Optimal Investment in Deferred Income Annuities

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Abstract

In this thesis we develop and analyze a technique for finding the optimal investment strategy for a deferred income annuity (DIA). We first present some initial background needed to understand what a DIA is. We then lay out a mathematical framework which will allow us to formalize the optimization process. The method and implementation of the optimization is explained, and the results are then analyzed. We then add an extra layer of complication to our model by allowing our optimal portfolio to contain more types of assets. The results of this new model are analyzed, and finally we mention possible extensions to this work.

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1 Introduction and Literature Review

A deferred income annuity (DIA) is a variation on the well understood life time annuity. The common aspect of all lifetime annuities is the paying of a lump sum of money in order to be guaranteed a steady income for the rest of ones life. The value and application of standard life time annuities is a very old topic, dating back to at least 1693 with a paper written by Edmond Halley, of Halley's comet fame.

This standard life time annuity is commonly refereed to as a single-premium income annuity, or SPIA. The DIA is different than a SPIA in one key element; The DIA is purchased at a certain point in time, and its income stream only starts after a certain delay period.

DIA have recently become more popular among retirement plan providers and participants. An article published by the Wall Street Journal in January of 2015 stated that sales of DIA's between 2013 and 2014 increased by 35%. This dramatic increase in volume has stirred a lot of interest in DIA's, and certainly in optimal investment strategies in them.

Furthermore, the Congressional Research Service [11] conducted a study in 2003 which found that 84.8 % of American workers ages 21 and up that had some sort of employer retirement plan are offered a lump sum in retirement. While this has some advantageous, such as high liquidity, there are major drawbacks that the retiree must face. The main risk is that without a steady stream of income, the retiree may outlive is lump sum of resources. It is for this reason that retirees should be scrupulous and devise a prudent retirement plan for themselves. The DIA we study in this thesis can be a key element of this much needed plan.

Over the last few decades, we have seen a shift in moving away from *Defined Benefit (DB)* pension plans to *Defined Contribution (DC)* pension plans [5]. This shift is usually attributed to the fact that many firms have not been able to fully fund the future liabilities of a DB plan, for a myriad of reasons – chief among them being the improvement in human longevity. In 2014, only 2% of the private sector workers participated in a DB pension plan. Whereas, in a DB plan, future income is guaranteed by the employer, no such guarantee is made in a DC plan. The employer contributes to the plan on behalf of the employee (and often times matches the additional employee contributions).

Prior to 2014, holding an income annuity within a retirement plan was not a tax effective strategy. In 2014, the Internal Revenue Service (IRS) revised its rules with respect to Required Minimum Distributions (RMD) and now with a Qualified Longevity Annuity Contract (QLAC), a plan participant may fund income annuities with assets from employer sponsored qualified plans, i.e. 401k plans. The income derived from these plans would not conflict with the RMD rules. Insurance companies offer DIA's, with the start of income deferred to one's actual retirement date. This then provides an opportunity to the plan participant to 'pensionize' some of her retirement assets thus partially replicating the benefits of a DB plan

We are interested in questions such as: should one purchase DIAs now, or wait till retirement to invest in annuities? If one does purchase them early in a lump sum up front, or should it be done gradually over time (dollar cost average into an annuity)? We are motivated by [15] whereby the authors examined optimal annuitization (using a single premium income annuity), investment and consumption strategies of a utility maximizing agent. In this thesis, we start with a similar approach but in the context of a DIA. Later we expand our model to control for asset allocation. Now it is important to state: a full discussion of DIA's would need to involve a term structure interest rate or some other approach to non-constant interest rates, as is performed by Huang, Milevsky and Young [9]. In this thesis we assume a constant interest rate. What we are examining here is not the true value of a DIA, but rather, under the assumption that we know the value of the DIA at any point in time, what the ideal strategy in purchasing the DIA is.

The standard present value discounting model which is the back bone of almost all pricing models is essentially ancient. The Babylonian Talmud (500 CE) discusses the notion of the time value of money.

The bulk of the work presented here is based off of the classic Merton portfolio problem. This problem is well documented in many different literature publications, but we opted to use Chris Rogers' *Optimal Investment* [17]. The first chapter in particular is a very informative introduction to the Hamilton Jacoby Bellman equation. An important paper relating to this topic is the one by Huang, Milevsky, and Young (2017)[9]. In it, they find the optimal investment strategy in a DIA in the setting that it is the only option available to the consumer. Other literature that deals with comparison of DIAs and standard stocks and bonds is Milevsky and Young (2007) [15] and Stabile (2006) [8]. The backgroung in actuarial concepts such as the Gompertz-Makeham law was taken from Milevsky's *The Calculus of Retirement Income* [14]. One can read more about stochastic mortality models in Lee and Carter (1992) [10].

2 Optimal DIA Strategy

2.1 Stochastic Mortality Model

An annuity is a financial instrument that a retiree may buy which produces a payout at certain intervals of time. There are many different types of annuities, with different payout schemes and deferral periods. For the purposes of this thesis, the annuities we will be dealing with will have *fixed* payouts at *fixed* intervals of time. Furthermore, once the annuity starts producing its payments, it will continue to do this at its fixed interval until the death of the retiree. So it stands to reason, that before we can even begin to discuss how to price such an annuity, we must develop a mathematical framework for the remainder of a human lifetime. To this end, we will employ very standard actuarial concepts to build our model. We rely mostly on The Calculus of Retirement Income [14] by Milevsky.

Let x be our retiree's current age. Then we define q_x as the probability of dying in the next year. So q_{60} is the probability of a 60 year old dying before he turns 61. Of course, as our intuition should indicate,

q_x gets larger as x gets larger, and will approach a value of 1. Of course q_x will be different depending on gender, demographic, and other factors. One obvious implication of this formulation is that $(1 - q_x)$ is the probability of the x year old surviving the year.

Now, this formulation is not robust enough. What we really want to have is a formula for the probability of an x year old surviving for another n years. We label this ${}_np_x$. We want to express ${}_np_x$ as a function of q_x. An x year old has a probability of surviving this year of $(1 - q_x)$. After surviving, he will be $x + 1$ years old. So his probability of surviving the year will be $(1 - q_{x+1})$. This will continue until he dies. So the probability of surviving those n consecutive years is

$$ {}_np_x = \prod_{i=0}^{n-1}(1 - q_{x+i}) \tag{1} $$

The flip side of this idea is the following: Let $F_x(t)$ be the probability of an x year old dying in the next t years. Clearly, an x year old will either survive the next t years or die in the next t years, so ${}_tp_x + F_x(t) = 1$ or ${}_tp_x = 1 - F_x(t)$.

We introduce some further notation. Let T_x be a random variable which is the remaining number of years left to live of an individual at age x. Let $f_x(t)$ be the probability density function of T_x. This means, for example, that if one wanted to know what the probability of a person dying in the next 10 years is when they are age 60, they would compute

$$ \int_0^{10} f_{60}(t)dt $$

Since ${}_tp_x$ is the probability of an x year old surviving for t years, its clear that the $Pr(T_x \geq t) = {}_tp_x$. That is to say, the probability of our x year old having a lifespan greater than $x + t$ years, is the probability of them surviving for another t years.

An important relationship we need to understand is the following. Say we wanted the expected value of T_x. Our standard rules of probability state:

$$ E[T_x] = \int_0^{\infty} tf_x(t)dt \tag{2} $$

This is true because $f_x(t)$ is the probability density function of T_x. But what is the intuition of this equation? It says that we are adding together the probability of the x year old living for another 0 years, the probability of the x year old living for another dt years, the probability of the x year old living another $2dt$ years, and so on. Based on this, equation 2 is equivalent to the following:

$$ E[T_x] = \int_0^{\infty} {}_tp_x dt \tag{3} $$

What we are going to do now is come up with a closed form equation for ${}_tp_x$ and then use the above formula to find a closed form equation for $E[T_x]$. Having an equation for expected life span is going to help us immensely when we want to find the fair price of an annuity which continues making payments until death.

We now make certain general assumptions to help us find the equation for $E[T_x]$. For any positive decreasing function, which ${}_tp_x$ is, we can write the following:

$$ {}_tp_x = \exp\{-\int_x^{x+t} \lambda(s)ds\} \tag{4} $$

Where λ is the instantaneous force of mortality (IFM), and is non negetive for $s \geq 0$. The idea behind this form is as follows: Since λ is positive, as we increase t (which means we want to see the probability of living longer and longer) the value of the integral increases, since we are adding more area from under the curve of the positive λ. As the value of the integral increases, the value of ${}_tp_x$ decreases, since we are taking an exponent of a negative number who's absolute value is getting larger. In other words, as we add on more and more time, we are adding on more and more opportunities to die.

Now we simply perform a simple change of variables to get

$$ {}_tp_x = \exp\{-\int_0^{t} \lambda(x+s)ds\} $$

Now we must ask ourselves the question of what λ should be. The first guess could be that λ is constant. This would arrive at the formula:

$$ {}_tp_x = e^{-\lambda t} $$

This formula has the property that the probability of survival from age x to t is independent of x. This cannot be true of humans. Instead, we will use the very popular and well known model called the Gompertz [1] - Makeham [12] (GoMa) law of mortality. It defines our λ as

$$ \lambda = \lambda_0 + \frac{1}{b}e^{\frac{x-m}{b}} \tag{5} $$

m is whats known as the modal value of life, and b is the dispersion coefficient. The λ_0 value is the constant rate of mortality at any age; the chance of accidental mortality that has nothing to do with age. This value is usually quite small. m is a location parameter and b is a scale parameter. See the appendix for a discussion of the exact value of these parameters and other parameters.

We can now find a closed form equation for ${}_tp_x$.

$$ {}_tp_x = \exp\{-\int_x^{x+t} \lambda_0 + \frac{1}{b}e^{\frac{s-m}{b}}ds\} \tag{6} $$

We can solve this explicitly to get:

$$ {}_tp_x = \exp\{-\lambda_0 t + (e^{\frac{x-m}{b}})(1-e^{\frac{t}{b}})\} \tag{7} $$

Finally, we can substitute equation 7 into equation 3:

$$ E[T_x] = \int_0^{\infty} \exp\{-\lambda_0 t + (e^{\frac{x-m}{b}})(1-e^{\frac{t}{b}})\}dt \tag{8} $$

This rather complicated integral can be solved to get the following:

$$E[T_x] = \frac{b\Gamma(-\lambda_0 b, \exp\frac{x-m}{b})}{e^{(m-x)\lambda_0 - \exp\frac{x-m}{b}}} \tag{9}$$

Where Γ is the so called incomplete Gamma function, which is defined as:

$$\Gamma(a, c) = \int_c^\infty t^{a-1}\exp(-t)dt \tag{10}$$

With an equation for $E[T_x]$, we now have the tools we need to find the price for different annuities.

