\underline{\tilde{M}}}_0 \cdot \underline{\underline{\tilde{X}}} - \underline{\underline{\tilde{K}}}_0 \cdot \underline{\underline{\tilde{X}}} \quad (2.154)$$

$$\underline{\underline{\tilde{M}}}_0 = \begin{bmatrix} p_1 \cdot Y_1 \cdot \underline{\underline{M}}_0 & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & p_2 \cdot Y_2 \cdot \underline{\underline{M}}_0 & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & p_n \cdot Y_n \cdot \underline{\underline{M}}_0 \end{bmatrix} \quad (2.155)$$

The analytical solution of the system can be achieved using Eq. 2.27 or 2.28.

After the achievement of the analytical solution, the parameters are functions of the random variables of modes components and circular frequencies:

$$a_s = a_s(\underline{\underline{X}}_1, \underline{\underline{X}}_2, \dots, \underline{\underline{X}}_n, Y_1, Y_2, \dots, Y_n) \quad (2.156)$$

for  $s = 1, 2, \dots, N$ .

All the parameters have the following form (see Eq. 2.27):

$$a_s = \frac{g_{1,s}(\underline{\underline{X}}_1, \underline{\underline{X}}_2, \dots, \underline{\underline{X}}_n, Y_1, Y_2, \dots, Y_n)}{g_{2,s}(\underline{\underline{X}}_1, \underline{\underline{X}}_2, \dots, \underline{\underline{X}}_n, Y_1, Y_2, \dots, Y_n)} \quad (2.157)$$

the rules of distributions propagation may be applied on Eq. 2.157 (Feller, 1966; Papoulis, 1991).

For  $g_{1,s}$  and  $g_{2,s}$ , first of all the multiplications must be performed and then the additions and subtractions following the subsequent rules:

1. if  $Z = X \cdot Y$

$$f_Z(z) = \int_{-\infty}^{+\infty} \frac{1}{|y|} \cdot f_{XY}\left(\frac{z}{y}, y\right) dy \quad (2.158)$$

in which  $f_Z(z)$  is the probability density function (pdf) of the resulting random variable  $Z$ ,  $f_{XY}(x, y)$  is the joint pdf of random variables  $X$  and  $Y$ .

2. if  $Z = X + Y$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(z - y, y) dy \quad (2.159)$$

in which  $f_Z(z)$  is the probability density function (pdf) of the resulting random variable  $Z$ ,  $f_{XY}(x, y)$  is the joint pdf of random variables  $X$  and  $Y$ .

If the two random variables  $X$  and  $Y$  are independent, Eq. 2.159 simplifies into:

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z-y) \cdot f_Y(y) dy \quad (2.159')$$

in which  $f_X(x)$  and  $f_X(y)$  are the pdf of random variables  $X$  and  $Y$ .

In the particular case  $Z = b_1 \cdot X_1 + b_2 \cdot X_2 + \dots + b_w \cdot X_w$  in which  $X_1 = N(\mu_1; \sigma_1^2)$ ,  $X_2 = N(\mu_2; \sigma_2^2)$ ,  $\dots$ ,  $X_w = N(\mu_w; \sigma_w^2)$  are all normally distributed independent random variables, the pdf of  $Z$  is still normally distributed:

$$Z = N(b_1 \cdot \mu_1 + b_2 \cdot \mu_2 + \dots + b_w \cdot \mu_w; b_1^2 \cdot \sigma_1^2 + b_2^2 \cdot \sigma_2^2 + \dots + b_w^2 \cdot \sigma_w^2) \quad (2.160)$$

Using several times Eqs. 2.158 and 2.159, in a concatenated way, one is able to achieve the distribution for the two functions  $g_{1,s}$  and  $g_{2,s}$ . After that, using the rule for the ratio between random variables (Feller, 1966; Papoulis, 1991), the parameters distributions are achieved:

$$a_s(z) = \frac{g_{1,s}(x)}{g_{2,s}(y)} \quad (2.161)$$

$$a_s(z) = \int_{-\infty}^{+\infty} |y| \cdot f_{XY}(z \cdot y, y) dy \quad (2.162)$$

for  $s = 1, 2, \dots, N$ .

In this way the parameters distributions are obtained from the distributions of the experimental outcomes without the need of a Monte Carlo analysis. The flowchart of the procedure is depicted in Figure 2.9 and a simple case will be analyzed in Appendix A.

### 2.11.1 Mean values and standard deviations of parameters

Knowing the parameters distributions, the mean values and standard deviations can be computed using the following relations:

$$\mu_{a,s} = \int_{-\infty}^{+\infty} z \cdot a_s(z) dz \quad (2.163)$$

$$\sigma_{a,s} = \sqrt{\int_{-\infty}^{+\infty} (z - \mu_{a,s})^2 \cdot a_s(z) dz} \quad (2.164)$$

for  $s = 1, 2, \dots, N$ .

## 2.12 Conclusions

In the present chapter, the definition of the new two steps algorithm, with uncertainties evaluation, has been developed starting from the definition of the system to be solved and of the new target function. The weighting functions needed by the procedure have also been introduced. The first step has firstly been studied, involving the solution of the system, without determinant equations, in a closed form. The comparison parameters must be then computed and, if these values satisfy the fixed thresholds, the procedure stops here.

A numerical example has been studied and the results, in terms of parameters values, frequencies error and MAC value, were very in good agreement with respect to the reference solutions.

The algorithm has then been improved describing the second step, in which the complete system of equations (with determinant equations) has to be faced. In this step a Trust-Region optimization is used in order to find the solution, using the previously computed one as starting point for the iterative procedure. The previously introduced numerical example has then been improved using the second step. The solution achieved is better with respect to the case of system without determinant equations.

The system of equations allows us to write, in closed form, the partial derivatives with respect to the frequencies and modes components. The standard deviations of parameters can therefore be achieved with direct formulation (using the theory of errors propagation). In this way, the uncertainties of parameters can be achieved, starting from known values of perturbations in frequencies and modes components.

The complete statistical analysis, utilizing the Monte Carlo procedure, has then been introduced.

Even in this case, a numerical example has been performed and a comparison between direct uncertainties evaluation and Monte Carlo procedure gave a very good results.

Moreover, procedure with DE-Q algorithm has also been introduced.

Eventually, a statistical distribution analysis of parameters, starting from known distributions for frequencies and modes components, has been performed.

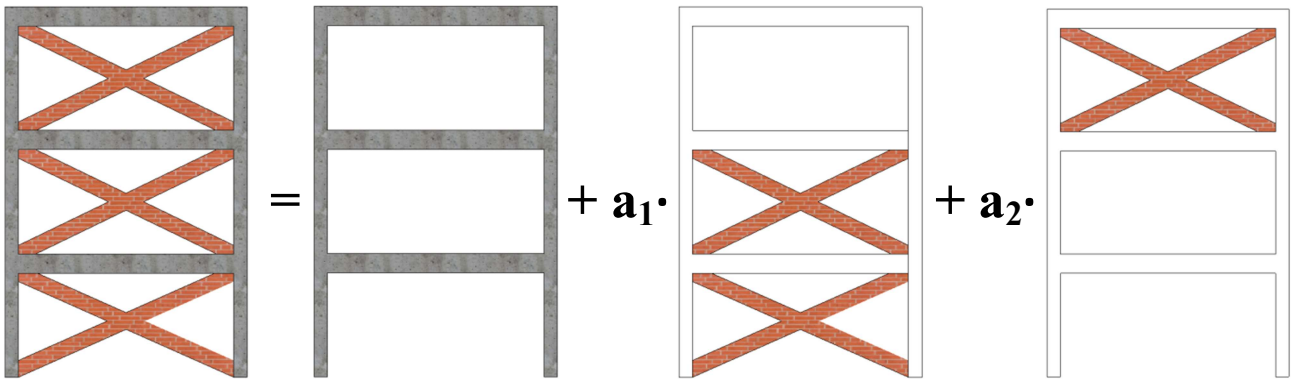
## Table of Chapter 2

	Parameter	Uncertainties evaluation	Monte Carlo Analysis
$a_1$	Standard Deviation	2.92	2.92
	CoV [%]	11.70	11.69
$a_2$	Standard Deviation	3.68	4.20
	CoV [%]	24.55	26.97

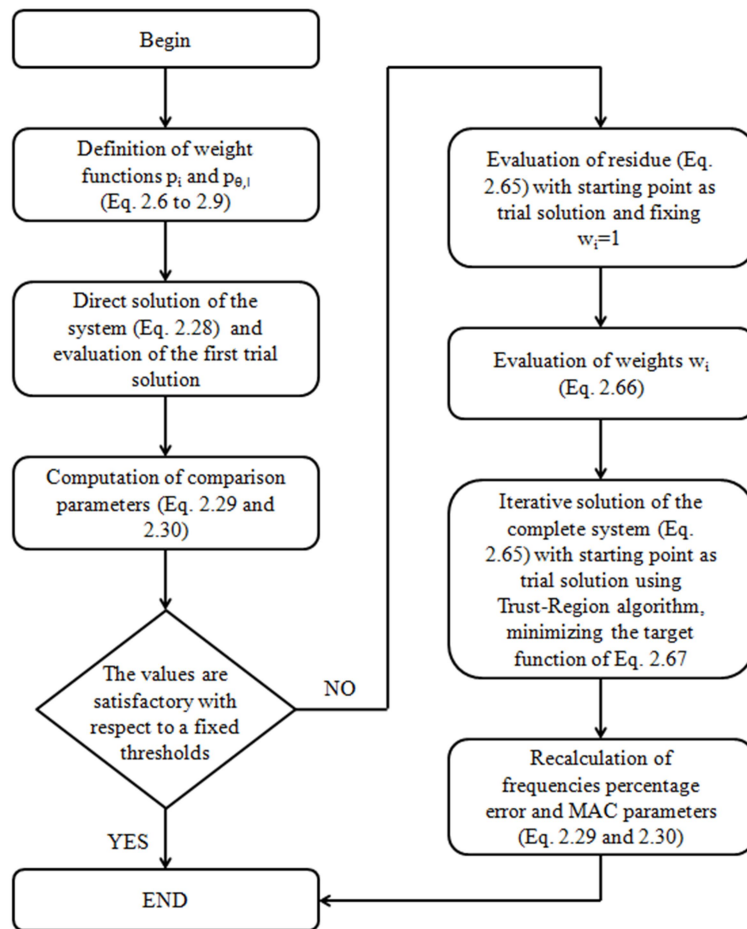
**Table 2.1:** Standard deviations and CoVs of parameters from uncertainties evaluation and Monte Carlo realizations.



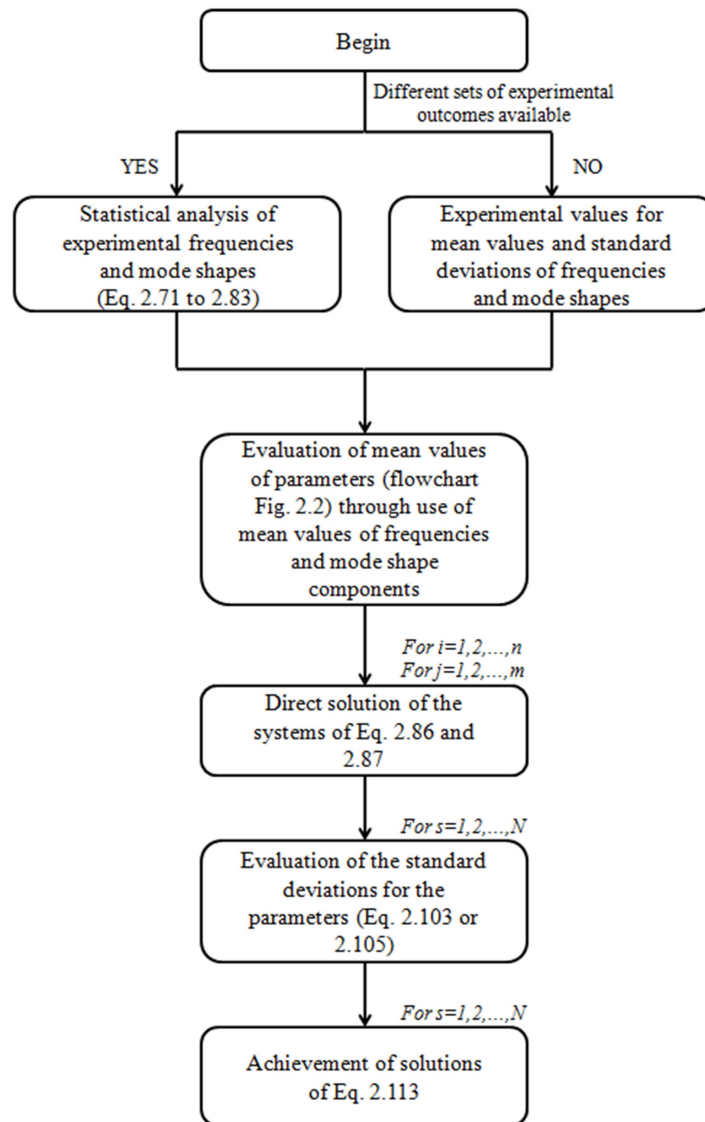
## Figures of Chapter 2



**Figure 2.1:** Two bays three storey infilled frame decomposition with two parameters.

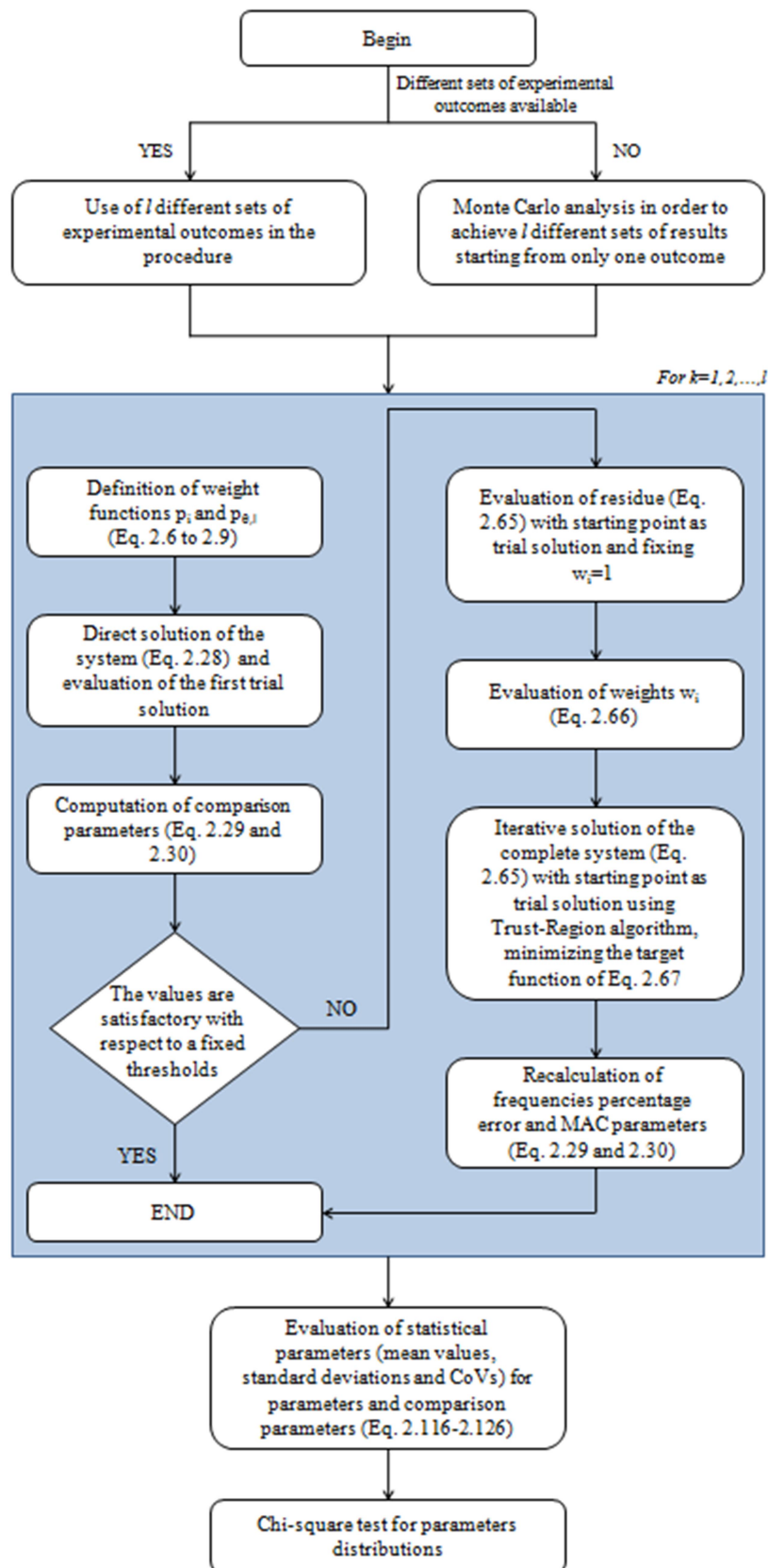


**Figure 2.2:** Flowchart of the two steps algorithm.

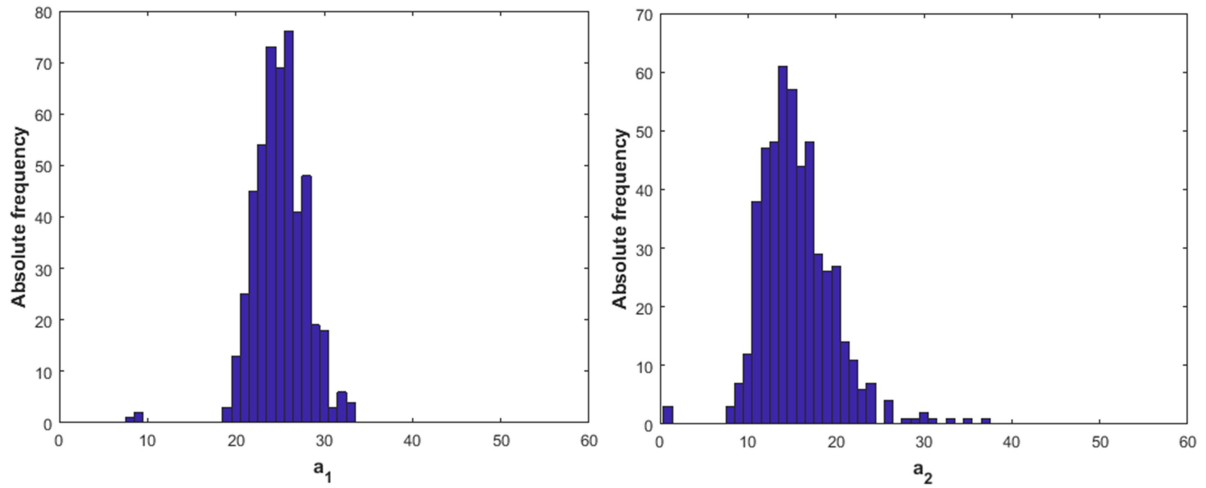


**Figure 2.3:** Flowchart of the two steps algorithm with uncertainties evaluation.

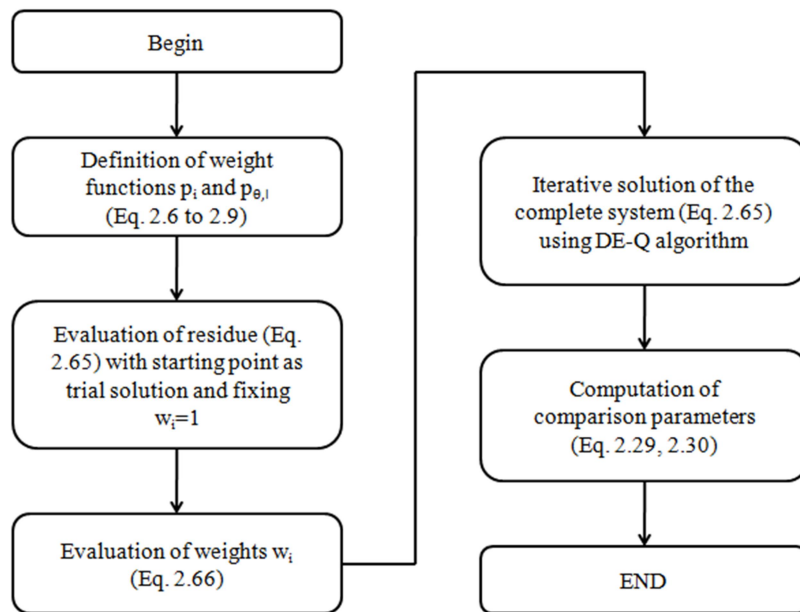




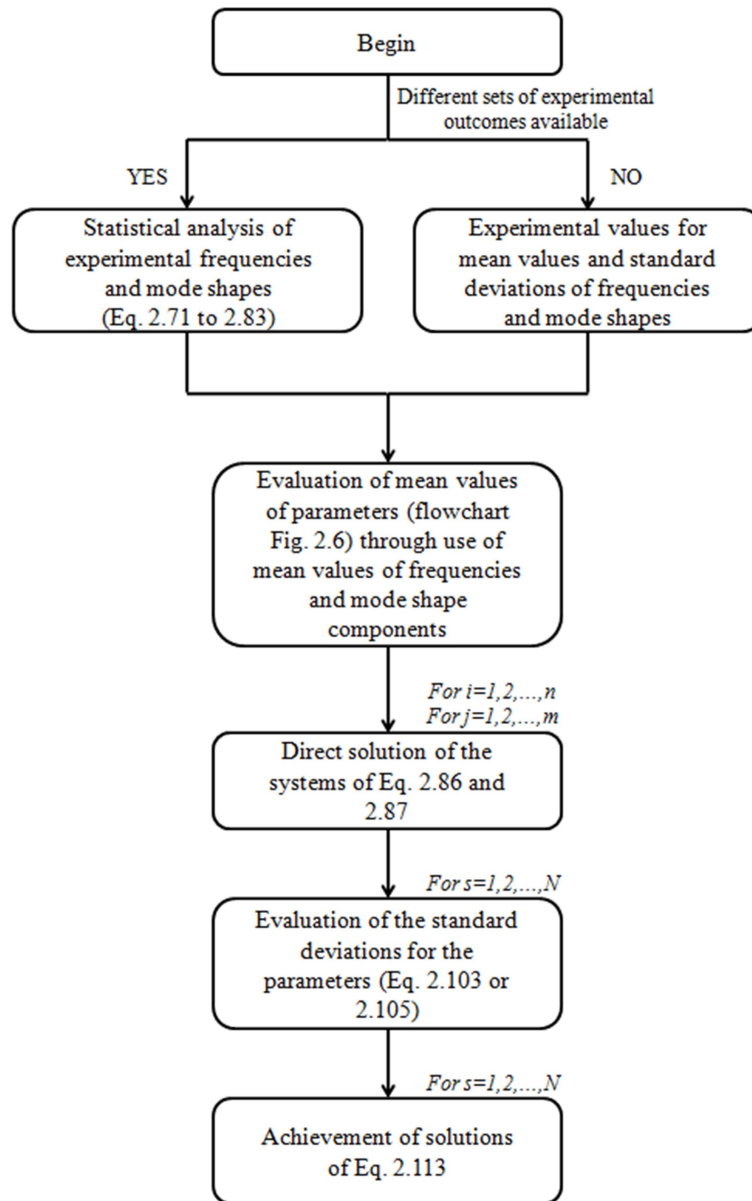
**Figure 2.4:** Flowchart of the two steps algorithm for complete statistical analysis of the procedure.



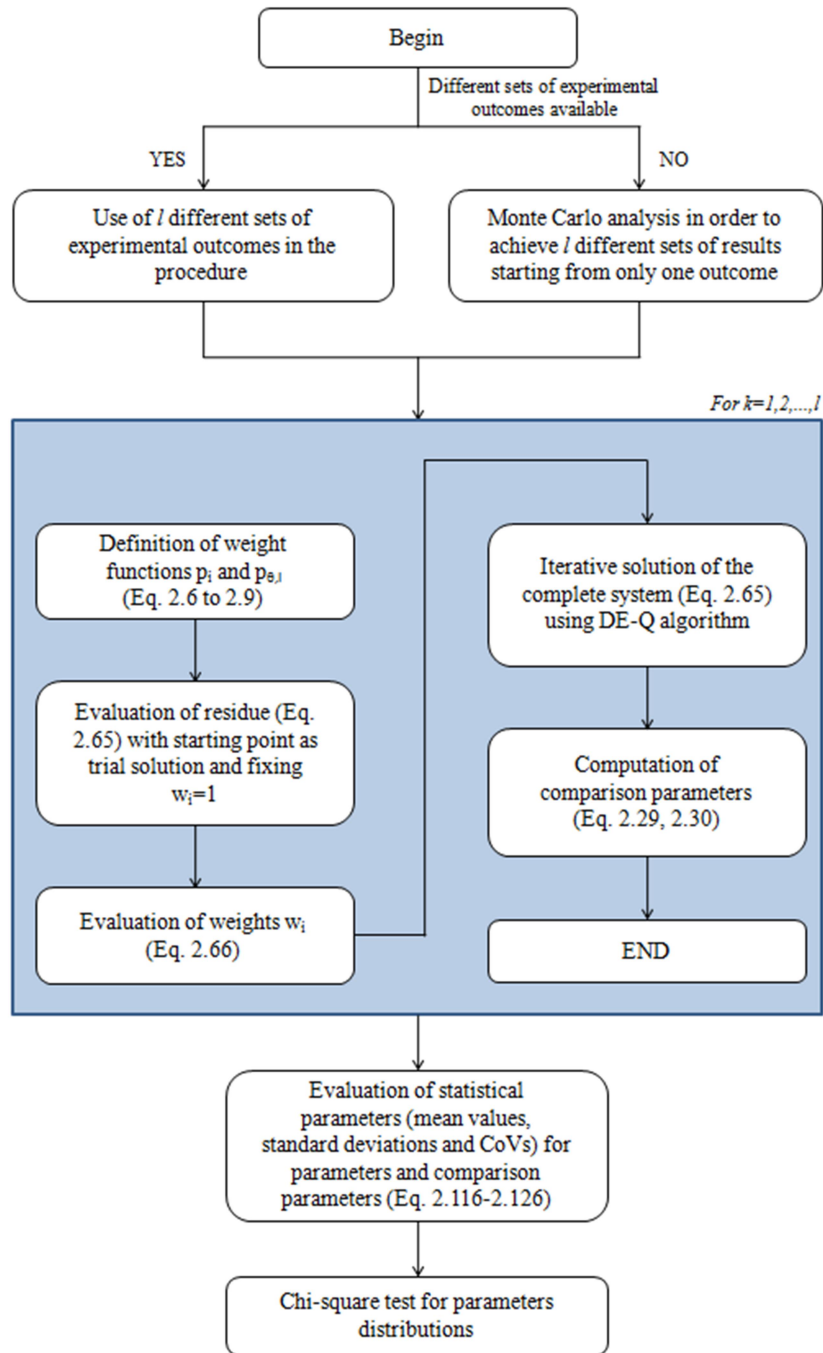
**Figure 2.5:** Frequencies histograms of parameters  $a_1, a_2$  from Monte Carlo analysis.



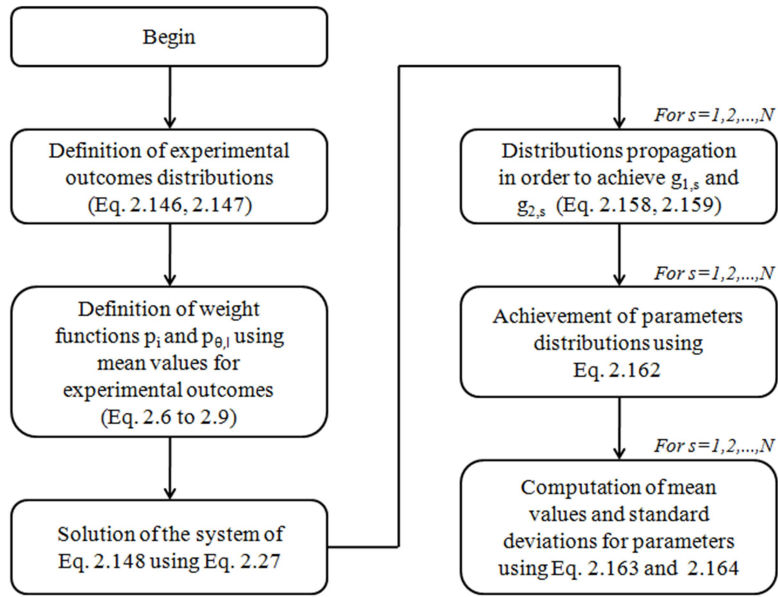
**Figure 2.6:** Flowchart of the algorithm with DE-Q procedure.



**Figure 2.7:** Flowchart of the algorithm with DE-Q procedure and uncertainties evaluation.



**Figure 2.8:** Flowchart of the algorithm with DE-Q procedure for complete statistical analysis.



**Figure 2.9:** Flowchart of the algorithm for the direct statistical distribution analysis.



## CHAPTER 3

### THEORETICAL BASIS

#### 3.1 Introduction

In this chapter will be analyzed the maximum number of parameters obtainable from the innovative procedure, from a theoretical point of view. Subsequently, the procedure with partial derivatives, in order to achieve the least squares solution, will be performed and the solution introduced in paragraph 2.3.2 will be demonstrated. After that, all the relations for the errors propagation, reported in paragraph 2.7, will be analyzed and demonstrated. Then, the linear independence of parameters and uniqueness of the solution will be investigated. The maximum number of parameters for non ideal case will be then analyzed. Eventually, description of the correctness of rigid diaphragm assumption will be performed from the theoretical standpoint.

#### 3.2 Maximum number of parameters

Starting from the general eigenvalues/eigenvectors problem, it can be demonstrated that the maximum number of parameters obtainable from inverse procedure follows the subsequent equation:

$$N = \sum_{i=1}^m i - \sum_{i=1}^{m-n} i \quad \text{or} \quad N = n \cdot (m+1) - \sum_{i=1}^n i \quad (3.1)$$

in which  $m$  is the size of the problem (number of rows or columns of matrices) and  $n$  is the number of known eigenvalues/eigenvectors,  $N$  the maximum number of parameters achievable.

##### 3.2.1 Demonstration

The problem can be written, for the  $i$ -th eigenvalue and eigenvector, as follows:

$$\left[ \underline{\underline{K}} - \omega_i^2 \cdot \underline{\underline{M}} \right] \cdot \underline{\underline{\varphi}}_i = \underline{\underline{0}} \quad (3.2)$$

in which  $\underline{\underline{K}}$  and  $\underline{\underline{M}}$  are the stiffness and mass matrix of the problem (of rank  $m$ ),  $\omega_i^2$  is the  $i$ -th square circular frequency ( $i$ -th eigenvalue) and  $\underline{\underline{\varphi}}_i$  is the  $i$ -th mode shape (eigenvector of length  $m$ );  $\underline{\underline{0}}$  is a vector (of length  $m$ ) with all null terms.

$\underline{\underline{K}}$  and  $\underline{\underline{M}}$  are symmetric and positive defined matrix:

$$\underline{\underline{K}}, \underline{\underline{M}} \in S_m(\mathfrak{R})$$

$$\underline{\underline{K}}, \underline{\underline{M}} \in D^+$$

$\underline{\underline{M}}$  can be decomposed in the following way:

$$\underline{\underline{M}} = \sqrt{\underline{\underline{M}}} \cdot \sqrt{\underline{\underline{M}}} \quad (3.3)$$

and therefore:

$$\underline{\underline{M}}^{-1} = \sqrt{\underline{\underline{M}}^{-1}} \cdot \sqrt{\underline{\underline{M}}^{-1}} \quad (3.4)$$

subsequently:

$$\sqrt{\underline{\underline{M}}} [\sqrt{\underline{\underline{M}}^{-1}} \cdot \underline{\underline{K}} \cdot \sqrt{\underline{\underline{M}}^{-1}} - \omega_i^2 \cdot \underline{\underline{I}}_m] \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_i = \underline{\underline{0}} \quad \forall i \quad (3.5)$$

in which  $\underline{\underline{I}}_m$  is the identity matrix of order  $m$ .

Then, it can be obtained:

$$[\sqrt{\underline{\underline{M}}^{-1}} \cdot \underline{\underline{K}} \cdot \sqrt{\underline{\underline{M}}^{-1}} - \omega_i^2 \cdot \underline{\underline{I}}_m] \cdot \underline{\underline{\hat{\varphi}}}_i = \underline{\underline{0}} \quad \forall i \quad (3.6)$$

in which  $\underline{\underline{\hat{\varphi}}}_i = \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_i$ ; changing notation:

$$\underline{\underline{\hat{K}}} = \sqrt{\underline{\underline{M}}^{-1}} \cdot \underline{\underline{K}} \cdot \sqrt{\underline{\underline{M}}^{-1}} \quad (3.7)$$

the new matrix is still symmetric and positive defined matrix (Casali et al., 2016; Lang, 2001 and Lang, 2005):

$$\underline{\underline{\hat{K}}} \in S_m(\mathfrak{R})$$

$$\underline{\underline{\hat{K}}} \in D^+$$

Then, the problem can be rewritten in this way:

$$[\underline{\underline{\hat{K}}} - \omega_i^2 \cdot \underline{\underline{I}}_m] \cdot \underline{\underline{\hat{\varphi}}}_i = \underline{\underline{0}} \quad \forall i \quad (3.8)$$

which is a standard eigenvalues problem.

Using the Spectral Theorem (Casali et al., 2016; Lang, 2001 and Lang, 2005), the new matrix can be decomposed using eigenvectors and eigenvalues from the problem of Eq. 3.8:

$$\underline{\underline{\hat{K}}} = \underline{\underline{\hat{P}}} \cdot \underline{\underline{\hat{A}}} \cdot \underline{\underline{\hat{P}}}^{-1} \quad (3.9)$$

in which  $\underline{\underline{\hat{P}}}$  is the matrix whose columns are the eigenvectors  $\underline{\underline{\hat{\varphi}}}_i$ ,  $\underline{\underline{\hat{A}}}$  is the matrix in which the diagonal presents the eigenvalues of the problem:



$$\underline{\underline{A}} = \begin{pmatrix} \omega_1^2 & 0 & \cdots & 0 \\ 0 & \omega_2^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_m^2 \end{pmatrix} \quad (3.10)$$

Considering that  $\hat{\underline{\underline{K}}}$  is symmetric, once again for the Spectral Theorem (Casali et al., 2016; Lang, 2001 and Lang, 2005), the matrix  $\hat{\underline{\underline{P}}}$  is orthogonal:

$$\hat{\underline{\underline{P}}} \in O_m(\mathfrak{R})$$

and also it can be imposed the normalization for the norms of the eigenvectors:

$$\langle \hat{\underline{\underline{\varphi}}}_i; \hat{\underline{\underline{\varphi}}}_j \rangle = \delta_{ij} \quad (3.11)$$

in which  $\delta_{ij}$  is the Kronecker delta. This equation can be rewritten as follows:

$$\langle \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_i; \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_j \rangle = \delta_{ij} \quad (3.11')$$

Subsequently, it can be written:

$$\hat{\underline{\underline{K}}} = \hat{\underline{\underline{P}}} \cdot \underline{\underline{A}} \cdot \hat{\underline{\underline{P}}}^T \quad (3.12)$$

$$\underline{\underline{K}} = \sqrt{\underline{\underline{M}}} \cdot \hat{\underline{\underline{K}}} \cdot \sqrt{\underline{\underline{M}}} \quad (3.13)$$

then:

$$\underline{\underline{K}} = \sqrt{\underline{\underline{M}}} \cdot \hat{\underline{\underline{P}}} \cdot \underline{\underline{A}} \cdot \hat{\underline{\underline{P}}}^T \cdot \sqrt{\underline{\underline{M}}} \quad (3.14)$$

changing the notation:

$$\underline{\underline{P}} = \sqrt{\underline{\underline{M}}} \cdot \hat{\underline{\underline{P}}} \quad (3.15)$$

in which  $\underline{\underline{P}}$  is the matrix whose i-th column is the eigenvector of the original generalized eigenvalues/eigenvectors problem, premultiplied for the  $\underline{\underline{M}}$  matrix,  $\underline{\underline{M}} \cdot \underline{\underline{\varphi}}_i$  (Eq. 3.2).

Lastly, the problem can be rewritten:

$$\underline{\underline{K}} = \underline{\underline{P}} \cdot \underline{\underline{A}} \cdot \underline{\underline{P}}^T \quad (3.16)$$

From the formulation of Eq. 3.14 is possible to compute the number of maximum parameters obtainable for a problem of size  $m$  and having at disposal  $n$  eigenvalues and eigenvectors.

Starting from the knowledge of  $n$  pairs of known eigenvalues and eigenvectors:

$$(\omega_1^2, \underline{\underline{\varphi}}_1); (\omega_2^2, \underline{\underline{\varphi}}_2); \cdots; (\omega_n^2, \underline{\underline{\varphi}}_n) \quad 1 \leq n \leq m \quad (3.17)$$

for Eq. 3.11'  $\{\sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_1, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_2, \cdots, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_n\}$  is a base of the subspace  $\mathfrak{R}^n$ .

In order to complete the base of the space of order  $m$ ,  $m-n$  vectors have to be defined. In this way, the new base of  $\mathfrak{R}^m$  is  $\{\sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_1, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_2, \dots, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_n, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}'_{n+1}, \dots, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}'_m\}$ . The matrices  $\underline{\underline{\hat{P}}}$  and  $\underline{\underline{A}}$  can be rewritten as follows:

$$\underline{\underline{\hat{P}}}' = \left[ \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_1, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_2, \dots, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}_n \mid \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}'_{n+1}, \dots, \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\varphi}}'_m \right] \quad (3.18)$$

$$\underline{\underline{A}}' = \begin{bmatrix} \underline{\underline{A}}_n & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{A}}'_{m-n} \end{bmatrix} \quad (3.19)$$

in which:

$$\underline{\underline{A}}_n = \begin{bmatrix} \omega_1^2 & 0 & \dots & 0 \\ 0 & \omega_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \omega_n^2 \end{bmatrix} \quad (3.20)$$

$\underline{\underline{A}}'_{m-n} \in S^{m-n}(\mathfrak{R})$  is a symmetric and positive defined matrix.  $\underline{\underline{K}}$ , therefore, can be rewritten:

$$\underline{\underline{K}} = \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\hat{P}}}' \cdot \begin{bmatrix} \underline{\underline{A}}_n & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{A}}'_{m-n} \end{bmatrix} \cdot \underline{\underline{\hat{P}}}'^T \cdot \sqrt{\underline{\underline{M}}} \quad (3.21)$$

$$\underline{\underline{K}} = \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\hat{P}}}' \cdot \begin{bmatrix} \underline{\underline{A}}_n & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}}_{m-n} \end{bmatrix} \cdot \underline{\underline{\hat{P}}}'^T \cdot \sqrt{\underline{\underline{M}}} + \sqrt{\underline{\underline{M}}} \cdot \underline{\underline{\hat{P}}}' \cdot \begin{bmatrix} \underline{\underline{0}}_n & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{A}}'_{m-n} \end{bmatrix} \cdot \underline{\underline{\hat{P}}}'^T \cdot \sqrt{\underline{\underline{M}}} \quad (3.22)$$

therefore  $\underline{\underline{K}}$  is an affine subspace of order  $\frac{(m-n) \cdot (m-n+1)}{2}$ , the same order of unknown matrix

$\underline{\underline{A}}'_{m-n}$ . Following the procedure outlined in Chapter 2, matrix  $\underline{\underline{K}}$  can be written as follows:

$$\underline{\underline{K}} = \underline{\underline{K}}_0 + a_1 \cdot \underline{\underline{K}}_{r,1} + a_2 \cdot \underline{\underline{K}}_{r,2} + \dots + a_N \cdot \underline{\underline{K}}_{r,N} \quad (3.23)$$

which is an affine subspace of order  $N$ .

Remembering the theorem about affine spaces, there is the following relation between subspaces of Eqs. 3.22 and 3.23

$$N + \frac{(m-n) \cdot (m-n+1)}{2} - \frac{m \cdot (m+1)}{2} \begin{cases} > 0 & \text{Infinite solutions} \\ = 0 & \text{Exact solution} \\ < 0 & \text{Optimal solution,} \\ & \text{least squares} \end{cases} \quad (3.24)$$

Therefore, the maximum number of parameters is:

$$N = \frac{m \cdot (m+1)}{2} - \frac{(m-n) \cdot (m-n+1)}{2} \quad (3.25)$$

or, rewriting it:

$$N = n \cdot m - \sum_{i=1}^n i + n \quad (3.26)$$

$$N = n \cdot (m + 1) - \sum_{i=1}^n i \quad (3.27)$$

$$N = \sum_{i=1}^m i - \sum_{i=1}^{m-n} i \quad (3.28)$$

### 3.2.2 Final remarks

The formula found for the maximum number of parameters (Eq. 3.1) refers to the maximum unknown quantities of the eigenvalues/eigenvectors problem without regard if the unknown parameters are in the stiffness or mass matrix. Therefore, parameters even in the mass matrix could be taken into account. Moreover, the maximum number can be achieved only if those parameters are chosen "linearly independent". This aspect will be clarified in paragraph 3.5.

### 3.3 Least squares solution of the problem

Starting from the system of Eq. 2.4, rewritten here for convenience:

$$\begin{cases} p_1 \cdot \left[ \underline{\underline{K}}_0 + \sum_{s=1}^q (a_s \underline{\underline{K}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{M}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{M}}_s \right) \right] \cdot \underline{\underline{\phi}}_1 = \underline{\underline{v}}_1 \\ \vdots \\ p_i \cdot \left[ \underline{\underline{K}}_0 + \sum_{s=1}^q (a_s \underline{\underline{K}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{M}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{M}}_s \right) \right] \cdot \underline{\underline{\phi}}_i = \underline{\underline{v}}_i \\ \vdots \\ p_n \cdot \left[ \underline{\underline{K}}_0 + \sum_{s=1}^q (a_s \underline{\underline{K}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{M}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N a_s \cdot \underline{\underline{M}}_s \right) \right] \cdot \underline{\underline{\phi}}_n = \underline{\underline{v}}_n \end{cases} \quad (3.29)$$

the following matrices and vectors can be defined, as already introduced in paragraph 2.3.2:

$$\underline{\underline{\tilde{K}}}_{r,s} = \begin{bmatrix} p_1 \cdot \underline{\underline{K}}_{r,s} & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & p_2 \cdot \underline{\underline{K}}_{r,s} & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & p_n \cdot \underline{\underline{K}}_{r,s} \end{bmatrix} \quad (3.30)$$

for  $s = 1, 2, \dots, q$ ;

$$\underline{\underline{\tilde{M}}}_s = \begin{bmatrix} -p_1 \cdot \omega_1^2 \cdot \underline{\underline{M}}_s & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & -p_2 \cdot \omega_2^2 \cdot \underline{\underline{M}}_s & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & -p_n \cdot \omega_n^2 \cdot \underline{\underline{M}}_s \end{bmatrix} \quad (3.31)$$

for  $s = q+1, q+2, \dots, N$ ;

$$\underline{\tilde{\varphi}} = \begin{bmatrix} \underline{\varphi}_1 \\ \underline{\varphi}_2 \\ \vdots \\ \underline{\varphi}_n \end{bmatrix} \quad (3.32)$$

$$\underline{\tilde{K}}_0 = \begin{bmatrix} p_1 \cdot \underline{K}_0 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & p_2 \cdot \underline{K}_0 & \cdots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \cdots & p_n \cdot \underline{K}_0 \end{bmatrix} \quad (3.33)$$

$$\underline{\tilde{M}}_0 = \begin{bmatrix} p_1 \cdot \omega_1^2 \cdot \underline{M}_0 & \underline{0} & \cdots & \underline{0} \\ \underline{0} & p_2 \cdot \omega_2^2 \cdot \underline{M}_0 & \cdots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \cdots & p_n \cdot \omega_n^2 \cdot \underline{M}_0 \end{bmatrix} \quad (3.34)$$

The least squares solution of Eq. 3.29 is equivalent to find the minimum of the subsequent function:

$$f(a_1, a_2, \dots, a_N) = \left\| \underline{\tilde{K}}_0 \cdot \underline{\tilde{\varphi}} + \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} - \underline{\tilde{M}}_0 \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} \right\|^2 \quad (3.35)$$

defining:

$$\underline{\tilde{\psi}} = \underline{\tilde{M}}_0 \cdot \underline{\tilde{\varphi}} - \underline{\tilde{K}}_0 \cdot \underline{\tilde{\varphi}} \quad (3.36)$$

the function becomes:

$$f(a_1, a_2, \dots, a_N) = \left\| \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\|^2 \quad (3.37)$$

or, that is the same:

$$f(a_1, a_2, \dots, a_N) = \left\langle \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}}; \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle \quad (3.38)$$

$\| \cdot \|$  and  $\langle \cdot \rangle$  mean the norm of the vector and the dot product between vectors, respectively.

Making the partial derivatives, remembering the properties of the derivatives themselves:

$$\frac{\partial \left\langle \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}}; \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle}{\partial a_s} = 2 \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle \quad (3.39)$$

for  $s = 1, 2, \dots, q$  and

$$\frac{\partial \left\langle \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}}; \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle}{\partial a_s} = 2 \cdot \left\langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \sum_{s=1}^q a_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N a_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle \quad (3.40)$$

for  $s = q+1, q+2, \dots, N$ .

The minimum solution is achieved if all the partial derivatives are null (stationary point):

$$\begin{cases} \frac{\partial f}{\partial a_1} = 0 \\ \vdots \\ \frac{\partial f}{\partial a_q} = 0 \\ \frac{\partial f}{\partial a_{q+1}} = 0 \\ \vdots \\ \frac{\partial f}{\partial a_N} = 0 \end{cases} \quad (3.41)$$

using the distributive properties of the dot product and remembering that:

$$\langle X; X \rangle = \|X\|^2 \quad (3.42)$$

The system can be rewritten as follows:

$$\begin{cases} a_1 \cdot \|\underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}\|^2 + \dots + a_q \cdot \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle + a_{q+1} \cdot \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle + \dots + a_N \cdot \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle = \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \vdots \\ a_1 \cdot \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle + \dots + a_q \cdot \|\underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}\|^2 + a_{q+1} \cdot \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle + \dots + a_N \cdot \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle = \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ a_1 \cdot \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle + \dots + a_q \cdot \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle + a_{q+1} \cdot \|\underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}\|^2 + \dots + a_N \cdot \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle = \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \vdots \\ a_1 \cdot \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle + \dots + a_q \cdot \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle + a_{q+1} \cdot \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle + \dots + a_N \cdot \|\underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}\|^2 = \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \end{cases} \quad (3.43)$$

Defining:

$$\underline{\tilde{A}} = \begin{bmatrix} \|\underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}\|^2 & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \dots & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \dots & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \|\underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}\|^2 & \dots & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \dots & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \dots & \vdots \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \dots & \|\underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}\|^2 & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \dots & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \dots & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \|\underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}\|^2 & \dots & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \dots & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \dots & \|\underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}\|^2 \end{bmatrix} \quad (3.44)$$

$$\underline{\tilde{b}} = \begin{bmatrix} \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \vdots \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \\ \vdots \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle \end{bmatrix} \quad (3.45)$$

the system can be written as follows:

$$\underline{\tilde{A}} \cdot \underline{a} = \underline{\tilde{b}} \quad (3.46)$$

which is no longer a overdetermined system ( $\underline{\tilde{A}}$  is a square matrix of order  $N$ ).

### 3.3.1 Uniqueness of the solution

Keeping in mind the rule for maximum number of parameters, analyzed in paragraph 3.2, the system of Eq. 3.46 takes unique solution if and only if:

$$\det(\underline{\tilde{A}}) \neq 0 \quad (3.47)$$

$\underline{\tilde{A}}$  is a symmetric matrix, because of the commutative property of the dot product, and also is semi-positive defined (as illustrated in the next paragraph). Therefore the condition of Eq. 3.47 is essential in order to assure the uniqueness of the solution.

Conjecturing that  $\underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \neq \underline{0}$ ,  $\underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \neq \underline{0}$ ,  $\dots$ ,  $\underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \neq \underline{0}$ ,  $\underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \neq \underline{0}$ ,  $\dots$ ,  $\underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \neq \underline{0}$ , Eq. 3.47 is not fulfilled if and only if the columns of the  $\underline{\tilde{A}}$  matrix are linearly dependents, and therefore if  $\exists(\alpha_2, \dots, \alpha_N) \neq \underline{0}$  for which:

$$\begin{bmatrix} \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \end{bmatrix} = \alpha_2 \cdot \begin{bmatrix} \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \end{bmatrix} + \dots + \alpha_q \cdot \begin{bmatrix} \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \end{bmatrix} + \alpha_{q+1} \cdot \begin{bmatrix} \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \end{bmatrix} + \dots + \alpha_N \cdot \begin{bmatrix} \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \end{bmatrix} \quad (3.48)$$

It is equivalent to say that the columns are linearly dependent if and only if  $\exists(\alpha_1, \dots, \alpha_N) \neq \underline{0}$  for which:

$$\begin{bmatrix} \left\langle \sum_{s=1}^q \alpha_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N \alpha_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \right\rangle \\ \left\langle \sum_{s=1}^q \alpha_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N \alpha_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \right\rangle \\ \vdots \\ \left\langle \sum_{s=1}^q \alpha_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N \alpha_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \right\rangle \\ \left\langle \sum_{s=1}^q \alpha_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N \alpha_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \right\rangle \\ \vdots \\ \left\langle \sum_{s=1}^q \alpha_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N \alpha_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \right\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3.49)$$

summing all the rows:

$$\left\langle \sum_{s=1}^q \alpha_s \cdot \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N \alpha_s \cdot \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \sum_{s=1}^q \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} + \sum_{s=q+1}^N \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} \right\rangle = 0 \quad (3.50)$$

Because of the two vectors in the dot product of Eq. 3.50 cannot be orthogonal, the linear dependence of the columns can be achieved with these relation:

$$\alpha_1 \cdot \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} + \alpha_2 \cdot \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} + \dots + \alpha_q \cdot \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} + \alpha_{q+1} \cdot \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} + \dots + \alpha_N \cdot \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} = \underline{0} \quad (3.51)$$

if  $\exists(\alpha_1, \dots, \alpha_N) \neq \underline{0}$ , Eq. 3.47 is not satisfied and the problem doesn't allow for unique solution.

