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# Functions generating $(m,M,\Psi)$-Schur-convex sums

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*Dedicated to Professor Karol Baron on his 70th birthday.*

**Abstract.** The notion of $(m, M, \Psi)$-Schur-convexity is introduced and functions generating $(m, M, \Psi)$-Schur-convex sums are investigated. An extension of the Hardy–Littlewood–Pólya majorization theorem is obtained. A counterpart of the result of Ng stating that a function generates $(m, M, \Psi)$-Schur-convex sums if and only if it is $(m, M, \psi)$-Wright-convex is proved and a characterization of $(m, M, \psi)$-Wright-convex functions is given.

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## 1. Introduction

Let $(X, \|\cdot\|)$ be a real normed space. Assume that $D$ is a convex subset of $X$ and $c$ is a positive constant. A function $f : D \to \mathbb{R}$ is called:

– *strongly convex with modulus $c$* if

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 \tag{1}$$

for all $x, y \in D$ and $t \in [0,1]$;

– *strongly Wright-convex with modulus $c$* if

$$f(tx + (1-t)y) + f((1-t)x + ty) \le f(x) + f(y) - 2ct(1-t)\|x-y\|^2 \tag{2}$$

for all $x, y \in D$ and $t \in [0,1]$;

– *strongly Jensen-convex with modulus $c$* if (1) is assumed only for $t = \frac{1}{2}$, that is

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} - \frac{c}{4}\|x-y\|^2, \quad x, y \in D. \tag{3}$$

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The usual concepts of convexity, Wright-convexity and Jensen-convexity correspond to the case $c = 0$, respectively. The notion of strongly convex functions was introduced by Polyak [22] and they play an important role in optimization theory and mathematical economics. Many properties and applications of them can be found in the literature (see, for instance, [10,15,19,22–24,27]). Let us also mention the paper [18] by the second author which is a survey article devoted to strongly convex functions and related classes of functions.

In [1] the first author introduced the following concepts of $(m, \psi)$-lower convex, $(M, \psi)$-upper convex and $(m, M, \psi)$-convex functions (see also [2–4]): Assume that $D$ is a convex subset of a real linear space $X$, $\psi : D \to \mathbb{R}$ is a convex function and $m, M \in \mathbb{R}$. A function $f : D \to \mathbb{R}$ is called $(m, \psi)$-*lower convex* ($(M, \psi)$-*upper convex*) if the function $f - m\psi$ (the function $M\psi - f$) is convex. We say that $f : D \to \mathbb{R}$ is $(m, M, \psi)$-*convex* if it is $(m, \psi)$-lower convex and $(M, \psi)$-upper convex. Denote the above classes of functions by:

$$\begin{aligned} \mathcal{L}(D, m, \psi) &= \{f : D \to \mathbb{R} \ \mid f - m\psi \text{ is convex}\}, \\ \mathcal{U}(D, M, \psi) &= \{f : D \to \mathbb{R} \ \mid M\psi - f \text{ is convex}\}, \\ \mathcal{B}(D, m, M, \psi) &= \mathcal{L}(D, m, \psi) \cap \mathcal{U}(D, M, \psi). \end{aligned}$$

Let us observe that if $f \in \mathcal{B}(D, m, M, \psi)$ then $f - m\psi$ and $M\psi - f$ are convex and then $(M - m)\psi$ is also convex, implying that $M \geq m$ whenever $\psi$ is not trivial (i.e. is not the zero function).

If $m > 0$ and $(X, \|\cdot\|)$ is an inner product space (that is, the norm $\|\cdot\|$ in $X$ is induced by an inner product: $\|x\| = \sqrt{\langle x, x\rangle}$) the notions of $(m, \|\cdot\|^2)$-lower convexity and strong convexity with modulus $m$ coincide. Namely, in this case the following characterization was proved in [19]: A function $f$ is strongly convex with modulus $c$ if and only if $f - c\|\cdot\|^2$ is convex (for $X = \mathbb{R}^n$ this result can be also found in [8, Prop. 1.1.2]). However, if $(X, \|\cdot\|)$ is not an inner product space, then the two notions are different. There are functions $f \in \mathcal{L}(D, m, \|\cdot\|^2)$ which are not strongly convex with modulus $m$, as well as there are functions strongly convex with modulus $m$ which do not belong to $\mathcal{L}(D, m, \|\cdot\|^2)$ (see the examples given in [6]).