2.2 Annuity Pricing

We start with the basics of Present Value(PV). Let R be the fixed annual investment rate. This means that if I put 1$ into an investment account, then I will have $\$1(1+R)$ next year, and $\$1(1+R)^2$ the year after that. If instead the rate R is compounded semi annually, then instead after one year, I will have $\$1(1+\frac{R}{2})^{1*2}$, and after two years I will have $\$1(1+\frac{R}{2})^{2*2}$. If the rate is compounded n times every year, then we have after t years that my one dollar becomes $\$1(1+\frac{R}{n})^{t*n}$. Its a well known limit that as $n \to \infty$, this expression becomes $\$1e^{Rt}$ Now what we really want is the reverse of this; if I know that I will receive \$1 next year, how much is that dollar worth now? The answer is simply the amount of money I would need to invest now at an interest rate of R to have it be worth \$1 in one year. Let PV be the present value of that \$1 . Then we write $PVe^{R1} = 1$, and we solve for $PV = e^{-R}$. Clearly for t years instead of one year,

$$PV = e^{-Rt} \tag{11}$$

If I were receiving payments from an annuity that I have invested in, and we suppose the payments are coming in in continuous time, then each unit of money I receive has to be discounted to the present. The present value of all the payments from the annuity are the present value of the annuity. Let a_x be the present value of an annuity for a person at age x. The payments come in every year until the person dies. Remembering that T_x is the random variable that represents how long an x year old will live, we can write the following:

$$a_x = \int_0^{T_x} e^{-Rt}dt \tag{12}$$

This integral is a function of a random variable. So we can ask what $E[a_x]$ is. Let us call this $\bar{a}_x$.

$$\bar{a}_x = E[\int_0^{T_x} e^{-Rt}dt] \tag{13}$$

We can rewrite this equation in the following form:

$$\bar{a}_x = E[\int_0^\infty e^{-Rt}\mathbb{1}_{\{T_x \geq t\}}dt] \tag{14}$$

Now we can bring the expectation value inside the integral, and apply to the only random portion, the $\mathbb{1}_{\{T_x \geq t\}}$. But we know that $E[\mathbb{1}_{\{T_x \geq t\}}] = Pr(T_x \geq t)$, and we know that this is equal to ${}_tp_x$. So we can write the following

$$\bar{a}_x = E[\int_0^{T_x} e^{-Rt}dt] = \int_0^{\infty} e^{-Rt}{}_tp_x dt \tag{15}$$

Substituting equation 7 into this, we get:

$$\bar{a}_x = \int_0^{\infty} e^{-Rt} \exp\{-\lambda_0 t + (e^{\frac{x-m}{b}})(1 - e^{\frac{t}{b}})\} \tag{16}$$

Using only basic calculus techniques, this expression can be simplified to find $\bar{a}_x$ explicitly.

$$\bar{a}_x = \frac{b\Gamma(-(\lambda_0 + R)b, \exp{(\frac{x-m}{b})})}{\exp{((m-x)(\lambda_0 + R) - \exp{(\frac{x-m}{b})})}} \tag{17}$$

Now the next step in laying our foundation is to find an equation for a deferred income annuity. That is to say, we need to know what the fair price of an annuity which I buy now, but only starts to make payments later, is. Let us say that the current age of our retiree is is x, and the annuity they are buying will make start to make payments at year x+u. Now, let us assume for now that if our retiree dies before they reach age x+u, then they do not get their investment back. Clearly this annuity should be less expensive than the standard annuity, since it will make less payments on average than the standard one, and it may not make any payments at all if the retiree dies before retirement. The formulation is simple. The value of this annuity will simply be the standard annuity that is at age x+u, i.e. a_{x+u}, with an extra coefficient to discount the price back from age x+u to age x, and an extra coefficient for the probability of surviving for the next u years. We label this new deferred income annuity ${}_u\bar{a}_x$ The result is:

$${}_u\bar{a}_x = \bar{a}_{x+u}({}_up_x)e^{-Ru} \tag{18}$$

Substituting equation 7 and 17 into the above equation, we get the following equation:

$${}_u\bar{a}_x = \frac{b\Gamma(-(\lambda_0 + R)b, \exp{(\frac{x-m+u}{b})})}{\exp{((m-x)(\lambda_0 + R) - \exp{(\frac{x-m}{b})})}} \tag{19}$$

In practise, we will hardly refer to this explicit closed form version of ${}_u\bar{a}_x$, but rather we will use numerical software to approximate it as every point in time.

The above equation is only for an annuity which does not have any refund if the retiree dies before retirement. In reality, most DIA's will have this feature, so we want to model this aspect of the DIA as well. Consider the following version of a DIA: each unit of DIA that the retiree buys is refunded to him should he die, at the value that it is worth on the day of death. For example, if the retiree buys two units of DIA (that is to say, he is now entitled to \$ 2 per year in retirement) at \$ 100, and then dies in one year when the DIA is worth \$ 110, then his progeny would receive \$220 .

With this refund scheme, there is essentially no risk of dying, since the entire sum of money deposited into the account is returned, with interest. This theoretical DIA will be priced at

$$ {}_u\bar{a}_x = \bar{a}_{x+u}e^{-Ru} \tag{20} $$

This is almost identical to the equation 12; we simply take away the ${}_up_x$ because we have taken away all notions of mortality from this model.

We now create the following model: the price of the DIA we use is a weighted average of the no refund DIA, and the full refund DIA. We introduced the parameter Q between 0 and 1, and define the price of the DIA as:

$$ {}_u\tilde{a}_x = \bar{a}_{x+u}e^{-Ru}[{}_up_x(1-Q)+Q] \tag{21} $$

As we shift Q from 0 to 1, we change the nature of the DIA. A Q value of 0 implies a DIA with no refund, while a Q value of 1 implies a DIA with a full refund. In case a refund is issued, the value that gets returned is only the portion of the waited average which is made up of the full refund DIA. That is to say, the refund, K is defined as:

$$ K_t = \bar{a}_{x+u}e^{-Ru}Q \tag{22} $$

Now that we have the equation for the fair DIA price, we can begin to find the optimal investment strategy.

2.3 Wealth dynamics

Before a fixed retirement date τ, i.e., $t < \tau$, the wealth dynamics for the retirement account W_t is given by

$$ dW_t = (\mu W_t + \nu - g_t)dt + \sigma W_t dB_t \tag{23} $$

where μ and σ are the drift and volatility of the investment portfolio, ν is a steady exogenous income stream to the retirement funds, and $g_t \geq 0$ is the rate for purchasing deferred annuities (DIA) that mature at retirement time τ. The meaning of this equation is the following; there are three factors that affect the value of ones wealth. The stock investment has a deterministic portion, $\mu W dt$, and a stochastic portion, $\sigma W dB_t$. The DIA portion is simply the rate of purchase one wishes to buy DIA at, and since we are purchasing (and not alloed to sell), this always reduces ones wealth, hence $-g_t dt$. Finally we have a constant income ν which is always increasing our wealth at each time step. In the event of the retiree dying before retirement, then they get back an amount of money equal to the value of a DIA at the time a death, times the number of units of DIA they own, times the weighted average term Q that was mentioned above. x is the starting age of the retiree when he starts considering DIA's, so this relationship is represented in the following equation:

$$ K_t = Q_{\tau-t}\tilde{a}_{x+t}I \tag{24} $$

The accumulation of DIA which generates income at retirement, denoted by I_t, is given by:

$$dI_t = \frac{g_t}{{}_{(\tau-t)}\bar{a}_{(x+t)}}dt \tag{25}$$

After the retirement, $t \geq \tau$, the wealth dynamics are given by

$$dW_t = (\mu W_t + I_t + \pi_t - c_t)dt + \sigma W_t dB_t \tag{26}$$

where π_t is the pension income rate and c_t is the consumption rate. The stock portion is the same as before. The DIA is now paying us at a rate of I_t, and the fixed pension acts similarly to the fixed income from before. Finally, c_t is how much we take out from our account at each time step to pay for our cost of living. Our control variables are g_t and c_t; that is to say, we can decide how much to invest in our DIA before we retire, and after we retire we can choose how much money to consume, but aside from those we cannot make any decisions about our portfolio.

This is how our wealth levels move through time. But we still need to concretize what we are trying to do. We now explore the values which we are trying to optimize and what variables we can change in order to achieve this optimization.

The basic question in pre-retirement ($t < \tau$) is what are the optimal choices for g_t while in post-retirement ($t \geq \tau$) we want to know the optimal c_t.

First we define the following function which has time, wealth, and income as its independent variables.

$$\tilde{J}(t, w, I) = \max_{g_s, c_s} \left[\int_0^\infty e^{-Rs} {}_s p_x u(c_s) \mathbb{1}_{s>\tau} + e^{-Rs} {}_s p_x \lambda_{x+s} u(Q_s) ds | W_t = w, I_t = I) \right] \tag{27}$$

$u(x)$ is a utility function and under a Constant Relativity Risk Aversion (CRRA) utility model, $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, where γ is the subjective risk aversion of a particular investor. ${}_s p_x$ and λ_{x+s} are the survival probability and mortality rate for an individual at time s, with initial age x at time 0. $Q_t = (w + K + \nu)$ for $t < \tau$, and $Q_t = (w + \pi)$ for $t \geq \tau$.

There are three main terms in this equation. The first $u(c_s)$ is the utility of consumption in post retirement. The next term is the utility of bequest. This is the utility that one gets should they die and leave their wealth to their next of kin. Its for this reason that this utility is multiplied by λ; it only applies if the retiree dies.

Next we define the following stochastic process:

$$J(t, w_t, I_t) = \max_{g_s, c_s} \left[\int_0^\infty e^{-Rs} {}_s p_x u(c_s) \mathbb{1}_{t>\tau} + e^{-Rs} {}_s p_x \lambda_{x+s} u(Q_s) ds | W_t = w, I_t = I) \right] \tag{28}$$

Note the subscripts on w and I to denote the fact that this equation involves stochastic processes.

Since this is a stochastic process for which we know all information about until time t, we can separate this integral into two portions, from 0 to t and from t to infinity. When we do this, we get:

$$J(t,w_t,I_t) = \max_{g_s,c_s} E\bigg[\int_0^t e^{-Rs}{}_sp_x u(c_s)\mathbb{1}_{t>\tau} + e^{-Rs}{}_sp_x\lambda_{x+s}u(Q_s)ds + e^{-Rt}{}_sp_x\tilde{J}(t,w_t,I_t)|W_t = w, I_t = I)\bigg] \tag{29}$$

It's important to understand that the integral portion of this equation is completely deterministic, since we are taking a conditional expectation given that we know all information about w, and I up to time t. The term outside the integral is stochastic.

We must now formulate a method of how to optimize this value function. What we will do is derive a Hamilton-Jacobi-Bellman equation for this value function. We follow the approach of L. C. G. Rogers in *Optimal Investment* [17].

Begin by taking an Ito expansion of J. Since the integral term is deterministic, we simply need to take the time derivative of it, and so we simply drop the integral from it. The non integral term is stochastic and so we need to take a full Ito expansion of it. Its important to recall that $\frac{d}{dt}{}_tp_x = -{}_tp_x\lambda_{x+t}$.

$$\begin{aligned} dJ = (e^{-Rt}{}_tp_x u(c_t)\mathbb{1}_{t>\tau} + e^{-Rt}{}_tp_x\lambda_{x+t}u(Q_t))dt \\ +(-Re^{-Rt}{}_tp_x\tilde{J} - \lambda_{x+t}e^{-Rt}{}_tp_x\tilde{J})dt \\ +e^{-Rt}{}_tp_x[\tilde{J}_t dt + \tilde{J}_w dw + \tfrac{1}{2}\tilde{J}_{ww} < dw > +\tilde{J}_I dI + \tfrac{1}{2}\tilde{J}_{II} < dI >] \end{aligned} \tag{30}$$

Let us find the values of $< dw >$, and $< dI >$.

$$< dw >= \sigma^2 w^2 dt$$

$$< dI >= 0$$

For $t < \tau$, we get:

$$\begin{aligned} dJ = (e^{-Rt}{}_tp_x\lambda_{x+t}u((w+K+\nu)))dt \\ +(-Re^{-Rt}{}_tp_x\tilde{J} - \lambda_{x+t}e^{-Rt}{}_tp_x\tilde{J})dt \\ +e^{-Rt}{}_tp_x[\tilde{J}_t dt + \tilde{J}_w((\mu W_t + \nu - g_t)dt + \sigma W_t dB_t) + \tfrac{1}{2}\tilde{J}_{ww}\sigma^2 w^2 dt + \tilde{J}_I \tfrac{g_t}{(\tau-t)\bar{a}_{(x+t)}}dt] \end{aligned} \tag{31}$$

Grouping all the dt and dB terms, we get:

$$\begin{aligned} dJ = \bigg[e^{-Rt}{}_tp_x\lambda_{x+t}u((w+K+\nu)) \\ +(-Re^{-Rt}{}_tp_x\tilde{J} - \lambda_{x+t}e^{-Rt}{}_tp_x\tilde{J}) \\ +e^{-Rt}{}_tp_x[\tilde{J}_t + \tilde{J}_w((\mu W_t + \nu - g_t)) + \tfrac{1}{2}\tilde{J}_{ww}\sigma^2 w^2 + \tilde{J}_I \tfrac{g_t}{(\tau-t)\bar{a}_{(x+t)}}]\bigg]dt + \sigma W_t dB_t \end{aligned} \tag{32}$$