Otherwise, if Eq. 3.51 is satisfied if and only if:

$$(\alpha_1, \alpha_2, \dots, \alpha_N) = \underline{0} \quad (3.52)$$

the problem has got only one solution. This is the *1<sup>st</sup> uniqueness condition*, that is equivalent to say that vectors  $\underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}, \dots, \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}, \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}, \dots, \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}$  are linearly independents (Casali et al., 2016; Lang, 2001).

If the condition in Eq. 3.51 is achieved for  $(\alpha_1, \alpha_2, \dots, \alpha_N) \neq \underline{0}$ , and therefore vectors  $\underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}, \dots, \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}, \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}, \dots, \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}$  are linearly dependents, also rows of vector  $\underline{\tilde{b}}$  are linearly dependents and

$$\text{rank}(\underline{\tilde{A}} | \underline{\tilde{b}}) = \text{rank}(\underline{\tilde{A}}) \quad (3.53)$$

in which  $\underline{\tilde{A}} | \underline{\tilde{b}}$  is the complete matrix of the system. Follows that, if the abovementioned vectors are linearly dependents, the system is never impossible and it always allows for infinite solutions.

Remembering the assumptions  $(\underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \neq \underline{0}, \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \neq \underline{0}, \dots, \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \neq \underline{0}, \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \neq \underline{0}, \dots, \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \neq \underline{0})$  made in the demonstration, other uniqueness conditions must be defined in order to avoid the case

in which one column of the  $\underline{\tilde{A}}$  matrix is already a null vector. Having noticed that in the columns there is always one constant vector in the dot product, the conditions, in order to avoid that one column is the null vector, can be written as follows:

$$\underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} \neq \underline{0} \quad (3.54)$$

for  $s = 1, 2, \dots, q$  and

$$\underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} \neq \underline{0} \quad (3.55)$$

for  $s = q+1, q+2, \dots, N$ .

These are the  $2^{nd}$  to  $(N+1)^{th}$  uniqueness conditions. These conditions are also needed by the first uniqueness condition because one null term of type  $\underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}$  or  $\underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}$  leads to the existence of one not null vector of Eq. 3.52 which satisfies Eq. 3.51.

The condition for the maximum number of parameters given in paragraph 3.2 is still valid and represent a sort of zero condition for uniqueness of the solution. If more than the maximum number of parameters are chosen, the problem is native undetermined and therefore no unique solution is reached.

### 3.3.2 Definiteness of Hessian matrix

In order to understand if the stationary point derived in paragraph 3.3 is a minimum, maximum or saddle point, the second derivatives, with respect to the parameters, must be computed and the Hessian matrix must be analyzed.

$$\frac{\partial^2 f}{\partial a_t \partial a_s} = \frac{\partial}{\partial a_t} \left( \frac{\partial f}{\partial a_s} \right) = \frac{\partial}{\partial a_t} \left( 2 \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \sum_{c=1}^q a_c \cdot \underline{\tilde{K}}_{r,c} \cdot \underline{\tilde{\varphi}} + \sum_{c=q+1}^N a_c \cdot \underline{\tilde{M}}_c \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle \right) = 2 \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,t} \cdot \underline{\tilde{\varphi}} \right\rangle \quad (3.56)$$

for  $s = 1, 2, \dots, q$ ,  $t = 1, 2, \dots, q$ ;

$$\frac{\partial^2 f}{\partial a_t \partial a_s} = \frac{\partial}{\partial a_t} \left( \frac{\partial f}{\partial a_s} \right) = \frac{\partial}{\partial a_t} \left( 2 \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \sum_{c=1}^q a_c \cdot \underline{\tilde{K}}_{r,c} \cdot \underline{\tilde{\varphi}} + \sum_{c=q+1}^N a_c \cdot \underline{\tilde{M}}_c \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle \right) = 2 \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_t \cdot \underline{\tilde{\varphi}} \right\rangle \quad (3.57)$$

for  $s = 1, 2, \dots, q$ ,  $t = q+1, q+2, \dots, N$  and

$$\frac{\partial^2 f}{\partial a_t \partial a_s} = \frac{\partial}{\partial a_t} \left( \frac{\partial f}{\partial a_s} \right) = \frac{\partial}{\partial a_t} \left( 2 \cdot \left\langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \sum_{c=1}^q a_c \cdot \underline{\tilde{K}}_{r,c} \cdot \underline{\tilde{\varphi}} + \sum_{c=q+1}^N a_c \cdot \underline{\tilde{M}}_c \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle \right) = 2 \cdot \left\langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_t \cdot \underline{\tilde{\varphi}} \right\rangle \quad (3.58)$$

for  $s = q+1, q+2, \dots, N$ ,  $t = q+1, q+2, \dots, N$ .

For the commutative property of the dot product, the Schwarz theorem is satisfy (Lanconelli, 2000; Lanconelli and Obrecht, 2001).



The Hessian matrix is as follows:

$$\underline{\underline{H}} = \begin{bmatrix} 2 \cdot \|\underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}\|^2 & 2 \cdot \langle \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} \rangle & \cdots & 2 \cdot \langle \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}} \rangle & 2 \cdot \langle \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}} \rangle & \cdots & 2 \cdot \langle \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}} \rangle \\ 2 \cdot \langle \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}} \rangle & 2 \cdot \|\underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}\|^2 & \cdots & 2 \cdot \langle \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}} \rangle & 2 \cdot \langle \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}} \rangle & \cdots & 2 \cdot \langle \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}} \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 2 \cdot \langle \underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}} \rangle & 2 \cdot \langle \underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} \rangle & \cdots & 2 \cdot \|\underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}}\|^2 & 2 \cdot \langle \underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}} \rangle & \cdots & 2 \cdot \langle \underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}} \rangle \\ 2 \cdot \langle \underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}} \rangle & 2 \cdot \langle \underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} \rangle & \cdots & 2 \cdot \langle \underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}} \rangle & 2 \cdot \|\underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}}\|^2 & \cdots & 2 \cdot \langle \underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}} \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 2 \cdot \langle \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}} \rangle & 2 \cdot \langle \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} \rangle & \cdots & 2 \cdot \langle \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,q} \cdot \underline{\underline{\varphi}} \rangle & 2 \cdot \langle \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_{q+1} \cdot \underline{\underline{\varphi}} \rangle & \cdots & 2 \cdot \|\underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}\|^2 \end{bmatrix} \quad (3.59)$$

therefore:

$$\underline{\underline{H}} = 2 \cdot \underline{\underline{A}} \quad (3.60)$$

In order to demonstrate the definiteness of the Hessian matrix, the definition of definiteness of matrices itself is used (Casali et al., 2016):

$$[x_1 \ x_2 \ \cdots \ x_N] \cdot \underline{\underline{H}} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} 2 \cdot x_1 \cdot \|\underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}\|^2 + 2 \cdot x_2 \cdot \langle \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} \rangle + \cdots + 2 \cdot x_N \cdot \langle \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}} \rangle \\ 2 \cdot x_1 \cdot \langle \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}} \rangle + 2 \cdot x_2 \cdot \|\underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}\|^2 + \cdots + 2 \cdot x_N \cdot \langle \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}} \rangle \\ \vdots \\ 2 \cdot x_1 \cdot \langle \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}} \rangle + 2 \cdot x_2 \cdot \langle \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} \rangle + \cdots + 2 \cdot x_N \cdot \|\underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}\|^2 \end{bmatrix}^T \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \quad (3.61)$$

$$= 2 \cdot x_1^2 \cdot \|\underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}\|^2 + 2 \cdot x_2^2 \cdot \|\underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}\|^2 + \cdots + 2 \cdot x_N^2 \cdot \|\underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}\|^2 + 4 \cdot x_1 \cdot x_2 \cdot \langle \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}}; \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} \rangle + \cdots + 4 \cdot x_2 \cdot x_N \cdot \langle \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}}; \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}} \rangle + \cdots = 2 \cdot \|x_1 \cdot \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}} + x_2 \cdot \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} + \cdots + x_N \cdot \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}\|^2$$

$$2 \cdot \|x_1 \cdot \underline{\underline{K}}_{r,1} \cdot \underline{\underline{\varphi}} + x_2 \cdot \underline{\underline{K}}_{r,2} \cdot \underline{\underline{\varphi}} + \cdots + x_N \cdot \underline{\underline{M}}_N \cdot \underline{\underline{\varphi}}\|^2 \geq 0 \quad \forall x_1, x_2, \dots, x_N \in \mathfrak{R} \quad (3.62)$$

Therefore the Hessian matrix  $\underline{\underline{H}}$  and the coefficient matrix  $\underline{\underline{A}}$  are semi-positive defined.

Knowing that  $\underline{\underline{A}}$  is semi positive defined, and knowing that all the eigenvalues of a semi positive defined matrix are greater or equal than 0:

$$\det(\underline{\tilde{A}}) = \det \begin{bmatrix} \|\underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}\|^2 & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \|\underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}\|^2 & \cdots & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \|\underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}\|^2 & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \|\underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}\|^2 & \cdots & \langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \rangle & \langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \rangle & \cdots & \|\underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}\|^2 \end{bmatrix} \geq 0 \quad (3.63)$$

which is a generalization of the Cauchy-Schwarz inequality.

With the  $1^{st}$  uniqueness condition,  $\det(\underline{\tilde{A}}) \neq 0$ , the matrices  $\underline{\tilde{A}}$  and  $\underline{\tilde{H}}$  are, eventually, positive defined:

$$\underline{\tilde{H}} \in S_N(\mathfrak{R}), D^+ \quad (3.64)$$

A unique stationary point associated to a positive defined Hessian matrix is a *global minimum point* for the function (Lanconelli, 2000; Lanconelli and Obrecht, 2001) and therefore the demonstration is completed.

### 3.3.3 Solution of the system

The solution for the system of Eq. 3.46 can be achieved, in closed relations, using the Cramer formulation (Casali et al., 2016):

$$a_s = \frac{\det(\underline{\tilde{A}}_s)}{\det(\underline{\tilde{A}})} \quad (3.65)$$

for  $s = 1, 2, \dots, N$ , in which  $\underline{\tilde{A}}_s$  is the  $\underline{\tilde{A}}$  matrix with the s-th column replaced by the  $\underline{\tilde{b}}$  vector (vector of known terms).

Otherwise the system can be solved through matrix inversion:

$$\underline{a} = \underline{\tilde{A}}^{-1} \cdot \underline{\tilde{b}} \quad (3.66)$$

### 3.4 Partial derivatives procedure for errors propagation

The procedure introduced in paragraph 2.7.2 is here developed in all the details.

Starting from system of Eq. 3.43, the partial derivatives with respect to  $\varphi_{ji}$  and  $\omega_i^2$  have to be computed. All the partial derivatives are computed with parameters values known and achieved using the procedure developed in paragraph 2.6, for sake of brevity the point in which the partial derivatives are computed will be omitted in the demonstration.

#### 3.4.1 Derivatives with respect to $\varphi_{ji}$

$$\left\{ \begin{array}{l} \frac{\partial}{\partial \varphi_{ji}} \left( a_1 \cdot \left\| \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \right\|^2 + \dots + a_q \cdot \left\langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \right\rangle + a_{q+1} \cdot \left\langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \right\rangle + \dots + a_N \cdot \left\langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \right\rangle \right) = \frac{\partial}{\partial \varphi_{ji}} \left( \left\langle \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \right\rangle \right) \\ \vdots \\ \frac{\partial}{\partial \varphi_{ji}} \left( a_1 \cdot \left\langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \right\rangle + \dots + a_q \cdot \left\| \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \right\|^2 + a_{q+1} \cdot \left\langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \right\rangle + \dots + a_N \cdot \left\langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \right\rangle \right) = \frac{\partial}{\partial \varphi_{ji}} \left( \left\langle \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \right\rangle \right) \\ \frac{\partial}{\partial \varphi_{ji}} \left( a_1 \cdot \left\langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \right\rangle + \dots + a_q \cdot \left\langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \right\rangle + a_{q+1} \cdot \left\| \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \right\|^2 + \dots + a_N \cdot \left\langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \right\rangle \right) = \frac{\partial}{\partial \varphi_{ji}} \left( \left\langle \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \right\rangle \right) \\ \vdots \\ \frac{\partial}{\partial \varphi_{ji}} \left( a_1 \cdot \left\langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \right\rangle + \dots + a_q \cdot \left\langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,q} \cdot \underline{\tilde{\varphi}} \right\rangle + a_{q+1} \cdot \left\langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{q+1} \cdot \underline{\tilde{\varphi}} \right\rangle + \dots + a_N \cdot \left\| \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \right\|^2 \right) = \frac{\partial}{\partial \varphi_{ji}} \left( \left\langle \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \right\rangle \right) \end{array} \right. \quad (3.67)$$

deriving the generic s-th row of the system, one can achieve the following relations:

- for  $s = 1, 2, \dots, q$ :

$$\begin{aligned} & \frac{\partial a_1}{\partial \varphi_{ji}} \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \right\rangle + a_1 \cdot \left\langle \underline{\tilde{K}}_{r,s}(:,j); \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \right\rangle + a_1 \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1}(:,j) \right\rangle + \frac{\partial a_2}{\partial \varphi_{ji}} \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \right\rangle + \\ & a_2 \cdot \left\langle \underline{\tilde{K}}_{r,s}(:,j); \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \right\rangle + a_2 \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2}(:,j) \right\rangle + \dots + \frac{\partial a_s}{\partial \varphi_{ji}} \cdot \left\| \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} \right\|^2 + 2 \cdot a_s \cdot \left\langle \underline{\tilde{K}}_{r,s}(:,j); \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} \right\rangle + \\ & \dots + \frac{\partial a_N}{\partial \varphi_{ji}} \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \right\rangle + a_N \cdot \left\langle \underline{\tilde{K}}_{r,s}(:,j); \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \right\rangle + a_N \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N(:,j) \right\rangle = \\ & = \left\langle \underline{\tilde{K}}_{r,s}(:,j); \underline{\tilde{\psi}} \right\rangle + \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \frac{\partial \underline{\tilde{\psi}}}{\partial \varphi_{ji}} \right\rangle \end{aligned} \quad (3.68)$$

with:

$$\frac{\partial \underline{\tilde{\psi}}}{\partial \varphi_{ji}} = \frac{\partial}{\partial \varphi_{ji}} \left( \underline{\tilde{M}}_0 \cdot \underline{\tilde{\varphi}} - \underline{\tilde{K}}_0 \cdot \underline{\tilde{\varphi}} \right) = \underline{\tilde{M}}_0(:,j) - \underline{\tilde{K}}_0(:,j) \quad (3.69)$$

$(:,j)$  means the j-th column of the matrix. Then Eq. 3.68 can be rewritten as follows:

$$\frac{\partial a_1}{\partial \varphi_{ji}} \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \right\rangle + \frac{\partial a_2}{\partial \varphi_{ji}} \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \right\rangle + \dots + \frac{\partial a_s}{\partial \varphi_{ji}} \cdot \left\| \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}} \right\|^2 + \dots + \frac{\partial a_N}{\partial \varphi_{ji}} \cdot \left\langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \right\rangle = \tilde{l}_{s,ji} \quad (3.70)$$

in which:

$$\begin{aligned}
\tilde{l}_{s,ji} = & -\sum_{k=1}^q a_k \cdot \left[ \langle \underline{\tilde{K}}_{r,s}(:,j); \underline{\tilde{K}}_{r,k} \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,k}(:,j) \rangle \right] - \\
& \sum_{k=q+1}^N a_k \cdot \left[ \langle \underline{\tilde{K}}_{r,s}(:,j); \underline{\tilde{M}}_k \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_k(:,j) \rangle \right] + \\
& \langle \underline{\tilde{K}}_{r,s}(:,j); \underline{\tilde{\psi}} \rangle + \langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_0(:,j) - \underline{\tilde{K}}_0(:,j) \rangle
\end{aligned} \tag{3.71}$$

– for  $s = q+1, q+2, \dots, N$ :

$$\frac{\partial a_1}{\partial \varphi_{ji}} \cdot \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle + \frac{\partial a_2}{\partial \varphi_{ji}} \cdot \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle + \dots + \frac{\partial a_s}{\partial \varphi_{ji}} \cdot \|\underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}\|^2 + \dots + \frac{\partial a_N}{\partial \varphi_{ji}} \cdot \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle = \tilde{l}_{s,ji} \tag{3.72}$$

in which:

$$\begin{aligned}
\tilde{l}_{s,ji} = & -\sum_{k=1}^q a_k \cdot \left[ \langle \underline{\tilde{M}}_s(:,j); \underline{\tilde{K}}_{r,k} \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,k}(:,j) \rangle \right] - \\
& \sum_{k=q+1}^N a_k \cdot \left[ \langle \underline{\tilde{M}}_s(:,j); \underline{\tilde{M}}_k \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_k(:,j) \rangle \right] + \\
& \langle \underline{\tilde{M}}_s(:,j); \underline{\tilde{\psi}} \rangle + \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_0(:,j) - \underline{\tilde{K}}_0(:,j) \rangle
\end{aligned} \tag{3.73}$$

The solution can therefore be written as follows:

$$\underline{\tilde{A}} \cdot \frac{\partial \underline{a}}{\partial \varphi_{ji}} = \underline{\tilde{l}}_{ji} \tag{3.74}$$

for  $i = 1, 2, \dots, n$ ;  $j = 1, 2, \dots, m$ ;  $\underline{\tilde{A}}$  already defined in Eq. 2.19;  $\frac{\partial \underline{a}}{\partial \varphi_{ji}}$ ,  $\underline{\tilde{l}}_{ji}$  defined as follows:

$$\frac{\partial \underline{a}}{\partial \varphi_{ji}} = \begin{bmatrix} \frac{\partial a_1}{\partial \varphi_{ji}} \\ \frac{\partial a_2}{\partial \varphi_{ji}} \\ \frac{\partial a_q}{\partial \varphi_{ji}} \\ \frac{\partial a_{q+1}}{\partial \varphi_{ji}} \\ \frac{\partial a_N}{\partial \varphi_{ji}} \end{bmatrix} \tag{3.75}$$

$$\tilde{\underline{l}}_{ji} = \begin{bmatrix} \tilde{l}_{1,ji} \\ \tilde{l}_{2,ji} \\ \vdots \\ \tilde{l}_{q,ji} \\ \tilde{l}_{q+1,ji} \\ \vdots \\ \tilde{l}_{N,ji} \end{bmatrix} \quad (3.76)$$

The uniqueness of the solution is ensured by uniqueness conditions of original system (paragraph 3.3.1) because the coefficient matrix is the same as for the original system ( $\tilde{\underline{A}}$  matrix).

### 3.4.2 Derivatives with respect to $\omega_i^2$

$$\begin{cases} \frac{\partial}{\partial \omega_i^2} \left( a_1 \cdot \left\| \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}} \right\|^2 + \dots + a_q \cdot \left\langle \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,q} \cdot \tilde{\underline{\varphi}} \right\rangle + a_{q+1} \cdot \left\langle \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_{q+1} \cdot \tilde{\underline{\varphi}} \right\rangle + \dots + a_N \cdot \left\langle \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}} \right\rangle \right) = \frac{\partial}{\partial \omega_i^2} \left( \left\langle \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{\psi}} \right\rangle \right) \\ \vdots \\ \frac{\partial}{\partial \omega_i^2} \left( a_1 \cdot \left\langle \tilde{\underline{K}}_{r,q} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}} \right\rangle + \dots + a_q \cdot \left\| \tilde{\underline{K}}_{r,q} \cdot \tilde{\underline{\varphi}} \right\|^2 + a_{q+1} \cdot \left\langle \tilde{\underline{K}}_{r,q} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_{q+1} \cdot \tilde{\underline{\varphi}} \right\rangle + \dots + a_N \cdot \left\langle \tilde{\underline{K}}_{r,q} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}} \right\rangle \right) = \frac{\partial}{\partial \omega_i^2} \left( \left\langle \tilde{\underline{K}}_{r,q} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{\psi}} \right\rangle \right) \\ \frac{\partial}{\partial \omega_i^2} \left( a_1 \cdot \left\langle \tilde{\underline{M}}_{q+1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}} \right\rangle + \dots + a_q \cdot \left\langle \tilde{\underline{M}}_{q+1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,q} \cdot \tilde{\underline{\varphi}} \right\rangle + a_{q+1} \cdot \left\| \tilde{\underline{M}}_{q+1} \cdot \tilde{\underline{\varphi}} \right\|^2 + \dots + a_N \cdot \left\langle \tilde{\underline{M}}_{q+1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}} \right\rangle \right) = \frac{\partial}{\partial \omega_i^2} \left( \left\langle \tilde{\underline{M}}_{q+1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{\psi}} \right\rangle \right) \\ \vdots \\ \frac{\partial}{\partial \omega_i^2} \left( a_1 \cdot \left\langle \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}} \right\rangle + \dots + a_q \cdot \left\langle \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,q} \cdot \tilde{\underline{\varphi}} \right\rangle + a_{q+1} \cdot \left\langle \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_{q+1} \cdot \tilde{\underline{\varphi}} \right\rangle + \dots + a_N \cdot \left\| \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}} \right\|^2 \right) = \frac{\partial}{\partial \omega_i^2} \left( \left\langle \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}}; \tilde{\underline{\psi}} \right\rangle \right) \end{cases} \quad (3.77)$$

deriving the generic s-th row of the system, one can achieve the following relations:

– for  $s = 1, 2, \dots, q$  :

$$\begin{aligned} & \frac{\partial a_1}{\partial \omega_i^2} \cdot \left\langle \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}} \right\rangle + \frac{\partial a_2}{\partial \omega_i^2} \cdot \left\langle \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,2} \cdot \tilde{\underline{\varphi}} \right\rangle + \dots + \frac{\partial a_s}{\partial \omega_i^2} \cdot \left\| \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}} \right\|^2 + \dots + \\ & + \frac{\partial a_N}{\partial \omega_i^2} \cdot \left\langle \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}} \right\rangle + a_N \cdot \left\langle \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_{N,i} \cdot \tilde{\underline{\varphi}} \right\rangle = \left\langle \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}}; \frac{\partial \tilde{\underline{\psi}}}{\partial \omega_i^2} \right\rangle \end{aligned} \quad (3.78)$$

with:

$$\frac{\partial \tilde{\underline{\psi}}}{\partial \omega_i^2} = \frac{\partial}{\partial \omega_i^2} \left( \tilde{\underline{M}}_0 \cdot \tilde{\underline{\varphi}} - \tilde{\underline{K}}_0 \cdot \tilde{\underline{\varphi}} \right) = \tilde{\underline{M}}_{0,i} \cdot \tilde{\underline{\varphi}} \quad (3.79)$$

Then Eq. 3.78 can be rewritten as follows:

$$\frac{\partial a_1}{\partial \omega_i^2} \cdot \left\langle \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,1} \cdot \tilde{\underline{\varphi}} \right\rangle + \frac{\partial a_2}{\partial \omega_i^2} \cdot \left\langle \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_{r,2} \cdot \tilde{\underline{\varphi}} \right\rangle + \dots + \frac{\partial a_s}{\partial \omega_i^2} \cdot \left\| \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}} \right\|^2 + \dots + \frac{\partial a_N}{\partial \omega_i^2} \cdot \left\langle \tilde{\underline{K}}_{r,s} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{M}}_N \cdot \tilde{\underline{\varphi}} \right\rangle = \tilde{r}_{s,i} \quad (3.80)$$

in which:

$$\tilde{r}_{s,i} = - \sum_{k=q+1}^N a_k \cdot \langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{k,i} \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{K}}_{r,s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{0,i} \cdot \underline{\tilde{\varphi}} \rangle \quad (3.81)$$

– for  $s = q+1, q+2, \dots, N$ :

$$\frac{\partial a_1}{\partial \omega_i^2} \cdot \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,1} \cdot \underline{\tilde{\varphi}} \rangle + \frac{\partial a_2}{\partial \omega_i^2} \cdot \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_{r,2} \cdot \underline{\tilde{\varphi}} \rangle + \dots + \frac{\partial a_s}{\partial \omega_i^2} \cdot \|\underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}\|^2 + \dots + \frac{\partial a_N}{\partial \omega_i^2} \cdot \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_N \cdot \underline{\tilde{\varphi}} \rangle = \tilde{r}_{s,i} \quad (3.82)$$

in which:

$$\tilde{r}_{s,i} = - \sum_{k=1}^q a_k \cdot \langle \underline{\tilde{K}}_{r,k} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{s,i} \cdot \underline{\tilde{\varphi}} \rangle - \sum_{k=q+1}^N a_k \cdot \left[ \langle \underline{\tilde{M}}_{k,i} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}} \rangle + \langle \underline{\tilde{M}}_k \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{s,i} \cdot \underline{\tilde{\varphi}} \rangle \right] + \langle \underline{\tilde{M}}_{s,i} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{\psi}} \rangle + \langle \underline{\tilde{M}}_s \cdot \underline{\tilde{\varphi}}; \underline{\tilde{M}}_{0,i} \cdot \underline{\tilde{\varphi}} \rangle \quad (3.83)$$

$$\underline{\tilde{M}}_{s,i} = \begin{bmatrix} -p_1 \cdot \underline{0} \cdot \underline{\tilde{M}}_s & \underline{0} & \dots & \underline{0} & \dots & \underline{0} \\ \underline{0} & -p_2 \cdot \underline{0} \cdot \underline{\tilde{M}}_s & \dots & \underline{0} & \dots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{0} & \underline{0} & \dots & -p_i \underline{\tilde{M}}_s & \dots & \underline{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \dots & \underline{0} & \dots & -p_n \cdot \underline{0} \cdot \underline{\tilde{M}}_s \end{bmatrix} \quad (3.84)$$

$$\underline{\tilde{M}}_{0,i} = \begin{bmatrix} p_1 \cdot \underline{0} \cdot \underline{\tilde{M}}_0 & \underline{0} & \dots & \underline{0} & \dots & \underline{0} \\ \underline{0} & p_2 \cdot \underline{0} \cdot \underline{\tilde{M}}_0 & \dots & \underline{0} & \dots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{0} & \underline{0} & \dots & p_i \underline{\tilde{M}}_0 & \dots & \underline{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \dots & \underline{0} & \dots & p_n \cdot \underline{0} \cdot \underline{\tilde{M}}_0 \end{bmatrix} \quad (3.85)$$

or, in the same way:

$$\underline{\tilde{M}}_{s,i} = \begin{bmatrix} \underline{0} & \underline{0} & \dots & \underline{0} & \dots & \underline{0} \\ \underline{0} & \underline{0} & \dots & \underline{0} & \dots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{0} & \underline{0} & \dots & -p_i \underline{\tilde{M}}_s & \dots & \underline{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \dots & \underline{0} & \dots & \underline{0} \end{bmatrix} \quad (3.84')$$

$$\underline{\tilde{M}}_{0,i} = \begin{bmatrix} \underline{0} & \underline{0} & \dots & \underline{0} & \dots & \underline{0} \\ \underline{0} & \underline{0} & \dots & \underline{0} & \dots & \underline{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{0} & \underline{0} & \dots & p_i \underline{\tilde{M}}_0 & \dots & \underline{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{0} & \underline{0} & \dots & \underline{0} & \dots & \underline{0} \end{bmatrix} \quad (3.85')$$

The solution can therefore be written as follows:

$$\underline{\tilde{A}} \cdot \frac{\partial \underline{a}}{\partial \omega_i^2} = \underline{\tilde{r}}_i \quad (3.86)$$

for  $i = 1, 2, \dots, n$ ;  $\underline{\tilde{A}}$  already defined in Eq. 2.19;  $\frac{\partial \underline{a}}{\partial \omega_i^2}$ ,  $\underline{\tilde{r}}_i$  defined as follows:

$$\frac{\partial \underline{a}}{\partial \omega_i^2} = \begin{bmatrix} \frac{\partial a_1}{\partial \omega_i^2} \\ \frac{\partial a_2}{\partial \omega_i^2} \\ \vdots \\ \frac{\partial a_q}{\partial \omega_i^2} \\ \frac{\partial a_{q+1}}{\partial \omega_i^2} \\ \vdots \\ \frac{\partial a_N}{\partial \omega_i^2} \end{bmatrix} \quad (3.87)$$

$$\underline{\tilde{r}}_i = \begin{bmatrix} \tilde{r}_{i,1} \\ \tilde{r}_{i,2} \\ \vdots \\ \tilde{r}_{i,q} \\ \tilde{r}_{i,q+1} \\ \vdots \\ \tilde{r}_{i,N} \end{bmatrix} \quad (3.88)$$

The uniqueness of the solution is ensured by uniqueness conditions of original system (paragraph 3.3.1) because the coefficient matrix is the same as the original system ( $\underline{\tilde{A}}$  matrix).

### 3.4.3 Solution of the systems

The systems of Eqs. 3.74 and 3.86 can be solve, in order to achieve all the partial derivatives, using the Cramer formulation (Casali et al., 2016):

$$\frac{\partial a_s}{\partial \varphi_{ji}} = \frac{\det(\underline{\tilde{A}}_{\underline{\tilde{r}}_{ji,s}})}{\det(\underline{\tilde{A}})} \quad (3.89)$$

for  $i = 1, 2, \dots, n$ ;  $j = 1, 2, \dots, m$ ;  $s = 1, 2, \dots, N$ ; in which  $\underline{\tilde{A}}_{\underline{\tilde{r}}_{ji,s}}$  is the  $\underline{\tilde{A}}$  matrix with the s-th column replaced by  $\underline{\tilde{r}}_{ji}$  vector;

$$\frac{\partial a_s}{\partial \omega_i^2} = \frac{\det(\underline{\underline{\tilde{A}}}_{\omega_i, s})}{\det(\underline{\underline{\tilde{A}}})} \quad (3.90)$$

for  $i=1,2,\dots,n$ ;  $s=1,2,\dots,N$ ; in which  $\underline{\underline{\tilde{A}}}_{\omega_i, s}$  is the  $\underline{\underline{\tilde{A}}}$  matrix with the s-th column replaced by  $\underline{\tilde{r}}_i$  vector.

Otherwise solving the determined system through matrix inversion:

$$\frac{\partial \underline{a}}{\partial \varphi_{ji}} = \underline{\underline{\tilde{A}}}^{-1} \cdot \underline{\tilde{l}}_{ji} \quad (3.91)$$

$$\frac{\partial \underline{a}}{\partial \omega_i^2} = \underline{\underline{\tilde{A}}}^{-1} \cdot \underline{\tilde{r}}_i \quad (3.92)$$

### 3.5 Linear independence of parameters and uniqueness of the solution

Starting from the considerations done in paragraph 3.2, the maximum number of parameters can be (theoretically) achieved only if the distribution of parameters themselves respects the criterion of linear independence and uniqueness of the solution.

In paragraph 3.3.1 the uniqueness conditions were achieved (Eqs. 3.51, 3.52, 3.54 and 3.55), reported here for convenience:

$$\alpha_1 \cdot \underline{\underline{\tilde{K}}}_{r,1} \cdot \underline{\tilde{\varphi}} + \alpha_2 \cdot \underline{\underline{\tilde{K}}}_{r,2} \cdot \underline{\tilde{\varphi}} + \dots + \alpha_q \cdot \underline{\underline{\tilde{K}}}_{r,q} \cdot \underline{\tilde{\varphi}} + \alpha_{q+1} \cdot \underline{\underline{\tilde{M}}}_{q+1} \cdot \underline{\tilde{\varphi}} + \dots + \alpha_N \cdot \underline{\underline{\tilde{M}}}_N \cdot \underline{\tilde{\varphi}} = \underline{0}$$

if:

$$(\alpha_1, \alpha_2, \dots, \alpha_N) = \underline{0}$$

$$\underline{\underline{\tilde{K}}}_{r,s} \cdot \underline{\tilde{\varphi}} \neq \underline{0}$$

for  $s=1,2,\dots,q$  and

$$\underline{\underline{\tilde{M}}}_s \cdot \underline{\tilde{\varphi}} \neq \underline{0}$$

for  $s=q+1, q+2, \dots, N$ .

In order to ensure the  $I^{st}$  uniqueness condition (Eqs. 3.51 and 3.52), one has to check that the N-ple  $(\alpha_1, \alpha_2, \dots, \alpha_N)$  which satisfies Eq. 3.51 is only the one in which all the parameters are null. If this happens, the initial definition of parameters  $a_1, a_2, \dots, a_N$  and the decomposition of the stiffness matrix are correct.



Otherwise, if there is one N-ple  $(\alpha_1, \alpha_2, \dots, \alpha_N)$  with at least one value not null, the problem is ill-conditioned and unique solution cannot be achieved.

The  $2^{nd}$  to  $(N+1)^{th}$  uniqueness conditions (Eqs. 3.54 and 3.55) have to be satisfied in order to ensure that the s-th parameter plays a role in the model updating.

With the usual modes vectors, Eqs. 3.51 and 3.52 simplify themselves and, in order to evaluate a *simplified  $I^{st}$  uniqueness condition*, the following criterion can be used:

$$\alpha_1 \cdot \underline{\underline{K}}_{r,1} + \alpha_2 \cdot \underline{\underline{K}}_{r,2} + \dots + \alpha_q \cdot \underline{\underline{K}}_{r,q} + \alpha_{q+1} \cdot \underline{\underline{M}}_{q+1} + \dots + \alpha_N \cdot \underline{\underline{M}}_N = \underline{\underline{0}} \quad (3.93)$$

if:

$$(\alpha_1, \alpha_2, \dots, \alpha_N) = \underline{\underline{0}} \quad (3.94)$$

One has to check that the N-ple  $(\alpha_1, \alpha_2, \dots, \alpha_N)$  which satisfy Eq. 3.93 is only the one in which all the parameters are null.

If the *simplified  $I^{st}$  uniqueness condition* is satisfied, the general  *$I^{st}$  uniqueness condition* is also satisfied and the uniqueness (if also Eqs. 3.54 and 3.55 are satisfied) is ensured.

### 3.6 Maximum number of parameters in real structures

Eq. 3.1 gives the maximum number of parameters in function of the number of experimental frequencies and modes shapes for ideal structures without truncation errors in the decimal digits or uncertainties in the experimental outcomes. All the tests performed present those problems and therefore Eq. 3.1 gives an upper-bound of the real maximum number of parameters achievable with the model updating.

The procedure which has to be used is as follows: starting from a real problem, the maximum number  $N$  of parameters given by Eq. 3.1 has to be found (knowing the number of experimental frequencies and modes available from tests) and the stiffness and mass matrices has to be decomposed (remembering Eqs. 3.51, 3.52, 3.54 and 3.55) in order to obtain the system of Eq. 2.65. After that, the procedure summarized in paragraph 2.7 has to be followed in order to obtain the mean values and CoVs of parameters themselves. All the CoVs must be checked with respect to thresholds and if:

$$CoV_{a,s} \leq \delta_{a,s} \quad (3.95)$$

for all  $a_1, a_2, \dots, a_N$ , with  $\delta_{a,1}, \delta_{a,2}, \dots, \delta_{a,N}$  thresholds, which depend upon the structure itself and what the parameter represents, the procedure stops and  $N$  is the maximum number of parameters for the non ideal structure too.

If the check fails, the parameter with the largest value of CoV must be fixed in other way (from literature proposals or through other techniques) and the procedure outlined above has to be repeated with a number of parameter decreased of one unit. Moreover, the parameters with maximum CoV (above the threshold) should likely be the one whose variation affects very little the dynamic behavior of the structure and therefore that parameter represents a sort of ill-conditioning of the problem.

This iterative procedure must be repeated as long as all the CoVs are below the thresholds. The largest number of parameters whose CoVs satisfy Eq. 3.95, is the maximum number of parameters achievable for the case under study. This procedure is summarized in the flowchart of Figure 3.2. In Appendix B will be presented a simple case in which this procedure is applied.

### 3.7 Correctness of rigid diaphragm assumption

From a dynamic test with shaker or through ambient vibration, the outcomes are frequencies and modes components at selected point of measurement. From that data, in existing buildings, one has to choose whether or not to use the rigid diaphragm configuration in the model for comparison purposes with the experimental data.

In the following is presented one criterion in order to evaluate if the choice of rigid diaphragm is acceptable or less.

Starting from the modal components at one storey (here we limit our study at only one storey but the following procedure has to be used at all the storey of the building), for sake of generality,  $l$  components of displacement in  $x$ -direction and  $p$  components in  $y$ -direction are considered (Fig. 3.1).

In the first phase, the three rigid diaphragm components (2 translations and 1 rotation) are computed. Sorting the mode components vector in order to get first the  $l$  components in  $x$ -direction and then the  $p$  components in  $y$ -direction (using a right-hand convention), the formula is as follows:

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_l \\ v_{l+1} \\ v_{l+2} \\ \vdots \\ v_{l+p} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -(y_1 - y_G) \\ 1 & 0 & -(y_2 - y_G) \\ \vdots & \vdots & \vdots \\ 1 & 0 & -(y_l - y_G) \\ 0 & 1 & (x_{l+1} - x_G) \\ 0 & 1 & (x_{l+2} - x_G) \\ \vdots & \vdots & \vdots \\ 0 & 1 & (x_{l+p} - x_G) \end{bmatrix} \cdot \begin{bmatrix} u_G \\ v_G \\ \vartheta_G \end{bmatrix} \quad (3.96)$$

in which  $u_1, u_2, \dots, u_l$  are the  $l$  components in  $x$ -direction and  $v_{l+1}, v_{l+2}, \dots, v_{l+p}$  are the  $p$  components in  $y$ -direction.  $y_1, y_2, \dots, y_l$  are the  $y$  coordinates of the points in which the  $x$ -direction modal components are measured,  $y_G$  is the  $y$  coordinate of the center of the mass,  $x_{l+1}, x_{l+2}, \dots, x_{l+p}$  are the  $x$  coordinates of the points in which the  $y$ -direction modal components are measured,  $x_G$  is the  $x$  coordinate of the center of the mass (in whatever reference system),  $u_G, v_G, \theta_G$  are the rigid diaphragm components (displacement in  $x$  direction, displacement in  $y$  direction and rotation about a vertical axes passing through the center of the mass).

Eq. 3.96 can be summarized:

$$\underline{u} = \underline{T} \cdot \underline{u}_G \quad (3.97)$$

Using the pseudo-inverse procedure, the rigid diaphragm components are computed (the pseudo-inverse procedure uses a least squares optimization).

$$\underline{u}_G = \underline{T}^+ \cdot \underline{u} \quad (3.98)$$

in which  $\underline{T}^+$  is the pseudo-inverse of matrix  $\underline{T}$ .

Having the rigid diaphragm components at disposal, one can obtain the modes components at each measured point following the inverse way, in order to compute a re-built  $\underline{u}$  vector, using the same above-defined  $\underline{T}$  matrix.

$$\underline{u}_{re-built} = \underline{T} \cdot \underline{u}_G \quad (3.99)$$

After that a comparison between  $\underline{u}$  and  $\underline{u}_{re-built}$  can be made through MAC, NMAC or NMD.

$$MAC(\underline{u}, \underline{u}_{re-built}) \quad (3.100)$$

If this parameter is close to 1, the two modes shapes are very close one another and therefore the rigid diaphragm assumption can be justified; conversely, if the value is lower than 0.90, the rigid diaphragm assumption should be avoided.

For buildings with  $m$  storey, the above procedure is the same, except that the  $\underline{T}$  matrix is a block diagonal matrix. The problem can be written in this way:

$$\begin{bmatrix} \underline{u}_1 \\ \underline{u}_2 \\ \vdots \\ \underline{u}_m \end{bmatrix} = \begin{bmatrix} \underline{T}_1 & 0 & \cdots & 0 \\ 0 & \underline{T}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \underline{T}_m \end{bmatrix} \cdot \begin{bmatrix} \underline{u}_{G1} \\ \underline{u}_{G2} \\ \vdots \\ \underline{u}_{Gm} \end{bmatrix} \quad (3.101)$$

in which  $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_m$  are the modes vectors at each storey,  $\underline{u}_{G1}, \underline{u}_{G2}, \dots, \underline{u}_{Gm}$  are the rigid diaphragm components at each storey and  $\underline{T}_1, \underline{T}_2, \dots, \underline{T}_m$  are the transformation matrices. Following the same procedure described above, a MAC vector can be defined in which

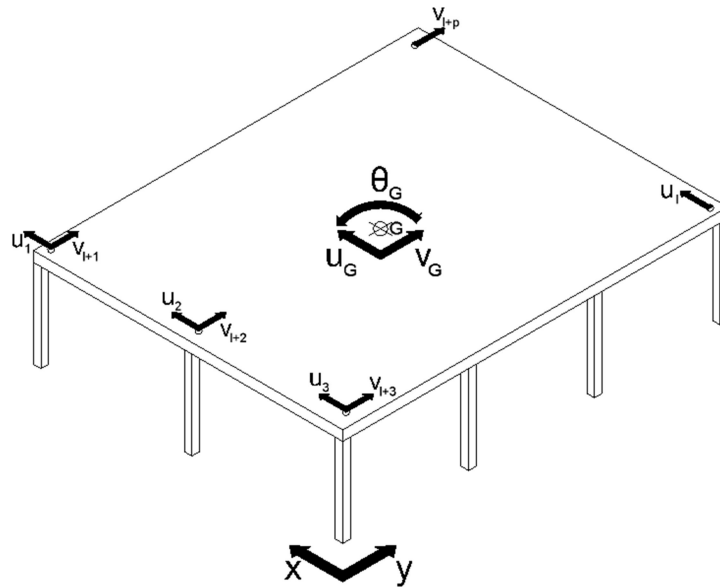
$$MAC(\underline{u}_i, \underline{u}_{re-built,i}) \quad (3.102)$$

is the i-th component of MAC vector containing the MAC value between the original mode shape at the i-th floor and the re-built vector at the i-th floor. The rigid diaphragm assumption can be justified if all the values are close to 1.

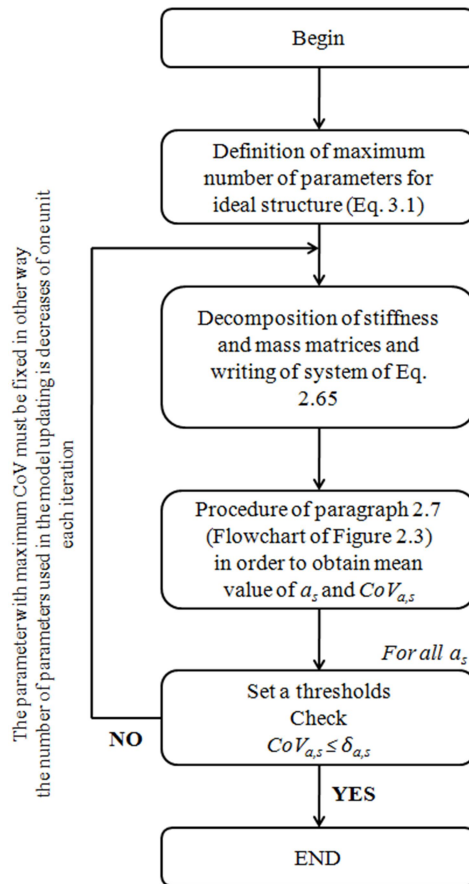
### 3.8 Conclusions

In the present Chapter, the theoretical basis of the procedure presented in Chapter 2 have been studied. In the first part, the maximum theoretical number of parameters has been investigated and a closed relation was given with the related demonstration. In the subsequent part, the procedure with partial derivatives has been studied and all the demonstrations of uniqueness of the solution and considerations regarding the stationary point have been performed. The procedure for the errors propagation has been then performed and relations for the partial derivatives, with respect to the experimental outcomes, have been achieved. After that the problem of ill-conditioning on the choice of the parameters has been examined and a criterion for understanding if one set of parameters can give unique solution, before running the procedure, has been achieved. The procedure for finding the maximum number of parameters for non ideal case has been then analyzed. Eventually, a criterion in order to evaluate the correctness of rigid diaphragm assumption has been developed.

## Figures of Chapter 3



**Figure 3.1:** General representation of rigid diaphragm components from  $l+p$  displacements.



**Figure 3.2:** Flowchart of the procedure for obtaining of the maximum number of parameters in real structures.



## CHAPTER 4

# PRELIMINARY SENSITIVITY ANALYSIS ON SIMPLE STRUCTURES AND COMPARISON BETWEEN PROCEDURES

### 4.1 Introduction

In this chapter will be performed a sensitivity analysis of some simple structures in order to assess the variability of parameters varying the frequencies and modes of vibration. In the first part, the analysis of 2-D infilled frames with and without the variation of modes shapes will be developed. In the second part the sensitivity analysis of more complex 3-D structures will be performed. The analysis of the results will be done for each case studying the variation of parameters as a function of the perturbations on frequencies and modes. In the last part, a comparison between the two algorithm and a statistical analysis of parameters will be carried out.

### 4.2 Procedure for the sensitivity analysis

The sensitivity analysis are performed on 2-D and 3-D sample structures with the same basic concept. The complete structure with fixed values of parameters is analyzed, frequencies and modes of vibration are obtained. These values are the "reference" ones for the sensitivity analysis. After that, perturbations are applied on frequencies, modes components or both depending upon the case through the *randn* MATLAB function that generates random normally distributed numbers with constant mean value and coefficient of variation. The mean values are set equal to the reference values of frequencies and modes components, in all the analysis; instead the CoVs are set equal to 2% or 5%. In order to take into account also possible errors on the original data, a bias equal to 2% is also included in the analysis.

Using these perturbed frequencies and modes components, a Monte Carlo procedure has been run for 100 realizations for each model and the new parameters are obtained, the mean values and CoVs of those parameters were computed in order to evaluate the stability of parameters themselves. The stiffness matrices were obtained using software OpenSEES (McKenna and Fenves, 2001; OpenSEES, 2016) for all the cases.

For these first analysis, the procedure with DE-Q algorithm was used (Flowchart of Figure 2.6).

### 4.3 2-D infilled frame sample

The first sample analyzed has been a 3 storey, 2 bays infilled framed structure. Only 3 DOFs (the horizontal displacements of the 3 storey) have been selected for this type of idealization (this sample structure is taken from specimen of Chapter 5.3). The assumption of shear-type frame has not been followed and a static condensation of rotational DOFs has been made in order to achieve the 3x3 stiffness and mass matrices. Therefore the maximum number of modes and frequencies achievable is 3. The view of the structure with the measures is depicted in Figure 4.1. For this sample, 3 parameters (the  $w/d$  ratio of the struts replacing the infills) has been used, the arrangements of them is depicted in Figure 4.2. In the 2 bays, it has been assumed the same type of infills and therefore the same values of parameters. Four different cases has been analyzed:

- Case A: Identification made with System of Eq. 2.3, using 3 modes and frequencies;
- Case B: Identification made with System of Eq. 2.3, using 2 modes and frequencies;
- Case C: Identification made with System of Eq. 2.64, using 3 modes and frequencies;
- Case D: Identification made with System of Eq. 2.64, using 2 modes and frequencies.

In all of them, a CoV of 5% in frequencies and modes components has been used, no bias was considered in the procedure.

The reference values of parameters, expressed as  $w/d$  ratio in percentage (where  $w$  is the width of the equivalent struts replacing the infill panel and  $d$  is the length of the diagonal), and for all the four cases, are listed in Table 4.1. The value of the thickness of the infill panels has been set to 18.8 cm and the values of the Young's modulus equal to 5410, 6820 and 6520 MPa for the ground, first and second storey infills respectively (these values are taken from Stavridis, 2009). The algorithm used for the sensitivity analysis is the one depicted in Figure 2.8 (procedure with DE-Q algorithm).

The analysis of the 100 realizations is made with the comparison between parameters mean values and reference ones, besides the CoVs of parameters themselves are analyzed. The mean values of the realizations are computed as follows:

$$a_i = \frac{\sum_{j=1}^s a_{j,i}}{s} \quad (4.1)$$

in which  $i$  stand for  $i$ -th parameter and  $s$  is the number of realizations in the Monte Carlo procedure.

The percentage errors in the parameters are computed with the following relation:

$$e_i = \frac{(a_i - a_{ref,i})}{a_{ref,i}} \cdot 100 \quad (4.2)$$



from the Monte Carlo procedure, the standard deviations  $\sigma$  of the realizations are also computed. From that value, the CoVs of parameters are computed as follows:

$$CoV_i = \frac{\sigma_i}{a_i} \cdot 100 \quad (4.3)$$

In Table 4.2 are summarized those results for all the four cases.

In Figures 4.3 to 4.6 are reported the histograms of the realizations.

