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AN OPERATOR EXTENSION OF CARTWRIGHT-FIELD INEQUALITY

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Abstract. We establish an operator extension of the following inequality, due to Cartwright and Field

$$\frac{1}{2}p\left(1-p\right)\frac{\left(b-a\right)^2}{\max\left\{a,b\right\}}\le\left(1-p\right)a+pb-a^{1-p}b^p\le\frac{1}{2}p\left(1-p\right)\frac{\left(b-a\right)^2}{\min\left\{a,b\right\}}$$
$$\left(a,b>0,\quad p\in\left[0,1\right]\right).$$

Applications of this inequality for self-adjoint operators, continuous fields of operators and unital fields of positive linear mapping are also given.

1. Introduction

As is customary, we reserve m, M for scalars. Other capital letters are used to denote general elements of the C^*-algebra $\mathcal{B}\left(\mathcal{H}\right)$ of all bounded linear operators acting on a Hilbert space $\left(\mathcal{H},\langle\cdot,\cdot\rangle\right)$. An operator $A\in\mathcal{B}\left(\mathcal{H}\right)$ is called positive if $\langle Ax,x\rangle\ge0$ for all $x\in\mathcal{H}$, and we then write $A\ge0$. For self-adjoint operators $A,B\in\mathcal{B}\left(\mathcal{H}\right)$ we say that $A\le B$ if $B-A\ge0$. The Gelfand map $f\left(t\right)\mapsto f\left(A\right)$ is an isometrical $*$-isomorphism between the C^*-algebra $C\left(sp\left(A\right)\right)$ of continuous functions on the spectrum $sp\left(A\right)$ of a self-adjoint operator A and the C^*-algebra generated by A and I. If $f,g\in C\left(sp\left(A\right)\right)$, then $f\left(t\right)\ge g\left(t\right)\ \ \left(t\in sp\left(A\right)\right)$ implies that $f\left(A\right)\ge g\left(A\right)$ (see for instance [12, p. 15]). A linear map ϕ is positive if $\phi\left(A\right)\ge0$ whenever $A\ge0$. It said to be unital if $\phi\left(I\right)=I$. Moreover, $C\left(\left[m,M\right]\right)$ is the set of all real valued continuous functions defined on an interval $\left[m,M\right]$. For more studies in this direction, we refer to [1].

Let A be a positive operator on $\mathcal{H}$. Then for any $x\in\mathcal{H}$ and a given positive real number p

$$\langle A^px,x\rangle\le\left\|x\right\|^{2\left(1-p\right)}\langle Ax,x\rangle^p,\ p\in\left(0,1\right] \tag{1.1}$$

and

$$\langle A^px,x\rangle\ge\left\|x\right\|^{2\left(1-p\right)}\langle Ax,x\rangle^p,\ p\ge1. \tag{1.2}$$

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McCarthy [9] proved the inequalities (1.1) and (1.2) by using the spectral resolution of A and the Hölder inequality. There is considerable amount of literature devoting to the study of Hölder-McCarthy inequality and reverses. In [5, Theorem 3] Fuji et al. obtained the following interesting ratio inequality that provides a reverse of the Hölder-McCarthy inequality (1.2) for self-adjoint operator A on $\mathcal{H}$ satisfying $0 < mI \leq A \leq MI$

$$\langle A^p x, x\rangle \leq \left(\frac{1}{p^{\frac{1}{p}} q^{\frac{1}{q}}} \frac{M^p - m^p}{(M-m)^{\frac{1}{p}} (mM^p - Mm^p)^{\frac{1}{p}}} \right)^p \langle Ax, x\rangle^p,$$

for any $x \in \mathcal{H}$ with $\|x\| = 1$, where $q = \frac{p}{p-1}, p > 1$.

Moreover, Dragomir in [4, Theorem 3.1] obtained the following reverse inequality under the same conditions

$$\begin{aligned} 0 &\leq \langle A^p x, x\rangle - \langle Ax, x\rangle^p \\ &\leq p \begin{cases} \frac{1}{2}(M-m)\left(\left\|A^{p-1}x\right\|^2 - \left\langle A^{p-1}x, x\right\rangle^2\right)^{\frac{1}{2}} \\ \frac{1}{2}\left(M^{p-1} - m^{p-1}\right)\left(\|Ax\|^2 - \langle Ax, x\rangle^2\right)^{\frac{1}{2}} \end{cases} \\ &\leq \frac{1}{4} p (M-m)\left(M^{p-1} - m^{p-1}\right), \end{aligned}$$

for any $x \in \mathcal{H}$ with $\|x\| = 1$, where $p > 1$. For other inequalities of Hölder-McCarthy type, we refer to [6, 7, 8] and references therein.