For $t \geq \tau$, we get:

$$dJ = (e^{-Rt}{}_tp_x u(c_t) + e^{-Rt}{}_tp_x \lambda_{x+t} u((w+\pi_t)))dt \tag{33}$$
$$+(-Re^{-Rt}{}_tp_x \tilde{J} - \lambda_{x+t} e^{-Rt}{}_tp_x \tilde{J})dt$$
$$+e^{-Rt}{}_tp_x[\tilde{J}_t dt + \tilde{J}_w(((\mu W_t + I_t + \pi_t - c_t))dt + \sigma W_t dB_t) + \tfrac{1}{2}\tilde{J}_{ww}\sigma^2 w^2 dt]$$

Grouping all the dt and dB terms, we get:

$$dJ = \Big[e^{-Rt}{}_tp_x u(c_t) + e^{-Rt}{}_tp_x \lambda_{x+t} u((w+\pi_t)) \tag{34}$$
$$+(-Re^{-Rt}{}_tp_x \tilde{J} - \lambda_{x+t} e^{-Rt}{}_tp_x \tilde{J})$$
$$+e^{-Rt}{}_tp_x[\tilde{J}_t + \tilde{J}_w(((\mu W_t + I_t + \pi_t - c_t))) + \tfrac{1}{2}\tilde{J}_{ww}\sigma^2 w^2]\Big] dt + \sigma W_t dB_t]$$

Now, in order to have an optimal value function, we must make sure that J is a martingale. To do this, we must set the deterministic drift term in the above equations equal to 0. Doing so, and after simplification, we get the following.

Hamilton-Jacobi-Bellman (HJB) Equation

For the optimal choices of $g_t \geq 0$ and $c_t \geq 0$, the value function satisfies the HJB equation:

$$0 = \sup_g \Big[\tilde{J}_t + (\mu w + \nu - g_t)\tilde{J}_w + \frac{1}{2}(\sigma w)^2 \tilde{J}_{ww} \tag{35}$$
$$+\frac{g_t}{{}_{(\tau-t)}\bar{a}_{(x+t)}}\tilde{J}_I + \lambda_{x+t} u(w+K+\nu) - (\rho + \lambda_{x+t})\tilde{J}\Big]$$

for $t < \tau$, i.e. *pre-retirement*, and

$$0 = \sup_c \Big[\tilde{J}_t + (\mu w + I_t + \pi_t - c_t)\tilde{J}_w + \frac{1}{2}(\sigma w)^2 \tilde{J}_{ww} \tag{36}$$
$$+\lambda_{x+t} U(w+\pi_t) + u(c_t) - (\rho + \lambda_{x+t})\tilde{J}\Big]$$

for $t \geq \tau$, i.e. *post-retirement.*

To find the optimal c_t in post-retirement is quite simple. Keeping in mind that $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$, we can take a simple derivative with respect to c_t and set it equal to zero, then solve for c_t to find the optimal c_t.

$$0 = -\tilde{J} + c_t^{-\gamma} \tag{37}$$
$$\rightarrow \quad c_t^* = \tilde{J}^{\frac{1}{\gamma}}$$

In pre retirement, finding the optimal g_t is not as trivial. Since all terms with g_t are linear in g_t, taking a derivative will eliminate all g_t terms. Instead we apply the following methodology.

We examine the variation:

$$\delta\mathcal{H} := \left(\frac{\tilde{J}_I}{_{(\tau-t)}\bar{a}_{(x+t)}} - \tilde{J}_w \right) \delta g_t \tag{38}$$

where

$$\mathcal{H} := -g_t \tilde{J}_w + \frac{g_t}{_{(\tau-t)}\bar{a}_{(x+t)}} \tilde{J}_I \tag{39}$$

Since δg_t is non-negative, the sign of $\left(\frac{J_I}{_{(\tau-t)}\bar{a}_{(x+t)}} - \tilde{J}_w \right)$ determines optimality.

If $\frac{\tilde{J}_I}{_{(\tau-t)}\bar{a}_{(x+t)}} - \tilde{J}_w > 0$, it is optimal to annuitize, since that makes $\delta\mathcal{H} > 0$. On the other hand, if $\frac{\tilde{J}_I}{_{(\tau-t)}\bar{a}_{(x+t)}} - \tilde{J}_w < 0$, it is optimal not to annuitize ($g_t = 0$) since annuitization makes $\delta\mathcal{H} < 0$, which decreases the value function. The annuitization boundary is therefore given by

$$\frac{\tilde{J}_I}{_{(\tau-t)}\bar{a}_{(x+t)}} = \tilde{J}_w \tag{40}$$

It is at this boundary that when we set the two PDE's equal to 0, and solve each for J, that we will find that both J values are the same. If we performed the same operation below the boundary, and the same operation above, we would see one equation being larger than the other, and vice versa. It is this phenomenon that allows to construct the following PDE:

$$\begin{aligned} \max \Big\{ &\tilde{J}_t + (\mu w + \nu)\tilde{J}_w + \frac{1}{2}(\sigma w)^2 \tilde{J}_{ww} + \lambda_{x+t} u(w + K_t + \nu) - (\rho + \lambda_{x+t})\tilde{J}, \\ &\tilde{J}_I - {}_{(\tau-t)}\bar{a}_{(x+t)} \tilde{J}_w \Big\} = 0 \end{aligned} \tag{41}$$

for $t < \tau$.

The meaning of the above equation is the following: We find two possible J values by setting each PDE equal to 0. Then we take the max of the two solved J values. The solution of the above equation provides the boundary between the annuitization & non-annuitization region and can be solved for all $t < \tau$. This boundary can be used by the retiree to develop an optimal strategy. Initially, the retiree will make sure that she stays at the annuitization boundary, ie., for a given wealth level, if she is inside the annuitization region, she will compute the amount g_t that is needed to bring her to the annuitization boundary. No action is required if she is outside the annuitization region since DIAs cannot be actively traded (or sold) by the retiree. The retiree re-evaluates her wealth, DIA levels, and repeats the process periodically.

We now have all the PDE's that we need to solve the value problem. We will now discuss the boundary and terminal conditions that we will use to solve the problem.

2.4 Boundary and Terminal Conditions

Post-retirement ($t \geq \tau$)

Terminal Condition

Let $t = \tilde{T}$ be the final time step in post retirement. At $t = \tilde{T}$, the Gompertz-Makeham *Instantaneous Force of Mortality*, λ_{x+t}, increases exponentially (i.e. the client is dead) and asymptotically, from (36), we have:

$$\lambda_{x+t} u(w + \pi) - \lambda_{x+t} \tilde{J} = 0,$$

which gives the terminal condition $\tilde{J} = u(w + \pi)$.

Boundary Condition

For maximum wealth in post retirement, we employ an asymptotic approximation at the boundary. We scale w in (36) by a suitably large wealth value w_m so that $W = w/w_m$ and $\delta = (\pi + I)/w_m \ll 1$. After rescaling $\tilde{J} = w_m^{1-\gamma} \Phi(t, W)$ and $c_t^* = w_m C_t^*$, the HJB equation becomes:

$$\begin{aligned} 0 &= \Phi_t + (\mu W + \delta - C_t^*)\Phi_W + \frac{1}{2}(\sigma W)^2 \Phi_{WW} \\ &\quad + \lambda_{x+t} u(W + \delta) + u(C_t^*) - (\rho + \lambda_{x+t})\Phi \end{aligned} \tag{42}$$

where

$$C_t^* = \Phi_W^{-\frac{1}{\gamma}}. \tag{43}$$

We now expand the value function as

$$\Phi = \Phi^{(0)}(t, W) + \delta \Phi^{(1)}(t, W) + \cdots$$

and the optimal consumption rate becomes $C_t^* = C_t^{(0)} + \delta C_t^{(1)} + \cdots$ where

$$C_t^{(0)} = (\Phi_W^{(0)})^{-\frac{1}{\gamma}}, \; C_t^{(1)} = -\frac{1}{\gamma}(\Phi_W^{(0)})^{-\frac{1}{\gamma}-1} \Phi_W^{(1)}.$$

Substituting these expressions into the HJB equation, and collecting the zeroth and first order terms in δ, we have

$$\begin{aligned} 0 &= \Phi_t^{(0)} + (\mu W - C_t^{(0)})\Phi_W^{(0)} + \frac{1}{2}(\sigma W)^2 \Phi_{WW}^{(0)} \\ &\quad + \lambda_{x+t} u(W) + u(C_t^{(0)}) - (\rho + \lambda_{x+t})\Phi^{(0)}, \end{aligned} \tag{44}$$

$$\begin{aligned} 0 &= \Phi_t^{(1)} + (\mu W - C_t^{(0)})\Phi_W^{(1)} + (1 - C_t^{(1)})\Phi_W^{(0)} + \frac{1}{2}(\sigma W)^2 \Phi_{WW}^{(1)} \\ &\quad + u'(C_t^{(0)}) C_t^{(1)} + \lambda_{x+t} u'(W) - (\rho + \lambda_{x+t})\Phi^{(1)}, \end{aligned} \tag{45}$$

with terminal conditions $\Phi^{(0)}(T, W) = u(W)$ and $\Phi^{(1)}(T, W) = u'(W)$.

Using CRRA utility, we seek a solution in the form of $\Phi^{(0)}(t, W) = h(t)u(W)$ and $\Phi^{(1)} = k(t)u'(W)$, where $h(t)$ and $k(t)$ are the solutions of the following ODEs

$$\frac{dh}{dt} - \left[(\gamma - 1)\mu + \rho + \lambda_{x+1} + \frac{\gamma(1-\gamma)\sigma^2}{2}\right] h + \gamma h^{\frac{\gamma-1}{\gamma}} + \lambda_{x+t} = 0 \tag{46}$$

with $h(T) =$ and

$$\frac{dk}{dt} - \left[\gamma\mu - \gamma h^{-\frac{1}{\gamma}} + \rho + \lambda_{x+1} - \frac{\sigma^2}{2}\gamma(1+\gamma)\right] k + \lambda_{x+t} + h = 0 \tag{47}$$

with $k(T) =$. Note $\Phi^{(1)} > 0$, the correction to the leading order approximation, is positive, which increases the value function.

We take our equation and set W equal to 0. We know that the two w terms are much smaller than the rest of the equation, so the rest of the terms must balance themselves.

$$0 = \tilde{J}_t + (I_t + \pi - c_t^*)\tilde{J}_w + \lambda_{x+t}u(\pi) + u(c_t^*) - (\rho + \lambda_{x+t})\tilde{J} \tag{48}$$

where $c^* = \tilde{J}_w^{-\frac{1}{\gamma}}$.

Pre-retirement $(t < \tau)$

Terminal Condition

The terminal condition is simply the last time step in the post retirement problem.

