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Research Article

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Trace inequalities of Shisha-Mond type for operators in Hilbert spaces

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Abstract: Some trace inequalities of Shisha-Mond type for operators in Hilbert spaces are provided. Applications in connection to Grüss inequality and for convex functions of selfadjoint operators are also given.

Keywords: Trace class operators, Hilbert-Schmidt operators, Schwarz inequality, Grüss inequality

MSC: 47A63, 47A99

1 Introduction

In 1967, Shisha and Mond [55, p. 301] proved the following reverse of Cauchy-Bunyakovsky-Schwarz inequality:

Theorem 1.1. *Let $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ be two positive n-tuples with*

$$0 < m \le \frac{a_k}{b_k} \le M < \infty \text{ for each } k \in \{1, \ldots, n\}, \tag{1}$$

then

$$0 \le \left(\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 \right)^{1/2} - \sum_{k=1}^{n} a_k b_k \le \frac{(M-m)^2}{4(M+m)} \sum_{k=1}^{n} b_k^2. \tag{2}$$

The equality holds in (2) if and only if there exists a subsequence $(k_1, ..., k_p)$ of $\{1, \ldots, n\}$ such that

$$\sum_{m=1}^{p} b_{k_m}^2 = \frac{M+3m}{4(M+m)} \sum_{k=1}^{n} b_k^2,$$

$\frac{a_{k_m}}{b_{k_m}} = M$ for every $m = 1, ..., p$ and $\frac{a_k}{b_k} = m$ for every k distinct from all k_m.

Recall some other classical reverses of Cauchy-Bunyakovsky-Schwarz inequality when bounds for each n-tuple are available.

Let $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ be two positive n*-tuples* with

$$0 < m_1 \le a_i \le M_1 < \infty \text{ and } 0 < m_2 \le b_i \le M_2 < \infty; \tag{3}$$

for each $i \in \{1, \ldots, n\}$, and some constants m_1, m_2, M_1, M_2.

The following reverses of the Cauchy-Bunyakovsky-Schwarz inequality for positive sequences of real numbers are well known:

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a) *Pólya-Szegö's inequality* [51]:

$$\frac{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2}{\left(\sum_{k=1}^{n} a_k b_k\right)^2} \le \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2.$$

b) *Shisha-Mond's inequality* [55]:

$$\frac{\sum_{k=1}^{n} a_k^2}{\sum_{k=1}^{n} a_k b_k} - \frac{\sum_{k=1}^{n} a_k b_k}{\sum_{k=1}^{n} b_k^2} \le \left[\left(\frac{M_1}{m_2}\right)^{\frac{1}{2}} - \left(\frac{m_1}{M_2}\right)^{\frac{1}{2}}\right]^2.$$

c) *Ozeki's inequality* [48]:

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \frac{n^2}{3}\left(M_1 M_2 - m_1 m_2\right)^2.$$

d) *Diaz-Metcalf's inequality* [17]:

$$\sum_{k=1}^{n} b_k^2 + \frac{m_2 M_2}{m_1 M_1}\sum_{k=1}^{n} a_k^2 \le \left(\frac{M_2}{m_1} + \frac{m_2}{M_1}\right)\sum_{k=1}^{n} a_k b_k.$$

If $\overline{\mathbf{w}} = (w_1, \ldots, w_n)$ is a positive sequence, then the following weighted inequalities also hold:

e) *Cassels' inequality* [58]. If the positive real sequences $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ satisfy the condition (1), then

$$\frac{\left(\sum_{k=1}^{n} w_k a_k^2\right)\left(\sum_{k=1}^{n} w_k b_k^2\right)}{\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2} \le \frac{(M+m)^2}{4mM}.$$

f) *Greub-Reinboldt's inequality* [38]. We have

$$\left(\sum_{k=1}^{n} w_k a_k^2\right)\left(\sum_{k=1}^{n} w_k b_k^2\right) \le \frac{(M_1 M_2 + m_1 m_2)^2}{4 m_1 m_2 M_1 M_2}\left(\sum_{k=1}^{n} w_k a_k b_k\right)^2,$$

provided $\overline{\mathbf{a}} = (a_1, \ldots, a_n)$ and $\overline{\mathbf{b}} = (b_1, \ldots, b_n)$ satisfy the condition (3).

g) *Generalized Diaz-Metcalf's inequality* [17], see also [46, p. 123]. If $u, v \in [0, 1]$ and $v \le u$, $u + v = 1$ and (1) holds, then one has the inequality

$$u\sum_{k=1}^{n} w_k b_k^2 + vMm\sum_{k=1}^{n} w_k a_k^2 \le (vm + uM)\sum_{k=1}^{n} w_k a_k b_k.$$

h) *Klamkin-McLenaghan's inequality* [40]. If $\overline{\mathbf{a}}$, $\overline{\mathbf{b}}$ satisfy (1), then

$$\left(\sum_{i=1}^{n} w_i a_i^2\right)\left(\sum_{i=1}^{n} w_i b_i^2\right) - \left(\sum_{i=1}^{n} w_i a_i b_i\right)^2 \le \left(M^{\frac{1}{2}} - m^{\frac{1}{2}}\right)^2 \sum_{i=1}^{n} w_i a_i b_i \sum_{i=1}^{n} w_i a_i^2. \tag{4}$$

For other recent results providing discrete reverse inequalities, see the monograph online [19].

The following reverse of Schwarz's inequality in inner product spaces holds [20].

