On the equivariant cohomology of isotropy actions

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Zusammenfassung

Sei G eine kompakte, zusammenhängende Lie Gruppe und $K \subseteq G$ eine abgeschlossene Untergruppe. Wir zeigen, dass die Isotropiewirkung von K auf G/K äquivariant formal ist und der Raum G/K formal im Sinne rationaler Homotopietheorie, falls es sich bei K um die Identitätskomponente des Schnitts der Fixpunktmengen zweier verschiedener Involutionen auf G handelt, G/K also ein $\mathbb{Z}_2 \times \mathbb{Z}_2$−symmetrischer Raum ist. Ist K die Identitätskomponente der Fixpunktmenge einer einzelnen Involution und $H \subseteq G$ eine abgeschlossene, zusammenhängende Untergruppe, die K enthält, so zeigen wir, dass auch die Wirkung von K auf G/H durch Linksmultiplikation äquivariant formal ist. Letztere Aussage ist äquivalent zum Hauptresultat in [6], wird hier aber mit anderen Mitteln bewiesen, nämlich durch Angabe eines algebraischen Modells für die äquivariante Kohomologie gewisser Wirkungen.

Abstract

Let G be a compact connected Lie group and $K \subseteq G$ a closed subgroup. We show that the isotropy action of K on G/K is equivariantly formal and that the space G/K is formal in the sense of rational homotopy theory whenever K is the identity component of the intersection of the fixed point sets of two distinct involutions on G, so that G/K is a $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric space. If K is the identity component of the fixed point set of a single involution and $H \subseteq G$ is a closed connected subgroup containing K, then we show that the action of K on G/H by left–multiplication is equivariantly formal. The latter statement is equivalent to the main result of [6], but is proved by different means, namely by providing an algebraic model for the equivariant cohomology of certain actions.

Contents

Chapter I.

Introduction

1. Introduction and background

This thesis is concerned with G–spaces, that is, topological spaces together with a continuous (left) action of a fixed (smooth) Lie group G, and a certain invariant associated with such spaces, their *equivariant cohomology.* To motivate its definition, consider the problem of assigning to a G–space X an invariant that gives the same answer on any G–space isomorphic to X but yet discerns as many distinct isomorphism classes of G–spaces as possible. Perhaps among the easiest such invariants that one might come up with (apart from the isomorphism class of X) is the cohomology $\mathrm{H}(X/G)$ of the orbit space X/G; of course, one might consider arbitrary coefficient groups, but here and thereafter we confine ourselves to singular real cohomology or to de Rham cohomology if the space under consideration happens to be a smooth manifold. In any case, it appears to be common understanding that $\mathrm{H}(X/G)$ is a reasonable invariant if the G–action is free, but less well–behaved for actions with non–trivial isotropy. A frequently given example of an action justifying this last statement is the action of the circle S^1 on the unit sphere S^2 by rotation about a fixed axis. This action has exactly two fixed points, namely the poles of the rotation axis, and its orbit space is homeomorphic to the closed unit interval, hence has trivial cohomology.

To overcome this difficulty one replaces X by what is now called the *Borel construction* and usually denoted X_G. Originally introduced in [2], this is the space $X_G := (EG\times X)/G$ obtained from a contractible space EG on which G acts freely (from the right), such as the total space in the universal G–bundle $EG \longrightarrow BG$ over the classifying space BG. The action of G on $EG \times X$ is the diagonal action, induced by the assignment $g.(e,x) = (eg^{-1}, gx)$ for $g \in G$ and $(e,x) \in EG\times X$, and the equivariant cohomology then is defined as $\mathrm{H}_G(X) := \mathrm{H}(X_G)$. Note that the G–action on $EG \times X$ is free. Another indication that $\mathrm{H}_G(X)$ is a useful invariant is that it can actually be computed in many situations: quite generally, if G acts locally freely on a space X, then the map $X_G \longrightarrow X/G$ induced by the quotient map $X \longrightarrow X/G$ yields an isomorphism $\mathrm{H}(X/G) \longrightarrow \mathrm{H}_G(X)$, cf. [12, Section C.2.1]. On the other hand, $\mathrm{H}_G(\cdot)$ satisfies the axioms of a generalized cohomology theory with morphisms replaced by G–equivariant morphisms, so that, for example, an equivariant Mayer–Vietoris sequence is available. In very much the same way as the Mayer–Vietoris sequence can be used to compute the ordinary cohomology of spheres, its equivariant counterpart can be utilized to compute the S^1–equivariant cohomology of the action on S^2 considered above, e. g. by means of the open cover consisting of the two open sets that one obtains by removing one of the poles of the rotation axis at a time. The conclusion now is that $\mathrm{H}_{S^1}(S^2) = \mathrm{H}(BS^1) \oplus \mathrm{H}(BS^1)$ in non–zero degrees, because for any Lie group G the equivariant cohomology of a single point is given by $\mathrm{H}_G(*) = \mathrm{H}(BG)$ and S^1 acts freely on S^2 outside its fixed point set.

The previous eaxmple can be written more concisely as $\mathrm{H}_{S^1}(S^2) = \mathrm{H}(BS^1) \otimes \mathrm{H}(S^2)$ (recall that the classifying space of S^1 is $\mathbb{C}P^\infty$, whose cohomology ring is a polynomial algebra in one variable of degree 2), and if one considers $\mathrm{H}_{S^1}(S^2)$ as a $\mathrm{H}(BS^1)$–module via the morphism of rings $\mathrm{H}_{S^1}(*) \longrightarrow \mathrm{H}_{S^1}(S^2)$ induced by the constant map $S^2 \longrightarrow \{*\}$, then this equality is even valid as $\mathrm{H}(BS^1)$–modules, showing that the S^1 action on S^2 is in fact *equivariantly formal.* This name was coined in [10] for actions of compact connected Lie groups G on topological spaces X, although its defining property, the collapse of the Serre spectral sequence associated with the fibration $X \hookrightarrow X_G \longrightarrow BG$ on the second page, was already investigated in [2], mostly for actions of tori and finite cyclic groups of prime order. It is also worth pointing out that for a general fibration $F \hookrightarrow E \longrightarrow B$ with connected fiber F and path–connected base B of finite type the degeneration of the associated Serre spectral sequence at the E_2–term is equivalent to surjectivity of the inclusion induced map $\mathrm{H}(E) \longrightarrow \mathrm{H}(F)$. In this situation, F is traditionally said to be *(totally) non–cohomologous to zero in E*, see [21, p. 148]. This shows the equivalence of the first two items in the following list of well–known characterizations of equivariant formality.

Proposition 1.1. Let G be a compact connected Lie group with maximal torus T and X a connected G–space. The following statements are equivalent.

(1) The G–action on X is equivariantly formal.

(2) Fiber inclusion of the fibration $X \hookrightarrow X_G \longrightarrow BG$ induces a surjection $\mathrm{H}_G(X) \longrightarrow \mathrm{H}(X)$.

(3) The T–action on X obtained by restriction of the G–action is equivariantly formal.

(4) The $\mathrm{H}(BG)$–module $\mathrm{H}_G(X)$ is free.

(5) We have an equality of total Betti numbers $\dim \mathrm{H}(X) = \dim \mathrm{H}\left(X^T\right)$, where X^T is the fixed point set of the induced T–action.

Actions on spaces with vanishing odd degree cohomology are equivariantly formal, as are symplectic manifolds with a Hamiltonian action [10, Theorem 14.1]. Further examples of equivariantly formal actions are *isotropy actions* on symmetric spaces [6] and, more generally, on homogeneous spaces G/K in which the subgroup K is the connected component of the fixed point set of an arbitrary Lie group automorphism on G, see [8]. Here, the isotropy action associated with a homogeneous space G/K is the action of K on G/K induced by left multiplication, that is, by the assignment $(k, gK) \mapsto kgK$ for all $k \in K$, $gK \in G/K$. Our main contribution with this thesis now is that we extend the list of actions which are known to be equivariantly formal by one more item.

In theorem II.1.2 below we will show that the isotropy action associated with G/K is equivariantly formal if K is the connected component of the common fixed point set of two distinct commuting involutions on G, in which case G/K is said to be a $\mathbb{Z}_2 \times \mathbb{Z}_2$*–symmetric space*, provided that none of the automorphisms is the identity map. The proof borrows some ideas from the proof of the main result of [8], which we therefore summarize in section 2. The key step is to construct a subgroup H of G which shares a maximal torus with K and for which the cohomology of G/H is more accessible than that of G/K, as then the isotropy action associated with G/H is equivariantly formal if and only if so is the isotropy action associated with G/K. Since eventually we want to be able to give a description of a maximal torus of K in terms of a maximal torus of G, we thus study in section II.2 the problem of reconstructing a maximal torus of G from a fixed maximal torus S of K. There is a general solution to this problem. Namely, upon fixing a reference torus T which is maximal in G and contains S, one finds that the complexification of the Lie algebra of the centralizer of S in G, which abstractly is the union of all maximal tori of G containing S, is the direct sum of the complexification $\mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{t}$ and the weight spaces of all $\mathfrak{g}^{\mathbb{C}}$–roots that vanish on $\mathfrak{s}$. While it is known that no such root exists if G/K is a symmetric space, certain $\mathfrak{g}^{\mathbb{C}}$–roots might (and in general will) restrict to zero on $\mathfrak{s}$ if G/K is $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric, even if the automorphisms defining K are both inner. Fortunately, however, the set of all such roots is *strongly orthogonal*, meaning that the sum of two elements of that set is not a root (see [16, p. 396]), and already sets of orthogonal roots in irreducible root systems can be classified up to application of a Weyl group element. This we have done in section II.4.

What makes this classification particularly useful is that in the present situation the maximal torus S of K is the intersection of the kernels of all roots vanishing on $\mathfrak{s}$ and the fixed point set on T of one of the automorphisms defining K. All of this data can be formulated in terms of the root system of $\mathfrak{g}^{\mathbb{C}}$ and the list of possible sets of roots vanishing on $\mathfrak{s}$ is further constrained by the requirement that the automorphisms defining K be involutive. At this point, one could thus go through the list of all possible candidates for S and verify that the subalgebra S acts in an equivariantly formal fashion on G/S. We proceed differently and show that we may sequentially modify the automorphisms defining K so as to almost always assume that one of them is an inner automorphism and that the semisimple part of the fixed point set of this inner automorphism realizes a subdiagram of the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$. Homogeneous spaces arising from such subgroups have tractable cohomology, which we determine in section II.5. Building on these results, in section II.6 we finally traverse the list of simple Lie groups, determine in each case the desired subgroup H, and show that the isotropy action of H on G/H is equivariantly formal.

Our second contribution, which actually is equivalent to the main theorem of [6], is theorem III.5.10. The statement here is that for every compact connected Lie group G and the connected component K of the fixed point set of any involution on G the action of K on G/H by left–multiplication is equivariantly formal whenever H is a closed connected subgroup of G that contains K. Of course, the novelty is not the statement itself, but rather its proof, as it relies on an algebraic model for the equivariant cohomology of the K–action on G/H which is solely built out of the Lie algebras of G, H, and K, and the inclusions of the latter two into the former. We note that such a model has been realized only very recently in [4, Sect. 3.1] using methods from rational homotopy theory, while our model is established by quite elementary means using the Cartan model for equivariant cohomology. The drawback of our method is that it only captures the $\mathrm{A}_{\mathfrak{k}}$–module structure of $\mathrm{H}_K(G/H)$, $\mathrm{A}_{\mathfrak{k}} \subseteq \mathrm{S}(\mathfrak{k}^*)$ the space of $\mathfrak{k}$–invariant polynomials on $\mathfrak{k}^*$, whereas the model given in [4] is isomorphic to $\mathrm{H}_K(G/H)$ via an isomorphism of $\mathrm{A}_{\mathfrak{k}}$–algebras. To explain this deficiency, consider an action of a compact connected Lie group G on a smooth manifold M. The basic observation we exploit to construct our model is that there is a sequence of vector subspaces $\Omega(M)^G$, $\mathrm{i}_{\mathfrak{g}}\,\Omega(M)^G$, $(\mathrm{i}_{\mathfrak{g}})^2\,\Omega(M)^G$, ... whose sum is stable under the differential on $\Omega(M)$; here, $\Omega(M)^G$ is the space of G–invariant forms on M and $\mathrm{i}_{\mathfrak{g}}$ denotes the image of the operator $\mathrm{i} : \mathfrak{g} \longrightarrow \mathrm{End}(\Omega(M))$, $X \mapsto \mathrm{i}_{\overline{X}}$, contracting a form with the vector field induced by $X \in \mathfrak{g}$. This leads to an additive, quasi–isomorphic model of $\Omega(M)$ and hence to a model of $\mathrm{H}_G(M)$ which is isomorphic as an $\mathrm{A}_{\mathfrak{g}}$–module.

Despite the lack of a ring structure our proof of theorem III.5.10, in contrast to the original proof in [6], does not rely on any classification result. Again, it has to be noted that a classification–free proof of the main theorem of [6] and even of [8, Theorem 1.1] was already achieved in [4, Theorem 7.8]. However, the proof presented in [4] uses K–theory and relies on a reduction to the case when G is simple, while our proof works equally well for simple and non– simple Lie groups and only uses the decomposition of $\mathfrak{g}$ into the eigenspaces of the involution defining K.

2. Previous results

Starting with this section we will almost exclusively consider isotropy actions on homogeneous spaces and be concerned with the question when such an action is equivariantly formal. It thus seems appropriate to make the following definition: given a compact connected Lie group G and a closed connected subgroup K, we say that the pair (G, K) is *equivariantly formal* if the action of K on G/K by left–multiplication is equivariantly formal; we also say that (G, K) is *formal* or a *Cartan pair* if the homogeneous space G/K is formal in the sense of rational homotopy theory, which means that there exist commutative differential graded $\mathbb{R}$–algebras $A_1, \dots, A_n$ and a chain of morphisms $\Omega(G/K) \longrightarrow A_1 \longleftarrow A_2 \longrightarrow \dots \longrightarrow A_n \longleftarrow \mathrm{H}(G/K)$, each of which induces an isomorphism on the level of cohomology. While this definition is valid for arbitrary (connected) manifolds, not just G/K, we prefer to use the following equivalent characterization of formality which is available in this particular situation: we recall from [11] that the space $\Omega(\mathfrak{g})^{\mathfrak{g}}$ of $\mathfrak{g}$–invariant forms on $\mathfrak{g}$ is an exterior algebra over an oddly graded subspace $P_{\mathfrak{g}} \subseteq \Omega(\mathfrak{g})^{\mathfrak{g}}$ of dimension $\mathrm{rank}\,\mathfrak{g}$, called *primitive space* of $\mathfrak{g}$, and that the *Samelson subspace* P of the pair $(\mathfrak{g}, \mathfrak{k})$ is the graded subspace of $P_{\mathfrak{g}}$ whose elements, considered as elements of $\mathrm{H}(\mathfrak{g})$, are contained in the image of the inclusion induced map $\Omega(\mathfrak{g}, \mathfrak{k}) \longrightarrow \Omega(\mathfrak{g})$. Then we have $\dim P \le \mathrm{rank}\, g - \mathrm{rank}\,\mathfrak{k}$, cf. [11, Theorem V, sect. 10.4], and the pair (G, K) is formal if and only if the previous inequality is actually an equality; see [11, Theorem VIII, sect. 10.4] for this and various other reformulations of formality.