If $M > 0$ and $f \in \mathcal{U}(D, M, \psi)$, then $f$ is a difference of two convex functions. Such functions are called *d.c. convex* or $\delta$-*convex* and play an important role in convex analysis (cf. e.g. [26] and the reference therein). Functions from the class $\mathcal{U}(D, M, \|\cdot\|^2)$ with $M > 0$ were also investigated in [13] under the name *approximately concave functions*.

In [5] Dragomir and Ionescu introduced the concept of $g$-convex dominated functions, where $g$ is a given convex function. Namely, a function $f$ is called $g$-*convex dominated*, if the functions $g + f$ and $g - f$ are convex. Note that this concept can be obtained as a particular case of $(m, M, \psi)$-convexity by choosing $m = -1$, $M = 1$ and $\psi = g$. Observe also (cf. [1]), that in the case where $I \subset \mathbb{R}$ is an open interval and $f, \psi : I \to \mathbb{R}$ are twice differentiable, $f \in \mathcal{B}(I, m, M, \psi)$ if and only if

$$m\psi''(t) \leq f''(t) \leq M\psi''(t), \quad \text{for all} \quad t \in I.$$

In particular, if $I \subset (0, \infty)$, $f : I \to \mathbb{R}$ is twice differentiable and $\psi(t) = -\ln t$, then $f \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$m \leq t^2 f''(t) \leq M, \quad \text{for all} \quad t \in I, \tag{4}$$

which is a convenient condition to verify in applications.

Let $I \subset \mathbb{R}$ be an interval and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in I^n$, where $n \geq 2$. Following I. Schur (cf. e.g. [12,25]) we say that $x$ *is majorized by* $y$, and write $x \preceq y$, if there exists a doubly stochastic $n \times n$ matrix $P$ (i.e. a matrix containing nonnegative elements with all rows and columns summing up to 1) such that $x = y \cdot P$. A function $F : I^n \to \mathbb{R}$ is said to be *Schur-convex* if $F(x) \leq F(y)$ whenever $x \preceq y$, $x, y \in I^n$.

It is known, by the classical works of Schur [25], Hardy et al. [7] and Karamata [9] that if a function $f : I \to \mathbb{R}$ is convex then it *generates Schur-convex sums*, that is the function $F : I^n \to \mathbb{R}$ defined by

$$F(x) = F(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n)$$

is Schur-convex. It is also known that the convexity of $f$ is a sufficient but not necessary condition under which $F$ is Schur-convex. A full characterization of functions generating Schur-convex sums was given by Ng [16]. Namely, he proved that a function $f : I \to \mathbb{R}$ generates Schur-convex sums if and only if it is Wright-convex (cf. also [17]). Recently Nikodem et al. [20] obtained similar results in connection with strong convexity in inner product spaces. Let us also mention the paper by Olbryś [21] in which delta Schur-convex mappings are investigated.

The aim of this paper is to present some generalizations and counterparts of the above mentioned results related to $(m, \psi)$-lower convexity, $(M, \psi)$-upper convexity and $(m, M, \psi)$-convexity. We introduce the notion of $(m, M, \Psi)$-Schur-convex functions and give a sufficient and necessary condition for a function $f$ to generate $(m, M, \Psi)$-Schur-convex sums. As a corollary we obtain a counterpart of the classical Hardy–Littlewood–Pólya majorization theorem. Finally we introduce the concept of $(m, M, \psi)$-Wright-convex functions, prove a representation theorem for them and present an Ng-type characterization of functions generating $(m, M, \Psi)$-Schur-convex sums. It is worth underlining, that our results concern a few different classes of functions related to convexity and are formulated in vector spaces, that is in a much more general setting than the original ones.

## 2. Main results

Let $X$ be a real vector space. Similarly as in the classical case we define majorization in the product space $X^n$. Namely, given two $n$–tuples $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in X^n$ we say that *$x$ is majorized by $y$*, written $x \preceq y$, if

$$(x_1, \ldots, x_n) = (y_1, \ldots, y_n) \cdot P$$

for some doubly stochastic $n \times n$ matrix $P$.