In 1978, Cartwright and Field [2] obtained the following inequality

$$\frac{1}{2} p (1-p) \frac{(b-a)^2}{\max\{a,b\}} \leq (1-p) a + pb - a^{1-p} b^p \leq \frac{1}{2} p (1-p) \frac{(b-a)^2}{\min\{a,b\}} \tag{1.3}$$

for any $a, b > 0$ and $p \in [0, 1]$. For some recent operator inequalities obtained via Cartwright and Field inequality see [3].

Several inequalities involving positive linear maps have been recently given in [10, 11]. Motivated by the above results, in this paper we continue and complement this research by proving some reverses of Hölder-McCarthy inequality involving unital positive linear map by use of Cartwright and Field inequality. In Section 2 of this paper, we generalize inequality (1.3). More precisely, we show new versions of reverse Hölder-McCarthy inequality (Theorem 2.1 and Proposition 2.2). We begin Section 3 by establishing an operator version of inequality (1.3) involving continuous fields of operators (see Theorem 3.1). It seems that the inequalities related to positive linear maps are useful in operator theory and mathematical physics and are interesting in their own right.

2. **Some Reverses of Operator Hölder-McCarthy Inequality**

We start our work with the following theorem.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ be a self-adjoint with $0 < mI \le A \le MI$ and Φ be a unital positive linear map on $\mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}\frac{p(1-p)}{M}\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle - \left\langle \Phi(A)x,x\right\rangle^2\right) &\le \left\langle \Phi(A)x,x\right\rangle - \left\langle \Phi\left(A^{1-p}\right)x,x\right\rangle\left\langle \Phi(A^p)x,x\right\rangle \\ &\le \frac{p(1-p)}{m}\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle - \left\langle \Phi(A)x,x\right\rangle^2\right)\end{aligned} \tag{2.1}$$

for each $x \in \mathcal{H}$ with $\|x\| = 1$ and $p \in [0,1]$.

Proof. If $a, b \in [m, M]$, then by Cartwright-Field inequality we have

$$\frac{1}{2M}p(1-p)(b-a)^2 \le (1-p)a + pb - a^{1-p}b^p \le \frac{1}{2m}p(1-p)(b-a)^2$$

for any $p \in [0,1]$.

Fix $a \in [m, M]$ and by using the operator functional calculus for A we get

$$\begin{aligned}\frac{1}{2M}p(1-p)\left(A^2 - 2aA + a^2 I\right) &\le (1-p)aI + pA - a^{1-p}A^p \\ &\le \frac{1}{2m}p(1-p)\left(A^2 - 2aA + a^2 I\right).\end{aligned}$$

Since Φ is unital positive linear map we have

$$\begin{aligned}\frac{1}{2M}p(1-p)\left(\Phi\left(A^2\right) - 2a\Phi(A) + a^2 I\right) &\le (1-p)aI + p\Phi(A) - a^{1-p}\Phi(A^p) \\ &\le \frac{1}{2m}p(1-p)\left(\Phi\left(A^2\right) - 2a\Phi(A) + a^2 I\right).\end{aligned} \tag{2.2}$$

Then for any $x \in \mathcal{H}$ with $\|x\| = 1$ we have from (2.2) that

$$\begin{aligned}&\frac{1}{2M}p(1-p)\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle - 2a\left\langle \Phi(A)x,x\right\rangle + a^2 I\right) \\ &\le (1-p)aI + p\left\langle \Phi(A)x,x\right\rangle - a^{1-p}\left\langle \Phi(A^p)x,x\right\rangle \\ &\le \frac{1}{2m}p(1-p)\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle - 2a\left\langle \Phi(A)x,x\right\rangle + a^2 I\right).\end{aligned} \tag{2.3}$$

By applying again functional calculus we deduce

$$\begin{aligned}&\frac{1}{2M}p(1-p)\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle I - 2A\left\langle \Phi(A)x,x\right\rangle + A^2\right) \\ &\le (1-p)A + p\left\langle \Phi(A)x,x\right\rangle I - A^{1-p}\left\langle \Phi(A^p)x,x\right\rangle \\ &\le \frac{1}{2m}p(1-p)\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle I - 2\left\langle \Phi(A)x,x\right\rangle A + A^2\right).\end{aligned}$$

Since Φ is unital positive linear map we have

$$\begin{aligned}&\frac{1}{2M}p(1-p)\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle I - 2\Phi(A)\left\langle \Phi(A)x,x\right\rangle + \Phi\left(A^2\right)\right) \\ &\le (1-p)\Phi(A) + p\left\langle \Phi(A)x,x\right\rangle I - \Phi\left(A^{1-p}\right)\left\langle \Phi(A^p)x,x\right\rangle \\ &\le \frac{1}{2m}p(1-p)\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle I - 2\Phi(A)\left\langle \Phi(A)x,x\right\rangle + \Phi\left(A^2\right)\right)\end{aligned}$$

for any $x \in \mathcal{H}$ with $\|x\| = 1$ and $p \in [0,1]$.