Boundary Conditions

Minimum wealth is always assumed to be a non-annuitization region. This is because the retiree is guaranteed to invest the very small amount of wealth they have into a risky asset to have any hope of growing their wealth. We take our non-annuitization equation and set W equal to 0. We know that the two w terms are much smaller than the rest of the equation, so the rest of the terms must balance themselves. Thus we get:

$$0 = \tilde{J}_t^{(1)} + (\nu)\tilde{J}_w^{(1)} + \lambda_{x+t}u({}_{(\tau-t)}\bar{a}_{(x+t)} \cdot I_t + \nu) - (\rho + \lambda_{x+t})\tilde{J}^{(1)} \tag{49}$$

At maximum wealth, we perform a type of asymptotic analysis on the non-annuitization equation. We assume that $\tilde{J}(t, w, I) = u(w)Q(t, I)$. With this assumption, we get that $\tilde{J}_w^{(1)} = \frac{(1-\gamma)}{w}u(w)Q(t, I)$ and $\tilde{J}_{ww}^{(1)} = \frac{-\gamma(1-\gamma)}{w^2}u(w)Q(t, I)$. Substituting this into both equations, and assuming that at maximum wealth $(\mu w + \nu) \approx \mu w$ and $(w + {}_{(\tau-t)}\bar{a}_{(x+t)} \cdot I_t + \nu) \approx w$, we get:

$$0 = Q_t^{(1)} + \mu(1-\gamma)Q^{(1)} - \frac{1}{2}\gamma(1-\gamma)(\sigma)^2 Q^{(1)} + \lambda_{x+t} - (\rho + \lambda_{x+t})Q^{(1)} \tag{50}$$

or

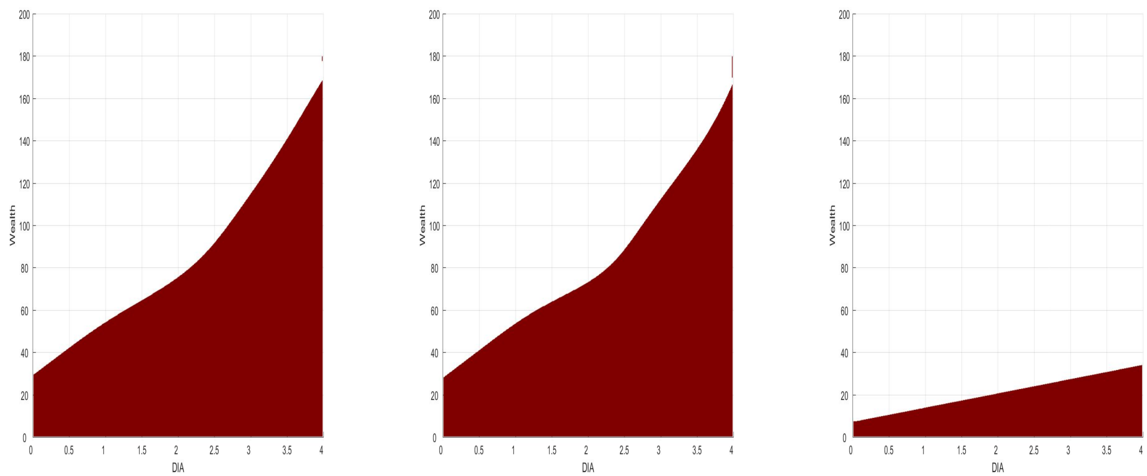
$$\tilde{J}_I^{(2)} - {}_{(\tau-t)}\bar{a}_{(x+t)}\tilde{J}_w^{(2)} = 0 \tag{51}$$

What we do is solve for Q at each income level, and then multiply by $u(W_{max})$ to find the value of $\tilde{J}^{(1)}$. What we do now is use a finite difference scheme for each PDE to create a numerical algorithm for solving the optimal J value. See the appendix for the discretization of the PDE's.

We solve the problem in two steps – Post-Retirement followed by Pre-Retirement. We use implicit finite difference methods to solve the partial differential equation (36) and an explicit scheme to solve equation (41). In pre retirement, we solve the two PDE's one grid point at a time, and compare the two values that they produce. The larger value is kept, and we also record in a binary manner which of the two value functions we kept. The following observations are of this binary graph.

2.5 Numerical Results

The baseline parameters we use are recorded and explained in the appendix. Keep in mind that the clear region is the annuitization region while the red region is the non-annuitization region. The indexing on our axis is scaled by the pension term π. This means, for instance, that at the grid point with W = 20, and I=1, the retiree has 20 times the value of their yearly pension in cash, and one times their yearly pension as an annuity income. We begin with a Q value of 1 (in reality, due to numerical issues, a Q value of 1.055 is used. The reason for this is because at Q = 1, the annuity and a risk free rate are effectively the same. Below, starting in figure 14, when we introduce the option of optimally investing in the risky and risk free rate, we want to be able to discern the difference between the risk free rate and the DIA, so we artificially increase the value of Q by a small amount to make the two recognizably different)



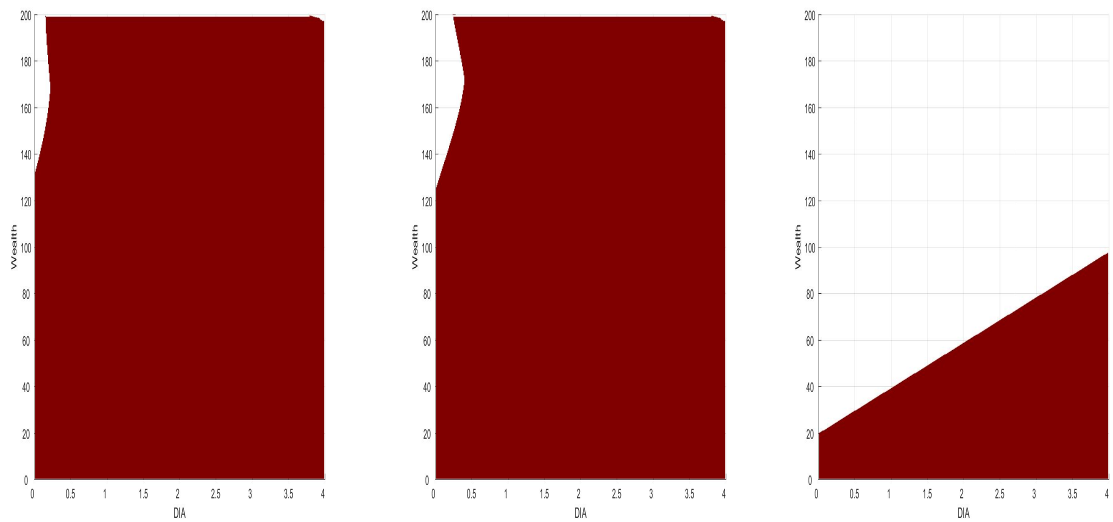
(a) Investment Profile at Age 62 **(b)** Investment Profile at Age 63 **(c)** Investment Profile at Age 65

Figure 1: Investment regimes at different times for baseline parameter values.

In Figure 1, we present results with baseline parameter values. We observe that for a reasonable μ, σ, and γ, the annuitization region (clear) gets larger when we move forward from age 62 to 65. At earlier ages, there is a larger chance of death before retirement, so the average retiree buys less at earlier ages. This is also in line with the standard idea that as a person gets older, they move away from risky assets and invest more in riskless ones (in this case, the DIA acts as the riskless asset).

Notice the significant change in the annuitization region which takes place in the final time step of pre-retirement (figure 1c). Since our model does not permit the retiree to purchase additional annuities in retirement, the final time-step is the last opportunity to do so. Furthermore, as the retiree enters into retirement, she now has to think about consumption. So where as before she did not need to worry about having a steady income, and simply having wealth was enough, she now must prioritize consumption, and therefore annuitization is more attractive at the last time step. Essentially, in both her options available and her priorities to be met, there is an abrupt change which is reflected in this abrupt change in graphs.

Next, we perturb the rate of return of the risky asset (μ) to see its impact on annuity allocation.

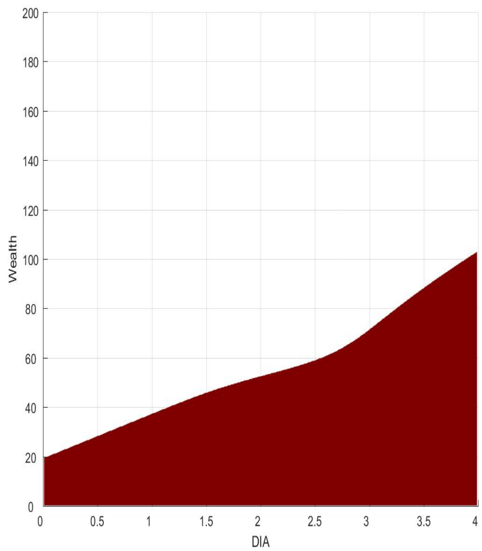


(a) Investment Profile at Age 62 **(b)** Investment Profile at Age 63 **(c)** Investment Profile at Age 65

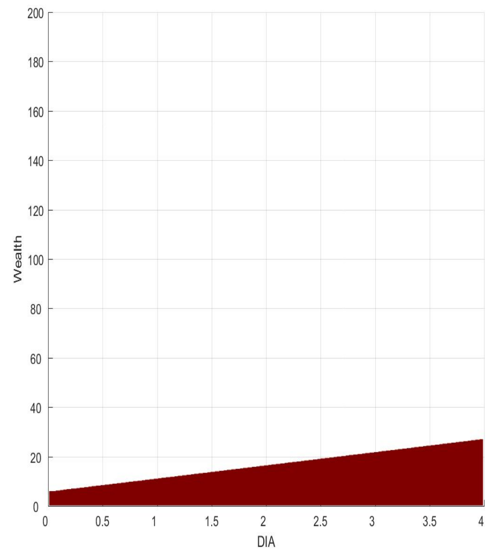
Figure 2: Investment regimes at different times for $\mu = 10\%$ (all other parameters remain unchanged).

With the μ values increased, we have made the stock investment more attractive, so therefore the annuitization region is smaller for a given time step.

Next we revert back to the default values for our parameters, but increase the value of the coefficient of risk-aversion (γ). In other words, for a more risk averse individual. As we expect, the annuitization region gets larger as can been seen when we compare Figure 1b and Figure 3a.



(a) Investment Profile at Age 63

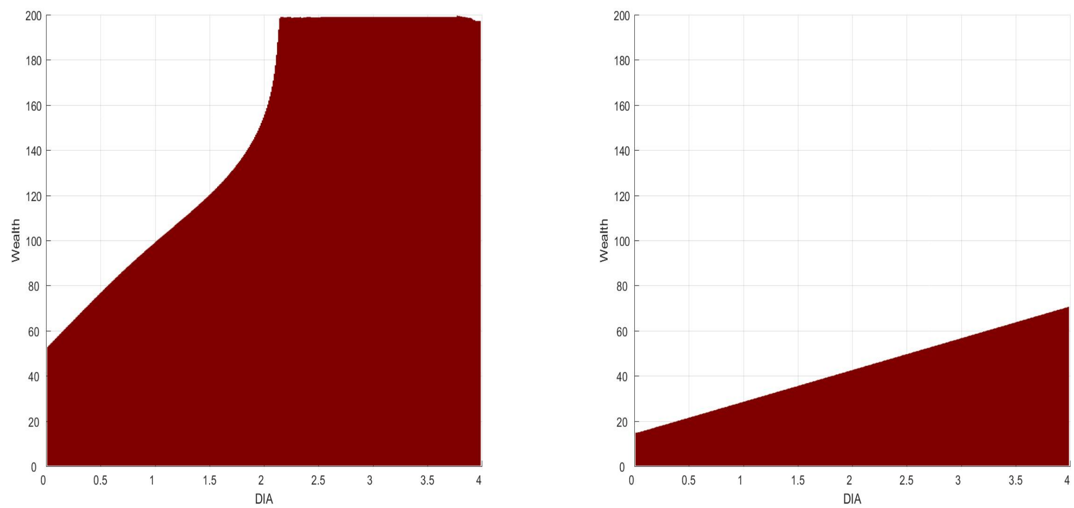


(b) Investment Profile at Age 65

Figure 3: Investment regimes at different times for γ=3.5 (all other parameters remain unchanged).

With a higher risk aversion, we of course see an increase in the annuitization region. The DIA acts as a risk free investment, so the risk averse retiree will prioritize it more.