#### 4.3.1 Analysis of results

The results listed in Tables 4.2 and depicted in Figures 4.3 to 4.6 show a great stability for the algorithm and the cases studied. The errors in the means reached the maximum value of 4.89% with no great differences among the four cases. The CoVs give very good results too. In all the cases except one (in which CoV is equal to 10.10%) the CoVs are less than the summation of the CoVs in frequencies and modes components. Moreover the sharpness of the histograms shows very low scattering about the mean values in the parameters realizations.

### 4.4 3-D infilled frame samples

With the same procedure described above, three 3-D infilled framed structures have been analyzed and a sensitivity analysis has been carried out for each of them. The three buildings present only 1 bay in the two horizontal directions and three storey. A rigid diaphragm assumption has been used in this type of buildings. Also in these cases, a static condensation has been made in order to achieve a problem size of 9 (2 translations and 1 rotation per storey).

The three cases are labeled as follows:

- AA: building asymmetric in both horizontal directions;
- AS: building asymmetric in one horizontal direction and symmetric in the other one;
- AAv: building asymmetric in both horizontal directions and with model error.

The last case highlights the possibility that one infill panel is modeled in the FE procedure but, actually, that panel doesn't play any role in the dynamic behavior of the structure (to take into account one model error).

The plan view of the structure is depicted in Figure 4.7. The four views for the three cases are depicted in Figures 4.8, 4.9 and 4.10.

From each case, 4 sub-cases are generated:

- 0: Without perturbations;

- 1: Frequencies and modes components are considered as independent random variables with CoVs equal to 2%;
- 2: Frequencies and modes components are considered as independent random variables with CoVs equal to 2% for frequencies and 5% for modes;
- 3: Frequencies and modes components are considered as independent random variables with CoVs equal to 2% for frequencies and 5% for modes, in addition a bias equal to 5% is considered in the first frequency.

Another subdivision is made in order to take into account different types of application of the perturbations themselves on modes components. The perturbations can be assigned directly on the rigid diaphragm modes components or to the displacement components of a sensor placed in a certain position on the structure and then, through Eq. 3.98, the rigid diaphragm components are reconstructed.

3 sub-cases are considered:

- I: Perturbations applied directly on the modes rigid diaphragm components;
- II: Perturbations applied on the sensors modes components, considering 4 sensors per storey (at four corners), 2 each horizontal directions;
- III: Perturbations applied on the sensors modes components, considering 8 sensors per storey (at four corners), 4 each horizontal directions.

For the concrete frame has been assumed a Young's modulus equal to 30000 MPa, for the masonry panels equal to 3000 MPa. The thickness of the infill panels has been set to 30 cm. 6 modes and frequencies have been used in the procedures, all of them are below 25 Hz. 4 parameters are considered in all the cases.

The cases are labeled with three indices; for instance Case AS-2-III means sample symmetric-asymmetric, with perturbations of 2% in frequencies and 5% in modes and perturbations applied on the sensors, considering 8 sensors per storey.

The algorithm used for the sensitivity analysis is the one depicted in Figure 2.8 (procedure with DE-Q algorithm) without taking into account the determinant equations.

In Table 4.3 are listed the reference values of parameters, calibrated using the proposals of Decanini and Fantin (Decanini and Fantin, 1987).

In Tables 4.4 to 4.8 are listed some results from the sensitivity analysis (Tondi et al., 2017).

In Figures from 4.11 to 4.13 are depicted the histograms of parameters for the cases AA-2-I, AS-2-I and AAv-2-I.

#### 4.4.1 Analysis of results

Analyzing the results, some general conclusions can be drawn: first of all, very little discrepancies are achieved in terms of errors and CoVs if no perturbations are assigned to frequencies and modes components (Table 4.4). This aspect indicates that the procedure doesn't create bias inside the mathematical formulation itself.

For case AA-1-I the mean errors are very low, the coefficients of variations are also low for the first parameter (with greatest value) and a little bit higher for the other three parameters. This is due to the fact that the first parameter rules the dynamic behavior of the structure and therefore the other ones present greater CoVs. Increasing the perturbation on modes components, the system behaves worse with greater mean values errors and CoVs. The introduction of one bias (equal to 5%) in the first frequency doesn't affect so much the procedure which is quite stable with respect to the frequencies perturbations (Table 4.5).

The same conclusions can be achieved, also, for the symmetric-asymmetric case (Table 4.6).

Changing the distribution of the perturbations on the mode shapes, from perturbations applied directly to rigid diaphragm components to perturbations on 4 and 8 sensors, the results are improved in terms of both mean values errors and CoVs. This is due to the fact that the perturbations on 4 or 8 sensors are "mediate" when they have been transformed into rigid diaphragm components and therefore errors and CoVs reached lesser values with respect to the case of direct rigid diaphragm components perturbations (Table 4.7).

The last sample, with modeling error, gives very good results and the modeling error is detected by the procedure in a very good way (Table 4.8).

### 4.5 Comparison between sensitivity analysis using the two procedures

Using the two different procedures outlined in Chapter 2 and depicted in Figures 2.4 and 2.8, a comparison is made with respect to the cases AA-2-I and AS-2-I. In Table 4.9 and 4.10 are summarized the mean values and CoVs for the two procedures and the two cases. For case AA-2-I one can see how the mean values of all the parameters are found with smaller error in the two steps procedure with Trust-Region updating with respect to the DE-Q procedure. Moreover, the CoVs of the two steps procedure are very smaller than the ones obtained with DE-Q algorithm. The second case AS-2-I achieves better results using two steps procedure for all the quantities (except for one mean value error that is a little bit smaller for DE-Q procedure); therefore, for the cases studied, the

new procedure is more stable with respect to face directly the entire system with a genetic algorithm.

In terms of computational effort and computational time, e.g. for 100 realizations and for case AA-2-I, the two steps procedure with Trust-Region updating needs 447 seconds to run, conversely the DE-Q algorithm, with the same thresholds set, needs 11676 seconds to run. The time saving using the first procedure is about 96%. In the case AS-2-I, the two steps procedure with Trust-Region updating needs 262 seconds to run, conversely the DE-Q algorithm, with the same thresholds set, needs 11898 seconds to run. Again, the time saving using the first procedure is about 98%.

#### **4.5.1 Comparison of sensitivity analysis between algorithms with and without determinant equations**

In order to understand the sensitivity of the model to the use of the incomplete set of Equations (Eq. 2.3 or 2.4) with respect to the use of the complete system of equations, adding the determinant equations (Eq. 2.64 or 2.65), one case is analyzed with both the systems using the two steps procedure with Trust-Region updating. The results are summarized in Table 4.11 in terms of mean values, errors and CoVs. One can see that the errors in the parameters decrease very much with the complete system and the CoV in the first parameter (the most important parameter in the dynamic behavior of the structure) decreases. The other CoVs have small increments. Therefore, the use of the determinant equations is not negligible in the procedure.

### **4.6 Analysis of the dependence upon the number of modes and statistical distribution analysis**

Starting from case AA-2-I (with CoV equal to 2% on frequencies and 5% on modes components), the dependence of the solution on the number of modes used in the system is analyzed. In Table 4.12 are summarized the results with 6 and 9 modes, obtained from Monte Carlo analysis with 100 realizations, using the two steps procedure of Figure 2.4. The procedure is stable with respect to the number of modes used and the solution is not so sensible passing from 6 modes to 9 modes.

The cases AA-2-I and AS-2-I, with 6 modes used in the algorithm, are also analyzed in order to figure out the statistical distributions of parameters starting from normally distributed perturbations on frequencies and modes components. For the comparison (Chapter 2), chi-square tests have been performed. The statistical distributions of parameters have been compared with five distributions:

- Beta distribution;

- Normal distribution;
- Lognormal distribution;
- T-student distribution;
- Weibull distribution.

All these distributions have been firstly fitted with the values of parameters from Monte Carlo procedure through a probability distribution fitting techniques (Cramer, 1946).

The chi-square values with evaluation of the levels of significance for the two cases are summarized in Table 4.13. The test is overcome if the value for the level of significance is greater than 0.05 (5%). One can see that the distribution changes from parameter to parameter and even from case to case. For AA-2-I the normal distribution overcomes, for all the parameters, the chi-square test (with 5% level of confidence) but other distributions for two parameters give better probability of goodness (for parameters 2 and 4, Weibull and Beta distributions respectively). For case AS-2-I, for some parameter, the normal distribution fails the test. In Table 4.14 the statistical distributions found are summarized. As anticipated in Chapter 2, no general rule can be given for statistical distributions of parameters from normally distributed perturbations on frequencies and modes components. In Figures 4.14 and 4.15 the histograms of the four parameters along with the main fitted distributions are depicted.

## 4.7 Conclusions

In this Chapter, the sensitivity analysis of simple structures have been performed in order to figure out the stability of the algorithms. In the first part, the sensitivity analysis of 2D structures is done and the results are very stable, in terms of errors in mean values and CoVs. Then, a 3D structures were analyzed utilizing two algorithm (DE-Q and two steps procedure with Trust-Region updating). The results are quite good for both the procedure, the two steps one yielded more stable results in terms of both errors in mean values and CoVs. Eventually, statistical analysis with chi-square tests have been performed in order to understand the parameters distribution starting from normal distribution for frequencies and modes components. Unfortunately, no general rule can be achieved for the statistical distributions themselves.



## Tables of Chapter 4

Case	$a_1$	$a_2$	$a_3$
A	0.250	0.200	0.100
B	0.250	0.200	0.100
C	0.250	0.200	0.100
D	0.250	0.200	0.100

**Table 4.1:** Reference values of parameters.

Parameter		A	B	C	D
$a_1$	Mean value	0.246	0.245	0.246	0.247
	Error [%]	1.61	2.16	1.83	1.41
	CoV [%]	6.06	6.94	6.02	6.64
$a_2$	Mean value	0.195	0.191	0.193	0.190
	Error [%]	2.77	4.57	3.39	4.89
	CoV [%]	8.18	9.13	7.59	10.10
$a_3$	Mean value	0.097	0.097	0.097	0.097
	Error [%]	2.93	2.63	3.23	3.38
	CoV [%]	9.28	8.63	9.73	9.27

**Table 4.2:** Mean values, errors and CoVs of parameters for 2-D sample.

Case	$a_1$	$a_2$	$a_3$	$a_4$
AA	0.257	0.069	0.058	0.076
AS	0.257	0.069	0.058	0.076
ASv	0.257	0.069	0.058	0.000

**Table 4.3:** Reference values of the equivalent strut widths.

Parameter		AA-0-0	AS-0-0	AAv-0-0
$a_1$	Mean value	0.257	0.257	0.257
	Error [%]	0.00	0.01	0.03
	CoV [%]	0.02	0.01	0.07
$a_2$	Mean value	0.069	0.069	0.069
	Error [%]	0.02	-0.02	0.05
	CoV [%]	0.02	0.06	0.11
$a_3$	Mean value	0.058	0.058	0.058
	Error [%]	0.03	-0.03	0.06
	CoV [%]	0.06	0.14	0.13
$a_4$	Mean value	0.076	0.076	0.000
	Error [%]	0.03	0.02	-
	CoV [%]	0.06	0.05	-

**Table 4.4:** Mean values, errors and CoVs for case AA-0-0, AS-0-0 and AAv-0-0.

Parameter		AA-1-I	AA-2-I	AA-3-I
$a_1$	Mean value	0.250	0.238	0.238
	Error [%]	-2.5	-7.2	-7.2
	CoV [%]	7.1	14.3	15.3
$a_2$	Mean value	0.069	0.067	0.066
	Error [%]	0.0	-3.8	-4.6
	CoV [%]	14.0	28.3	33.5
$a_3$	Mean value	0.057	0.057	0.052
	Error [%]	-2.9	-3.1	-11.5
	CoV [%]	27.8	49.6	46.8
$a_4$	Mean value	0.077	0.071	0.070
	Error [%]	2.0	-6.4	-8.0
	CoV [%]	18.0	36.7	36.9

**Table 4.5:** Mean values, errors and CoVs for case AA-1-I, AA-2-I and AA-3-I.

Parameter		AS-1-I	AS-2-I	AS-3-I
$a_1$	Mean value	0.246	0.225	0.213
	Error [%]	-4.3	-12.3	-17.0
	CoV [%]	9.7	30.2	30.4
$a_2$	Mean value	0.066	0.084	0.078
	Error [%]	-4.4	21.5	11.8
	CoV [%]	24.3	59.0	64.0
$a_3$	Mean value	0.059	0.087	0.085
	Error [%]	0.3	49.2	45.1
	CoV [%]	41.0	80.0	81.4
$a_4$	Mean value	0.072	0.061	0.059
	Error [%]	-4.4	-19.0	-22.7
	CoV [%]	16.9	49.7	52.8

**Table 4.6:** Mean values, errors and CoVs for case AS-1-I, AS-2-I and AS-3-I.

Parameter		AA-2-I	AA-2-II	AA-2-III
$a_1$	Mean value	0.238	0.243	0.249
	Error [%]	-7.2	-5.5	-3.3
	CoV [%]	14.3	11.4	9.4
$a_2$	Mean value	0.067	0.065	0.067
	Error [%]	-3.8	-7.8	-3.9
	CoV [%]	28.3	23.7	18.1
$a_3$	Mean value	0.057	0.053	0.056
	Error [%]	-3.1	-10.1	-4.7
	CoV [%]	49.6	45.0	33.3
$a_4$	Mean value	0.071	0.069	0.071
	Error [%]	-6.4	-10.0	-6.8
	CoV [%]	36.7	34.8	24.9

**Table 4.7:** Mean values, errors and CoVs for case AA-2-I, AA-2-II and AA-2-III.



Parameter		AAv-2-I	AAv-2-II	AAv-2-III
$a_1$	Mean value	0.228	0.234	0.246
	Error [%]	-11.3	-9.0	-4.5
	CoV [%]	13.1	11.3	7.3
$a_2$	Mean value	0.073	0.068	0.069
	Error [%]	4.4	-3.2	-1.3
	CoV [%]	28.5	22.6	18.3
$a_3$	Mean value	0.054	0.053	0.055
	Error [%]	-8.0	-9.2	-5.9
	CoV [%]	24.3	18.1	15.6
$a_4$	Mean value	0.022	0.018	0.010
	Error [%]	-	-	-
	CoV [%]	-	-	-

**Table 4.8:** Mean values, errors and CoVs for case AAv-2-I, AAv-2-II and AAv-2-III.

Parameter		AA-2-I DE-Q	AA-2-I two steps procedure
$a_1$	Mean value	0.231	0.252
	Error [%]	-10.2	-1.9
	CoV [%]	12.4	7.2
$a_2$	Mean value	0.069	0.069
	Error [%]	-1.9	-0.8
	CoV [%]	28.3	14.7
$a_3$	Mean value	0.057	0.059
	Error [%]	-2.5	0.4
	CoV [%]	47.0	11.3
$a_4$	Mean value	0.075	0.076
	Error [%]	-1.8	-0.2
	CoV [%]	26.8	11.3

**Table 4.9:** Mean values, errors and CoVs for case AA-2-I with DE-Q and two steps procedures.

Parameter		AS-2-I DE-Q	AS-2-I two steps procedure
$a_1$	Mean value	0.223	0.257
	Error [%]	-13.3	0.0
	CoV [%]	13.5	4.6
$a_2$	Mean value	0.067	0.065
	Error [%]	-4.8	-6.7
	CoV [%]	35.1	13.9
$a_3$	Mean value	0.063	0.057
	Error [%]	6.9	-3.6
	CoV [%]	56.7	32.5
$a_4$	Mean value	0.064	0.081
	Error [%]	-16.7	6.0
	CoV [%]	22.8	22.0

**Table 4.10:** Mean values, errors and CoVs for case AS-2-I with DE-Q and two steps procedures.

Parameter		AA-2-I without det eq.	AA-2-I with det eq.
$a_1$	Mean value	0.219	0.252
	Error [%]	-14.8	-1.9
	CoV [%]	10.3	7.2
$a_2$	Mean value	0.072	0.069
	Error [%]	2.3	-0.8
	CoV [%]	8.9	14.7
$a_3$	Mean value	0.061	0.059
	Error [%]	3.3	0.4
	CoV [%]	9.3	11.3
$a_4$	Mean value	0.073	0.076
	Error [%]	-4.2	-0.2
	CoV [%]	8.5	11.3

**Table 4.11:** Mean values, errors and CoVs for case AA-2-I with and without determinant equations using two steps procedure (for the system without determinant equations only the first step is needed).

Parameter		AA-2-I - 6 modes	AA-2-I - 9 modes
$a_1$	Mean value	0.252	0.257
	Error [%]	-1.9	-0.3
	CoV [%]	7.2	5.7
$a_2$	Mean value	0.069	0.073
	Error [%]	-0.8	4.1
	CoV [%]	14.7	12.1
$a_3$	Mean value	0.059	0.060
	Error [%]	0.4	1.5
	CoV [%]	11.3	12.3
$a_4$	Mean value	0.076	0.079
	Error [%]	-0.2	3.4
	CoV [%]	11.3	9.9

**Table 4.12:** Mean values, errors and CoVs for case AA-2-I with 6 and 9 modes.

Parameter		Level of confidence	
		AA-2-I	AS-2-I
$a_1$ Levels of significance	Beta	0.50 - 0.60	$\approx 0.25$
	Normal	$\approx \mathbf{0.62}$	$\approx \mathbf{0.29}$
	Lognormal	0.50 - 0.60	$< 0.20$
	T-student	0.50 - 0.60	$< 0.20$
	Weibull	$\approx 0.60$	$\approx 0.27$
$a_2$ Levels of significance	Beta	$< 0.05$	$< 0.10$
	Normal	0.05 - 0.10	0.10 - 0.50
	Lognormal	$< 0.05$	$< 0.05$
	T-student	0.05 - 0.10	0.10 - 0.50
	Weibull	$\mathbf{0.10 - 0.50}$	$> \mathbf{0.50}$
$a_3$ Levels of significance	Beta	$> 0.60$	$< 0.05$
	Normal	$> \mathbf{0.60}$	$< 0.05$
	Lognormal	$> 0.60$	$< 0.05$
	T-student	$> 0.60$	$< 0.05$
	Weibull	0.50 - 0.6	$> \mathbf{0.05}$
$a_4$ Levels of significance	Beta	$\approx \mathbf{0.14}$	$< 0.10$
	Normal	$\approx 0.13$	$< 0.05$
	Lognormal	$\approx 0.13$	$> \mathbf{0.10}$
	T-student	0.05 - 0.10	$< 0.05$
	Weibull	$< 0.05$	$< 0.05$

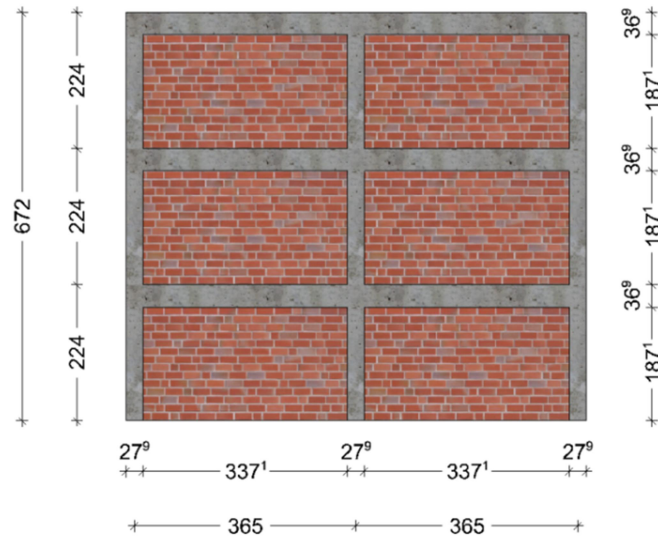
**Table 4.13:** Chi-square tests for cases AA-2-I and AS-2-I.

Parameter		AA-2-I	AS-2-I
$a_1$	Distribution	Normal	Normal
$a_2$	Distribution	Weibull	Weibull
$a_3$	Distribution	Normal	Weibull
$a_4$	Distribution	Beta	Lognormal

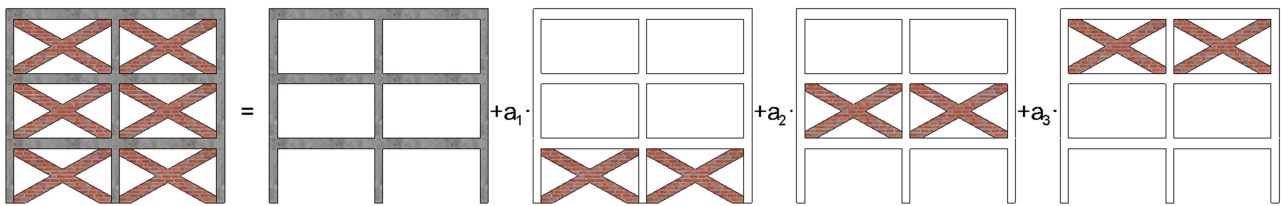
**Table 4.14:** Statistical distributions of parameters for cases AA-2-I and AS-2-I.



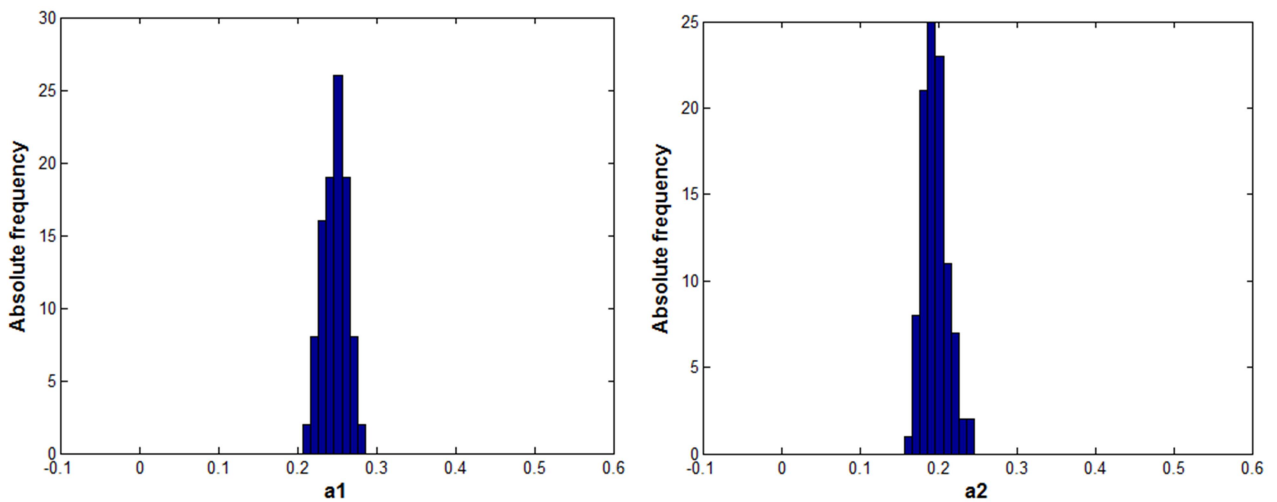
## Figures of Chapter 4

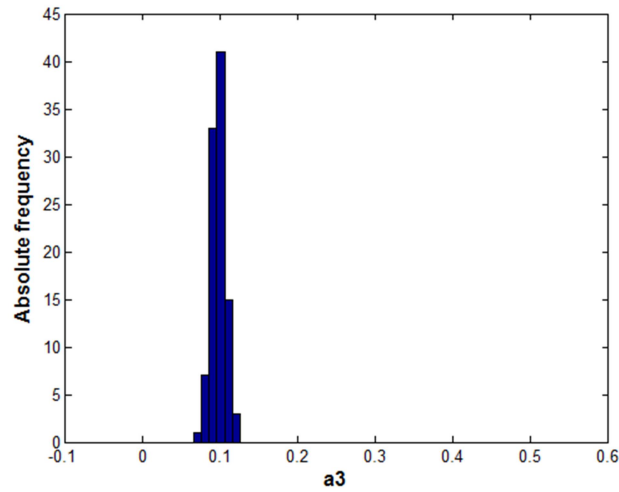


**Figure 4.1:** View of the 2-D sample structure.

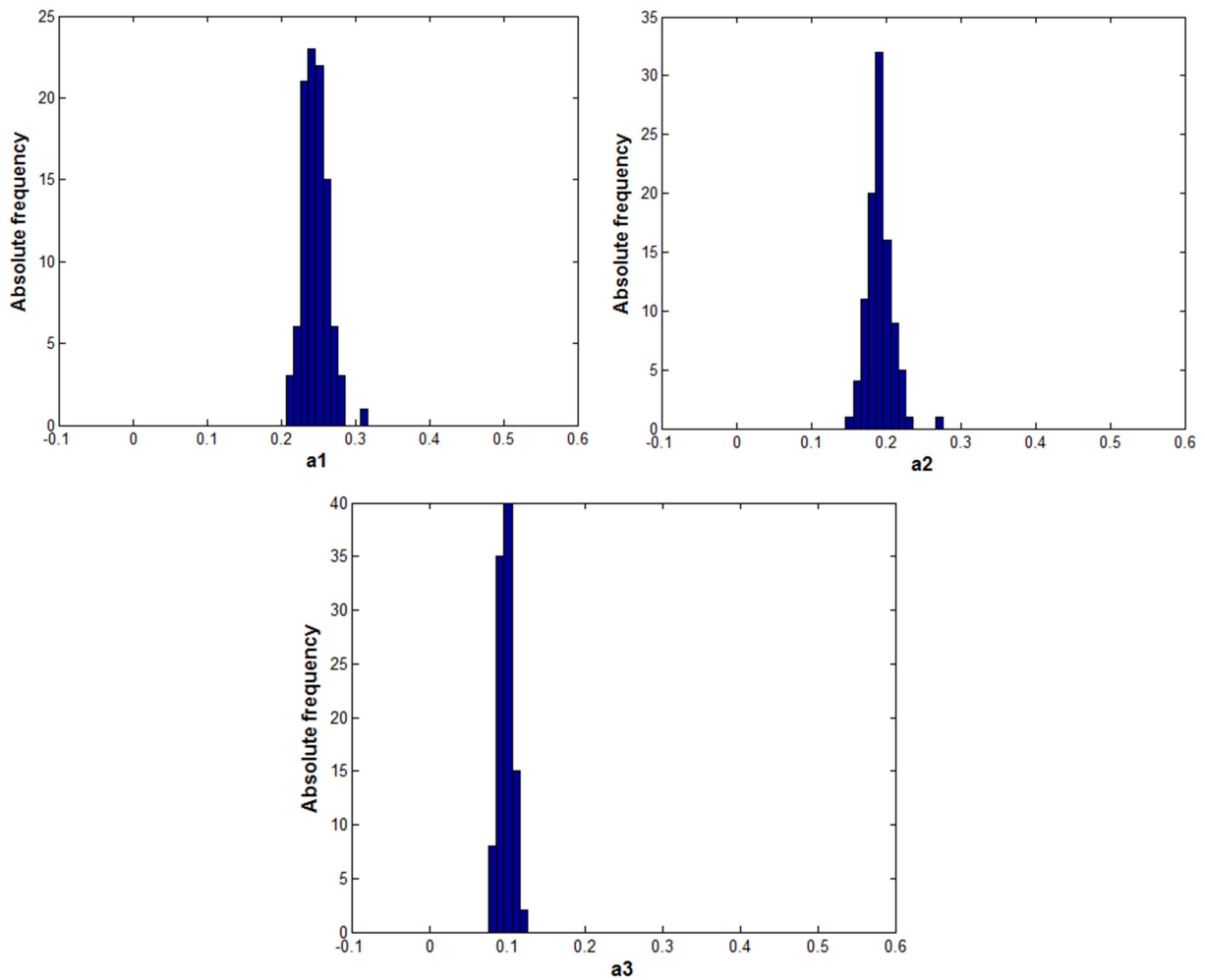


**Figure 4.2:** Parameters arrangement.

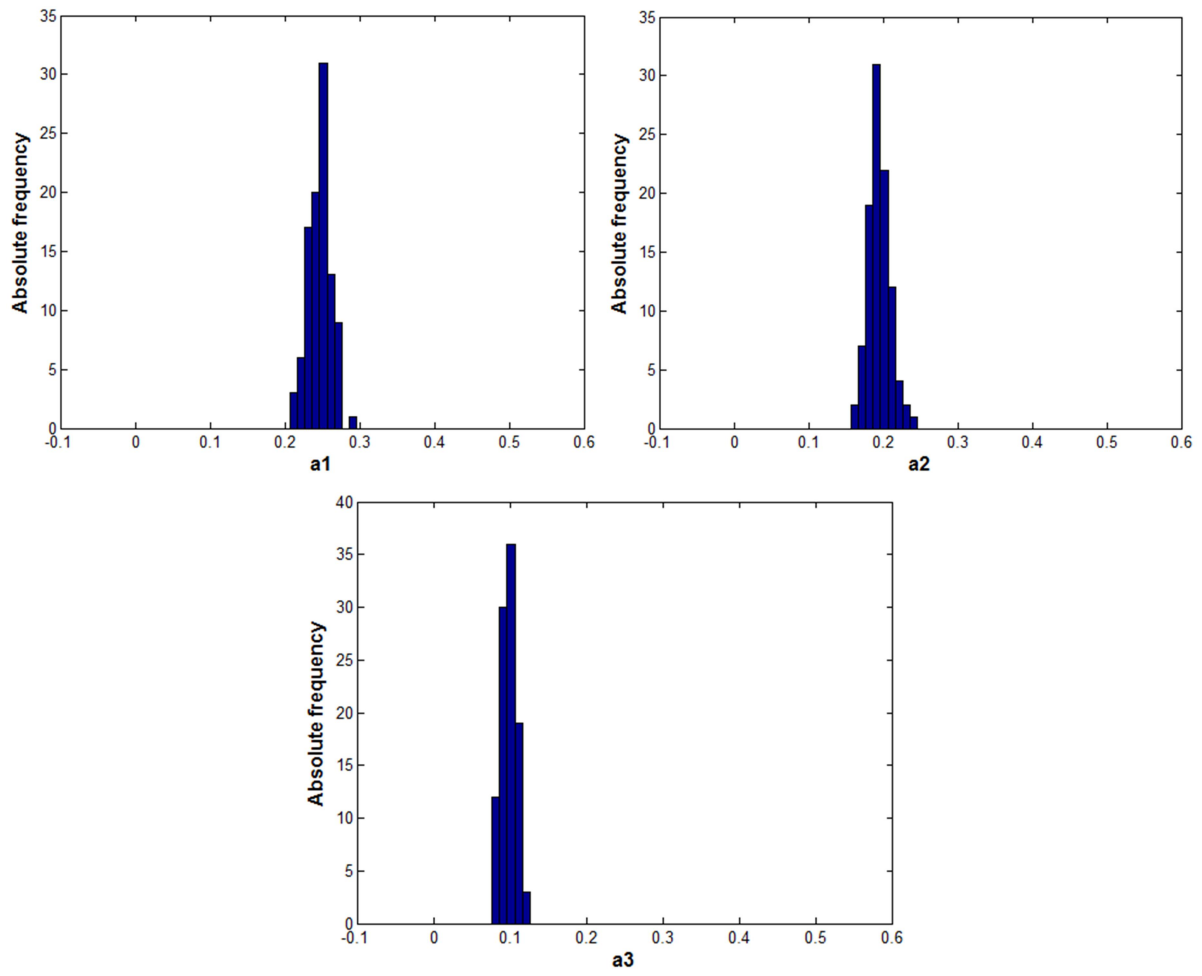




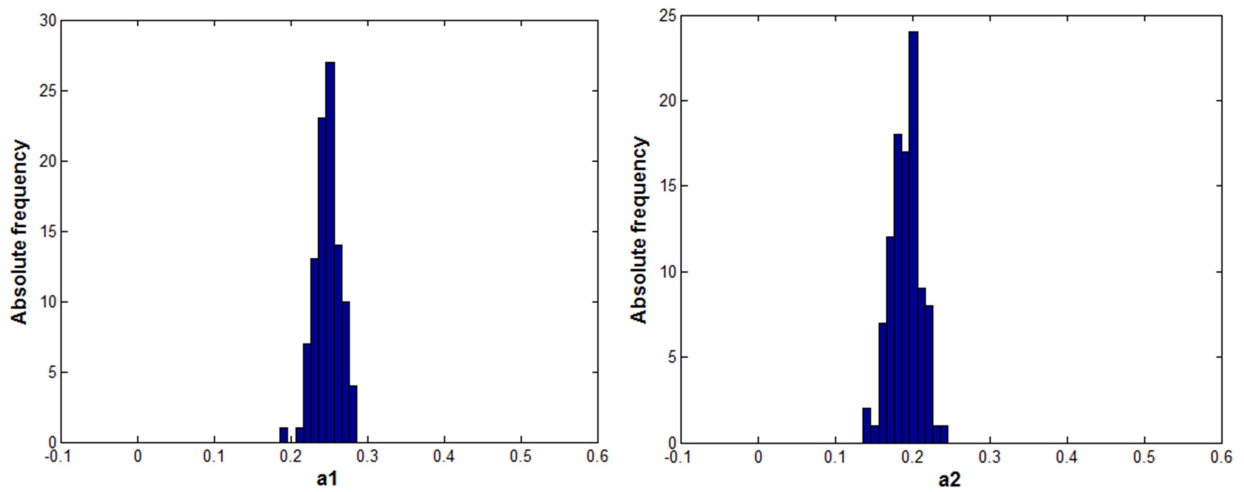
**Figure 4.3:** Frequencies histograms of parameters  $a_1, a_2, a_3$  for 2-D sample - Case A.



**Figure 4.4:** Frequencies histograms of parameters  $a_1, a_2, a_3$  for 2-D sample - Case B.



**Figure 4.5:** Frequencies histograms of parameters  $a_1$ ,  $a_2$ ,  $a_3$  for 2-D sample - Case C.



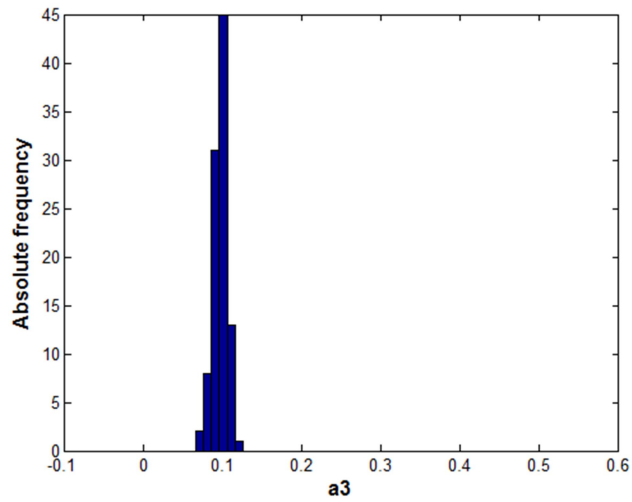


Figure 4.6: Frequencies histograms of parameters  $a_1$ ,  $a_2$ ,  $a_3$  for 2-D sample - Case D.

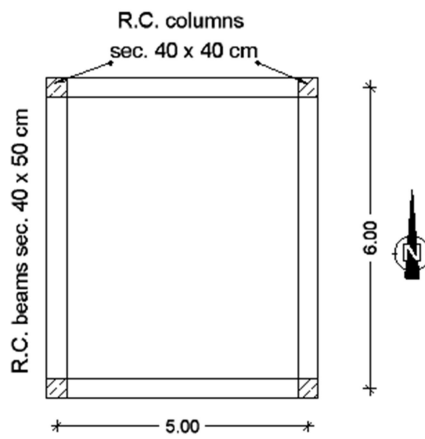


Figure 4.7: Plan view of 3-D samples.

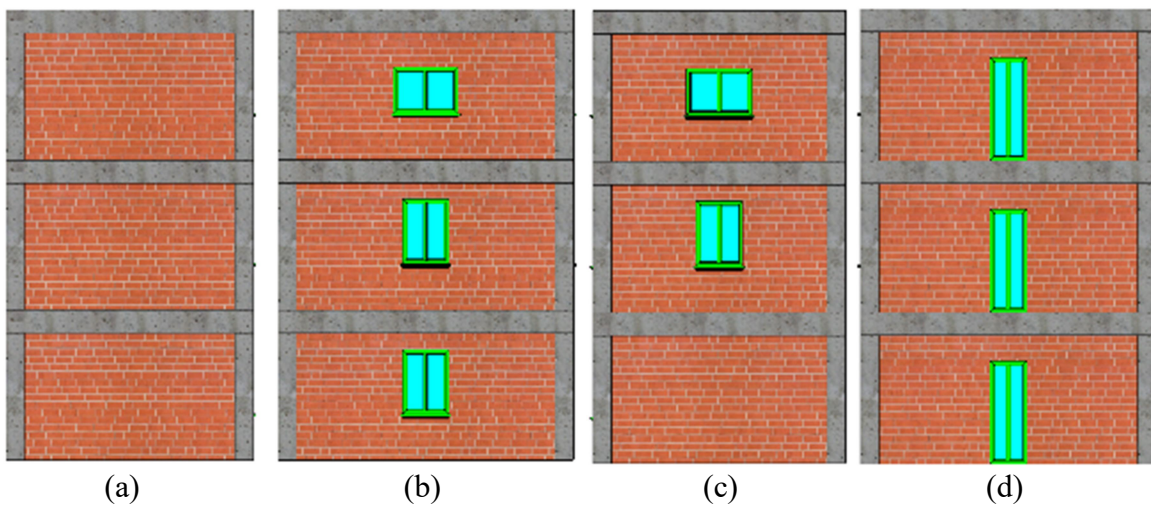
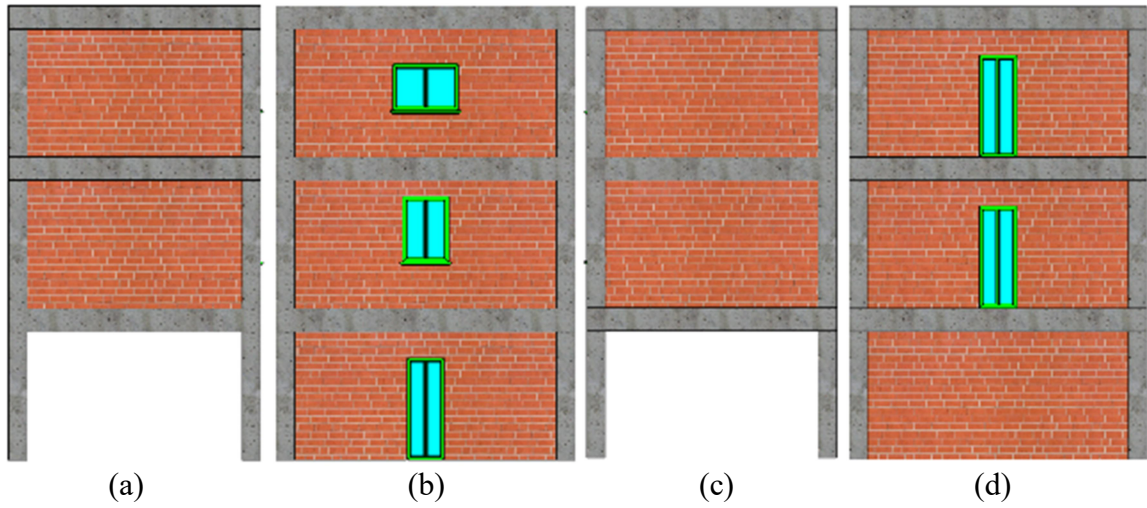
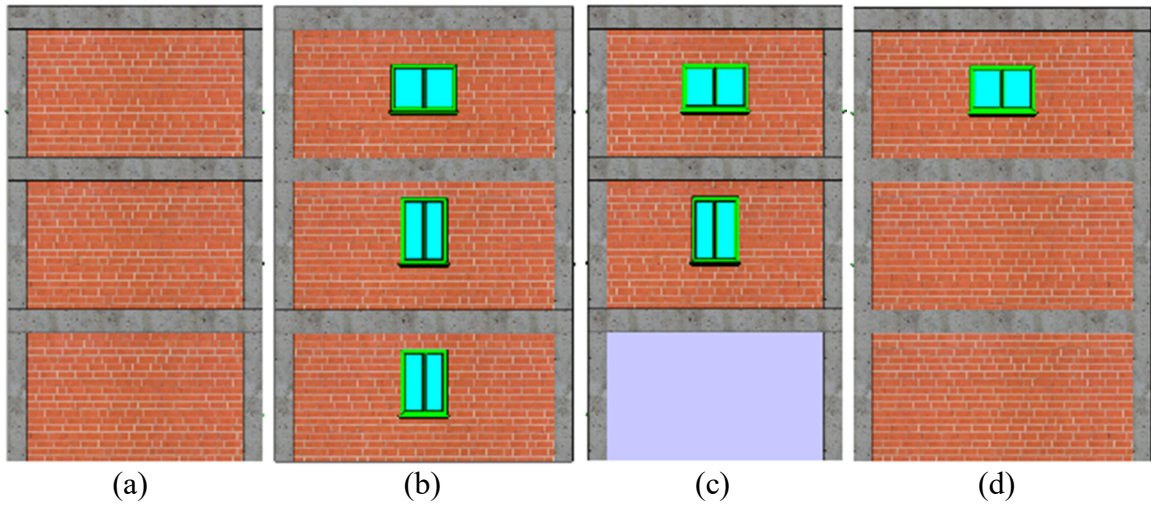


Figure 4.8: Views of 3-D samples - Case AA: (a) North elevation, (b) West elevation, (c) South elevation, (d) East elevation.

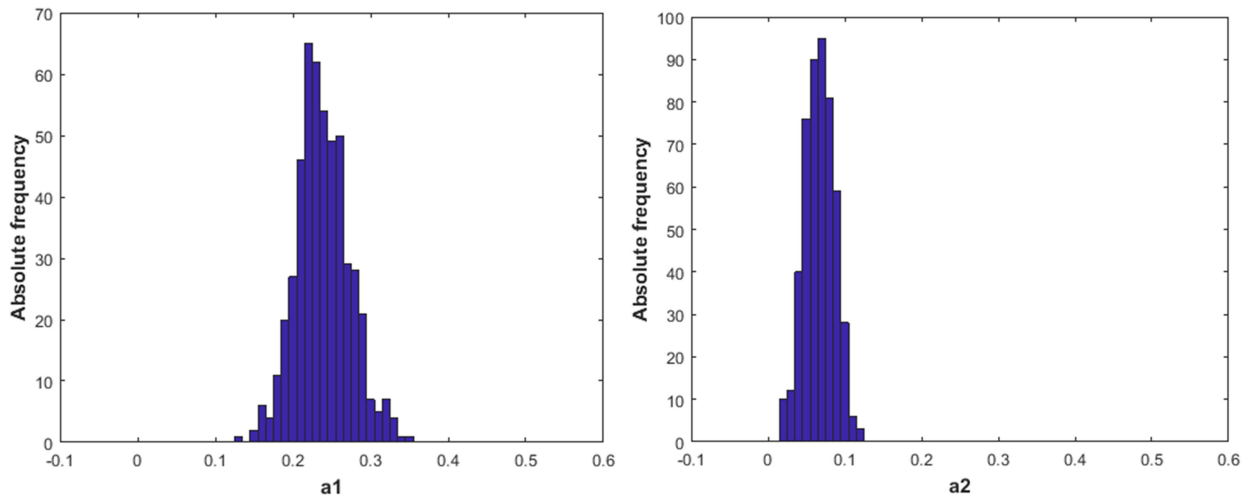


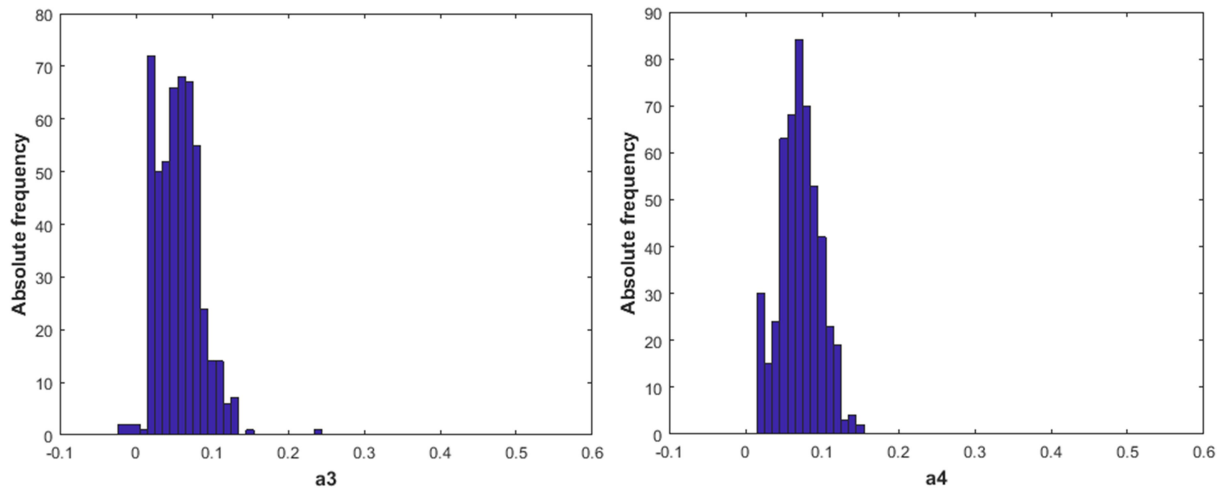


**Figure 4.9:** Views of 3-D samples - Case AS: (a) North elevation, (b) West elevation, (c) South elevation, (d) East elevation.

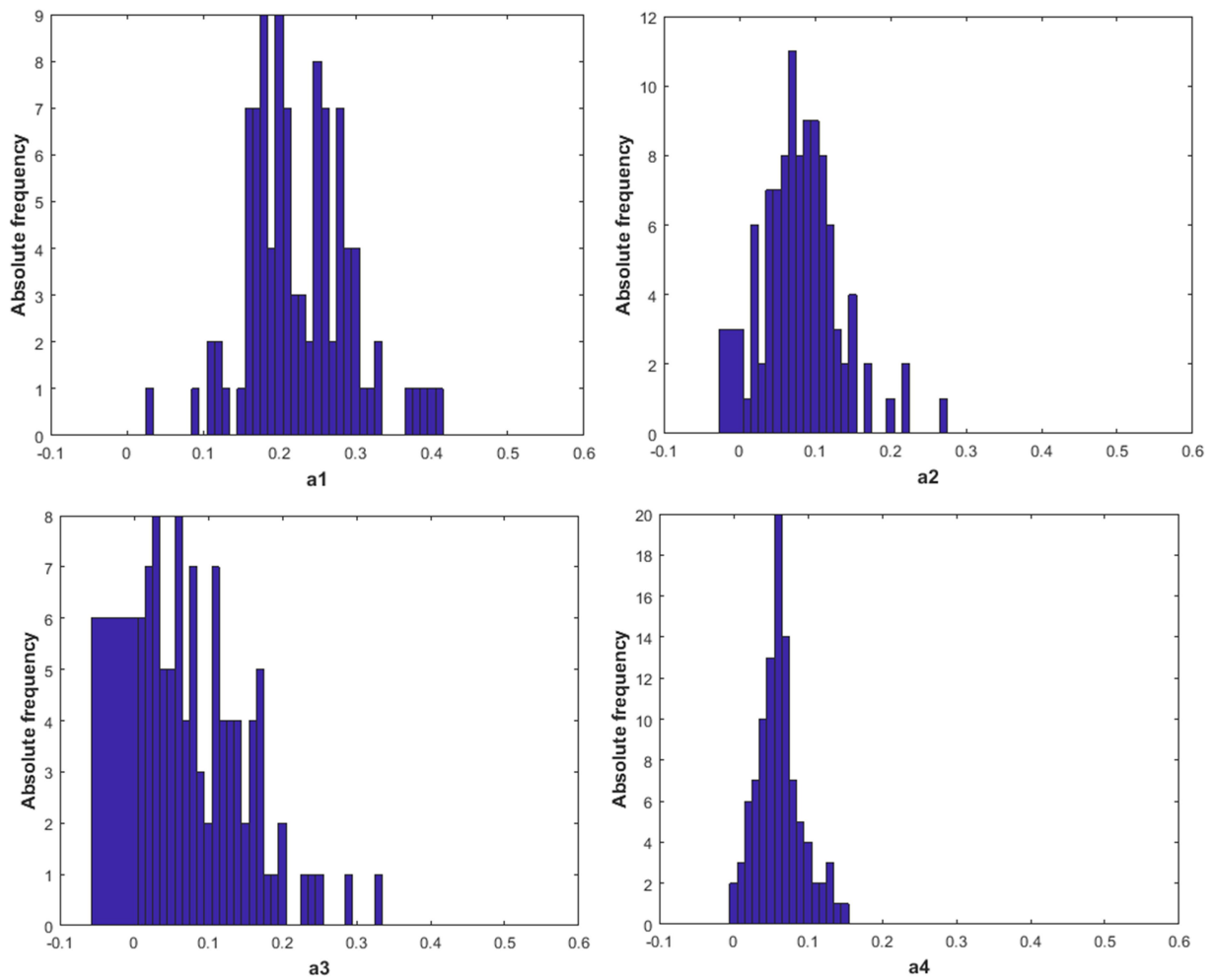


**Figure 4.10:** Views of 3-D samples - Case AAv: (a) North elevation, (b) West elevation, (c) South elevation, (d) East elevation - The blue infill is the model error.