Therefore

$$\frac{p(1-p)}{M}\left(\left\langle\Phi\left(A^{2}\right)x,x\right\rangle-\left\langle\Phi\left(A\right)x,x\right\rangle^{2}\right)\leq\left\langle\Phi\left(A\right)x,x\right\rangle-\left\langle\Phi\left(A^{1-p}\right)x,x\right\rangle\left\langle\Phi\left(A^{p}\right)x,x\right\rangle$$
$$\leq\frac{p(1-p)}{m}\left(\left\langle\Phi\left(A^{2}\right)x,x\right\rangle-\left\langle\Phi\left(A\right)x,x\right\rangle^{2}\right)$$

for any $x\in\mathcal{H}$ with $\|x\|=1$ and $p\in[0,1]$. □

Remark. *Let Φ be a identity positive linear map in (2.1), then we have the following reverses of McCarty inequality*

$$\frac{p(1-p)}{M}\left(\left\langle A^{2}x,x\right\rangle-\left\langle Ax,x\right\rangle^{2}\right)\leq\left\langle Ax,x\right\rangle-\left\langle A^{1-p}x,x\right\rangle\left\langle A^{p}x,x\right\rangle$$
$$\leq\frac{p(1-p)}{m}\left(\left\langle A^{2}x,x\right\rangle-\left\langle Ax,x\right\rangle^{2}\right)\tag{2.4}$$

for each $x\in\mathcal{H}$ with $\|x\|=1$ and $p\in[0,1]$. Choose $p=\frac{1}{2}$ in (2.4), then

$$\frac{1}{4M}\left(\left\langle A^{2}x,x\right\rangle-\left\langle Ax,x\right\rangle^{2}\right)\leq\left\langle Ax,x\right\rangle-\left\langle A^{\frac{1}{2}}x,x\right\rangle^{2}\leq\frac{1}{4m}\left(\left\langle A^{2}x,x\right\rangle-\left\langle Ax,x\right\rangle^{2}\right)$$

for each $x\in\mathcal{H}$ with $\|x\|=1$.

On the other hand, if A is a self-adjoint operator satisfying $0<mI\leq A\leq MI$, then for every $x\in\mathcal{H}$ with $\|x\|=1$ we have

$$\left\langle A^{2}x,x\right\rangle-\left\langle Ax,x\right\rangle^{2}\leq\frac{(M-m)^{2}}{4}.$$

Based on this fact and the inequality (2.4), we deduce

$$\left\langle Ax,x\right\rangle-\left\langle A^{\frac{1}{2}}x,x\right\rangle^{2}\leq\frac{(M-m)^{2}}{16m}$$

for each $x\in\mathcal{H}$ with $\|x\|=1$.

Remark. *All assumptions as in Theorem 2.1. Another interesting case for Theorem 2.1 is to apply functional calculus for self-adjoint operator B in (2.3). In fact the following inequality holds for each $x\in\mathcal{H}$ with $\|x\|=1$ and $p\in[0,1]$.*

$$\frac{1}{2M}p(1-p)\left(\left\langle\Phi\left(A^{2}\right)x,x\right\rangle-2\left\langle\Phi\left(B\right)x,x\right\rangle\left\langle\Phi\left(A\right)x,x\right\rangle+\left\langle\Phi\left(B^{2}\right)x,x\right\rangle\right)$$
$$\leq(1-p)\left\langle\Phi\left(B\right)x,x\right\rangle+p\left\langle\Phi\left(A\right)x,x\right\rangle-\left\langle\Phi\left(B^{1-p}\right)x,x\right\rangle\left\langle\Phi\left(A^{p}\right)x,x\right\rangle$$
$$\leq\frac{1}{2m}p(1-p)\left(\left\langle\Phi\left(A^{2}\right)x,x\right\rangle-2\left\langle\Phi\left(B\right)x,x\right\rangle\left\langle\Phi\left(A\right)x,x\right\rangle+\left\langle\Phi\left(B^{2}\right)x,x\right\rangle\right).$$

In the following, we give a reverse of AM-GM inequality involving operator norm. In fact, when the operators A and B are bounded above and below by constants we have the following result as well:

