

# FUSION SYSTEMS WITH SOME SPORADIC $J$ -COMPONENTS

JUSTIN LYND AND JULIANNE RAINBOLT

ABSTRACT. Aschbacher’s program for the classification of simple fusion systems of “odd” type at the prime 2 has two main stages: the classification of 2-fusion systems of subintrinsic component type and the classification of 2-fusion systems of  $J$ -component type. We make a contribution to the latter stage by classifying 2-fusion systems with a  $J$ -component isomorphic to the 2-fusion systems of several sporadic groups under the assumption that the centralizer of this component is cyclic.

## 1. INTRODUCTION

The Dichotomy Theorem for saturated fusion systems [AKO11, II 14.3] partitions the class of saturated 2-fusion systems into the fusion systems of characteristic 2-type and the fusion systems of component type. This is a much cleaner statement than the corresponding statement for finite simple groups, and it has a much shorter proof. In the last few years, M. Aschbacher has begun work on a program to give a classification of a large subclass of the 2-fusion systems of component type. A memoir setting down the outline and first steps of such a program is forthcoming [Asc16], but see [Asc15] for a survey of some of its contents. The immediate goal is to give a simpler proof of roughly half of the classification of the finite simple groups by carrying out most of the work in the category of saturated 2-fusion systems.

Let  $\mathcal{F}$  be a saturated fusion system over a finite 2-group  $S$ , of which the standard example is the fusion system  $\mathcal{F}_S(G)$ , where  $G$  is a finite group and  $S$  is a Sylow 2-subgroup of  $G$ . A *component* is a subnormal, quasisimple subsystem. The system is said to be of *component type* if some involution centralizer in  $\mathcal{F}$  has a component. The 2-fusion systems of *odd type* consist of those of *subintrinsic component type* and those of  *$J$ -component type*. This is a proper subclass of the 2-fusion systems of component type. In focusing attention on this restricted class, one is expected to avoid several difficulties in the treatment of *standard form* problems like the ones considered in this paper. By carrying out the work in fusion systems, it is expected that certain difficulties within the classification of simple groups of component type can be avoided, including the necessity of proving Thompson’s  $B$ -conjecture.

We refer to [Asc16] for the definition of a fusion system of subintrinsic component type, as it is not needed in this paper. The fusion system  $\mathcal{F}$  is said to be of  *$J$ -component type* if it is not of subintrinsic component type, and there is a (fully centralized) involution  $x \in S$  such that the 2-rank of  $C_S(x)$  is equal to the 2-rank of  $S$ , and  $C_{\mathcal{F}}(x)$  has a component. We shall call such a component in an involution centralizer a  *$J$ -component*.

---

*Date:* March 13, 2018.

*Key words and phrases.* fusion systems, sporadic groups, involution centralizer, components.

The research of the first author was partially supported by NSA Young Investigator Grant H98230-14-1-0312 and was supported by an AMS-Simons grant which allowed for travel related to this work.

In this paper, we classify saturated 2-fusion systems having a  $J$ -component isomorphic to the 2-fusion system of  $M_{23}$ ,  $J_3$ ,  $McL$ , or  $Ly$  under the assumption that the centralizer of the component is a cyclic 2-group. A similar problem for the fusion system of  $L_2(q)$ ,  $q \equiv \pm 1 \pmod{8}$  was treated in [Lyn15] under stronger hypotheses.

**Theorem 1.1.** *Let  $\mathcal{F}$  be a saturated fusion system over the finite 2-group  $S$ . Suppose that  $x \in S$  is a fully centralized involution such that  $F^*(C_{\mathcal{F}}(x)) \cong Q \times \mathcal{K}$ , where  $\mathcal{K}$  is the 2-fusion system of  $M_{23}$ ,  $J_3$ ,  $McL$ , or  $Ly$ , and where  $Q$  is a cyclic 2-group. Assume further that  $m(C_S(x)) = m(S)$ . Then  $\mathcal{K}$  is a component of  $\mathcal{F}$ .*

Here  $F^*(C_{\mathcal{F}}(x))$  is the generalized Fitting subsystem of the centralizer  $C_{\mathcal{F}}(x)$  [Asc11], and  $m(S) := m_2(S)$  is the 2-rank of  $S$  – that is, the largest rank of an elementary abelian 2-subgroup of  $S$ . We mention that any fusion system having an involution centralizer with a component isomorphic to  $McL$  or  $Ly$  is necessarily of subintrinsic component type by [Asc16, 6.3.5]. This means that, when restricted to those components, Theorem 1.1 gives a result weaker than is needed to fit into the subintrinsic type portion of Aschbacher’s program. However, we have included  $McL$  and  $Ly$  here because our arguments apply equally well in each of the four cases.

There is no almost simple group with an involution centralizer having any of these simple groups as a component, but the wreath product  $G = (K_1 \times K_2)\langle x \rangle$  with  $K_1^x = K_2$  always has  $C_G(x) = \langle x \rangle \times K$  with  $K$  a component that is diagonally embedded in  $K_1 \times K_2$ . The strategy for the proof of Theorem 1.1 is to locate a suitable elementary abelian subgroup  $F$  in the Sylow 2-subgroup of  $\mathcal{K}$ , and then to show that the normalizer in  $S$  of  $E := \langle x \rangle F$  has at least twice the rank as that of  $F$ . Thus, the aim is to force a resemblance with the wreath product, in which  $N_G(\langle x \rangle F)$  modulo core is an extension of  $F_1 \times F_2$  (with  $F_i$  the projection of  $F$  onto the  $i$ th factor) by  $\langle x \rangle \times \text{Aut}_K(F)$ . Lemma 3.2 is important for getting control of the extension of  $E$  determined by  $N_{\mathcal{F}}(E)$  in order to carry out this argument.

**Acknowledgements.** We would like to thank the Department of Mathematics and Statistics at Saint Louis University and the Departments of Mathematics at Rutgers University and Ohio State University for their hospitality and support during mutual visits of the authors. We would also like to thank R. Solomon and R. Lyons for helpful discussions, and an anonymous referee for their comments and suggestions.

## 2. BACKGROUND ON FUSION SYSTEMS

We assume some familiarity with notions regarding saturated fusion systems as can be found in [AKO11] or [Cra11], although some items are recalled below. Most of our notation is standard.

Whenever  $G$  is a group, we write  $G^{\#}$  for the set of nonidentity elements of  $G$ . If we wish to indicate that  $G$  is a split extension of a group  $A \trianglelefteq G$  by a group  $B$ , then we will write  $G = A \cdot B$ . For  $g \in G$ , denote by  $c_g$  the conjugation homomorphism  $c_g: x \mapsto x^g$  and its restrictions. Morphisms in fusion systems are written on the right and in the exponent. That is, we write  $x^\varphi$  (or  $P^\varphi$ ) for the image of an element  $x$  (or subgroup  $P$ ) of  $S$  under a morphism  $\varphi$  in a fusion system, by analogy with the more standard exponential notation for conjugation in a group.