Theorem 1.2 (Dragomir, 2003, [20]). *Let $A, a \in \mathbb{C}$ and $x, y \in H$, where H is a complex inner product space with the inner product $\langle \cdot, \cdot \rangle$. If*

$$\operatorname{Re}\langle Ay - x, x - ay\rangle \ge 0, \tag{5}$$

or equivalently,

$$\left\| x - \frac{a + A}{2}\cdot y \right\| \le \frac{1}{2}|A - a|\,\|y\|, \tag{6}$$

holds, then we have the inequality

$$0 \le \|x\|^2 \|y\|^2 - |\langle x, y\rangle|^2 \le \frac{1}{4} |A - a|^2 \|y\|^4 . \tag{7}$$

The constant $\frac{1}{4}$ *is sharp in (7).*

In 1935, G. Grüss [39] proved the following integral inequality which gives an approximation of the integral mean of the product in terms of the product of the integrals means as follows:

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{b-a} \int_a^b f(x) dx \cdot \frac{1}{b-a} \int_a^b g(x) dx \right| \le \frac{1}{4} (\Phi - \phi)(\Gamma - \gamma), \tag{8}$$

where $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$ and satisfy the condition

$$\phi \le f(x) \le \Phi, \gamma \le g(x) \le \Gamma \tag{9}$$

for each $x \in [a, b]$, where ϕ, Φ, γ, Γ are given real constants.

Moreover, the constant $\frac{1}{4}$ is sharp in the sense that it cannot be replaced by a smaller one.

In [22], in order to generalize the Grüss integral inequality in abstract structures the author has proved the following inequality in inner product spaces.

Theorem 1.3 (Dragomir, 1999, [22]). *Let* $(H, \langle \cdot, \cdot \rangle)$ *be an inner product space over* $\mathbb{K}$ $(\mathbb{K} = \mathbb{R}, \mathbb{C})$ *and* $e \in H$, $\|e\| = 1$. *If* $\varphi, \gamma, \Phi, \Gamma$ *are real or complex numbers and* x, y *are vectors in* H *such that the conditions*

$$\mathrm{Re} \langle \Phi e - x, x - \varphi e\rangle \ge 0 \text{ and } \mathrm{Re} \langle \Gamma e - y, y - \gamma e\rangle \ge 0 \tag{10}$$

hold, then we have the inequality

$$|\langle x, y\rangle - \langle x, e\rangle \langle e, y\rangle| \le \frac{1}{4} |\Phi - \varphi| |\Gamma - \gamma| . \tag{11}$$

The constant $\frac{1}{4}$ *is best possible in the sense that it can not be replaced by a smaller constant.*

For other results of this type, see the recent monograph [25] and the references therein.

For other Grüss type results for integral and sums see the papers [1]-[3], [8]-[10], [11]-[13], [21]-[28], [35], [49], [62] and the references therein.

In order to state some reverses of Schwarz and Grüss type inequalities for trace operators on complex Hilbert spaces we need some preparations as follows.

2 Some facts on trace of operators

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $\{e_i\}_{i \in I}$ an *orthonormal basis* of H. We say that $A \in \mathcal{B}(H)$ is a *Hilbert-Schmidt operator* if

$$\sum_{i \in I} \|Ae_i\|^2 < \infty. \tag{12}$$

It is well know that, if $\{e_i\}_{i \in I}$ and $\{f_j\}_{j \in J}$ are orthonormal bases for H and $A \in \mathcal{B}(H)$ then

$$\sum_{i \in I} \|Ae_i\|^2 = \sum_{j \in I} \|Af_j\|^2 = \sum_{j \in I} \|A^* f_j\|^2 \tag{13}$$

showing that the definition (12) is independent of the orthonormal basis and A is a Hilbert-Schmidt operator iff A^* is a Hilbert-Schmidt operator.

Let $\mathcal{B}_2(H)$ the set of Hilbert-Schmidt operators in $\mathcal{B}(H)$. For $A \in \mathcal{B}_2(H)$ we define

$$\|A\|_2 := \left(\sum_{i \in I} \|Ae_i\|^2 \right)^{1/2} \tag{14}$$

for $\{e_i\}_{i \in I}$ an orthonormal basis of H. This definition does not depend on the choice of the orthonormal basis.

Using the triangle inequality in $l^2(I)$, one checks that $\mathcal{B}_2(H)$ is a *vector space* and that $\|\cdot\|_2$ is a norm on $\mathcal{B}_2(H)$, which is usually called in the literature as the *Hilbert-Schmidt norm*.

Denote *the modulus* of an operator $A \in \mathcal{B}(H)$ by $|A| := (A^*A)^{1/2}$.

Because $\||A|x\| = \|Ax\|$ for all $x \in H$, A is Hilbert-Schmidt iff $|A|$ is Hilbert-Schmidt and $\|A\|_2 = \||A|\|_2$. From (13) we have that if $A \in \mathcal{B}_2(H)$, then $A^* \in \mathcal{B}_2(H)$ and $\|A\|_2 = \|A^*\|_2$.

The following theorem collects some of the most important properties of Hilbert-Schmidt operators:

Theorem 2.1. *We have*

(i) $(\mathcal{B}_2(H), \|\cdot\|_2)$ *is a Hilbert space with inner product*

$$\langle A, B\rangle_2 := \sum_{i \in I} \langle Ae_i, Be_i\rangle = \sum_{i \in I} \langle B^*Ae_i, e_i\rangle \tag{15}$$

and the definition does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$;

(ii) We have the inequalities

$$\|A\| \le \|A\|_2 \tag{16}$$

for any $A \in \mathcal{B}_2(H)$ *and*

$$\|AT\|_2, \|TA\|_2 \le \|T\| \|A\|_2 \tag{17}$$

for any $A \in \mathcal{B}_2(H)$ *and* $T \in \mathcal{B}(H)$;

(iii) $\mathcal{B}_2(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_2(H)\mathcal{B}(H) \subseteq \mathcal{B}_2(H);$$

(iv) $\mathcal{B}_{fin}(H)$, *the space of operators of finite rank, is a dense subspace of* $\mathcal{B}_2(H)$;

(v) $\mathcal{B}_2(H) \subseteq \mathcal{K}(H)$, *where* $\mathcal{K}(H)$ *denotes the algebra of compact operators on* H.