Remark. *Assume that $0<mI\leq B\leq A\leq MI$, then*

$$m-M\leq\|\Phi(A)\|-\|\Phi(B)\|\leq M-m$$

therefore

$$\left(\|\Phi(A)\|-\|\Phi(B)\|\right)^{2}\leq(M-m)^{2}.$$

If we take $a=\|\Phi(A)\|$ and $b=\|\Phi(B)\|$ in (1.3), we get

$$\begin{aligned}\frac{p(1-p)}{2M}(\|\Phi(B)\|-\|\Phi(A)\|)^2 &\le (1-p)\|\Phi(A)\|+p\|\Phi(B)\|-\|\Phi(A)\|^{1-p}\|\Phi(B)\|^p\\ &\le \frac{p(1-p)}{2m}(\|\Phi(B)\|-\|\Phi(A)\|)^2.\end{aligned} \tag{2.5}$$

Put $p=\frac{1}{2}$ in (2.5), then

$$\begin{aligned}\frac{1}{8M}(\|\Phi(B)\|-\|\Phi(A)\|)^2 &\le \frac{\|\Phi(A)\|+\|\Phi(B)\|}{2}-(\|\Phi(A)\|\,\|\Phi(B)\|)^{\frac{1}{2}}\\ &\le \frac{1}{8m}(\|\Phi(B)\|-\|\Phi(A)\|)^2.\end{aligned} \tag{2.6}$$

By the inequality (2.6) we deduce

$$\frac{\|\Phi(A)\|+\|\Phi(B)\|}{2}-(\|\Phi(A)\|\,\|\Phi(B)\|)^{\frac{1}{2}} \le \frac{(M-m)^2}{8m}. \tag{2.7}$$

Remark. *There are several examples of normalized positive linear maps. But for our application we consider among them $\Phi: M_2(\mathbb{C}) \to M_2(\mathbb{C})$, which $\Phi(A)=\left(\frac{1}{2}tr(A)\right)1_{\mathcal{H}}$ for all Hermitian matrices $A\in M_2(\mathbb{C})$. From inequality (2.7), we have*

$$\frac{|tr(A)|+|tr(B)|}{2}-(|tr(A)|\,|tr(B)|)^{\frac{1}{2}} \le \frac{(M-m)^2}{4m}. \tag{2.8}$$

Remark. *Taking $\Phi(A)=\sum\limits_{i=1}^{n}X_i^*AX_i$ and $\Phi(B)=\sum\limits_{i=1}^{n}X_i^*BX_i$ with $X_i^*X_i=I$ in (2.7), therefore*

$$\frac{\left\|\sum\limits_{i=1}^{n}X_i^*AX_i\right\|+\left\|\sum\limits_{i=1}^{n}X_i^*BX_i\right\|}{2}-\left(\left\|\sum_{i=1}^{n}X_i^*A^{\frac{1}{2}}X_i\right\|\left\|\sum_{i=1}^{n}X_i^*B^{\frac{1}{2}}X_i\right\|\right)\le\frac{(M-m)^2}{8m}.$$

Now, from a different view point we may state:

Proposition 2.2. *Let $A,B\in\mathcal{B}(\mathcal{H})$ be self-adjoint with $0<mI\le A,B\le MI$ and Φ be a unital positive linear map on $\mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}&\frac{p(1-p)}{2M}\left(\langle\Phi(A)x,x\rangle^2-2\langle\Phi(A)x,x\rangle\langle\Phi(B)x,x\rangle+\left\langle\Phi\left(B^2\right)x,x\right\rangle\right)\\ &\le(1-p)\langle\Phi(B)x,x\rangle+p\langle\Phi(A)x,x\rangle-\left\langle\Phi\left(B^{1-p}\right)x,x\right\rangle\langle\Phi(A)x,x\rangle^p\\ &\le\frac{p(1-p)}{2m}\left(\langle\Phi(A)x,x\rangle^2-2\langle\Phi(A)x,x\rangle\langle\Phi(B)x,x\rangle+\left\langle\Phi\left(B^2\right)x,x\right\rangle\right)\end{aligned} \tag{2.9}$$

for each $x\in\mathcal{H}$ with $\|x\|=1$ and $p\in[0,1]$.

Proof. If we take $b=\langle\Phi(A)x,x\rangle$ in (1.3) and then applying functional calculus for the operator B we get

$$\begin{aligned}&\frac{p(1-p)}{2M}\left(\langle\Phi(A)x,x\rangle^2I-2\langle\Phi(A)x,x\rangle B+B^2\right)\\ &\le(1-p)B+p\langle\Phi(A)x,x\rangle I-B^{1-p}\langle\Phi(A)x,x\rangle^p\\ &\le\frac{p(1-p)}{2m}\left(\langle\Phi(A)x,x\rangle^2I-2\langle\Phi(A)x,x\rangle B+B^2\right)\end{aligned}$$

for each $x \in \mathcal{H}$ with $\|x\| = 1$.