**2.1. Terminology and basic properties.** Throughout this section, fix a saturated fusion system  $\mathcal{F}$  over the  $p$ -group  $S$ . We will sometimes refer to  $S$  as the *Sylow subgroup* of  $\mathcal{F}$ . For a subgroup  $P \leq S$ , we write  $\text{Aut}_{\mathcal{F}}(P)$  for  $\text{Hom}_{\mathcal{F}}(P, P)$ , and  $\text{Out}_{\mathcal{F}}(P)$  for  $\text{Aut}_{\mathcal{F}}(P)/\text{Inn}(P)$ . Whenever two subgroups or elements of  $S$  are isomorphic in  $\mathcal{F}$ , we say that they are  $\mathcal{F}$ -conjugate. Write  $P^{\mathcal{F}}$  for the set of  $\mathcal{F}$ -conjugates of  $P$ . If  $\mathcal{E}$  is a subsystem of  $\mathcal{F}$  on the subgroup  $T \leq S$  and  $\alpha: T \rightarrow S$  is a morphism in  $\mathcal{F}$ , the *conjugate* of  $\mathcal{E}$  by  $\alpha$  is the subsystem  $\mathcal{E}^{\alpha}$  over  $T^{\alpha}$  with morphisms  $\varphi^{\alpha} := \alpha^{-1}\varphi\alpha$  for  $\varphi$  a morphism in  $\mathcal{E}$ .

We first recall some of the terminology for subgroups and common subsystems in a fusion system.

**Definition 2.1.** Fix a saturated fusion system over the  $p$ -group  $S$ , and let  $P \leq S$ . Then

- $P$  is *fully  $\mathcal{F}$ -centralized* if  $|C_S(P)| \geq |C_S(Q)|$  for all  $Q \in P^{\mathcal{F}}$ ,
- $P$  is *fully  $\mathcal{F}$ -normalized* if  $|N_S(P)| \geq |N_S(Q)|$  for all  $Q \in P^{\mathcal{F}}$ ,
- $P$  is  *$\mathcal{F}$ -centric* if  $C_S(Q) \leq Q$  for all  $Q \in P^{\mathcal{F}}$ ,
- $P$  is  *$\mathcal{F}$ -radical* if  $O_p(\text{Out}_{\mathcal{F}}(P)) = 1$ ,
- $P$  is *weakly  $\mathcal{F}$ -closed* if  $P^{\mathcal{F}} = \{P\}$ ,
- the *centralizer* of  $P$  in  $\mathcal{F}$  is the fusion system  $C_{\mathcal{F}}(P)$  over  $C_S(P)$  with morphisms those  $\varphi \in \text{Hom}_{\mathcal{F}}(Q, R)$  such that there is an extension  $\tilde{\varphi} \in \text{Hom}_{\mathcal{F}}(PQ, PR)$  that restricts to the identity on  $P$ ,
- the *normalizer* of  $P$  in  $\mathcal{F}$  is the fusion system  $N_{\mathcal{F}}(P)$  over  $N_S(P)$  with morphisms those  $\varphi \in \text{Hom}_{\mathcal{F}}(Q, R)$  such that there is an extension  $\tilde{\varphi}: PQ \rightarrow PR$  in  $\mathcal{F}$  such that  $P^{\tilde{\varphi}} = P$ .

We write  $\mathcal{F}^f$  and  $\mathcal{F}^c$  for the collections of fully  $\mathcal{F}$ -normalized and  $\mathcal{F}$ -centric, respectively, and we write  $\mathcal{F}^{fc}$  for the intersection of these two collections.

Sometimes we refer to an *element*  $x$  of  $S$  as being fully  $\mathcal{F}$ -centralized, when we actually mean that the group  $\langle x \rangle$  is fully  $\mathcal{F}$ -centralized, especially when  $x$  is an involution. For example, this was done in the the statement of the theorem in the introduction.

Whenever  $P \leq S$ , we write  $\mathfrak{A}(P)$  for the set of  $\alpha \in \text{Hom}_{\mathcal{F}}(N_S(P), S)$  such that  $P^{\alpha}$  is fully  $\mathcal{F}$ -normalized.

**Lemma 2.2.** *For each  $P \leq S$ ,  $\mathfrak{A}(P)$  is not empty. Moreover, for each  $Q \in P^{\mathcal{F}} \cap \mathcal{F}^f$ , there is  $\alpha \in \mathfrak{A}(P)$  with  $P^{\alpha} = Q$ .*

*Proof.* This is [BLO03, A.2(b)], applied with  $K = \text{Aut}(P)$ . □

By a result of Puig, the centralizer  $C_{\mathcal{F}}(P)$  is saturated if  $P$  is fully  $\mathcal{F}$ -centralized, and the normalizer  $N_{\mathcal{F}}(P)$  is saturated if  $P$  is fully  $\mathcal{F}$ -normalized. We write  $O_p(\mathcal{F})$  for the (unique) largest subgroup  $P$  of  $S$  satisfying  $N_{\mathcal{F}}(P) = \mathcal{F}$ , and  $Z(\mathcal{F})$  for the (unique) largest subgroup  $P$  of  $S$  satisfying  $C_{\mathcal{F}}(P) = \mathcal{F}$ . We note that if  $\mathcal{F} = \mathcal{F}_S(G)$  for some finite group  $G$  with Sylow  $p$ -subgroup  $S$ , then  $O_p(G) \leq S$  is normal in  $\mathcal{F}$  so that  $O_p(G) \leq O_p(\mathcal{F})$ , but the converse does not hold in general.

**2.2. The Model Theorem.** A subgroup  $P \leq S$  is  $\mathcal{F}$ -centric if and only if  $C_S(P) \leq P$  and  $P$  is fully  $\mathcal{F}$ -centralized [AKO11, I.3.1]. If  $P$  is  $\mathcal{F}$ -centric and fully  $\mathcal{F}$ -normalized, then the normalizer fusion system  $\mathcal{M} := N_{\mathcal{F}}(P)$  is *constrained* – that is,  $O_p(\mathcal{M})$  is  $\mathcal{M}$ -centric. By the Model Theorem [AKO11, Proposition III.5.10], there is then a unique finite group  $M$  up to isomorphism having Sylow  $p$ -subgroup  $N_S(P)$  and such that  $O_{p'}(M) = 1$ ,  $O_p(M) = O_p(\mathcal{M})$ , and  $\mathcal{F}_{N_S(P)}(M) \cong \mathcal{M}$ . Then  $M$  is said to be a *model* for  $\mathcal{M}$  in this case.

**2.3. Tame fusion systems.** The main hypothesis of Theorem 1.1 is that the generalized Fitting subsystem of the involution centralizer  $\mathcal{C}$  is the fusion system of a finite group  $Q \times K$ , where  $Q$  is a cyclic 2-group, and  $K$  is simple. In this situation,  $C_{\mathcal{F}}(x)$  is itself the fusion system of a finite group  $C$  with  $F^*(C) = Q \times K$ , where  $K \cong M_{23}, McL, J_3,$  or  $Ly$ , since each of these simple groups *tamely realizes* its 2-fusion system [AOV12, Oli16b]. Roughly, a finite group *tamely realizes* its fusion system if every automorphism of its fusion system is induced by an automorphism of the group. Moreover, a fusion system is said to be *tame* if there is some finite group that *tamely realizes* it. We refer to [AOV12] for more details.