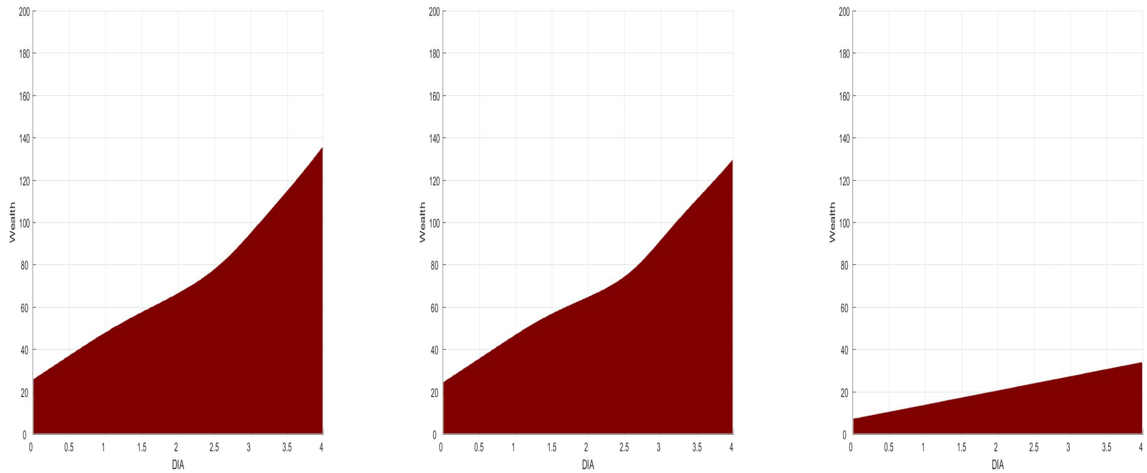
In Figure 4 we plot the annuitization & non-annuitization regimes for an increased rate of return of the risky asset (μ) and increased coefficient of risk aversion (γ). Observe the increased investment region (red colour) now that investment returns have gone up (Figure 3a vs. Figure 4a).



(a) Investment Profile at Age 63 **(b)** Investment Profile at Age 65

Figure 4: Investment regimes at different times for parameter values of $\mu = 10\%$ and γ=3.5 (all other parameters remain unchanged).

Now we will look at another set of results with the Q value lowered to 0.7. We first present, once again, the baseline parameters:

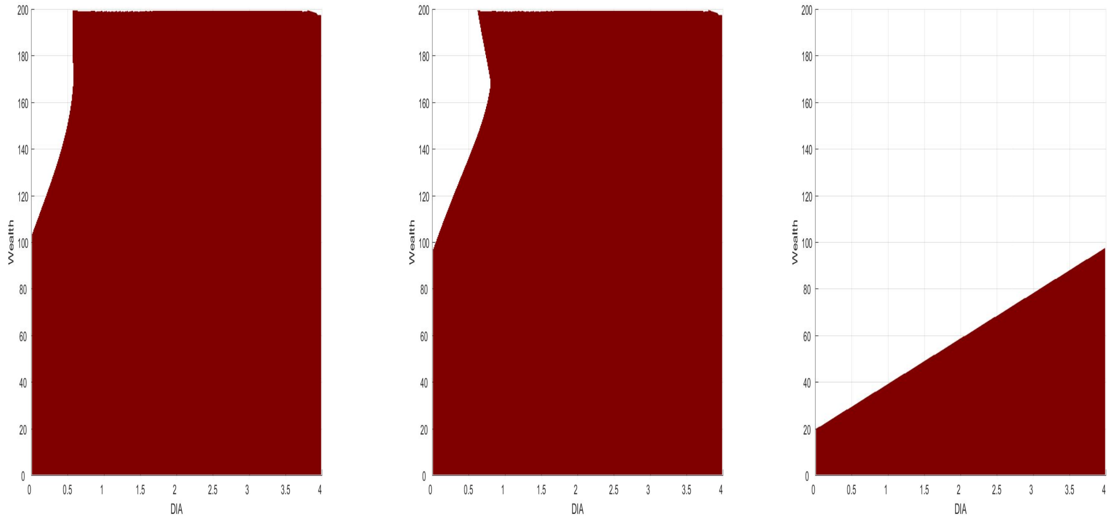


(a) Investment Profile at Age 62 **(b)** Investment Profile at Age 63 **(c)** Investment Profile at Age 65

Figure 5: Investment regimes at different times for baseline parameter values.

When we compare figure 1 and figure 5, we can see that a lower Q value will increase the clear annuitizaiton region. This is of course the intuitive result, since a lower Q value creates more mortality credits and therefore a more desirable (i.e. less costly) product.

The next images are for an increased μ value.

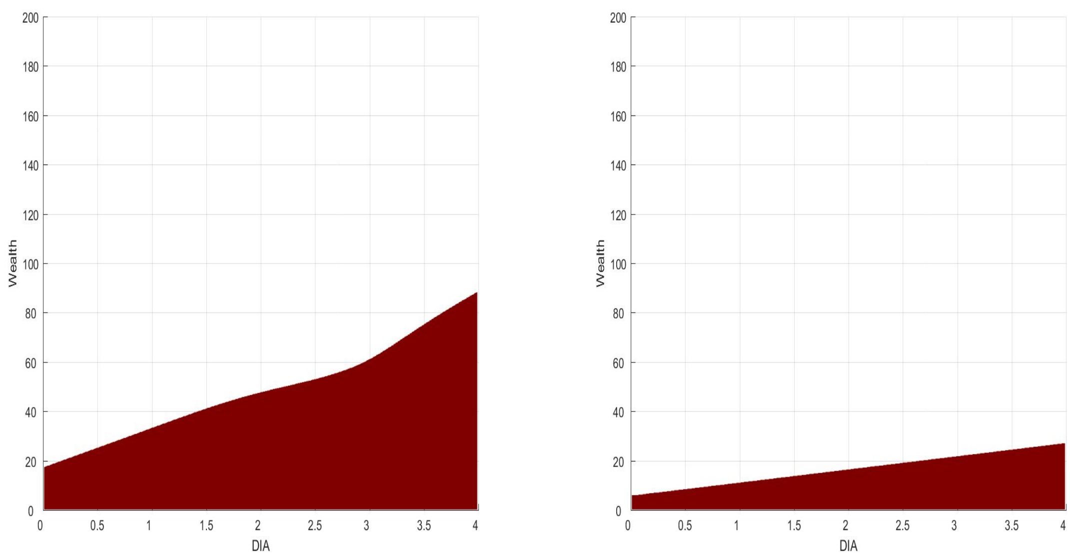


(a) Investment Profile at Age 62 **(b)** Investment Profile at Age 63 **(c)** Investment Profile at Age 65

Figure 6: Investment regimes at different times for $\mu = 10\%$ (all other parameters remain unchanged).

Again, when we compare figure 2 and figure 6, we have the obvious result that a lower Q value results in a larger annuitization region.

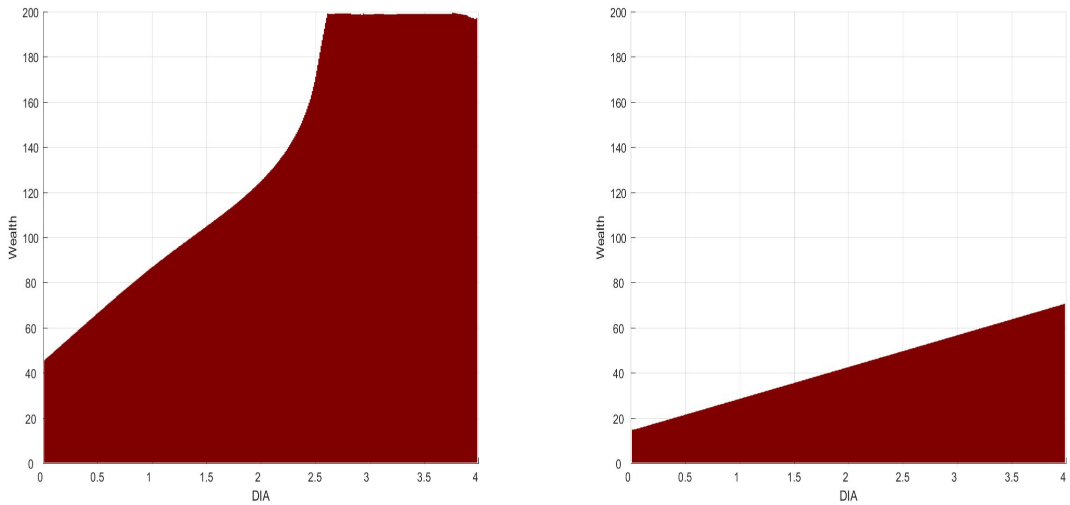
We look at results for a larger γ value.



(a) Investment Profile at Age 63 **(b)** Investment Profile at Age 65

Figure 7: Investment regimes at different times for γ=3.5 (all other parameters remain unchanged).

And finally, we look at results for a larger μ and larger γ.



(a) Investment Profile at Age 63

(b) Investment Profile at Age 65

Figure 8: Investment regimes at different times for parameter values of $\mu = 10\%$ and γ=3.5 (all other parameters remain unchanged).

One more set of figures that is worth looking at is a comparison of multiple regime diagrams. We now present figures which have only the outline of the regime boundaries, but with multiple boundaries in each figure.

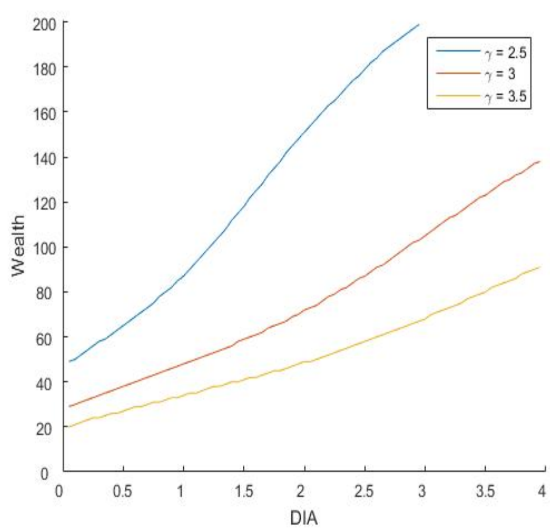


Figure 9: Multiple regime boundaries at age 55 with varying γ values

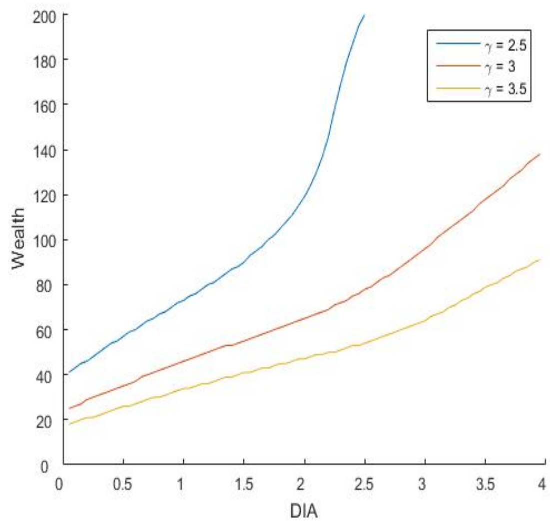


Figure 10: Multiple regime boundaries at age 61 with varying γ values

We can also do the same thing but with varying ages.