**Figure 4.11:** Frequencies histograms of parameters  $a_1, a_2, a_3, a_4$  for 3-D sample - Case AA-2-I.



**Figure 4.12:** Frequencies histograms of parameters  $a_1, a_2, a_3, a_4$  for 3-D sample - Case AS-2-I.

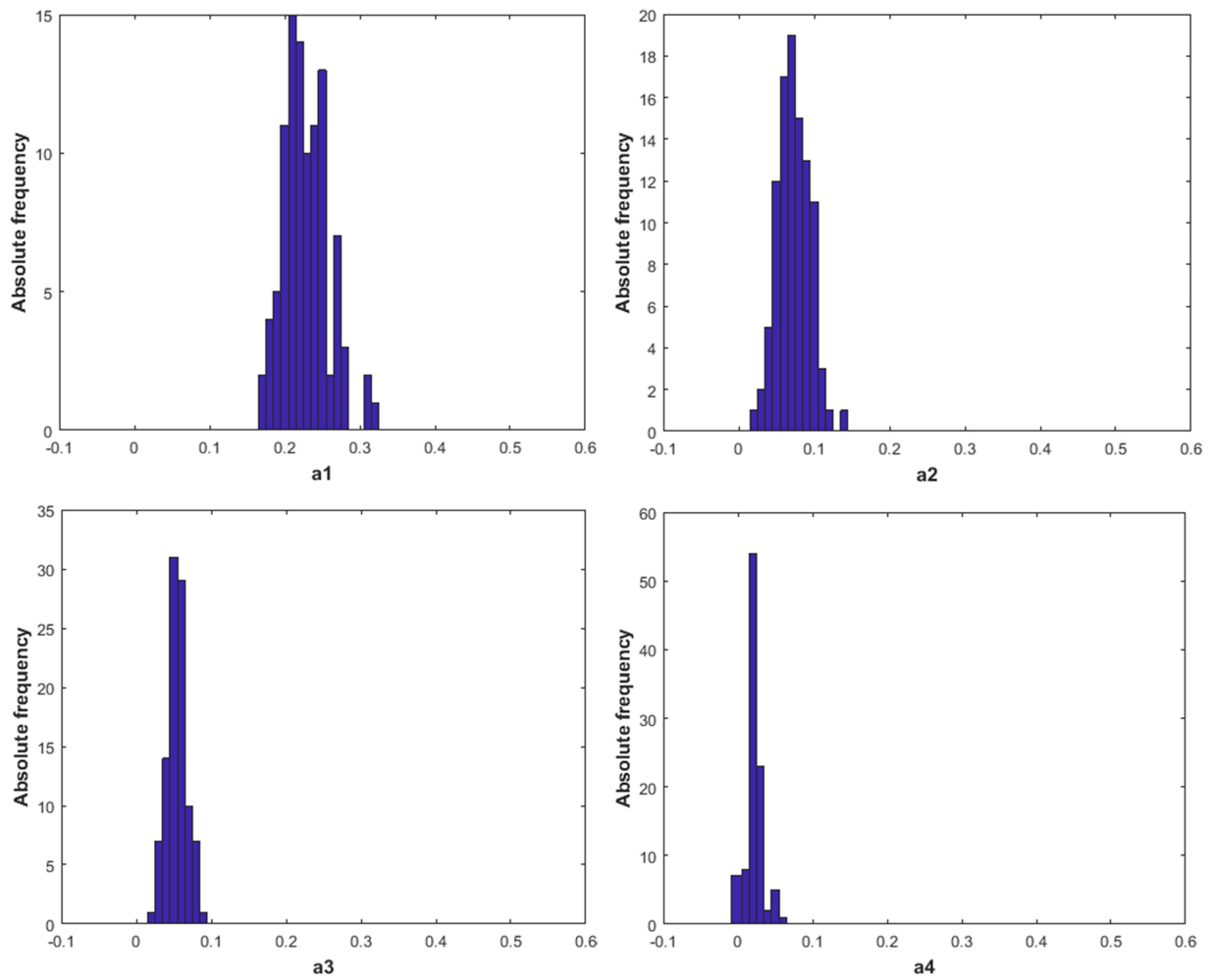
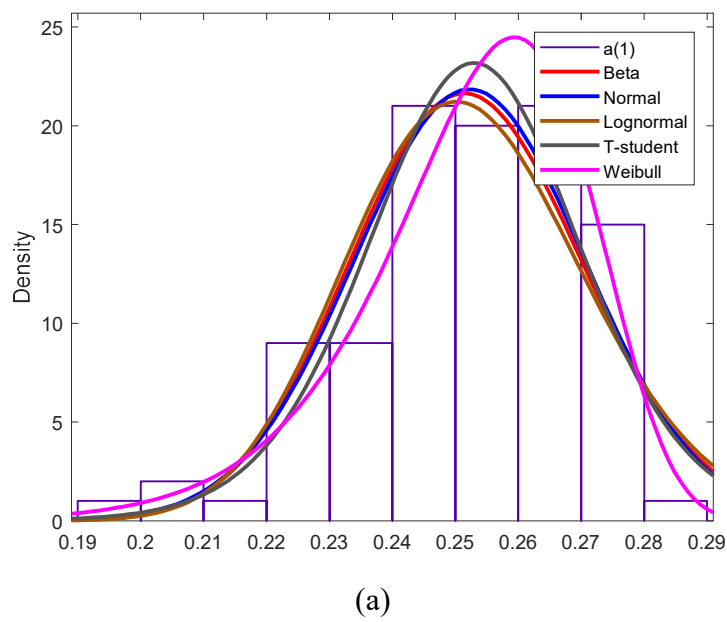
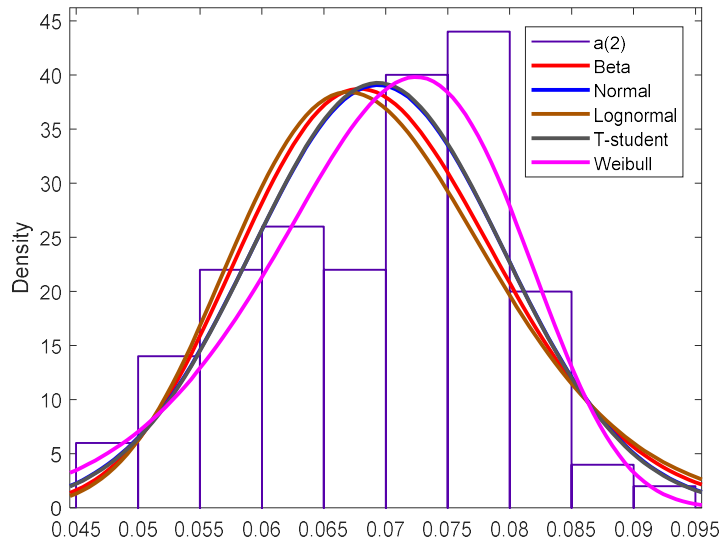
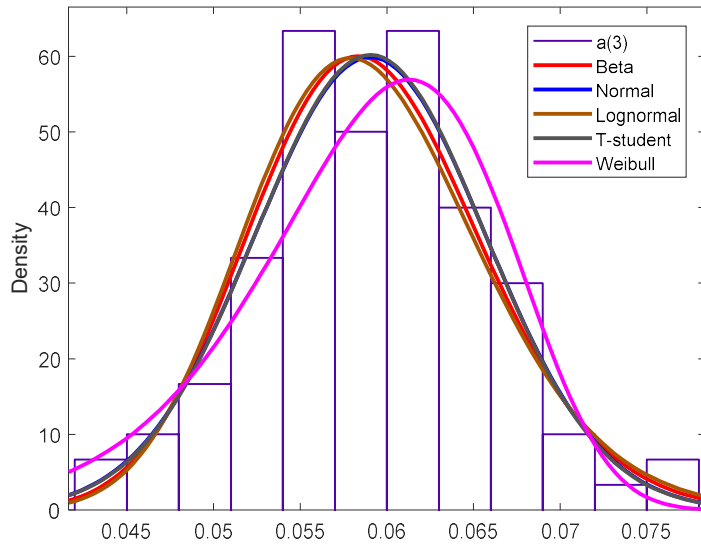


Figure 4.13: Frequencies histograms of parameters  $a_1, a_2, a_3, a_4$  for 3-D sample - Case AA $v$ -2-I.

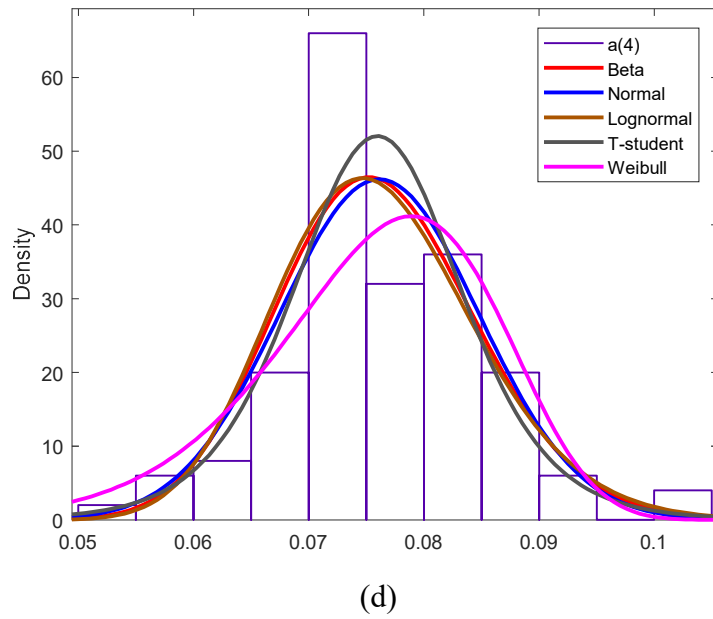




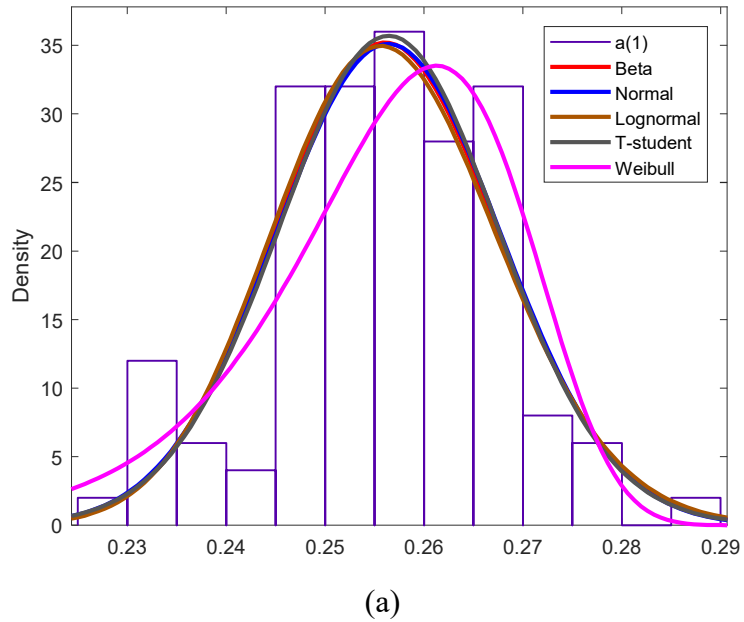
(b)

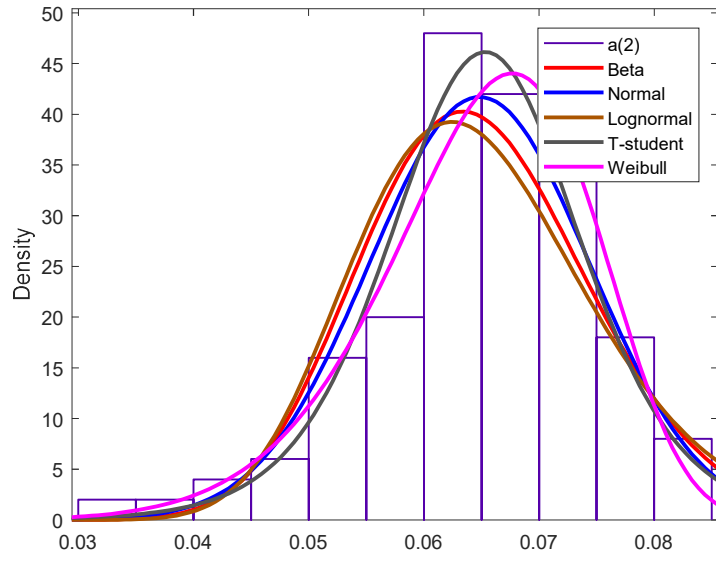


(c)

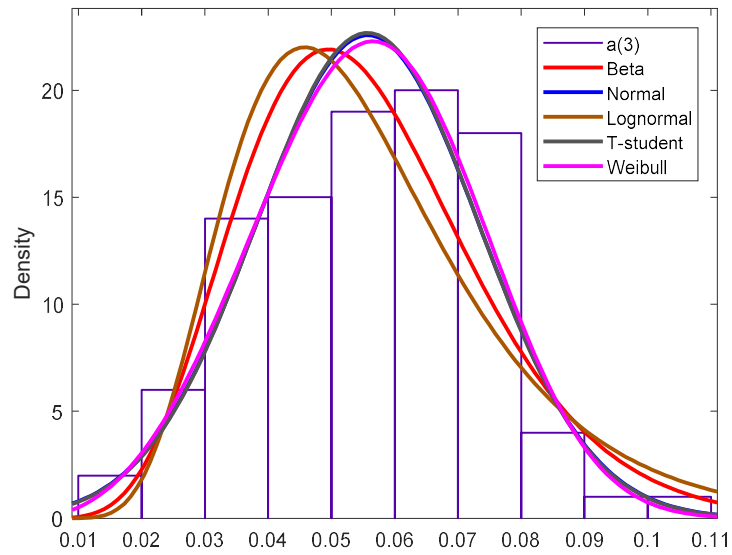


**Figure 4.14:** Frequencies histograms of parameters  $a_1, a_2, a_3, a_4$  for 3-D sample - Case AA-2-I along with main fitted distributions: (a) parameter  $a_1$ , (b) parameter  $a_2$ , (c) parameter  $a_3$ , (d) parameter  $a_4$ .

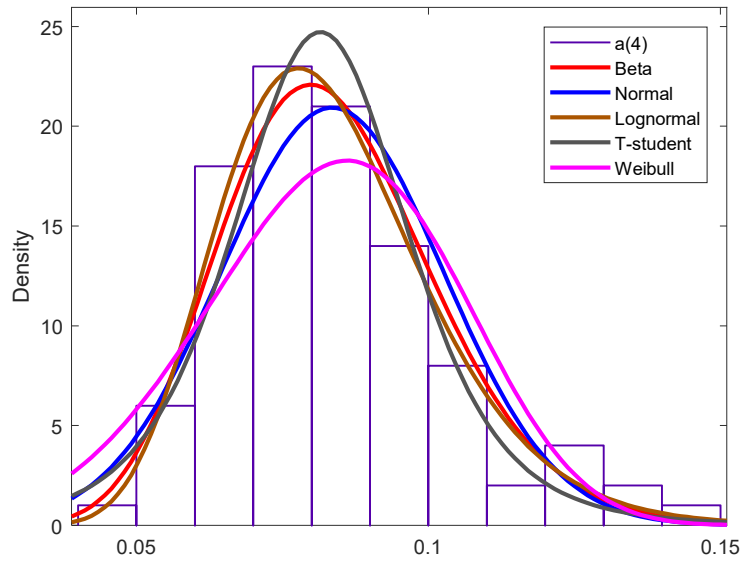




(b)



(c)



(d)

**Figure 4.15:** Frequencies histograms of parameters  $a_1, a_2, a_3, a_4$  for 3-D sample - Case AS-2-I along with main fitted distributions: (a) parameter  $a_1$ , (b) parameter  $a_2$ , (c) parameter  $a_3$ , (d) parameter  $a_4$ .





## CHAPTER 5

# DAMAGE IDENTIFICATION OF EXISTING INFILLED R.C. FRAMES

### 5.1 Introduction

In this Chapter, the damage identification of two existing infilled R.C. frames will be performed, for which the frequencies and mode shapes are available from experimental tests. The first step will be the definition of the damage function in order to evaluate the percentage of damage at fixed damage state with respect to the initial state of the structure before the test. After that two different damage identification will be performed on two structures in paragraphs 5.3 and 5.4. The model updating was performed using both the two steps algorithm and the DE-Q one. The results are practically the same for both the structures analyzed.

### 5.2 Damage parameter

The damage percentage parameter, for the  $j$ -th parameter and for the  $i$ -th damage state, following a consolidated literature technique (e.g. Moaveni et al., 2013), can be defined as follows:

$$E_j(DS_i) = \frac{a_j(DS_0) - a_j(DS_i)}{a_j(DS_0)} \cdot 100 \quad (5.1)$$

with this definition, the damage parameter is able to range from 0% (undamaged parameter) to 100% (total damaged parameter).

For the damage identification, from damage state to damage state, the boundary constraints of the DE-Q algorithm were updated every time in order to take into account the fact that parameters must decrease, increasing the damage state.

### 5.3 UCSD sample

#### 5.3.1 Description of the 3-storey infilled R.C. frame

The infilled frame considered here is a two-thirds-scale model of an exterior frame of a prototype structure, designed by Stavridis (Stavridis, 2009) to have non-ductile reinforcing details representative of the 1920s R.C. construction in California. The plan view of the prototype structure

and the elevation view of the exterior frame are presented in Figures 5.1 and 5.2 respectively. In these pictures, the dimensions are for the prototype structure, the specimen is a two-thirds-scale of it. The design of the specimen and the cross sections of columns and beams are reported in Figures 5.3 and 5.4 respectively. The design was based on the allowable stress design approach, considering only gravity loads in accordance with engineering practice of that era. However, the design was based on properties of contemporary construction materials, which were used for the construction of this specimen. The frame is infilled with three wythes unreinforced masonry walls on the exterior. Such structural systems can be found in many existing older buildings in the western United States, including pre-1930s buildings in California. This type of construction is also common in many regions of the world with high seismicity, such as the Mediterranean and Latin America regions.

The specimen tested on the large outdoor shake table at UCSD is shown in Figure 5.5. The structure included slabs that simulated the scaled gravity mass of the external frame of the prototype accounting for the two-thirds-length scale factor. Because the prototype structure has infill walls only in its exterior frames, the exterior frames are significantly stiffer and stronger than the interior frames. Consequently, their tributary seismic mass is significantly larger than the gravity mass as illustrated in Figure 5.1 for the exterior frame along column line A, which was modeled by the test specimen. The test specimen did not have additional gravity load-carrying systems. Therefore, it was decided that the mass carried by the specimen should accurately represent the gravity mass to induce the same vertical stresses as those experienced by the R.C. columns and infill walls of the prototype. To account for the effect of the seismic mass not included in the specimen, the input ground acceleration time histories had to be scaled in time and amplitude (Stavridis, 2009) to satisfy the similitude requirement for the seismic forces. It should be pointed out that the ground motion levels referred to in the subsequent sections are always with respect to the full-scale prototype structure. Two steel towers were erected on the shake table on the north and south sides of the test specimen to prevent a potential out-of-plane collapse of the structure during severe shaking. These towers did not interact with the structure during the tests as they were placed with a 2-cm gap from the specimen. Further details on the design and configuration of the specimen and the shake table tests can be found in Stavridis, 2009 and Stavridis et al., 2011.

### **5.3.2 Instrumentation layout**

The specimen and steel towers were extensively instrumented with an array of 265 sensors, including 135 strain gauges, 71 string potentiometers and linear variable differential transformers

(LVDTs), and 59 uniaxial accelerometers. The accelerometers were used to measure the accelerations along the x, y, and z directions, with x being the direction of the base excitation (longitudinal), y the transverse (out-of-plane) direction, and z the vertical direction. In this study, the measured response from three longitudinal, three vertical, and two transversal acceleration channels on each floor are used (total of 24 channels) to identify the modal parameters of the test structure. The locations of the accelerometers at each floor level are shown in Figure 5.6. The measured acceleration responses were sampled at 240 Hz resulting in a Nyquist frequency of 120 Hz, which is significantly higher than the modal frequencies of interest in this study ( $< 60$  Hz). Before applying the system identification method to the measured data, all acceleration time histories were band-pass filtered between 0.5 and 70 Hz using a high-order (1024) finite impulse response filter.

### 5.3.3 Dynamic tests performed

The specimen was subjected to a sequence of 44 dynamic tests including ambient-vibration tests, free-vibration tests, and forced vibration tests (white-noise and seismic base excitations). The main events and tests are shown in Table 5.1. The testing sequence consisted of earthquake ground motions of increasing intensity. Before and after each earthquake record, low-amplitude white-noise base excitation tests were performed to provide data for the model updating and damage identification. The input ground motions were obtained by scaling the time and amplitude of the ground acceleration time history recorded along the NS direction at the Gilroy 3 station during the 1989 Loma Prieta earthquake. For structures with a fundamental frequency close to that of the infilled frame studied here, the Gilroy 3 motion scaled at 67% corresponds to a moderate design level earthquake for Seismic Design Category D, while the original (unscaled) motion corresponds to a maximum considered earthquake (MCE). The MCE event selected as the reference base motion intensity in this study has spectral accelerations,  $S_s = 1.5g$  and  $S_l = 0.6g$ , and represents the worst-case scenario for San Diego and a moderate scenario for the Los Angeles area (Stavridis, 2009). In the Table, 7 different damage states are pointed out, damage state DS0 represent the reference state for undamaged structure, corresponding to the uncracked state of the structure before its exposure to the first base excitation, while damage states DS1 to DS7 correspond to the conditions of the structure after it was subjected to different levels of the Gilroy earthquake.

Using these tests, a system identification was made by Moaveni (Moaveni et al., 2013), frequencies and mode shapes were achieved for all the damage states. 4 modes were identified (2 translational, 1 rotational and 1 coupled modes were found), for this study only the two translational modes are

taken into account. The polar plot of the two modes for damage state DS0 and a graphical representation are reported in Figure 5.7. The frequencies obtained for all the damage states are listed in Table 5.2. From the polar plot one can see how the modes are almost real modes.

### 5.3.4 Model updating

For the purpose of model updating, a stick model of the structure has been created. Only translational degrees of freedom are considered in the analysis. The mass are concentrated in the nodes of the stick model and the parameters to obtain are defined as follows:

$$a_i = \frac{K_i}{3 \cdot \frac{b_c \cdot h_c^3}{l_c^3}} \quad (5.2)$$

in which  $K_i$  is the storey stiffness at the  $i$ -th floor,  $b_c$  and  $h_c$  are the base and width of the columns section (both equal for all the columns to 0.279 m),  $l_c$  is the columns height (equal for all the columns to 2.060 m).  $K_i$  was divided by the term in Eq. 5.2 only to limit the values of parameters themselves. All the geometrical features and values for masses are taken from Stavridis, 2009. Figure 5.8 depicts the stick model (y axis is normalized with respect to the interstorey height).

In Figures 5.9, 5.10 and 5.11 are depicted the parameters values, the storey stiffness and the damage parameters for the seven damage states. In Table 5.3 are listed the experimental and numerical frequencies along with the MAC values. In Figure 5.12 a comparison between experimental and numerical modal shapes is reported.

### 5.3.5 Analysis of results

Analyzing the results one can see how the first mode is, for all damage states, found by the procedure with MAC values equal to 1.00. The second mode has always MAC values greater than 0.90 and therefore is found with acceptable precision. The frequencies are matched with very good agreement up to DS6 in which the agreement is poorer. This is due to the fact that for DS6 and DS7 the damages undergone by the structure were severe and a refined model, with respect to the stick model used here, should be used.

The damage parameters can capture the huge damages at the first storey, mainly in DS6 and DS7, and even some damages in the second storey at DS7. This matches with experimental evidences after the tests, especially, after Gil120, extensive damages are found at the ground storey and severe damage at the first storey, as well.

The match between experimental and numerical modal shapes is practically perfect for the first mode, is poorer for the second one especially for DS6 which is the worst FE identified mode.

## 5.4 El-Centro building

### 5.4.1 Description of the structure

As reported in Yousefianmoghdam et al., 2015, the structure under study is a two-story reinforced concrete moment frame structure with a basement, located in El Centro, CA, US shown in Figure 5.13. The building was constructed in the 1920s and it was typical of the construction practice in California in that era. It had sustained damage during the Imperial Valley Earthquakes of 1940 and 1979, and the 1987 West Westmoreland Earthquake. The exterior frames and infills of the ground floor were repaired and retrofitted after the first three earthquakes. However, the structure was red tagged and evacuated after the 2010 Baja California Earthquake due to extensive damage in the first story. With all non-structural components removed from the building, only its structural members, including the RC frame and the infill walls, was in place during the tests. The structure was demolished after the completion of the tests as it could not be repaired cost-effectively considering the economy in the area. Figure 5.14 shows the 26 m by 32 m plan view of the structure at the first floor level. The plan was rectangular except for the west exterior frame which was curved. Plan view of the first floor was similar to that shown in Figure 5.14 except that the exterior infills on the northern side had one frame recess to allow for a pedestrian side walk passage as shown in Figure 5.13. On the south, there was a one-story wooden structure attached to the main building as indicated in Figure 5.14.

The structure comprised of six reinforced concrete frames in the north-south direction connected by arch-type joists in the east-west direction. The dimensions and reinforcement details of the joists are shown in Figure 5.14. The interior columns were 40 cm diameter circular. The exterior ones were 40 cm by 40 cm square except for the columns in south side and the ground story columns in north side which were 40 cm diameter circular.

The ground story of the structure had been repaired and retrofitted in the late 1980s after the 1987 earthquake. The retrofit had focused on the strengthening of the masonry infills of the ground story. As a result, there were three types of infill in this floor: reinforced concrete, unreinforced masonry and combinations of the two. The exterior frames in the basement had reinforced concrete walls with openings near the top, while the first story had a masonry infill of two independent wythes in all the exterior frames. The wythes had a distance of 10 cm and the gap in between was filled with a

powder; most probably for insulation reasons. The stiffness and strength discontinuity between the ground and first stories resulted in severe damage of the first story infills and frames in the north, west and south bays during the 2010 Baja California earthquake. The east side on the first floor that had a solid infills and the entire first story that was strengthened did not develop any visible cracks.

#### **5.4.2 Induced damages**

The exterior infill walls were part of the lateral load resisting system and their removal could affect the lateral stiffness and strength of the structure. Four infill walls were removed at three stages introducing four levels of damage to the building (DS1, DS2, DS3 and DS4). The first damage state (DS1) was the initial condition of the structure prior to the experiment, which included a wall already removed in the bay A3-A4 of the first floor. This wall was removed prior to the test to allow the sliding of the shaker inside. The second damage state (DS2) resulted once the infill in D6-E6 bay in the first story (Y-direction) was removed. The third damage state (DS3) resulted from the removal of the infill in E6-F6 bay in the same exterior frame in the first floor. The fourth and last damage state (DS4) was introduced after the removal of infills in F6-G6 and G5-G6 in the first floor. The walls removal procedure is summarized in Table 5.4 and the locations, along with the sequence, of the removed walls are shown in Figure 5.15.

#### **5.4.3 Dynamic testing**

A total of 97 sensors including accelerometers, string pots and LVDTs were installed on the building to measure accelerations and displacements. To measure the accelerations, 21 uniaxial and 39 triaxial force-balance accelerometers were utilized. The accelerometers were installed close to the four corners and the center of the ground and the first floor and the roof. In every location, they measured the acceleration in two horizontal directions and one vertical direction (X, Y and Z), so that 15 acceleration measurements were obtained at each level as shown in Figure 5.16. The X direction corresponds to the east-west direction with positive measurements being towards the west and the Y direction corresponds to the north-south direction with positive towards the north. Moreover, two triaxial accelerometers were installed at the north-west and south-east corner of the basement, while three additional uniaxial accelerometers were mounted on the extension building at its north and west sides. Eventually, two triaxial accelerometers were installed on the ground close to the structure at the north and west side of it. For the purposes of this work, only the horizontal DOFs of the structure have been taken into account, at ground, first and roof levels. Therefore 10 mode shape components per story have been used with total amount of 30 components. The sensors

were connected to a data logger with data sampling rate of 200 Hz and were synchronized by GPS timing.

Two type of tests were conducted: Ambient Vibration (using wind vibrations) and Forced Vibration with shaker. A series of shake tests were performed on the structure to investigate its dynamic properties in the quasi-linear and nonlinear ranges of behavior. The experiments were conducted using two mobile shakers owned and operated by NEES@UCLA. Initially a small, linear, one-man portable shaker was used to identify the natural frequencies of the building. The other shaker, was a mobile eccentric mass shaker with a force capacity of 445 kN (100 kips). The shaker could produce harmonic excitations within a range of frequencies (0-5.5 Hz). The shaker was mounted on the second floor at the north-west corner and bolted to the concrete slab (Figure 5.15). The excitations produced by the latter shaker were sine sweeps and sine steps which were used to excite both the X and Y directions.

Force vibration and ambient vibration tests were conducted in all the damage states, before and after the wall removal. A total of 26 force vibration tests were performed and 166 ambient vibration recordings (80 hours) were obtained in the four damage states over a four-day period.

For the purposes of this work only the ambient vibration tests have been considered because of forced vibration ones overcame the linear range of the structure.

#### **5.4.4 System identification**

Modal parameters of the test structure were estimated at each damage state from the ambient measurements. The system identification was performed using time-domain (NExT-ERA) and frequency-domain (Peak picking) methods (Ferrar and James, 1997; Bendat and Piersol, 1980).

First of all the NExT-ERA method is discussed. The natural excitation technique combined with the eigensystem realization algorithm (NExT-ERA) was used to identify the modal parameters of the building. The data cleaning process included: (1) filtering between 0.5 and 7 Hz with a Finite Impulse Response (FIR) band-pass filter, and (2) down-sampling the data from 200 Hz to 50 Hz. Based on the length of available data in each damage state, the ambient acceleration measurements were divided into 30 sets for DS1, 5 sets for DS2, 2 sets for DS3, and 20 sets for DS4. Each set of data corresponds to approximately 10 minutes of measurements for DS1-DS3 and 5 minutes for DS4. The system identification algorithm was applied for all 57 datasets. For each set, the signal was divided into 4 Hamming windows with 50% overlap to compute the cross power spectral density. SW-X on 1st floor and SW-X on roof were chosen as two reference channels for computing cross-correlation functions which were then used as free vibration data and fed into the

ERA method. The order of ERA was chosen manually for each of the 57 sets based on the stabilization diagrams. The modal parameters (natural frequencies, damping ratios and mode shapes) were then identified for each set of DS1 and DS4.

The peak-picking method was also used to find the natural frequencies and mode shapes of the building at different damage states. The transfer functions between all of the accelerometers recordings (outputs) and a reference accelerometer were computed from their power spectral densities using one set of data in each damage state, with each set corresponding to approximately 20 minutes of measurements for DS1, DS2, and DS4 and 10 minutes of measurements for DS3. As reference, the accelerometers at north-west corner of the second floor measuring in either the X direction or Y direction depending on the direction considered, were selected. In the next step, the peak and the corresponding frequencies were estimated from the transfer function between the signals at the roof and at the reference location. The mode-shape components were then estimated using the values of the transfer functions between all the locations and the reference accelerometer at the identified frequency. The damping ratios of the reference channels were also found and averaged using the half power bandwidth method on their power spectral densities. Power-cross spectral densities of the acceleration measurements were estimated using the Welch method and averaged over Hanning windows of 8192 data points with 50% window overlap. The same window, window function, and overlap was used to find the Fourier transform of the measurements.

The identified frequencies are summarized in Table 5.5 for both NExT-ERA and peak picking methods. The identified frequencies are decreasing for both modes as expected. The rate of change is different for different damage states showing different contribution of each removed wall on the overall stiffness due to the location but also due to the prior damage.

The first mode involved motion mainly in the X (east-west) direction, however the west corners of the building also moved in the Y (north-south) direction. This happened because of the damage in the west infills that shifted the center of rigidity towards the undamaged east well. As a result, a combination of translational and torsional motion was introduced. The second mode was mainly torsional with the center of rotation close to the east side of the building. As in the first mode, the movement in the X direction at the south side is more than the north side one. The infill walls at the east side appeared undamaged and as a result, stiffer than the damaged walls at the west side. Furthermore, there was a stairway shaft close to the north-east corner of the building having reinforced concrete walls which provided lateral stiffness. Hence, the center of the rigidity moved toward the north-east corner of the structure. Figure 5.17 depicts the deformed shapes of the structure at roof level for both modes in DS1 and DS4. The comparison of these two damage states



in mode 1 indicates that the structure tended to displace more in the Y direction after the wall removal, which was expected because three walls were removed in this direction.

For the purposes of this work the NExT-ERA identified frequencies and mode shapes are used in order to perform the model updating.

#### 5.4.5 Model updating for 4 parameters case

For the goal of model updating of the structure, in order to figure out the contribution in stiffness of the infills, the structure was modeled with software OpenSEES (McKenna and Fenves, 2001; OpenSEES, 2016). All the columns and beams were modeled using 2D Beam-Column elements with modulus of elasticity equal to 9997 MPa (1450 ksi), obtained through experimental tests. As usual, the infills were replaced by equivalent struts (linear truss elements). The four parameters used in the model updating were the  $E \cdot A$  for the struts replacing the infill panels at the first story. The distribution of parameters is depicted in Figure 5.18. The struts stiffness for infill panels at the ground story and in the underground were calibrated using the relations proposed by Stavridis (Stavridis, 2009) taking into account a reduction factor in order to reduce the stiffness because of the openings and the already present damages in the infill panels. The reduction factor was defined as follows:

$$\theta = \alpha \cdot \beta \quad (5.3)$$

in which  $\alpha$  is the stiffness loss due to the damage of the infill which is a value between 0 and 1, with 1 corresponding to healthy and 0 to totally damaged walls, respectively. This parameter was estimated based on engineering judgment after the inspection of the walls. The parameter  $\beta$  is the stiffness reduction due to the infill opening(s), estimated from the formula proposed by Stavridis (Stavridis, 2009) for masonry infills:

$$\beta = (1 - 1.6 \cdot \frac{A_{Opening}}{A_{Total}}) \quad (5.4)$$

where  $\frac{A_{Opening}}{A_{Total}}$  is the ratio of the opening area to the gross area of the infill.

In order to perform the model updating, the 10 modes components per story are condensed into 3 rigid diaphragm components (translations in X and Y direction and the rotation about the center of the mass). The assumption of rigid diaphragm (at least for the ambient vibration case) has been revealed correct and the comparison between original modes components and re-built ones (following the procedure of paragraph 3.7) gave very good results (Table 5.6). Moreover, having noticed that the two modes gave practically the same value of  $p_i$ , both of them were set equal to 1.

The model updating has been done for all the damage states and the results achieved, in terms of parameters values, are listed in Table 5.7; frequencies, errors and MAC values are summarized in Table 5.8. The parameters values are depicted in Figure 5.19, the evolutions of the damage parameters are listed in Table 5.9 and depicted in Figure 5.20. The comparison between updated parameters and the ones obtained with calibration (following the procedure described by Stavridis, 2009) is presented in Figure 5.21. The modes components along with the experimental ones are depicted in Figure 5.22. In the last Figure, the comparison is made with respect to the modes components at the location of the accelerometers, therefore 10 graphs are depicted (5 accelerometers per 2 displacement components).

#### **5.4.6 Analysis of results for 4 parameters case**

The results from the model updating are very in good agreement with the experimental outcomes. The MAC values are almost 1.00 for all modes and in all the damage states. The errors in frequencies are very low, with maximum value equal to 1.56%. The damage parameters match very well the state of the structure after each test and is able to point out the walls removal. The damage parameters for  $a_1$  and  $a_2$  are practically 0% for all the damage states because of no walls are removed in the north and east elevation. The damage parameter for  $a_3$  is 0% up to DS4 in which one wall was removed from the south elevation. The damage parameter for  $a_4$  can figure out the progressive walls removal in the west elevation.

#### **5.4.7 Sensitivity analysis for 4 parameters case**

The model updating made in the previous paragraphs was done using the frequencies and modes shapes from NExT-ERA procedure. A set of experimental outcomes were also obtained through peak-picking for 15 minutes acquisition windows. For DS1 285 different values of frequencies and modes components were obtained (for a total time of measurements of about 71 hours); for DS2 18 different data windows; for DS3 3 data windows and for DS4 11 ones. The procedure outlined previously was applied for each data windows acquired and the results are given in terms of mean values, standard deviations and CoVs of parameters, mean values, standard deviations and CoVs for the frequencies and MAC values. The mean values for experimental frequencies, along with standard deviations and CoVs are reported in Table 5.10. The mean values, standard deviations and CoVs of parameters are listed in Table 5.11. The mean values, standard deviations and CoVs for computed frequencies (after parameters achievement) along with the errors between mean values of experimental frequencies and mean values of the numerical ones are listed

in Table 5.12. In the same Tables, the MAC values along with their CoVs are listed. In Figure 5.23 the histograms of parameters are depicted for DS1 case. In Figure 5.24 the mean values of parameters for all the damage states along with values with  $\pm\sigma$  are depicted.

The results achieved are in very good agreement with the ones obtained before (one run analysis) for all the parameters and damage states. Also the walls removal is understood well by the procedure, the first two parameters remain practically unchanged from DS1 to DS4, the third has significant change only between DS3 and DS4; the last one changes at each damage state, as already found with other procedure. The CoVs in the parameters assume acceptable values. The mean values of numerical frequencies are very close to the mean values of the experimental ones with errors quite negligible. The mean values of MAC coefficients are always greater than 0.92 with small CoVs.

#### **5.4.8 Model updating for 8 parameters case**

With the same procedure outlined in paragraph 5.4.5, the structure was updated considering 8 parameters, adding the ground story infills too. The parameters arrangement is shown in Figure 5.25. Once again, the infills were replaced by equivalent struts and, again, the parameters are the  $E \cdot A$  for the struts replacing the infills at the ground and first storey. The problem didn't have a single solution, therefore the starting point has been given from the already achieved solution for the 4 parameters case. The results in terms of parameters values are presented in Table 5.13. In Table 5.14 are listed the numerical frequencies, errors with respect to the experimental ones and MAC values. In Table 5.15 the damage parameters are listed.

In Figure 5.26 the bar graph of parameters values is depicted. In Figure 5.27 the damage parameters are also reported. In Figure 5.28 a comparison between undamaged structure parameters and DS1 ones is given. In Figure 5.29 the damage parameters between undamaged structure and damage state DS1 are also depicted.

#### **5.4.9 Analysis of results for 8 parameters case and final remarks**

The results achieved updating 8 parameters are in good agreement with respect to the ones obtained updating only the infills at the first storey (4 parameters case). The values of parameters at the first storey are practically the same in both 4 parameters and 8 parameters cases. The damage parameters for the infills at the first storey give the same results as well. Using 8 parameters some improvement in terms of frequencies errors and MAC values are achieved (those values are very good also for the 4 parameters case). Moreover, the procedure can capture also the more infills

stiffness at the ground storey. The main problem is that some damages are found at the ground storey. Those damages didn't have any visual confirmation in the structure. They can be loss in stiffness due to shaker tests which led to decreases in the stiffness and not in the strength of the infill panels. Nonetheless, using the procedure with 8 parameters only for DS1 and then keeping constant the infill panels values at the ground storey for damage states DS2, DS3 and DS4; the results achieved are listed in Table 5.16 in terms of frequencies, errors and MAC values.

Comparing Tables 5.14 and 5.16 one can see that the updating of the ground storey infills has negligible influence on the global behavior of the structure at damage states DS2, DS3 and DS4.

## 5.5 Conclusions

In the present Chapter, two real structures have been analyzed in order to achieve the model updating starting from experimental outcomes (frequencies and modes components). The first structure was a three storey infilled frames with two bays. This structure was analyzed using a stick model and the parameters updated were the storey stiffness. The second one was an actual two storey building. The parameters updated were the  $E \cdot A$  of struts replacing the infill panels. First of all only the infills at the first storey were updated. After that a sensitivity analysis was carried out. Eventually, a 8 parameters case (the parameters were the struts replacing the infill panels at the ground and first storey) has been analyzed.

## Tables of Chapter 5

Test number	Test date	Test description	Damage state
5	03/11/2008	0.03g RMS WN, 5 min	DS0
8	03/11/2008	20% Gilroy EQ	
9	03/11/2008	0.03g RMS WN, 5 min	DS1
12	06/11/2008	40% Gilroy EQ	
13	06/11/2008	0.03g RMS WN, 5 min	DS2
21	10/11/2008	67% Gilroy EQ (DE)	
25	12/11/2008	0.04g RMS WN, 5 min	DS3
26	12/11/2008	67% Gilroy EQ (DE)	
27	12/11/2008	0.04g RMS WN, 5 min	DS4
28	12/11/2008	83% Gilroy EQ	
29	12/11/2008	0.04g RMS WN, 5 min	DS5
33	13/11/2008	91% Gilroy EQ	
34	13/11/2008	0.04g RMS WN, 5 min	
35	13/11/2008	100% Gilroy EQ (MCE)	
36	13/11/2008	0.04g RMS WN, 5 min	DS6
40	18/11/2008	120% Gilroy EQ	
41	18/11/2008	0.04g RMS WN, 5 min	DS7

Note: WN = white noise base excitation; EQ = earthquake base excitation;  
DE = design earthquake; MCE = maximum considered earthquake.

**Table 5.1:** Dynamic tests performed in the specimen (from Moaveni et al.,2013).

Damage state	Frequencies [Hz]	
	Mode 1-L	Mode 2-L
DS0	18.18	41.22
DS1	18.11	41.09
DS2	17.99	41.56
DS3	16.74	40.21
DS4	15.93	38.56
DS5	14.78	35.50
DS6	8.47	27.34
DS7	5.34	22.57

**Table 5.2:** Identified frequencies from white noise tests.

Damage state		Frequencies [Hz]			MAC
		Experimental	Numerical	Error [%]	
DS0	Mode 1	18.18	18.18	0.00	1.00
	Mode 2	41.22	41.86	1.54	0.96
DS1	Mode 1	18.11	18.10	-0.05	1.00
	Mode 2	41.09	41.14	0.12	0.95
DS2	Mode 1	17.99	17.98	-0.02	1.00
	Mode 2	41.56	40.75	-1.95	0.95
DS3	Mode 1	16.74	16.74	0.00	1.00
	Mode 2	40.21	38.32	-4.69	0.95
DS4	Mode 1	15.93	15.93	0.00	1.00
	Mode 2	38.56	36.86	-4.41	0.94

DS5	Mode 1	14.78	14.78	0.00	1.00
	Mode 2	35.50	35.35	-0.44	0.94
DS6	Mode 1	8.47	8.13	-4.04	1.00
	Mode 2	27.34	31.35	14.68	0.90
DS7	Mode 1	5.34	4.78	-10.48	1.00
	Mode 2	22.57	21.49	-4.80	0.99

**Table 5.3:** Comparison between experimental and numerical frequencies and MAC values.

Damage state	Demolished wall(s)
DS0	-
DS1	A3-A4 (1 <sup>st</sup> floor)
DS2	D6-E6 (1 <sup>st</sup> floor)
DS3	E6-F6 (1 <sup>st</sup> floor)
DS4	F6-G6 and G5-G6 (1 <sup>st</sup> floor)

**Table 5.4:** Walls demolition sequence and resulted damage state.

Damage state	Mode 1 - Frequencies [Hz]		Mode 2 - Frequencies [Hz]		Mode 1 - Damping Ratio [%]		Mode 2 - Damping Ratio [%]	
	Peak-Picking	NExT-ERA	Peak-Picking	NExT-ERA	Peak-Picking	NExT-ERA	Peak-Picking	NExT-ERA
DS1	2.29	2.26	3.32	3.37	1.7	1.6	3.1	2.3
DS2	2.17	2.14	3.03	3.08	2.2	1.3	2.1	2.0
DS3	2.12	2.07	3.00	2.96	2.4	2.0	2.5	2.7
DS4	2.05	1.97	2.81	2.72	1.4	1.6	2.5	2.7

**Table 5.5:** Summary of system identification results (Yousefianmoghadam et al., 2015).

Damage state	MAC	
DS1	Mode 1	0.998
	Mode 2	0.997
DS2	Mode 1	0.998
	Mode 2	0.998
DS3	Mode 1	0.998
	Mode 2	0.998
DS4	Mode 1	0.997
	Mode 2	0.998

**Table 5.6:** Comparison between modes components and re-built modes components.

Damage state	Parameters values [ $\times 10^4$ ] [kN]			
	$a_1$	$a_2$	$a_3$	$a_4$
DS1	4.34	37.25	17.33	11.35
DS2	4.26	37.25	17.33	6.65
DS3	4.26	37.25	17.33	4.73
DS4	4.26	36.43	13.66	3.52

**Table 5.7:** Values of parameters for 4 parameters case.

Damage state		Frequencies [Hz]			MAC
		Experimental	Numerical	Error [%]	
DS1	Mode 1	2.26	2.27	0.54	0.99
	Mode 2	3.37	3.38	0.36	0.99
DS2	Mode 1	2.14	2.16	1.12	0.99
	Mode 2	3.07	3.19	0.92	0.99
DS3	Mode 1	2.06	2.08	1.10	0.98
	Mode 2	2.96	2.99	0.92	0.97
DS4	Mode 1	1.96	1.99	1.56	0.98
	Mode 2	2.72	2.75	1.25	0.99

**Table 5.8:** Comparison between experimental and numerical frequencies and MAC values for 4 parameters case.

Parameter	Damage parameters [%]			
	DS1	DS2	DS3	DS4
E <sub>1</sub>	0.0	1.7	1.7	1.7
E <sub>2</sub>	0.0	0.0	0.0	2.2
E <sub>3</sub>	0.0	0.0	0.0	21.2
E <sub>4</sub>	0.0	41.4	58.3	69.0

**Table 5.9:** Damage parameters for 4 parameters case.

Damage state		Experimental frequencies [Hz]		
		Mean value	Standard deviation	CoV [%]
DS1	Mode 1	2.29	0.07	3.24
	Mode 2	3.34	0.09	2.56
DS2	Mode 1	2.11	0.08	3.93
	Mode 2	3.04	0.12	4.06
DS3	Mode 1	2.05	0.00	0.00
	Mode 2	2.93	0.00	0.00
DS4	Mode 1	1.97	0.05	2.74
	Mode 2	2.75	0.13	4.71

**Table 5.10:** Mean values, standard deviations and CoVs for experimental frequencies.

Damage state		Parameters values [ $\times 10^4$ ] [kN]			
		$a_1$	$a_2$	$a_3$	$a_4$
DS1	Mean	4.61	35.52	18.93	11.33
	S. deviation	1.55	8.16	3.18	1.63
	CoV [%]	33.61	22.96	16.80	14.38
DS2	Mean	4.73	33.08	17.94	6.51
	S. deviation	2.36	9.69	0.98	1.42
	CoV [%]	49.81	29.28	5.44	21.87
DS3	Mean	4.42	36.82	17.58	4.34
	S. deviation	0.16	1.19	0.09	0.28
	CoV [%]	3.73	3.23	0.49	6.34
DS4	Mean	4.81	34.04	12.75	4.37
	S. deviation	1.26	7.59	3.15	2.56
	CoV [%]	26.12	22.30	24.69	58.69

**Table 5.11:** Mean values, standard deviations and CoV values of parameters.

Damage state		Numerical frequencies [Hz]				MAC	
		Mean value	Standard deviation	CoV [%]	Error [%]	Mean value	CoV [%]
DS1	Mode 1	2.30	0.08	3.42	0.64	0.97	10.26
	Mode 2	3.37	0.09	2.79	0.72	0.98	6.75
DS2	Mode 1	2.15	0.08	3.79	1.67	0.94	23.47
	Mode 2	3.06	0.11	3.69	0.80	0.94	20.67
DS3	Mode 1	2.08	0.00	0.06	1.43	0.99	0.40
	Mode 2	2.97	0.01	0.20	1.43	0.99	0.23
DS4	Mode 1	1.99	0.05	3.41	1.44	0.97	4.94
	Mode 2	2.78	0.13	3.37	1.04	0.92	19.17

**Table 5.12:** Mean values, standard deviations, CoVs and errors for frequencies and MAC values.

Damage state	Parameters values [ $\times 10^4$ ] [kN]							
	$a_1$	$a_2$	$a_3$	$a_4$	$a_5$	$a_6$	$a_7$	$a_8$
DS1	315.38	188.86	244.83	145.03	4.06	37.25	17.33	10.03
DS2	315.38	173.78	191.04	139.63	4.06	37.25	17.33	5.70
DS3	315.38	124.69	188.62	139.63	4.06	35.61	17.33	4.19
DS4	315.36	124.69	188.62	139.63	4.06	35.61	13.83	2.92

**Table 5.13:** Values of parameters for 8 parameters case.

Damage state		Frequencies [Hz]			MAC
		Experimental	Numerical	Error [%]	
DS1	Mode 1	2.26	2.26	0.10	0.99
	Mode 2	3.37	3.37	0.07	0.99
DS2	Mode 1	2.14	2.13	-0.16	0.99
	Mode 2	3.07	3.08	0.35	0.99
DS3	Mode 1	2.06	2.06	-0.12	0.98
	Mode 2	2.96	2.98	0.56	0.99



DS4	Mode 1	1.96	1.96	-0.10	0.99
	Mode 2	2.72	2.74	0.75	0.99

**Table 5.14:** Comparison between experimental and numerical frequencies and MAC values for 8 parameters case.

Parameter	Damage parameters [%]			
	DS1	DS2	DS3	DS4
E <sub>1</sub>	0.0	0.0	0.0	0.0
E <sub>2</sub>	0.0	8.0	34.0	34.0
E <sub>3</sub>	0.0	22.0	23.0	21.2
E <sub>4</sub>	0.0	3.7	3.7	3.7
E <sub>5</sub>	0.0	0.0	0.0	0.0
E <sub>6</sub>	0.0	0.0	4.4	4.4
E <sub>7</sub>	0.0	0.0	0.0	20.2
E <sub>8</sub>	0.0	43.1	58.2	70.9

**Table 5.15:** Damage parameters for 8 parameters case.

Damage state		Frequencies [Hz]			MAC
		Experimental	Numerical	Error [%]	
DS1	Mode 1	2.26	2.26	0.10	0.99
	Mode 2	3.37	3.37	0.07	0.99
DS2	Mode 1	2.14	2.14	-0.09	0.99
	Mode 2	3.07	3.10	0.81	0.99
DS3	Mode 1	2.06	2.06	-0.02	0.98
	Mode 2	2.96	3.00	1.31	0.99
DS4	Mode 1	1.96	1.96	-0.01	0.99
	Mode 2	2.72	2.75	1.34	0.99

**Table 5.16:** Comparison between experimental and numerical frequencies and MAC values for 8 parameters case without updating the ground storey parameters for DS2, DS3 and DS4.