These preliminary notions being introduced, we briefly summarize the proof of the main result in [8] and show how [8] is related to [7].

Theorem 2.1 ([8, Theorem 1.1]). Let G be a compact connected Lie group and $K \subseteq G$ the identity component of the fixed point set of an automorphism on G. Then the pair (G, K) is (equivariantly) formal.

Note that according to [4, Theorem A] an equivariantly formal pair (G, K) with both G and K connected is necessarily formal as well. That formality of a pair (G, K) does not necessarily enforce equivariant formality of (G, K) is shown in [8, Example 3.7].

The proof of theorem 2.1 given in [8] can be divided into two major steps: the first step is to show that it suffices to consider pairs (G, K) satisfying the assumptions of theorem 2.1 and for which G is simple. In the second step one actually proves theorem 2.1 for simple groups G. Both steps crucially rely on the following general principle.

Theorem 2.2 ([4, Theorem 2.2]). Let K and H be equal rank closed connected subgroups of a compact connected Lie group G and such that $H \subseteq K$. Then (G, K) is equivariantly formal if and only if so is (G, H).

A proof of theorem 2.2 is also contained in [8, Proposition 3.5] under the additional hypothesis that the pairs (G, K) and (G, H) are formal. Since by [22, p. 212] the pair (G, K) is formal if and only if so is (G, H), it follows from [4, Theorem A] that this seemingly more restrictive setting is actually equivalent to the general situation considered in theorem 2.2; the proof of the first item of [8, Proposition 3.5], which essentially states that formality of (G, K) is equivalent to that of (G, H), is erroneous though[1].

The most important consequence of theorem 2.2 is that whenever H and K are closed connected subgroups of a compact connected Lie group G and T is a maximal torus of both H and K, then the pair (G, K) is equivariantly formal if and only if (G, H) is equivariantly formal, because this property is satisfied by either one of the pairs if and only if it is satisfied by the pair (G, T). In this way one can reduce the question of equivariant formality of pairs (G, K) as in theorem 2.1 and with G simple to pairs for which K is the identity component of the fixed point set of a finite–order automorphism. The homogeneous space G/K arising from such a pair (G, K) is called a *k–symmetric space* ($k \geq 0$ the order of the automorphism defining K) or *generalized symmetric space*, and the question whether or not (G, K) is equivariantly formal was already answered affirmatively in [7]. In fact, by [7, Proposition 3.7] K shares a maximal torus with a subgroup H dubbed "folded subgroup" in [7], because its Dynkin diagram is obtained from the Dynkin diagram of G by a process commonly called folding, and it was observed in [7, Theorem 5.5] that *H is (totally) non–cohomologous to zero in G*, that is, the fiber inclusion in the fibration $H \hookrightarrow G \longrightarrow G/H$ induces a surjection in cohomology. That (G, H) is formal then is a classical result (cf. [11, Corollary I, sect. 10.19]) and equivariant formality follows from

Proposition 2.3 ([7, Proposition 2.6]). Let G be a compact connected Lie group, K a closed connected subgroup. If K is totally non–cohomologous to zero in G, then (G, K) is equivariantly formal.

The question of (equivariant) formality being settled for pairs in which the ambient group is simple, we return to the general situation considered in theorem 2.1. One now observes that whenever (G, K) and (G', K') are two pairs of compact and connected Lie groups such that there is an isomorphism of Lie algebra pairs $(\mathfrak{g}, \mathfrak{k}) \longrightarrow (\mathfrak{g}', \mathfrak{k}')$, then (G, K) is (equivariantly) formal if and only if so is (G', K'), cf. [7, Corollary 2.4]. Thus, we call a Lie algebra pair $(\mathfrak{u}_0, \mathfrak{h}_0)$ *equivariantly formal* if there exists a compact connected Lie group U and a closed connected subgroup H such that (U, H) is equivariantly formal and $(\mathfrak{u}, \mathfrak{h})$ is isomorphic to $(\mathfrak{u}_0, \mathfrak{h}_0)$, for then any other compact connected Lie group pair with matching Lie algebras is equivariantly formal as well. Passing to the level of Lie algebras, we denote by σ the automorphism on $\mathfrak{g}$ whose fixed point set is $\mathfrak{k}$. Then $\mathfrak{g}$ decomposes as a direct sum of σ–invariant subalgebras $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ which are minimal in the sense that none of them contains a non–trivial proper σ–invariant subalgebra, $\mathfrak{k}$ decomposes accordingly as the direct sum of the subalgebras $\mathfrak{g}_1 \cap \mathfrak{k}, \ldots, \mathfrak{g}_n \cap \mathfrak{k}$, and it only remains to check that each of the pairs $(\mathfrak{g}_i, \mathfrak{g}_i \cap \mathfrak{k})$ is (equivariantly) formal. This is indeed the case: the pair $(\mathfrak{g}_i, \mathfrak{g}_i \cap \mathfrak{k})$ is isomorphic to a Lie algebra pair $\big(\mathfrak{u} \oplus \ldots \oplus \mathfrak{u}, \Delta(\mathfrak{f})\big)$, where $\mathfrak{u}$ is a compact simple Lie algebra and $\Delta(\mathfrak{f})$ is the diagonal embedding of the fixed point set $\mathfrak{f}$ of an automorphism on $\mathfrak{u}$, and $\Delta(\mathfrak{u})$ is totally non–cohomologous to zero in $\mathfrak{g}$; these two facts together imply that $(\mathfrak{g}_i, \mathfrak{g}_i \cap \mathfrak{k})$ is (equivariantly) formal, see [8, Section 5] for more details.

[1] Namely, instead of the displayed equation in the proof of the first part of [8, Proposition 3.5] one has to consider an equation of the form $\tau(\omega)|_{\mathfrak{t}} = \sum_i f_i|_{\mathfrak{t}} \cdot g_i$ with f_i polynomials in the image of the transgression and g_i non–constant polynomials invariant under the Weyl group of H. Averaging both sides over the Weyl group of K gives the desired conclusion.

Chapter II.

$\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric spaces

1. $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric spaces

There is yet another generalization of symmetric spaces that also incorporates the notion of k–symmetric spaces, the so–called Γ–symmetric spaces introduced in [19].

Definition 1.1. Let Γ be a finite Abelian group, G a connected Lie group, and $K \subseteq G$ a closed subgroup. The homogeneous space G/K is called Γ*–symmetric* if there exists an injective group homomorphism $\Gamma \hookrightarrow \mathrm{Aut}(G)$ such that $(G^\Gamma)_0 \subseteq K \subseteq G^\Gamma$, where G^Γ is the common fixed point set of the automorphisms $\Gamma \subseteq \mathrm{Aut}(G)$.

Since every finite Abelian group is a product of cyclic groups, the above definition can be rephrased by saying that a homogeneous space G/K with G connected and $K \subseteq G$ closed is $\Gamma = \mathbb{Z}_{k_1} \times \ldots \times \mathbb{Z}_{k_\ell}$–symmetric if there exist ℓ distinct commuting automorphisms $\sigma_1, \ldots, \sigma_\ell$ of G, with σ_i of order k_i, such that

$$(G^{\sigma_1} \cap \ldots \cap G^{\sigma_\ell})_0 \subseteq K \subseteq (G^{\sigma_1} \cap \ldots \cap G^{\sigma_\ell}).$$

Theorem 1.2. Let G be a compact connected Lie group, σ_1 and σ_2 two involutions on G, and suppose that G/K is a $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric space, where $K = (G^{\sigma_1} \cap G^{\sigma_2})_0$. Then the pair (G, K) is (equivariantly) formal.

We note that the classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric spaces G/K with G a simple Lie group was achieved in [1] and [17], but while we do make use of the classification of simple Lie algebras and finite–order automorphisms thereon, our proof of theorem 1.2 does not rely on the classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$–symmetric spaces.

Recall (cf. [14, p. 130]) that a Lie algebra $\mathfrak{g}$ is compact, if so is the connected subgroup of $\mathrm{Aut}(\mathfrak{g})$ with Lie algebra $\{\mathrm{ad}_X \mid X \in \mathfrak{g}\}$. According to [14, Corollary 6.7, chap. II] this is the case if and only if there is a compact Lie group with Lie algebra (isomorphic to) $\mathfrak{g}$. If $\mathfrak{g}$ is compact and semisimple, then every connected Lie group with Lie algebra $\mathfrak{g}$ is compact (see [14, Theorem 6.9, chap. II]), and we call a subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ *compact*, if the connected subgroup $H \subseteq G$ with Lie algebra $\mathfrak{h}$ is compact, where G is the simply–connected Lie group with Lie algebra $\mathfrak{g}$. For the sequel and for the proof of theorem 1.2 it will be convienent to introduce the following relation on the set of all compact subalgebras of a compact semisimple Lie algebra $\mathfrak{g}$: two such subalgebras $\mathfrak{h}, \mathfrak{k} \subseteq \mathfrak{g}$ are related, if there exists a sequence of compact subalgebras $\mathfrak{m}_0, \ldots, \mathfrak{m}_{k+1}$ of $\mathfrak{g}$ such that $\mathfrak{m}_0 = \mathfrak{h}$, $\mathfrak{m}_{k+1} = \mathfrak{k}$ and if for all $i = 0, \ldots, k$ the subalgebras $\mathfrak{m}_i$ and $\mathfrak{m}_{i+1}$ share a common maximal torus, that is, if there exists a maximal torus $\mathfrak{s} \subseteq \mathfrak{m}_i$ which also is maximal torus of $\mathfrak{m}_{i+1}$. This defines an equivalence relation and we denote the equivalence class of a subalgebra $\mathfrak{k}$ by $[\mathfrak{k}]_{\mathrm{f}}$. Note that if $\mathfrak{k} \subseteq \mathfrak{g}$ is a compact subalgebra, then the pair $(\mathfrak{g}, \mathfrak{k})$ is (equivariantly) formal if and only if there exists a subalgebra $\mathfrak{h} \in [\mathfrak{k}]_{\mathrm{f}}$ such that $(\mathfrak{g}, \mathfrak{h})$ is so. Now theorem 1.2 will be a consequence of

Theorem 1.3. In addition to the hypotheses of theorem 1.2 assume that G is simple. Then there exists a compact subalgebra $\mathfrak{h} \in [\mathfrak{k}]_{\mathrm{f}}$ which is totally non–cohomologous to zero in $\mathfrak{g}$.

Proof of theorem 1.2 using theorem 1.3. Let $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ be the decomposition of the semisimple part of $\mathfrak{g}$ into its simple ideals and consider the subgroup $\Gamma = \{\mathrm{id}_{\mathfrak{g}}, \sigma_1, \sigma_2, \sigma_1\sigma_2\}$ inside the group of Lie algebra automorphisms of $\mathfrak{g}$. It is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ and acts naturally on $\mathcal{I} := \{\mathfrak{g}_1, \ldots, \mathfrak{g}_m\}$. Moreover, as was already observed in [8, Section 5], it will suffice to check that for each i the pair $(\mathfrak{m}, \mathfrak{m} \cap \mathfrak{k})$, where $\mathfrak{m} = \sum_{\gamma \in \Gamma} \gamma(\mathfrak{g}_i)$, is (equivariantly) formal.

Set $\mathfrak{h} := \mathfrak{g}_i$ and choose representatives $\gamma_1\Gamma_{\mathfrak{h}}, \dots, \gamma_p\Gamma_{\mathfrak{h}}$ for each class in $\Gamma/\Gamma_{\mathfrak{h}}$, where $\Gamma_{\mathfrak{h}}$ is the isotropy subgroup at $\mathfrak{h}$ of the action of Γ on $\mathcal{I}$, $p = |\Gamma/\Gamma_{\mathfrak{h}}|$, and $\gamma_1 = \mathrm{id}_{\mathfrak{g}}$. Then an isomorphism of Lie algebras is given by the map

$$\Phi : \mathfrak{h} \oplus \dots \oplus \mathfrak{h} \longrightarrow \mathfrak{m}, (X_1, \dots, X_p) \mapsto \gamma_1(X_1) + \dots + \gamma_p(X_p),$$

because $\gamma_s(\mathfrak{h})$ and $\gamma_t(\mathfrak{h})$ are distinct ideals of $[\mathfrak{g}, \mathfrak{g}]$ for $s \neq t$ and $\mathfrak{m} = \bigoplus_{s=1}^{p} \gamma_s(\mathfrak{h})$. Moreover, if $\mathfrak{f} \subseteq \mathfrak{h}$ is the common fixed point set of all elements in $\Gamma_{\mathfrak{h}}$, then Φ maps $\Delta(\mathfrak{f})$, the diagonal embedding of $\mathfrak{f}$, isomorphically onto $\mathfrak{m} \cap \mathfrak{k}$: in fact, any element $\gamma \in \Gamma$ permutes $\Gamma/\Gamma_{\mathfrak{h}}$, so there exist a permutation π on $\{1, \dots, p\}$ and elements $\gamma_s' \in \Gamma_{\mathfrak{h}}$ for each s such that $\gamma\gamma_t = \gamma_{\pi(t)}\gamma_t'$ for all t. Then we have, for all $X \in \mathfrak{f}$:

$$\gamma(\Phi(X, \dots, X)) = \sum_{s=1}^{p} \gamma\gamma_s(X) = \sum_{s=1}^{p} \gamma_{\pi(s)}(X) = \Phi(X, \dots, X).$$

To prove the converse inclusion, note that if $\Phi(X_1, \dots, X_p)$ is fixed by some γ_i, then $X_i = X_1$, because we chose $\gamma_1 = \mathrm{id}$ and because $\gamma_i\gamma_j(X_j) \in \mathfrak{h}$ only holds if $i = j$. Hence, if $\Phi(X_1, \dots, X_p)$ is fixed by all elements of Γ, then $X_1 = X_2 = \dots = X_p$ and also $X_1 \in \mathfrak{f}$, because every $\gamma \in \Gamma_{\mathfrak{h}}$ leaves $\mathfrak{h}$ invariant.