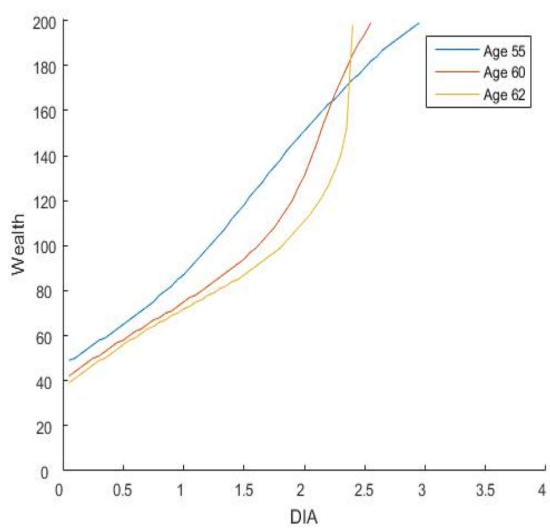


Figure 11: Multiple regime boundaries with baseline parameters with varying ages

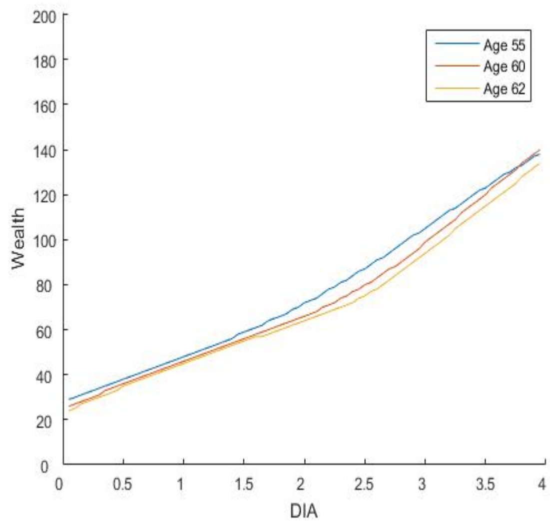


Figure 12: Multiple regime boundaries at $\gamma = 2.5$ with varying ages

It is worth elaborating on how a possible investor would use these graphs. Say the market aligns itself with the base line parameter values that we use, and say the investor has a gamma value of 3. At any point in time, the investor would look at her current wealth and income levels, and if she found herself in an annuitization region, would annuitize until she reaches the boundary between the two regions. Of course, the cost of the annuity at time t would be ${}_{\tau-t}\tilde{a}_{x+t}$. So for every dW that the retiree uses, he would get $\frac{1}{{}_{\tau-t}\tilde{a}_{x+t}}$ units of income. So we can draw a straight line from any point in the annuitization region to the boundary with a slope of $-{}_{\tau-t}\tilde{a}_{x+t}$. The following is just one example of this line drawn on the regime graph.

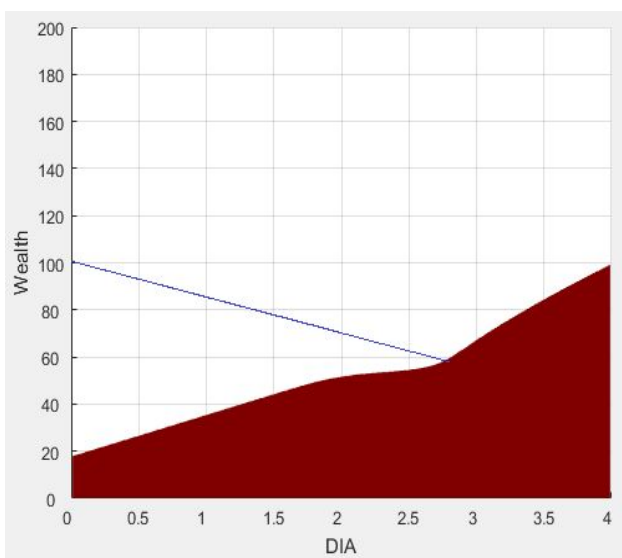


Figure 13: Investment Profile at Age 55 with the Characteristic Line Included

3 Optimal DIA Strategy and Asset Allocation

3.1 Problem Description

So far we have analyzed our problem whereby asset allocation is constant, i.e. the retiree does not in any way adjust the allocation to risky asset. In reality this is not true – the retiree is always allowed to choose her asset allocation.

We want to introduce the choice of asset allocation by allowing the retiree to invest in an optimal combination of risky and riskless assets. That is to say, we will compute at each time step the optimal mix of risky and riskless asset, and assume that instead of investing simply in the risky asset, retirees will instead invest in this optimal mix.

We now introduce a riskless investment for which the rate of return is r. Furthermore, we introduce the variable α which should range from 0 to 1 to indicate what proportion of our investment portfolio is allocated to risky asset. The rate of return and volatility of this new portfolio are $\alpha\mu + (1-\alpha)r$ and $\alpha\sigma$, respectively.

To reformulate our equations, we simply replace every μ and σ term in the previous section, with our new terms for rate of return and volatility. Our pre-retirement wealth process is:

$$dW_t = ((\alpha\mu + (1-\alpha)r)W_t + \nu - g_t)dt + \sigma\alpha W_t dB_t \tag{52}$$

And our post-retirement wealth process is:

$$dW_t = ((\alpha\mu + (1-\alpha)r)W_t + I_t + \pi_t - c_t)dt + \sigma\alpha W_t dB_t \tag{53}$$

Our value function J remains unchanged, and so we can apply the same procedure as above to find:

$$\begin{aligned} 0 \quad = \quad \sup_g \Bigg[& J_t + ((\alpha\mu + (1-\alpha)r)w + \nu - g_t)J_w + \frac{1}{2}(\sigma\alpha w)^2 J_{ww} \\ & + \frac{g_t}{{}_{(\tau-t)}\bar{a}_{(x+t)}} J_I + \lambda_{x+t}U(w+K+\nu) - (\rho + \lambda_{x+t})J \Bigg] \end{aligned} \tag{54}$$

for $t < \tau$, and

$$\begin{aligned} 0 \quad = \quad \sup_c \Bigg[& J_t + ((\alpha\mu + (1-\alpha)r)w + I_t + \pi_t - c_t)J_w + \frac{1}{2}(\sigma\alpha w)^2 J_{ww} \\ & + \lambda_{x+t}U(w+\pi_t) + U(c_t) - (\rho + \lambda_{x+t})J \Bigg] \end{aligned} \tag{55}$$

for $t \geq \tau$.

From here, we can follow the exact same procedure as before (equations (38) through (41)). One detail we must deal with is finding the optimal α. This can be done simply by taking the first derivative with respect to α in both equations and setting it equal to zero. Solving for alpha we get:

$$\alpha^* = -\frac{J_{ww}}{J_w}\frac{\mu - r}{w\sigma^2}$$

In post-retirement, we solve:

$$
\begin{aligned}
0 &= \sup_c \Big[J_t + ((\alpha\mu + (1-\alpha)r)w + I_t + 1 - c_t^*) J_w + \frac{1}{2}(\sigma\alpha w)^2 J_{ww} \\
&\quad + \lambda_{x+t} U(w+1) + U(c_t^*) - (\rho + \lambda_{x+t}) J \Big]
\end{aligned} \tag{56}
$$

and in pre-retirment, we either solve the PDE

$$
0 = J_t^{(1)} + ((\alpha\mu + (1-\alpha)r)w + \nu) J_w^{(1)} + \frac{1}{2}(\sigma\alpha w)^2 J_{ww}^{(1)} + \lambda_{x+t} U(w + K + \nu) - (\rho + \lambda_{x+t}) J^{(1)} \tag{57}
$$

or

$$
J_I^{(2)} - {}_{(\tau-t)}\bar{a}_{(x+t)} J_w^{(2)} = 0 \tag{58}
$$

The method we employ to solve these new equations is precisely the same as the previous section. We reformulate the finite difference equation with new coefficients as stated above and adjust the terminal and boundary conditions.

Mathematically speaking, while there is nothing stopping us from having an allocation to risky asset which is greater than 100%, there is typically a restriction put in place, in the context of most retirement portfolios, that one cannot borrow money and leverage their portfolio. Likewise, short positions (i.e. $\alpha < 0$) within retirement accounts is also prohibited. Therefore, we constrain $0 \leq \alpha \leq 1$. Whenever we calculate the optimal α and we see that it is greater than 1, we force the α value to be equal to 1.
Now in order to clearly compare the previous results with these new ones, we only allow optimal allocation in pre retirement. This way we will know that any changes we see will be a result of the change in the investment strategy in pre retirement.

Boundary and Terminal Conditions

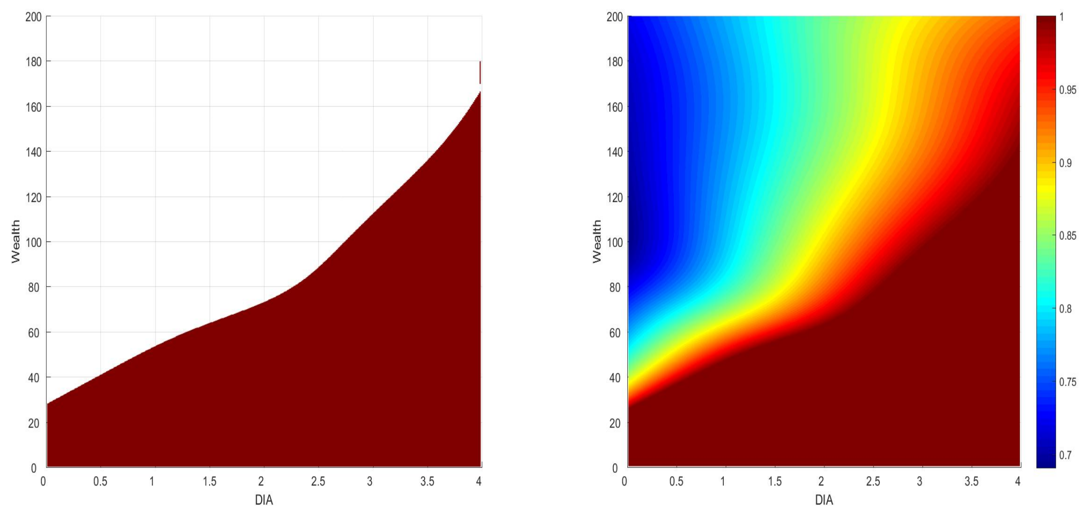
The boundary conditions in post retirement are, of course, the same, since we do not allow optimal allocation in post retirement and therefore use the same set up as before. In pre retirement, we simply adjust our return and volatility to have the new correct values. Aside from that, the boundary conditions are completely identical.

3.2 Numerical Results

We first outline some expected results. All things being equal, we should see that the non-annuitization region should get bigger in this new optimal asset allocation configuration. The reason for this is intuitive; the conditions of annuitization have remained the same, that is to say, with the addition of optimal allocation, we have not made the option of annuitization any better or worse. However, the option of non-annuitization has been made better by the allocation. So, we should see the non-annuitization region (red) get bigger when compared to the fixed allocation version.

When reading the graphs, recall that the part of the graph which is clear is the annuitization region, and the color banded region is the alpha profile. The color bar indicates what percentage of the portfolio is made up of the risky asset

Again we first look at results for Q=1 (which in implementation is set to 1.055, for the reasons mentioned above). We start with baseline parameters compare previous results. (Figures 1 through 8). As before, we first look at baseline parameters.

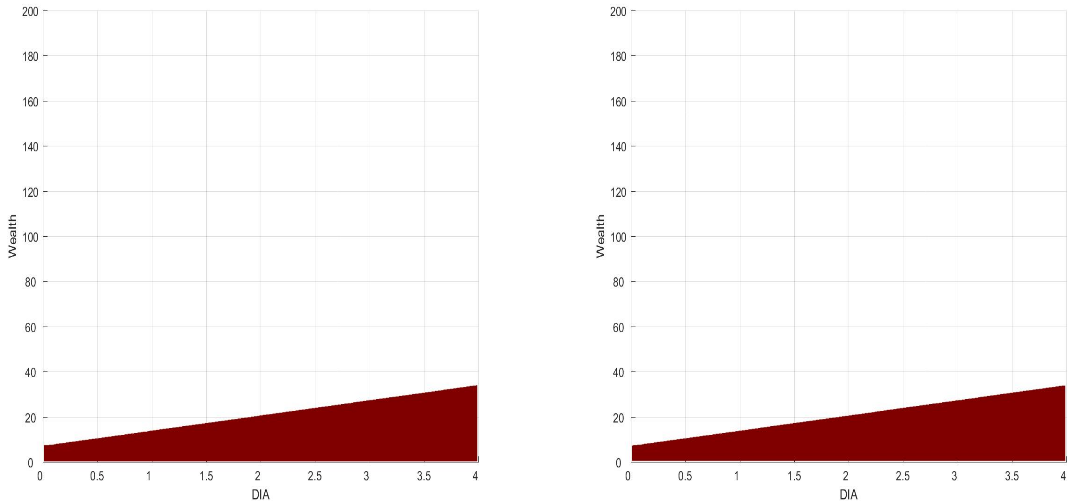


(a) Investment Profile at Age 63 without the Ability to Optimize Allocation

(b) Investment Profile at Age 63 with the Ability to Optimize Allocation

Figure 14: Investment regimes at age 63 for baseline parameter values.