The importance of tameness in the context of standard form problems was pointed out in [Lyn15, §§1.5]. The discussion there is centered around the notion of strong tameness, which was needed for proofs of the results of [AOV12], but the contents of [Oli13, GL16] imply that a fusion system is tame if and only if it is strongly tame. Recently, Oliver has established the following useful corollary of the results in [AOV12], which we state for our setup here.

**Theorem 2.3** ([Oli16a, Corollary 2.5]). *Let  $\mathcal{C}$  be a saturated fusion system over a 2-group. Assume that  $F^*(\mathcal{C}) = O_2(\mathcal{C})\mathcal{K}$ , where  $\mathcal{K}$  is simple and tamely realized by a finite simple group  $K$ . Then  $\mathcal{C}$  is tamely realized by a finite group  $C$  such that  $F^*(C) = O_2(C)K$ .*

Note that, upon application of Theorem 2.3 to the involution centralizer  $\mathcal{C} = C_{\mathcal{F}}(x)$  in Theorem 1.1, we have  $O_2(C) = Q = O_2(\mathcal{C})$ . Indeed,  $O_2(C) \leq Q$  since  $O_2(C)$  is normal in  $\mathcal{C}$ , and one sees that  $Q = C_S(K)$  by combining Lemma 1.12(c) of [Lyn15] with Lemma 3.7(c) below. However,  $O_2(C)$  is normal and self-centralizing in  $C_C(K)$  by properties of the generalized Fitting subgroup, so that  $C_C(K)/O_2(C)$  is a group of outer automorphisms of the cyclic 2-group  $O_2(C)$ , and so is itself a 2-group. It follows that  $C_C(K) = C_S(K)$  is a normal 2-subgroup of  $C$  (since  $K \trianglelefteq C$ ), and hence  $Q = C_S(K) \leq O_2(C)$ .

Thus, the effect of Theorem 2.3 for our purposes is that we may work in the group  $C$ , where  $Q$  is a normal subgroup. In particular, in the setup of the Theorem 1.1, the quotient  $C/Q$  is isomorphic to a subgroup of  $\text{Aut}(K)$  containing  $\text{Inn}(K)$ , where  $K$  is one of the simple groups appearing in Theorem 1.1.

### 3. STRUCTURE OF THE COMPONENTS

In this section, we recall some properties of the simple systems appearing in Theorem 1.1 that are required for the remainder.

**Lemma 3.1.** *Let  $G$  be  $A_7$  or  $GL_2(4)$ , and  $V$  a faithful  $\mathbf{F}_2[G]$ -module of dimension 4. Then*

- (a)  $G$  acts transitively on the nonzero vectors of  $V$ ,
- (b)  $C_{GL(V)}(G) \leq G$ ,
- (c)  $H^1(G, V) = 0$ , and
- (d) if  $G$  acts on a homocyclic 2-group  $Y$  with  $\Omega_1(Y) = V$ , then  $Y = V$ .

*Proof.* In each case,  $V$  is irreducible. There is a unique such module for  $GL_2(4) \cong C_3 \times A_5$ , namely the natural  $\mathbf{F}_4[G]$ -module considered as a module over  $\mathbf{F}_2$ , and thus (a) holds in this case. The module for  $A_7$  is unique up to taking duals; clearly points (a) and (b) are independent of the choice between these two modules, and (c) is independent of such a choice by [Asc00, §17]. Note that  $A_7$  acts transitively on  $V^\#$ , which can be seen by noting that a Sylow 7-subgroup acts with exactly one fixed point on  $V^\#$ , and a Sylow 5-subgroup acts with no fixed points. Point (b) holds for  $G = A_7$  by absolute irreducibility. Similarly for

$G = GL_2(4)$ , one has that  $C_{GL(V)}(G) = \text{End}_{\mathbf{F}_2[G]}(V)^\times = \mathbf{F}_4^\times$ , and so (b) follows in this case as  $Z(G) \cong C_3$ . Point (c) for  $G = GL_2(4)$  holds because, by coprime action,  $Z(G)$  (and so  $G$ ) has a fixed point on any 5-dimensional module containing  $V$  as a submodule (see [Asc00, §17]). Point (c) for  $G = A_7$  holds, for example, by applying [AG72] with  $\mathcal{L} = \{L_0, L_1, L_2\}$ , where  $L_1 = C_G((1, 2, 3))$ ,  $L_2 = C_G((4, 5, 6))$ , and  $L_0 = L_1 \cap L_2$ . The  $L_i$  indeed satisfy the hypotheses of that theorem:  $H^0(L_i, V) = 0$  because each Sylow 3-subgroup of  $GL_4(2) \geq G$  has no nontrivial fixed point on  $V$ , and  $H^1(L_i, V) = 0$  using a similar argument via coprime action as above.

We now turn to (d), which follows from a special case of a result of G. Higman [Hig68, Theorem 8.2]. This says that if  $SL_2(4)$  acts faithfully on a homocyclic 2-group  $Y$  in which an element of order 3 acts without fixed points on  $Y^\#$ , then  $Y$  is elementary abelian. In case  $G = GL_2(4)$ ,  $V$  is the natural module for  $G$  and certainly  $SL_2(4) \leq G$  has (with respect to an appropriate basis) a diagonal element of order 3 acting without fixed points. In the case  $G = A_7$ , we have  $G \leq A_8 \cong GL_4(2)$ , and the action of  $G$  on  $V$  is the restriction of the natural (or dual) action of  $GL_4(2)$ . Restriction of either one to  $GL_2(4) \cong C_3 \times SL_2(4) \cong C_3 \times A_5$  shows that  $SL_2(4)$  is embedded in  $G$  as an  $A_5$  moving 5 points in the natural permutation action. So  $SL_2(4)$  is contained in  $G$  up to conjugacy, and as before, it has an element of order 3 acting without fixed points on  $V^\#$ . Hence, (d) holds in this case as well by Higman's Theorem.  $\square$

For a vector space  $E$  over the field with two elements, the next lemma examines under rather strong hypotheses the structure of extensions of  $E$  by certain subgroups of the stabilizer in  $GL(E)$  of a hyperplane.

**Lemma 3.2.** *Let  $E \cong E_{2^{n+1}}$  ( $n \geq 3$ ),  $A = \text{Aut}(E)$ ,  $V \leq E$  with  $V \cong E_{2^n}$ ,  $P = N_A(V)$ , and  $U = O_2(P)$ . Let  $L$  be a complement to  $U$  in  $P$  acting decomposably on  $E$ ,  $x \in E - V$  the fixed point for the action of  $L$ , and  $G \leq L$ . Let  $H$  be an extension of  $E$  by  $UG$  with the given action and let  $X$  be the preimage in  $H$  of  $U$  under the quotient map  $H \rightarrow UG$ . Assume that*

- (a)  $G$  acts transitively on  $V^\#$ ;
- (b)  $C_{GL(V)}(G) \leq G$ ; and
- (c)  $H^1(G, V) = 0$ .