Since Φ is unital positive linear map we have

$$\begin{aligned}
&\frac{p(1-p)}{2M}\left(\langle \Phi(A)x,x\rangle^2 I - 2\langle \Phi(A)x,x\rangle \Phi(B) + \Phi\left(B^2\right)\right)\\
&\le (1-p)\Phi(B) + p\langle \Phi(A)x,x\rangle I - \Phi\left(B^{1-p}\right)\langle \Phi(A)x,x\rangle^p\\
&\le \frac{p(1-p)}{2m}\left(\langle \Phi(A)x,x\rangle^2 I - 2\langle \Phi(A)x,x\rangle \Phi(B) + \Phi\left(B^2\right)\right).
\end{aligned}$$

Whence

$$\begin{aligned}
&\frac{p(1-p)}{2M}\left(\langle \Phi(A)x,x\rangle^2 - 2\langle \Phi(A)x,x\rangle\langle \Phi(B)x,x\rangle + \left\langle \Phi\left(B^2\right)x,x\right\rangle\right)\\
&\le (1-p)\langle \Phi(B)x,x\rangle + p\langle \Phi(A)x,x\rangle - \left\langle \Phi\left(B^{1-p}\right)x,x\right\rangle\langle \Phi(A)x,x\rangle^p \qquad (2.10)\\
&\le \frac{p(1-p)}{2m}\left(\langle \Phi(A)x,x\rangle^2 - 2\langle \Phi(A)x,x\rangle\langle \Phi(B)x,x\rangle + \left\langle \Phi\left(B^2\right)x,x\right\rangle\right)
\end{aligned}$$

for each $x \in \mathcal{H}$ with $\|x\| = 1$. □

Remark. *If we take $B = A$ in Proposition 2.2, we get*

$$\begin{aligned}
\frac{p(1-p)}{2M}\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle - \langle \Phi(A)x,x\rangle^2\right) &\le \langle \Phi(A)x,x\rangle - \left\langle \Phi\left(A^{1-p}\right)x,x\right\rangle\langle \Phi(A)x,x\rangle^p\\
&\le \frac{p(1-p)}{2m}\left(\left\langle \Phi\left(A^2\right)x,x\right\rangle - \langle \Phi(A)x,x\rangle^2\right).
\end{aligned}$$

3. **Continuous Field of Operators**

Let T be a locally compact Hausdorff space, and let $\mathfrak{A}$ be a C^*-algebra of operators on a Hilbert space $\mathcal{H}$. We say that a field $(A_t)_{t\in T}$ of operators in $\mathfrak{A}$ is continuous if the function $t \to A_t$ is a norm continuous on T. If in addition μ is a Radon measure on T and the function $t \to \|A_t\|$ is integrable, then we can form the Bochner integral $\int_T x_t d\mu(t)$, which is the unique element in $\mathfrak{A}$ such that $\varphi\left(\int_T A_t d\mu(t)\right) = \int_T \varphi(A_t)\, d\mu(t)$ for every linear functional φ in the norm dual $\mathfrak{A}^*$.

Let $(\phi_t)_{t\in T}$ be a field of positive linear mappings $\Phi_t : \mathfrak{A} \to \mathfrak{B}$ from $\mathfrak{A}$ to another C^*-algebra $\mathfrak{B}$ of operators on a Hilbert space $\mathcal{K}$. We say that such a field is continuous if the function $t \to \Phi_t(A)$ is continuous for every $A \in \mathfrak{A}$. If in addition the C^*-algebras are unital and $\Phi_t(1)$ is integrable with integral 1, we say that $(\Phi_t)_{t\in T}$ is unital.

Utilizing the above facts we prove our main result of this section.

Theorem 3.1. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be C^*-algebras of operators containing I, T be a locally compact Hausdorff space equipped with a bounded Radon measure μ such that $\mu(T) = 1$. Also let $(A_t)_{t\in T}$ be a bounded continuous field of positive elements in $\mathfrak{A}$ and $(\Phi_t)_{t\in T}$ be a field of positive linear mappings $\Phi_t : \mathfrak{A} \to \mathfrak{B}$ defined on T.*