In what follows we will assume that $D$ is a convex subset of a real vector space $X$, $\psi : D \to \mathbb{R}$ is a convex function and $m, M \in \mathbb{R}$. For any $n \geq 2$ define $\Psi_n : D^n \to \mathbb{R}$ by

$$\Psi_n(x_1, \ldots, x_n) = \psi(x_1) + \cdots + \psi(x_n), \quad x_1, \ldots, x_n \in D. \tag{5}$$

We say that a function $F : D^n \to \mathbb{R}$ is $(m, M, \Psi_n)$-*Schur-convex* if for all $x, y \in D^n$

$$x \preceq y \implies F(x) \leq F(y) - m\big(\Psi_n(y) - \Psi_n(x)\big) \tag{6}$$

and

$$x \preceq y \implies F(x) \geq F(y) - M\big(\Psi_n(y) - \Psi_n(x)\big). \tag{7}$$

If only condition (6) [condition (7)] is satisfied, we say that $F$ is $(m, \Psi_n)$-lower ($(M, \Psi_n)$-upper) Schur-convex.

Note that if $x \preceq y$ then $\Psi_n(x) \leq \Psi_n(y)$. It follows from the fact that the function $\psi$ is convex and so it generates Schur-convex sums $\Psi_n$.

Given a function $f : D \to \mathbb{R}$ and an integer $n \geq 2$ we define the function $F_n : D^n \to \mathbb{R}$ by

$$F_n(x_1, \ldots, x_n) = f(x_1) + \cdots + f(x_n), \quad x_1, \ldots, x_n \in D. \tag{8}$$

Now, let $D$ be a convex subset of a real vector space $X$, and let $m, M \in \mathbb{R}$. Assume that $\psi : D \to \mathbb{R}$ is a convex function and $\Psi_n : D^n \to \mathbb{R}$ is defined by (5). We will prove now that $(m, M, \psi)$-convex functions generate $(m, M, \Psi_n)$-Schur-convex sums.

**Theorem 1.** (i) *If $f \in \mathcal{L}(D, m, \psi)$, then the function $F_n$ defined by* (8) *is $(m, \Psi_n)$-lower Schur-convex;*

(ii) *If $f \in \mathcal{U}(D, M, \psi)$, then the function $F_n$ defined by* (8) *is $(M, \Psi_n)$-upper Schur-convex;*

(iii) *If $f \in \mathcal{B}(D, m, M, \psi)$, then the function $F_n$ defined by* (8) *is $(m, M, \Psi_n)$-Schur-convex.*

*Proof.* To prove (i) fix $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $D^n$ with $x \preceq y$. There exists a doubly stochastic $n \times n$ matrix $P = [t_{ij}]$ such that $x = y \cdot P$. Then

$$x_j = \sum_{i=1}^{n} t_{ij} y_i, \quad j = 1, \ldots, n.$$

Since $f \in \mathcal{L}(D, m, \psi)$, the function $g = f - m\psi$ is convex and hence

$$
\begin{aligned}
g(x_1) + \cdots + g(x_n) &= \sum_{j=1}^{n} g\Big(\sum_{i=1}^{n} t_{ij} y_i\Big) \le \sum_{j=1}^{n}\sum_{i=1}^{n} t_{ij} g(y_i) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij} g(y_i) = \sum_{i=1}^{n} g(y_i) \sum_{j=1}^{n} t_{ij} = g(y_1) + \cdots + g(y_n).
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
F_n(x) &= f(x_1) + \cdots + f(x_n) \\
&= g(x_1) + \cdots + g(x_n) + m\big(\psi(x_1) + \cdots + \psi(x_n)\big) \\
&\le g(y_1) + \cdots + g(y_n) + m\big(\psi(x_1) + \cdots + \psi(x_n)\big) \\
&= f(y_1) + \cdots + f(y_n) - m\big(\psi(y_1) + \cdots + \psi(y_n)\big) \\
&\quad + m\big(\psi(x_1) + \cdots + \psi(x_n)\big) \\
&= F_n(y) - m\big(\Psi_n(y) - \Psi_n(x)\big).
\end{aligned}
$$

This shows that $F_n$ satisfies (6), i.e. it is $(m, \Psi_n)$-lower Schur-convex.

The proof of part (ii) is similar. Since $f \in \mathcal{U}(D, M, \psi)$, the function $h = M\psi - f$ is convex. Hence, for $x$ and $y$ as previously, we have

$$
\begin{aligned}
F_n(x) &= f(x_1) + \cdots + f(x_n) \\
&= +M\big(\psi(x_1) + \cdots + \psi(x_n)\big) - h(x_1) - \cdots - h(x_n) \\
&\ge M\big(\psi(x_1) + \cdots + \psi(x_n)\big) - h(y_1) - \cdots - h(y_n) \\
&= M\big(\psi(x_1) + \cdots + \psi(x_n)\big) - M\big(\psi(y_1) + \cdots + \psi(y_n)\big) \\
&\quad + f(y_1) + \cdots + f(y_n) \\
&= F_n(y) - M\big(\Psi_n(y) - \Psi_n(x)\big).
\end{aligned}
$$

Part (iii) follows from (i) and (ii). □

As an immediate consequence of the above theorem, we obtain the following counterpart of the classical Hardy–Littlewood–Pólya majorization theorem [7].