If $\{e_i\}_{i \in I}$ an orthonormal basis of H, we say that $A \in \mathcal{B}(H)$ is *trace class* if

$$\|A\|_1 := \sum_{i \in I} \langle |A| e_i, e_i\rangle < \infty. \tag{18}$$

The definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote by $\mathcal{B}_1(H)$ the set of trace class operators in $\mathcal{B}(H)$.

The following proposition holds:

Proposition 2.2. *If* $A \in \mathcal{B}(H)$, *then the following are equivalent:*

(i) $A \in \mathcal{B}_1(H)$;

(ii) $|A|^{1/2} \in \mathcal{B}_2(H)$;

(ii) A *(or* $|A|$*) is the product of two elements of* $\mathcal{B}_2(H)$.

The following properties are also well known:

Theorem 2.3. *With the above notations:*

(i) We have

$$\|A\|_1 = \|A^*\|_1 \text{ and } \|A\|_2 \le \|A\|_1 \tag{19}$$

for any $A \in \mathcal{B}_1(H)$;

(ii) $\mathcal{B}_1(H)$ *is an operator ideal in* $\mathcal{B}(H)$, *i.e.*

$$\mathcal{B}(H)\mathcal{B}_1(H)\mathcal{B}(H) \subseteq \mathcal{B}_1(H);$$

(iii) We have

$$\mathcal{B}_2(H)\mathcal{B}_2(H) = \mathcal{B}_1(H);$$

(iv) We have

$$\|A\|_1 = \sup\{\langle A, B\rangle_2 \mid B \in \mathcal{B}_2(H),\ \|B\| \le 1\};$$

(v) $(\mathcal{B}_1(H), \|\cdot\|_1)$ *is a Banach space.*

(iv) We have the following isometric isomorphisms

$$\mathcal{B}_1(H) \cong K(H)^* \text{ and } \mathcal{B}_1(H)^* \cong \mathcal{B}(H),$$

where $K(H)^*$ *is the dual space of* $K(H)$ *and* $\mathcal{B}_1(H)^*$ *is the dual space of* $\mathcal{B}_1(H)$.

We define the *trace* of a trace class operator $A \in \mathcal{B}_1(H)$ to be

$$\operatorname{tr}(A) := \sum_{i\in I} \langle Ae_i, e_i\rangle, \tag{20}$$

where $\{e_i\}_{i\in I}$ an orthonormal basis of H. Note that this coincides with the usual definition of the trace if H is finite-dimensional. We observe that the series (20) converges absolutely and it is independent from the choice of basis.

The following result collects some properties of the trace:

Theorem 2.4. *We have*

(i) If $A \in \mathcal{B}_1(H)$ *then* $A^* \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(A^*) = \overline{\operatorname{tr}(A)}; \tag{21}$$

(ii) If $A \in \mathcal{B}_1(H)$ *and* $T \in \mathcal{B}(H)$, *then* $AT, TA \in \mathcal{B}_1(H)$ *and*

$$\operatorname{tr}(AT) = \operatorname{tr}(TA) \text{ and } |\operatorname{tr}(AT)| \le \|A\|_1 \|T\|; \tag{22}$$

(iii) $\operatorname{tr}(\cdot)$ *is a bounded linear functional on* $\mathcal{B}_1(H)$ *with* $\|\operatorname{tr}\| = 1$;

(iv) If $A, B \in \mathcal{B}_2(H)$ *then* $AB, BA \in \mathcal{B}_1(H)$ *and* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$;

(v) $\mathcal{B}_{fin}(H)$ *is a dense subspace of* $\mathcal{B}_1(H)$.

Utilising the trace notation we obviously have that

$$\langle A, B\rangle_2 = \operatorname{tr}(B^*A) = \operatorname{tr}(AB^*) \text{ and } \|A\|_2^2 = \operatorname{tr}(A^*A) = \operatorname{tr}\left(|A|^2\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

The following Hölder's type inequality has been obtained by Ruskai in [52]

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^{1/\alpha}\right)\right]^{\alpha} \left[\operatorname{tr}\left(|B|^{1/(1-\alpha)}\right)\right]^{1-\alpha} \tag{23}$$

where $\alpha \in (0,1)$ and $A, B \in \mathcal{B}(H)$ with $|A|^{1/\alpha}, |B|^{1/(1-\alpha)} \in \mathcal{B}_1(H)$.

In particular, for $\alpha = \frac{1}{2}$ we get the Schwarz inequality

$$|\operatorname{tr}(AB)| \le \operatorname{tr}(|AB|) \le \left[\operatorname{tr}\left(|A|^2\right)\right]^{1/2} \left[\operatorname{tr}\left(|B|^2\right)\right]^{1/2} \tag{24}$$

with $A, B \in \mathcal{B}_2(H)$.

For the theory of trace functionals and their applications the reader is referred to [56].

For some classical trace inequalities see [14], [16], [47] and [61], which are continuations of the work of Bellman [5]. For related works the reader can refer to [4], [6], [14], [36], [41], [42], [44], [53] and [57].

We denote by

$$\mathcal{B}_1^+(H) := \{P :\ P \in \mathcal{B}_1(H),\ P \text{ and is selfadjoint and } P \ge 0\}.$$

We obtained recently the following result [33]:

Theorem 2.5. *For any $A, C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality*

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&\le \inf_{\lambda\in\mathbb{C}} \|A - \lambda \cdot 1_H\| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\
&\le \inf_{\lambda\in\mathbb{C}} \|A - \lambda \cdot 1_H\| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2},
\end{aligned} \tag{25}$$

where $\|\cdot\|$ is the operator norm.