Thus, it will suffice to check that $\left(\bigoplus_{s=1}^{p} \mathfrak{h}, \Delta(\mathfrak{f})\right)$ is (equivariantly) formal. But an orbit of Γ is either of length 1, 2, or 4, and if $p = 1$, then $\mathfrak{f}$ is just the common fixed point set of σ_1 and σ_2, whence the pair in question is (equivariantly) formal by theorem 1.3. If $p = 2$, then $\Gamma_{\mathfrak{h}}$ contains one non–trivial element σ, so $\mathfrak{f} = \mathfrak{h}^\sigma$ is the fixed point set of an involution, and it was observed in [8, Section 5] that $(\mathfrak{h} \oplus \mathfrak{h}, \Delta(\mathfrak{f}))$ is (equivariantly) formal in this case as well: indeed, if we choose $\mathfrak{n} \in [\mathfrak{f}]_{\mathfrak{f}}$ to be totally non–cohomologous to zero in $\mathfrak{h}$, which is possible by [8, Section 4] or [7, Theorem 5.5], then $\Delta(\mathfrak{n})$ is totally non–cohomologous to zero in $\mathfrak{h} \oplus \mathfrak{h}$ as well and $\Delta(\mathfrak{n}) \in [\Delta(\mathfrak{f})]_{\mathfrak{f}}$. Finally, if $p = 4$, then $\Gamma_{\mathfrak{h}}$ is trivial, whence $\mathfrak{f} = \mathfrak{h}$. As is well–known, $\Delta(\mathfrak{h})$ is totally non–cohomologous to zero in $\mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h} \oplus \mathfrak{h}$. □

2. Preliminaries

Let G be a compact connected Lie group and σ a finite–order automorphism on G. It follows from [14, Lemma 5.3, chap. X], that the centralizer $\mathrm{Z}_{\mathfrak{g}}(\mathfrak{s})$ in $\mathfrak{g}$ of any maximal torus $\mathfrak{s}$ of $\mathfrak{g}^\sigma$ is a maximal torus of $\mathfrak{g}$, and hence the unique maximal torus of $\mathfrak{g}$ containing $\mathfrak{s}$. Thus, if $\sigma_1, \dots, \sigma_\ell$ are commuting automorphisms of G, then there is a maximal torus of $\mathfrak{g}$ which is invariant for all σ_i, $i = 1, \dots, \ell$. In fact, put $\sigma_{\ell+1} = \mathrm{id}_G$ and suppose that for some i, $1 \leq i \leq \ell$, $\mathfrak{t}_i$ is a maximal torus of $\mathfrak{k}_i$, where

$$\mathfrak{k}_i := \mathfrak{g}^{\sigma_i} \cap \dots \cap \mathfrak{g}^{\sigma_{\ell+1}},$$

and that $\mathfrak{t}_i$ is invariant under $\sigma_1, \dots, \sigma_{\ell+1}$; such a torus exists for $i = 1$, because $\mathfrak{k}_1$ is the common fixed point set of $\sigma_1, \dots, \sigma_\ell$, whence any maximal torus of $\mathfrak{k}_1$ is fixed by each σ_j. Since all σ_j commute, σ_i then restricts to a finite–order automorphism $\sigma_i : \mathfrak{k}_{i+1} \longrightarrow \mathfrak{k}_{i+1}$ with fixed point set $\mathfrak{k}_i$. As $\mathfrak{k}_{i+1}$ is the common fixed point set of $\sigma_{i+1}, \dots, \sigma_\ell$ and thus the Lie algebra of a compact Lie group, we conclude that $\mathfrak{t}_{i+1} = \mathrm{Z}_{\mathfrak{k}_{i+1}}(\mathfrak{t}_i)$ is a maximal torus of $\mathfrak{k}_{i+1}$. By definition, $\mathfrak{t}_{i+1}$ is fixed by $\sigma_{i+1}, \dots, \sigma_\ell$, and if $j \leq i$, then $\sigma_j(\mathfrak{t}_{i+1})$ is a maximal torus of $\mathfrak{k}_{i+1}$ containing $\mathfrak{t}_i$, hence must be equal to $\mathfrak{t}_{i+1}$. Continuing in this way, we eventually obtain a maximal torus $\mathfrak{t}_{\ell+1}$ of $\mathfrak{k}_{\ell+1} = \mathfrak{g}$ with $\sigma_j(\mathfrak{t}_{\ell+1}) = \mathfrak{t}_{\ell+1}$ for all $j = 1, \dots, \ell$.

Proposition 2.1. Let G be a compact connected Lie group, $\mathfrak{a} \subseteq \mathfrak{g}$ an Abelian subalgebra, and $\mathfrak{t}$ a maximal torus of $\mathfrak{g}$ containing $\mathfrak{a}$. Denote by $\Delta \subseteq (\mathfrak{t}^{\mathbb{C}})^*$ the set of roots with respect to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$ of $\mathfrak{g}^{\mathbb{C}}$ and by $\Gamma \subseteq \Delta$ the set of roots vanishing on $\mathfrak{a}$. Then, as a vector space,

$$\mathrm{N}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{a}) = \mathrm{Z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{a}) = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Gamma} \mathfrak{g}_\alpha^{\mathbb{C}}.$$

Proof. That $\mathfrak{t}^{\mathbb{C}}$ is contained in $\mathrm{Z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{a})$ is true because $\mathfrak{t}$ is Abelian. Now choose $\alpha \in \Gamma$ as well as $X \in \mathfrak{g}_{\pm\alpha}^{\mathbb{C}}$. By definition, for every $Y \in \mathfrak{a}$:

$$[Y, X] = \pm\alpha(Y)X = 0,$$

hence $\mathfrak{g}^{\mathbb{C}}_{\pm\alpha}$ is contained in $\mathrm{Z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{a})$. Conversely, let $N \in \mathrm{N}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{a})$, and write

$$N = X_0 + \sum_{\alpha\in\Delta} X_\alpha,$$

where $X_0 \in \mathfrak{t}^{\mathbb{C}}$ and $X_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha$. For $Y \in \mathfrak{a}$ we have

$$\mathfrak{t}^{\mathbb{C}} \supseteq \mathfrak{a} \ni [Y, N] = \sum_{\alpha\in\Delta} \alpha(Y)X_\alpha \in \bigoplus_{\alpha\in\Delta} \mathfrak{g}^{\mathbb{C}}_\alpha,$$

which is only possible if $\alpha(Y)X_\alpha = 0$ for all $\alpha \in \Delta$. Hence, if $X_\alpha \neq 0$, then $\mathfrak{a} \subseteq \ker\alpha$ and $\alpha \in \Gamma$. We have shown:

$$\mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Gamma} \mathfrak{g}^{\mathbb{C}}_\alpha \subseteq \mathrm{Z}_{\mathfrak{g}}(\mathfrak{a}) \subseteq \mathrm{N}_{\mathfrak{g}}(\mathfrak{a}) \subseteq \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Gamma 0} \mathfrak{g}^{\mathbb{C}}_\alpha. \qquad \square$$

For the remainder of this section we fix a compact connected Lie group G, two commuting involutions σ_1 and σ_2 on G (not necessarily different), and an Ad–invariant negative definite inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ for which σ_1 and σ_2 are isometries. Note that any negative definite Ad–invariant inner product $(\cdot,\cdot)$ on $\mathfrak{g}$ gives rise to such an inner product: just take

$$(\cdot,\cdot) + \sigma_1^*(\cdot,\cdot) + \sigma_2^*(\cdot,\cdot) + (\sigma_1\sigma_2)^*(\cdot,\cdot).$$

Moreover, we put $K_1 := (G^{\sigma_1})_0$, $K_2 := (G^{\sigma_2})_0$, and choose a maximal torus $S \subseteq (G^{\sigma_1} \cap G^{\sigma_2})_0$. According to our previous observations, $T_1 = \mathrm{Z}_{K_1}(S)$ then is a maximal torus in K_1 and $T := \mathrm{Z}_G(T_1)$ is a maximal torus in G. Let Δ be the $\mathfrak{g}^{\mathbb{C}}$–roots with respect to $\mathfrak{t}^{\mathbb{C}}$, Δ^+ a choice of positive roots, $\Gamma \subseteq \Delta$ the set of roots vanishing on $\mathfrak{s}$, and $\Gamma^+ := \Gamma \cap \Delta^+$. We also set $\tau\alpha := \alpha\circ\tau$ whenever α is a root and τ is an automorphism on $\mathfrak{g}$ leaving $\mathfrak{t}$ invariant.

Proposition 2.2. Let $\mathfrak{g} = \mathfrak{k}_1 \oplus \mathfrak{p}_1$ be the decomposition of $\mathfrak{g}$ into the 1– and (-1)–eigenspaces of σ_1. Then

(1) the root space $\mathfrak{g}^{\mathbb{C}}_\alpha$ is contained in $\mathfrak{p}_1^{\mathbb{C}}$ for all $\alpha \in \Gamma$;

(2) if $\alpha \in \Gamma$, then $\sigma_1\alpha = \alpha$ and

(3) $\sigma_2\alpha = -\alpha$;

(4) any two roots $\alpha, \beta \in \Gamma$ are *strongly orthogonal*, that is, neither $\alpha + \beta$ nor $\alpha - \beta$ is a root;

(5) denoting for a root α by $H_\alpha \in$ it the element with $\langle H_\alpha, \cdot\,\rangle = \alpha$, we have

$$\mathfrak{t}^{\mathbb{C}} = \bigcap_{\alpha\in\Gamma^+} \ker\alpha \oplus \bigoplus_{\alpha\in\Gamma^+} \mathbb{C}H_\alpha,$$

and any two summands in this decomposition are mutually orthogonal with respect to $\langle\cdot,\cdot\rangle$.

Proof.

(1) Pick $\alpha \in \Gamma$ and note that $\sigma_1\alpha$ still vanishes on $\mathfrak{s}$. Thus, the space $U := \mathfrak{g}^{\mathbb{C}}_\alpha + \mathfrak{g}^{\mathbb{C}}_{\sigma_1\alpha}$ is σ_1–invariant, and so decomposes as the direct sum $U = (U \cap \mathfrak{k}_1^{\mathbb{C}}) \oplus (U \cap \mathfrak{p}_1^{\mathbb{C}})$. Now proposition 2.1 implies that

$$U \cap \mathfrak{k}_1^{\mathbb{C}} \subseteq \mathrm{Z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}) \cap \mathfrak{k}_1^{\mathbb{C}} = \mathrm{Z}_{\mathfrak{k}_1^{\mathbb{C}}}(\mathfrak{s}) = \mathfrak{t}_1^{\mathbb{C}} \subseteq \mathfrak{t}^{\mathbb{C}},$$

and since $U \cap \mathfrak{t}^{\mathbb{C}} = \{0\}$, it follows that $U \cap \mathfrak{k}_1^{\mathbb{C}} = \{0\}$ as well. Thus, $U \subseteq \mathfrak{p}_1^{\mathbb{C}}$.

(2) We have just seen that given $\alpha \in \Gamma$ the root space $\mathfrak{g}_\alpha^{\mathbb{C}}$ is contained in $\mathfrak{p}_1^{\mathbb{C}}$. So, if we pick $E_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}$ and $E_{-\alpha} \in \mathfrak{g}_{-\alpha}^{\mathbb{C}}$, then $[E_\alpha, E_{-\alpha}] \in \mathfrak{k}_1^{\mathbb{C}}$. We may assume that $\langle E_\alpha, E_{-\alpha}\rangle = 1$, and then $H_\alpha = [E_\alpha, E_{-\alpha}]$ for the element $H_\alpha \in i\mathfrak{t}$ with $\langle H_\alpha, \cdot\,\rangle = \alpha$. Therefore,

$$\alpha = \langle H_\alpha, \cdot\,\rangle = \langle \sigma_1(H_\alpha), \cdot\,\rangle \circ \sigma_1 = \langle H_\alpha, \cdot\,\rangle \circ \sigma_1 = \sigma_1 \alpha.$$

(3) According to the previous item, $H_\alpha \in \mathfrak{k}_1^{\mathbb{C}}$, and since σ_1 and σ_2 commute, $\sigma_2(H_\alpha)$ must be contained in $\mathfrak{k}_1^{\mathbb{C}}$ as well. Therefore,

$$H_\alpha + \sigma_2(H_\alpha) \in \mathfrak{t}^{\mathbb{C}} \cap \mathfrak{k}_1^{\mathbb{C}} \cap \mathfrak{k}_2^{\mathbb{C}} = \mathfrak{s}^{\mathbb{C}}.$$

Now for $Y \in \mathfrak{s}^{\mathbb{C}}$ we compute

$$\langle Y, H_\alpha + \sigma_2(H_\alpha)\rangle = 2\langle Y, H_\alpha\rangle = 2\alpha(Y) = 0.$$

But $\langle \cdot, \cdot \rangle$ is non–degenerate on $\mathfrak{s}^{\mathbb{C}}$, hence we must have $H_\alpha + \sigma_2(H_\alpha) = 0$, which is equivalent to saying that $\sigma_2\alpha = -\alpha$, because σ_2 is an isometry of $\langle \cdot, \cdot \rangle$.

(4) Let $\alpha, \beta \in \Gamma$ and suppose that $\alpha + \beta$ was a root for a contradiction. We could choose non–zero root vectors $X_\alpha \in \mathfrak{g}_\alpha^{\mathbb{C}}$ and $X_\beta \in \mathfrak{g}_\beta^{\mathbb{C}}$, and then $[X_\alpha, X_\beta] \in \mathfrak{g}_{\alpha+\beta}^{\mathbb{C}}$ would be a non–zero root vector as well. But $\mathrm{Z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s})$ is a Lie algebra and X_α, X_β are elements of $\mathrm{Z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s})$, so according to the first item

$$[X_\alpha, X_\beta] \in \mathfrak{k}_1^{\mathbb{C}} \cap \mathrm{Z}_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}) = \mathfrak{t}_1^{\mathbb{C}} \subseteq \mathfrak{t}^{\mathbb{C}},$$

which is impossible. Therefore, $\alpha + \beta$ is not a root.

(5) Let $\alpha, \beta \in \Gamma^+$ be two distinct roots. It is well known (cf. [16, Proposition 2.48, sect. II.5]) that the α–string containing β, that is, the subset of $\Delta \cup \{0\}$ consisting of elements $\beta + n\alpha$ with $n \in \mathbb{Z}$, has no gaps and that the integers $p, q \geq 0$ such that $(\beta + n\alpha \in \Delta \cup \{0\}) \iff (-p \leq n \leq q)$ satisfy $p - q = 2\langle \alpha, \beta\rangle / \langle \alpha, \alpha\rangle$. Since neither $\alpha + \beta$ nor $\alpha - \beta$ is a root, we hence must have

$$0 = \langle \alpha, \beta \rangle = \langle H_\alpha, H_\beta\rangle.$$

In particular, the elements H_α, $\alpha \in \Gamma^+$, are linearly independent. Now let

$$U = \bigoplus_{\alpha \in \Gamma^+} \mathbb{C} H_\alpha \text{ and } U' = \bigcap_{\alpha \in \Gamma^+} \ker \alpha.$$

Then the equation $\alpha(Y) = \langle H_\alpha, Y\rangle$ for $Y \in \mathfrak{t}^{\mathbb{C}}$ shows that $U' = \mathfrak{t}^{\mathbb{C}} \cap U^\perp$, and so $\mathfrak{t}^{\mathbb{C}} = U \oplus U'$. □

3. Automorphisms

We continue to use the notation of the previous section. Given $\alpha \in \Delta$, denote by $s_{H_\alpha} : i\mathfrak{t} \longrightarrow i\mathfrak{t}$ the reflection along the hyperplane orthogonal to H_α, i.e. the map

$$s_{H_\alpha}(X) = X - \frac{2\langle H_\alpha, X\rangle}{\langle H_\alpha, H_\alpha\rangle}.$$

Since the elements of Γ are mutually orthogonal, we immediately have

Proposition 3.1. The members of $\{s_{H_\alpha} \mid \alpha \in \Gamma^+\}$ commute pairwise.