## Figures of Chapter 5

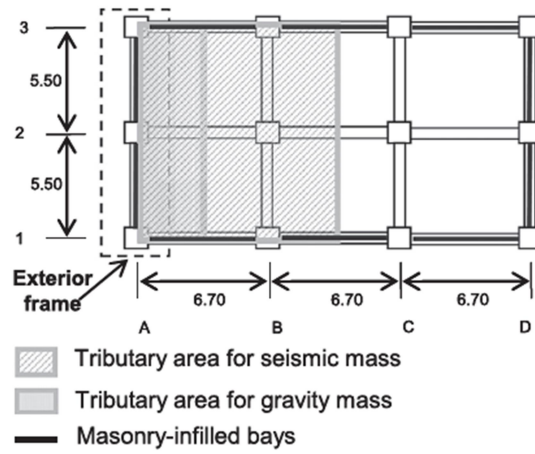


Figure 5.1: Plan view of the prototype structure (from Moaveni et al.,2013).

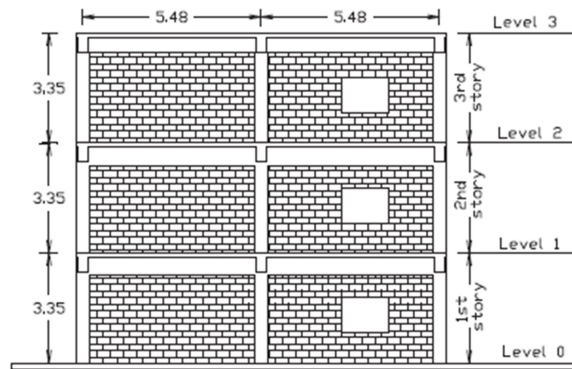


Figure 5.2: Elevation view of the prototype structure (from Moaveni et al.,2013).

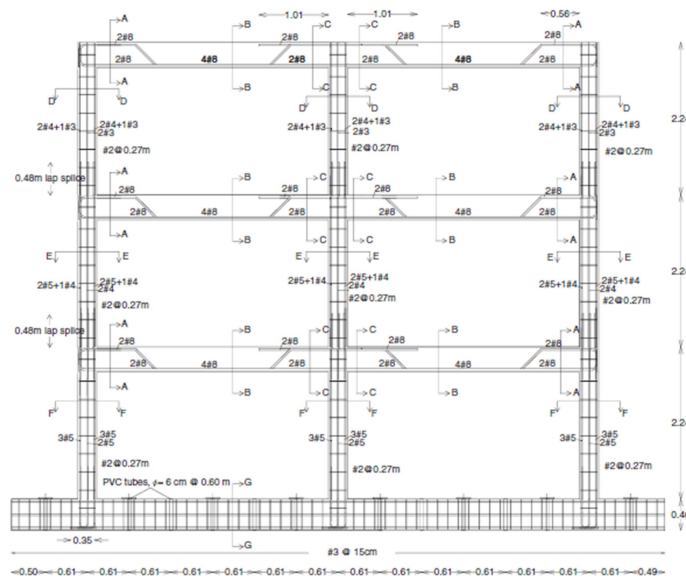
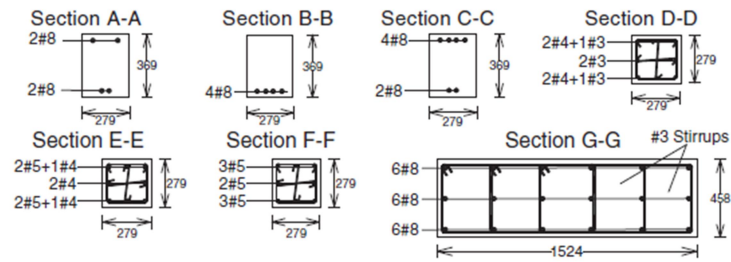


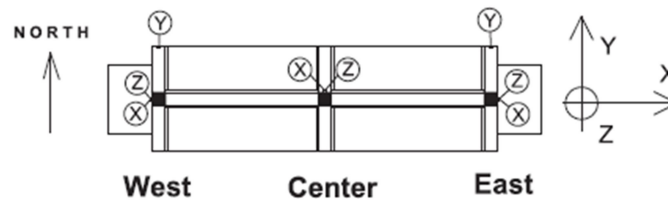
Figure 5.3: Design of the tree storey specimen (dimensions in m) (from Stavridis et al., 2011).



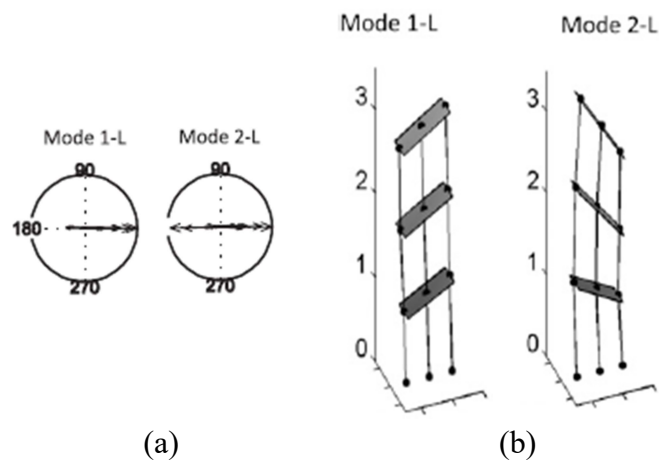
**Figure 5.4:** Cross sections of R.C. members (dimensions in mm) (from Stavridis et al., 2011).



**Figure 5.5:** Front view of the specimen (from Moaveni et al., 2013).



**Figure 5.6:** Location of accelerometers at each floor level (from Moaveni et al., 2013).



**Figure 5.7:** Modes for damage state DS0: (a) Polar plot representation for complex mode shapes; (b) Vibration mode shapes (from Moaveni et al., 2013).

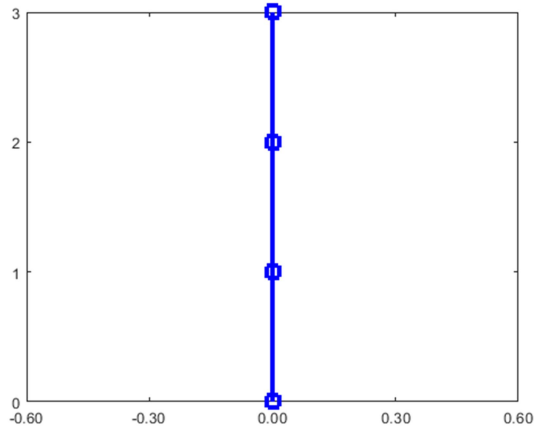


Figure 5.8: Stick model representation.

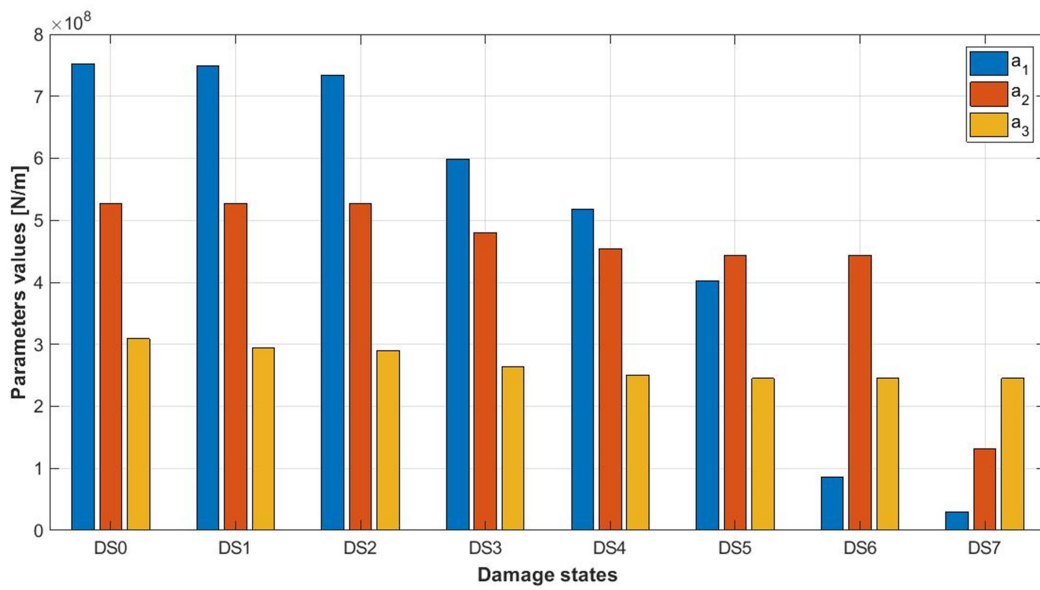


Figure 5.9: Parameters values for all damage states.

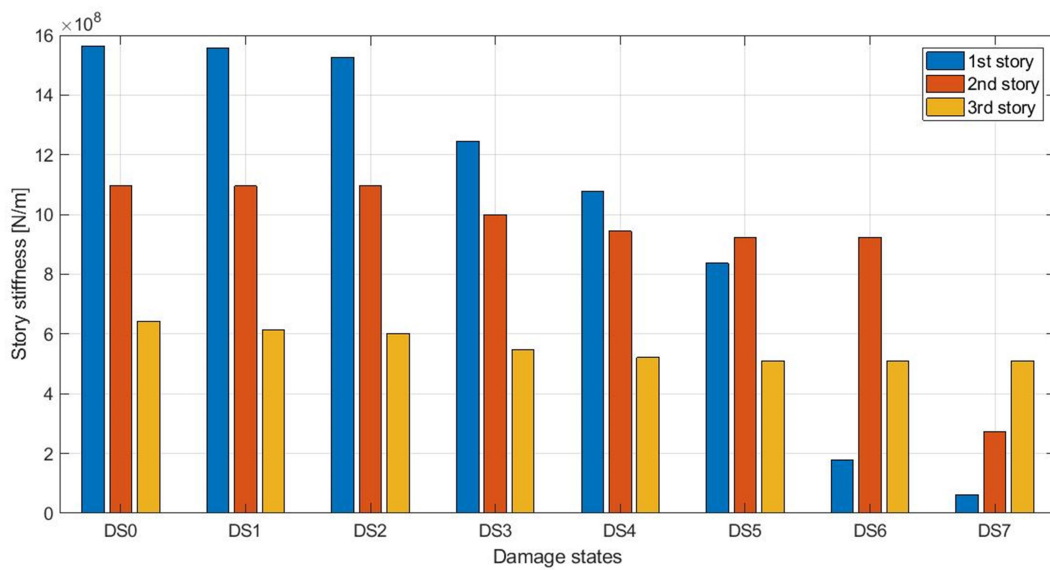
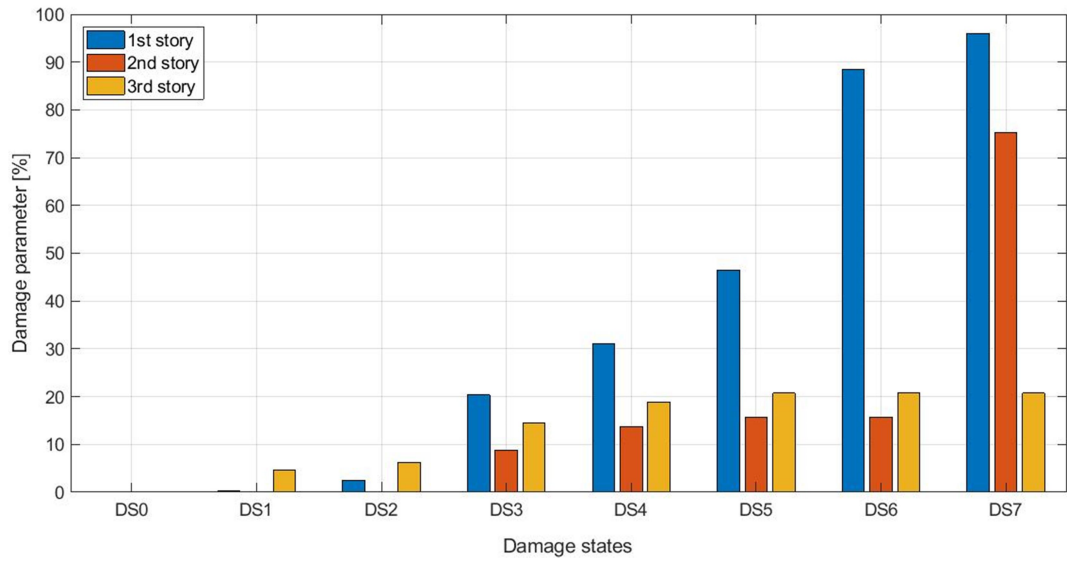
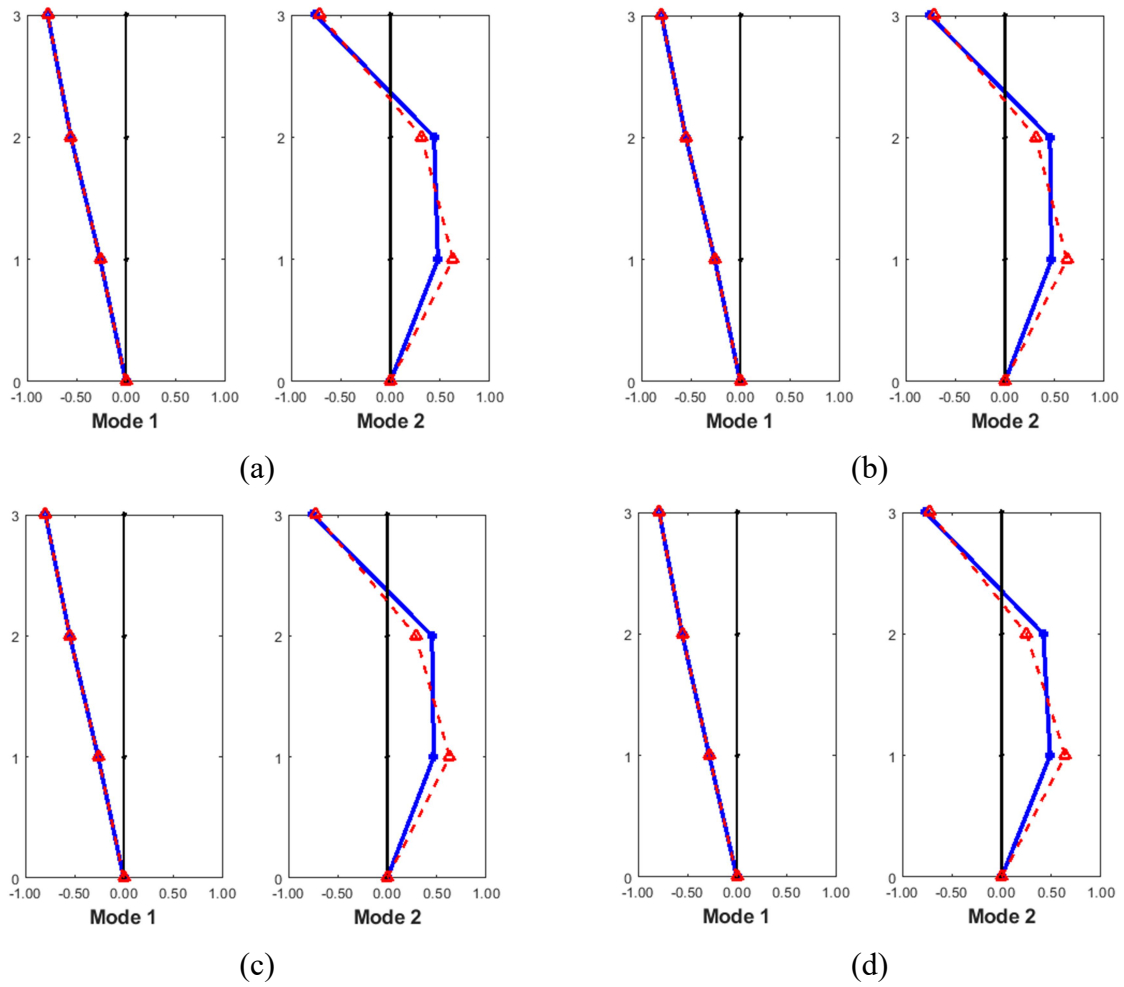
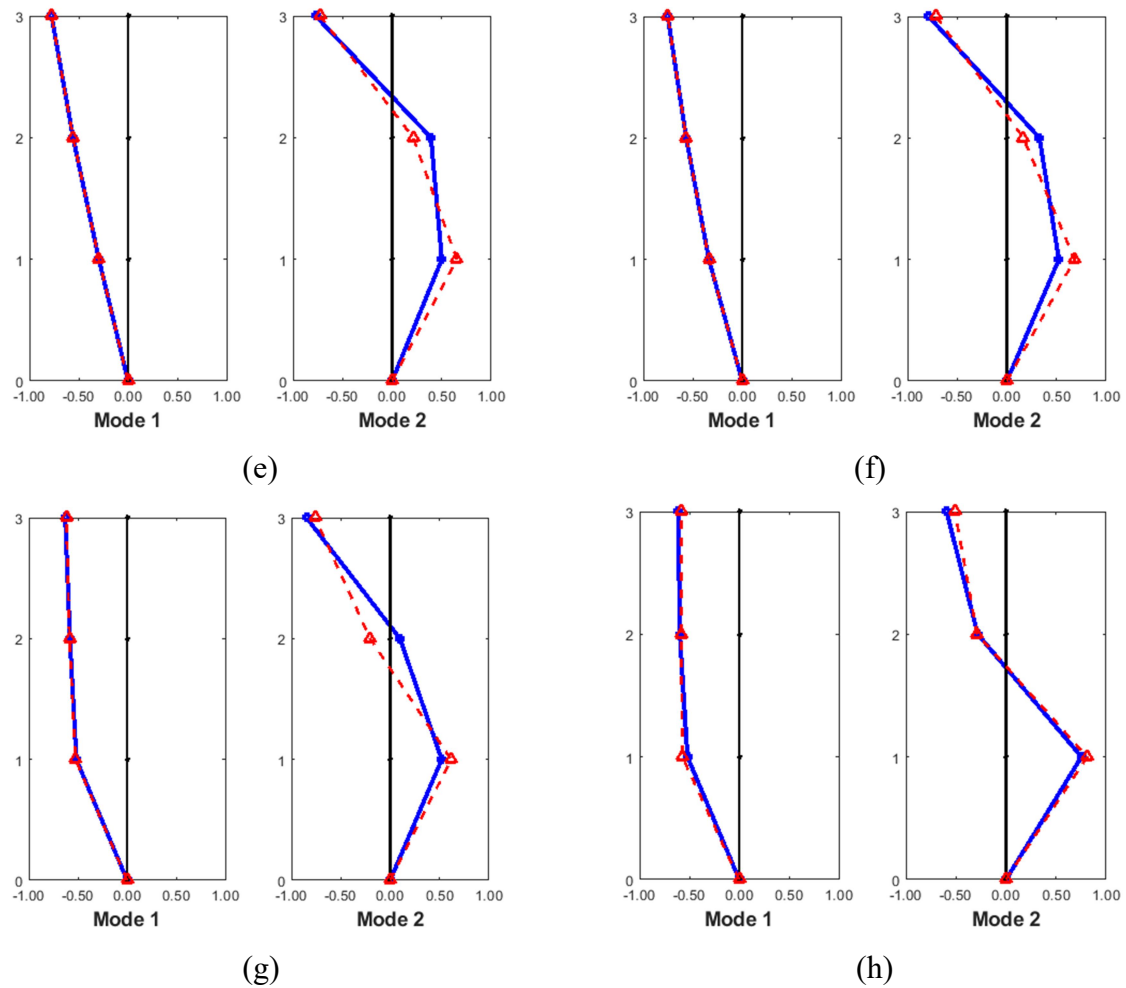


Figure 5.10: Story stiffness for all damage states.

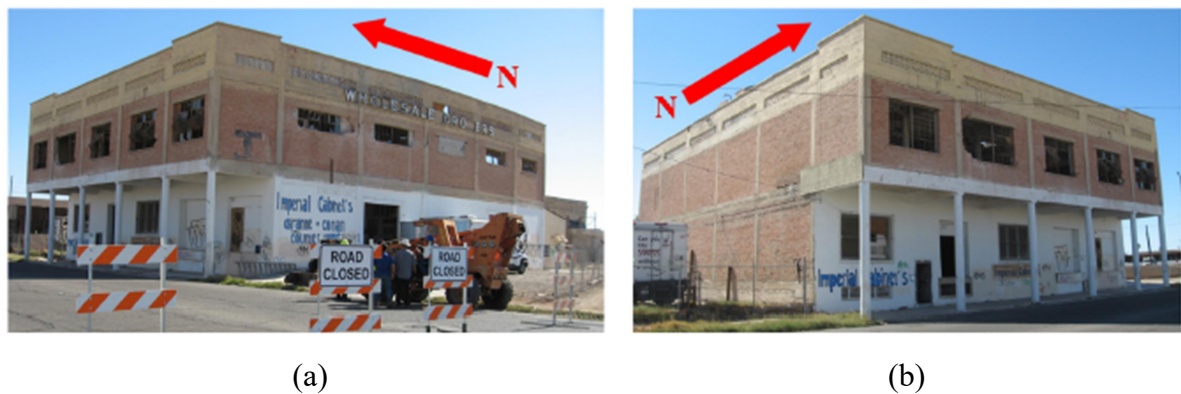


**Figure 5.11:** Damage parameters for all damage states.





**Figure 5.12:** Comparison between experimental and numerical mode shapes (the red lines are the experimental mode shapes, the blue lines are the numerical ones): (a) DS0; (b) DS1; (c) DS2; (d) DS3; (e) DS4; (f) DS5; (g) DS6; (h) DS7.



**Figure 5.13:** Views of the structure under study: (a) north-west view; (b) north-east view (Song et al., 2017).

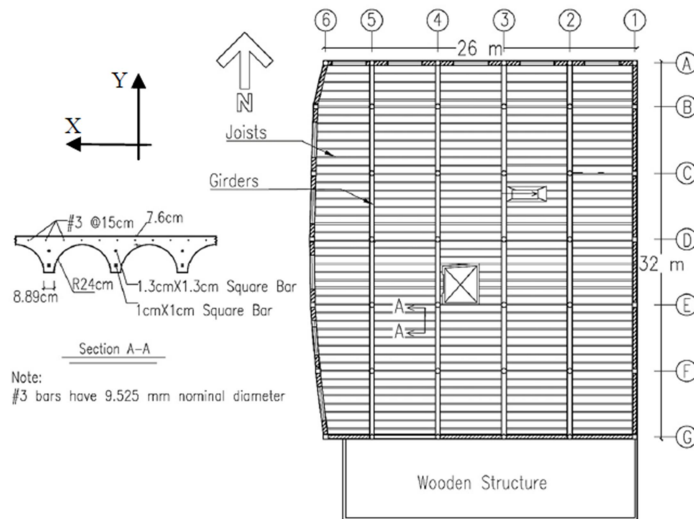
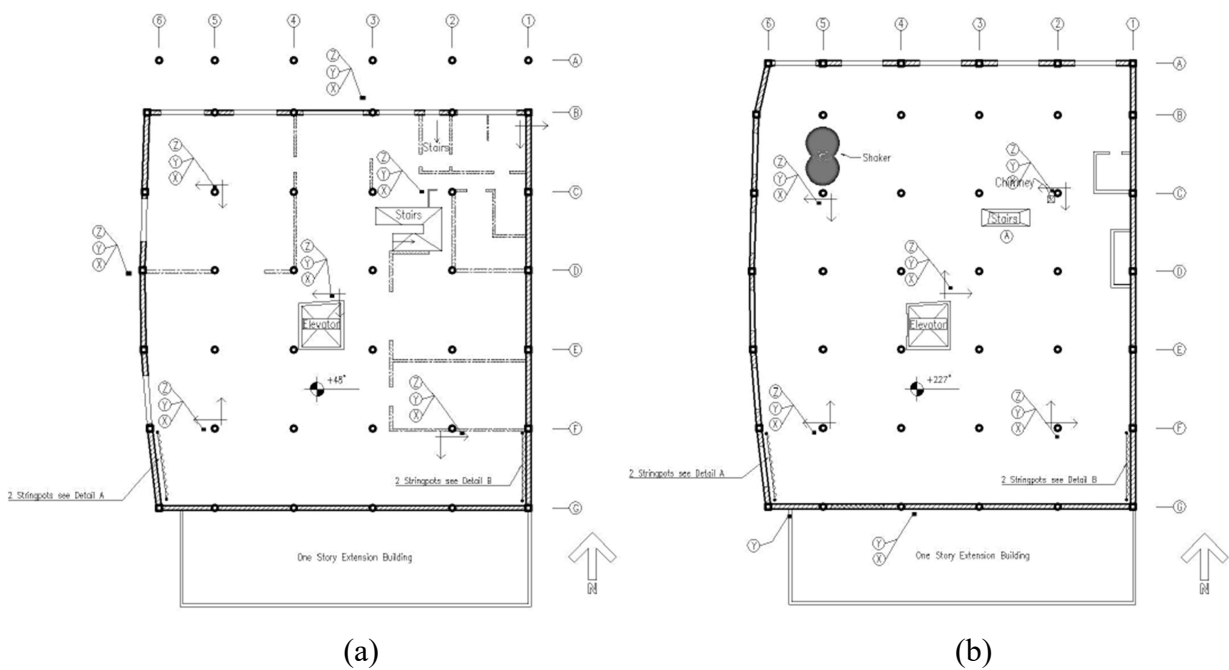


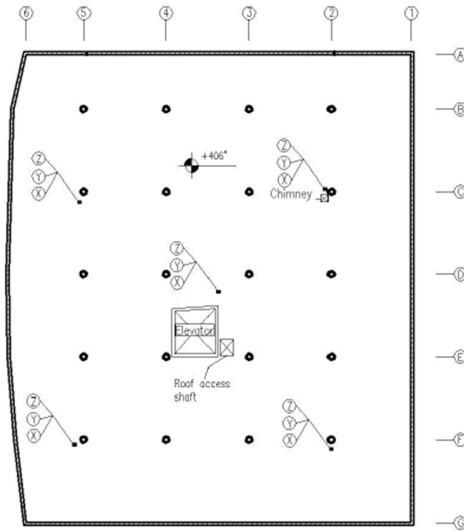
Figure 5.14: First floor plan view with joists details (Yousefianmoghadam et al., 2015).



Figure 5.15: Walls demolition sequence and resulted damage state (Yousefianmoghadam et al., 2015).







(c)

Figure 5.16: Structure instrumentations: (a) ground floor; (b) First floor; (c) Roof level.

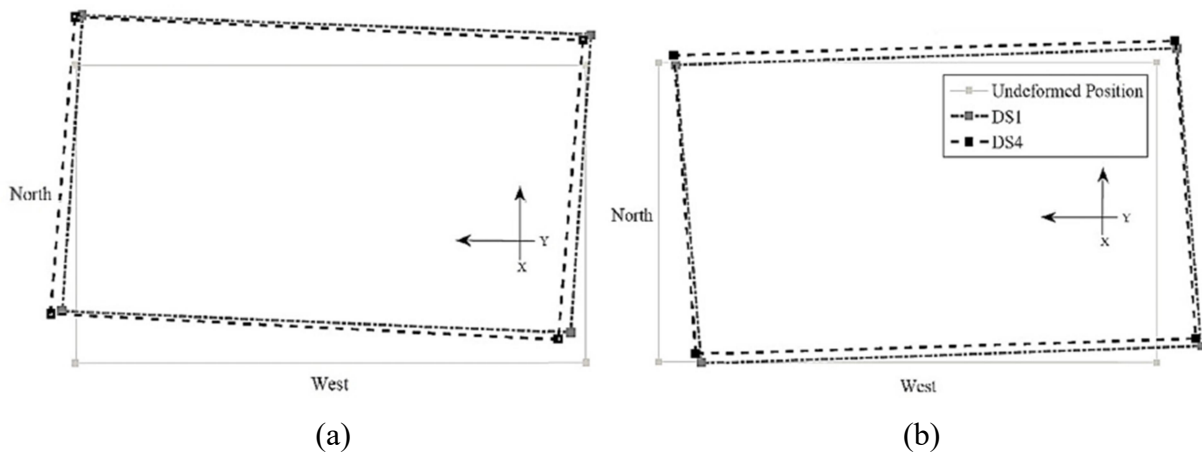
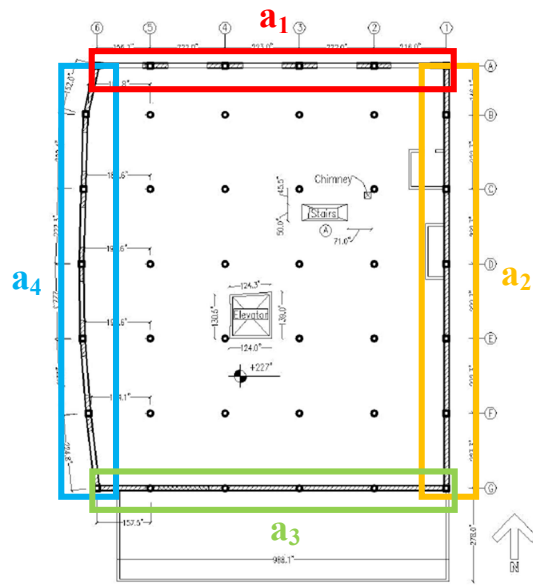
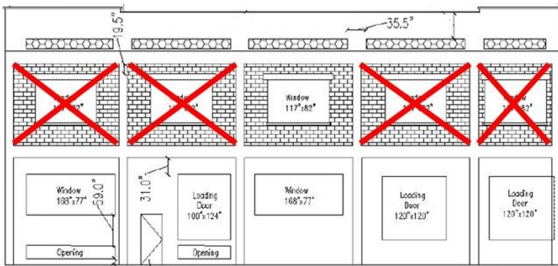


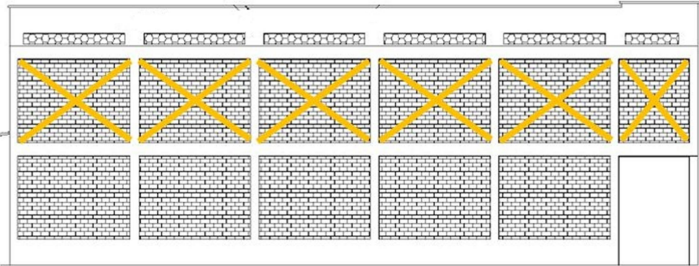
Figure 5.17: Mode shapes for roof level: (a) Mode 1; (b) Mode 2 (Yousefianmoghadam et al., 2015).



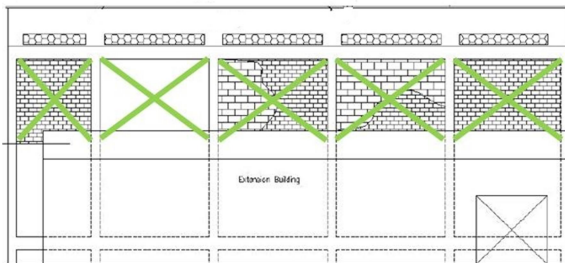
(a)



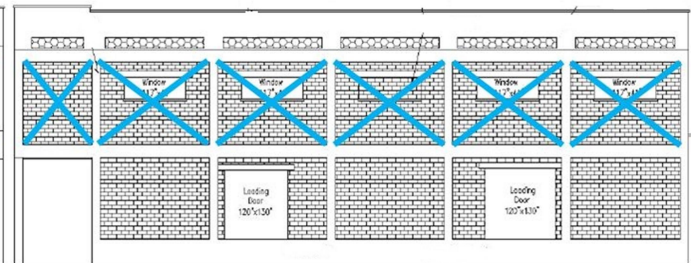
(b)



(c)

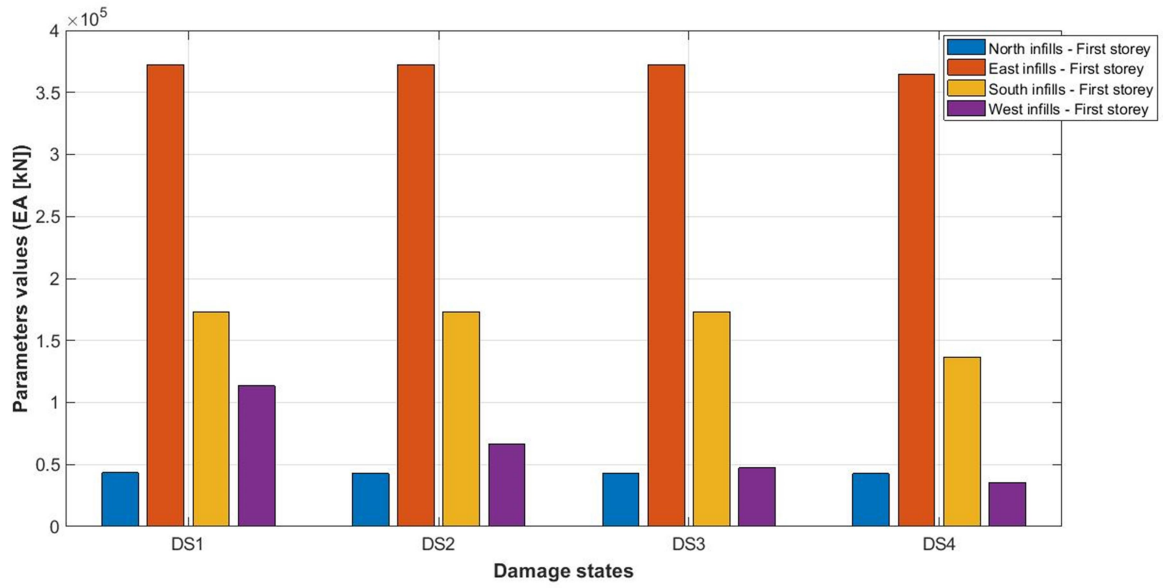


(d)

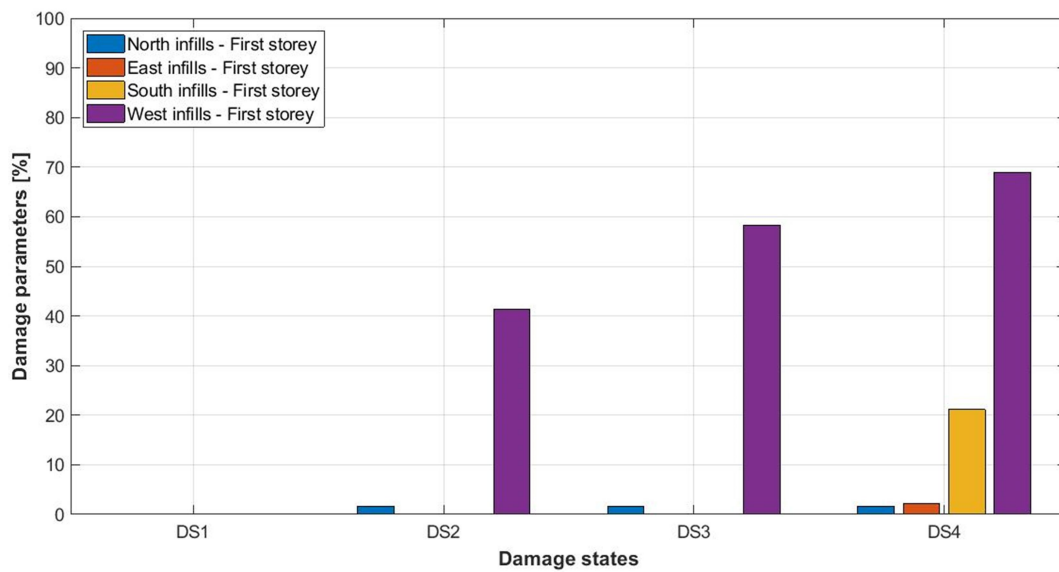


(e)

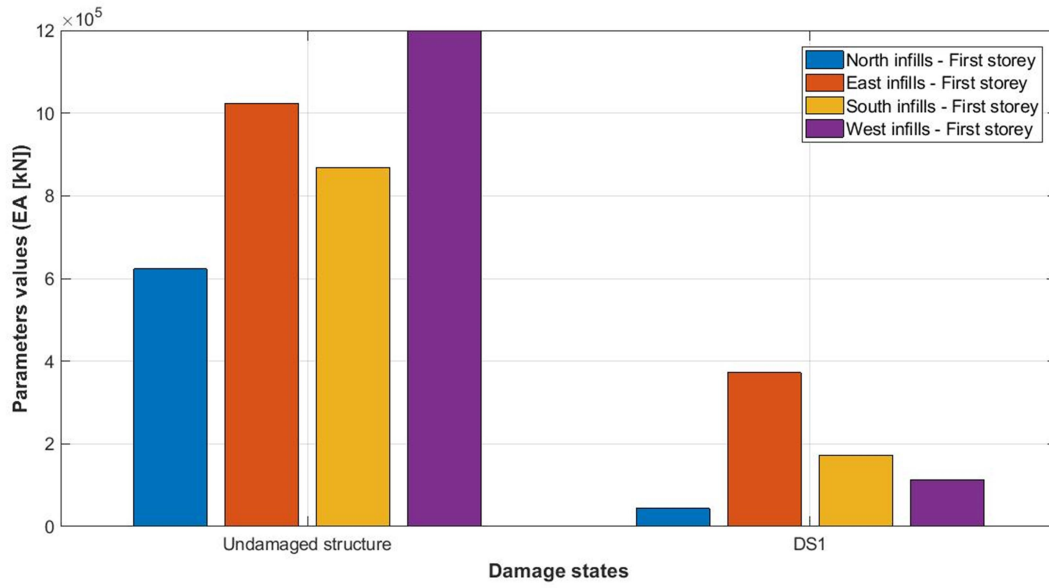
**Figure 5.18:** Parameters definition for 4 parameters case: (a) Plan view of the first storey; (b) North elevation, 1<sup>st</sup> parameter (in red); (c) East elevation, 2<sup>nd</sup> parameter (in orange); (d) South elevation, 3<sup>rd</sup> parameter (in green); (e) West elevation, 4<sup>th</sup> parameter (in blue).



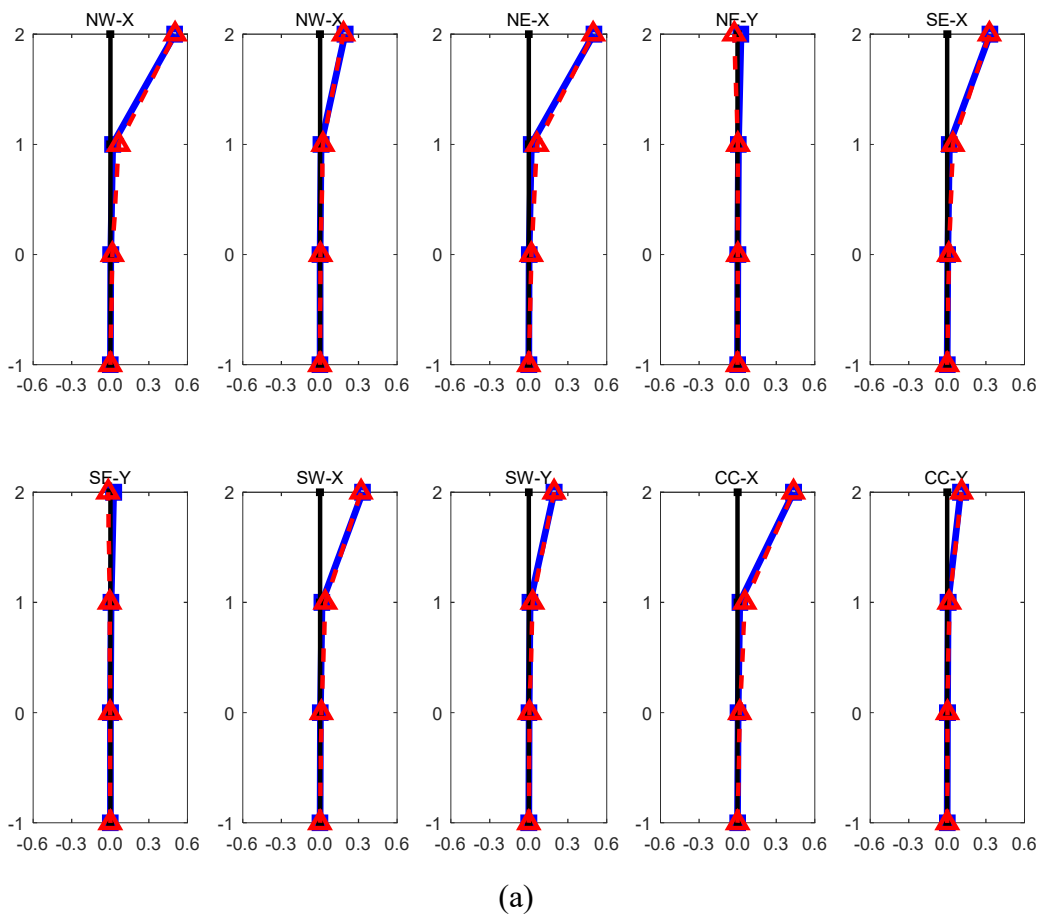
**Figure 5.19:** Parameters values for all the damage states for 4 parameters case.