We also have [33]:

Corollary 2.6. *Let $\alpha, \beta \in \mathbb{C}$ and $A \in B(H)$ such that*

$$\left\|A - \frac{\alpha+\beta}{2}\cdot 1_H\right\| \le \frac{1}{2}|\beta-\alpha|.$$

For any $C \in \mathcal{B}(H)$ and $P \in \mathcal{B}_1^+(H) \setminus \{0\}$ we have the inequality

$$\begin{aligned}
&\left|\frac{\operatorname{tr}(PAC)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right| \\
&\le \frac{1}{2}|\beta-\alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\
&\le \frac{1}{2}|\beta-\alpha| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2}.
\end{aligned} \tag{26}$$

In particular, if $C \in \mathcal{B}(H)$ is such that

$$\left\|C - \frac{\alpha+\beta}{2}\cdot 1_H\right\| \le \frac{1}{2}|\beta-\alpha|,$$

then

$$\begin{aligned}
0 &\le \frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2 \\
&\le \frac{1}{2}|\beta-\alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\
&\le \frac{1}{2}|\beta-\alpha| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2} \le \frac{1}{4}|\beta-\alpha|^2.
\end{aligned} \tag{27}$$

Also

$$\begin{aligned}
&\left|\frac{\operatorname{tr}\left(PC^2\right)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right)^2\right| \\
&\le \frac{1}{2}|\beta-\alpha| \frac{1}{\operatorname{tr}(P)} \operatorname{tr}\left(\left|\left(C - \frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)} 1_H\right) P\right|\right) \\
&\le \frac{1}{2}|\beta-\alpha| \left[\frac{\operatorname{tr}\left(P|C|^2\right)}{\operatorname{tr}(P)} - \left|\frac{\operatorname{tr}(PC)}{\operatorname{tr}(P)}\right|^2\right]^{1/2} \le \frac{1}{4}|\beta-\alpha|^2.
\end{aligned} \tag{28}$$

For other related results see [33].

3 Shisha-Mond type trace inequalities

For two given operators $T, U \in B(H)$ and two given scalars $\alpha, \beta \in \mathbb{C}$ consider the transform

$$\mathcal{C}_{\alpha,\beta}(T,U) = \left(T^* - \bar{\alpha}U^*\right)(\beta U - T).$$

This transform generalizes the transform

$$\mathcal{C}_{\alpha,\beta}(T) := \left(T^* - \bar{\alpha}1_H\right)(\beta 1_H - T) = \mathcal{C}_{\alpha,\beta}(T, 1_H),$$

where 1_H is the identity operator, which has been introduced in [31] in order to provide some generalizations of the well known Kantorovich inequality for operators in Hilbert spaces.

We recall that a bounded linear operator T on the complex Hilbert space $(H, \langle\cdot,\cdot\rangle)$ is called *accretive* if $\operatorname{Re}\langle Ty, y\rangle \geq 0$ for any $y \in H$.

Utilizing the following identity

$$\begin{aligned}\operatorname{Re}\langle \mathcal{C}_{\alpha,\beta}(T,U)x, x\rangle &= \operatorname{Re}\langle \mathcal{C}_{\beta,\alpha}(T,U)x, x\rangle \\ &= \frac{1}{4}|\beta-\alpha|^2\|Ux\|^2 - \left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\|^2 \\ &= \frac{1}{4}|\beta-\alpha|^2\left\langle |U|^2 x, x\right\rangle - \left\langle \left|T - \frac{\alpha+\beta}{2}\cdot U\right|^2 x, x\right\rangle\end{aligned} \tag{29}$$

that holds for any scalars α, β and any vector $x \in H$, we can give a simple characterization result that is useful in the following:

Lemma 3.1. *For $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$ the following statements are equivalent:*

(i) The transform $\mathcal{C}_{\alpha,\beta}(T,U)$ (or, equivalently, $\mathcal{C}_{\beta,\alpha}(T,U)$) is accretive;

(ii) We have the norm inequality

$$\left\|Tx - \frac{\alpha+\beta}{2}\cdot Ux\right\| \leq \frac{1}{2}|\beta-\alpha|\,\|Ux\| \tag{30}$$

for any $x \in H$;

(iii) We have the following inequality in the operator order

$$\left|T - \frac{\alpha+\beta}{2}\cdot U\right|^2 \leq \frac{1}{4}|\beta-\alpha|^2|U|^2.$$

As a consequence of the above lemma we can state:

Corollary 3.2. *Let $\alpha, \beta \in \mathbb{C}$ and $T, U \in B(H)$. If $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive, then*

$$\left\|T - \frac{\alpha+\beta}{2}\cdot U\right\| \leq \frac{1}{2}|\beta-\alpha|\,\|U\|. \tag{31}$$

Remark 3.3. *In order to give examples of linear operators $T, U \in B(H)$ and numbers $\alpha, \beta \in \mathbb{C}$ such that the transform $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive, it suffices to select two bounded linear operator S and V and the complex numbers z, w ($w \neq 0$) with the property that $\|Sx - zVx\| \leq |w|\,\|Vx\|$ for any $x \in H$, and, by choosing $T = S$, $U = V$, $\alpha = \frac{1}{2}(z+w)$ and $\beta = \frac{1}{2}(z-w)$ we observe that T and U satisfy (30), i.e., $\mathcal{C}_{\alpha,\beta}(T,U)$ is accretive.*

The following result is useful in the sequel:

Lemma 3.4. *Let, either $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ and $\gamma, \Gamma \in \mathbb{C}$. Then*

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^* - \overline{\gamma}B^*\right)(\Gamma B - A)\right]\right) \geq 0 \tag{32}$$

if and only if

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)\le\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right). \tag{33}$$

To simplify the writing, we the say that (A,B) satisfies the P-(γ,Γ)-trace property.