*Then there is a subgroup  $Y$  of  $X$  that is elementary abelian or homocyclic of order  $2^{2n}$  and a  $G$ -invariant complement to  $\langle x \rangle$  in  $X$ .*

*Proof.* Let  $\bar{X} = X/V$ . Since the commutator map

$$(3.3) \quad [x, -] \text{ determines a } G\text{-equivariant linear isomorphism } X/E \rightarrow V,$$

$G$  is transitive on the nonzero vectors of  $X/E$  by (a), and  $\langle \bar{x} \rangle \leq Z(\bar{X})$ . Hence if  $\bar{X}$  is not elementary abelian, then it is extraspecial with center  $\langle \bar{x} \rangle$ , and  $G$  preserves the squaring map  $\bar{X} \rightarrow Z(\bar{X})$ . This is not the case, because  $G$  is transitive on the nonzero vectors of  $X/E$  and  $n \geq 3$ . Therefore,

$$(3.4) \quad \bar{X} \text{ is elementary abelian.}$$

Assumption (c) now yields that there is a  $G$ -invariant complement  $\bar{Y}$  to  $\langle \bar{x} \rangle$ . Let  $Y$  be the preimage of  $\bar{Y}$  in  $X$ . We claim that  $Y$  is abelian; assume on the contrary. Then  $[Y, Y]$  and  $\bar{U}^1(Y)$  are contained in  $V$  since  $\bar{Y}$  is elementary abelian, and by assumption, neither of

these are trivial. Similarly,  $V$  is contained in  $Z(Y)$ , which is not  $Y$ . Therefore,

$$(3.5) \quad V = [Y, Y] = \Phi(Y) = \mathcal{U}^1(Y) = Z(Y),$$

by (a) and (3.3).

By (3.5), the squaring map  $\bar{Y} \rightarrow V$  is  $G$ -equivariant linear isomorphism; let  $\sqrt{\quad}$  be its inverse. Then the map  $\xi: V \rightarrow V$  given by  $\xi(v) = [x, \sqrt{v}]$  is a linear isomorphism commuting with the action of  $G$ , and so  $\xi \in \rho(G)$  by (b), where  $\rho: G \rightarrow GL(V)$  is the structure map. Let  $g \in G$  map to  $\xi^{-1}$  under  $\rho$ . Then  $y \mapsto [x, y^g]$  is the squaring map. This means that for each  $y \in Y$ , we have  $y^{g^x} = y^2 y^g$ . Hence, for each pair  $w, y \in Y$ ,

$$w^2 y^2 w^g y^g = w^{g^x} y^{g^x} = (wy)^{g^x} = (wy)^2 w^g y^g$$

which gives  $w^2 y^2 = (wy)^2 = w^2 y^2 [y, w]$ . Thus  $Y$  is abelian after all. It follows that  $\Omega_1(Y) = V$  or  $Y$  by (a), and this completes the proof of the lemma.  $\square$

We now examine the structure of the simple systems occupying the role of  $\mathcal{K}$  in Theorem 1.1. Let  $T_0$  be a 2-group isomorphic to a Sylow 2-subgroup of  $L_3(4)$ . This is generated by involutions  $t_1, t_2, a_1, a_2, b_1$ , and  $b_2$  such that  $Z(T_0) = \langle t_1, t_2 \rangle$  and with additional defining relations:

$$[a_1, a_2] = [b_1, b_2] = 1, \quad [a_1, b_1] = [a_2, b_2] = t_1, \quad [a_2, b_1] = t_2, \quad [a_1, b_2] = t_1 t_2.$$

A Sylow subgroup of  $M_{23}$  or  $McL$  is isomorphic to a Sylow 2-subgroup of an extension of  $L_3(4)$  by a field automorphism; this is a semidirect product  $T_0 \langle f \rangle$  with

$$f^2 = 1, \quad [a_1, f] = [a_2, f] = a_1 a_2, \quad [b_1, f] = [b_2, f] = b_1 b_2, \quad [t_2, f] = t_1.$$

A Sylow subgroup of  $J_3$  is isomorphic to a Sylow 2-subgroup of  $L_3(4)$  extended by a unitary (i.e., graph-field) automorphism; this is a semidirect product  $T_0 \langle u \rangle$  with

$$u^2 = 1, \quad [a_1, u] = [u, b_1] = a_1 b_1, \quad [a_2, u] = [u, b_2] = a_2 b_2, \quad [t_2, u] = t_1.$$

A Sylow subgroup of  $Ly$  is isomorphic to a Sylow 2-subgroup of  $\text{Aut}(L_3(4))$ ; this is a semidirect product  $T_0 \langle f, u \rangle$  with  $[f, u] = 1$  and the relations above.

Denote by  $T_1$  a 2-group isomorphic to one of  $T_0 \langle f \rangle$ ,  $T_0 \langle u \rangle$ , or  $T_0 \langle f, u \rangle$ . Recall that the *Thompson subgroup*  $J(P)$  of a finite  $p$ -group  $P$  is the subgroup generated by the elementary abelian subgroups of  $P$  of largest order.

**Lemma 3.6.** *Let  $K$  be  $M_{23}$ ,  $McL$ ,  $J_3$ , or  $Ly$ , with Sylow 2-subgroup  $T_1$  as above. Then*

- (a)  $Z(T_1) = \langle t_1 \rangle$  is of order 2;
- (b)  $\mathcal{A}(T_1) = \{F_1, F_2\}$  where  $F_1 = \langle t_1, t_2, a_1, a_2 \rangle$  and  $F_2 = \langle t_1, t_2, b_1, b_2 \rangle$ , so that  $J(T_1) = T_0$ . Also, after suitable choice of notation, one of the following holds:
  - (i)  $K = M_{23}$ ,  $McL$ , or  $Ly$ , and  $\text{Aut}_K(F_1) \cong A_7$ , or
  - (ii)  $K = J_3$  and  $\text{Aut}_K(F_1) \cong GL_2(4)$ .
- (c) There is  $F \in \mathcal{A}(T_1)$  such that the pair  $(\text{Aut}_K(F), F)$  satisfies assumptions (a)-(c) of Lemma 3.2 in the role of  $(G, V)$ .
- (d) All involutions in  $\langle t_1, t_2 \rangle$  are  $\text{Aut}_K(J(T_1))$ -conjugate.