Then

$$\begin{aligned}
&\frac{1}{2}p\left(1-p\right)c\left(m,M\right)\int_T \Phi_t\left(I\right)d\mu\left(t\right)\\
&\le \frac{1}{2}\frac{p\left(1-p\right)}{\max\left\{M,1\right\}}\left(\int_T \Phi_t\left(A_t^2\right)d\mu\left(t\right)-2\int_T \Phi_t\left(A_t\right)d\mu\left(t\right)+\int_T \Phi_t\left(I\right)d\mu\left(t\right)\right)\\
&\le \left(1-p\right)\int_T \Phi_t\left(I\right)d\mu\left(t\right)+p\int_T \Phi_t\left(A_t\right)d\mu\left(t\right)-\int_T \Phi_t\left(A_t^p\right)d\mu\left(t\right)\\
&\le \frac{1}{2}\frac{p\left(1-p\right)}{\min\left\{m,1\right\}}\left(\int_T \Phi_t\left(A_t^2\right)d\mu\left(t\right)-2\int_T \Phi_t\left(A_t\right)d\mu\left(t\right)+\int_T \Phi_t\left(I\right)d\mu\left(t\right)\right)\\
&\le \frac{1}{2}p\left(1-p\right)C\left(m,M\right)\int_T \Phi_t\left(I\right)d\mu\left(t\right)
\end{aligned}$$

for every $p\in[0,1]$, *where*

$$c\left(m,M\right)=\begin{cases}\left(M-1\right)^2 & \text{if } M<1\\ 0 & \text{if } m\le 1\le M\\ \frac{\left(m-1\right)^2}{M} & \text{if } 1<m\end{cases}$$

and

$$C\left(m,M\right)=\begin{cases}\frac{\left(m-1\right)^2}{m} & \text{if } M<1\\ \frac{1}{m}\max\left\{\left(m-1\right)^2,\left(M-1\right)^2\right\} & \text{if } m\le 1\le M\\ \left(M-1\right)^2 & \text{if } 1<m\end{cases}.$$

Proof. If we write the inequality (1.3) for $a=1$ and $b=z$ we get

$$\frac{1}{2}p\left(1-p\right)\frac{\left(z-1\right)^2}{\max\left\{z,1\right\}}\le 1-p+pz-z^p\le \frac{1}{2}p\left(1-p\right)\frac{\left(z-1\right)^2}{\min\left\{z,1\right\}} \tag{3.1}$$

for any $z>0$ and for any $p\in[0,1]$.

If $z\in[m,M]\subset(0,\infty)$, then $\max\{z,1\}\le\max\{M,1\}$ and $\min\{m,1\}\le\min\{x,1\}$ and by (3.1) we get

$$\begin{aligned}
\frac{1}{2}p\left(1-p\right)\frac{\min_{z\in[m,M]}\left(z-1\right)^2}{\max\left\{M,1\right\}}&\le \frac{1}{2}p\left(1-p\right)\frac{\left(z-1\right)^2}{\max\left\{M,1\right\}}\\
&\le 1-p+pz-z^p\\
&\le \frac{1}{2}p\left(1-p\right)\frac{\left(z-1\right)^2}{\min\left\{m,1\right\}}\\
&\le \frac{1}{2}p\left(1-p\right)\frac{\max_{z\in[m,M]}\left(z-1\right)^2}{\min\left\{m,1\right\}}
\end{aligned} \tag{3.2}$$

for any $z>0$ and for any $p\in[0,1]$.

Observe that

$$\min_{z\in[m,M]}\left(z-1\right)^2=\begin{cases}\left(M-1\right)^2 & \text{if } M<1\\ 0 & \text{if } m\le 1\le M\\ \left(m-1\right)^2 & \text{if } 1<m\end{cases}$$

and

$$\max_{z\in[m,M]}(z-1)^2=\begin{cases}(m-1)^2 & \text{if } M<1\\ \max\left\{(m-1)^2,(M-1)^2\right\} & \text{if } m\le 1\le M\\ (M-1)^2 & \text{if } 1<m\end{cases}.$$

Then

$$\frac{\min_{z\in[m,M]}(z-1)^2}{\max\{M,1\}}=\begin{cases}(M-1)^2 & \text{if } M<1\\ 0 & \text{if } m\le 1\le M\\ \frac{(m-1)^2}{M} & \text{if } 1<m\end{cases}\equiv c(m,M)$$

and

$$\frac{\max_{z\in[m,M]}(z-1)^2}{\min\{m,1\}}=\begin{cases}\frac{(m-1)^2}{m} & \text{if } M<1\\ \frac{1}{m}\max\left\{(m-1)^2,(M-1)^2\right\} & \text{if } m\le 1\le M\\ (M-1)^2 & \text{if } 1<m\end{cases}\equiv C(m,M).$$

Using the inequality (3.2) we have

$$\begin{aligned}\frac{1}{2}p(1-p)\,c(m,M)&\le\frac{1}{2}p(1-p)\frac{(z-1)^2}{\max\{M,1\}}\\&\le 1-p+pz-z^p\\&\le\frac{1}{2}p(1-p)\frac{(z-1)^2}{\min\{m,1\}}\\&\le\frac{1}{2}p(1-p)\,C(m,M)\end{aligned}\tag{3.3}$$

for any $z>0$ and for any $p\in[0,1]$.