Proof. Doing the calculation, we have the equality

$$\frac{1}{4}|\Gamma-\gamma|^2P|B|^2-P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2=P\left[-|A|^2+\frac{\overline{\gamma+\Gamma}}{2}B^*A+\frac{\gamma+\Gamma}{2}A^*B-\operatorname{Re}(\Gamma\overline{\gamma})|B|^2\right] \tag{34}$$

for any bounded operators A, B, P and the complex numbers $\gamma,\Gamma\in\mathbb{C}$.

Taking the trace in (34) we get after some simple manipulation

$$\begin{aligned}&\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)\\&=-\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}(\Gamma\overline{\gamma})\operatorname{tr}\left(P|B|^2\right)\\&+\operatorname{Re}\left[\overline{\gamma}\operatorname{tr}(PB^*A)\right]+\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}\right].\end{aligned} \tag{35}$$

Since

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}B^*\right)(\Gamma B-A)\right]\right)=\operatorname{Re}\left[\Gamma\overline{\operatorname{tr}(PB^*A)}+\overline{\gamma}\operatorname{tr}(PB^*A)\right]-\operatorname{tr}\left(P|B|^2\right)\operatorname{Re}(\overline{\gamma}\Gamma)-\operatorname{tr}\left(P|A|^2\right),$$

then we get

$$\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right)-\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)=\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}B^*\right)(\Gamma B-A)\right]\right), \tag{36}$$

which proves the desired equivalence. □

Corollary 3.5. *Let, either $P\in\mathcal{B}_+(H)$, $A,B\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, $A,B\in\mathcal{B}(H)$ and $\gamma,\Gamma\in\mathbb{C}$. If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ is accretive, then (A,B) satisfies the P-(γ,Γ)-trace property.*

We have the following result:

Theorem 3.6. *Let, either $P\in\mathcal{B}_+(H)$, $A,B\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, $A,B\in\mathcal{B}(H)$ and $\gamma,\Gamma\in\mathbb{C}$ with $\Gamma+\gamma\neq0$.*

(i) If (A,B) satisfies the P-(γ,Γ)-trace property, then we have

$$\begin{aligned}\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}&\le\frac{\operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\operatorname{tr}(PB^*A)+\operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\operatorname{tr}(PB^*A)}{|\Gamma+\gamma|}\\&+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|B|^2\right)\\&\le\left|\operatorname{tr}(PB^*A)\right|+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|B|^2\right).\end{aligned} \tag{37}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A,B)$ is accretive, then the inequality (37) also holds.

Proof. (i) If (A,B) satisfies the P-(γ,Γ)-trace property, then

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}B\right|^2\right)\le\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right)$$

that is equivalent to

$$\operatorname{tr}\left(P|A|^2\right)-\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}(PB^*A)\right]+\frac{1}{4}|\Gamma+\gamma|^2\operatorname{tr}\left(P|B|^2\right)\le\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right),$$

which implies that

$$\operatorname{tr}\left(P|A|^2\right)+\frac{1}{4}|\Gamma+\gamma|^2\operatorname{tr}\left(P|B|^2\right)\le\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]+\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right). \tag{38}$$

Making use of the elementary inequality

$$2\sqrt{pq}\le p+q,\ \ p,q\ge 0,$$

we also have

$$|\Gamma+\gamma|\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\le\operatorname{tr}\left(P|A|^2\right)+\frac{1}{4}|\Gamma+\gamma|^2\operatorname{tr}\left(P|B|^2\right). \tag{39}$$

Utilising (38) and (39) we get

$$|\Gamma+\gamma|\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)\right]^{1/2}\le\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]+\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}\left(P|B|^2\right). \tag{40}$$

Dividing by $|\Gamma+\gamma|>0$ and observing that

$$\operatorname{Re}\left[\left(\overline{\gamma}+\overline{\Gamma}\right)\operatorname{tr}\left(PB^*A\right)\right]=\operatorname{Re}(\gamma+\Gamma)\operatorname{Re}\operatorname{tr}\left(PB^*A\right)+\operatorname{Im}(\gamma+\Gamma)\operatorname{Im}\operatorname{tr}\left(PB^*A\right)$$

we get the first inequality in (37).

The second inequality in (37) is obvious by Schwarz inequality

$$(ab+cd)^2\le\left(a^2+c^2\right)\left(b^2+d^2\right),\ a,b,c,d\in\mathbb{R}.$$

The (ii) is obvious from (i). □

Remark 3.7. *We observe that the inequality between the first and last term in (37) is equivalent to*

$$0\le\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}-\left|\operatorname{tr}\left(PB^*A\right)\right|\le\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|B|^2\right). \tag{41}$$

Corollary 3.8. *Let, either $P\in\mathcal{B}_+(H)$, $A\in\mathcal{B}_2(H)$ or $P\in\mathcal{B}_1^+(H)$, $A\in\mathcal{B}(H)$ and $\gamma,\Gamma\in\mathbb{C}$ with $\gamma+\Gamma\neq 0$.*

(i) If A satisfies the P-(γ,Γ)-trace property, namely

$$\operatorname{Re}\left(\operatorname{tr}\left[P\left(A^*-\overline{\gamma}1_H\right)\left(\Gamma 1_H-A\right)\right]\right)\ge 0 \tag{42}$$

or, equivalently

$$\operatorname{tr}\left(P\left|A-\frac{\gamma+\Gamma}{2}1_H\right|^2\right)\le\frac{1}{4}|\Gamma-\gamma|^2\operatorname{tr}(P), \tag{43}$$

then we have

$$\begin{aligned}\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)}}&\le\frac{\operatorname{Re}(\gamma+\Gamma)\frac{\operatorname{Re}\operatorname{tr}(PA)}{\operatorname{tr}(P)}+\operatorname{Im}(\gamma+\Gamma)\frac{\operatorname{Im}\operatorname{tr}(PA)}{\operatorname{tr}(P)}}{|\Gamma+\gamma|}+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\\&\le\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|+\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}.\end{aligned} \tag{44}$$

(ii) If the transform $\mathcal{C}_{\gamma,\Gamma}(A)$ is accretive, then the inequality (37) also holds.