Note that proposition 2.2 suggests that σ_2 acts as a product of hyperplane reflections on a certain subspace of $\mathfrak{t}$. This subspace will be a proper subspace in general, but if σ_2 is an inner autormophism, then it actually is all of $\mathfrak{t}$. We shall show that under some mild assumptions on σ_1 the maximal torus $\mathfrak{s}$ of $\mathfrak{k}_1 \cap \mathfrak{k}_2$ can in fact be recovered from Γ.

Proposition 3.2. Suppose that $\sigma_2 = c_n \circ \nu$ holds for some element $n \in G$ and some automorphism ν on G that fixes $\mathfrak{t}_1$ pointwise. Then

(1) the element n is contained in $\mathrm{Z}_G(S) \cap \mathrm{N}_G(T_1)$,

(2) the maximal torus $\mathfrak{t}$ is ν–invariant,

(3) $\sigma_2|_{i\mathfrak{t}} = \prod_{\alpha\in\Gamma^+}(s_{H_\alpha})\circ(\nu|_{i\mathfrak{t}})$.

Proof.

(1) By assumption, T_1 is contained in the 1–eigenspace of ν and $\sigma_2 = c_n \circ \nu$. Since S is contained in the 1–eigenspace of σ_2 and T_1 is σ_2–invariant, the same statement is true with c_n in place of σ_2.

(2) Just note that $\nu(T)$ is a maximal torus of G containing T_1, so $\nu(T) = T$.

(3) We already observed that n centralizes S and it is a well–known fact (see [16, Corollary 4.51, sect. IV.5]) that centralizers of tori are connected, so, according to proposition 2.1, we may express n as $n = \exp(X)$, where $X = X_0 + X_\Gamma$ for certain elements $X_0 \in \mathfrak{t}$ and $X_\Gamma \in \bigoplus_{\alpha\in\Gamma}\mathfrak{g}^{\mathbb{C}}_\alpha$. In particular, if $Y \in L$, $L := \bigcap_{\alpha\in\Gamma^+}\ker\alpha$, then $[Y, X] = 0$. Thus, Ad_n fixes $L \cap i\mathfrak{t}$ pointwise, as do the elements s_{H_α} with $\alpha \in \Gamma^+$. On the other hand, if $\beta \in \Gamma^+$ is arbitrary, then $H_\beta \subseteq i\mathfrak{t}_1$ by proposition 2.2, so

$$\mathrm{Ad}_n(H_\beta) = \sigma_2(H_\beta) = -H_\beta = \left(\prod_{\alpha\in\Gamma^+} s_{H_\alpha}\right)(H_\beta).$$

Therefore, Ad_n restricts to $\prod_{\alpha\in\Gamma^+} s_{H_\alpha}$ on $i\mathfrak{t}$, whence the ν–invariance of $\mathfrak{t}$ implies the claim. □

Corollary 3.3. Suppose that $\sigma_2 = c_n \circ \nu$ and $\mathfrak{t}_1 \subseteq \mathfrak{g}^\nu$, and put $L := \bigcap_{\alpha\in\Gamma^+}\ker\alpha$. Then

$$i\mathfrak{t} \cap L = (i\mathfrak{s}) \oplus i(\mathfrak{t} \cap \mathfrak{p}_1),\ i\mathfrak{t}_1 = (i\mathfrak{s}) \oplus \bigoplus_{\alpha\in\Gamma^+}\mathbb{R}H_\alpha, \text{ and } \operatorname{rank}(\mathfrak{k}_1 \cap \mathfrak{k}_2) = \operatorname{rank}(\mathfrak{k}_1) - |\Gamma^+|.$$

Proof. We know from proposition 2.2 that $i\mathfrak{t} = (i\mathfrak{t} \cap L) \oplus \bigoplus_{\alpha\in\Gamma^+}\mathbb{R}H_\alpha$ is a decomposition into two σ_1–invariant subspaces and that $\bigoplus_{\alpha\in\Gamma^+}\mathbb{R}H_\alpha$ is entirely contained in $i\mathfrak{t}_1$. Thus, we must have $i\mathfrak{t} \cap L = (i\mathfrak{t}_1 \cap L) \oplus i(\mathfrak{t} \cap \mathfrak{p}_1)$ and $i\mathfrak{t}_1 = (i\mathfrak{t}_1 \cap L) \oplus \bigoplus_{\alpha\in\Gamma^+}\mathbb{R}H_\alpha$. Now recall that $(i\mathfrak{t}_1)^{\sigma_2} = i\mathfrak{s}$, while $(i\mathfrak{t}_1)^{\sigma_2} = i\mathfrak{t}_1 \cap L$ holds by proposition 3.2. □

If $\mathfrak{g}$ is simple the condition that σ_2 is a composition of an inner automorphism and an automorphism fixing $\mathfrak{t}_1$ is not too restrictive: in fact, we will see later that if σ_1 is an outer automorphisms, then, except for Lie algebras of type D_4, we may assume that $\sigma_1 = c_t \circ \tau$ and $\sigma_2 = c_n \circ \tau$ or that $\sigma_1 = c_t \circ \tau$ and $\sigma_2 = c_n$ for some involution $\tau : G \longrightarrow G$ and elements $t \in T_1$, $n \in \mathrm{N}_H(T_1) \cdot T$, where $H = (G^\tau)_0$.

The following propositions state that in this case we may trade $t \in T_1$ for some element $t' \in T$ to first assume that $n \in H$ and that $\sigma_2 = c_n \circ \tau$; afterwards we may replace σ_1 by an inner automorphism.

Proposition 3.4. Suppose that $\sigma_1 = c_t \circ \tau$ and $\sigma_2 = c_n \circ \nu$, where τ is an involution, $\nu = \tau$ or $\nu = \mathrm{id}_G$, $t \in T_1$, and $n \in \mathrm{N}_H(T_1) \cdot T$, with $H = (G^\tau)_0$. Then there exist elements $t' \in T$ and $h \in \mathrm{N}_H(T_1)$ such that $c_{t'} \circ \tau$ and $c_h \circ \tau$ are commuting involutions whose common fixed point set has $\mathfrak{s}$ as a maximal torus.

Proof. First suppose that $\nu = \tau$. Then we choose $q \in \exp(\mathfrak{t} \cap \mathfrak{p}_1)$, $h \in \mathrm{N}_H(T_1)$ with $n = hq$ and set $L := \bigcap_{\alpha\in\Gamma^+}\ker\alpha$. Note that $\mathfrak{t} \cap L$ decomposes, by corollary 3.3, as $\mathfrak{t} \cap L = \mathfrak{s} \oplus (\mathfrak{t} \cap \mathfrak{p}_1)$ and that the elements of $\mathfrak{t} \cap L$ are fixed by Ad_h, because $\sigma_1|_\mathfrak{t} = \tau|_\mathfrak{t}$ and hence proposition 3.2 applies. So if we pick $Y \in \mathfrak{t} \cap \mathfrak{p}_1$ with $q = \exp(Y)$ and put $r = \exp(Y/2)$, then $c_{r^{-1}} \circ \sigma_2 \circ c_r$ is an involution, $q = r^2$, and $\tau(r) = r^{-1}$. Therefore, we have

$$c_{r^{-1}} \circ \sigma_2 \circ c_r = c_{hqr^{-1}} \circ \tau \circ c_r = c_{hqr^{-1}} \circ c_{r^{-1}} \circ \tau = c_h \circ \tau;$$

similarly, $c_{r^{-1}}\circ\sigma_1\circ c_r = c_{q^{-1}t}\circ\tau$. Thus, $c_{q^{-1}t}\circ\tau$ and $c_h\circ\tau$ are two commuting involutions. Since their common fixed point subalgebra is conjugate to $\mathfrak{k}_1 \cap \mathfrak{k}_2$ via $\mathrm{Ad}_{r^{-1}}$ and $\mathrm{Ad}_{r^{-1}}$ fixes $\mathfrak{s}$, the claim follows.

Now assume that $\nu = \mathrm{id}_G$. Choose a decomposition $n = hq$ as before and use corollary 3.3 to additionally find $s \in \exp(\mathfrak{s})$, $a \in \exp(\bigoplus_{\alpha\in\Gamma^+}\mathbb{R}(\mathrm{i}H_\alpha))$ with $t = sa$. We will show that $\mu := c_{ns}\circ\tau$ is an involution, that μ commutes with σ_1, and that $\mathfrak{s}$ is a maximal torus of $\mathfrak{g}^{\sigma_1} \cap \mathfrak{g}^{\mu}$. The previous case then implies the claim, because $ns \in \mathrm{N}_H(T_1)\cdot T$. To begin with, we assert that $(c_s)^2 = (c_q)^2$; indeed, c_h and c_n coincide on $\mathfrak{t}$, whence we have $c_h(a) = a^{-1}$ and $c_h(s) = s$ (cf. proposition 3.2), so this follows from

$$c_n = \sigma_1\circ c_n\circ\sigma_1 = c_t\circ c_{\tau(n)}\circ c_t = c_s\circ c_a\circ c_h\circ c_{q^{-1}}\circ c_t = c_h\circ c_s\circ c_{a^{-1}}\circ c_{q^{-1}}\circ c_t$$

together with the commutativity of q, s, and a. Also note that h, q, and s commute with each other and that H contains s. These observations imply that μ is an involution commuting with σ_1, since

$$\mu^2 = c_{ns}\circ\tau\circ c_{ns}\circ\tau = c_{ns}\circ c_{hq^{-1}s} = (c_h)^2\circ(c_s)^2 = (c_n)^2 = \mathrm{id}$$

and since c_s, τ, and c_n commute with σ_1. Finally, note that any maximal torus of $\mathfrak{g}^{\sigma_1} \cap \mathfrak{g}^{\mu}$ containing $\mathfrak{s}$ is a subset of $\mathrm{Z}_{\mathfrak{g}}(\mathfrak{s})$ and that by propositions 2.1 and 2.2 σ_1 only fixes $\mathfrak{t}_1$ on $\mathrm{Z}_{\mathfrak{g}}(\mathfrak{s})$. Then $\mathfrak{s}$ must be a maximal torus of $\mathfrak{g}^{\sigma_1} \cap \mathfrak{g}^{\mu}$, as $\mathfrak{t}^{\mathbb{C}}$ and $\bigoplus_{\alpha\in\Gamma}\mathfrak{g}^{\mathbb{C}}_\alpha$ are μ–invariant subspaces and $\mu|_{\mathfrak{t}_1} = \sigma_2|_{\mathfrak{t}_1}$ only fixes $\mathfrak{s}$. ☐

Proposition 3.5. Suppose that $\sigma_1 = c_t\circ\tau$ and that $\sigma_2 = c_h\circ\tau$, where τ is an involution, t is contained in T, and h is an element of $\mathrm{N}_H(T_1)$, with $H = (G^\tau)_0$. Let $\Pi_{\mathrm{odd}} \subseteq \Pi$ be the set of all roots $\beta \in \Pi$ for which the integer $\sum_{\alpha\in\Gamma^+} 2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle$ is odd. Then $\tau(\alpha) \neq \alpha$ for all $\alpha \in \Pi_{\mathrm{odd}}$.

Lemma 3.6. Under the assumptions of proposition 3.5 we have $\mathfrak{g}^{\mathbb{C}}_\alpha \subseteq \mathfrak{h}^{\mathbb{C}}$ for each root $\alpha \in \Gamma$.

Proof. Observe that the requirements of proposition 3.2 are met, so h is an element of $\mathrm{Z}_G(S) \cap H = \mathrm{Z}_H(S)$. Since $\mathrm{Z}_H(S)$ is connected, we may express h as $h = \exp(Z)$ for some element $Z \in \mathrm{Z}_{\mathfrak{h}}(\mathfrak{s}) = \mathrm{Z}_{\mathfrak{g}}(\mathfrak{s}) \cap \mathfrak{h}$, say $Z = Z_0 + \sum_{\alpha\in\Gamma^+} Z_\alpha$, with $Z_0 \in \mathfrak{t}$ and $Z_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha \oplus \mathfrak{g}^{\mathbb{C}}_{-\alpha}$ for each root $\alpha \in \Gamma^+$. Recall that σ_1 coincides with τ on $\mathfrak{t}$, because c_t is the identity on $\mathfrak{t}$, so as σ_1 fixes each root $\alpha \in \Gamma^+$, τ fixes each element of Γ^+ too. Therefore, $\mathfrak{g}^{\mathbb{C}}_\alpha$ and $\mathfrak{g}^{\mathbb{C}}_{-\alpha}$ are eigenspaces of τ, whence Z_α necessarily vanishes if $\mathfrak{g}^{\mathbb{C}}_\alpha \not\subseteq \mathfrak{h}^{\mathbb{C}}$. However, if $\beta \in \Gamma^+$ was a root with $Z_\beta = 0$, then, as the elements of Γ are strongly orthogonal, we also would have $[Z, H_\beta] = 0$, and hence $\mathrm{Ad}_h(H_\beta) = H_\beta$. But this is impossible, because we know from proposition 3.2 that $\mathrm{Ad}_h(H_\beta) = -H_\beta$. Consequently, $Z_\beta \neq 0$ and $\mathfrak{g}^{\mathbb{C}}_\beta \subseteq \mathfrak{h}^{\mathbb{C}}$. ☐

Proof of proposition 3.5. The decomposition $\mathfrak{t} = \mathfrak{s}\oplus\mathfrak{s}'$, with $\mathfrak{s}' = \bigoplus_{\alpha\in\Gamma^+}\mathbb{R}(\mathrm{i}H_\alpha)\oplus(\mathfrak{t}\cap\mathfrak{p}_1)$ yields a decomposition $k = k_+k_-$ for every element $k \in T$, where $k_+ \in \exp(\mathfrak{s})$ and $k_- \in \exp(\mathfrak{s}')$. Moreover, σ_2 restricts to id on $\mathfrak{s}$ and to $(-\mathrm{id})$ on $\mathfrak{s}'$, so the condition that $c_k\circ\tau$ commutes with σ_2 can be rephrased as

$$c_k\circ\tau = \sigma_2\circ c_k\circ\tau\circ(\sigma_2)^{-1} \iff c_k\circ\tau = c_{\sigma_2(k)}\circ(c_h)^2\circ\tau \iff (c_{k_-})^2 = (c_h)^2;$$

but c_h is an involution, because c_h commutes with τ and σ_2 is an involution, so $c_k\circ\tau$ commutes with σ_2 if and only if c_{k_-} is an involution. In particular, if we let $t = t_+t_-$, then c_{t_-} is an involution.