**Corollary 2.** *Let $I \subset \mathbb{R}$ be an interval and $n \ge 2$. Assume that $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in I^n$ satisfy:*

(a) $x_1 \le \cdots \le x_n,\ y_1 \le \cdots \le y_n$;
(b) $y_1 + \cdots + y_k \le x_1 + \cdots + x_k,\quad k = 1, \ldots, n-1$;
(c) $y_1 + \cdots + y_n = x_1 + \cdots + x_n$.

*Assume also that $f, \psi : I \to \mathbb{R}$ and $\psi$ is convex.*

(i) *If $f \in \mathcal{L}(D, m, \psi)$, then*

$$f(x_1) + \cdots + f(x_n) \le f(y_1) + \cdots + f(y_n) - m\big(\Psi_n(y) - \Psi_n(x)\big);$$

(ii) *If* $f \in \mathcal{U}(D, M, \psi)$*, then*

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n) - M\big(\Psi_n(y) - \Psi_n(x)\big);$$

(iii) *If* $f \in \mathcal{B}(D, m, M, \psi)$*, then*

$$\begin{aligned} f(y_1) + \cdots + f(y_n) - M\big(\Psi_n(y) - \Psi_n(x)\big) &\leq f(x_1) + \cdots + f(x_n) \\ &\leq f(y_1) + \cdots + f(y_n) - m\big(\Psi_n(y) - \Psi_n(x)\big). \end{aligned}$$

*Proof.* Note that assumptions (a)–(c) imply $x \preceq y$ (see e.g. [12]) and apply Theorem 1. □

*Remark* 3. Specifying the functions $\psi$ and $f$ in Corollary 2 above, one can get various analytic inequalities. For example, if $I \subset (0, \infty)$ and $f \in \mathcal{B}(I, m, M, -\ln)$, then for all $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in I^n$ satisfying conditions (a)–(c), we get

$$m \ln \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right) \leq \sum_{i=1}^{n} f\left(y_i\right) - \sum_{i=1}^{n} f\left(x_i\right) \leq M \ln \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right),$$

or, equivalently,

$$\prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{m} \leq \frac{\exp\left[\sum_{i=1}^{n} f\left(y_i\right)\right]}{\exp\left[\sum_{i=1}^{n} f\left(x_i\right)\right]} \leq \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{M}. \tag{9}$$

If we take, for instance, $I = [k, K] \subset (0, \infty)$ and $f\left(t\right) = \frac{1}{p(p-1)} t^p$, with $p > 0$, $p \neq 1$, then $t^2 f''\left(t\right) = t^p \in \left[k^p, K^p\right]$, which means [cf. (4)] that $f \in \mathcal{B}(I, k^p, K^p, -\ln)$. Therefore, by (9), we then have

$$\prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{p(p-1)k^p} \leq \frac{\exp\left(\sum_{i=1}^{n} y_i^p\right)}{\exp\left(\sum_{i=1}^{n} x_i^p\right)} \leq \prod_{i=1}^{n} \left(\frac{x_i}{y_i}\right)^{p(p-1)K^p}.$$

One can give other examples by choosing $f\left(t\right) = t^q$ with $q < 0$, $f\left(t\right) = t \ln t$, etc.

We say that a function $f : D \to \mathbb{R}$ is $(m, \psi)$-*lower Jensen-convex* ($(M, \psi)$-*upper Jensen-convex*) if the function $f - m\psi$ (the function $M\psi - f$) is Jensen-convex, i.e. satisfies (3) with $c = 0$. We say that $f : D \to \mathbb{R}$ is $(m, M, \psi)$-*Jensen-convex* if it is $(m, \psi)$-lower Jensen-convex and $(M, \psi)$-upper Jensen-convex.

In the next theorem we show that functions generating $(m, M, \Psi_n)$-Schur-convex sums must be $(m, M, \psi)$-Jensen–convex.