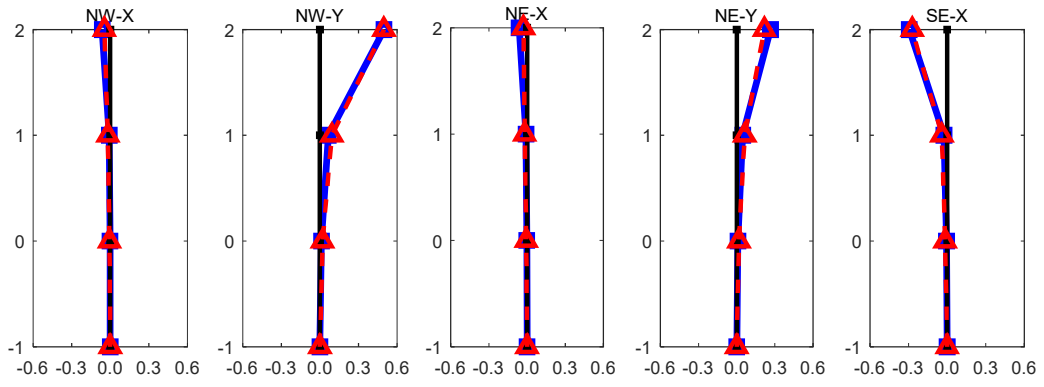


**Figure 5.20:** Damage parameters for all the damage states for 4 parameters case.

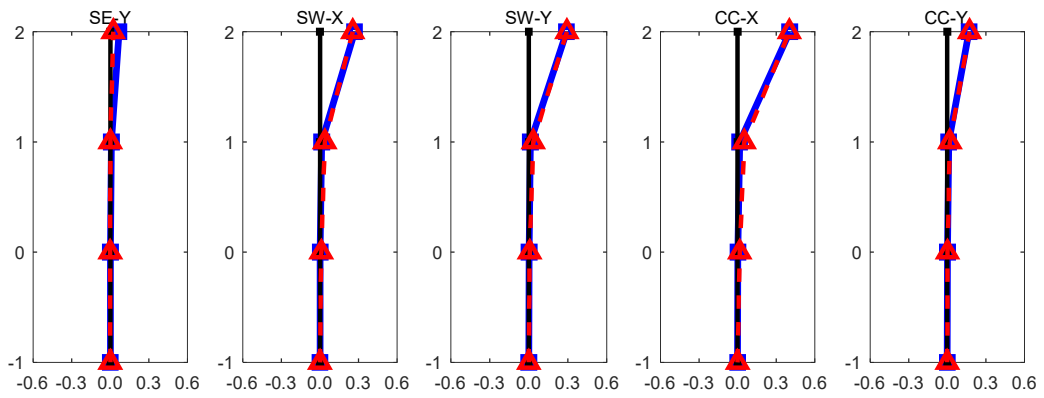
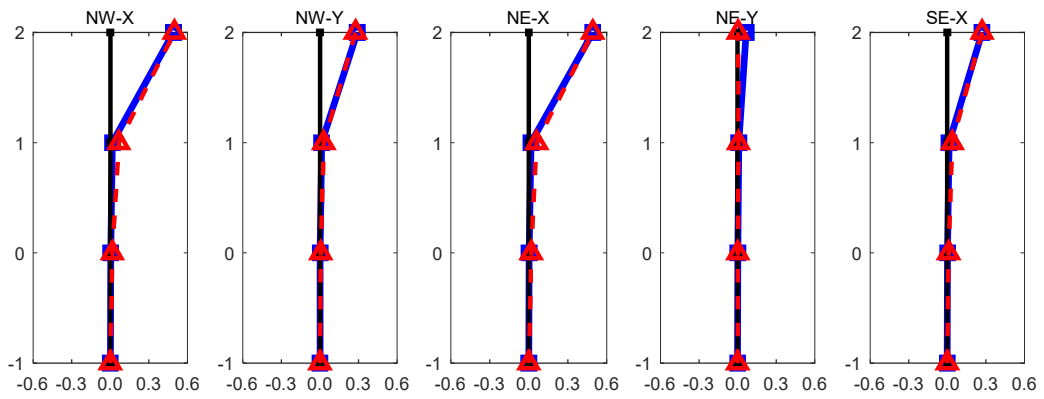
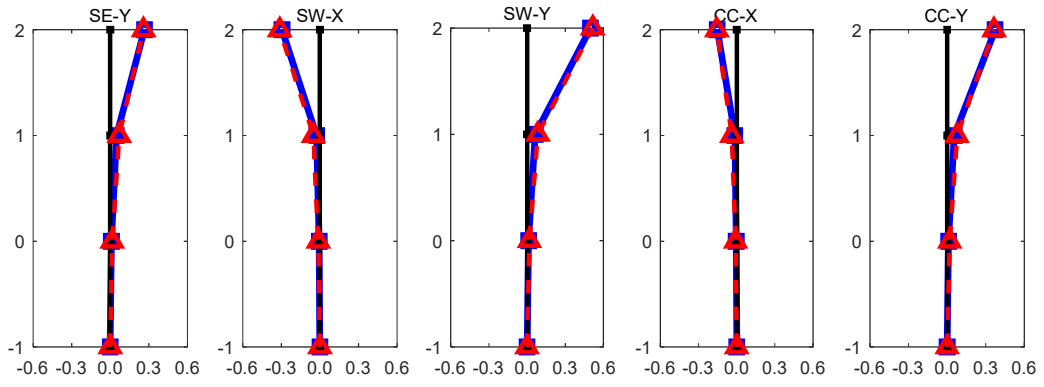


**Figure 5.21:** Comparison between parameters for undamaged structure and DS1 for 4 parameters case.

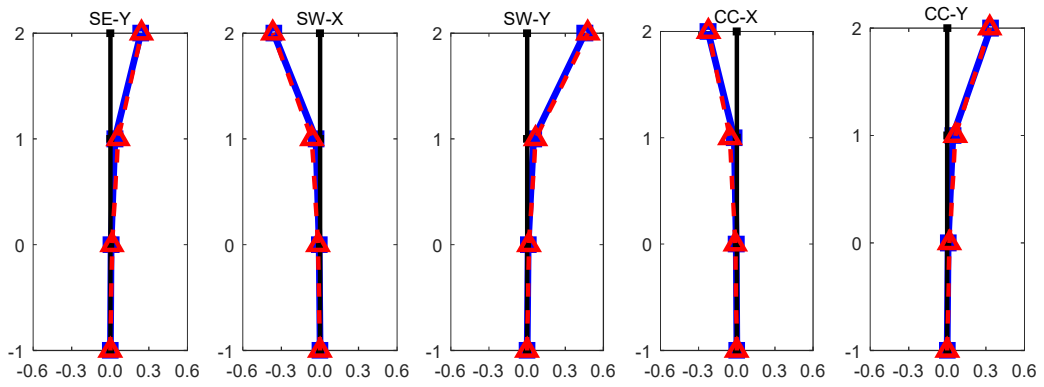
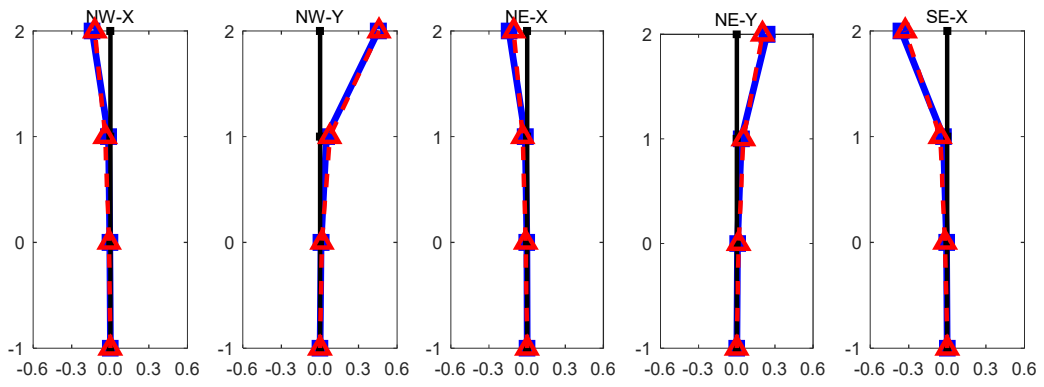




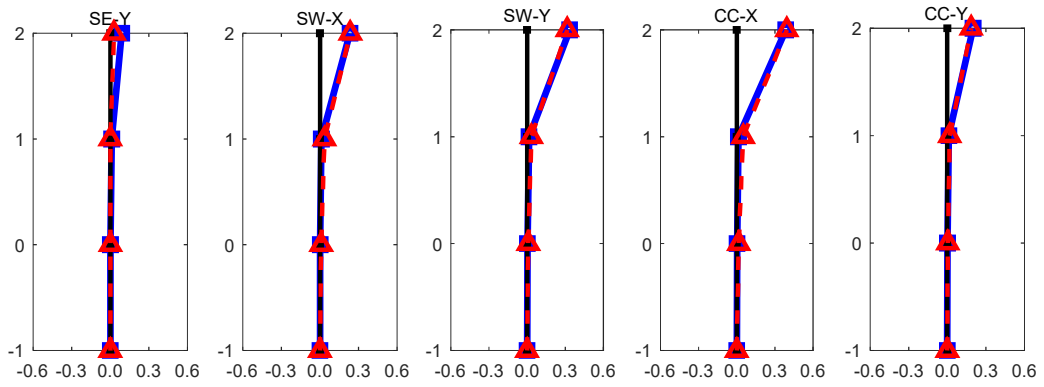
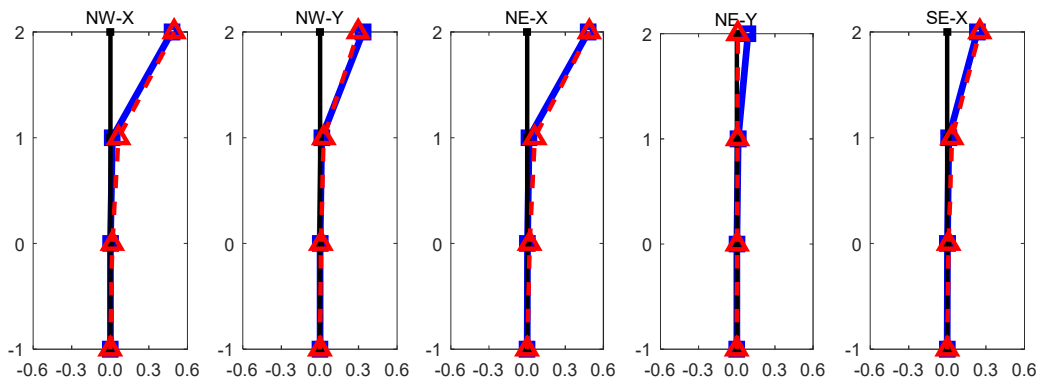
(b)



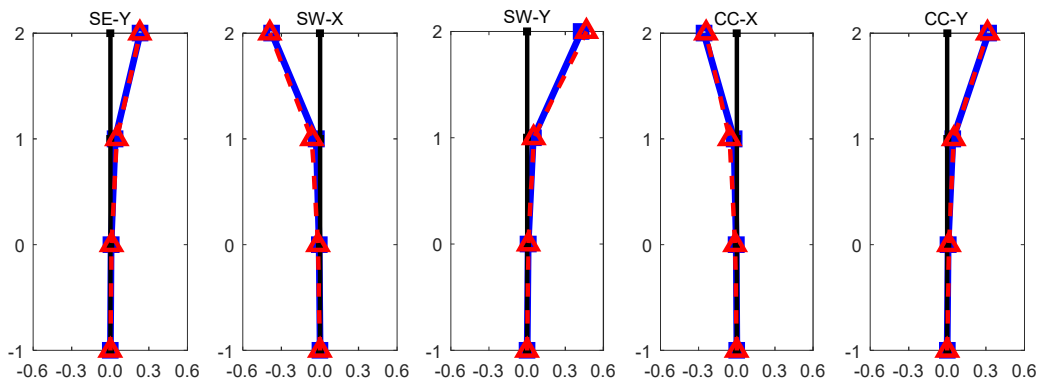
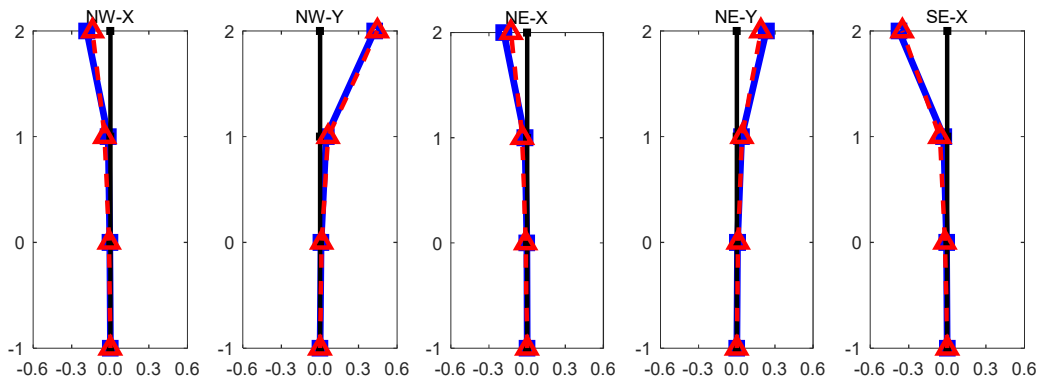
(c)



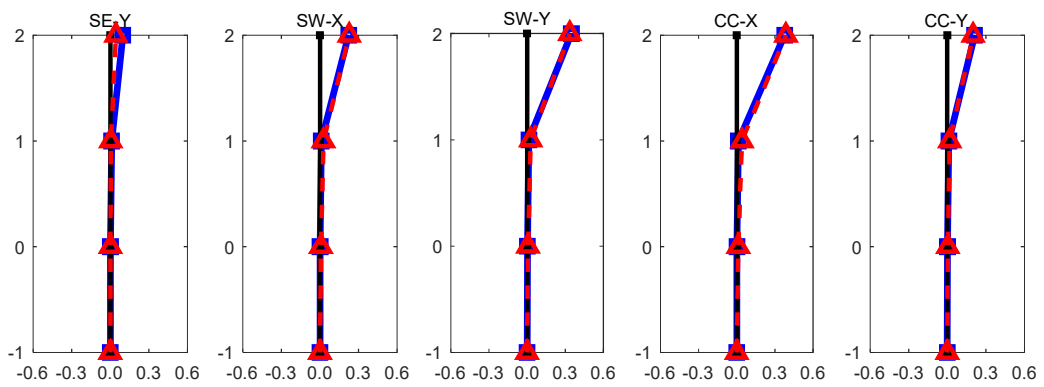
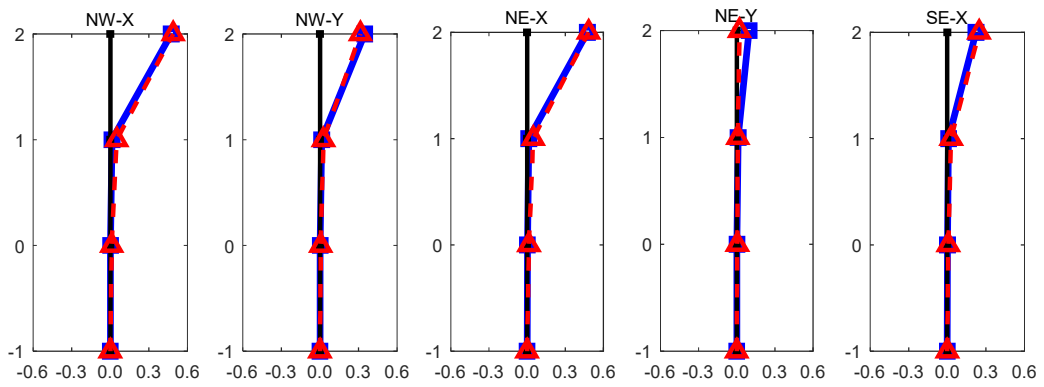
(d)



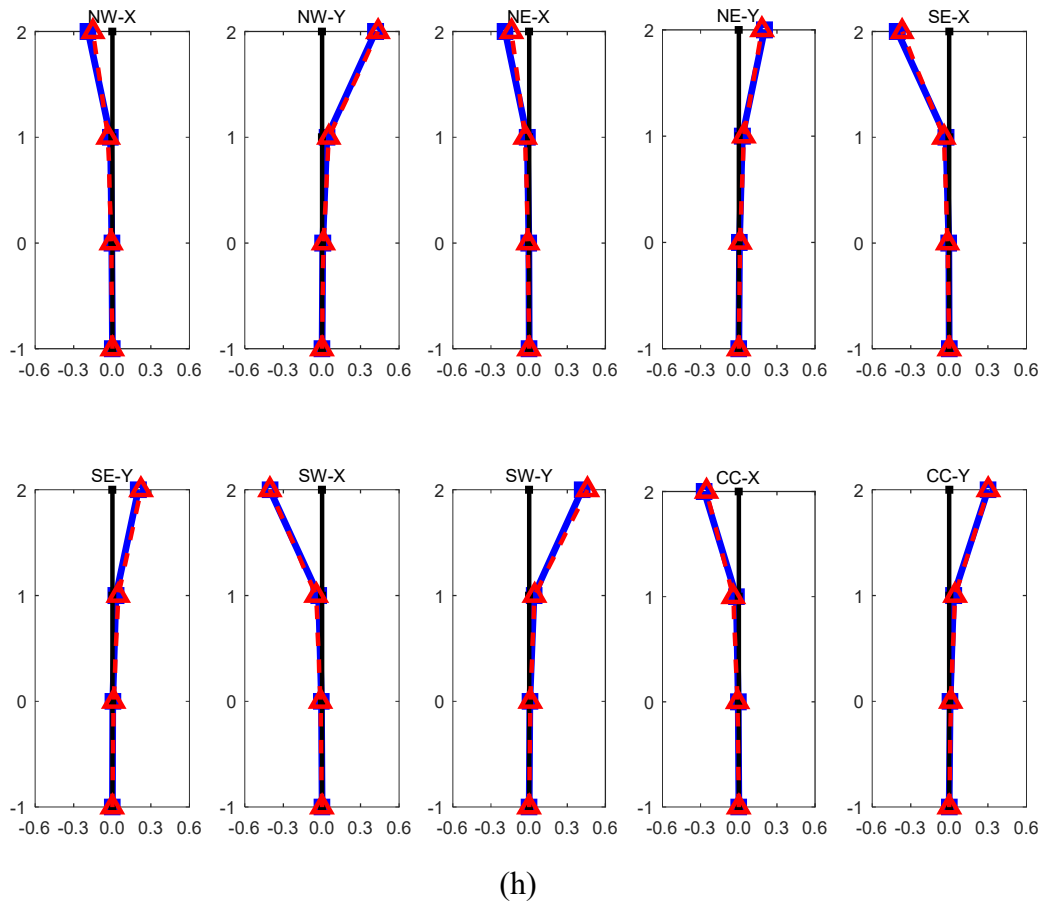
(e)



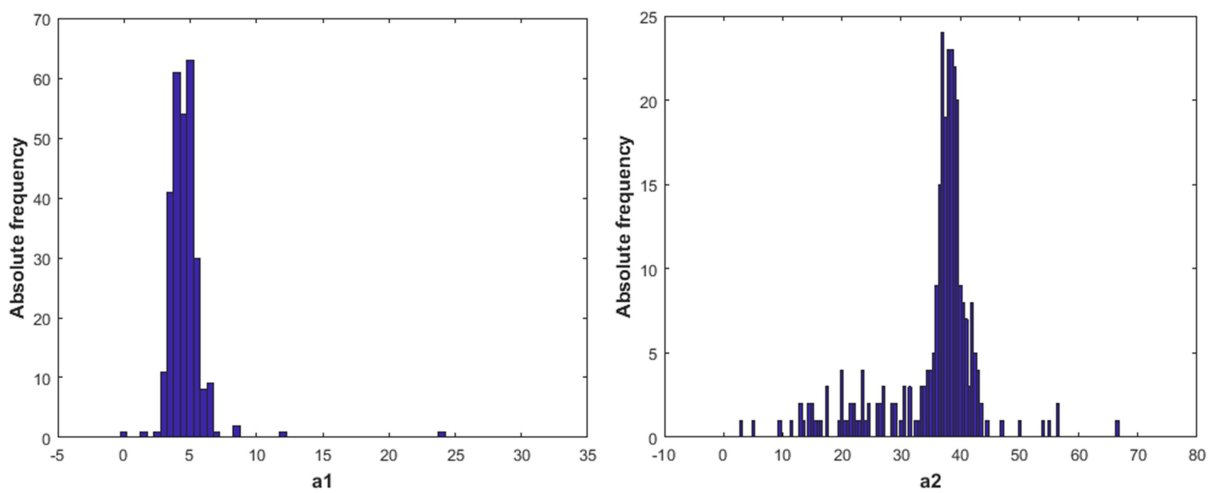
(f)



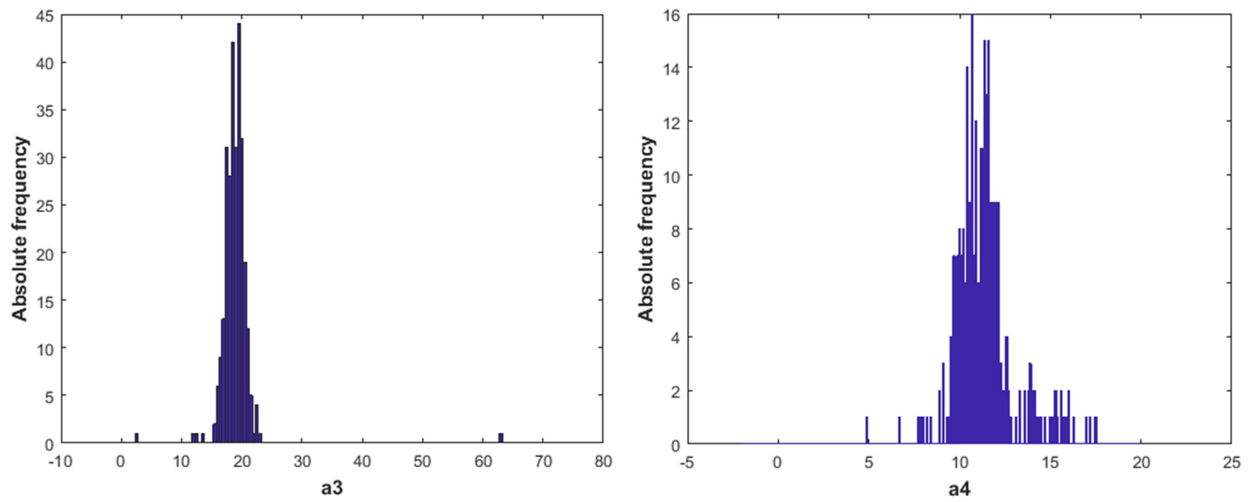
(g)



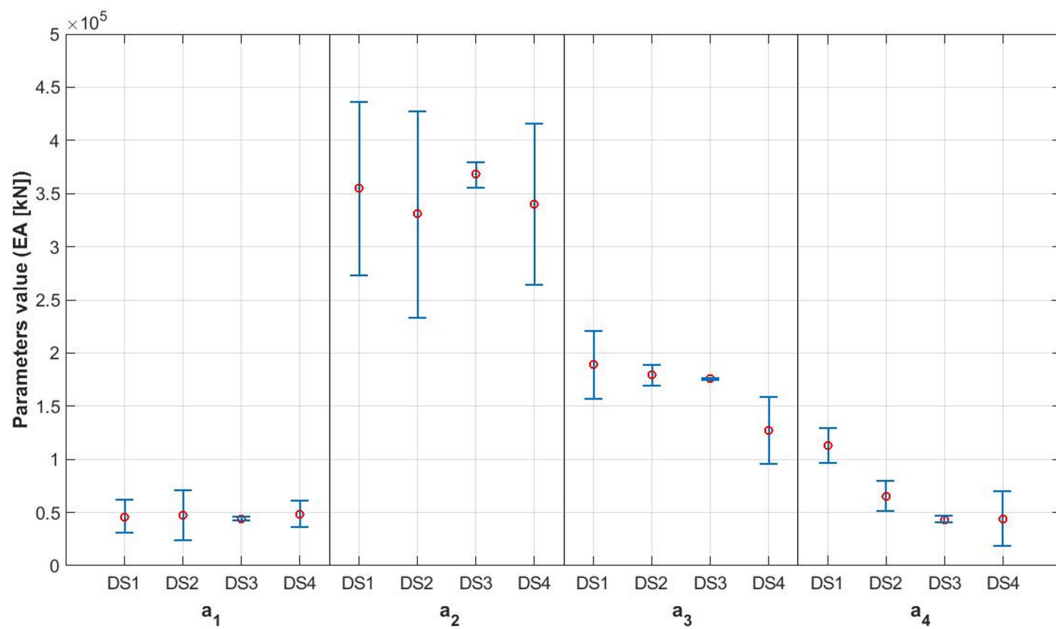
**Figure 5.22:** Comparison between experimental and numerical mode shapes (the red lines are the experimental mode shapes, the blue lines are the numerical ones): (a) DS1 - Mode 1; (b) DS1 - Mode 2; (c) DS2 - Mode 1; (d) DS2 - Mode 2; (e) DS3 - Mode 1; (f) DS3 - Mode 2; (g) DS4 - Mode 1; (h) DS4 - Mode 2.





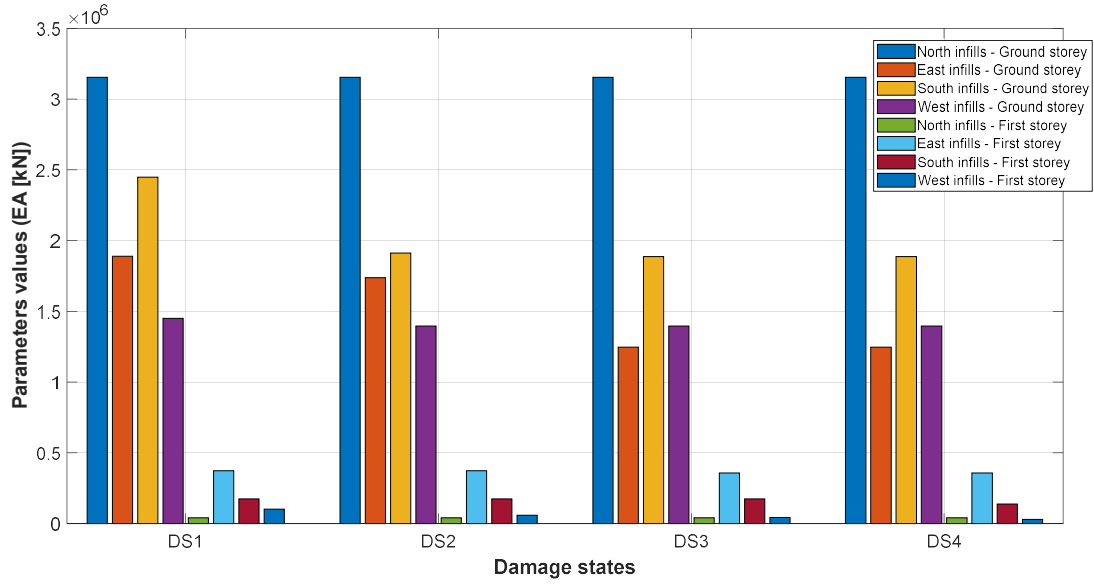


**Figure 5.23:** Frequencies histograms of parameters  $a_1$ ,  $a_2$ ,  $a_3$  and  $a_4$  for damage state DS1.

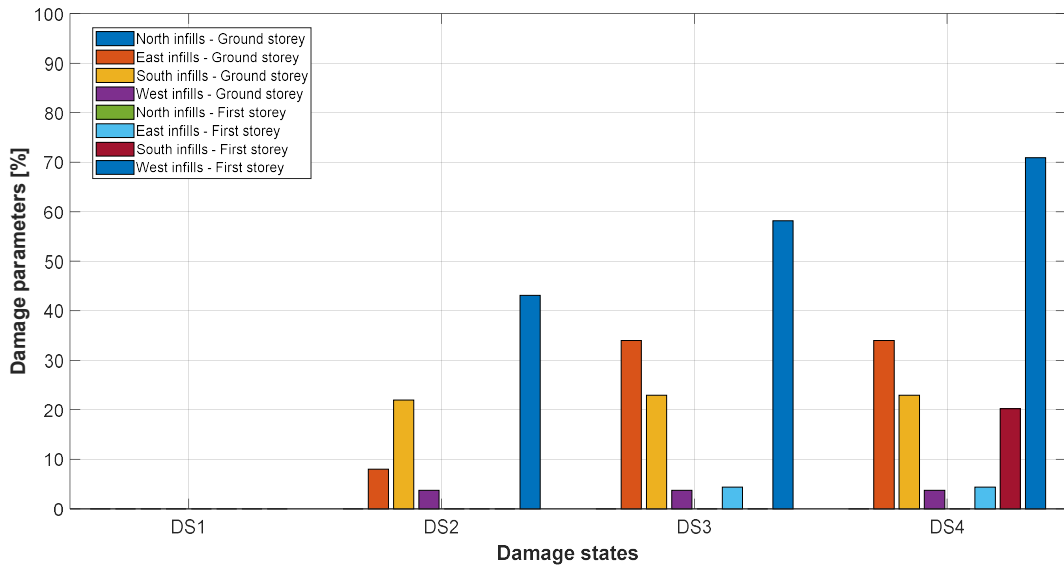


**Figure 5.24:** Mean values of parameters along with standard deviations for all the damage states.

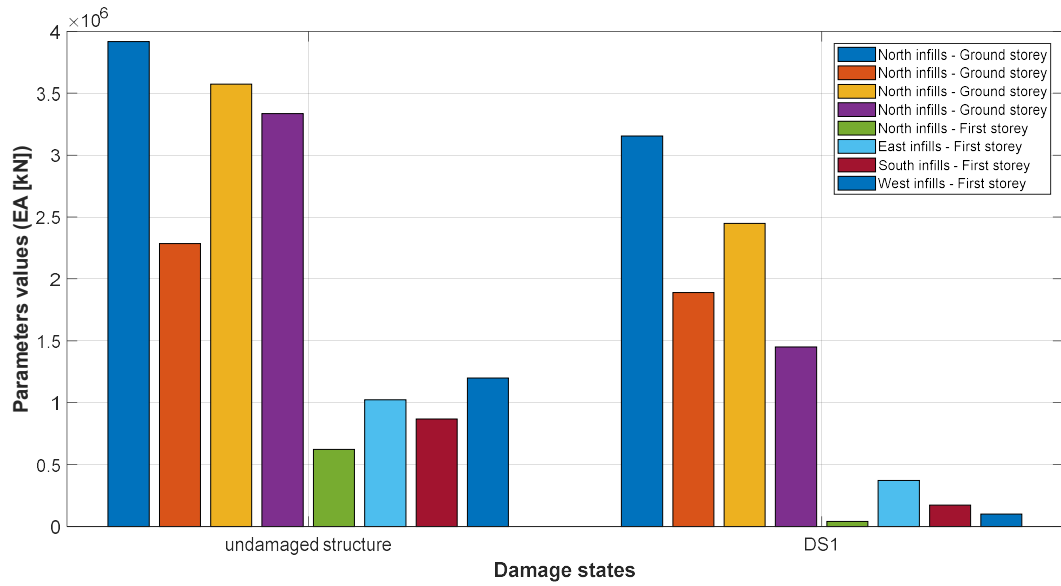




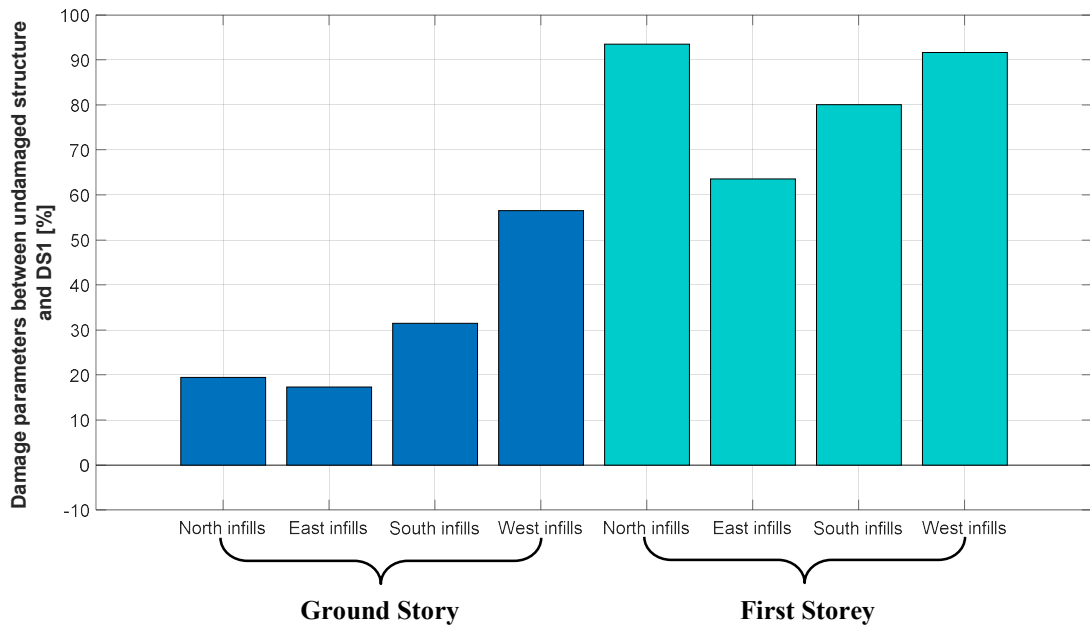
**Figure 5.26:** Parameters values for all the damage states for 8 parameters case.



**Figure 5.27:** Damage parameters for all the damage states for 8 parameters case.



**Figure 5.28:** Comparison between parameters for undamaged structure and DS1 for 8 parameters case.



**Figure 5.29:** Damage parameters between undamaged structure and DS1 for 8 parameters case.

## CHAPTER 6

### GENERALIZATION OF THE PROCEDURE TO NON-LINEAR PARAMETERS IN MODEL UPDATING

#### 6.1 Introduction

In this Chapter the generalization of the procedure developed in Chapter 2 will be done in order to take into account non-linear parameters. In the first part, the case in which the matrices decomposition can be done (and therefore the parameters are unrelated each other) will be analyzed. In the second part the general case in which the parameters are connected each other will be also treated. In the last part, a generalization taking into account the viscous damping for classically damped system will be studied.

#### 6.2 Unrelated non-linear parameters

In this first case, unrelated non-linear parameters are taken into account. Since the parameters are unrelated, the matrices decomposition can be done also in this case. The general system to be solved, from Eq. 2.65, can be modified in the following way:

$$\left\{ \begin{array}{l}
 p_1 \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (f_s(a_s) \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N f_s(a_s) \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_1 = \underline{\underline{\boldsymbol{v}}}_1 \\
 \vdots \\
 p_i \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (f_s(a_s) \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N f_s(a_s) \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_i = \underline{\underline{\boldsymbol{v}}}_i \\
 \vdots \\
 p_n \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (f_s(a_s) \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N f_s(a_s) \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_n = \underline{\underline{\boldsymbol{v}}}_n \\
 \hline
 p_1 \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (f_s(a_s) \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N f_s(a_s) \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_1 = v_1 \\
 \vdots \\
 p_i \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (f_s(a_s) \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N f_s(a_s) \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_i = v_i \\
 \vdots \\
 p_n \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (f_s(a_s) \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N f_s(a_s) \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_n = v_n
 \end{array} \right. \quad (6.1)$$

in which  $f_s(a_s)$  is a non-linear function of parameter  $a_s$  (for  $s = 1, 2, \dots, N$ ).

Defining the auxiliary variables in the following way:

$$t_s = f_s(a_s) \quad (6.2)$$

for  $s = 1, 2, \dots, N$ , the system can be rewritten as follows:

$$\left\{ \begin{array}{l} p_1 \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_1 = \underline{\underline{\boldsymbol{\nu}}}_1 \\ \vdots \\ p_i \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_i = \underline{\underline{\boldsymbol{\nu}}}_i \\ \vdots \\ p_n \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_n = \underline{\underline{\boldsymbol{\nu}}}_n \end{array} \right. \quad (6.3)$$

which is the transformation of Eq. 2.4, solvable with closed formulation and:

$$\left\{ \begin{array}{l} p_1 \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_1 = \underline{\underline{\boldsymbol{\nu}}}_1 \\ \vdots \\ p_i \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_i = \underline{\underline{\boldsymbol{\nu}}}_i \\ \vdots \\ p_n \cdot \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] \cdot \underline{\underline{\boldsymbol{\varphi}}}_n = \underline{\underline{\boldsymbol{\nu}}}_n \end{array} \right. \quad (6.4)$$

$$\left\{ \begin{array}{l} p_1 \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_1^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_1^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_1 = \nu_1 \\ \vdots \\ p_i \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_i^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_i^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_i = \nu_i \\ \vdots \\ p_n \cdot \det \left[ \underline{\underline{\mathbf{K}}}_0 + \sum_{s=1}^q (t_s \cdot \underline{\underline{\mathbf{K}}}_{r,s}) - \omega_n^2 \cdot \underline{\underline{\mathbf{M}}}_0 - \omega_n^2 \cdot \left( \sum_{s=q+1}^N t_s \cdot \underline{\underline{\mathbf{M}}}_s \right) \right] / w_n = \nu_n \end{array} \right.$$

which is the transformation of Eq. 6.1. Therefore all the results from Chapter 2 and 3 for the original system of Eq. 2.65 can be applied to this case, having  $t_s$  as unknown instead of  $a_s$ .

In this way all the  $t_s$  and  $\sigma_{t,s}$  values can be achieved.

After that, in order to find the  $a_s$  parameters values, the following relation can be used:

$$a_s = f_s^{-1}(t_s) \quad (6.5)$$

for  $s = 1, 2, \dots, N$ ; which gives unique solution if and only if  $f_s$  is a bi-univocal function. Otherwise criteria must be defined in order to choose the right solution of Eq. 6.5. The solution of Eq. 6.5 can be performed, in some cases, in iterative way depending on the definition of the function  $f_s$ .

In order to find the standard deviations of parameters  $a_s$ , the following criteria can be used (Lanconelli, 1998):

$$t_s \pm \sigma_{t,s} = f_s(a_s \pm \sigma_{a,s}) \quad (6.6)$$

$$t_s \pm \sigma_{t,s} \cong f_s(a_s) \pm \frac{df_s}{da_s}(a_s) \cdot \sigma_{a,s} \quad (6.7)$$

$$\sigma_{t,s} \cong \frac{df_s}{da_s}(a_s) \cdot \sigma_{a,s} \quad (6.8)$$

$$\sigma_{a,s} \cong \frac{\sigma_{t,s}}{\frac{df_s}{da_s}(a_s)} \quad (6.9)$$

for  $s = 1, 2, \dots, N$ ; in which  $a_s$  is the parameter value computed using Eq. 6.5. Eq. 6.9 gives, if the higher order terms are not so important, a good estimation of the standard deviations  $\sigma_{a,s}$  of parameters  $a_s$ . Better approximations can be achieved solving the following equation (Lanconelli, 1998):

$$\sum_{l=1}^c \frac{1}{l!} \cdot \frac{d^l f_s}{da_s^l}(a_s) \cdot \sigma_{a,s}^l - \sigma_{t,s} \cong 0 \quad (6.10)$$

for  $s = 1, 2, \dots, N$ ; in which  $c$  is the order of the Taylor's polynomial taken into account in the procedure. Eq. 6.10 can be solved in numerical way (through Trust-Region algorithm, having as unknowns  $\sigma_{a,s}$ ) starting from the estimation done in Eq. 6.9.

After that, the CoVs of parameters can be computed using Eq. 2.106.

The flowcharts for two steps algorithm, two steps algorithm with uncertainties evaluation and complete statistical analysis are depicted in Figures 6.1, 6.2 and 6.3 respectively.

## 6.3 Related non-linear parameters

### 6.3.1 Solution of the system

If the parameters are related each other, and therefore the stiffness and mass matrices decomposition cannot be performed, the entire system of equations can be written as follows:

$$\left\{ \begin{array}{l} p_1 \cdot [\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_1^2 \cdot \underline{\mathbf{M}}_0 - \omega_1^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] \cdot \underline{\boldsymbol{\varphi}}_1 = \underline{\boldsymbol{v}}_1 \\ \vdots \\ p_i \cdot [\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_i^2 \cdot \underline{\mathbf{M}}_0 - \omega_i^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] \cdot \underline{\boldsymbol{\varphi}}_i = \underline{\boldsymbol{v}}_i \\ \vdots \\ p_n \cdot [\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_n^2 \cdot \underline{\mathbf{M}}_0 - \omega_n^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] \cdot \underline{\boldsymbol{\varphi}}_n = \underline{\boldsymbol{v}}_n \\ p_1 \cdot \det[\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_1^2 \cdot \underline{\mathbf{M}}_0 - \omega_1^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] / w_1 = v_1 \\ \vdots \\ p_i \cdot \det[\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_i^2 \cdot \underline{\mathbf{M}}_0 - \omega_i^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] / w_i = v_i \\ \vdots \\ p_n \cdot \det[\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_n^2 \cdot \underline{\mathbf{M}}_0 - \omega_n^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] / w_n = v_n \end{array} \right. \quad (6.11)$$

This system must be, unfortunately, solved with iterative algorithm which faces directly the entire system of Eq. 6.11. Therefore the DE-Q algorithm (which avoids the achievement of local minimum of the function) must be used to achieve the solution. The parameters are inside the matrices  $\underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N)$  and  $\underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)$  which are functions of the parameters themselves. In this way,  $a_1, a_2, \dots, a_N$  are found.

### 6.3.2 Uncertainties evaluation

Starting from Eq. 2.4, the system can be modified in the following way:

$$\left\{ \begin{array}{l} p_1 \cdot [\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_1^2 \cdot \underline{\mathbf{M}}_0 - \omega_1^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] \cdot \underline{\boldsymbol{\varphi}}_1 = \underline{\boldsymbol{v}}_1 \\ \vdots \\ p_i \cdot [\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_i^2 \cdot \underline{\mathbf{M}}_0 - \omega_i^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] \cdot \underline{\boldsymbol{\varphi}}_i = \underline{\boldsymbol{v}}_i \\ \vdots \\ p_n \cdot [\underline{\mathbf{K}}_0 + \underline{\mathbf{K}}_r(a_1, a_2, \dots, a_N) - \omega_n^2 \cdot \underline{\mathbf{M}}_0 - \omega_n^2 \cdot \underline{\mathbf{M}}_r(a_1, a_2, \dots, a_N)] \cdot \underline{\boldsymbol{\varphi}}_n = \underline{\boldsymbol{v}}_n \end{array} \right. \quad (6.12)$$

the least squares solution of that system can be achieved using the procedure introduced in paragraphs 2.3.2 and 3.3, finding the minimum of the subsequent function:

$$f(a_1, a_2, \dots, a_N) = \left\| \underline{\tilde{\mathbf{K}}}_0 \cdot \tilde{\boldsymbol{\varphi}} + \underline{\tilde{\mathbf{K}}}_r(a_1, a_2, \dots, a_N) \cdot \tilde{\boldsymbol{\varphi}} - \underline{\tilde{\mathbf{M}}}_0 \cdot \tilde{\boldsymbol{\varphi}} + \underline{\tilde{\mathbf{M}}}_r(a_1, a_2, \dots, a_N) \cdot \tilde{\boldsymbol{\varphi}} \right\|^2 \quad (6.13)$$

with:



$$\underline{\underline{\tilde{K}}}_r(a_1, a_2, \dots, a_N) = \begin{bmatrix} p_1 \cdot \underline{\underline{K}}_r(a_1, a_2, \dots, a_N) & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & p_2 \cdot \underline{\underline{K}}_r(a_1, a_2, \dots, a_N) & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & p_n \cdot \underline{\underline{K}}_r(a_1, a_2, \dots, a_N) \end{bmatrix} \quad (6.14)$$

$$\underline{\underline{\tilde{M}}}_r(a_1, a_2, \dots, a_N) = \begin{bmatrix} -p_1 \cdot \omega_1^2 \cdot \underline{\underline{M}}_r(a_1, a_2, \dots, a_N) & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & -p_2 \cdot \omega_2^2 \cdot \underline{\underline{M}}_r(a_1, a_2, \dots, a_N) & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & -p_n \cdot \omega_n^2 \cdot \underline{\underline{M}}_r(a_1, a_2, \dots, a_N) \end{bmatrix} \quad (6.15)$$

$$\underline{\underline{\tilde{\varphi}}} = \begin{bmatrix} \underline{\underline{\varphi}}_1 \\ \underline{\underline{\varphi}}_2 \\ \vdots \\ \underline{\underline{\varphi}}_n \end{bmatrix} \quad (6.16)$$

$$\underline{\underline{\tilde{K}}}_0 = \begin{bmatrix} p_1 \cdot \underline{\underline{K}}_0 & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & p_2 \cdot \underline{\underline{K}}_0 & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & p_n \cdot \underline{\underline{K}}_0 \end{bmatrix} \quad (6.17)$$

$$\underline{\underline{\tilde{M}}}_0 = \begin{bmatrix} p_1 \cdot \omega_1^2 \cdot \underline{\underline{M}}_0 & \underline{\underline{0}} & \cdots & \underline{\underline{0}} \\ \underline{\underline{0}} & p_2 \cdot \omega_2^2 \cdot \underline{\underline{M}}_0 & \cdots & \underline{\underline{0}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{\underline{0}} & \underline{\underline{0}} & \cdots & p_n \cdot \omega_n^2 \cdot \underline{\underline{M}}_0 \end{bmatrix} \quad (6.18)$$

Following the procedure of paragraph 3.3, the partial derivatives, with respect to the parameters, can be computed:

$$\frac{\partial \langle \underline{\underline{\tilde{K}}}_r \cdot \underline{\underline{\tilde{\varphi}}} + \underline{\underline{\tilde{M}}}_r \cdot \underline{\underline{\tilde{\varphi}}} - \underline{\underline{\tilde{\psi}}}; \underline{\underline{\tilde{K}}}_r \cdot \underline{\underline{\tilde{\varphi}}} + \underline{\underline{\tilde{M}}}_r \cdot \underline{\underline{\tilde{\varphi}}} - \underline{\underline{\tilde{\psi}}} \rangle}{\partial a_s} = 2 \cdot \left\langle \frac{\partial \underline{\underline{\tilde{K}}}_r}{\partial a_s} \cdot \underline{\underline{\tilde{\varphi}}} + \frac{\partial \underline{\underline{\tilde{M}}}_r}{\partial a_s} \cdot \underline{\underline{\tilde{\varphi}}}; \underline{\underline{\tilde{K}}}_r \cdot \underline{\underline{\tilde{\varphi}}} + \underline{\underline{\tilde{M}}}_r \cdot \underline{\underline{\tilde{\varphi}}} - \underline{\underline{\tilde{\psi}}} \right\rangle \quad (6.19)$$

for  $s = 1, 2, \dots, N$ , in which:

$$\underline{\underline{\tilde{\psi}}} = \underline{\underline{\tilde{M}}}_0 \cdot \underline{\underline{\tilde{\varphi}}} - \underline{\underline{\tilde{K}}}_0 \cdot \underline{\underline{\tilde{\varphi}}} \quad (6.20)$$

The minimum solution can be achieved if all the partial derivatives are null (stationary point):

$$\begin{cases} \frac{\partial f}{\partial a_1} = 0 \\ \vdots \\ \frac{\partial f}{\partial a_q} = 0 \\ \frac{\partial f}{\partial a_{q+1}} = 0 \\ \vdots \\ \frac{\partial f}{\partial a_N} = 0 \end{cases} \quad (6.21)$$

therefore:

$$\begin{cases} \left\langle \frac{\partial \tilde{\underline{K}}_r}{\partial a_1} \cdot \tilde{\underline{\varphi}} + \frac{\partial \tilde{\underline{M}}_r}{\partial a_1} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_r \cdot \tilde{\underline{\varphi}} + \tilde{\underline{M}}_r \cdot \tilde{\underline{\varphi}} - \tilde{\underline{\psi}} \right\rangle = 0 \\ \vdots \\ \left\langle \frac{\partial \tilde{\underline{K}}_r}{\partial a_q} \cdot \tilde{\underline{\varphi}} + \frac{\partial \tilde{\underline{M}}_r}{\partial a_q} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_r \cdot \tilde{\underline{\varphi}} + \tilde{\underline{M}}_r \cdot \tilde{\underline{\varphi}} - \tilde{\underline{\psi}} \right\rangle = 0 \\ \left\langle \frac{\partial \tilde{\underline{K}}_r}{\partial a_{q+1}} \cdot \tilde{\underline{\varphi}} + \frac{\partial \tilde{\underline{M}}_r}{\partial a_{q+1}} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_r \cdot \tilde{\underline{\varphi}} + \tilde{\underline{M}}_r \cdot \tilde{\underline{\varphi}} - \tilde{\underline{\psi}} \right\rangle = 0 \\ \vdots \\ \left\langle \frac{\partial \tilde{\underline{K}}_r}{\partial a_N} \cdot \tilde{\underline{\varphi}} + \frac{\partial \tilde{\underline{M}}_r}{\partial a_N} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_r \cdot \tilde{\underline{\varphi}} + \tilde{\underline{M}}_r \cdot \tilde{\underline{\varphi}} - \tilde{\underline{\psi}} \right\rangle = 0 \end{cases} \quad (6.22)$$

in which  $\frac{\partial \tilde{\underline{K}}_r}{\partial a_s}$  and  $\frac{\partial \tilde{\underline{M}}_r}{\partial a_s}$  are element by element partial derivatives. From these equations, the partial derivatives, with respect to the experimental outcomes, can be performed:

– With respect to  $\varphi_{ji}$ :

$$\frac{\partial}{\partial \varphi_{ji}} \left( \left\langle \frac{\partial \tilde{\underline{K}}_r}{\partial a_s} \cdot \tilde{\underline{\varphi}} + \frac{\partial \tilde{\underline{M}}_r}{\partial a_s} \cdot \tilde{\underline{\varphi}}; \tilde{\underline{K}}_r \cdot \tilde{\underline{\varphi}} + \tilde{\underline{M}}_r \cdot \tilde{\underline{\varphi}} - \tilde{\underline{\psi}} \right\rangle \right) = 0 \quad (6.23)$$

for  $s = 1, 2, \dots, N$ . Developing the calculations:

$$\begin{aligned} & \left\langle \frac{\partial^2 \tilde{\underline{K}}_r}{\partial \varphi_{ji} \partial a_s} \cdot \tilde{\underline{\varphi}} + \frac{\partial \tilde{\underline{K}}_r}{\partial a_s}(:, j) + \frac{\partial^2 \tilde{\underline{M}}_r}{\partial \varphi_{ji} \partial a_s} \cdot \tilde{\underline{\varphi}} + \frac{\partial \tilde{\underline{M}}_r}{\partial a_s}(:, j); \tilde{\underline{K}}_r \cdot \tilde{\underline{\varphi}} + \tilde{\underline{M}}_r \cdot \tilde{\underline{\varphi}} - \tilde{\underline{\psi}} \right\rangle + \\ & + \left\langle \frac{\partial \tilde{\underline{K}}_r}{\partial a_s} \cdot \tilde{\underline{\varphi}} + \frac{\partial \tilde{\underline{M}}_r}{\partial a_s} \cdot \tilde{\underline{\varphi}}; \frac{\partial \tilde{\underline{K}}_r}{\partial \varphi_{ji}} \cdot \tilde{\underline{\varphi}} + \tilde{\underline{K}}_r(:, j) + \frac{\partial \tilde{\underline{M}}_r}{\partial \varphi_{ji}} \cdot \tilde{\underline{\varphi}} + \tilde{\underline{M}}_r(:, j) + \tilde{\underline{K}}_0(:, j) - \tilde{\underline{M}}_0(:, j) \right\rangle = 0 \end{aligned} \quad (6.24)$$

in which  $(:, j)$  means the j-th column of the matrix. The s-th equation of the system is as follows:

$$\begin{aligned}
& \left\langle \frac{\partial^2 \underline{\tilde{K}}_r}{\partial \varphi_{ji} \partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial^2 \underline{\tilde{M}}_r}{\partial \varphi_{ji} \partial a_s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_r \cdot \underline{\tilde{\varphi}} + \underline{\tilde{M}}_r \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle + \left\langle \frac{\partial \underline{\tilde{K}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}}; \frac{\partial \underline{\tilde{K}}_r}{\partial \varphi_{ji}} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial \varphi_{ji}} \cdot \underline{\tilde{\varphi}} \right\rangle = \\
& = - \left\langle \frac{\partial \underline{\tilde{K}}_r}{\partial a_s}(:, j) + \frac{\partial \underline{\tilde{M}}_r}{\partial a_s}(:, j); \underline{\tilde{K}}_r \cdot \underline{\tilde{\varphi}} + \underline{\tilde{M}}_r \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle - \\
& \left\langle \frac{\partial \underline{\tilde{K}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_r(:, j) + \underline{\tilde{M}}_r(:, j) + \underline{\tilde{K}}_0(:, j) - \underline{\tilde{M}}_0(:, j) \right\rangle
\end{aligned} \tag{6.25}$$

for  $s = 1, 2, \dots, N$ . Solving the system in which the s-th equation is of the form of Eq. 6.25,

all  $\frac{\partial a_1}{\partial \varphi_{ji}}, \frac{\partial a_2}{\partial \varphi_{ji}}, \dots, \frac{\partial a_N}{\partial \varphi_{ji}}$  can be achieved, for  $i = 1, 2, \dots, n$ ;  $j = 1, 2, \dots, m$ .

– With respect to  $\omega_i^2$ :

$$\frac{\partial}{\partial \omega_i^2} \left( \left\langle \frac{\partial \underline{\tilde{K}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_r \cdot \underline{\tilde{\varphi}} + \underline{\tilde{M}}_r \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle \right) = 0 \tag{6.26}$$

for  $s = 1, 2, \dots, N$ . Developing the calculations:

$$\begin{aligned}
& \left\langle \frac{\partial^2 \underline{\tilde{K}}_r}{\partial \omega_i^2 \partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial^2 \underline{\tilde{M}}_r}{\partial \omega_i^2 \partial a_s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_r \cdot \underline{\tilde{\varphi}} + \underline{\tilde{M}}_r \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle + \\
& + \left\langle \frac{\partial \underline{\tilde{K}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}}; \frac{\partial \underline{\tilde{K}}_r}{\partial \omega_i^2} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial \omega_i^2} \cdot \underline{\tilde{\varphi}} - \frac{\partial \underline{\tilde{M}}_0}{\partial \omega_i^2} \cdot \underline{\tilde{\varphi}} \right\rangle = 0
\end{aligned} \tag{6.27}$$

in which:

$$\frac{\partial \underline{\tilde{M}}_0}{\partial \omega_i^2} = \begin{bmatrix} p_1 \cdot \underline{\mathbf{0}} \cdot \underline{\tilde{M}}_0 & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & p_2 \cdot \underline{\mathbf{0}} \cdot \underline{\tilde{M}}_0 & \dots & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \dots & p_i \underline{\tilde{M}}_0 & \dots & \underline{\mathbf{0}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} & \dots & p_n \cdot \underline{\mathbf{0}} \cdot \underline{\tilde{M}}_0 \end{bmatrix} \tag{6.28}$$

or, in the same way:

$$\frac{\partial \underline{\tilde{M}}_0}{\partial \omega_i^2} = \begin{bmatrix} \underline{\mathbf{0}} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \dots & p_i \underline{\tilde{M}}_0 & \dots & \underline{\mathbf{0}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \underline{\mathbf{0}} & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} & \dots & \underline{\mathbf{0}} \end{bmatrix} \tag{6.28'}$$

The s-th equation of the system is as follows:

$$\begin{aligned} \left\langle \frac{\partial^2 \underline{\tilde{K}}_r}{\partial \omega_i^2 \partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial^2 \underline{\tilde{M}}_r}{\partial \omega_i^2 \partial a_s} \cdot \underline{\tilde{\varphi}}; \underline{\tilde{K}}_r \cdot \underline{\tilde{\varphi}} + \underline{\tilde{M}}_r \cdot \underline{\tilde{\varphi}} - \underline{\tilde{\psi}} \right\rangle + \left\langle \frac{\partial \underline{\tilde{K}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}}; \frac{\partial \underline{\tilde{K}}_r}{\partial \omega_i^2} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial \omega_i^2} \cdot \underline{\tilde{\varphi}} \right\rangle = \\ = \left\langle \frac{\partial \underline{\tilde{K}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}} + \frac{\partial \underline{\tilde{M}}_r}{\partial a_s} \cdot \underline{\tilde{\varphi}}; \frac{\partial \underline{\tilde{M}}_0}{\partial \omega_i^2} \cdot \underline{\tilde{\varphi}} \right\rangle \end{aligned} \quad (6.29)$$

for  $s = 1, 2, \dots, N$ . Solving the system in which the s-th equation is of the form of Eq. 6.29,

all  $\frac{\partial a_1}{\partial \omega_i^2}, \frac{\partial a_2}{\partial \omega_i^2}, \dots, \frac{\partial a_N}{\partial \omega_i^2}$  can be achieved, for  $i = 1, 2, \dots, n$ .