*Proof.* Point (a) holds by inspection of the relations above. Now  $F_1$  and  $F_2$  are the elementary abelian subgroups of  $T_0$  of maximal rank, and so to prove (b), it suffices to show that each elementary abelian subgroup of maximal rank in  $T_1$  is contained in  $T_0$ . Set  $L := L_3(4)$ , and identify  $\text{Inn}(L)$  with  $L$ . Write  $\text{Inndiag}(L) \geq L$  for the group of inner-diagonal automorphisms

of  $L$ . Then  $\text{Inndiag}(L)$  contains  $L$  with index 3, corresponding to the size of the center of the universal version  $SL_3(4)$  of  $L$  [GLS98, Theorem 2.5.12(c)]. Also,  $\text{Aut}(L)$  is a split extension of  $\text{Inndiag}(L)$  by  $\Phi_L \Gamma_L = \Phi_L \times \Gamma_L$ , where  $\Phi_L = \langle \varphi \rangle \cong C_2$  is generated by a field automorphism of  $L$ , and  $\Gamma_L = \langle \gamma \rangle \cong C_2$  is generated by a graph automorphism [GLS98, Theorem 2.5.12]. By [GLS98, Theorems 4.9.1, 4.9.2], each involution of  $\text{Aut}(L) - L$  is  $\text{Aut}(L)$ -conjugate to  $\varphi$ ,  $\varphi\gamma$ , or  $\gamma$ , and the centralizers in  $L$  of these automorphisms are isomorphic to  $L_3(2)$ ,  $U_3(2) \cong (C_3 \times C_3)Q_8$ , and  $Sp_2(4) \cong A_5$ , respectively, again by those theorems. These centralizers have 2-ranks 2, 1, and 2, respectively. Since  $T_0$  has 2-rank 4, this shows that  $J(T_1) \leq T_0$ . From the relations used in defining  $T_0$ , each involution in  $T_0$  is contained in one of  $F_1$  or  $F_2$ , so we conclude that  $\mathcal{A}(T_1) = \mathcal{A}(T_0) = \{F_1, F_2\}$ . The description of the automizers in (b) follows from [Fin76a, Table 1] for  $M_{23}$  and  $McL$ , [Fin76b, Lemma 3.7] for  $J_3$ , and [Wil84] for  $Ly$ . Now point (c) follows from (b) and Lemma 3.1, and point (d) follows from (c) and Burnside's fusion theorem (i.e. the statement that the automizer of a weakly  $K$ -closed subgroup of  $T_1$  (which is  $J(T_1)$  in this case) controls the  $K$ -conjugacy in its center).  $\square$

**Lemma 3.7.** *Let  $K$  be one of the sporadic groups  $M_{23}$ ,  $McL$ ,  $J_3$ , or  $Ly$ , and let  $T_1$  be a Sylow 2-subgroup of  $K$ . Then*

- (a)  $\text{Out}(K) = 1$  if  $K \cong M_{23}$  or  $Ly$ , and  $\text{Out}(K) \cong C_2$  otherwise; and
- (b) for each involution  $\alpha \in \text{Aut}(K) - \text{Inn}(K)$ ,
  - (i)  $C_K(\alpha) \cong M_{11}$  if  $K \cong McL$ ,
  - (ii)  $C_K(\alpha) \cong L_2(17)$  if  $K \cong J_3$ ; and
- (c) each automorphism of  $K$  centralizing a member of  $\mathcal{A}(T_1)$  is inner.

*Proof.* Points (a) and (b) follow by inspection of Table 5.3 of [GLS98]. By Lemma 3.6(b), the 2-rank of  $K$  is 4, while each of the centralizers in (b)(i-ii) is of 2-rank 2, so (c) holds.  $\square$

#### 4. PRELIMINARY LEMMAS

We now begin in this section the proof of Theorem 1.1, and so we fix the notation and hypotheses that will hold throughout the remainder of the paper.

Let  $\mathcal{F}$  be a saturated fusion system over the 2-group  $S$ , and let  $x \in S$  be an involution. Assume that  $\langle x \rangle$  is fully  $\mathcal{F}$ -centralized, that  $m(C_S(x)) = m(S)$ , and that  $F^*(C_{\mathcal{F}}(x)) = Q \times \mathcal{K}$ , where  $Q$  is cyclic. Set  $\mathcal{C} = C_{\mathcal{F}}(x)$  and  $T = C_S(x)$ , so that  $\mathcal{C}$  is a saturated fusion system over  $T$  by the remark just after Lemma 2.2. Let  $T_1$  be the Sylow subgroup of  $\mathcal{C}$ , and set

$$R := Q \times T_1 \leq T.$$

Assume that  $\mathcal{K}$  is the fusion system of one of the sporadic groups  $K = M_{23}$ ,  $J_3$ ,  $McL$ , or  $Ly$ . Since  $K$  tamely realizes  $\mathcal{K}$  in each case, the quotient

$$(4.1) \quad T/R \text{ induces a 2-group of outer automorphisms of } K$$

by Theorem 2.3. Arguing by contradiction, we assume

$$\mathcal{K} \text{ is not a component of } \mathcal{F}.$$

We fix the presentation in Section 3 for  $T_1$  in whichever case is applicable, and we note that  $\Omega_1(Q) = \langle x \rangle$  by assumption on  $Q$ .

**Lemma 4.2.** *Notation may be chosen so that  $T$  and  $J(T)$  are fully  $\mathcal{F}$ -normalized.*

*Proof.* We repeatedly use Lemma 2.2. Let  $\alpha \in \mathfrak{A}(T)$ . Then as  $|C_S(x^\alpha)| \geq |T^\alpha| = |T| = |C_S(x)|$ , we have that  $x^\alpha$  is still fully  $\mathcal{F}$ -centralized. Thus we may assume that  $T$  is fully  $\mathcal{F}$ -normalized after replacing  $x$ ,  $T$ , and  $J(T)$  by their conjugates under  $\alpha$ . Now let  $\beta \in \mathfrak{A}(J(T))$ . Then as  $N_S(T) = N_{N_S(J(T))}(T)$ , it follows that

$$|N_S(T^\beta)| = |N_{N_S(J(T)^\beta)}(T^\beta)| \geq |N_{N_S(J(T))^\beta}(T^\beta)| = |N_{N_S(J(T))}(T)^\beta| = |N_S(T)^\beta| = |N_S(T)|$$

and so equality holds because  $T$  is fully  $\mathcal{F}$ -normalized. Hence  $T^\beta$  is still fully  $\mathcal{F}$ -normalized. As before,  $x^\beta$  is still fully  $\mathcal{F}$ -centralized.  $\square$

**Lemma 4.3.**  $\langle x \rangle$  is not weakly  $\mathcal{F}$ -closed in  $T$ .

*Proof.* Assume on the contrary, in which case  $T = S$ , otherwise  $N_S(T)$  contains  $T$  properly and moves  $x$ . It follows that  $x \in Z(S)$ , which is contained in the center of every  $\mathcal{F}$ -centric subgroup. Hence  $x \in Z(\mathcal{F})$  by Alperin's fusion theorem [BLO03, Theorem A.10], and we conclude that  $\mathcal{K}$  is a component of  $\mathcal{C} = \mathcal{F}$ , contrary to hypothesis.  $\square$

**Lemma 4.4.** The following hold.

- (a)  $J(T) = J(R) = \langle x \rangle \times J(T_1)$ ; and
- (b)  $C_T(T_1) = Q\langle t_1 \rangle$ .