At Q = 1, the DIA has no mortality credits, and therefore it is indistinguishable from the risk free asset, except for the fact that the purchase of a DIA is irreversible. Therefore, a retiree who wishes to purchase a DIA ought to wait until retirement to purchase a no delay DIA (effectively, a SPIA). This is because there is absolutely no incentive to purchase the DIA early. Let us see what happens in the last time step.



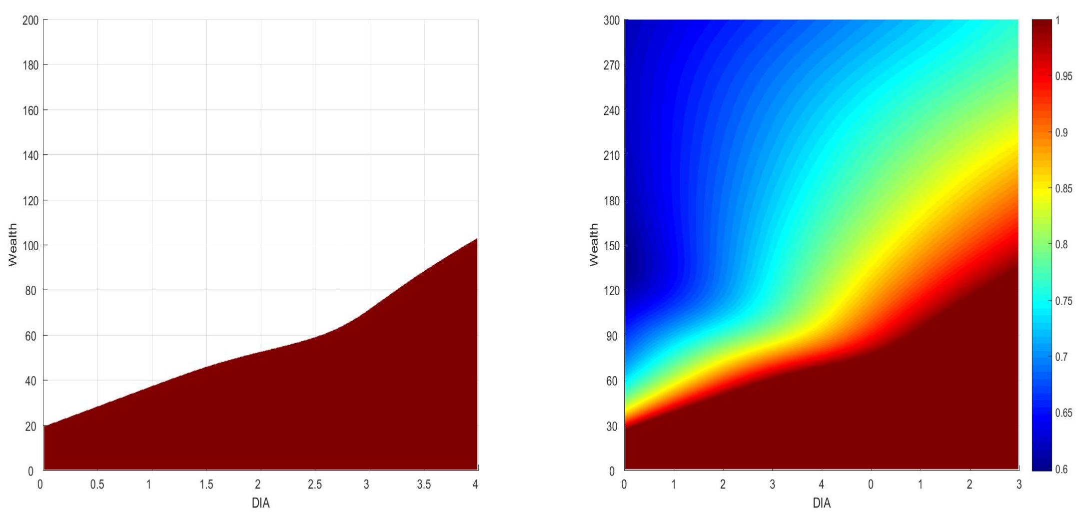
(a) Investment Profile at Age 65 without the Ability to Optimize Allocation

(b) Investment Profile at Age 65 with the Ability to Optimize Allocation

Figure 15: Investment regimes at age 65 for baseline parameter values.

At this final time step, we see the purchase of DIA is nearly identical weather or not there is the option to optimally invest.

Next we look at an increased γ value.



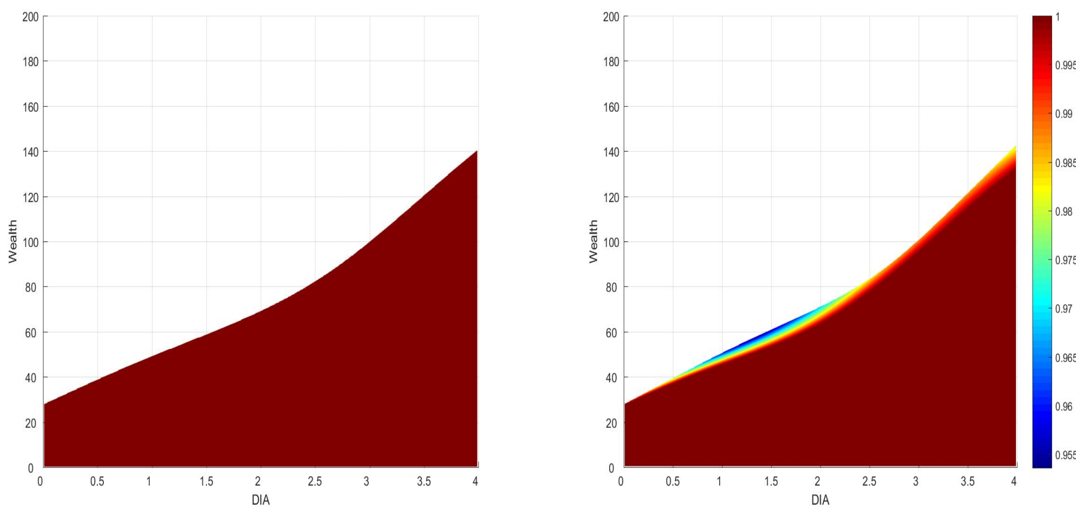
(a) Investment Profile at Age 63 without the Ability to Optimize Allocation

(b) Investment Profile at Age 63 with the Ability to Optimize Allocation

Figure 16: Investment regimes at age 63 for $\gamma = 3.5$ (all other parameters remain unchanged)

Once again, for a Q value of 1, we see no purchasing of DIA before retirement. It is worth mentioning however, that since we now have a higher risk aversion, the area in which we take advantage of the risk free asset, and the degree with which we take advantage of the risk free asset, is larger.

Now let us move on to a Q value of 0.7, as before. Remeber that this means there we are introducing mortality credits, and therefore are incentivizing the purchase of DIA early. We will now see that there are situations where both the risk free asset and the DIA are viable in a time step, depending on ones wealth and income level. We begin with baseline parameters.

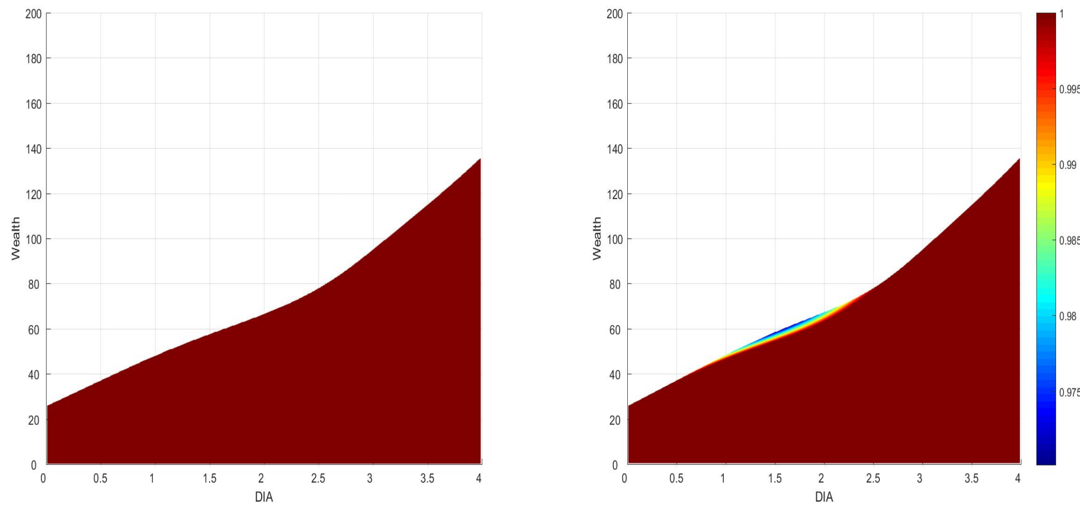


(a) Investment Profile at Age 60 without the Ability to Optimize Allocation

(b) Investment Profile at Age 60 with the Ability to Optimize Allocation

Figure 17: Investment regimes at age 60 for baseline parameter values.

We see at age 60 there is a region in which the retiree invests in the risk free asset, and also a region where they purchase DIA. Let us now see the same graphs but for later time.



(a) Investment Profile at Age 62 without the Ability to Optimize Allocation

(b) Investment Profile at Age 62 with the Ability to Optimize Allocation

Figure 18: Investment regimes at age 62 for baseline parameter values.

We can see here that the region of risk free investing has gotten smaller. As the retiree gets older, they move away from the risk free rate and instead use the DIA more and more as their proxy for a risk free rate. For reference, we also present the optimally investment profile at age 55:

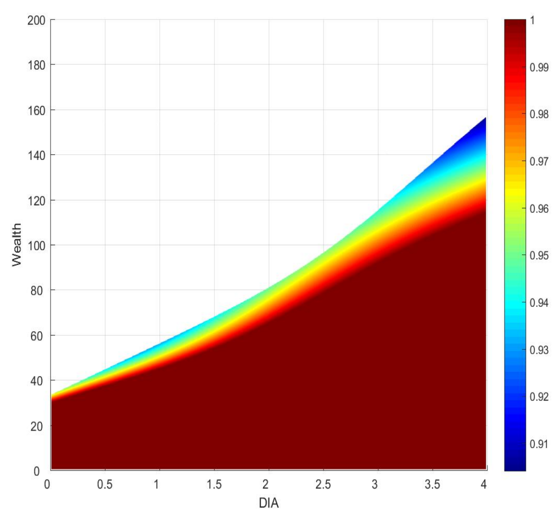
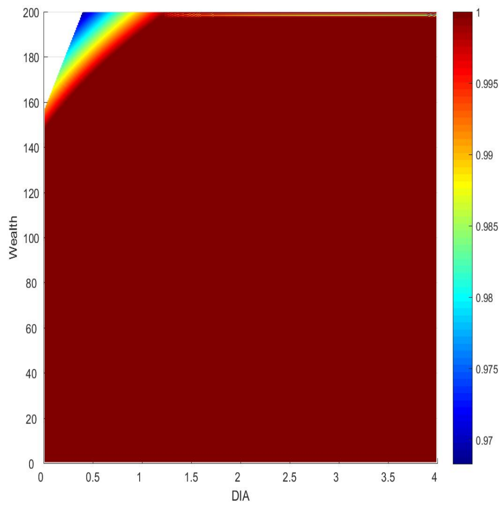
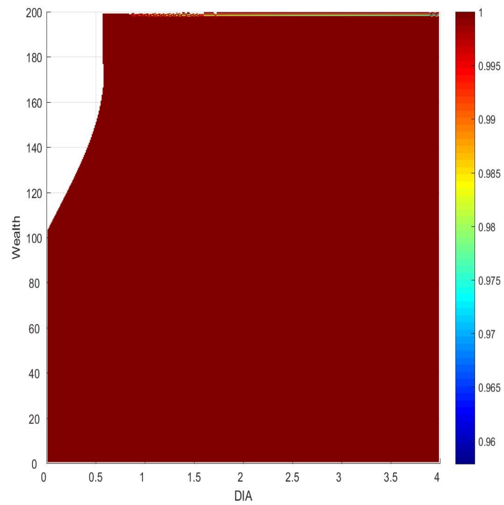


Figure 19: Investment Profile at Age 55 with the Ability to Optimize Allocation with baseline parameters

Next we increase the value of μ. From now on we only present the results from the optimal investment model at different time step.



(a) Investment Profile at Age 55 with the Ability to Optimize Allocation



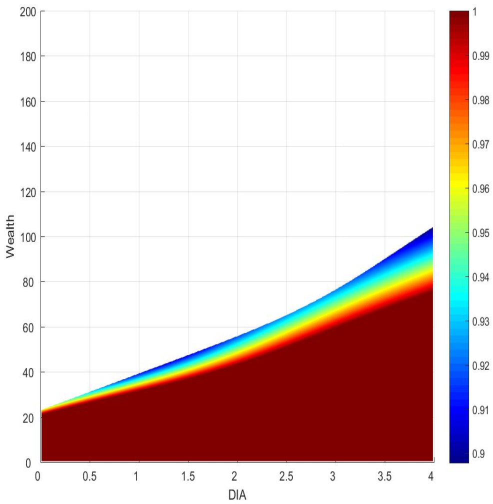
(b) Investment Profile at Age 62 with the Ability to Optimize Allocation

Figure 20: Investment regimes for $\mu = 10\%$ (all other parameters remain unchanged).