**Theorem 4.** *Let* $f : D \to \mathbb{R}$*.*

(i) *If for some* $n \geq 2$ *the function* $F_n$ *defined by* (8) *is* $(m, \Psi_n)$*-lower Schur-convex, then* $f$ *is* $(m, \psi)$*-lower Jensen-convex;*

(ii) *If for some* $n \geq 2$ *the function* $F_n$ *defined by* (8) *is* $(M, \Psi_n)$*-upper Schur-convex, then* $f$ *is* $(M, \psi)$*-upper Jensen-convex;*

(iii) *If for some* $n \geq 2$ *the function* $F_n$ *defined by* (8) *is* $(m, M, \Psi_n)$*-Schur-convex, then* $f$ *is* $(m, M, \psi)$*- Jensen-convex.*

*Proof.* To prove (i) take $y_1, y_2 \in D$ and put $x_1 = x_2 = \frac{1}{2}(y_1 + y_2)$. Consider the points

$$y = (y_1, y_2, y_2, \ldots, y_2), \quad x = (x_1, x_2, y_2, \ldots, y_2)$$

(if $n = 2$, then we take $y = (y_1, y_2)$, $x = (x_1, x_2)$). One can check easily that $x \preceq y$. Therefore, by (6),

$$F_n(x) \le F_n(y) - m\big(\Psi_n(y) - \Psi_n(x)\big),$$

that is

$$2f\Big(\frac{y_1 + y_2}{2}\Big) \le f(y_1) + f(y_2) - m\Big(\psi(y_1) + \psi(y_2) - 2\psi\big(\frac{y_1 + y_2}{2}\big)\Big).$$

Hence, for $g = f - m\psi$ we have

$$\begin{aligned} 2g\Big(\frac{y_1 + y_2}{2}\Big) &= 2f\Big(\frac{y_1 + y_2}{2}\Big) - 2m\psi\Big(\frac{y_1 + y_2}{2}\Big) \\ &\le f(y_1) + f(y_2) - m\big((\psi(y_1) + \psi(y_2)\big) = g(y_1) + g(y_2), \end{aligned}$$

which means that $f$ is $(m, \psi)$-lower Jensen-convex.
The proof of part (ii) is similar. Part (iii) follows from (i) and (ii). □

We say that a function $f : D \to \mathbb{R}$ is $(m, \psi)$-*lower Wright-convex* ($(M, \psi)$-*upper Wright-convex*) if the function $f - m\psi$ (the function $M\psi - f$) is Wright-convex, i.e. satisfies (2) with $c = 0$. We say that $f : D \to \mathbb{R}$ is $(m, M, \psi)$-*Wright-convex* if it is $(m, \psi)$-lower Wright-convex and $(M, \psi)$-upper Wright-convex.

As was shown above in Theorems 1 and 2, if a function $f : D \to \mathbb{R}$ is $(m, M, \psi)$-convex, then for every $n \ge 2$ the corresponding function $F_n$ defined by (8) is $(m, M, \Psi_n)$-Schur-convex and if for some $n \ge 2$ the function $F_n$ is $(m, M, \Psi_n)$-Schur-convex, then $f$ is $(m, M, \psi)$-Jensen-convex. The next theorem characterizes all the functions $f$ for which $F_n$ are $(m, M, \Psi_n)$- Schur–convex. It is a counterpart of the result of Ng [16] on functions generating Schur–convex sums.

Recall also that a subset $D$ of a vector space $X$ is said to be *algebraically open* if for every $x \in D$ and for every $y \in X$ there exists $\varepsilon > 0$ such that

$$\{ty + (1 - t)x \ \mid t \in (-\varepsilon, \varepsilon)\} \subset D.$$

**Theorem 5.** *Let $f : D \to \mathbb{R}$, where $D$ is an algebraically open convex subset of a vector space $X$. Then:*

(i) *If $f$ is $(m, \psi)$-lower Wright-convex, then for every $n \ge 2$ the function $F_n$ defined by (8) is $(m, \Psi_n)$-lower Schur-convex. Conversely, if for some $n \ge 2$ the function $F_n$ is $(m, \Psi_n)$-lower Schur-convex, then $f$ is $(m, \psi)$-lower Wright-convex;*

(ii) *If $f$ is $(M, \psi)$-upper Wright-convex, then for every $n \ge 2$ the function $F_n$ defined by (8) is $(M, \Psi_n)$-upper Schur-convex. Conversely, if for some $n \ge 2$ the function $F_n$ is $(M, \Psi_n)$-upper Schur-convex, then $f$ is $(M, \psi)$-upper Wright-convex;*