*Proof.* Suppose (a) does not hold. Choose  $A \in \mathcal{A}(T)$  with  $A \not\leq \langle x \rangle \times J(T_1)$ . Then  $A \not\leq R$  by the structure of  $R$ , and  $A$  acts nontrivially on  $K$  by (4.1). In particular,  $\text{Out}(K) = 2$  and  $m(C_R(A)) \leq 3$  by Lemma 3.7. Hence,  $m(A) \leq 4$  while  $m(R) = 5$ . This contradicts the choice of  $A$  and establishes (a). By Lemma 3.6(a),  $C_R(T_1) = Q\langle t_1 \rangle$ . Also,  $C_T(T_1) = C_R(T_1)$  by part (a), so (b) is also established.  $\square$

**Lemma 4.5.** If  $T = S$ , then  $\Omega_1 Z(S) = \langle x, t_1 \rangle$ . If  $T < S$ , then  $\Omega_1(Z(S)) = \langle t_1 \rangle$ .

*Proof.* Note first that  $\Omega_1(Z(S)) \leq T$ . By Lemma 3.7(c) and (4.1),  $\Omega_1(Z(S)) \leq R$ , and so  $\Omega_1 Z(S) \leq \Omega_1(Z(R)) = \langle x, t_1 \rangle$  by Lemma 3.6(a). Thus, the lemma holds in case  $T = S$ . In case  $T < S$ , let  $a \in N_S(T) - T$  with  $a^2 \in T$ . Note that  $[J(T), J(T)] = [J(T_1), J(T_1)] = \langle t_1, t_2 \rangle$  by Lemma 4.4(a). So  $a$  normalizes  $\Omega_1 Z(T) = \langle x, t_1 \rangle$  and  $\Omega_1 Z(T) \cap [J(T), J(T)] = \langle x, t_1 \rangle \cap \langle t_1, t_2 \rangle = \langle t_1 \rangle$ , but  $a$  does not centralize  $x$ . Thus,  $\Omega_1 Z(S) = \langle t_1 \rangle$  as claimed.  $\square$

**Lemma 4.6.** The following hold.

- (a)  $x$  is not  $\mathcal{F}$ -conjugate to  $t_1$ ; and
- (b)  $x$  is conjugate to  $xt_1$  if and only if  $T < S$ , and in this case  $x$  is  $N_S(T)$ -conjugate to  $xt_1$ .

*Proof.* For part (a), let  $\varphi \in \mathcal{F}$  with  $x^\varphi \in Z(T)$ . Assume first that  $T = S$ . By the extension axiom,  $\varphi$  extends to  $\tilde{\varphi} \in \text{Aut}_{\mathcal{F}}(T)$ , which restricts to an automorphism of  $J(T)$ . Lemma 4.4(a) shows that  $x \notin [J(T), J(T)]$ , while  $t_1 \in [J(T), J(T)]$  from Lemma 3.6(d). Hence,  $x^\varphi \neq t_1$ ; this shows  $x$  is not  $\mathcal{F}$ -conjugate to  $t_1$  in this case.

Now assume that  $T < S$ . Then  $x \notin Z(S)$ , whereas  $Z(S) = \langle t_1 \rangle$  by Lemma 4.5. Since  $\langle x \rangle$  is fully  $\mathcal{F}$ -centralized by assumption, we conclude that  $x$  is not  $\mathcal{F}$ -conjugate to  $t_1$  in this case either. This completes the proof of (a).

If  $T < S$ , then  $x$  is  $N_S(T)$ -conjugate to  $xt_1$  by (a), while if  $T = S$ , then (a) and Burnside's fusion theorem imply that  $\langle x \rangle$  is weakly  $\mathcal{F}$ -closed in  $\Omega_1(Z(S)) = \langle x, t_1 \rangle$ . Thus, (b) holds.  $\square$



## 5. THE 2-CENTRAL CASE

In this section it is shown that  $T < S$ ; that is,  $x$  is not 2-central. We continue the notation set at the beginning of Section 4.

**Lemma 5.1.** *If  $T = S$ , then  $\langle x \rangle$  is weakly  $\mathcal{F}$ -closed in  $R$ .*

*Proof.* Assume  $T = S$ . Then  $\Omega_1(Z(S)) = \langle x, t_1 \rangle$  by Lemma 4.5. Using Burnside's fusion theorem and assumption, we see from Lemma 4.6 that

$$(5.2) \quad \langle x \rangle \text{ is weakly } \mathcal{F}\text{-closed in } \langle x, t_1 \rangle.$$

By inspection of [GLS98, Table 5.3],  $K$  has one class of involutions. Thus, there are exactly three  $\mathcal{C}$ -classes of involutions, namely  $\{x\}$ ,  $(xt_1)^{\mathcal{C}}$ , and  $t_1^{\mathcal{C}}$ . The lemma therefore holds by (5.2).  $\square$

**Lemma 5.3.** *If  $T = R$ , then  $T < S$ . In particular,  $T < S$  in case  $\mathcal{K}$  is the fusion system of  $M_{23}$  or  $Ly$ .*

*Proof.* Assume  $T = R$ , and also to the contrary that  $T = S$ . By Lemma 5.1,  $\langle x \rangle \leq Z(S)$  is weakly  $\mathcal{F}$ -closed, and so is fixed by each automorphism of each  $\mathcal{F}$ -centric subgroup. Therefore,  $\langle x \rangle \leq Z(\mathcal{F})$ , and so  $\mathcal{K}$  is a component of  $\mathcal{C} = \mathcal{F}$ , contrary to assumption. The last statement follows then follows from Lemma 3.7(a).  $\square$

**Lemma 5.4.** *Assume  $\mathcal{K}$  is the fusion system of  $McL$  or  $J_3$ . Then  $T < S$ .*

*Proof.* Assume  $T = S$ . Then  $R < T$  by Lemma 5.3. Fix  $f \in x^{\mathcal{F}} \cap (T - R)$ . The extension  $\mathcal{K}\langle f \rangle := \mathcal{K}T_1\langle f \rangle$  of  $\mathcal{K}$  is defined by [Asc11, §8], and  $\mathcal{K}\langle f \rangle/Q$  is the 2-fusion system of  $\text{Aut}(McL)$  or  $\text{Aut}(J_3)$  by Theorem 2.3. Thus,  $|T : R| = 2$  by Lemma 3.7(b).

Conjugating in  $\mathcal{K}\langle f \rangle$  if necessary, we may assume  $\langle f \rangle$  is fully  $\mathcal{K}\langle f \rangle$ -centralized. By Lemma 3.7(b), all involutions of  $T_1\langle f \rangle - T_1$  are  $\mathcal{K}\langle f \rangle$ -conjugate, and  $C_{\mathcal{K}\langle f \rangle}(f) \cong \langle f \rangle \times \mathcal{F}_2(M_{11})$  or  $\langle f \rangle \times \mathcal{F}_2(L_2(17))$ . In particular,  $C_{T_1}(f)$  is semidihedral or dihedral, respectively, of order 16 and with center  $\langle t_1 \rangle$ . Fix a four subgroup  $V \leq C_{T_1}(f)$ . Then  $f$  is conjugate to  $ft_1$  (for example, by an element in the normalizer of  $C_{T_1\langle f \rangle}(f)$  in  $T_1\langle f \rangle$ ), and hence is  $\mathcal{F}$ -conjugate to each element of  $fV$  by the structures of  $\mathcal{F}_2(M_{11})$  and  $\mathcal{F}_2(L_2(17))$ .