Once all the partial derivatives have been obtained, the procedure of paragraph 2.7.3 has to be followed in order to obtain the standard deviations of parameters:

$$\sigma_{a,s} \cong \sqrt{\nabla a_s^T \cdot \underline{\underline{C}} \cdot \nabla a_s} \quad (6.30)$$

for  $s = 1, 2, \dots, N$ , in which:

$$\nabla a_s = \begin{bmatrix} \frac{\partial a_s}{\partial \varphi_{11}} \\ \frac{\partial a_s}{\partial \varphi_{21}} \\ \vdots \\ \frac{\partial a_s}{\partial \varphi_{mn}} \\ \frac{\partial a_s}{\partial \omega_1^2} \\ \vdots \\ \frac{\partial a_s}{\partial \omega_n^2} \end{bmatrix} \quad (6.31)$$

and

$$\underline{\underline{C}} = \begin{bmatrix} \sigma_{\varphi,11}^2 & \text{cov}(\varphi_{11}; \varphi_{21}) & \cdots & \text{cov}(\varphi_{11}; \varphi_{mn}) & \text{cov}(\varphi_{11}; \omega_1^2) & \cdots & \text{cov}(\varphi_{11}; \omega_n^2) \\ \text{cov}(\varphi_{21}; \varphi_{11}) & \sigma_{\varphi,21}^2 & \cdots & \text{cov}(\varphi_{21}; \varphi_{mn}) & \text{cov}(\varphi_{21}; \omega_1^2) & \cdots & \text{cov}(\varphi_{21}; \omega_n^2) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ \text{cov}(\varphi_{mn}; \varphi_{11}) & \text{cov}(\varphi_{mn}; \varphi_{21}) & \cdots & \sigma_{\varphi,mn}^2 & \text{cov}(\varphi_{mn}; \omega_1^2) & \cdots & \text{cov}(\varphi_{mn}; \omega_n^2) \\ \text{cov}(\omega_1^2; \varphi_{11}) & \text{cov}(\omega_1^2; \varphi_{21}) & \cdots & \text{cov}(\omega_1^2; \varphi_{mn}) & \sigma_{\omega^2,1}^2 & \cdots & \text{cov}(\omega_1^2; \omega_n^2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{cov}(\omega_n^2; \varphi_{11}) & \text{cov}(\omega_n^2; \varphi_{21}) & \cdots & \text{cov}(\omega_n^2; \varphi_{mn}) & \text{cov}(\omega_n^2; \omega_1^2) & \cdots & \sigma_{\omega^2,n}^2 \end{bmatrix} \quad (6.32)$$

The solution can therefore be written as follows:

$$\tilde{a}_s = a_s \pm \sigma_{a,s} \quad (6.33)$$

for  $s = 1, 2, \dots, N$ .

### 6.3.3 Complete Statistical analysis

The procedure introduced in paragraph 2.8 can be used for the complete statistical analysis in which the two steps procedure with Trust-Region updating has to be replaced by the iterative solution with DE-Q algorithm of system of Eq. 6.11.

The flowcharts of the procedure for the parameters analysis, the parameters analysis with uncertainties evaluation and the complete statistical analysis are depicted in Figures 6.4, 6.5 and 6.6 respectively.

An example of parameters analysis with uncertainties evaluation will be presented in Appendix C.

## 6.4 System with damping

If one wants to take into account the damping in the structure, the system of Eq. 6.11 must be changed according to the type of damping presents in the structure. Only the case of viscous damping for classically damped structures will be analyzed because in the other cases the damping matrix must be known (this matrix can be computed only with very huge approximations).

### 6.4.1 System with viscous damping - classically damped system

If the system is classically damped, the modal shapes for undamped structure and the ones for damped structure are the same (Ewins, 2000; Chopra, 2016). The natural circular frequencies follow the subsequent criterion (Chopra, 2016):

$$\omega_{iD} = \omega_i \cdot \sqrt{1 - \xi_i^2} \quad (6.34)$$

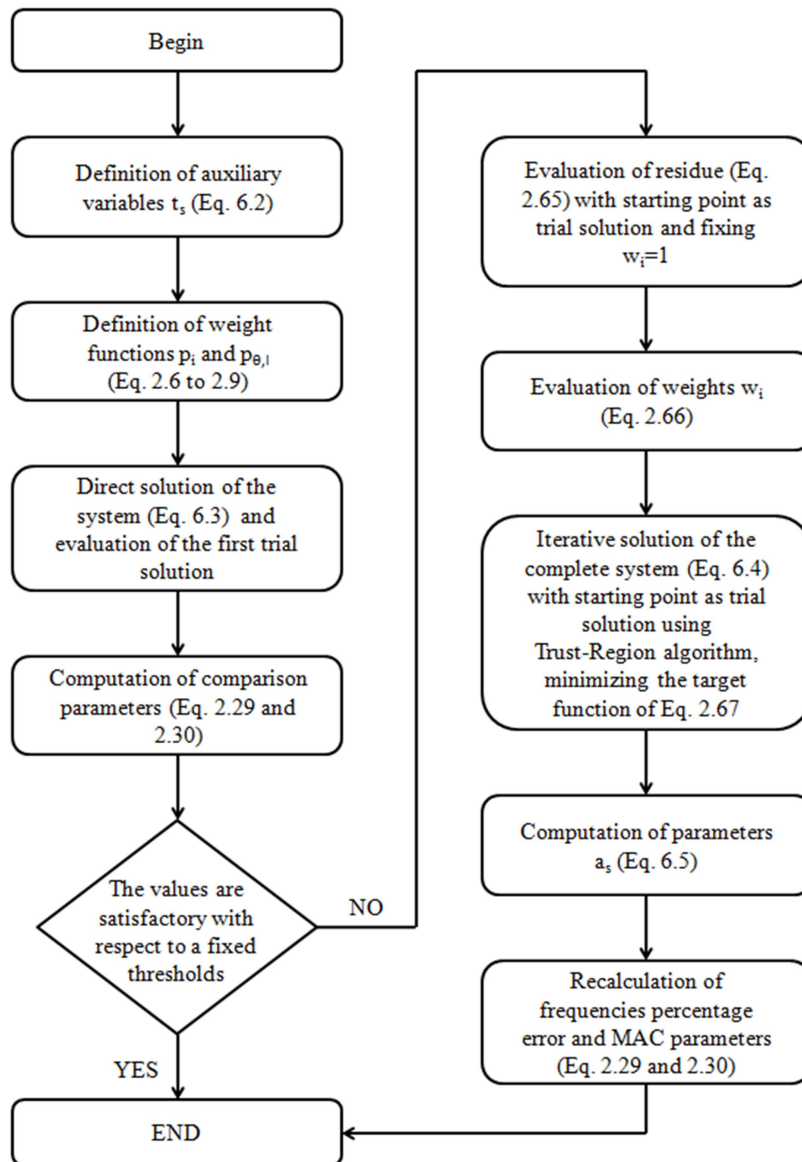
in which  $\omega_{iD}$  is the i-th damped natural circular frequency,  $\omega_i$  is the i-th undamped natural circular frequency and  $\xi_i$  is the damping ratio for the i-th mode. For the usual values of  $\xi_i$  for civil structure,  $\omega_{iD} \cong \omega_i$  (Chopra, 2016) and therefore, for classically damped system, the damping doesn't play a role for free vibrations. System of Eq. 6.11 is therefore still valid.

## 6.5 Conclusions

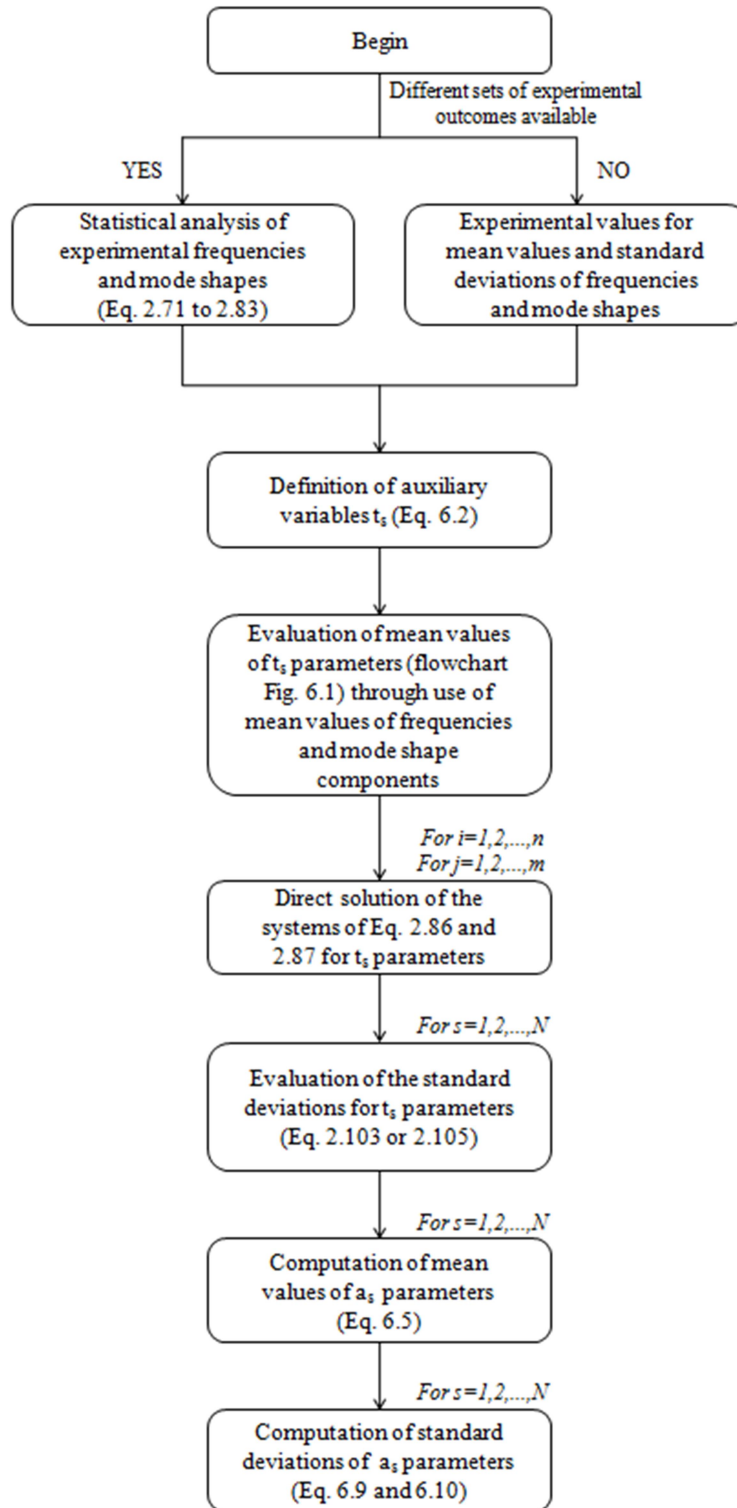
In this Chapter the generalization of the procedures developed in Chapter 2 has been done in order to take into account non-linear parameters. In the first part, the case in which the parameters are unrelated each other and the decomposition of matrices can be performed has been analyzed. In the second part the more general case in which the parameters are related has been analyzed, an

application of it is reported in Appendix C. In the last part, a generalization taking into account the viscous damping for classically damped system has been studied.

## Figures of Chapter 6

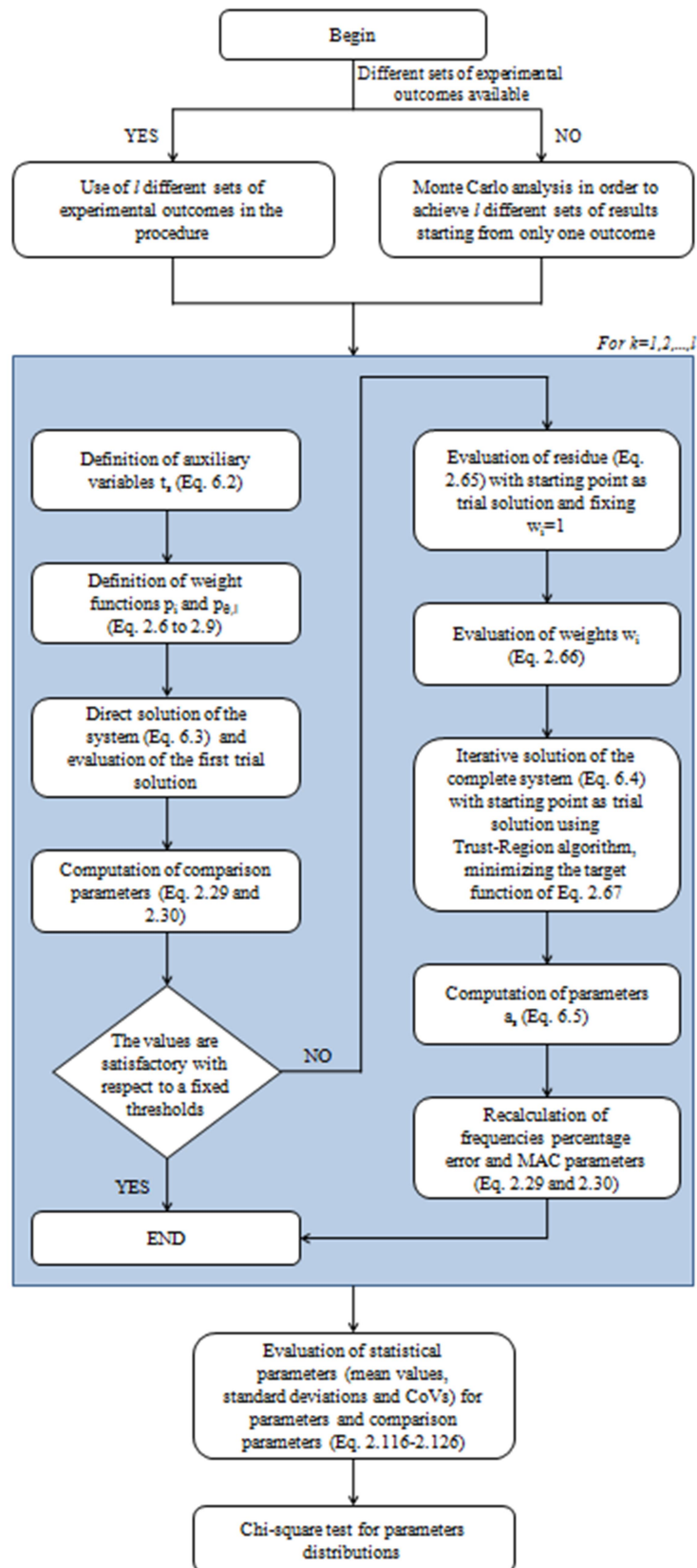


**Figure 6.1:** Flowchart of the two steps algorithm for unrelated non-linear parameters.

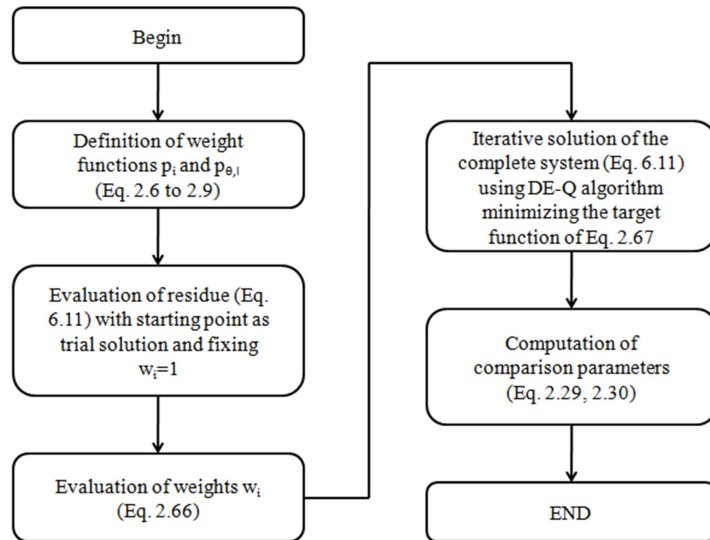


**Figure 6.2:** Flowchart of the two steps algorithm with uncertainties evaluation for unrelated non-linear parameters.

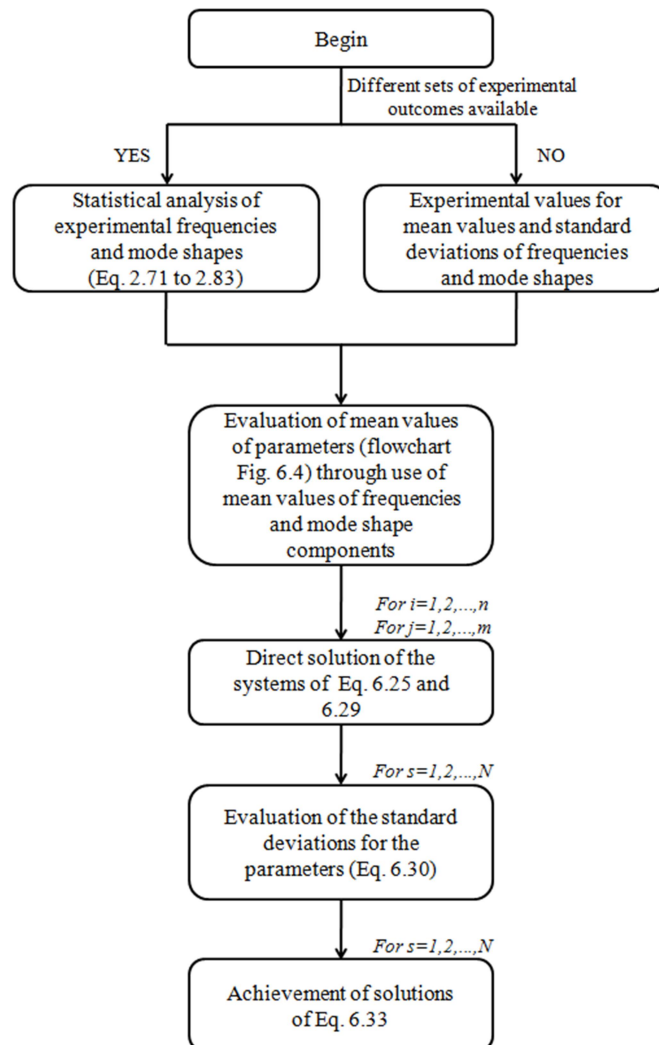




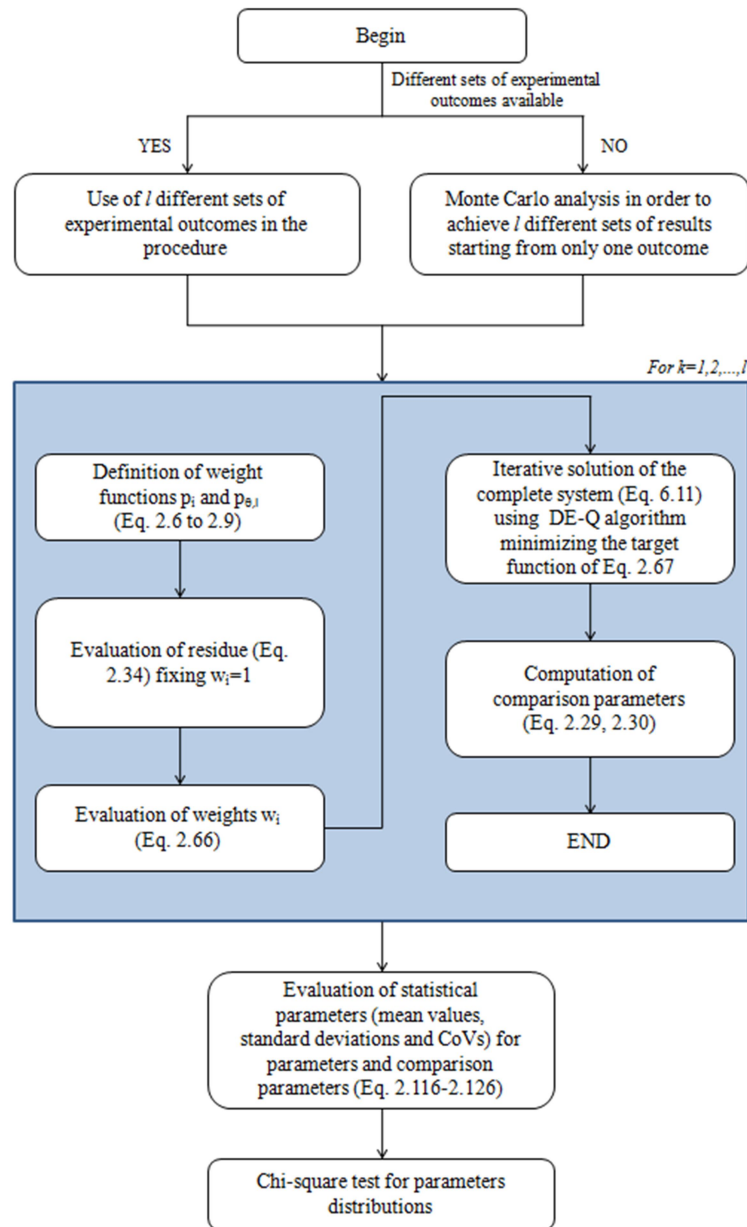
**Figure 6.3:** Flowchart of the two steps algorithm for complete statistical analysis for unrelated non-linear parameters.



**Figure 6.4:** Flowchart of the algorithm with DE-Q procedure for related non-linear parameters.



**Figure 6.5:** Flowchart of the algorithm with DE-Q procedure and uncertainties evaluation for related non-linear parameters.



**Figure 6.6:** Flowchart of the algorithm with DE-Q procedure for complete statistical analysis for related non-linear parameters.



## CHAPTER 7

# APPLICATION OF THE PROCEDURE FOR THE RETROFITTING OF EXISTING R.C. STRUCTURES

### 7.1 Introduction

In this Chapter, the algorithm is used in order to perform a first trial retrofitting of existing R.C. structures. In the first part the algorithm will be introduced. After that two case studies will be analyzed in order to achieve the values of parameters for reaching a first trial retrofitting of the structures. Eventually, the procedure for unrelated non-linear parameters and related non-linear ones will be introduced.

### 7.2 Description of the procedure

In order to perform a first trial retrofitting of existing structures, the algorithm outlined in Chapter 2 can be used. The main goal of this procedure is to uncouple the modal shapes in order to obtain a centering of the center of rigidity on the center of mass and reaching the regularity in height. This procedure can give to the designer a first trial values of parameters (e.g. the thickness of some R.C. walls used as retrofitting) in order to achieve, first of all, the uncoupling of the modal shapes.

To perform these goals, the algorithm in closed form introduced in paragraph 2.3 (Eq. 2.4 with solution of Eq. 2.16 or 2.28) can be used.

The procedure has to follow the subsequent steps:

- From the initial structure, the computation of natural frequencies must be done;
- A set of parameters (remembering the maximum number of parameters given in Chapter 3) must be defined;
- Definition of stiffness and mass matrices needed by the Equations 2.4;
- Check the goodness-of-definition of parameters (paragraph 3.5);
- Modes components definition (usually the ones listed in Table 7.1);
- Rotational equations weighting functions definition (Eq. 2.9);
- Solution in closed form of Eq. 2.4 through Eq. 2.16 or 2.28;
- Check the presence of unrealistic negative parameters;

- If some parameters achieve negative values (instead of expected positive ones), these parameters must be set to null value and the procedure must be re-run again with smaller number of parameters.

The flowchart of the procedure is depicted in Figure 7.1.

## 7.3 Case 1: Symmetric-Asymmetric structure

### 7.3.1 Description of the structure

The sample structure used for the first case is a three storey, three bays in X direction, one bay in Y direction, R.C. frame with R.C. walls at ground and first storey (Figures. 7.2 and 7.3). This structure is symmetric in X direction and asymmetric in Y direction. The sources of asymmetry are only due to the R.C. walls. For all the structural members the modulus of elasticity for the concrete is fixed at the value of 30000 MPa. The spans in the X direction are 5 m long, the one in the Y direction is 6 m long. The interstorey height is equal to 3 m. All the columns have 40 cm by 40 cm square sections, the beams have 40 cm by 50 cm rectangular sections.

The R.C. walls are modeled as equivalent struts. For the two walls in the X direction a value of  $\frac{w}{d} = 0.25$  is fixed (being  $w$  the width of the struts and  $d$  the length of the diagonal of the R.C. wall), with thickness  $t$  equal to the dimension of the columns. For the walls in Y direction, a value of  $\frac{w}{d} = 0.35$  is fixed. Three parameters (the  $\frac{w}{d}$  ratio for the new walls used in the retrofitting) are considered; two walls per storey in X direction and one wall per storey in Y direction (Figure 7.2).

### 7.3.2 Description of the mode shape vectors and results

Three mode shapes are used for the procedure of Eq. 2.4. The modes components used (remembering they have 9 rigid diaphragm components) are listed in Table 7.1. Running the procedure, the parameters achieved are listed in Table 7.2. As expected, the two parameters for walls in X direction give negative values because of the system is already symmetric in X direction (therefore uncoupled with respect to Y direction and rotation about the center of the mass). The two parameters have to be set to null values. The other parameter assumes the correct value in order to symmetrise the structure also in Y direction. The original natural frequencies and the ones after retrofitting are listed in Table 7.3. One can see how the procedure is quite insensible about the

initial values of the natural frequencies (because of the determinant equations in Eq. 2.65 are not taken into account for this procedure).

Changing the mode shape vectors (preserving the uncoupling of modes, as in Table 7.4), the solution doesn't change. Moreover, the procedure is almost insensitive also to the change of the sequence of modes. If before the retrofitting the first mode is a predominant X direction mode, after the procedure the first mode could be a Y direction one.

## 7.4 Case 2: Totally asymmetric structure

### 7.4.1 Description of the structure

The second structure studied is, once again, a three storey, three bays in X direction, one bay in Y direction, R.C. frame with R.C. walls at ground and first storey (Figs. 7.2 and 7.4). This structure is asymmetric in both X and Y directions. The sources of asymmetry are only due to the R.C. walls. For all the structural members the modulus of elasticity for the concrete is fixed at the value of 30000 MPa. All the geometrical features are the same as the previous case.

The R.C. walls are modeled as equivalent struts. For the walls in the X direction a value of

$\frac{w}{d} = 0.25$  is fixed. For the walls in Y direction, a value of  $\frac{w}{d} = 0.35$  is chosen. Three parameters

(the  $\frac{w}{d}$  ratio for the new walls used in the retrofitting) are considered; one wall per storey in X

direction (for ground and first storey) and one wall per storey in Y direction (once again, for ground and first storey); another parameter is chosen at the second floor (Figure 7.4), this parameter should achieve null value in the iterative procedure (it was chosen as parameter in order to understand if the procedure can capture wrong choices in the parameters definition).

### 7.4.2 Results

Three mode shapes are used for the procedure of Eq. 2.4. The mode components used (remembering they have 9 rigid diaphragm components) are listed in Table 7.1. Running the procedure, the parameters achieved are listed in Table 7.5. As expected, the first two parameters achieve the correct values and the third parameter (the modeling error) reaches a null value. No negative values are achieved and therefore the modification of them is not necessary. After the procedure the structure produces perfectly uncoupled mode shapes.

The original natural frequencies and the ones after retrofitting are listed in Table 7.6. One can see again how the procedure is quite regardless about the initial values of the natural frequencies. Changing the mode shape vectors, the solution doesn't change. Moreover, the procedure is again almost insensitive also to the change in the sequence of modes.

## 7.5 Non-linear parameters

### 7.5.1 Unrelated parameters

If unrelated non-linear parameters are used in the procedure, the algorithm must be changed according to the following steps:

- From the initial structure, the computation of natural frequencies must be done;
- A set of parameters (remembering the maximum number of parameters given in Chapter 3) must be defined;
- Definition of stiffness and mass matrices needed by the Equations 6.3;
- The auxiliary variables (Eq. 6.2) must be defined;
- Check the goodness-of-definition of parameters (paragraph 3.5);
- Modes components definition (usually the ones listed in Table 7.1);
- Rotational equations weighting functions definition (Eq. 2.9);
- Solution in closed form of Eq. 6.3 through Eq. 2.16 or Eq. 2.28;
- The original parameters must be computed (Eq. 6.5);
- Check the presence of unrealistic negative parameters;
- If some parameters achieve negative values (instead of expected positive ones), these parameters must be set to null values and the procedure must be re-run again with smaller number of parameters.

The flowchart of the procedure is depicted in Figure 7.5.

### 7.5.2 Related parameters

If related non-linear parameters are used in the procedure, the algorithm must be changed according to the following steps:

- From the initial structure, the computation of natural frequencies must be done;
- A set of parameters (remembering the maximum number of parameters given in Chapter 3) must be defined;
- Modes components definition (usually the ones listed in Table 7.1);



- Rotational equations weighting functions definition (Eq. 2.9);
- Solution of the system of Eq. 6.12 through iterative DE-Q algorithm;
- Check the presence of unrealistic negative parameters;
- If some parameters achieve negative values (instead of expected positive ones), these parameters must be set to null values and the procedure must be re-run again with smaller number of parameters.

The flowchart of the procedure is depicted in Figure 7.6.

## 7.6 Conclusions

Starting from two sample structures, of which the parameters values are known (called expected values), in order to obtain the modes uncoupling, the procedure has been run and the expected values have been obtained for both symmetric-asymmetric case and asymmetric one even with the numerical procedure. The procedure is quite insensitive to the change of the natural frequencies and, moreover, to different choices of the modes with uncoupled components. In the last part of the Chapter, the cases of non-linear parameters have been analyzed.



## Tables of Chapter 7

Component	Mode components		
	Mode 1	Mode 2	Mode 2
X-1 <sup>st</sup> storey	0.33	0	0
Y-1 <sup>st</sup> storey	0	0.33	0
$\theta$ -1 <sup>st</sup> storey	0	0	0.33
X-2 <sup>nd</sup> storey	0.66	0	0
Y-2 <sup>nd</sup> storey	0	0.66	0
$\theta$ -2 <sup>nd</sup> storey	0	0	0.66
X-3 <sup>rd</sup> storey	1.00	0	0
Y-3 <sup>rd</sup> storey	0	1.00	0
$\theta$ -3 <sup>rd</sup> storey	0	0	1.00

**Table 7.1:** Modes components chosen for the procedure

Parameter	w/d values		
	Expected values	Procedure values	Rectified values
a <sub>1</sub>	0.0	-0.39	0.0
a <sub>2</sub>	0.0	-0.39	0.0
a <sub>3</sub>	0.35	0.35	0.35

**Table 7.2:** w/d values for the 3 parameters - Case 1

Mode	Frequencies [Hz]	
	Original structure	After retrofitting
Mode 1	3.55	5.51
Mode 2	5.78	5.78
Mode 3	6.48	7.18

**Table 7.3:** Frequencies before and after retrofitting - Case 1

Components	Mode components		
	Mode 1	Mode 2	Mode 2
X-1 <sup>st</sup> storey	0.25	0	0
Y-1 <sup>st</sup> storey	0	0.25	0
$\theta$ -1 <sup>st</sup> storey	0	0	0.25
X-2 <sup>nd</sup> storey	0.50	0	0
Y-2 <sup>nd</sup> storey	0	0.50	0
$\theta$ -2 <sup>nd</sup> storey	0	0	0.50
X-3 <sup>rd</sup> storey	1.00	0	0
Y-3 <sup>rd</sup> storey	0	1.00	0
$\theta$ -3 <sup>rd</sup> storey	0	0	1.00

**Table 7.4:** Modified modes components chosen for the procedure

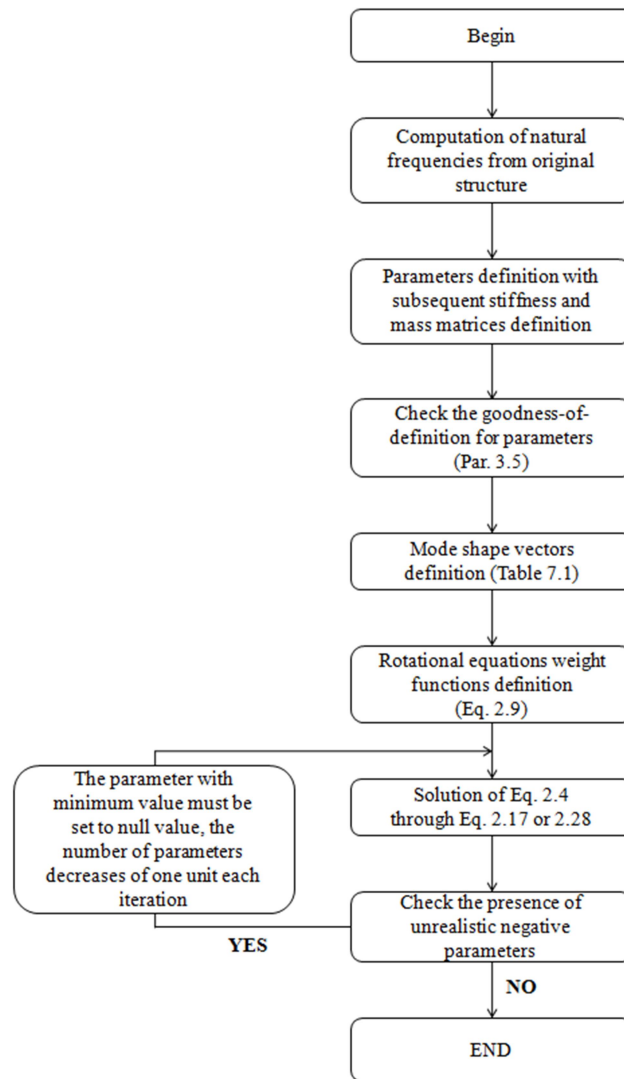
Parameter	w/d values	
	Expected values	Procedure values
$a_1$	0.25	0.25
$a_2$	0.35	0.35
$a_3$	0.0	0.0

**Table 7.5:** w/d values for the 3 parameters - Case 2

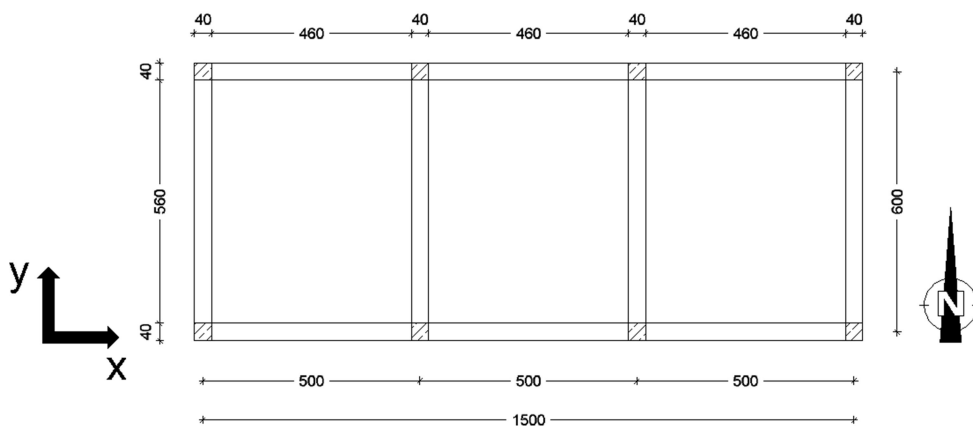
Mode	Frequencies [Hz]	
	Original structure	After retrofitting
Mode 1	2.38	3.59
Mode 2	5.32	5.79
Mode 3	6.34	6.50

**Table 7.6:** Frequencies before and after retrofitting - Case 2

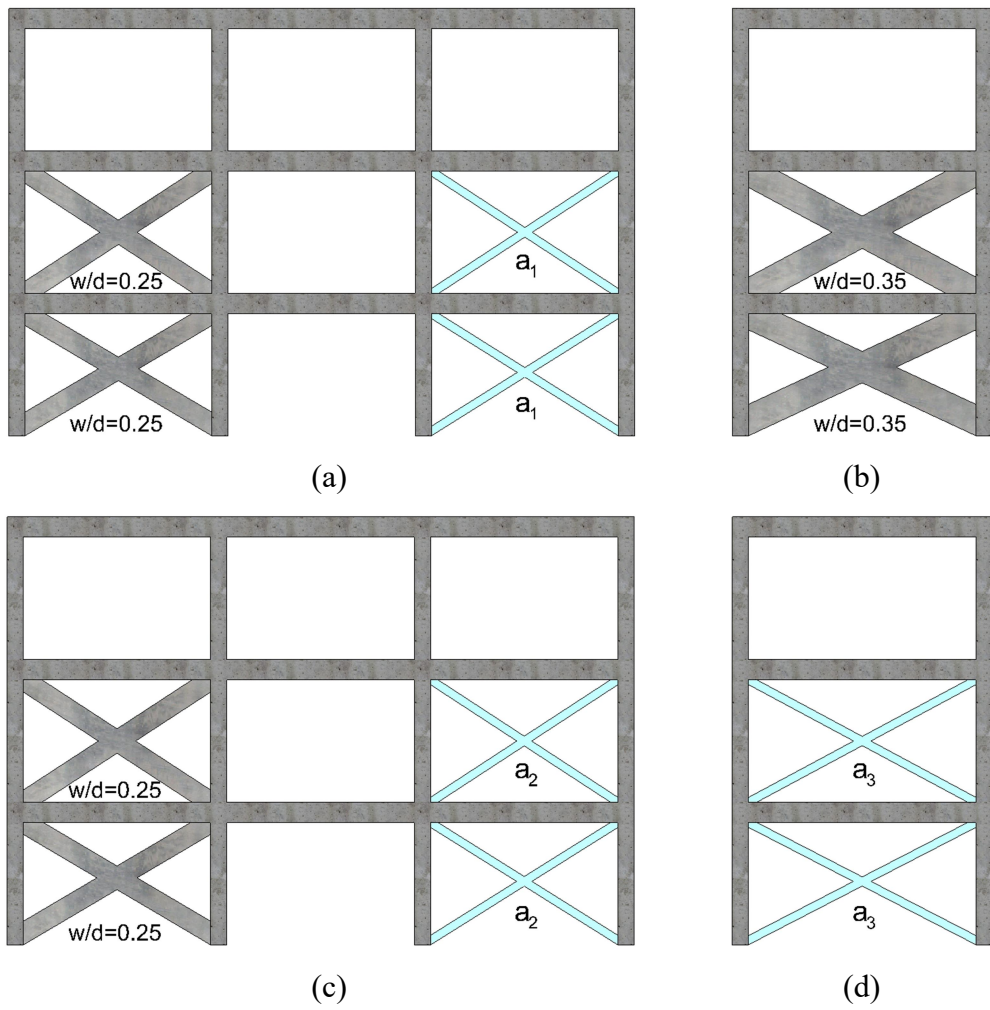
## Figures of Chapter 7



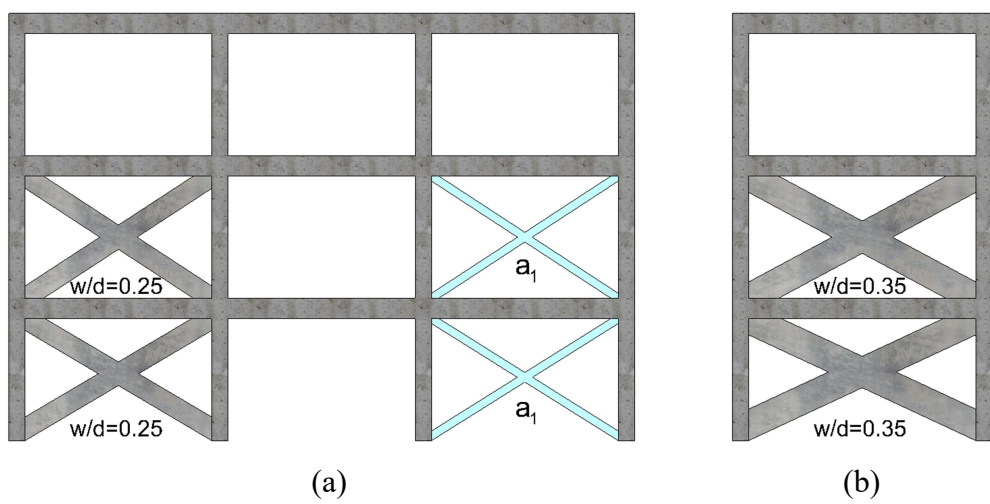
**Figure 7.1:** Flowchart of the procedure.

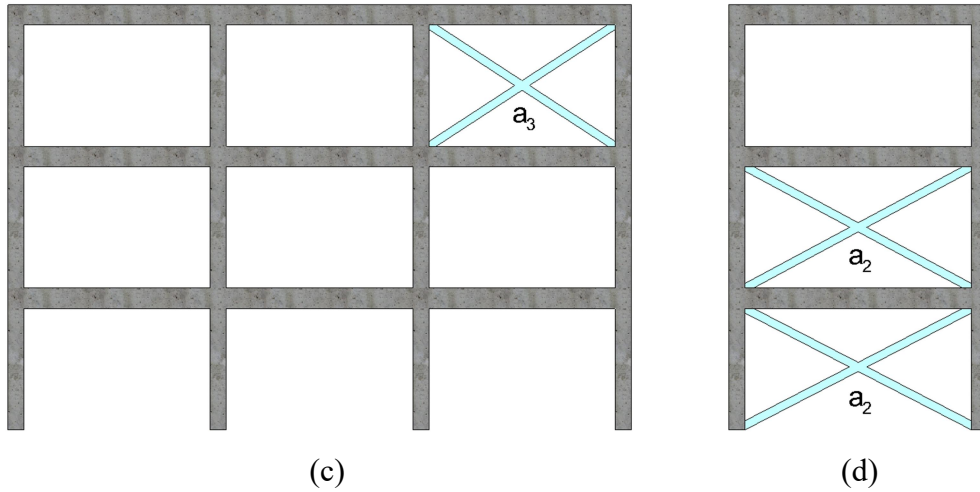


**Figure 7.2:** Plan view for Case 1 (dimensions in cm).

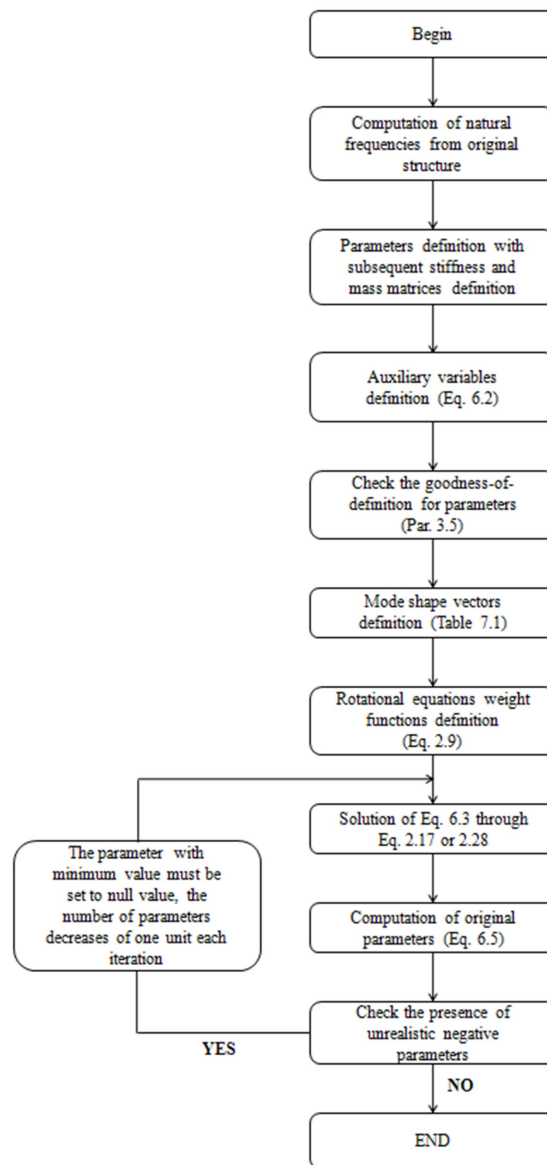


**Figure 7.3:** Elevation views for Case 1: (a) South elevation; (b) East elevation; (c) North elevation; (d) West elevation.

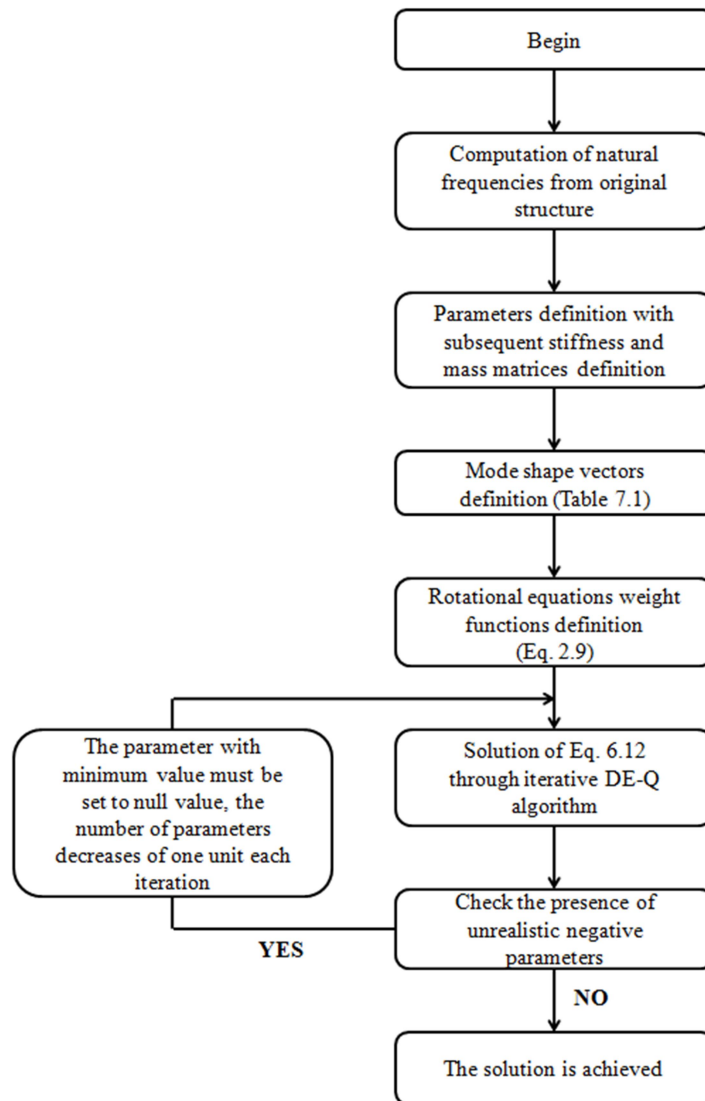




**Figure 7.4:** Elevation views for Case 2: (a) South elevation; (b) East elevation; (c) North elevation; (d) West elevation.



**Figure 7.5:** Flowchart of procedure for unrelated non-linear parameters.



**Figure 7.6:** Flowchart of procedure for related non-linear parameters.



## CHAPTER 8

### SUMMARY AND CONCLUSIONS

In the first part of this work, the definition of the innovative procedure was given starting from the eigenvalues/eigenvectors problem. The algorithm relies on a two steps procedure with uncertainties assessment. The first step is a closed-form evaluation of the first trial parameters using the eigenvalues/eigenvectors problem without considering the determinant equations. The closed solution was achieved exploiting the partial derivatives of the problem (as pointed out in Chapter 2 and 3). The comparison parameters (percentage frequencies errors and MAC) have to be computed. If those values satisfy fixed thresholds, the procedure stops with the first step, otherwise the second one is required. In order to improve the agreement between experimental and numerical frequencies, the second step (which considers also the determinant equations) has to be run in iterative way, having as starting point the trial solution computed in the first step. After the second one, the comparison parameters have to be recomputed and the procedure stops. With this algorithm the partial derivatives, with respect to the experimental outcomes, can be computed in closed form too. This fact allows us to perform the errors propagation and we can achieve the parameters standard deviations from the knowledge of the standard deviations in frequencies and modes components. Therefore the two steps algorithm allows us to achieve the uncertainties in the parameters themselves (flowchart of Figures 2.2 and 2.3).

After that, a second algorithm was described relying on genetic algorithm with response surface methodology (the so-called DE-Q algorithm) in which the entire problem is faced completely in iterative way. Also in this case the direct uncertainties evaluation can be performed in closed form (flowchart of Figures 2.6 and 2.7).

A complete statistical analysis were also achieved for both the procedures outlined above (flowcharts of Figure 2.4 for two steps procedure and Figure 2.8 for DE-Q one).

In the last part of Chapter 2, the distribution analysis of parameters has also been introduced.