(iii) *If* $f$ *is* $(m, M, \psi)$*- Wright-convex, then for every* $n \geq 2$ *the function* $F_n$ *defined by* (8) *is* $(m, M, \Psi_n)$*- Schur-convex. Conversely, if for some* $n \geq 2$ *the function* $F_n$ *is* $(m, M, \Psi_n)$*- Schur-convex, then* $f$ *is* $(m, M, \psi)$*-Wright-convex.*

*Proof.* To prove (i) assume that $f$ is $(m, \psi)$-lower Wright-convex and fix an $n \geq 2$. Since the function $g = f - m\psi$ is Wright-convex, it is of the form $g = g_1 + a$, where $g_1$ is convex and $a$ is additive (cf. [11]; here the assumption that $D$ is algebraically open is needed). Therefore it generates Schur-convex sums. Thus, for $x = (x_1, \ldots, x_n) \preceq y = (y_1, \ldots, y_n)$, we have

$$g(x_1) + \cdots + g(x_n) \leq g(y_1) + \cdots + g(y_n).$$

Hence

$$\begin{aligned} &f(x_1) + \cdots + f(x_n) - m\big(\psi(x_1) + \cdots + \psi(x_n)\big) \\ &\quad \leq g(y_1) + \cdots + g(y_n) - m\big(\psi(y_1) + \cdots + \psi(y_n)\big), \end{aligned}$$

which means that

$$F_n(x) \leq F_n(y) - m\big(\Psi_n(y) - \Psi_n(x)\big),$$

that is $F_n$ is $(m, \Psi_n)$-lower Schur-convex. Now, assume that for some $n \geq 2$ the function $F_n$ is $(m, \Psi_n)$-lower Schur-convex. Take $y_1, y_2 \in D$ and $t \in (0, 1)$. Put

$$x_1 = ty_1 + (1-t)y_2, \quad x_2 = (1-t)y_1 + ty_2$$

and, if $n > 2$, take additionally $x_i = y_i = z \in D$ for $i = 3, \ldots, n$. Then $x = (x_1, \ldots, x_n) \preceq y = (y_1, \ldots, y_n)$. Therefore, by (6),

$$F_n(x) \leq F_n(y) - m\big(\Psi_n(y) - \Psi_n(x)\big),$$

that is

$$\begin{aligned} &f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) \leq f(y_1) + f(y_2) - m\big(\psi(y_1) \\ &\quad + \psi(y_2) - \psi(x_1) - \psi(x_2)\big). \end{aligned}$$

Hence, for $g = f - m\psi$ we get

$$\begin{aligned} &g(ty_1 + (1-t)y_2) + g((1-t)y_1 + ty_2) \\ &\quad = f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) - m\psi(ty_1 + (1-t)y_2) \\ &\qquad - m\psi((1-t)y_1 + ty_2) \\ &\quad \leq f(y_1) + f(y_2) - m\psi(y_1) - m\psi(y_2) = g(y_1) + g(y_2). \end{aligned}$$

Thus $g$ is Wright-convex, which means that $f$ is $(m, \psi)$-lower Wright-convex. The proof of part (ii) is similar. Part (iii) follows from (i) and (ii). □

*Remark* 6. In the special case where $(X, \|\cdot\|)$ is an inner product space, $\psi = \|\cdot\|^2$ and $m = c > 0$, parts (i) of the above Theorems 1, 4, 5 reduce to the results obtained in [20] for strong Schur-convexity. For $m = 0$ and $X = \mathbb{R}^n$ they coincide with the Ng theorem [16].

Finally, we give a representation theorem for $(m, M, \psi)$-Wright-convex functions. It is known (and easy to check) that every convex function is Wright-convex, and every Wright-convex function is Jensen-convex, but not the converse (some examples can be found in [18]). In [16] Ng proved that a function $f$ defined on a convex subset of $\mathbb{R}^n$ is Wright-convex if and only if it can be represented in the form $f = f_1 + a$, where $f_1$ is a convex function and $a$ is an additive function (see also [18]). Kominek [11] extended that result to functions defined on algebraically open subset of a vector space. An analogous result for strongly Wright-convex functions was obtained in [14]. In the next theorem we give a similar representation for $(m, M, \psi)$-Wright-convex functions. In the proof we will use the following fact:

**Lemma 7.** *Assume that $f, g : D \to \mathbb{R}$ are convex functions, $a : X \to \mathbb{R}$ is additive and $a(x) = f(x) - g(x)$ for all $x \in D$. Then $a$ is an affine function on $D$.*