With this characterization at hand we can show that no root in Π_{odd} is fixed by τ: let us further decompose t_- as $t_- = qr$, where $q \in \exp(\mathfrak{t}\cap\mathfrak{p}_1)$, $r = \exp(Z)$, and $Z = \sum_{\alpha\in\Gamma^+} t_\alpha \mathrm{i}\pi/\langle\alpha,\alpha\rangle H_\alpha$ for certain real numbers t_α. Recalling that each element $\beta \in \Gamma$ is contained in the (-1)–eigenspace of σ_1, but in the fixed point set of τ, and that $\mathfrak{s} \oplus (\mathfrak{t}\cap\mathfrak{p}_1)$ is the common kernel of the elements of Γ on $\mathfrak{t}$, we find that

$$-\mathrm{id}_{\mathfrak{g}^{\mathbb{C}}_\beta} = \sigma_1|_{\mathfrak{g}^{\mathbb{C}}_\beta} = \mathrm{Ad}_r|_{\mathfrak{g}^{\mathbb{C}}_\beta} = e^{\mathrm{i}\pi t_\beta}\,\mathrm{id};$$

so, $(t_\beta - 1) \in 2\mathbb{Z}$. On the other hand, if $\beta \in \Pi$ with $\tau(\beta) = \beta$ is arbitrary, then Ad_q restricts to $\pm$ id on $\mathfrak{g}^{\mathbb{C}}_\beta$, because

$\mathfrak{t} \cap \mathfrak{p}_1$ is the (-1)–eigenspace of τ on $\mathfrak{t}$. Combined with the fact that c_{t_-} is an involution this gives

$$\mathrm{id}_{\mathfrak{g}^{\mathbb{C}}_\beta} = \left(\mathrm{Ad}_{qr}|_{\mathfrak{g}^{\mathbb{C}}_\beta}\right)^2 = \left(\mathrm{Ad}_r|_{\mathfrak{g}^{\mathbb{C}}_\beta}\right)^2 = (-1)^{\sum_{\alpha\in\Gamma^+}\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}}\cdot\mathrm{id},$$

because $2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle$ is an integer and $(t_\alpha - 1)$ is an even number. Therefore, $\beta \in \Pi_{\mathrm{odd}}$. □

Corollary 3.7. In addition to the hypotheses of proposition 3.5 assume that $\mathfrak{g}$ is semisimple. Let $\Pi_{\mathrm{even}} = \Pi\backslash\Pi_{\mathrm{odd}}$ and choose, for each $\alpha \in \Pi_{\mathrm{odd}}$, $\epsilon_\alpha \in \{\pm 1\}$ with $\epsilon_\alpha = -\epsilon_{\tau(\alpha)}$. There exists $p \in \exp(\mathfrak{t}\cap\mathfrak{p}_1)$ such that

(1) Ad_p is equal to $(\epsilon_\alpha \mathrm{i})\cdot\mathrm{id}$ on $\mathfrak{g}^{\mathbb{C}}_\alpha$ and to the identity on $\mathfrak{g}^{\mathbb{C}}_\beta$ for all $\alpha\in\Pi_{\mathrm{odd}}$, $\beta\in\Pi_{\mathrm{even}}$,

(2) the automorphism $\nu = c_{p\exp(X)}$, where $X = \sum_{\alpha\in\Gamma^+} \mathrm{i}\pi/\langle\alpha,\alpha\rangle H_\alpha$, is an involution, and

(3) σ_2 commutes with ν and $\mathfrak{s}$ is a maximal torus of $\mathfrak{g}^\nu \cap \mathfrak{g}^{\sigma_2}$.

Proof. Choose $Y \in \mathfrak{t}$ such that $\alpha(Y) = 0$ for all $\alpha \in \Pi_{\mathrm{even}}$ and such that $\alpha(Y) = \epsilon_\alpha \mathrm{i}\pi/2$ for all roots $\alpha \in \Pi_{\mathrm{odd}}$; this is possible, because the restrictions of the elements of Π constitute a basis of $(\mathrm{i}\mathfrak{t})^*$. Then Y is necessarily contained in $\mathfrak{t}\cap\mathfrak{p}_1$, because $\alpha(Y + \tau(Y))$ vanishes for all α by choice of the integers ϵ_β, $\beta\in\Pi_{\mathrm{odd}}$. We set $p := \exp(Y)$ and observe that Ad_p indeed is equal to $(\epsilon_\alpha\mathrm{i})\cdot\mathrm{id}$ on $\mathfrak{g}^{\mathbb{C}}_\alpha$, if $\alpha\in\Pi_{\mathrm{odd}}$, and to id else. Thus, for each simple root $\alpha\in\Pi$ the maps $(\mathrm{Ad}_p)^2$ and $(\mathrm{Ad}_{\exp(X)})^2$ coincide on $\mathfrak{g}^{\mathbb{C}}_\alpha$ and are equal to id or $(-\mathrm{id})$, so $\nu = \mathrm{Ad}_{p\exp(X)}$ is an involution. Moreover, ν commutes with σ_2, because $\nu\circ\sigma_2 = \nu^{-1}\circ\sigma_2$.

Hence, it remains to show that $\mathfrak{s}$ is a maximal torus of $\mathfrak{g}^\nu\cap\mathfrak{g}^{\sigma_2}$, and to this end it suffices to verify the maximality of $\mathfrak{s}$. However, we already know that the complexification of $\mathrm{Z}_{\mathfrak{g}}(\mathfrak{s})$ is the sum of the σ_1– and σ_2–invariant subspaces $\mathfrak{t}^{\mathbb{C}}$ and $\bigoplus_{\alpha\in\Gamma}\mathfrak{g}^{\mathbb{C}}_\alpha$. By construction, Ad_p equals id on the latter space, because $\tau(\alpha) = \alpha$ for $\alpha\in\Gamma$, while $\mathrm{Ad}_{\exp(X)}$ is just $(-\mathrm{id})$ by proposition 2.2; hence ν only fixes $\mathfrak{t}$ in $\mathrm{Z}_{\mathfrak{g}}(\mathfrak{s})$, and the fixed point set of σ_2 on $\mathfrak{t}$ is precisely $\mathfrak{s}$, because $\mathfrak{t}_1 = \mathfrak{t}^\tau$. Thus, only $\mathfrak{s}$ is fixed by both ν and σ_2 in $\mathrm{Z}_{\mathfrak{g}}(\mathfrak{s})$. □

4. Normal forms for strongly orthogonal roots

4.1. Abstract normal forms

In the previous sections we learned that for a suitable choice of Cartan subalgebra the set of roots vanishing on a maximal torus of the joint fixed point subalgebra of two commuting inner involutions is strongly orthogonal and satisfies a certain involutivity condition. The purpose of this section is to establish a normal form for all sets of roots satisfying these properties.

Recall (cf. [16, p. 149]) that an *(abstract) root system* $(V,\langle\cdot,\cdot\rangle,\Delta)$ consists of a finite–dimensional Euclidean vector space $(V,\langle\cdot,\cdot\rangle)$ together with a non–empty set $\Delta\subseteq V$ of non–zero vectors such that

(1) $V = \mathrm{span}_{\mathbb{R}}\,\Delta$,

(2) for each $\alpha\in\Delta$ the reflection

$$s_\alpha : V \longrightarrow V,\ v \mapsto v - \frac{2\langle\alpha, v\rangle}{\langle\alpha,\alpha\rangle}\alpha,$$

maps Δ into itself, and

(3) the number $2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle$ is an integer whenever α and β are elements of Δ.

A root system Δ is *reduced* if $\alpha\in\Delta$ implies that $2\alpha\notin\Delta$. It is called *reducible* if there exists a non–trivial disjoint decomposition $\Delta = \Delta'\sqcup\Delta''$ such that $\langle\alpha',\alpha''\rangle = 0$ for all $\alpha'\in\Delta'$ and $\alpha''\in\Delta''$. If no such decomposition exists, then Δ is *irreducible*.

Definition 4.1. Let Δ be a root system in the Euclidean vector space $(V,\langle\cdot,\cdot\rangle)$.

(1) A pair (U,Ω) is a *root subsystem* of Δ if

   a) $\Omega \subseteq \Delta$ is non–empty,

   b) $U = \operatorname{span}_{\mathbb{R}} \Omega$, and

   c) $s_\alpha(\Omega) \subseteq \Omega$ for all $\alpha \in \Omega$.

(2) The *root subsystem of* Δ *spanned by* S, $S \subseteq \Delta$ a non–empty set, is the pair $(\operatorname{span}_{\mathbb{R}} S, \Delta \cap \operatorname{span}_{\mathbb{Z}} S)$.

Remark 4.2. Let Δ be a root system.

(1) If (U,Ω) is a root subsystem of Δ, then $(U, \langle\cdot,\cdot\rangle|_{U\times U}, \Omega)$ is a root system. If $S \subseteq \Delta$ is a non–empty subset, then the root subsystem spanned by S is a root subsytem in the sense of definition 4.1.

(2) Let (U,Ω) be a root subsystem of Δ. We can identify the Weyl group $W(\Omega)$ of Ω, which by definition is a subgroup of $\mathrm{O}(U, \langle\cdot,\cdot\rangle|_{U\times U})$, with a subgroup $W(\Omega,\Delta)$ of the Weyl group $W(\Delta)$ of Δ, where

$$W(\Omega,\Delta) := \left\{ w \in W(\Delta) \mid w = s_{\alpha_1} \circ \ldots \circ s_{\alpha_k},\ \alpha_i \in \Omega \right\} \subseteq \mathrm{O}(V, \langle\cdot,\cdot\rangle).$$

In fact, the map $p : W(\Omega,\Delta) \longrightarrow W(\Omega)$ restricting an element $w \in W(\Omega,\Delta)$ to U is a homomorphism of groups. Moreover, if $w \in W(\Omega)$, say with $w = t_{\alpha_1} \circ \ldots \circ t_{\alpha_k}$, where $\alpha_i \in \Omega$ and $t_{\alpha_i} : U \longrightarrow U$ denotes reflection along the hyperplane in U perpendicular to α_i, then

$$p(s_{\alpha_1} \circ \ldots \circ s_{\alpha_k}) = w;$$

and if $w \in \ker p$, then $w = \mathrm{id}_V$, because $p(w) = \mathrm{id}_U$ and $w(v') = v'$ for all $v' \in U^\perp$ by definition.

Recall that any choice of positive roots Δ^+ in a root system Δ determines a set of simple roots $\Pi \subseteq \Delta^+$, and that any root α can be uniquely written as $\alpha = \sum_{\beta\in\Pi} m_\beta \beta$ for integers m_β of the same sign. The number $\sum_{\beta\in\Pi} m_\beta$ is commonly referred to as the *level* of the root α.

Proposition 4.3. Let Δ be a reduced irreducible root system, $\Delta^+ \subseteq \Delta$ a choice of positive roots, and $\alpha_0 \in \Delta$. There exists a unique root δ of maximal level in the orbit $W \cdot \alpha_0$ of the Weyl group $W = W(\Delta)$, and this root satisfies $\langle \delta, \alpha \rangle \geq 0$ for all $\alpha \in \Delta^+$.

Proof. Choose any root δ of maximal level in $W \cdot \alpha_0 = \{ w(\alpha_0) \mid w \in W \}$. If $\alpha \in \Delta^+$ is a root with $\langle \delta, \alpha \rangle < 0$, then $s_\alpha(\delta)$ is a root having higher level than δ and still is contained in $W \cdot \alpha_0$, which is impossible. Therefore, we have $\langle \delta, \alpha \rangle \geq 0$ for any positive root α. In order to prove the uniqueness statement, let $\Pi \subseteq \Delta^+$ be the simple roots associated with the given choice of positivity and note that δ is positive, so we may write

$$\delta = \sum_{\alpha\in\Pi} m_\alpha \alpha,$$

with $m_\alpha \in \mathbb{Z}_{\geq 0}$. We claim that each of the integers m_α is non–zero. For if this was not the case, then $\Pi = \Pi' \cup \Pi''$ with $\Pi' = \{ \alpha \mid m_\alpha = 0 \}$ and $\Pi'' = \{ \alpha \mid m_\alpha > 0 \}$ would be a non–trivial disjoint union. Moreover, for any $\beta \in \Pi'$ we would have

$$\langle \delta, \beta \rangle = \sum_{\alpha\in\Pi''} m_\alpha \langle \alpha, \beta \rangle,$$

and the right hand side is non–positive, because the inner product of two distinct simple roots already is non–positive. By what we have just shown, $\langle \delta, \beta \rangle \geq 0$, and so $\langle \delta, \beta \rangle = 0$ and hence $\langle \alpha, \beta \rangle = 0$ would have to hold for all $\alpha \in \Pi''$ and $\beta \in \Pi'$. But this is impossible, because we are assuming Δ to be irreducible. Now let $\gamma \in W \cdot \alpha_0$ be another root of maximal level. The same line of reasoning as before also applies to γ and shows that $\gamma = \sum_{\alpha\in\Pi} n_\alpha \alpha$ for integers $n_\alpha > 0$. In particular, since there is some simple root $\beta \in \Pi$ with $\langle \delta, \beta \rangle > 0$, we also must have $\langle \delta, \gamma \rangle > 0$. Therefore, $\delta - \gamma$ is either a positive or a negative root (or 0), and since

$$\delta - \gamma = \sum_{\alpha\in\Pi} (m_\alpha - n_\alpha)\alpha,$$

it follows that $(m_\alpha - n_\alpha)_{\alpha\in\Pi}$ is either a sequence of non–negative or non–positive integers. But δ and γ have the same level, that is, $\sum_{\alpha\in\Pi} m_\alpha = \sum_{\alpha\in\Pi} n_\alpha$, and therefore $m_\alpha = n_\alpha$ for all $\alpha \in \Pi$. □

Let Δ be a reduced irreducible root system and Δ^+ a choice of positive roots. A well–known consequence of the classification of such root systems is that any two simple roots of the same length are contained in the same Weyl group orbit. On the other hand, every root is contained in the Weyl group orbit of a simple root (see [16, Proposition 2.62, sect. II.6]), so if L is the length of a root in Δ^+, then by proposition 4.3 we may unambiguously speak of the highest root (with respect to the level) of length L.