(iii) We have

$$0\le\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)}{\operatorname{tr}(P)}}-\left|\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right|\le\frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}. \tag{45}$$

Remark 3.9. *The case of selfadjoint operators is as follows.*

Let A, B be selfadjoint operators and either $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ and $m, M \in \mathbb{R}$ with $m + M \neq 0$.

(i) If (A, B) satisfies the P-(m, M)-trace property, then we have

$$\sqrt{\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right)} \leq \operatorname{Re}\operatorname{tr}(PBA) + \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right) \leq |\operatorname{tr}(PBA)| + \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right) \tag{46}$$

and

$$0 \leq \sqrt{\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right)} - \operatorname{Re}\operatorname{tr}(PBA) \leq \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right).$$

(ii) If the transform $\mathcal{C}_{m,M}(A, B)$ is accretive, then the inequality (46) also holds.

(iii) If $(A - mB)(MB - A) \geq 0$, then (46) is valid.

We observe that the inequality (46) in the case when $M > m > 0$ is the operator trace inequality version of Shisha-Mond inequality (1) from Introduction.

Corollary 3.10. *Let A, B be selfadjoint operators and either $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ and $m, M \in \mathbb{R}$ with $m + M \neq 0$.*

(i) If (A, B) satisfies the P-(m, M)-trace property, then we have

$$\left(\sqrt{\operatorname{tr}\left(PA^2\right)} + \sqrt{\operatorname{tr}\left(PB^2\right)}\right)^2 - \operatorname{tr}\left(P(A+B)^2\right) \leq \frac{(M-m)^2}{4|M+m|}\operatorname{tr}\left(PB^2\right) \tag{47}$$

and

$$\sqrt{\operatorname{tr}\left(PA^2\right)} + \sqrt{\operatorname{tr}\left(PB^2\right)} - \sqrt{\operatorname{tr}\left(P(A+B)^2\right)} \leq \frac{\sqrt{2}}{2}\frac{M-m}{\sqrt{|M+m|}}\sqrt{\operatorname{tr}\left(PB^2\right)}. \tag{48}$$

Proof. Observe that

$$\left(\sqrt{\operatorname{tr}\left(PA^2\right)} + \sqrt{\operatorname{tr}\left(PB^2\right)}\right)^2 - \operatorname{tr}\left(P(A+B)^2\right) = 2\left(\sqrt{\operatorname{tr}\left(PA^2\right)\operatorname{tr}\left(PB^2\right)} - \operatorname{Re}\operatorname{tr}(PBA)\right).$$

Utilising (46) we deduce (47).

The inequality (48) follows from (47). □

4 Trace inequalities of Grüss type

Let P be a selfadjoint operator with $P \geq 0$. The functional $\langle\cdot,\cdot\rangle_{2,P}$ defined by

$$\langle A, B\rangle_{2,P} := \operatorname{tr}\left(PB^*A\right) = \operatorname{tr}\left(APB^*\right) = \operatorname{tr}\left(B^*AP\right)$$

is a *nonnegative Hermitian form* on $\mathcal{B}_2(H)$, i.e. $\langle\cdot,\cdot\rangle_{2,P}$ satisfies the properties:

(h) $\langle A, A\rangle_{2,P} \geq 0$ for any $A \in \mathcal{B}_2(H)$;

(hh) $\langle\cdot,\cdot\rangle_{2,P}$ is linear in the first variable;

(hhh) $\langle B, A\rangle_{2,P} = \overline{\langle A, B\rangle_{2,P}}$ for any $A, B \in \mathcal{B}_2(H)$.

Using the properties of the trace we also have the following representations

$$\|A\|_{2,P}^2 := \operatorname{tr}\left(P|A|^2\right) = \operatorname{tr}\left(APA^*\right) = \operatorname{tr}\left(|A|^2 P\right)$$

and

$$\langle A, B\rangle_{2,P} = \operatorname{tr}\left(APB^*\right) = \operatorname{tr}\left(B^*AP\right)$$

for any $A, B \in \mathcal{B}_2(H)$.

The same definitions can be considered if $P \in \mathcal{B}_1^+(H)$ and $A, B \in \mathcal{B}(H)$.

We have the following Grüss type inequality:

Theorem 4.1. *Let, either $P \in \mathcal{B}_+(H)$, $A, B, C \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, $A, B, C \in \mathcal{B}(H)$ with $P|A|^2$, $P|B|^2$, $P|C|^2 \neq 0$ and $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ with $\gamma + \Gamma \neq 0$, $\delta + \Delta \neq 0$. If (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ)-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}\left(P|C|^2\right)} - \frac{\operatorname{tr}(PC^*A)}{\operatorname{tr}\left(P|C|^2\right)}\frac{\operatorname{tr}(PB^*C)}{\operatorname{tr}\left(P|C|^2\right)}\right|^2 \tag{49}$$
$$\leq \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}{\left[\operatorname{tr}\left(P|C|^2\right)\right]^2}}.$$

Proof. We prove in the case that $P \in \mathcal{B}_+(H)$ and $A, B, C \in \mathcal{B}_2(H)$.

Making use of the Schwarz inequality for the nonnegative hermitian form $\langle\cdot,\cdot\rangle_{2,P}$ we have

$$\left|\langle A, B\rangle_{2,P}\right|^2 \leq \langle A, A\rangle_{2,P}\langle B, B\rangle_{2,P}$$

for any $A, B \in \mathcal{B}_2(H)$.