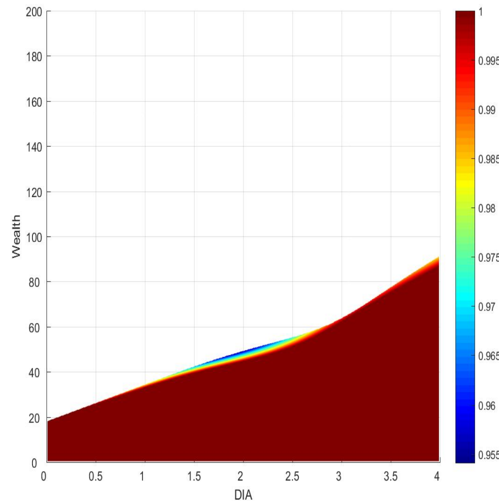
Note that near the top boundary there is some numerical noise which should be disregarded. It can easily be gotten rid of by adjusting the grid size and maximum wealth value.

We see that the increased rate of return in the risky asset makes the annnuitization region much smaller, and the amount that we invest in the risk free asset much less. When one looks at the color index on the right of each graph, one sees that the most the retiree will invest in the risk free asset is 3%, and this is only at the earliest possible age of 55; by age 62, there is no more investing in the risk free asset. Comparing that with values in figure 18b, we see there that approximately the same investment in the risk free asset of up to 3% takes place, but at age 62.

The following are results with in increased γ value.



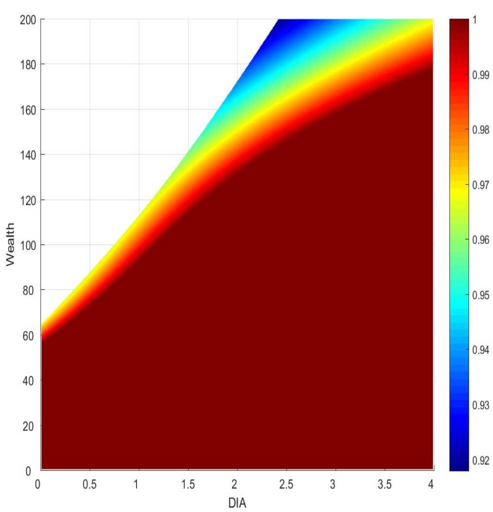
(a) Investment Profile at Age 55 with the Ability to Optimize Allocation



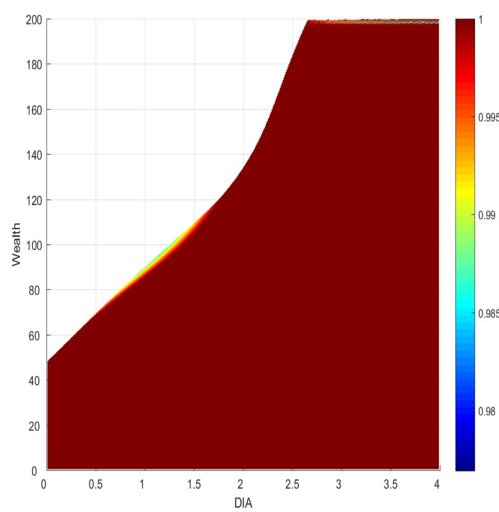
(b) Investment Profile at Age 62 with the Ability to Optimize Allocation

Figure 21: Investment regimes for $\gamma = 3.5$ (all other parameters remain unchanged).

As we expect, we have both a larger annuitization region, and more investing in the risk free asset. Finally, we look at results with both an increased γ and an increased μ:



(a) Investment Profile at Age 55 with the Ability to Optimize Allocation



(b) Investment Profile at Age 62 with the Ability to Optimize Allocation

Figure 22: Investment regimes for $\gamma = 3.5$ and $\mu = 10\%$ (all other parameters remain unchanged).

In general, we seem to always have some sort of asset allocation at early ages, but at later ages we have much less or no asset allocation. As well, we get larger annuitization regions as the retiree gets older.

4 Conclusion

In this thesis we explored the feasibility of optimally purchasing DIAs within a retirement investment plan. We developed an optimal allocation strategy using the principles of stochastic control theory and dynamic programming. We are able to demonstrate the wealth & income levels at which it would be optimal to annuitize a portion of the investments into a DIA.

Our intention was to answer the following questions: should one purchase DIAs now, or wait till retirement to invest in annuities? If one does purchase them early in a lump sum up front, or should it be done gradually over time?

Under fixed asset allocation, we saw that investing in the DIA is optimal under the correct combination of parameter values and timing. We saw that the annuitization region becomes larger for older ages. This is in line with the common wisdom of moving into risk free assets later in life. When we adjust parameters, we see results that are inline with the intuitive understanding of what these parameters should do. An increased μ will make the annuitization region smaller. An increased γ will make the annuitization region larger.

We saw that as time moved forward, the annuitization region got larger under any set of parameters. From this we could make the conclusion that the DIA should be bought in small increments as the retiree gets older and older. At the final time step, however, we did see a more dramatic increase in the annuitization region. This leads us to the following conclusion: The DIA should be bought in small increments throughout ones investment process, until one reaches retirement, at which point a large purchase of the DIA should be made.

While the DIA allocation region is larger under the fixed allocation strategy it is so because the DIA allocation becomes a proxy to fixed-income allocation. Under dynamic asset allocation, the allocation to DIA is less appealing. We saw that under optimal allocation, the annuitization region got smaller for certain parameter values and times, and stayed the same for certain parameters and times. There was never a scenario where the annuitization region became larger.

Furthermore, we also saw that there are scenarios where both the DIA and the risk free asset were both in one regime profile. This tells us that the DIA is not simply a proxy for the risk free rate, but that each has its own merits that may be taken advantage of at different times, wealth level and income levels.

Our model can be enhanced further by accounting for the actual cash refund as oppose to the commuted value. Our intuition is that the annuitization region will become smaller, all else being equal. Since DIAs are driven by the combined effects of mortality and prevailing interest rates, incorporating a stochastic DIA model in our framework could possibly be more insightful. As well, a term structure interest rate model would of course be more realistic.

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Appendices

Appendix A - Parameter Values and Grid Size

To generate our results we need capital market assumptions (r, μ, σ), mortality assumptions (λ_0, m, b), and a measure of risk aversion (γ). Our capital market assumptions are based on the 2017 Long-Term Capital Market Assumptions from J.P. Morgan Asset Management. Specifically, we use the returns assumption of US Large Cap as a proxy for risky assets, i.e. $\mu = 7.25\%$ and $\sigma = 14.75\%$, and US Short Duration Government Treasury as a proxy for risk-free return, i.e. $r = 3.25\%$. Of course these are just the values we start with. For simplicity, we used slightly adjusted values for implimentation purposes; $\mu = 8\%$ and $\sigma = 16\%$. We adjust these parameters and compare different results. We do not adjust ν and π. We simply set them to 1 for all simulations.

We fit the parameters of the Gompertz-Makeham model (λ_0, m, b) to the RP2014 table from the Society of Actuaries with projection scale table MP2014 applied for 2017. For males, the Gompertz-Makeham parameters are $\lambda_0 = 0.003148$, m = 89.5, b = 8.6, and for female, the mortality parameters are $\lambda_0 = 0.001982$, m = 91.6, and b = 8.5. We assume that the subjective discount rate ρ is equal to the risk-free rate r. These values are generally held fixed and never adjusted.

A variety of studies have estimated the value of γ. One of the earliest papers is the work by Friend and Blume (1975) [7], which has withstood the test of time and provides an empirical justification for constant relative risk aversion, estimates the value of γ to be between 1 and 2. Feldstein and Ranguelova (2001) [6]; Mitchel et al. (1999) [16] in the economics literature have employed values of less than 3. Mankiw and Zeldas (1991) [13]; Blake and Burrows (2001)[2]; Campbell and Viceira (2002)[4] suggest that risk aversion levels might be higher. On the other hand, to avoid the problem of picking a γ value, Browne et al. (2003) [3] invert the Merton optimum to solve for γ. However, any formulaic approach requires that we have the client's complete financial balance sheet inclusive of financial and real assets. In this thesis, we opt to use a γ value of 3 as the baseline value.

In pre-retirement, we use a W grid between 10^{-3} to 300 which is broken into 200 grid points. In port-retirement, we use a W grid between 10^{-3} to 3000 which is broken into 500 grid points. The reason we don't use a minimum W value of 0 is to avoid any issues of division by 0. We use income values from 0 to 6 with 60 grid points. Each time step is equal to $\frac{10}{50000}$ years. These values can be adjusted without a problem. They are simply being used here since they seem to produce reliable and stable results for most appropriate parameter values.

Appendix B - Finite Difference Equations Derivations

We now present the formulation for the post retirement problem and pre retirement problem. The post retirement problem is solved implicitly.

Let

$$J(t, w, I) = J(t_n, w_k, I_q) = J^n_{k,q}$$

$$\Delta t = \frac{T_{max} - T_{Min}}{N}, \Delta w = \frac{W_{max} - W_{Min}}{K}, \Delta I = \frac{I_{max} - I_{Min}}{Q}$$

This is the formal equations for the set up mentioned in the previous paragraph. We begin with standard first order approximations for J_t, J_w, and J_{ww}.

$$J_t = \frac{J^{n+1}_{k,q} - J^n_{k,q}}{\Delta t}$$

$$J_w = \frac{J^n_{k,q} - J^n_{k-1,q}}{\Delta w}, \text{ For a backward difference approximation;}$$

$$J_w = \frac{J^n_{k+1,q} - J^n_{k,q}}{\Delta w}, \text{ For a forward difference approximation.}$$

We will use both the forward and the backward difference approximation when we employ a clever upwinding technique. We will see this soon.

$$J_{ww} = \frac{J^n_{k+1,q} - 2J^n_{k,q} + J^n_{k-1,q}}{\Delta w^2}$$

Post Retirement

We now substitute these approximations into the post-retirement PDE (36). We get:

$$\begin{aligned} 0 \quad = \quad & \frac{J^{n+1}_{k,q} - J^n_{k,q}}{\Delta t} + (\mu w + I_q + 1 - c^*_t)J_w + \frac{1}{2}(\sigma w)^2 \frac{J^n_{k+1,q} - 2J^n_{k,q} + J^n_{k-1,q}}{\Delta w^2} \\ & + \lambda_{x+t} u(w+1) + u(c^*_t) - (\rho + \lambda_{x+t})J^n_{k,q} \end{aligned} \tag{59}$$

Now, for the J_w term, we will perform a forward difference if its coefficient is negative, and a backwards difference if its positive. This technique is known as up winding.

Now one can isolate for the J^{n+1} in terms of J^n at different wealth levels. But since we have a terminal condition for J at the final time step, we are actually moving backwards in time and therefore are looking for J^n. So we simply rearrange the equation to solve for J^n in terms of J^{n+1}. This is easy to do once we implement a matrix-vector notation and simply have to compute an inverse matrix. This is known as an implicit scheme.

Pre Retirement

For the pre retirement problem, we need to also approximate the J_I term.

$$J_I = \frac{J^n_{k,q} - J^n_{k,q-1}}{\Delta I}$$

Now we substitute our approximations into our 2 pre-retirement PDE (41).

$$\begin{aligned} 0 \quad = \quad & \frac{J^{n+1}_{k,q} - J^n_{k,q}}{\Delta t} + (\mu w + \nu)\frac{J^{n+1}_{k,q} - J^{n+1}_{k-1,q}}{\Delta w} && (60) \\ & + \frac{1}{2}(\sigma w)^2 \frac{J^{+1}_{k+1,q} - 2J^{n+1}_{k,q} + J^{n+1}_{k-1,q}}{\Delta w^2} && (61) \\ & + \lambda_{x+t} u(w + I_q \bar{a} + \nu) - (\rho + \lambda_{x+t}) J^{n+1}_{k,q} && (62) \end{aligned}$$

and

$$0 = \frac{J^n_{k,q} - J^n_{k,q-1}}{\Delta I} - \bar{a}\frac{J^n_{k,q} - J^n_{k-1,q}}{\Delta w} \tag{63}$$

For the preretirement problem, we must solve the system explicitly, so we will rearrange each equation so that $J^n_{k,q}$ is isolated.