Now let $\Pi \subseteq \Delta^+$ be the simple roots and $\Gamma \subseteq \Delta$ a non–empty set of (not necessarily strongly) orthogonal roots such that $\Gamma = (-\Gamma)$. We further suppose that all elements of Γ are of the same length $L > 0$ and put $\Gamma^+ = \Gamma \cap \Delta^+$. We claim that there is a way to describe the possible elements that Γ may contain, up to application of a Weyl group element. To this end, let us introduce some notation for non–empty subsets $A \subseteq \Pi$ that we will make use of in the sequel. Given such a set A we write Δ_A to denote the root subsystem of Δ spanned by A and we put $\Delta_A^+ = \Delta_A \cap \Delta^+$, which is a notion of positivity with simple roots A. Moreover, we call A *irreducible* if Δ_A is irreducible, and refer to a non–empty subset $A' \subseteq A$ as an *irreducible component* of A if A' is maximal (with respect to inclusion) among all irreducible subsets of A. Note that A decomposes as $A = A_1 \cup \ldots \cup A_p$, where each A_i is an irreducible component of A and the members of A_i are orthogonal to A_j for all $i \neq j$. Finally, if A is irreducible and admits roots of length L, then we write $\delta(A)$ to denote the highest root of length L in Δ_A (with respect to Δ_A^+).

Next, we recursively define a family $(\mathcal{A}_i)_{i=0,\ldots,n}$ of non–empty subsets of $\mathcal{P}(\Pi)$ (the power set of Π) as follows. We put $\mathcal{A}_0 := \{\Pi\}$ and suppose that for some $k \geq 0$ the sets $\mathcal{A}_0, \ldots, \mathcal{A}_k$ are already defined. Then a non–empty subset $A \subseteq \Pi$ is contained in $\mathcal{A}_{k+1}$ if and only if

(1) Δ_A is irreducible and admits roots of length L,

(2) there exists a (possibly empty) set $B \subseteq \Pi$ whose members are orthogonal to each member of A and a set $v(A) \in \mathcal{A}_k$ such that

$$\delta(v(A))^\perp \cap v(A) = B \cup A;$$

in other words, A is an irreducible component of $\delta(v(A))^\perp \cap v(A)$ that admits roots of length L. We put $n := k$ if no such A exists and call $\mathcal{A}_0, \ldots, \mathcal{A}_n$ the *normal form tree* for (Δ, Δ^+) and L.

Remark 4.4. Closely related to the normal form tree construed above is the so–called cascade of strongly orthogonal roots defined in [18, Section 1]: indeed, if $A \in \mathcal{A}_i$ for some $i > 1$, then in the notation of [18] $\delta(A)$ is an offspring of $\delta(v(A))$. If $A_0, \ldots, A_i$ are such that $A_i \in \mathcal{A}_i$, then $\{\delta(A_0), \ldots, \delta(A_i)\}$ is called a chain cascade in [18].

Proposition 4.5. *Any two distinct sets $\mathcal{A}_i$, $\mathcal{A}_j$ are disjoint and Δ_A, $\Delta_{A'}$ are perpendicular for all $A, A' \in \mathcal{A}_k$ with $A \neq A'$. Moreover, for $A \in \mathcal{A}_{k+1}$ the element $v(A)$ is the only set in $\mathcal{A}_k$ with $A \cap v(A) \neq \emptyset$.*

Remark 4.6. Thus, we may define a graph with vertices the elements of $\mathcal{A}_0 \cup \ldots \cup \mathcal{A}_n$, where A, A' are connected by an edge if and only if $A = v(A')$. The resulting graph is a tree, hence the name.

Proof. We first show by induction on $k = 0, \ldots, n$ that $v(A)$ is the only set in $\mathcal{A}_k$ intersecting $A \in \mathcal{A}_{k+1}$ non–trivially and that $A, A' \in \mathcal{A}_k$ have non–trivial intersection only if $A' = A$. This is immediate if $k = 0$, because $\mathcal{A}_0 = \{\Pi\}$, so suppose that the induction hypothesis has been established for some natural number $k \geq 0$.

Choose $A, A' \in \mathcal{A}_{k+1}$ arbitrarily and note that by the induction assumption $v(A)$ and $v(A')$ are the unique sets in $\mathcal{A}_k$ with $A \cap v(A) \neq \emptyset$ and $A' \cap v(A') \neq \emptyset$. Hence, if $A \cap A'$ is non–empty, then, since $A \subseteq v(A)$ and $A' \subseteq v(A')$ holds by definition, also $v(A) \cap v(A')$ is non–empty, so by the induction assumption we must have $v(A) = v(A')$. The defining property of $v(A)$ is that $\delta(v(A))^{\perp} \cap v(A) = B \cup A$ holds for some subset $B \subseteq \Pi$ whose members are orthogonal to each member of A. Therefore,

$$\Delta_{A'} = \big(\Delta_{A'} \cap \operatorname{span}_{\mathbb{Z}}(A' \cap B)\big) \cup \big(\Delta_{A'} \cap \operatorname{span}_{\mathbb{Z}}(A' \cap A)\big)$$

is a decomposition into two sets whose members are mutually orthogonal, whence by irreducibility of $\Delta_{A'}$ we must have $\Delta_{A'} \subseteq \operatorname{span}_{\mathbb{Z}}(A \cap A')$. Thus, $A' \cap B$ is empty and $A' \subseteq A$. Exchanging the roles of A and A' we conclude that $A = A'$, so two sets in $\mathcal{A}_{k+1}$ intersect non–trivially only if they are equal. To finish the induction step, just note that if $A \in \mathcal{A}_{k+2}$ is arbitrary and $B \in \mathcal{A}_{k+1}$ intersects A non–trivially, then also $v(A) \cap B \neq \emptyset$, because $A \subseteq v(A)$, so by what we have just shown $B = v(A)$.

Now suppose that $A, A' \in \mathcal{A}_k$ are two distinct sets and let $j \geq 0$ be the smallest integer such that $v^{j+1}(A) = v^{j+1}(A')$. By definition we have $\delta(v^{j+1}(A))^{\perp} \cap v^{j+1}(A) = B \cup v^j(A)$ for some set B which is perpendicular to $v^j(A)$ and hence intersects $v^j(A)$ trivially. Since we just showed that $v^j(A)$ intersects $v^j(A')$ trivially as well, we conclude that $v^j(A')$ must be contained in B. Thus, $v^j(A')$ is perpendicular to $v^j(A)$, whence A and A' are perpendicular too, because $A \subseteq v^j(A)$ and $A' \subseteq v^j(A')$. Finally, suppose that A is contained in $\mathcal{A}_k \cap \mathcal{A}_{k+j}$ for integers $k \geq 0$ and $j \geq 1$. Then $v^k(A) \in \mathcal{A}_0 \cap \mathcal{A}_j$, whence $v^k(A) = \Pi$. This is impossible, however, because each element of $\mathcal{A}_j$ is a proper subset of Π. □

Corollary 4.7. For $B \in \mathcal{A}_k$, and all $A \in \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_k$ such that $A \neq B$ we have $B \subseteq \delta(A)^{\perp}$.

Proof. If $A \in \mathcal{A}_k$, the statement follows readily from proposition 4.5, so we suppose that $A \in \mathcal{A}_{k-j}$ for some $j \geq 1$. If A is different from $v^j(B)$, then even $v^j(B)$ and A are perpendicular. If A is equal to $v^j(B)$, then $v^{j-1}(B) \subseteq \delta(A)^{\perp}$ holds by definition, so $B \subseteq v^{j-1}(B)$ is perpendicular to $\delta(A)$. □

Corollary 4.8. Let $B \in \mathcal{A}_k$. Any $w \in W(B)$ permutes the members of $\{\Delta_A \mid A \in \mathcal{A}_m\}$, if $m \leq k$.

Proof. Fix some $j \geq 0$ and put $\ell := k - j$. If $A \in \mathcal{A}_\ell$ is different from $v^j(B)$, then A and $v^j(B)$ are perpendicular, whence so are A and B. Since w is a product of reflections s_α with $\alpha \in B$, w hence fixes A and Δ_A in this case. On the other hand, if $A = v^j(B)$, but $j > 0$, let $C_1, \ldots, C_p \subseteq \Pi$ be the irreducible components of $\delta(A)^{\perp} \cap A$. Note that C_i is contained in $\mathcal{A}_{\ell+1}$ if and only if Δ_{C_i} admits roots of length L, so we may further assume that for some $s \geq 1$ the sets $C_1, \ldots, C_s$ contain roots of length L, while $C_{s+1}, \ldots, C_p$ do not, and that $v^{j-1}(B) = C_1$. Now observe that the root subsystem spanned by $\delta(A)^{\perp} \cap A$ is precisely $\delta(A)^{\perp} \cap \Delta_A$. Indeed, any root $\alpha \in \Delta_A$ is a $\mathbb{Z}_{\geq 0}$– or $\mathbb{Z}_{\leq 0}$–linear combination of elements in A, so if $\langle \delta(A), \alpha \rangle = 0$, then α must be a linear combination of elements in $\delta(A)^{\perp} \cap A$, because $\langle \delta(A), \beta \rangle \geq 0$ holds for all $\beta \in A$ by proposition 4.3. Hence, we have

$$\delta(A)^{\perp} \cap \Delta_A = \Delta_{C_1} \cup \ldots \cup \Delta_{C_p}.$$

Also note that B is perpendicular to $\delta(A)$, but contained in Δ_A, so w leaves $\delta(A)^{\perp} \cap \Delta_A$ invariant. Hence, since w is an isometry and Δ_{C_i} is irreducible, we must have $w(\Delta_{C_i}) \in \{\Delta_{C_1}, \ldots, \Delta_{C_p}\}$ for each i. Moreover, if Δ_{C_i} admits roots of length L, then so does $w(\Delta_{C_i})$, whence w even permutes the set $\{\Delta_{C_1}, \ldots, \Delta_{C_s}\}$. □

Theorem 4.9. There exists a Weyl group element $w \in W(\Delta)$ such that

(1) $w(\Gamma) \cap \Delta^+ \subseteq \{\delta(A) \mid A \in \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_n\}$ and

(2) if $\delta(A)$ is contained in $w(\Gamma) \cap \Delta^+$, then either $A = \Pi$ or $\delta(v(A))$ is contained in $w(\Gamma)$.

Lemma 4.10. If $\alpha \in \Delta_A$, $A \in \mathcal{A}_k$, is perpendicular to $\delta(A)$, then $\alpha \in \Delta_{A'}$ for some irreducible component A' of $\delta(A)^{\perp} \cap A$. If in addition α is of length L, then $k < n$ and A' is contained in $\mathcal{A}_{k+1}$.

Proof. Express α as $\alpha = \sum_{\beta\in A} m_\beta \beta$ for integers $(m_\beta)_{\beta\in A}$ of the same sign. Since $\langle \alpha, \delta(A)\rangle = 0$ holds by assumption, we conclude that only those coefficients m_β with $\langle \delta(A), \beta\rangle = 0$ can be non−zero, and since α is a root, some m_β must be non−zero. Hence, $\delta(A)^\perp \cap A$ is non−empty and $\delta(A)^\perp \cap A = C_1 \cup \ldots \cup C_p$, where $C_1, \ldots, C_p$ are the irreducible components. Thus, if $\beta \in C_i$ for some i and some β with $m_\beta \neq 0$, then also $\alpha \in \Delta_{C_i}$. Moreover, if α is of length L, then Δ_{C_i} admits roots of length L, so $C_i \in \mathcal{A}_{k+1}$ and $k < n$. □

Proof of theorem 4.9. Put $\mathcal{A}_{n+1} := \emptyset$ and denote for each $k = -1, \ldots, n$ by $\delta(\mathcal{A}_{\leq k})$ the set $\{\delta(A) \,|\, A \in \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_k\}$. We inductively prove that for $k = -1, \ldots, n$ there exists an element $w \in W(\Delta)$ such that

(1) every element in $(w(\Gamma) \cap \Delta^+) \setminus \delta(\mathcal{A}_{\leq k})$ is contained in Δ_A for some $A \in \mathcal{A}_{k+1}$ and

(2) $\delta(v(A)) \in w(\Gamma)$ whenever $\alpha \in w(\Gamma) \cap \Delta_A$ for some $A \in \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{k+1}$.

For $k = -1$ the set $\delta(\mathcal{A}_{\leq k})$ is empty and $\mathcal{A}_0 = \{\Pi\}$, so we may take $w = \mathrm{id}$ in this case. Now suppose that the induction hypothesis holds for some number $k \leq n$, so there exists $w \in W(\Delta)$ verifying the two properties above. In particular, there exist elements $A_1, \ldots, A_p \in \mathcal{A}_{k+1}$ such that each element of $\Gamma' := (w(\Gamma) \cap \Delta^+) \setminus \delta(\mathcal{A}_{\leq k})$ is contained in some $\Delta_{A_1}, \ldots, \Delta_{A_p}$, and we may assume p to be the minimal number of elements required to satisfy this property. Thus, we may choose an element $\gamma_i \in \Gamma' \cap \Delta_{A_i}$ for each $i = 1, \ldots, p$. Since Δ_{A_i} is reduced and irreducible, all roots of the same length are contained in one Weyl group orbit, so there exists an element $w_i \in W(\Delta_{A_i})$ such that $w_i(\gamma_i)$ is the highest root of Δ_{A_i} having length L, that is, $w_i(\gamma_i) = \delta(A_i)$. Now consider the element $w' := w_1 \circ \ldots \circ w_p$. We know from proposition 4.5 that w' leaves each of the root systems Δ_{A_j} invariant, because each w_i is a product of root reflections s_α with $\alpha \in A_i$. The same reasoning combined with corollary 4.7 shows that w_i fixes $\delta(A_j)$ for all $i \neq j$ and also all roots in $\delta(\mathcal{A}_{\leq k})$. Hence, w' fixes the elements in $\delta(\mathcal{A}_{\leq k})$, so if we put $\tilde{w} := w' \circ w$, then the set $\tilde{w}(\Gamma) \cap \Delta^+$ fully contains $w(\Gamma) \cap \delta(\mathcal{A}_{\leq k})$ and all of the roots $\delta(A_1), \ldots, \delta(A_p)$. Moreover, each root α in $(\tilde{w}(\Gamma) \cap \Delta^+) \setminus \delta(\mathcal{A}_{\leq k+1})$ is contained in some Δ_{A_i}, because the same is true for $(w')^{-1}(\alpha) \in \Gamma'$. Since the roots in Γ are pairwise orthogonal, such an α hence is orthogonal to $\delta(A_i)$, because $w'(\gamma_i) = \delta(A_i)$, and therefore already contained in Δ_A for some $A \in \mathcal{A}_{k+2}$ by lemma 4.10; in particular, no such α exists if $k = n - 1$. It remains to verify the second property, so suppose that we are given a positive root $\alpha \in \tilde{w}(\Gamma) \cap \Delta_B$ for some $B \in \mathcal{A}_1 \cup \ldots \cup \mathcal{A}_{k+2}$. We already know from the induction assumption that either $\alpha \in \delta(\mathcal{A}_{\leq k})$ or $\alpha \in \Delta_{A_i}$ must hold, and if $\alpha \in \delta(\mathcal{A}_{\leq k})$, then B must be contained in $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$ by corollary 4.7. Since w' fixes $\delta(\mathcal{A}_{\leq k})$ pointwise, the induction statement for k shows that $\delta(v(B))$ must be contained in $\tilde{w}(\Gamma)$ if $\alpha \in \delta(\mathcal{A}_{\leq k})$. If $\alpha \in \Delta_{A_i}$ for some i and $B \in \mathcal{A}_{k+1-j}$ for some $j \geq 0$, then Δ_B and $\Delta_{v^j(A_i)}$ intersect non−trivially, hence B and $v^j(A_i)$ must be equal by proposition 4.5. Moreover, $(w')^{-1}(\alpha)$ and α both are contained in $\Delta_{v^j(A_i)}$, because w' leaves invariant Δ_{A_i}, so by corollary 4.8 $(w')^{-1}$ must leave $\Delta_{v^j(A_i)}$ and Δ_B invariant as well. Therefore, $(w')^{-1}(\alpha)$ is contained in $w(\Gamma) \cap \Delta_B$, whence by induction assumption $\delta(v(B)) \in \delta(\mathcal{A}_{\leq k})$ is contained in $w(\Gamma)$ and also $\tilde{w}(\Gamma)$. The final case to consider is that α is an element of some Δ_{A_i}, but that $B \in \mathcal{A}_{k+2}$. Then $A_i = v(B)$, and $\delta(A_i)$ is contained in $\tilde{w}(\Gamma)$ by construction. □