Using functional calculus and taking $z=A_t$, from (3.3) follows

$$\begin{aligned}\frac{1}{2}p(1-p)\,c(m,M)\,I&\le\frac{1}{2}p(1-p)\frac{(A_t-I)^2}{\max\{M,1\}}\\&\le(1-p)I+pA_t-A_t^p\\&\le\frac{1}{2}p(1-p)\frac{(A_t-I)^2}{\min\{m,1\}}\\&\le\frac{1}{2}p(1-p)\,C(m,M)\,I.\end{aligned}\tag{3.4}$$

Applying the positive linear mapping Φ_t and integrating, we obtain

$$\begin{aligned}&\frac{1}{2}p(1-p)\,c(m,M)\int_T\Phi_t(I)d\mu(t)\\&\le\frac{1}{2}\frac{p(1-p)}{\max\{M,1\}}\left(\int_T\Phi_t\left(A_t^2\right)d\mu(t)-2\int_T\Phi_t(A_t)\,d\mu(t)+\int_T\Phi_t(I)\,d\mu(t)\right)\\&\le(1-p)\int_T\Phi_t(I)d\mu(t)+p\int_T\Phi_t(A_t)\,d\mu(t)-\int_T\Phi_t\left(A_t^p\right)d\mu(t)\\&\le\frac{1}{2}\frac{p(1-p)}{\min\{m,1\}}\left(\int_T\Phi_t\left(A_t^2\right)d\mu(t)-2\int_T\Phi_t(A_t)\,d\mu(t)+\int_T\Phi_t(I)\,d\mu(t)\right)\\&\le\frac{1}{2}p(1-p)\,C(m,M)\int_T\Phi_t(I)d\mu(t).\end{aligned}$$

Hence Theorem 3.1 is proved. □

A discrete version of Theorem 3.1 is the following result obtained by taking $T = \{1, \ldots, n\}$.

Corollary 3.2. *Let $A_1, \ldots, A_n$ be positive operators acting on a Hilbert space $\mathcal{H}$ and $\Phi_i\,(i = 1, \ldots, n)$ be positive linear mappings on $\mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}\frac{1}{2}p\,(1-p)\,c\,(m,M)\sum_{i=1}^{n}\Phi_i\,(I) &\le \frac{1}{2}\frac{p\,(1-p)}{\max\{M,1\}}\left(\sum_{i=1}^{n}\Phi_i\left(A_i^2\right) - 2\sum_{i=1}^{n}\Phi_i\,(A_i) + \sum_{i=1}^{n}\Phi_i\,(I)\right)\\ &\le (1-p)\sum_{i=1}^{n}\Phi_i\,(I) + p\sum_{i=1}^{n}\Phi_i\,(A_i) - \sum_{i=1}^{n}\Phi_i\left(A_i^p\right)\\ &\le \frac{1}{2}\frac{p\,(1-p)}{\min\{m,1\}}\left(\sum_{i=1}^{n}\Phi_i\left(A_i^2\right) - 2\sum_{i=1}^{n}\Phi_i\,(A_i) + \sum_{i=1}^{n}\Phi_i\,(I)\right)\\ &\le \frac{1}{2}p\,(1-p)\,C\,(m,M)\sum_{i=1}^{n}\Phi_i\,(I).\end{aligned}$$

By setting $\Phi_t\,(A) = X_i^* A X_i$ in Corollary 3.2, we find the following result.

Corollary 3.3. *Let $A_1, \ldots, A_n$ be positive operators acting on a Hilbert space $\mathcal{H}$ with $A_i \ge 0$ and $X_1, \ldots, X_n \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}\frac{1}{2}p\,(1-p)\,c\,(m,M)\sum_{i=1}^{n}X_i^*X_i &\le \frac{1}{2}\frac{p\,(1-p)}{\max\{M,1\}}\left(\sum_{i=1}^{n}X_i^*A_i^2X_i - 2\sum_{i=1}^{n}X_i^*A_iX_i + \sum_{i=1}^{n}X_i^*X_i\right)\\ &\le (1-p)\sum_{i=1}^{n}X_i^*X_i + p\sum_{i=1}^{n}X_i^*A_iX_i - \sum_{i=1}^{n}X_i^*A_i^pX_i\\ &\le \frac{1}{2}\frac{p\,(1-p)}{\min\{m,1\}}\left(\sum_{i=1}^{n}X_i^*A_i^2X_i - 2\sum_{i=1}^{n}X_i^*A_iX_i + \sum_{i=1}^{n}X_i^*X_i\right)\\ &\le \frac{1}{2}p\,(1-p)\,C\,(m,M)\sum_{i=1}^{n}X_i^*X_i.\end{aligned}$$