Fix  $\alpha \in \text{Hom}_{\mathcal{F}}(\langle f \rangle, \langle x \rangle)$ . By the extension axiom,  $\alpha$  extends to a morphism, which we also call  $\alpha$ , defined on  $\langle f \rangle V \leq C_T(f)$ . Therefore,  $x$  is  $\mathcal{F}$ -conjugate to each element in  $xV^{\alpha}$ . Now the intersection  $xV^{\alpha} \cap R = V^{\alpha} \cap R$  is nontrivial because  $|T : R| = 2$ , so as  $x$  is not itself in  $V^{\alpha}$ , we see that  $x$  has a distinct conjugate in  $R$ . This contradicts Lemma 5.1 and completes the proof.  $\square$

## 6. PROOF OF THEOREM 1.1

Continue the notation and hypotheses set at the beginning of Section 4. In addition, we fix  $F \in \mathcal{A}(T_1)$  satisfying assumptions (a)-(c) of Lemma 3.2, as guaranteed by Lemma 3.6(c), and set  $E := \langle x \rangle F$ . Then  $E \in \mathcal{A}(T)$  by Lemma 4.4(a), and so

$$(6.1) \quad m(T) = 5.$$

In this section, we finish the proof of Theorem 1.1, by showing that the hypotheses of Lemma 3.2 hold for a model of the normalizer in  $\mathcal{F}$  of an appropriate  $\mathcal{F}$ -conjugate of  $E$ . Via Lemma 3.1(d), this forces the 2-rank of  $S$  to be at least 8, contrary to the hypothesis that  $m(S) = m(T)$ .

By Lemmas 5.3 and 5.4,

$$T < S.$$

**Lemma 6.2.**  $|\text{Aut}_{\mathcal{F}}(T) : \text{Aut}_{\mathcal{C}}(T)| = 2$ .

*Proof.* Represent  $\text{Aut}_{\mathcal{F}}(T)$  on  $\Omega_1 Z(T) = \langle x, t_1 \rangle$  and apply Lemma 4.6.  $\square$

**Lemma 6.3.** *The following hold.*

- (a)  $|\text{Aut}_{\mathcal{F}}(J(T)) : \text{Aut}_{\mathcal{C}}(J(T))| = 4$  and
- (b)  $x^{\text{Aut}_{\mathcal{F}}(E)} = xF$ , and so  $|\text{Aut}_{\mathcal{F}}(E) : \text{Aut}_{\mathcal{C}}(E)| = 16$ .

*Proof.* Represent  $\text{Aut}_{\mathcal{F}}(J(T))$  on  $\Omega_1 Z(J(T)) = \langle x, t_1, t_2 \rangle$ . Now  $\text{Aut}_{\mathcal{C}}(J(T)) = C_{\text{Aut}_{\mathcal{F}}(J(T))}(x)$ , and the former is transitive on  $\Omega_1 ZJ(T_1)^{\#}$  by Lemma 3.6(d). Also, since  $x$  is  $N_S(T)$ -conjugate to  $xt_1$ , we conclude from Lemma 4.6 that  $x^{\text{Aut}_{\mathcal{F}}(J(T))} = xZJ(T_1)$  is of size 4. Thus, (a) holds.

Similarly to (a), we have  $\text{Aut}_{\mathcal{C}}(E) = C_{\text{Aut}_{\mathcal{F}}(E)}(x)$ , and the former is transitive on  $F^{\#}$  by choice of  $F$ . From Lemma 4.4(a) and Lemma 3.6(a),  $|\mathcal{A}(T)| = 2$ . By part (a) and Lemma 4.2,  $|N_S(J(T)) : T| = 4$ . Representing  $N_S(J(T))$  on  $\mathcal{A}(T)$ , we see that the kernel has index at most 2, so there is an element of  $N_S(J(T)) - T$  that normalizes  $E$ . In particular,  $N_T(E) < N_S(E)$ , and so  $x$  is  $\text{Aut}_{\mathcal{F}}(E)$ -conjugate to a member of  $xF^{\#}$ . Now by choice of  $F$ , another appeal to Lemma 4.6 yields that  $x^{\text{Aut}_{\mathcal{F}}(E)} = Fx$  has size 16, which establishes (b).  $\square$

**Lemma 6.4.** *The following hold:*

- (a)  $Q = \langle x \rangle$ .
- (b)  $E$  is  $\mathcal{F}$ -centric.

*Proof.* Suppose on the contrary that  $Q > \langle x \rangle$  and choose  $w \in Q$  with  $w^2 = x$ . Fix  $a \in N_S(T) - T$  such that  $a^2 \in T$ . Then  $x^a = xt_1$  and also  $(w^a)^2 = xt_1$ . Further  $\langle w^a \rangle$  is normal in  $T$  since  $\langle w \rangle$  is. Thus,  $[\langle w^a \rangle, T_1] \leq \langle w^a \rangle \cap T_1 = 1$ , whereas  $C_T(T_1) = Q\langle t_1 \rangle$  by Lemma 4.4(b). It follows that  $\langle xt_1 \rangle = \mathcal{U}^1(\langle w^a \rangle) \leq \Omega_1 \mathcal{U}^1(Q\langle t_1 \rangle) = \langle x \rangle$ , a contradiction that establishes (a).

Let  $E_0$  be one of the two elementary abelian subgroups of rank 5 in  $T$ , and set  $F_0 = E_0 \cap J(T_1)$ . Then  $E_0 = \langle x \rangle F_0$  contains  $x$ , and so  $C_S(E_0) = C_T(E_0)$ . By Lemma 3.7(c),  $C_T(E_0) = C_R(E_0) = QF_0$ . Hence  $C_S(E_0) = QF_0 = E_0$  by part (a).

We can now prove (b). Fix  $\alpha \in \mathfrak{A}(E)$ . Since  $\langle x \rangle$  is fully centralized, the restriction of  $\alpha^{-1}$  to  $\langle x^\alpha \rangle$  has an extension  $\beta : C_S(x^\alpha) \rightarrow C_S(x) = T$ , which is defined on  $C_S(E^\alpha)$ . Thus, setting  $E_0 := C_S(E^\alpha)^\beta \leq T$ , we see from the previous paragraph that  $|C_S(E)| = |C_S(E_0)| = |E_0| = |C_S(E^\alpha)|$ , so that  $C_S(E^\alpha) = E^\alpha$ . As  $E^\alpha$  is fully  $\mathcal{F}$ -normalized and contains its centralizer in  $S$ , this means that  $E$  is  $\mathcal{F}$ -centric, as claimed.  $\square$

Since we will be working in  $N_{\mathcal{F}}(E)$  for the remainder, we may assume, after replacing  $E$  by an  $\mathcal{F}$ -conjugate if necessary, that  $E$  is fully  $\mathcal{F}$ -normalized. Hence  $E \in \mathcal{F}^{fc}$  by Lemma 6.4(b). Fix a model  $H$  for  $N_{\mathcal{F}}(E)$  (cf. §§2.2).