4.2. Normal forms for simply laced root systems

Withis this section, we fix a reduced irreducible root system Δ whose roots are all of the same length, a set of positive roots Δ^+ with corresponding simple roots Π, and a non−empty set of strongly orthogonal roots $\Gamma \subseteq \Delta$. As before, we also set $\Gamma^+ = \Gamma \cap \Delta^+$ and we additionally suppose that the integer

$$p(\alpha) := p(\Delta, \Gamma, \alpha) := \sum_{\beta\in\Gamma^+} \frac{2\langle \alpha, \beta\rangle}{\langle \beta, \beta\rangle}$$

is even for all roots α. Note that if $w \in W(\Delta)$ is arbitrary, then $p(\Delta, w(\Gamma), \alpha)$ still is even, because this number is equal to $p(w^{-1}(\alpha))$. Hence, we may use theorem 4.9 to assume that Γ^+ is contained in $\{\delta_A \,|\, A \in \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_n\}$ and that each $\delta(v(A))$ is contained in Γ whenever $\delta(A)$ is an element of Γ and $A \neq \Pi$.

Example 4.11 (Normal form for A_r). It will be convenient to associate with any reduced irreducible root system Ω with positive roots Ω^+ and simple roots Φ a *modified Dynkin diagram*. By this we shall mean the graph with vertices $\Phi \cup \{\delta_c\}_c$, where δ_c denotes the highest root of length c and c ranges over all root lengths in Ω, and whose edge set is built according to the rules of an ordinary Dynkin diagram. The resulting diagram for root systems of type A_r, $r \geq 1$, is given in figure 1. If Δ is of type A_r and we label the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$

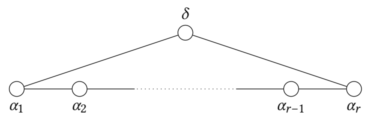

FIGURE 1. Modified Dynkin diagram for root systems of type A_r, $r \geq 1$. The highest root is $\delta = \alpha_1 + \dots + \alpha_r$.

as in figure 1, we can immediately read off the sets $\mathcal{A}_0, \dots, \mathcal{A}_n$. In fact,

$$\mathcal{A}_0 = \{\Pi\},\ \mathcal{A}_1 = \big\{\{\alpha_2, \dots, \alpha_{r-1}\}\big\},\ \dots,\ \mathcal{A}_i = \big\{\{\alpha_{i+1}, \dots, \alpha_{r-i}\}\big\},\ \dots$$

so $\Gamma^+ = \{\delta(A_0), \dots, \delta(A_q)\}$ for some $q < \lceil r/2 \rceil$, where $A_i = \{\alpha_{i+1}, \dots, \alpha_{r-i}\}$. However, the constraint $p(\alpha) \in 2\mathbb{Z}$ can only be satisfied if r is odd and $q = (r-1)/2$, for otherwise $\alpha_{q+1} - \delta(A_q)$ is a root and $p(\alpha_{q+1}) = 1$. Therefore, $r = 2k+1$ and Γ^+ is equal to $\{\delta_1, \dots, \delta_{k+1}\}$, where $\delta_i = \alpha_i + \dots + \alpha_{r-i+1}$.

Example 4.12 (Normal form for D_r). Suppose that Δ is of type D_r, $r \geq 4$, and enumerate the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ as in figure 2. We first assume that $r = 2k+1$ is odd. Then we have, for $i \geq 1$:

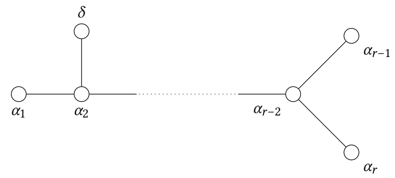

FIGURE 2. Modified Dynkin diagram for root systems of type D_r, $r \geq 4$. The highest root is $\delta = \alpha_1 + 2\alpha_2 + \dots + 2\alpha_{r-2} + \alpha_{r-1} + \alpha_r$.

$$\dots,\ \mathcal{A}_i = \big\{\{\alpha_{2i-1}\}, \{\alpha_{2i+1}, \dots, \alpha_r\}\big\},\ \dots,\ \mathcal{A}_{k-1} = \big\{\{\alpha_{2k-3}\}, \{\alpha_{2k-1}, \alpha_{2k}, \alpha_{2k+1}\}\big\},\ \mathcal{A}_k = \big\{\{\alpha_{2k-1}\}\big\}.$$

Thus, if we let $A_i = \{\alpha_i, \dots, \alpha_r\}$, then there exists a maximal integer $1 \leq m \leq k$ such that Γ^+ contains the element $\delta(A_{2m-1})$, and then Γ^+ will also contain $\delta(A_1), \delta(A_3), \dots, \delta(A_{2m-3})$, because $v(A_{2i+1}) = A_{2i-1}$. No element α_{2i-1} with $m < i \leq k$ can be contained in Γ^+, for otherwise we could choose i maximal with $\alpha_{2i-1} \in \Gamma^+$, and then α_{2i-1} is the only element of Γ^+ not perpendicular to α_{2i}, whence $p(\alpha_{2i}) = -1$. Similarly, if α_{2i-1} is contained in Γ^+ for some $1 < i$, then α_{2i-3} is contained in Γ^+ as well, for otherwise $p(\alpha_{2i-2}) = -1$ would hold. On the other hand, α_{2m-1} must be contained in Γ^+ to ensure $p(\alpha_{2m}) \in 2\mathbb{Z}$, hence Γ^+ is equal to $\{\alpha_1, \delta(A_1), \alpha_3, \delta(A_3), \dots, \alpha_{2m-1}, \delta(A_{2m-1})\}$, for some $1 \leq m \leq k$. Now suppose that $r = 2k$. This time we have

$$\mathcal{A}_i = \big\{\{\alpha_{2i-1}\}, \{\alpha_{2i+1}, \dots, \alpha_r\}\big\} \text{ for } i < k-1 \text{ and } \mathcal{A}_{k-1} = \big\{\{\alpha_{2k-3}\}, \{\alpha_{2k-1}\}, \{\alpha_{2k}\}\big\}.$$

We again let $A_i = \{\alpha_i, \dots, \alpha_r\}$ and define $1 \leq m \leq k-2$ to be the maximal integer such that Γ^+ contains $\delta(A_{2m-1})$. If $m < k-2$, then the same argument as in the case of odd rank shows that Γ^+ is equal to $\{\alpha_{2i-1}, \delta(A_{2i-1}) \mid i \leq m\}$. If $m = k-2$, then an odd (in particular non–zero) number of elements of $\{\alpha_{2k-3}, \alpha_{2k-1}, \alpha_{2k}\}$ must be contained in Γ^+, for otherwise $p(\alpha_{2k-2})$ is not even, and if α_{2k-3} is contained in Γ^+, then the same reasoning as in the previous case shows that $\alpha_1, \dots, \alpha_{2k-3}$ actually are contained in Γ^+. For later reference, let us summarize all the cases we discussed: Γ^+ is equal to one of the sets

$$\{\alpha_1, \delta_1, \alpha_3, \delta_3, \dots, \alpha_{2m-1}, \delta_{2m-1}\},\ \{\delta_1, \delta_3, \dots, \delta_{r-3}, \gamma\} \text{ or } \{\alpha_1, \delta_1, \alpha_3, \delta_3, \dots, \delta_{r-3}, \alpha_{r-1}, \alpha_r\},$$

where $2m - 1 < r - 2$, γ is either α_{r-1} or α_r, and $\delta_i = \alpha_i + 2(\alpha_{i+1} + \dots + \alpha_{r-2}) + \alpha_{r-1} + \alpha_r$; moreover, the last two cases can only occur if r is even.

Example 4.13 (Normal form for E_6)**.** We assume that Δ is of type E_6 and enumerate the simple roots as in figure 3. Note that the root subsystem spanned by $\Pi \setminus \{\alpha_6\}$ is of type A_5. Hence we can immediately deduce

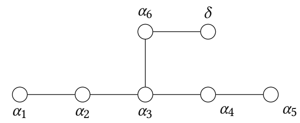

Figure 3. Modified Dynkin diagram for root systems of type E_6. The highest root is $\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$.

from example 4.11 that Γ^+ is equal to $\{\delta, \delta_1, \delta_2, \alpha_3\}$, where $\delta = \alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+2\alpha_6$, $\delta_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$, and $\delta_2 = \alpha_2 + \alpha_3 + \alpha_4$.

Example 4.14 (Normal form for E_7)**.** Suppose that Δ is of type E_7 and that the simple roots are enumerated as in figure 4. The root subsystem spanned by $\Pi \setminus \{\alpha_7\}$ is of type D_6, so Γ^+ must be the union of $\{\delta(\Pi)\}$ and one

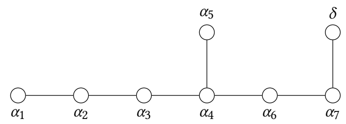

Figure 4. Modified Dynkin diagram for root systems of type E_7. The highest root is $\delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7$.

of the sets that we determined in example 4.12. However, in order for $p(\alpha_7)$ to be even, there must be an odd number of roots in $\Gamma^+ \setminus \{\delta(\Pi)\}$ which are non–perpendicular to α_7, and this only leaves the possibilities

$$\{\delta, \alpha_1, \delta_1\},\ \{\delta, \delta_1, \delta_3, \alpha_6\}, \text{ or } \{\delta, \alpha_1, \delta_1, \alpha_3, \delta_3, \alpha_5, \alpha_6\}$$

for Γ^+, where $\delta = \delta(\Pi)$ and $\delta_i = \alpha_i + 2(\alpha_{i+1} + \dots + \alpha_4) + \alpha_5 + \alpha_6$.

Example 4.15 (Normal form for E_8)**.** The case that Δ is of type E_8 can be treated similarly as in example 4.14. In fact, enumerate the simple roots as in figure 5 and observe that $\Pi \setminus \{\alpha_8\}$ spans a root subsystem of type E_7.

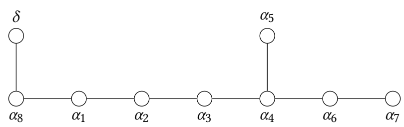

Figure 5. Modified Dynkin diagram for root systems of type E_8, $r \geq 1$. The highest root is $\delta = 3\alpha_1 + 4\alpha_2 + 5\alpha_3 + 6\alpha_4 + 3\alpha_5 + 4\alpha_6 + 2\alpha_7 + 2\alpha_8$.

So, $\Gamma^+ \setminus \{\delta\}$, where δ is the highest root of Δ^+, must contain an odd number of roots that are non–perpendicular to α_8, as otherwise $p(\alpha_8)$ would not be even, and this shows that Γ^+ is equal to

$$\{\delta, \delta', \alpha_1, \delta_1\} \text{ or } \{\delta, \delta', \alpha_1, \delta_1, \alpha_3, \delta_3, \alpha_5, \alpha_6\},$$

where $\delta' = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 2\alpha_5 + 3\alpha_6 + 2\alpha_7$ and $\delta_i = \alpha_i + 2(\alpha_{i+1} + \dots + \alpha_4) + \alpha_5 + \alpha_6$.

4.3. Normal forms for non–simply laced root systems

As in the previous section, we fix a reduced irreducible root system Δ, a set of positive roots Δ^+, and the associated simple roots Π. As already pointed out earlier, it is a consequence of the classification of irreducible root systems that Δ admits at most two different root lengths, and we assume that different root lengths do occur in Δ. The purpose of this section is to provide normal forms for sets Γ consisting of strongly orthogonal roots in Δ in case that the elements of Γ are not necessarily all of the same length. We begin with some slightly more general considerations.

Proposition 4.16. Let δ be the highest long root in Δ and suppose that $\delta^\perp \cap \Pi = A \cup A'$ for non–empty sets A, A' with the property that each element in A is orthogonal to A'. Then Δ_A and $\Delta_{A'}$ are irreducible, and there exist long roots $\alpha_0, \alpha_1 \in \Pi$ such that $\delta^\perp \cap \Pi = \Pi \setminus \{\alpha_0\}$ and such that $A = \{\alpha_1\}$ or $A' = \{\alpha_1\}$.