Let $C \in \mathcal{B}_2(H)$, $C \neq 0$. Define the mapping $[\cdot,\cdot]_{2,P,C} : \mathcal{B}_2(H) \times \mathcal{B}_2(H) \to \mathbb{C}$ by

$$[A, B]_{2,P,C} := \langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}.$$

Observe that $[\cdot,\cdot]_{2,P,C}$ is a nonnegative Hermitian form on $\mathcal{B}_2(H)$ and by Schwarz inequality we also have

$$\left|\langle A, B\rangle_{2,P}\|C\|_{2,P}^2 - \langle A, C\rangle_{2,P}\langle C, B\rangle_{2,P}\right|^2$$
$$\leq \left[\|A\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle A, C\rangle_{2,P}\right|^2\right]\left[\|B\|_{2,P}^2\|C\|_{2,P}^2 - \left|\langle B, C\rangle_{2,P}\right|^2\right]$$

for any $A, B \in \mathcal{B}_2(H)$, namely

$$\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right) - \operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \tag{50}$$
$$\leq \left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - \left|\operatorname{tr}(PC^*A)\right|^2\right]$$
$$\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right) - \left|\operatorname{tr}(PB^*C)\right|^2\right],$$

where for the last term we used the equality $\left|\langle B, C\rangle_{2,P}\right|^2 = \left|\langle C, B\rangle_{2,P}\right|^2$.

Since (A, C) has the trace P-(λ, Γ)-property and (B, C) has the trace P-(δ, Δ) -property, then by (41) we have

$$0 \leq \sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} - \left|\operatorname{tr}(PC^*A)\right| \leq \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)$$

and

$$0 \leq \sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)} - \left|\operatorname{tr}(PC^*B)\right| \leq \frac{1}{4}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)$$

which imply

$$0 \leq \operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right) - \left|\operatorname{tr}(PC^*A)\right|^2 \tag{51}$$
$$\leq \frac{1}{4}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)\left(\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} + \left|\operatorname{tr}(PC^*A)\right|\right)$$

$$\le \frac{1}{2}\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)}$$

and

$$\begin{aligned}0 &\le \operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-\left|\operatorname{tr}(PB^*C)\right|^2 \\ &\le \frac{1}{4}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\left(\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}+\left|\operatorname{tr}(PC^*B)\right|\right) \\ &\le \frac{1}{2}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}.\end{aligned} \tag{52}$$

If we multiply the inequalities (51) and (52) we get

$$\begin{aligned}&\left[\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)-\left|\operatorname{tr}(PC^*A)\right|^2\right] \\ &\times\left[\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)-\left|\operatorname{tr}(PB^*C)\right|^2\right] \\ &\le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} \\ &\times\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}.\end{aligned} \tag{53}$$

If we use (50) and (53) we get

$$\begin{aligned}&\left|\operatorname{tr}(PB^*A)\operatorname{tr}\left(P|C|^2\right)-\operatorname{tr}(PC^*A)\operatorname{tr}(PB^*C)\right|^2 \\ &\le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|C|^2\right)} \\ &\times\operatorname{tr}\left(P|C|^2\right)\sqrt{\operatorname{tr}\left(P|B|^2\right)\operatorname{tr}\left(P|C|^2\right)}.\end{aligned} \tag{54}$$

Since $P|C|^2 \neq 0$ then by (54) we get the desired result (49). □

Corollary 4.2. *Let, either $P \in \mathcal{B}_+(H)$, $A, B \in \mathcal{B}_2$ or $P \in \mathcal{B}_1^+(H)$, $A, B \in \mathcal{B}(H)$ with $P|A|^2, P|B|^2 \neq 0$ and $\lambda, \Gamma, \delta, \Delta \in \mathbb{C}$ with $\gamma+\Gamma \neq 0$, $\delta+\Delta \neq 0$. If A has the trace P-(λ,Γ)-property and B has the trace P-(δ,Δ)-property, then*

$$\left|\frac{\operatorname{tr}(PB^*A)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB^*)}{\operatorname{tr}(P)}\right|^2 \le \frac{1}{4}\cdot\frac{|\Gamma-\gamma|^2}{|\Gamma+\gamma|}\frac{|\Delta-\delta|^2}{|\Delta+\delta|}\sqrt{\frac{\operatorname{tr}\left(P|A|^2\right)\operatorname{tr}\left(P|B|^2\right)}{[\operatorname{tr}(P)]^2}}. \tag{55}$$

The case of selfadjoint operators is useful for applications.

Remark 4.3. *Assume that A, B, C are selfadjoint operators. If, either $P \in \mathcal{B}_+(H)$, $A, B, C \in \mathcal{B}_2(H)$ or $P \in \mathcal{B}_1^+(H)$, $A, B, C \in \mathcal{B}(H)$ with $PA^2, PB^2, PC^2 \neq 0$ and $m, M, n, N \in \mathbb{R}$ with $m+M, n+N \neq 0$. If (A,C) has the trace P-(m,M)-property and (B,C) has the trace P-(n,N)-property, then*

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(PC^2)}-\frac{\operatorname{tr}(PCA)}{\operatorname{tr}(PC^2)}\frac{\operatorname{tr}(PBC)}{\operatorname{tr}(PC^2)}\right|^2 \le \frac{1}{4}\cdot\frac{(M-m)^2}{|M+m|}\frac{(N-n)^2}{|N+n|}\sqrt{\frac{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)}{[\operatorname{tr}(PC^2)]^2}}. \tag{56}$$

If A has the trace P-(k,K)-property and B has the trace P-(l,L)-property, then

$$\left|\frac{\operatorname{tr}(PBA)}{\operatorname{tr}(P)}-\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\frac{\operatorname{tr}(PB)}{\operatorname{tr}(P)}\right|^2 \le \frac{1}{4}\cdot\frac{(K-k)^2}{|K+k|}\frac{(L-l)^2}{|L+l|}\sqrt{\frac{\operatorname{tr}(PA^2)\operatorname{tr}(PB^2)}{[\operatorname{tr}(P)]^2}}, \tag{57}$$

where $k+K, l+L \neq 0$.