Corollary 3.4. *If in addition we take $\sum_{i=1}^{n} X_i^*X_i = I$, in Corollary 3.3 we deduce*

$$\begin{aligned}\frac{1}{2}p\,(1-p)\,c\,(m,M)\,I &\le \frac{1}{2}\frac{p\,(1-p)}{\max\{M,1\}}\left(\sum_{i=1}^{n}X_i^*(A_i-I)^2X_i\right)\\ &\le \sum_{i=1}^{n}X_i^*\left((1-p) + pA_i - A_i^p\right)X_i\\ &\le \frac{1}{2}\frac{p\,(1-p)}{\min\{m,1\}}\left(X_i^*(A_i-I)^2X_i\right)\\ &\le \frac{1}{2}p\,(1-p)\,C\,(m,M)\,I.\end{aligned}$$

Remark. *If we replace A_t by $\Phi_t(A_t)$ in* (3.4) *and then integrating, we obtain*

$$\begin{aligned}
&\frac{1}{2}p\left(1-p\right)c\left(m,M\right)\int_T \Phi_t\left(I\right)d\mu\left(t\right) \\
&\le \frac{1}{2}\frac{p\left(1-p\right)}{\max\left\{M,1\right\}}\left(\int_T \Phi_t^2\left(A_t\right)d\mu\left(t\right)-2\int_T \Phi_t\left(A_t\right)d\mu\left(t\right)+\int_T \Phi_t\left(I\right)d\mu\left(t\right)\right) \\
&\le \left(1-p\right)\int_T \Phi_t\left(I\right)d\mu\left(t\right)+p\int_T \Phi_t\left(A_t\right)d\mu\left(t\right)-\int_T \Phi_t^p\left(A_t\right)d\mu\left(t\right) \\
&\le \frac{1}{2}\frac{p\left(1-p\right)}{\min\left\{m,1\right\}}\left(\int_T \Phi_t^2\left(A_t\right)d\mu\left(t\right)-2\int_T \Phi_t\left(A_t\right)d\mu\left(t\right)+\int_T \Phi_t\left(I\right)d\mu\left(t\right)\right) \\
&\le \frac{1}{2}p\left(1-p\right)C\left(m,M\right)\int_T \Phi_t\left(I\right)d\mu\left(t\right).
\end{aligned}$$

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References

[1] R. Bhatia, *Positive Definite Matrices*, Princeton University Press, 2007.
[2] D. I. Cartwright, M. J. Field, *A refinement of the arithmetic mean-geometric mean inequality*, Proc. Amer. Math. Soc. **71** (1978), 36–38.
[3] S.S. Dragomir, *Additive refinements and reverses of Young's operator inequality via a result of Cartwright and Field*, Preprint RGMIA Res. Rep. Coll. **19** (2016), Art. 6.
[4] ———, *Some reverses of the Jensen inequality for functions of self-adjoint operators in Hilbert spaces*, J. Inequal. Appl., Volume 2010, Article ID 496821, 15 pages.
[5] M. Fuji, S. Izumino, R. Nakamoto, Y. Seo, *Operator inequalities related to Cauchy-Schwarz and Hölder-McCarthy inequalities*, Nihonkai Math. J. **8** (1997), 117-122.
[6] M. Fujii, S. Izumino, R. Nakamoto, *Classes of operators determined by the Heinz-Kato Furuta inequality and the Hölder-McCarthy inequality*, Nihonkai Math. J. **5**(1) (1994), 61–67.
[7] T. Furuta, *Extensions of Hölder-McCarthy and Kantorovich inequalities and their applications*, Proc. Japan Acad. Ser. A Math. Sci. **73**(3) (1997), 38–41.
[8] ———, *Operator inequalities associated with Hölder-McCarthy and Kantorovich inequalities*, J. Inequal. Appl. **2**(2) (1998), 137-148.
[9] C.A. McCarthy, c_p, Israel J. Math. **5** (1967), 249–271.
[10] H.R. Moradi, M.E. Omidvar, S.S. Dragomir, *An operator extension of Čebyšev inequality*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat. **25** (2) (2017), 135–147
[11] H.R. Moradi, M.E. Omidvar, S.S. Dragomir, M.K. Anwary, *More operator inequalities for positive linear maps*, Bull. Math. Anal. Appl. **9**(7) (2017), 65–73.
[12] G. J. Murphy, *C^*-Algebras and Operator Theory*, Academic Press, San Diego, 1990.

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