*Proof.* Fix $x, y \in D$ and consider the function $\varphi : [0,1] \to \mathbb{R}$ defined by

$$\varphi(s) = a(sx + (1-s)y) = f(sx + (1-s)y) - g(sx + (1-s)y), \ \ s \in [0,1].$$

As a difference of convex functions on $[0,1]$, $\varphi$ is continuous on $(0,1)$. Fix any $t \in (0,1)$ and take a sequence $(q_n)$ of rational numbers in $(0,1)$ tending to $t$. By the additivity of $a$ we have

$$a(q_n x + (1-q_n)y) = q_n a(x) + (1-q_n)a(y),$$

whence

$$\varphi(q_n) = q_n a(x) + (1-q_n)a(y).$$

Going to the limit we get

$$\varphi(t) = ta(x) + (1-t)a(y).$$

Hence

$$a(tx + (1-t)y) = ta(x) + (1-t)a(y),$$

which proves that $a$ is affine on $D$. □

**Theorem 8.** *Let $f : D \to \mathbb{R}$, where $D$ is an algebraically open convex subset of a vector space $X$. Then:*

*(i) $f$ is $(m, \psi)$-lower Wright-convex if and only if $f = g_1 + a_1$, where $g_1 \in \mathcal{L}(D, m, \psi)$ and $a_1 : X \to \mathbb{R}$ is additive;*
*(ii) $f$ is $(M, \psi)$-upper Wright-convex if and only if $f = g_2 + a_2$, where $g_2 \in \mathcal{U}(D, M, \psi)$ and $a_2 : X \to \mathbb{R}$ is additive;*
*(iii) $f$ is $(m, M, \psi)$- Wright-convex if and only if $f = g + a$, where $g \in \mathcal{B}(D, m, M, \psi)$ and $a : X \to \mathbb{R}$ is additive.*

*Proof.* To prove (i) assume first that $f$ is $(m,\psi)$-lower Wright-convex, that is $h = f - m\psi$ is Wright-convex. By the Ng representation theorem [16] (extended by Kominek [11] to functions defined on algebraically open domains), there exist a convex function $h_1 : D \to \mathbb{R}$ and an additive function $a_1 : X \to \mathbb{R}$ such that $h = h_1 + a_1$ on $D$. Then $g_1 = h_1 + m\psi$ belongs to $\mathcal{L}(D,m,\psi)$ and

$$f = h + m\psi = h_1 + a_1 + m\psi = g_1 + a_1,$$

which was to be proved. Conversely, if $f = g_1 + a_1$ with some $g_1 \in \mathcal{L}(D,m,\psi)$ and $a_1$ additive, then $f - m\psi = g_1 - m\psi + a_1$ is Wright-convex as a sum of a convex function and an additive function. This shows that $f$ is $(m,\psi)$-lower Wright-convex.

The proof of part (ii) is analogous.

Part (iii). If $f = g + a$, where $g \in \mathcal{B}(D,m,M,\psi)$ and $a : X \to \mathbb{R}$ is additive, then, by (i) and (ii) $f$ is $(m,\psi)$-lower Wright-convex and $(M,\psi)$-upper Wright-convex. Consequently, it is $(m,M,\psi)$-Wright-convex.

The proof in the opposite direction is more delicate. If $f$ is $(m,M,\psi)$-Wright-convex, then $f - m\psi$ and $M\psi - f$ are Wright-convex. Then

$$f - m\psi = h_1 + a_1 \quad \text{and} \quad M\psi - f = h_2 + a_2$$

with some convex functions $h_1, h_2$ and additive functions $a_1, a_2$. Hence

$$a_1 + a_2 = (M - m)\psi - (h_1 + h_2)$$

which, by Lemma 5, implies that $A = a_1 + a_2$ is affine. Denote $a = a_1$ and $g = f - a$. Then

$$g - m\psi = f - a - m\psi = h_1,$$

which implies that $g \in \mathcal{L}(D,m,\psi)$ because $h_1$ is convex. Also

$$M\psi - g = M\psi - f + a = h_2 + a_2 + a = h_2 + A,$$

which implies that $g \in \mathcal{U}(D,m,\psi)$ because $h_2 + A$ is convex. Thus $g \in \mathcal{B}(D,m,\psi)$ and $f = g + a$, which finishes the proof. □

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## References

[1] Dragomir, S.S.: On the reverse of Jessen's inequality for isotonic linear functionals. J. Inequal. Pure Appl. Math. **2**(3), 1–13 (2001). (Art. 36)