**Lemma 6.5.**  *$H$  satisfies the hypotheses of Lemma 3.2.*

*Proof.* Set  $\tilde{G} = \text{Aut}_{\mathcal{C}}(E)$  and observe that  $\text{Aut}_{\mathcal{F}}(E) \cong H/E$  by Lemma 6.4(b). Thus  $\tilde{G}$  contains  $G := A_7$  or  $GL_2(4)$  with index 1 or 2. As  $G$  acts transitively on  $F^{\#}$  and centralizes  $x$ , it follows from Lemma 6.3(b) that  $xF$  and  $F^{\#}$  are the orbits of  $\text{Aut}_{\mathcal{F}}(E)$  on  $E^{\#}$ . Hence

$\text{Aut}_{\mathcal{F}}(E) \leq N_{\text{Aut}(E)}(F)$ , a nontrivial split extension of an elementary abelian 2-group  $U$  of order 16 by  $GL(F)$  with the standard action.

We claim that  $U \leq \text{Aut}_{\mathcal{F}}(E)$ . Suppose that this is not the case. Now  $G$  acts transitively on  $F^{\#}$  and the commutator map  $[x, -]$  defines an isomorphism of  $G$ -modules from  $U$  to  $F$ , so  $G$  acts transitively on  $U^{\#}$ . Since  $U$  is normalized by  $\text{Aut}_{\mathcal{F}}(E)$ , we see that  $\text{Aut}_{\mathcal{F}}(E) \cap U = 1$ . In particular,  $\text{Aut}_{\mathcal{F}}(E)$  embeds into  $GL(F)$ . Now  $G \cong A_7$  or  $GL_2(4)$  in the cases under consideration, and by Lemma 6.3(b),  $\text{Aut}_{\mathcal{F}}(E)$  is therefore a subgroup of  $GL(F)$  containing  $G$  with index 16 or 32. However,  $A_7$  has index 8 in  $GL_4(2)$ , and  $GL_2(4)$  is contained with index 2 in a unique maximal subgroup of  $GL_4(2)$ , a contradiction. Therefore,  $U \leq \text{Aut}_{\mathcal{F}}(E)$  as claimed.

It has thus been shown that  $\text{Aut}_{\mathcal{F}}(E)$  contains a subgroup with index 1 or 2 that is a split extension of  $U = O_2(N_{\text{Aut}(E)}(F))$  by  $G$ . Thus,  $H$  has a subgroup of index 1 or 2 that is an extension of  $E$  by  $UG$ . Assumptions (a)-(c) of Lemma 3.2 hold via Lemma 3.6(c) by the choice of  $F$ .  $\square$

*Proof of Theorem 1.1.* Keep the notation of the proof of Lemma 6.5. By that lemma and Lemma 3.2, there is a  $G$ -complement  $Y$  to  $\langle x \rangle$  in  $O_2(H)$  that is homocyclic of order  $2^8$  with  $\Omega_1(Y) = F$ , or elementary abelian of order  $2^8$ . Now  $G$  is isomorphic to  $A_7$  or  $GL_2(4)$  with faithful action on  $F$ , so the former case is impossible by Lemma 3.1. Hence,  $m_2(T) = 5 < 8 \leq m_2(S)$ , contrary to hypothesis.  $\square$

## REFERENCES

- [AG72] J. L. Alperin and Daniel Gorenstein, *A vanishing theorem for cohomology*, Proc. Amer. Math. Soc. **32** (1972), 87–88. MR 0291293
- [AKO11] Michael Aschbacher, Radha Kessar, and Bob Oliver, *Fusion systems in algebra and topology*, London Mathematical Society Lecture Note Series, vol. 391, Cambridge University Press, Cambridge, 2011. MR 2848834
- [AOV12] Kasper K. S. Andersen, Bob Oliver, and Joana Ventura, *Reduced, tame and exotic fusion systems*, Proc. Lond. Math. Soc. (3) **105** (2012), no. 1, 87–152. MR 2948790
- [Asc00] Michael Aschbacher, *Finite group theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 10, Cambridge University Press, Cambridge, 2000. MR 1777008 (2001c:20001)
- [Asc11] ———, *The generalized Fitting subsystem of a fusion system*, Mem. Amer. Math. Soc. **209** (2011), no. 986, vi+110. MR 2752788
- [Asc15] ———, *Classifying finite simple groups and 2-fusion systems*, ICCM Not. **3** (2015), no. 1, 35–42. MR 3385504
- [Asc16] ———, *On fusion systems of component type*, preprint (2016), 228pp.
- [BLO03] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), no. 4, 779–856 (electronic).
- [Cra11] David A. Craven, *The theory of fusion systems*, Cambridge Studies in Advanced Mathematics, vol. 131, Cambridge University Press, Cambridge, 2011, An algebraic approach. MR 2808319
- [Fin76a] Larry Finkelstein, *Finite groups with a standard component isomorphic to  $M_{23}$* , J. Algebra **40** (1976), no. 2, 541–555. MR 0414700 (54 #2795)
- [Fin76b] ———, *Finite groups with a standard component isomorphic to HJ or HJM*, J. Algebra **43** (1976), no. 1, 61–114. MR 0427450 (55 #482)
- [GL16] George Glauberman and Justin Lynd, *Control of fixed points and existence and uniqueness of centric linking systems*, Invent. Math. **206** (2016), no. 2, 441–484.
- [GLS98] Daniel Gorenstein, Richard Lyons, and Ronald Solomon, *The classification of the finite simple groups. Number 3. Part I. Chapter A*, Mathematical Surveys and Monographs, vol. 40, American Mathematical Society, Providence, RI, 1998, Almost simple  $K$ -groups. MR 1490581 (98j:20011)

- [Hig68] Graham Higman, *Odd characterizations of finite simple groups*, Lecture Notes of University of Michigan. Ann Arbor (1968), 76pp.
- [Lyn15] Justin Lynd, *A characterization of the 2-fusion system of  $L_4(q)$* , J. Algebra **428** (2015), 315–356. MR 3314296
- [Oli13] Bob Oliver, *Existence and uniqueness of linking systems: Chermak’s proof via obstruction theory*, Acta Math. **211** (2013), no. 1, 141–175. MR 3118306
- [Oli16a] ———, *Reductions to simple fusion systems*, Bulletin of the London Mathematical Society **48** (2016), no. 6, 923–934.
- [Oli16b] ———, *Tameness of fusion systems of sporadic simple groups*, preprint (2016), 29pp., arXiv:1604.05681v2.
- [Wil84] Robert A. Wilson, *The subgroup structure of the Lyons group*, Math. Proc. Cambridge Philos. Soc. **95** (1984), no. 3, 403–409. MR 755827

INSTITUTE OF MATHEMATICS, UNIVERSITY OF ABERDEEN, FRASER NOBLE BUILDING, ABERDEEN AB24 3UE

*E-mail address:* justin.lynd@abdn.ac.uk

DEPARTMENT OF MATHEMATICS AND STATISTICS, SAINT LOUIS UNIVERSITY, 220 NORTH GRAND BLVD., SAINT LOUIS, MO 63103

*E-mail address:* rainbolt@slu.edu