The relations and the demonstrations for the maximum number of parameters achievable by the procedure (Eq. 3.1) has been given as a function of the natural frequencies and mode shapes available from tests. All the computations for the least squares solution and the uncertainties evaluation have been given in paragraphs 3.3 and 3.4 respectively. The goodness-of-definition of parameters themselves and the uniqueness of the solution have been then analyzed and relations

(Eqs. 3.51, 3.52, 3.54 and 3.55) have been found. After that the analysis of maximum number of parameters for non-ideal case has been studied and a procedure (outlined in flowchart of Figure 3.2) has been found. Moreover, a procedure for testing the rigid diaphragm assumptions for experimental tests on structures has been studied.

After the definition of the target functions and the procedures in order to obtain the values of parameters, several sensitivity analysis (using the Monte Carlo procedure) have been carried out in order to test the stability of the algorithm itself. Firstly, a 2-D infilled framed structure has been studied and the results are listed in Table 4.2. The results are very stable because the CoVs of parameters are always lesser than the sum of the perturbations on frequencies and modes components. After that a 3-D infilled framed structures have been analyzed varying the parameters arrangement, the perturbations on frequencies and modes components, the perturbations on mode shapes (directly on rigid diaphragm components or on accelerometers placed on the structures). The results are listed in Tables 4.4 to 4.8. Even in this case the results are stable for all the cases analyzed. Subsequently a comparison between procedure with DE-Q algorithm and two steps algorithm with Trust-region updating has been done for two cases. The results from two steps procedure are better, both in term of mean values and CoVs, with respect to the ones obtained using DE-Q algorithm. The time saving using the two steps algorithm is very important and is about 95% less than the time required by the DE-Q procedure. The comparison between procedures with or without determinant equations was performed and, unfortunately, the procedure without determinant equations led to worse results especially for the most important parameter, which play the most important role in the dynamic behavior of the structure. A comparison between procedures with 6 or 9 modes utilized led to improvements on results but not so significant. From the distribution analysis, unfortunately, no general rules could be achieved for parameters distribution starting from normally distributed perturbations on frequencies and modes components.

The procedures were then used in order to perform the damage assessment for two real structures of which the experimental frequencies and modes components were at disposal. The first one is a two-dimensional three storey two bays infilled frame, tested at UCSD, CA, US, through shake table (which induced the damages). For this structure a stick model has been used and the parameters to achieve were the storey stiffness. The results in terms of frequencies errors and MAC coefficients are listed in Table 5.3, the storey stiffness, the damage parameter and the mode shapes are depicted in Figures 5.10, 5.11 and 5.12. The errors in frequencies are quite low (except for DS6) and MAC coefficients are always greater than 0.90. The second structure analyzed was a two storey three-dimensional infilled framed structure located in El-Centro, CA, US, subjected to ambient vibration

and forced vibration tests. The damages were artificially introduced through infills removal. The parameters to update were the stiffness of infills at the first storey (4 parameters case) and at the ground and first storey (8 parameters case). The infills were replaced by equivalent struts. The 4 parameters case gave very good results in terms of frequencies errors and MAC values (Table 5.8). Also the damage parameters is able to point out the wall removals with great accuracy. Having at disposal a lot of data windows of acquisition for ambient vibration tests, sensitivity analysis have been carried out and parameters along with standard deviations (and CoVs) were obtained and depicted in Figure 5.24. The results are in agreement with the previous ones and, also in this case, the infills removals can be captured by the procedure. Eventually, a 8 parameters case has been analyzed and the results are listed in Tables 5.14 and 5.15. The results were improved respect the 4 parameters case. The parameters at the ground storey (the infills at that storey) were, unfortunately, quite insensitive to the model updating as summarized in Table 5.16.

Subsequently, in Chapter 7, the generalizations of the procedures in order to take into account non-linear parameters were performed. In paragraph 7.2, unrelated non-linear parameters, in which the stiffness and mass matrices can be decomposed in the same way as for the original system, have been studied. Another generalization was made for related non-linear parameters in which the entire problem must be faced in iterative way. The uncertainties evaluation can be performed for both unrelated and related non-linear parameters exploiting the partial derivatives and errors propagation. In the last part of Chapter 7, the analysis of viscous damping for classically damped structures have been performed.

The last part of this work is focused on finding the first trial values of parameters in order to perform a retrofitting of existing structures. Using the algorithm already described in the closed relation (Eq. 2.4 with solution of Eq. 2.16 or 2.28) the uncoupling of mode shapes can be performed. Two sample structures have been analyzed: symmetric-asymmetric three storey frame structure, totally asymmetric one. For these structures the solutions, in order to uncouple the mode shapes, have been achieved with hand calculations and labeled as expected solutions. The solutions achieved by the procedure were the same as the expected ones and the procedure produced the uncoupling of the modes. Generalizations for unrelated and related parameters have been performed and presented in paragraph 7.5.



# APPENDIX A

## EXAMPLE OF DIRECT DISTRIBUTION ANALYSIS

### A.1 Introduction

In this Appendix one simple case of direct distribution analysis of parameters, using the relations of paragraph 2.11, will be analyzed. A case with matrices of grade 2 and only one parameter and one mode is considered.

### A.2 Numerical example

The matrices are defined as follows:

$$K_0 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \quad (\text{A.1})$$

$$K_{r1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{A.2})$$

$$M_0 = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix} \quad (\text{A.3})$$

It is conjectured that the mode components are normally distributed with 5% of CoVs (the Gaussian distributions will be introduced with mean value and variance as parameters of the distribution itself), the frequency is conversely assumed as deterministic quantity:

$$\omega_1^2 = 25 \quad (\text{A.4})$$

$$\underline{\varphi} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad (\text{A.5})$$

$$X_1 = N(0.4472; 5.0000 \cdot 10^{-4}) \quad (\text{A.6})$$

$$X_2 = N(0.8944; 2.0000 \cdot 10^{-4}) \quad (\text{A.7})$$

Starting from Eq. 2.18, the solution of the problem of Eq. 2.4 can be written as follows:

$$\|\underline{K}_{r1} \cdot \underline{\varphi}\|^2 \cdot a_1 = \langle \underline{K}_{r1} \cdot \underline{\varphi}; \underline{\psi} \rangle \quad (\text{A.8})$$

$$a_1 = \frac{\langle \underline{K}_{r1} \cdot \underline{\varphi}; \underline{\psi} \rangle}{\|\underline{K}_{r1} \cdot \underline{\varphi}\|^2} \quad (\text{A.9})$$

in which:

$$\underline{\psi} = \omega_1^2 \underline{M}_0 \cdot \underline{\varphi} - \underline{K}_0 \cdot \underline{\varphi} = 25 \cdot \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} - \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} -1.5 \cdot X_1 + X_2 \\ X_1 - 0.5 \cdot X_2 \end{bmatrix} \quad (\text{A.10})$$

$$\underline{K}_{r1} \cdot \underline{\varphi} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ 0 \end{bmatrix} \quad (\text{A.11})$$

$$\|\underline{K}_{r1} \cdot \underline{\varphi}\|^2 = [X_1 \quad 0] \cdot \begin{bmatrix} X_1 \\ 0 \end{bmatrix} = X_1^2 \quad (\text{A.12})$$

$$\langle \underline{K}_{r1} \cdot \underline{\varphi}; \underline{\psi} \rangle = [X_1 \quad 0] \cdot \begin{bmatrix} -1.5 \cdot X_1 + X_2 \\ X_1 - 0.5 \cdot X_2 \end{bmatrix} = -1.5 \cdot X_1^2 + X_1 \cdot X_2 \quad (\text{A.13})$$

$$a_1 = \frac{-1.5 \cdot X_1^2 + X_1 \cdot X_2}{X_1^2} = \frac{-1.5 \cdot X_1 + X_2}{X_1} \quad (\text{A.14})$$

The solution is written as a function of the distributions of the two mode components. Now the distribution propagation relations (Eqs. 2.158, 2.159 and 2.162) must be used. The numerator presents the sum of two normally distributed independent random variables, therefore the distribution of the sum is:

$$Y(y) = -1.5 \cdot X_1 + X_2 = N(-1.5 \cdot 0.4472 + 0.8944; (-1.5)^2 \cdot 5.0000 \cdot 10^{-4} + 2.0000 \cdot 10^{-3})$$

$$Y(y) = N(0.2236; 3.1250 \cdot 10^{-3}) \quad (\text{A.15})$$

Therefore:

$$a_1 = \frac{Y(y)}{X_1(x)} \quad (\text{A.16})$$

Eq. 2.162 must be then applied. The joint pdf for the case of normally distributed random variables ( $X(x)$  and  $Y(y)$ ) is as follows:

$$f_{XY} = \frac{1}{2 \cdot \pi \cdot \sigma_X \cdot \sigma_Y \cdot \sqrt{1 - \rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)} \left[ \frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho \frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X \sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right]} \quad (\text{A.17})$$

in which  $\sigma_X$  and  $\sigma_Y$  are the standard deviations,  $\mu_X$  and  $\mu_Y$  the mean values of the  $X(x)$  and  $Y(y)$  random variables respectively,  $\rho$  is the correlation coefficient between  $X(x)$  and  $Y(y)$ . If the two random variables are independent ( $\rho=0$ , as for the case analyzed), Eq. A.17 simplifies itself in:

$$f_{XY} = \frac{1}{2 \cdot \pi \cdot \sigma_X \cdot \sigma_Y} \cdot e^{-\frac{1}{2} \left[ \frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right]} \quad (\text{A.18})$$

For parameter  $a_1$ , the distribution is achieved solving the integral of Eq. 2.162 (remembering Eqs. 2.165 and A.16):

$$a_1(z) = \int_{-\infty}^{+\infty} |x| \cdot f_{YX_1}(z \cdot x, x) dx \quad (\text{A.19})$$

$$a_1(z) = \int_{-\infty}^{+\infty} |x| \cdot \frac{1}{2 \cdot \pi \cdot \sqrt{5.0000 \cdot 10^{-4}} \cdot \sqrt{3.1250 \cdot 10^{-3}}} \cdot e^{-\frac{1}{2} \left[ \frac{(x-0.4472)^2}{5.0000 \cdot 10^{-4}} + \frac{(y-0.2236)^2}{3.1250 \cdot 10^{-3}} \right]} dx \quad (\text{A.20})$$

Integral of Eq. A.20 is solved numerically with software MATLAB.

The reference value of the parameter (from which the mode shape components and the circular frequency were computed), the mean value, variance, standard deviation and CoV from the distribution propagation are listed in Table A.1. In the same Table are also listed the mean value, variance, standard deviation and CoV from Monte Carlo analysis.

In Figure A.1 the analytical distribution along with calibrated Gaussian and Cauchy distributions are depicted. The Gaussian distribution fits very well the analytical one, the Chi-square test confirmed this results (level of confidence greater than 10%).

In Figure A.2 the analytical distribution along with the normalized histogram from Monte Carlo analysis are depicted. The agreement is satisfactory.

### A.3 Conclusions

The distribution propagation is used, in this Appendix, in order to evaluate the distribution of one parameter starting from normal distributions in the mode shape components. The results obtained are in very good agreement with respect to the ones obtained through Monte Carlo analysis.





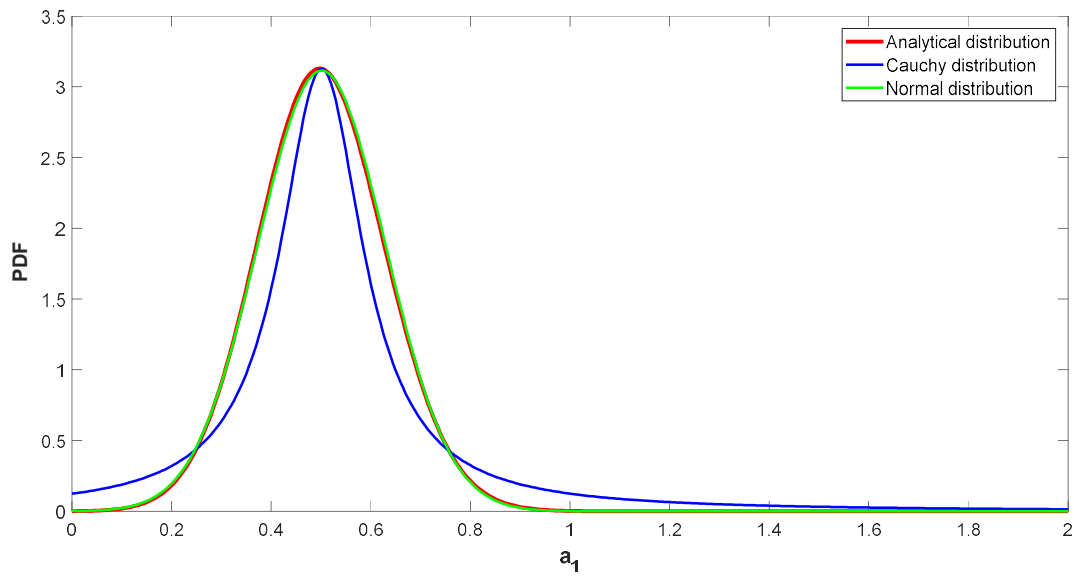
## Table of Appendix A

	Parameter	Reference value	Distribution propagation	Monte Carlo Analysis
$a_1$	Mean value	0.5000	0.5013	0.5074
	Variance	-	0.0165	0.0204
	Standard Deviation	-	0.1284	0.1429
	CoV [%]	-	25.62	28.16

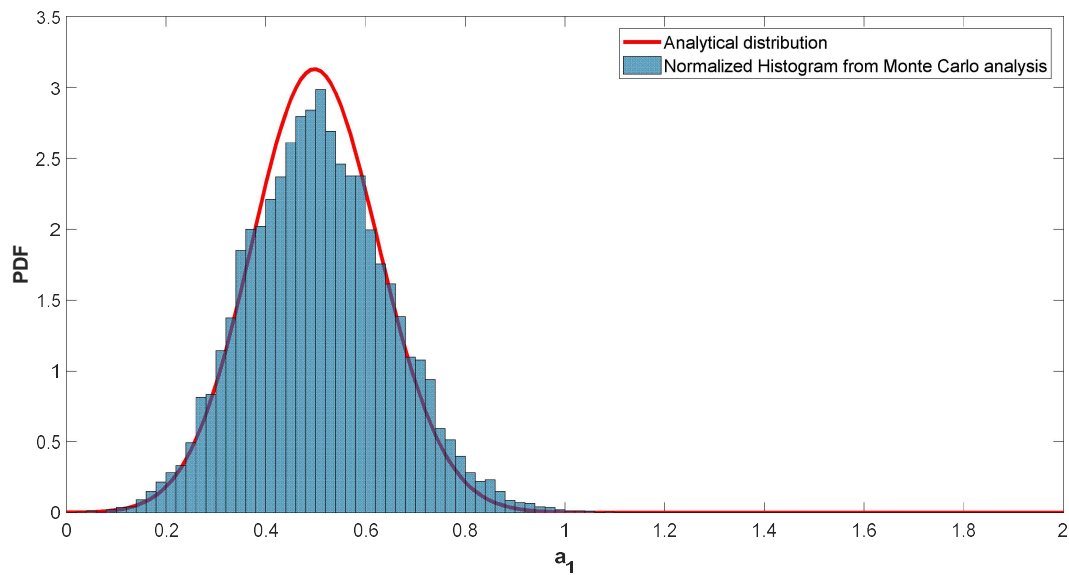
**Table A.1:** Mean values, variances, standard deviations and CoVs of parameter from distribution propagation and Monte Carlo procedure.



## Figures of Appendix A



**Figure A.1:** Analytical distribution along with calibrated Gaussian and Cauchy distributions.



**Figure A.2:** Analytical distribution along with normalized frequencies histogram from Monte Carlo analysis.



## APPENDIX B

### MAXIMUM NUMBER OF PARAMETERS

#### B.1 Introduction

In this Appendix the maximum number of parameters for a sample case, starting from the upper bound given by Eq. 3.1, will be found. The number of parameters will be reduced in order to make sure that all the CoVs are below a fixed thresholds, following the procedure outlined in paragraph 3.6.

#### B.2 Numerical example

The case analyzed has got matrices of grade 3, all the modes and frequencies are considered (3 eigenvalues and 3 eigenvectors used), therefore Eq. 3.1 gives the following upper bound for the number of parameters:

$$N = n \cdot (m + 1) - \sum_{i=1}^n i = 3 \cdot (3 + 1) - (1 + 2 + 3) = 12 - 6 = 6 \quad (\text{B.1})$$

6 is therefore the upper bound for the number of parameters.

The matrices are defined in the following way:

$$\underline{\underline{K}}_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad (\text{B.2})$$

$$\underline{\underline{K}}_{r1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{B.3})$$

$$\underline{\underline{K}}_{r2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \quad (\text{B.4})$$

$$\underline{\underline{K}}_{r3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{B.5})$$

$$\underline{\underline{K}}_{r4} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{B.6})$$

$$\underline{\underline{K}}_{r5} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{B.7})$$

$$\underline{\underline{K}}_{r6} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \quad (\text{B.8})$$

$$\underline{\underline{M}}_0 = \begin{bmatrix} 0.002 & 0 & 0 \\ 0 & 0.002 & 0 \\ 0 & 0 & 0.002 \end{bmatrix} \quad (\text{B.9})$$

The reference values of parameters, in order to compute the frequencies and mode shape vectors, are as follows:

$$\underline{a} = \begin{bmatrix} 12 \\ 5 \\ 10 \\ 4 \\ 6 \\ 0.2 \end{bmatrix} \quad (\text{B.10})$$

The procedure of paragraph 3.6 is followed, computing the CoVs for all the parameters, fixing one parameter if the related CoV is greater than a fixed threshold and running again the procedure. The threshold is fixed, for all the parameters, as follows:

$$\delta_a = 12\% \quad (\text{B.11})$$

In Table B.1 are listed the CoVs for different number of parameters updated. One can see how already for 4 parameters the procedure gives, for this particular case, good results. Therefore, for the fixed threshold, 4 is the maximum number of parameters.

The CoVs are not monotonically decreasing if the number of parameters used is reduced but the maximum value of CoVs is instead monotonically reduced iteration after iteration.

### B.3 Conclusions

The procedure used in this Appendix allows us to compute the maximum number of parameters achievable after the definition of the thresholds for the CoVs. For the case studied, the

upper bound for the number of parameters is 6 but the procedure can achieve only 4 parameters with CoVs below the fixed threshold. Moreover, the CoVs associated to small parameters values are usually greater than that of larger parameters values (as already found for other cases in Chapter 4).





## Table of Appendix B

Parameter	CoV [%]				
	6 parameters	5 parameters	4 parameters	3 parameters	2 parameters
	case	case	case	case	case
$a_1$	7.09	9.09	7.00	8.99	8.23
$a_2$	7.98	9.13	8.21	8.84	8.31
$a_3$	8.56	9.84	8.58	9.71	-
$a_4$	11.05	11.39	11.25	-	-
$a_5$	15.46	18.24	-	-	-
$a_6$	216.03	-	-	-	-

**Table B.1:** CoVs of parameters changing the number of parameters updated.



## APPENDIX C

### RELATED NON-LINEAR PARAMETER CASE

#### C.1 Introduction

A quite simple case of related non-linear parameter is studied in this Appendix. The structure studied is a two storey, one bay reinforced concrete building and the parameter to identify is the interstorey height. The analysis in order to compute the value of the parameter will be performed and then a statistical analysis will be carried out. Eventually, a comparison with Monte Carlo realizations will be reported.

#### C.2 Theoretical relations

##### C.2.1 Definition of solving system for chosen parameter

The sample structure under study is a two storey, one bay reinforced concrete shear-type frame (in Figure C.1 is depicted the structure analyzed). Only one parameter, one frequency and mode shape are considered, therefore the system of Eq. 6.11 simplifies into:

$$\begin{cases} [\underline{\underline{K}}_0 + \underline{\underline{K}}_r(a_1) - \omega_1^2 \cdot \underline{\underline{M}}_0] \cdot \underline{\underline{\phi}}_1 = \underline{\underline{v}}_1 \\ \det([\underline{\underline{K}}_0 + \underline{\underline{K}}_r(a_1) - \omega_1^2 \cdot \underline{\underline{M}}_0]) / w_1 = v_1 \end{cases} \quad (C.1)$$

in which:

$$\underline{\underline{K}}_r(a_1) = 12 \cdot E_c \cdot J \cdot \begin{bmatrix} \frac{2}{a_1^3} + \frac{2}{(H-a_1)^3} & -\frac{2}{(H-a_1)^3} \\ -\frac{2}{(H-a_1)^3} & \frac{2}{(H-a_1)^3} \end{bmatrix} \quad (C.2)$$

in which  $J$  is the second moment of inertia of the columns,  $E_c$  is the modulus of elasticity for the concrete,  $H$  is the total height of the frame and  $a_1$  is the parameter to update.

$$\underline{\underline{K}}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (C.3)$$

$$\underline{\underline{M}}_0 = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad (C.4)$$

being  $m$  the lumped masses at the storey level.  $\underline{\varphi}_1$  and  $\omega_1^2$  was found solving the eigenvalues problem for the subsequent reference value of the parameter:  $a_{1,ref} = 3000mm$ . All the weighting functions are defined following the criteria outlined in Chapter 2.

### C.2.2 Definition of solving system for partial derivatives

Starting from the theoretical results outlined in paragraph 6.3.2, for the case under study Eq. 6.25 can be simplified:

$$\left\langle \frac{\partial^2 \underline{K}_r}{\partial \varphi_{ji} \partial a_1} \cdot \underline{\varphi}; \underline{K}_r \cdot \underline{\varphi} - \underline{\psi} \right\rangle + \left\langle \frac{\partial \underline{K}_r}{\partial a_1} \cdot \underline{\varphi}; \frac{\partial \underline{K}_r}{\partial \varphi_{ji}} \cdot \underline{\varphi} \right\rangle = - \left\langle \frac{\partial \underline{K}_r}{\partial a_1}(:,j); \underline{K}_r \cdot \underline{\varphi} - \underline{\psi} \right\rangle - \left\langle \frac{\partial \underline{K}_r}{\partial a_1} \cdot \underline{\varphi}; \underline{K}_r(:,j) - \omega_i^2 \cdot \underline{M}_0(:,j) \right\rangle \quad (C.5)$$

for  $i = 1$ ;  $j = 1, 2$ . Solving the system of Eq. C.5,  $\frac{\partial a_1}{\partial \varphi_{11}}, \frac{\partial a_1}{\partial \varphi_{21}}$  can be achieved.

$$\underline{\psi} = \omega_1^2 \cdot \underline{M}_0 \cdot \underline{\varphi} - \underline{K}_0 \cdot \underline{\varphi} = \omega_1^2 \cdot \underline{M}_0 \cdot \underline{\varphi} \quad (C.6)$$

The partial derivatives of the matrices, with respect to  $\varphi_{ji}$ , are as follows:

$$\frac{\partial \tilde{\underline{K}}_r}{\partial a_1} = 12 \cdot E_c \cdot J \cdot \begin{bmatrix} -\frac{6}{a_1^4} + \frac{6}{(H-a_1)^4} & -\frac{6}{(H-a_1)^4} \\ -\frac{6}{(H-a_1)^4} & \frac{6}{(H-a_1)^4} \end{bmatrix} \quad (C.7)$$

$$\frac{\partial \tilde{\underline{K}}_r}{\partial \varphi_{ji}} = 12 \cdot E_c \cdot J \cdot \frac{\partial a_1}{\partial \varphi_{ji}} \cdot \begin{bmatrix} -\frac{6}{a_1^4} + \frac{6}{(H-a_1)^4} & -\frac{6}{(H-a_1)^4} \\ -\frac{6}{(H-a_1)^4} & \frac{6}{(H-a_1)^4} \end{bmatrix} \quad (C.8)$$

$$\frac{\partial^2 \tilde{\underline{K}}_r}{\partial \varphi_{ji} \partial a_1} = 12 \cdot E_c \cdot J \cdot \frac{\partial a_1}{\partial \varphi_{ji}} \cdot \begin{bmatrix} \frac{24}{a_1^5} + \frac{24}{(H-a_1)^5} & -\frac{24}{(H-a_1)^5} \\ -\frac{24}{(H-a_1)^5} & \frac{24}{(H-a_1)^5} \end{bmatrix} \quad (C.9)$$

For the case under study, Eq. 6.29 can be simplified:

$$\left\langle \frac{\partial^2 \underline{K}_r}{\partial \omega_i^2 \partial a_s} \cdot \underline{\varphi}; \underline{K}_r \cdot \underline{\varphi} - \underline{\psi} \right\rangle + \left\langle \frac{\partial \underline{K}_r}{\partial a_s} \cdot \underline{\varphi}; \frac{\partial \underline{K}_r}{\partial \omega_i^2} \cdot \underline{\varphi} \right\rangle = \left\langle \frac{\partial \underline{K}_r}{\partial a_s} \cdot \underline{\varphi}; \underline{M}_0 \cdot \underline{\varphi} \right\rangle \quad (C.10)$$

for  $i = 1$ . Solving the system of Eq. C.10,  $\frac{\partial a_1}{\partial \omega_1^2}$  can be achieved.

The partial derivatives of the matrices, with respect to  $\omega_1^2$ , are as follows:

$$\frac{\partial \underline{\tilde{K}}_r}{\partial \omega_1^2} = 12 \cdot E_c \cdot J \cdot \frac{\partial a_1}{\partial \omega_1^2} \cdot \begin{bmatrix} -\frac{6}{a_1^4} + \frac{6}{(H-a_1)^4} & -\frac{6}{(H-a_1)^4} \\ -\frac{6}{(H-a_1)^4} & \frac{6}{(H-a_1)^4} \end{bmatrix} \quad (\text{C.11})$$

$$\frac{\partial^2 \underline{\tilde{K}}_r}{\partial \omega_1^2 \partial a_1} = 12 \cdot E_c \cdot J \cdot \frac{\partial a_1}{\partial \omega_1^2} \cdot \begin{bmatrix} \frac{24}{a_1^5} + \frac{24}{(H-a_1)^5} & -\frac{24}{(H-a_1)^5} \\ -\frac{24}{(H-a_1)^5} & \frac{24}{(H-a_1)^5} \end{bmatrix} \quad (\text{C.12})$$

Once all the partial derivatives have been obtained, the procedure of paragraph 2.7.3 has to be followed in order to obtain the standard deviations, considering that all the experimental outcomes are independent each other, Eq. 6.30 becomes:

$$\sigma_{a,1} \cong \sqrt{\left(\frac{\partial a_1}{\partial \varphi_{11}}\right)^2 \cdot \sigma_{\varphi,11}^2 + \left(\frac{\partial a_1}{\partial \varphi_{21}}\right)^2 \cdot \sigma_{\varphi,21}^2 + \left(\frac{\partial a_1}{\partial \omega_1^2}\right)^2 \cdot \sigma_{\omega^2,1}^2} \quad (\text{C.13})$$

### C.3 Numerical example

The sample structure under study has got columns with sections of 40 cm by 40 cm (having therefore second moment of inertia  $J = 2.13 \cdot 10^9 \text{ mm}^4$ ), modulus of elasticity ( $E_c$ ) equal to 30000 MPa and lumped mass at the storey level equal to 18000 Kg ( $m = 18t$ ). The total height of the frame is equal to 6.50 m ( $H = 6500 \text{ mm}$ ) and the interstorey height is assumed the parameter to update ( $a_1$ ). The matrices have the following definition ( $E_c$  in MPa,  $J$  in  $\text{mm}^4$ ,  $H$  in  $\text{mm}$ , masses in  $10^{-3} \cdot \text{Kg}$ ):

$$\underline{K}_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (\text{C.14})$$

$$\underline{K}_r(a_1) = 7.68 \cdot 10^{14} \cdot \begin{bmatrix} \frac{2}{a_1^3} + \frac{2}{(6500-a_1)^3} & -\frac{2}{(6500-a_1)^3} \\ -\frac{2}{(6500-a_1)^3} & \frac{2}{(6500-a_1)^3} \end{bmatrix} \quad (\text{C.15})$$

$$\underline{M}_0 = \begin{bmatrix} 18.00 & 0 \\ 0 & 18.00 \end{bmatrix} \quad (\text{C.16})$$

Running the procedure, solving system of Eq. C.1, the solution is achieved and listed in Table C.1, this solution is labeled as the mean value for the parameter itself.

Using the procedure for the uncertainties evaluation outlined in paragraph 6.3.2, the partial derivatives of the parameter must be computed using Eqs. C.5 and C.10. The values for the standard

deviations of the frequency and the mode components are fixed in order to give a 5% CoVs. Having the values of the partial derivatives and the standard deviations, Eq. C.13 can be used in order to obtain the standard deviation of the parameter. This quantity, along with variance and CoV, are listed in Table C.1.

A Monte Carlo analysis, with 300 realizations, was then performed using a normally distributed frequency and mode shape components and giving a 5% CoV to all these quantities. The mean value, the standard deviation, variance and CoV of the parameter are also listed in Table C.1.

The target function  $H$  (Eq. 2.67), as a function of the unknown parameter, is depicted in Figure C.2. The figure shows the non-linearity of the function and the presence of a minimum at  $a_1 = 3000mm$  which is the same value as for the reference solution.

In Figure C.3 the frequencies histogram for the Monte Carlo analysis is depicted.

## C.4 Conclusions

The case presented in this Appendix needs the definition of a non-linear parameter which cannot allow us for the use of the stiffness matrix decomposition. The general procedure, outlined in paragraph 6.3, is followed in order to achieve the parameter value (then compared to reference solution) and to achieve the standard deviation of the parameter itself. The parameter value was achieved properly by the procedure, as indicated in Table C.1, and the uncertainties evaluation gave results very similar to the ones obtained through Monte Carlo procedure, with errors in terms of mean value, standard deviation and CoV negligible. These results are also listed in Table C.1. The procedure is also very stable with CoV of the parameter approximately equal to 2% starting from CoVs equal to 5% in the frequency and in the two mode shape components.

## Table of Appendix C

	Parameter	Reference value	Uncertainties evaluation	Monte Carlo Analysis
$a_1$	Mean value	3000	3000	3001
	Variance	-	3697	3858
	Standard Deviation	-	60.80	62.12
	CoV [%]	-	2.03	2.07

**Table C.1:** Mean values, variances, standard deviations and CoVs of parameter from uncertainties evaluation and Monte Carlo procedure.





## Figures of Appendix C

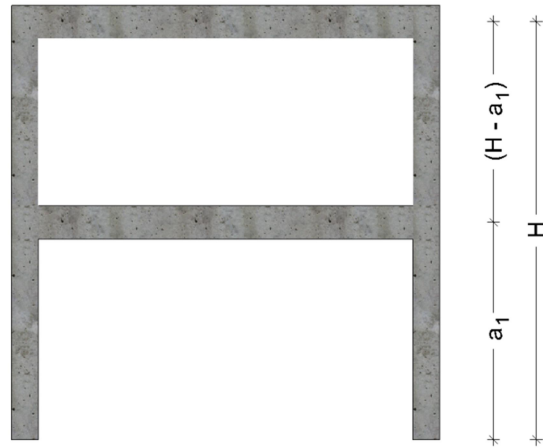


Figure C.1: View of the sample structure.

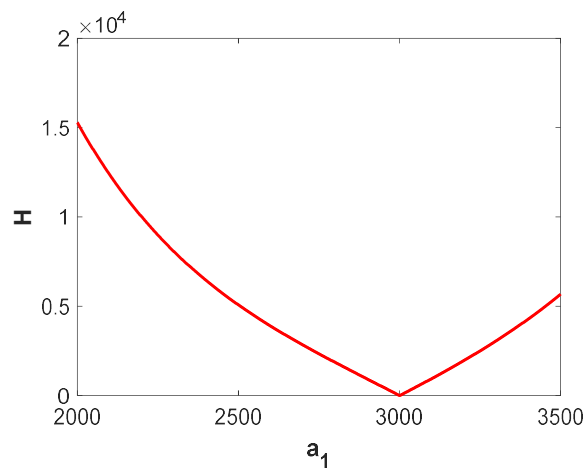


Figure C.2: Target function  $H$  for not perturbed system.

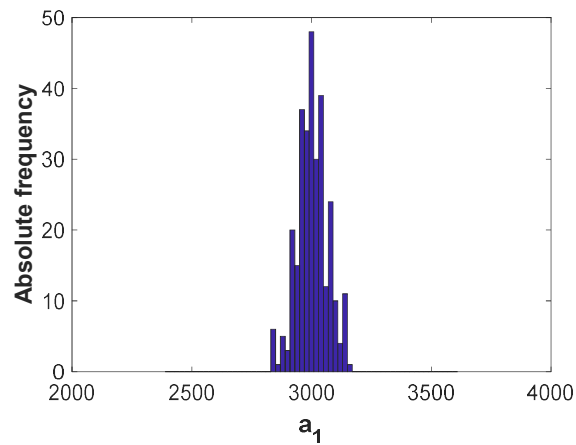


Figure C.3: Frequencies histogram of parameter  $a_1$  from Monte Carlo analysis.



## REFERENCES

- Allemang, R.J., 1984. "Experimental modal analysis bibliography", Proc. IMAC2.
- Allemang, R.J., 2003. "The Modal Assurance Criterion - Twenty Years of Use and Abuse", IMAC-XX, Proc. 20<sup>th</sup> International Modal Analysis Conference, Los Angeles, CA, February 2002.
- Alotto, P., Gaggero, M. and Molinari, G., 1997. "A design of experiment and statistical approach to enhance the generalized response surface method in the optimization of multim minima problems", IEEE Transaction on Magnetics 33(2),1896-1899, 1997.
- Amato, G., Cavaleri, L., Fossetti, M. and Papia, M., 2008. "Infilled frames: influence of vertical load on the equivalent diagonal strut model", Proceedings of the 14<sup>th</sup> World Conference on Earthquake Engineering, Beijing, China.
- Asteris, P.G., Antoniou, S.T., Sophianopoulos, D.S. and Chrysostomou, C.Z., 2011. "Mathematical Macromodeling of Infilled frames: State of the Art", ASCE Journal of structural engineering, pp. 1508-1517.
- Bazan, E. and Meli, R., 1980. "Seismic analysis of structures with masonry walls", Proceedings of the 7<sup>th</sup> WCEE, 5, pp. 633-640, Istanbul, Turkey.
- Beck, J.L. and Katafygiotis, L.S., 1998. "Updating Models and their Uncertainties. I: Bayesian Statistical Framework", Journal of Engineering Mechanics, Vol. 124, No. 4.
- Bendat, J.S. and Piersol, A.G., 1980. "Engineering applications of correlation and spectral analysis", Wiley-Interscience: New York.
- Berman, A. and Nagy, E.J., 1983. "Improvement of a large analytical model using test data", AIAA Journal, 21(8), pp. 1168-1173.
- Branch, M.A., Coleman, T.F. and Li, Y, 1999. "A Subspace, Interior and Conjugate Gradient Method for Large-Scale Bound-Constrained Minimization Problems", SIAM Journal on Scientific Computing, Vol. 21, Number 1, pp. 1-23.
- Byrd, R.H., Schnabel, R.B. and Shultz, G.A., 1988. "Approximate Solution of the Trust Region Problem by Minimization over Two-Dimensional Subspaces", Mathematical Programming, Vol. 40, pp. 247-263.
- Campione, G., Cavaleri, L., Macaluso, G. and Amato, G., 2014. "Evaluation of infilled frames: an update in-plane-stiffness macro model considering the effects of vertical loads", Bull. Earthquake Eng., Springer.
- Casali, M.R., Gagliardi, C. and Grasselli, L., 2002. "Geometria", Progetto Leonardo, Bologna. (In Italian).

- Cavaleri, L., Fossetti, M. and Papia, M., 2005. "Infilled frames: developments in the evaluation of the cyclic behavior under lateral loads", *Structural Engineering and Mechanics*, 21(4), pp. 469-494.
- Ching, J., Muto, M. and Beck, J.L., 2006. "Structural Model Updating and Health Monitoring with Incomplete Modal Data Using Gibbs Sampler", *Computer-Aided Civil and Infrastructure Engineering*, 21 (2006), 242-257.
- Chopra, A.K., 2016. "Dynamics of Structures: Theory and Applications to Earthquake Engineering, Pearson College Division, Fifth edition.
- Chrysostomou, C.Z., Gergely, P. and Abel, J.F., 2002. "A six-strut model for nonlinear dynamic analysis of steel infilled frames", *International J. of Struct. Stability and Dyn.*, 2(3), pp. 335-353.
- Coleman, T. F. and Li, Y., 1996. "An interior, trust region approach for nonlinear minimization subject to bounds", *SIAM J.Optim.*, 6(2), 418–445.
- Coleman, T.F. and Verma, A., 2001. "A preconditioned Conjugate Gradient Approach to Linear Equality Constrained Minimization", *Computational Optimization and Applications*, Vol. 20, Number 1, pp. 61-72.
- Conn, A.R., Gould, N.I.M. and Toint, P.L., 2000. "Trust-Region Methods", *MOS-SIAM Series on Optimization*.
- Cramér, H., 1946. "Mathematical methods of statistics", Princeton University Press.
- Crisafulli, F.J., Carr, A.J. and Park, R., 2000. "Analytical modeling of infilled frame structures - a general review", *Bulletin of the New Zealand society for earthquake engineering*, 33(1), pp. 30-47.
- Crisafulli, F.J. and Carr, A.J., 2007. "Proposed macro-model for the analysis of infilled frame structures", *Bulletin of the New Zealand society for earthquake engineering*, 40(2), pp. 30-47.
- Decanini, L. and Fantin, G.E., 1987. "Modelos simplificados de la mamposteria incluida en porticos. Caracteristicas de rigidez y resistencia lateral en estado limite", *Proceedings of Jornadas Argentinas de Ingegneria Estructural*, pp. 817-836. (In Spanish).
- Durrani, A.J. and Luo, Y.H., 1994. "Seismic Retrofit of Flat-Slab Buildings with Masonry Infills", *Proceedings of the NCEER Workshop on Seismic Response of Masonry Infills*, Technical Report NCEER-94-0004, D.P. Abrams, pp. 1-8, San Francisco, California, USA.
- El-Dakhkhni, W., Elgaaly, M. and Hamid, A., 2003. "Three-strut model for concrete masonry-infilled steel frames", *ASCE Journal of structural engineering*, 129(2), 177.

- Evans, M., Hastings, N. and Peacock, B., 2000. "Statistical Distributions", Hoboken, NJ, Wiley-Interscience.
- Ewins, D.J., Silva, J.M.M. and Maleci, G., 1980. "Vibration analysis of a helicopter with externally-attached carrier structure", *Shock and Vibration Bulletin*, 50(2), pp. 155-171.
- Ewins, D.J., 2000. "Modal Testing: Theory and Practice", New York, John Wiley & Sons.
- Feller, W., 1966. "An Introduction to Probability Theory and Its Applications", Volume II, John Wiley & Sons.
- FEMA-274, 1997. "NEHRP commentary on the guidelines for the seismic rehabilitation of buildings", Washington, DC, USA.
- FEMA-306, 1998. "Evaluation of earthquake damaged concrete and masonry wall buildings: Basic procedures manual", Washington, DC, USA.
- Ferrar, C. and James III, G., 1997. "System identification from ambient vibration measurements on a bridge", *Journal of Sound and Vibration*, 205(1), pp. 1-18.
- Friswell, M.I. and Mottershead, J.E., 1995. "Finite element model updating in structural dynamics", Kluwer Academic Publishers, Dordrecht, The Netherlands.
- Goldberg, D.E., 1989. "Genetic Algorithms in Search, Optimization and Machine Learning", Addison-Wesley.
- He, J., 1987. "Identification of Structural Dynamics Characteristics", Ph.D. Thesis, Imperial College of Science, Technology and Medicine, University of London.
- Hendry, A.W., 1981. "*Structural Brickwork*", MacMillan Press, Ltd., Ed. London.
- Holmes, M., 1961. "Steel frames with brickwork and concrete infilling", *Institution of Civil Engineers*, 19(4), pp. 473-478, London.
- Imregun, M. and Visser, W.J., 1991. "A review of model updating techniques", *Shock & Vibration Digest*, 3(1), pp. 9-20.
- Kaymax, I. and McMahon, C.A., 2005. "A response surface method based on weighted regression for structural reliability analysis", *Probabilistic Engineering Mechanics* 20(1), 11-17, 2005.
- Khuri, A. and Cornell, J.A., 1996. "Response Surfaces. Designs and Analyses", Marcel Dekker Inc., New York.
- Lanconelli, E., 1998. "Lezioni di Analisi Matematica 1", Pitagora Editrice Bologna. (In Italian).
- Lanconelli, E., 2000. "Lezioni di Analisi Matematica 2, Prima Parte", Pitagora Editrice Bologna. (In Italian).
- Lanconelli, E. and Obrecht, E., 2001. "Calcolo differenziale per funzioni di più variabili", Pitagora Editrice Bologna. (In Italian).

- Lang, S., 2001. "Undergraduate algebra", Springer, 2nd edition.
- Lang, S., 2005. "Algebra", Springer, 3rd edition.
- Liauw, T.C. and Kwan, K.H., 1984. "Nonlinear behavior of non-integral infilled frames", *Comput. Struct.*, 18, pp. 551-560.
- Lin, R.M., 1991. "Identification of the Dynamic Characteristics of Non-linear Structures", Ph.D. Thesis, Imperial College of Science, Technology and Medicine, University of London.
- Mainstone, R.J., 1971. "On the stiffness and strengths of infilled frames", *Proceedings of the Institution of Civil Engineers*, 49(2), pp. 57-90, Garston, Watford, UK.
- Mainstone, R.J., 1974. "Supplementary note on the stiffness and strengths of infilled frames", *Current Paper CP 13/74*, Garston, Watford, UK.
- Mares, C., Friswell, M. I. and Mottershead, J. E., 2002. "Model updating using robust estimation", *Mech. Syst. Signal Process.*, 16(1), 169–183.
- MathWorks, 2005. "Matlab: High performance numeric computation and visualization software, User's Guide", MathWorks Inc., Natick, MA, United States.
- MathWorks, 2017. MATLAB and Statistical Toolbox, Release 2017b, MathWorks Inc., Natick, MA, United States.  
[https://www.mathworks.com/products/new\\_products/release2017b.html](https://www.mathworks.com/products/new_products/release2017b.html) (website)
- McKenna, F. and Fenves, G., 2001. "The OpenSEES Command Language Manual: version 1.2", Pacific Earthquake Engineering Center, University of California, Berkeley.
- Moaveni, B., He, X., Conte, J. P. and De Callafon, R. A., 2008. "Damage identification of a composite beam using finite element model updating", *J. Comput. Aided Civil Infrastructure Eng.*, 23(5), 339–359.
- Moaveni, B., He, X., Conte, J. P. and Restrepo, J. I., 2010. "Damage identification study of a seven-story full-scale building slice tested on the UCSD-NEES shake table" *Struct. Saf.*, 32(5), 347–356.
- Moaveni, B., Stavridis, A., Lombaert, G., Conte, J.P. and Shing, P.B., 2013. "Finite-Element Model Updating for Assessment of Progressive Damage in a 3-Story Infilled RC Frame", *J. Struct. Eng.*, 139(10): 1665-1674.
- Moré, J.J. and Sorensen, D.C., 1983. "Computing a Trust Region step", *SIAM Journal on Scientific and Statistical Computing*, Vol. 3, pp. 553-573.
- Mottershead, J.E. and Friswell, M.I., 1993. "Model Updating in structural dynamics: a survey", *Journal Sound and Vibration*, 167(2), pp. 347-375.

- Muto, M. and Beck, J.L., 2008. "Bayesian Updating and Model Class Selection for Hysteretic Structural Models Using Stochastic Simulation", *Journal of Vibration and Control*, 14(1-2), 7-34.
- Myers, R.H. and Montgomery, D.C., 1995. "Response Surface Methodology: process and product optimization using designed experiments", John Wiley & Sons, New York.
- Papoulis, A., 1991. "Probability, Random Variables and Stochastic Processes", third edition, McGraw Hill International Editions.
- Paulay, T. and Priestley, M.J.N., 1992. "Seismic Design of Reinforced Concrete and Masonry Buildings", New York, John Wiley & Sons.
- Peeter, B. and De Roeck, G., 1999. "Reference-based stochastic subspace identification for output-only modal analysis", *Mech. Syst. Signal Process*, 6(3), pp. 855-868.
- Polyakov, S.V., 1960. "On the interaction between masonry filler walls and enclosing frame when loading in the plane of the wall", *Earthquake Engineering*, EERI, Oakland, CA, pp. 36-42.
- OpenSEES, 2016. <http://opensees.berkeley.edu> (website).
- Ross, S.M., 2014. "Introduction to Probability and Statistics for Engineers and Scientists", Academic Press, Fifth Edition.
- Savoia, M. and Vincenzi, L., 2008. "Differential evolution algorithm for dynamic structural identification", *Journal of Earthquake Engineering* 12(5), 800-821.
- Song, M., Yousefianmoghadam, S. Mohammadi, M.E., Moaveni, B., Stavridis, A. and Wood, R.L., 2017. "An application of fine element model updating for damage assessment of a two-story reinforced concrete building and comparison with lidar", *Structural Health Monitoring* 1-22.
- Sorensen, D.C., 1994. "Minimization of a Large Scale Quadratic Function Subject to an Ellipsoidal Constraint", Department of Computational and Applied Mathematics, Rice University, Technical Report TR94-27.
- Stafford Smith, B., 1967. "Methods for Predicting the Lateral Stiffness and Strength of Multi-Storey Infilled Frames", *Buil. Sci.*, Vol. 2, pp.247-257.
- Stavridis, A., 2009. "Analytical and experimental seismic performance assessment of masonry-infilled RC frames", Ph.D. dissertation, Dept. of Structural Engineering, Univ. of California, San Diego.
- Stavridis, A., Koutromanos, I, and Shing, P.B., 2011. "Shake-table tests of a three-story reinforced frame with masonry infill walls", *Earthquake Engineering and Structural Dynamics*, 41(6): 1089-1108.

- Steihaug, T., 1983. "The Conjugate Gradient Method and Trust Regions in Large Scale Optimization", *SIAM Journal on Numerical Analysis*, Vol. 20, pp. 626-637.
- Storn, R. and Price, K., 1997. "Differential Evolution – A Simple and Efficient Heuristic for global Optimization over Continuous Spaces", *Journal of Global Optimization* 11(4), 341-359.
- Tarque, N., Candido, L., Camata, G. and Spacone, E., 2015. "Masonry infilled frame structures: state-of-the-art review of numerical modeling", *Earthquake and Structures*, 8(1).
- Tassios, T.P., 1984. "Masonry Infill and R/C Walls Under Cyclic Actions", *Proc. 3<sup>rd</sup> International Symposium on Wall Structures*, Warsaw, Poland.
- Taylor, J.R., 1996. "An introduction to Error Analysis: The Study of Uncertainties in Physical Measurements", *University Science Books*.
- Teughels, A., 2003. "Inverse modelling of civil engineering structures based on operational modal data", Ph.D. dissertation, Dept. of Civil Engineering, K. U. Leuven, Leuven, Belgium.
- Teughels, A. and De Roeck, G., 2005. "Damage detection and parameter identification by finite element model updating", *Arch. Comput. Meth. Eng.*, 12(2), 123–164.
- Thiruvengadam, V., 1985. "On the natural frequencies of infilled frames", *Earthquake Eng. Struct. Dyn.*, 31(2), pp. 44-46.
- Tondi, M., Bovo, M. and Savoia, M., 2018. "Influence of modeling infills in the identification of the dynamic behavior of a R.C. framed buildings", Under review.
- Tondi, M., Bovo, M., Bassoli, E., Vincenzi, L. and Savoia, M., 2017. "Identificazione della Rigidezza dei Tamponamenti in Strutture Intelaiate Mediante Analisi Inversa", *ANIDIS Conference, Pistoia (2017)*, Italy. (In Italian).
- Vanderplaats, G.N., 1984. "*Numerical Optimization Techniques for Engineering Design*", McGraw-Hill.
- Vincenzi, L. and Savoia, M., 2010. "Improving the speed performance of an Evolutionary Algorithm by a second-order cost function approximation" *Proceedings of the 2<sup>nd</sup> International Conference on Engineering Optimization*, Lisbon, Portugal.
- Vincenzi, L., Bassoli, E. and Gambarelli, P., 2013. "Dynamic behavior of a steel footbridge under pedestrian loads", *Proceedings of the 11<sup>th</sup> International Conference on Vibration Problems*, Lisbon, Portugal.
- Vincenzi, L. and Savoia, M., 2015. "Coupling response surface and differential evolution for parameter identification problems", *Computer-Aided Civil and Infrastructure Engineering* 30(5), 376-393.



- Vincenzi, L. and Gambarelli, P., 2017. "A proper infill sampling strategy for improving the speed performance of a Surrogate-Assisted Evolutionary Algorithm", *Computers and Structures* 178, 58-70.
- Visser, W.J., 1992. "Updating Structural Dynamic Models Using Frequency Response Data", Ph.D. Thesis, Imperial College of Science, Technology and Medicine, University of London.
- Yousenfianmoghadam, S., Song, M., Stavridis, A. and Moaveni, B., 2015. "System Identification of a Two-Story Infilled RC Building in Different Damage States", *Proceedings of the improving the seismic performance of existing buildings and other structures 2015*, San Francisco, CA, 10–12 December 2015, pp. 607–618. Reston, VA: ASCE.