[2] Dragomir, S.S.: Some inequalities for $(m,M)$-convex mappings and applications for Csiszár Φ-divergence in information theory. Math. J. Ibaraki Univ. **33**, 35–50 (2001)

[3] Dragomir, S.S.: On the Jessen's inequality for isotonic linear functionals. Nonlinear Anal. Forum **7**(2), 139–151 (2002)

[4] Dragomir, S.S.: A survey on Jessen's type inequalities for positive functionals. In: Pardalos, P.M., et al. (eds.) Nonlinear Analysis. Springer Optimmization and Its Applications 68, pp. 177–232. Springer, New York (2012)

[5] Dragomir, S.S., Ionescu, N.M.: On some inequalities for convex-dominated functions. L'Anal. Num. Théor. L'Approx. **19**(1), 21–27 (1990)

[6] Dragomir, S.S., Nikodem, K.: Jensen's and Hermite–Hadamard's inequalities for lower and strongly convex functions on normed spaces. Bull. Iranian Math. Soc. **(in press)**

[7] Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, Cambridge (1952)

[8] Hiriart-Urruty, J.B., Lemaréchal, C.: Fundamentals of Convex Analysis. Springer, Berlin (2001)

[9] Karamata, J.: Sur une inégalité rélative aux fonctions convexes. Publ. Math. Univ. Belgrade **1**, 145–148 (1932)

[10] Klaričić, Bakula M., Nikodem, K.: On the converse Jensen inequality for strongly convex functions. J. Math. Anal. Appl. **434**, 516–522 (2016)

[11] Kominek, Z.: On additive and convex functionals. Radovi Mat. **3**, 267–279 (1987)

[12] Marshall, A.W., Olkin, I.: Inequalities: Theory of Majorization and Its Applications, Mathematics in Science and Engineering 143. Academic Press Inc., New York (1979)

[13] Merentes, N., Nikodem, K.: Strong convexity and separation theorems. Aequat. Math. **90**, 47–55 (2016)

[14] Merentes, N., Nikodem, K., Rivas, S.: Remarks on strongly Wright-convex functions. Ann. Polon. Math. **103**(3), 271–278 (2011)

[15] Montrucchio, L.: Lipschitz continuous policy functions for strongly concave optimization problems. J. Math. Econ. **16**, 259–273 (1987)

[16] Ng C. T.: Functions generating Schur-convex sums, In: Walter, W. (ed.) General Inequalities 5 (Oberwolfach, 1986), Internat. Ser. Numer. Math. vol. 80, pp. 433–438. Birkhäuser Verlag, Basel–Boston (1987)

[17] Nikodem, K.: On some class of midconvex functions. Ann. Polon. Math. **72**, 145–151 (1989)

[18] Nikodem, K.: On strongly convex functions and related classes of functions. In: Rassias, T.M (ed.) Handbook of Functional Equations. Functional Inequalities. Springer Optimization and Its Application 95, pp. 365–405. Springer, New York (2014).

[19] Nikodem, K., Páles, Zs: Characterizations of inner product spaces by strongly convex functions. Banach J. Math. Anal. **5**(1), 83–87 (2011)

[20] Nikodem, K., Rajba, T., Wąsowicz, Sz: Functions generating strongly Schur-convex sums. In: Bandle, C., et al. (eds.) Inequalities and Applications 2010, International Series of Numerical Mathematics 161, pp. 175–182. Springer, New York (2012)

[21] Olbryś, A.: On delta Schur-convex mappings. Publ. Math. Debrecen **86**, 313–323 (2015)

[22] Polyak, B.T.: Existence theorems and convergence of minimizing sequences in extremum problems with restrictions. Sov. Math. Dokl. **7**, 72–75 (1966)

[23] Rajba, T., Wąsowicz, Sz: Probabilistic characterization of strong convexity. Opusc. Math. **31**(1), 97–103 (2011)

[24] Roberts, A.W., Varberg, D.E.: Convex Functions. Academic Press, New York (1973)

[25] Schur, I.: Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie. Sitzungsber. Berl. Math. Ges. **22**, 9–20 (1923)

[26] Veselý, L., Zajíček, L.: Delta-convex mappings between Banach spaces and applications. Dissert. Math. **289**, PWN, Warszawa (1989)

[27] Vial, J.P.: Strong convexity of sets and functions. J. Math. Econ. **9**, 187–205 (1982)

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