5 Applications for convex functions

In the paper [34] we obtained amongst other the following reverse of the Jensen trace inequality:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, then we have

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \\ &\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} \\ &\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \frac{\operatorname{tr}\left(P\left|A - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \\ \\ \frac{1}{2}(M - m) \frac{\operatorname{tr}\left(P\left|f'(A) - \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)} 1_H\right|\right)}{\operatorname{tr}(P)} \end{cases} \\ &\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \left[\frac{\operatorname{tr}(PA^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}(P[f'(A)]^2)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}\left[f'(M) - f'(m)\right](M - m). \end{aligned} \tag{58}$$

Let $\mathcal{M}_n(\mathbb{C})$ be the space of all square matrices of order n with complex elements and $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If f is a continuously differentiable convex function on $[m, M]$, then by taking $P = I_n$ in (58) we get

$$\begin{aligned} 0 &\leq \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \\ &\leq \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n} \\ &\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \frac{\operatorname{tr}\left(\left|A - \frac{\operatorname{tr}(A)}{n} 1_H\right|\right)}{n} \\ \\ \frac{1}{2}(M - m) \frac{\operatorname{tr}\left(\left|f'(A) - \frac{\operatorname{tr}(f'(A))}{n} 1_H\right|\right)}{n} \end{cases} \\ &\leq \begin{cases} \frac{1}{2}[f'(M) - f'(m)] \left[\frac{\operatorname{tr}(A^2)}{n} - \left(\frac{\operatorname{tr}(A)}{n}\right)^2\right]^{1/2} \\ \\ \frac{1}{2}(M - m) \left[\frac{\operatorname{tr}([f'(A)]^2)}{n} - \left(\frac{\operatorname{tr}(f'(A))}{n}\right)^2\right]^{1/2} \end{cases} \\ &\leq \frac{1}{4}\left[f'(M) - f'(m)\right](M - m). \end{aligned} \tag{59}$$

The following reverse inequality also holds:

Proposition 5.1. *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m + M \neq 0$. If f is a continuously differentiable convex function on $[m, M]$ with $f'(m) + f'(M) \neq 0$ and $P \in \mathcal{B}_1(H) \setminus \{0\}$, $P \geq 0$, then we have*

$$0 \leq \frac{\operatorname{tr}(Pf(A))}{\operatorname{tr}(P)} - f\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{60}$$

$$\leq \frac{\operatorname{tr}(Pf'(A)A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(Pf'(A))}{\operatorname{tr}(P)}$$

$$\leq \frac{1}{2} \cdot \frac{|M-m|\,|f'(M)-f'(m)|}{\sqrt{|m+M|}\sqrt{|f'(m)+f'(M)|}} \sqrt[4]{\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(P\left[f'(A)\right]^2\right)}{\operatorname{tr}(P)}}.$$

The proof follows by the inequality (57) and the details are omitted,

Let $A \in \mathcal{M}_n(\mathbb{C})$ be a Hermitian matrix such that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m + M \neq 0$. If f is a continuously differentiable convex function on $[m, M]$ with $f'(m) + f'(M) \neq 0$ then by taking $P = I_n$ in (60) we get

$$0 \leq \frac{\operatorname{tr}(f(A))}{n} - f\left(\frac{\operatorname{tr}(A)}{n}\right) \tag{61}$$

$$\leq \frac{\operatorname{tr}(f'(A)A)}{n} - \frac{\operatorname{tr}(A)}{n} \cdot \frac{\operatorname{tr}(f'(A))}{n}$$

$$\leq \frac{1}{2} \cdot \frac{|M-m|\,|f'(M)-f'(m)|}{\sqrt{|m+M|}\sqrt{|f'(m)+f'(M)|}} \sqrt[4]{\frac{\operatorname{tr}\left(A^2\right)}{n} \frac{\operatorname{tr}\left(\left[f'(A)\right]^2\right)}{n}}.$$

We consider the power function $f : (0, \infty) \to (0, \infty)$, $f(t) = t^r$ with $t \in \mathbb{R} \setminus \{0\}$. For $r \in (-\infty, 0) \cup [1, \infty)$, f is convex while for $r \in (0, 1)$, f is concave.

Let $r \geq 1$ and A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $0 < m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then

$$0 \leq \frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right)^r \tag{62}$$

$$\leq r\left[\frac{\operatorname{tr}(PA^r)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}\left(PA^{r-1}\right)}{\operatorname{tr}(P)}\right]$$

$$\leq \frac{1}{2} r \frac{(M-m)\left(M^{r-1}-m^{r-1}\right)}{(m+M)^{1/2}\left(m^{r-1}+M^{r-1}\right)^{1/2}} \sqrt[4]{\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}\left(PA^{2(p-1)}\right)}{\operatorname{tr}(P)}}.$$

Consider the convex function $f : \mathbb{R} \to (0, \infty)$, $f(t) = \exp t$ and let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m < M$. If $P \in \mathcal{B}_1^+(H) \setminus \{0\}$, then using (60) we have

$$0 \leq \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)} - \exp\left(\frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)}\right) \tag{63}$$

$$\leq \frac{\operatorname{tr}(PA \exp A)}{\operatorname{tr}(P)} - \frac{\operatorname{tr}(PA)}{\operatorname{tr}(P)} \cdot \frac{\operatorname{tr}(P \exp A)}{\operatorname{tr}(P)}$$

$$\leq \frac{1}{2} \frac{|M-m|\,(\exp(M)-\exp(m))}{\sqrt{|m+M|}\sqrt{\exp m + \exp M}} \sqrt[4]{\frac{\operatorname{tr}\left(PA^2\right)}{\operatorname{tr}(P)} \frac{\operatorname{tr}(P \exp(2A))}{\operatorname{tr}(P)}}.$$

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