Proof. Express δ as $\delta = \sum_{\alpha\in\Pi} m_\alpha \alpha$ and consider the equation

$$2 = \frac{2\langle \delta, \delta\rangle}{\langle \delta, \delta\rangle} = \sum_{\alpha\in\Pi} m_\alpha \frac{2\langle \delta, \alpha\rangle}{\langle \delta, \delta\rangle};$$

it implies, due to the non–negativity of $\langle \delta, \alpha\rangle$ for each $\alpha \in \Pi$, that at most two summands in the right hand sum can be non–zero, and we first show that there is actually only one non–zero summand. Suppose that there exist $\beta_1, \beta_2 \in \Pi$ with $\langle \delta, \beta_i\rangle > 0$ for a contradiction and observe that $m_{\beta_1} = m_{\beta_2} = 1$. Now for any root $\beta \in \Pi$ we have

$$q(\beta) := \frac{2\langle \beta, \delta\rangle}{\langle \beta, \beta\rangle} = 2m_\beta + \sum_{\alpha\in\Pi\setminus\{\beta\}} m_\alpha \frac{2\langle \beta, \alpha\rangle}{\langle \beta, \beta\rangle}$$

and $q(\beta)$ is determined by the β–string containing δ. Since $\delta - 2\beta_i$ is neither a $\mathbb{Z}_{\geq 0}$– nor a $\mathbb{Z}_{\leq 0}$–linear combination of elements in Π and hence not a root, and since $m_\alpha > 0$ for all $\alpha \in \Pi$, we conclude that $q(\beta_i) = 1$ and that there is exactly one element $\gamma_i \in \Pi$ with $\langle \beta_i, \gamma_i\rangle \neq 0$. In particular, there can be no decomposition $\Pi \setminus \{\beta_1\} = B \cup B'$ for non–empty sets B, B' such that $B' \subseteq B^\perp$, for if $\gamma_1 \in B$, say, then also $\Pi = (B \cup \{\beta_1\}) \cup B'$ would be a decomposition into the orthogonal sets $B \cup \{\beta_1\}$ and B'. For the same reason $\Pi \setminus \{\beta_1, \beta_2\}$ does not admit a non–trivial orthogonal decomposition either, and this contradicts our assumptions, because

$$\Pi \setminus \{\beta_1, \beta_2\} = \delta^\perp \cap \Pi = A \cup A'.$$

Therefore, there exists exactly one root $\alpha_0 \in \Pi$ with $\langle \delta, \alpha_0\rangle > 0$ and $\delta^\perp \cap \Pi = \Pi \setminus \{\alpha_0\}$, and since δ is a long root, which implies that $2\langle \delta, \alpha_0\rangle/\langle \delta, \delta\rangle = 1$, this root has $m_{\alpha_0} = 2$. Also note that $\delta - 3\alpha_0$ is not a root, whence either $q(\alpha_0) = 1$ or $q(\alpha_0) = 2$. However, if $q(\alpha_0) = 2$ would hold, then $\delta - 2\alpha_0$ would be a root and could be expressed as $\delta - 2\alpha_0 = \beta + \beta'$ for elements $\beta \in \Delta_A \cup \{0\}$ and $\beta' \in \Delta_{A'} \cup \{0\}$. But since $m_\alpha > 0$ for all $\alpha \in \Pi$, the elements β and β' both are non–trivial, which is impossible, because A and A' are mutually orthogonal. Thus, $q(\alpha_0) = 1$ and α_0 also is a long root, so the explicit expression for $q(\alpha_0)$ given above shows that there are at most three simple roots different from α_0 which are non–perpendicular to α_0. But if there was only one simple root β with $\langle \alpha_0, \beta\rangle \neq 0$, then a similar argument as already provided earlier would show that $\Pi \setminus \{\alpha_0\}$ admits no non–trivial orthogonal decomposition. To exclude the case that there are three roots, we observe that if $\beta \in \Pi$ is a root which is different from α_0, has $m_\beta = 1$, and is non–perpendicular to α_0, then α_0 is the only root $\alpha \in \Pi \setminus \{\beta\}$ with $\langle \alpha, \beta\rangle \neq 0$, because any other such root would contribute a summand $m_\alpha 2\langle \beta, \alpha\rangle/\langle \beta, \beta\rangle < 0$ to $q(\beta)$, which is impossible because $m_{\alpha_0} = 2$ and $q(\beta) = 0$. Moreover, in this case $2\langle \beta, \alpha_0\rangle/\langle \beta, \beta\rangle$ is equal to 1, which is equivalent to saying that β is a long root. In particular, if there were three simple roots β_1, β_2, and β_3 different from α_0 satisfying $\langle \beta_i, \alpha_0\rangle \neq 0$, then necessarily $m_{\beta_i} = 1$, because $q(\alpha_0) = 1$, and α_0 would be the only root not perpendicular to β_i. Hence, $\Pi \setminus \{\alpha_0, \beta_1, \beta_2, \beta_3\}$ would not admit any non–trivial orthogonal decomposition, which by irreducibility of Δ would only be possible if $\Pi = \{\alpha_0, \beta_1, \beta_2, \beta_3\}$. But then Π would only consist of long roots and hence not admit two different root lengths. Therefore, there are exactly two roots $\alpha_1, \alpha_2 \in \Pi$ which are non–perpendicular to α_0, and if suitably enumerated they satisfy $m_{\alpha_1} = 1$ and $m_{\alpha_2} = 2$. As just observed, α_1

then is a long root and α_0 is the only root not perpendicular to α_1. Thus, $\Pi \setminus \{\alpha_1\}$ and hence also $\Pi \setminus \{\alpha_0, \alpha_1\}$ admits no non–trivial orthogonal decomposition, so either $A = \{\alpha_1\}$ and $A' = \Pi \setminus \{\alpha_0, \alpha_1\}$ or $A = \Pi \setminus \{\alpha_0, \alpha_1\}$ and $A' = \{\alpha_1\}$ has to hold. □

For the remainder of this section we fix a reduced irreducible root system Δ admitting two root lengths L_{long} and L_{short}, positive roots Δ^+, and denote by Π the simple roots. Let $\mathcal{A}_0, \dots, \mathcal{A}_n$ be the normal form tree for (Δ, Δ^+) and L_{long} constructed in section 4.1. An inductive proof using proposition 4.16 then shows that either $\mathcal{A}_k$ consists of a single element or that $\mathcal{A}_k$ contains two elements, one of which consists of a single long root. In particular, each $\mathcal{A}_k$ contains exactly one set A_k such that Δ_{A_k} might admit short roots.

Now if $\Gamma \subseteq \Delta$ with $\Gamma = (-\Gamma)$ is a non–empty subset consisting of orthogonal roots, then we may provide a normal form for Γ as follows. Let Γ_{long} and Γ_{short} be the subsets of Γ containing all long and short roots, respectively. If one of Γ_{long} or Γ_{short} is empty, then we may use theorem 4.9 to obtain a normal form for Γ. Otherwise, we may still use theorem 4.9 to assume that $\Gamma_{\text{long}} \cap \Delta^+$ is contained in $\{\delta(A) \,|\, A \in \mathcal{A}_0 \cup \dots \cup \mathcal{A}_n\}$. Note that $A_0 = \Pi$ and let $\ell \geq 0$ be the maximal integer such that Γ_{short} is contained in $\delta(A_\ell)^\perp$. Since Γ_{short} is orthogonal to $\delta(A_\ell), \delta(v(A_\ell)), \dots, \delta(v^\ell(A_\ell))$, lemma 4.10 implies that Γ_{short} is fully contained in $\Omega := \Delta_\Phi$ for some irreducible component Φ of $\delta(A_\ell)^\perp \cap A_\ell$. Observe that no element $\delta(A)$ with $A \in \mathcal{A}_{\ell+k}$ and $k \geq 2$ can be contained in Γ_{long}. In fact, if this was the case, then $v^k(A) = A_\ell$ would have to hold, because $\mathcal{A}_\ell \setminus \{A_\ell\}$ contains at most one more set and this set consists of a single root. Similarly, $v^{k-1}(A)$ cannot consist of a single element, because $v^{k-2}(A)$ is non–empty, so $v^{k-1}(A)$ must be equal to $\Phi = A_{\ell+1}$. But then $\delta(A) \in \Gamma_{\text{long}}$ would also imply $\delta(A_{\ell+1}) \in \Gamma_{\text{long}}$, and Γ_{short} would be contained in $\delta(A_{\ell+1})^\perp$ by lemma 4.10, contradicting the choice of ℓ. Consequently, as Φ is perpendicular to all elements in $\mathcal{A}_{\ell+1}$ different from Φ, it follows from corollary 4.7 that each element $w \in W(\Omega) \subseteq W(\Delta_{A_\ell})$ fixes Γ_{long} pointwise. Moreover, if $\mathcal{B}_0, \dots, \mathcal{B}_m$ is the normal form tree for (Ω, Ω^+) and L_{short}, where $\Omega^+ = \Omega \cap \Delta^+$ are the positive roots with corresponding simple roots Φ, then according to theorem 4.9 there exists $w \in W(\Omega)$ such that $w(\Gamma_{\text{short}}) \cap \Omega^+$ is contained in $\{\gamma(B) \,|\, B \in \mathcal{B}_0 \cup \dots \cup \mathcal{B}_m\}$, where $\gamma(B)$ denotes the highest short root in Ω_B. In summary, we have shown

Theorem 4.17. There exist a Weyl group element $w \in W(\Delta)$, an integer $\ell \geq 0$, a set $A_0 \in \mathcal{A}_\ell$, and an irreducible component Φ of $\delta(A_0)^\perp \cap A_0$ with the following properties: if $\mathcal{B}_0, \dots, \mathcal{B}_m \subseteq \mathcal{P}(\Phi)$ is the normal form tree for $\Omega := \Delta_\Phi$ and the short root length in Δ, then

(1) $w(\Gamma_{\text{long}}) \cap \Delta^+$ is contained in $\{\delta(A) \,|\, A \in \mathcal{A}_0 \cup \dots \cup \mathcal{A}_{\ell+1}\}$,

(2) $w(\Gamma_{\text{short}}) \cap \Delta^+$ is contained in $\{\gamma(B) \,|\, B \in \mathcal{B}_0 \cup \dots \cup \mathcal{B}_m\}$,

(3) if $\delta(A) \in w(\Gamma)$ for some $A \in \mathcal{A}_1 \cup \dots \cup \mathcal{A}_{\ell+1}$, then also $\delta(v(A)) \in w(\Gamma)$,

(4) $\delta(A_0) \in w(\Gamma)$, and if $\gamma(B) \in w(\Gamma)$ for some $B \in \mathcal{B}_1 \cup \dots \cup \mathcal{B}_m$, then also $\gamma(v(B)) \in w(\Gamma)$.

Fix a set of strongly orthogonal roots Γ with $\Gamma = (-\Gamma)$ and suppose that the integer $p(\alpha)$ introduced earlier, $\alpha \in \Pi$, is even. In the following, we explicitly determine normal forms for Γ in case that Γ satisfies the conclusions of theorem 4.9 or theorem 4.17 with $w = \mathrm{id}$.

Example 4.18 (Normal form for B_r)**.** Suppose that Δ is of type B_r, $r \geq 2$, and enumerate the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ as in figure 6. Let us further suppose that Γ only consists of short roots first. Then according to figure 6 the normal form tree $\mathcal{B}_0, \dots, \mathcal{B}_m$ for the short roots is given by $\mathcal{B}_i = \{B_i\}$, where $B_i = \{\alpha_{i+1}, \dots, \alpha_r\}$ for $i \leq r-1$, so $m = r-1$. Note, however, that the difference $\gamma(B_i) - \gamma(B_j)$ of two highest short roots with $i > j$ is a root again, but that the roots in Γ are assumed to be strongly orthogonal, which is why Γ^+ can only consist of the highest short root $\gamma(\Pi)$. Now suppose that Γ only consists of long roots and write $r = 2k+1$ or $r = 2k$. Then the normal form tree $\mathcal{A}_0, \dots, \mathcal{A}_n$ is given by

$$\mathcal{A}_i = \big\{ \{\alpha_{2i-1}\}, \{\alpha_{2i+1}, \dots, \alpha_r\} \big\}, \text{ if } 1 \leq i \leq k-1, \text{ and } \mathcal{A}_k = \big\{ \{\alpha_{2k-1}\} \big\};$$

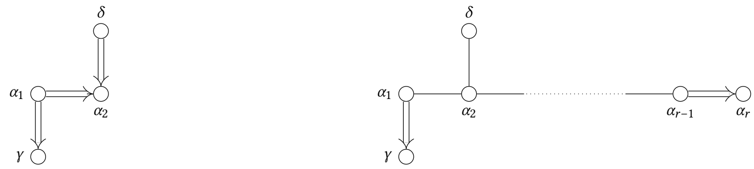

Diagram for $r = 2$. Diagram for $r \geq 3$.

FIGURE 6. Modified Dynkin diagrams for root systems of types B_r, $r \geq 2$. The highest root is $\delta = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_r$, the highest short root is $\gamma = \alpha_1 + \ldots + \alpha_r$.

note that the above formula is indeed valid in case $r = 2k+1$, because α_{2k+1} is a short root, so $\{\alpha_{2k+1}\} \notin \mathcal{A}_k$ in this case. Hence, if we put $A_i := \{\alpha_i, \ldots, \alpha_r\}$ for $1 \leq i \leq r$, then the situation is analogous to that of example 4.12, whence there exists an integer $m \leq k$ such that Γ^+ is equal to $\{\alpha_{2i-1}, \delta(A_{2i-1}) \mid i \leq m\}$. Finally, suppose that Γ contains both long and short roots. Then each element in Γ_{long} is equal to $\delta(A)$ for some $A \in \mathcal{A}_k$ and there exist $\ell \geq 0$, $A' \in \mathcal{A}_\ell$ such that each element of Γ_{short} is equal to $\gamma(B)$ for some $B \in \mathcal{B}_t$, where $\mathcal{B}_0, \ldots, \mathcal{B}_m$ now is the normal form tree for Δ_Φ and Φ is an irreducible component of $\delta(A')^\perp \cap A'$. In particular, $\Delta_{A'}$ must admit short roots, whence either $A' = A_{2i-1}$ for some i or $A' = \{\alpha_r\}$. In any case it follows that $\Gamma_{\text{short}} \cap \Delta^+$ only contains one highest short root $\gamma = \gamma(B)$ for some irreducible set $B \subseteq \Pi$. However, the integer $2\langle \gamma, \alpha \rangle / \langle \gamma, \gamma \rangle$ is even for all simple roots α, no matter if $\alpha \in B$ or not, whence the parity of $p(\Delta, \Gamma_{\text{long}}, \alpha)$ and $p(\alpha)$ is the same for all $\alpha \in \Pi$. In particular, $p(\Delta, \Gamma_{\text{long}}, \cdot)$ must be an even function, so $\Gamma_{\text{long}} \cap \Delta^+$ must be equal to $\{\alpha_1, \delta(A_1), \ldots, \alpha_{2m-1}, \delta(A_{2m-1})\}$, where $m \leq k$ and $k = \lfloor r/2 \rfloor$. Then $\Gamma_{\text{short}} \cap \Delta^+ = \{\gamma(B)\}$, with B the irreducible component of $\delta(A_{2m-1})^\perp \cap A_{2m-1}$ admitting short roots. All of these cases can be summarized as follows: Γ^+ is equal to

$$\{\gamma_1\},\ \{\alpha_1, \delta_1, \alpha_3, \delta_3, \ldots, \alpha_m, \delta_m\}, \text{ or } \{\alpha_1, \delta_1, \alpha_3, \delta_3, \ldots, \alpha_m, \delta_m, \gamma_{m+2}\},$$

where $m < r$ is an odd number, $\delta_i = \alpha_i + 2(\alpha_{i+1} + \ldots + \alpha_r)$, and $\gamma_i = \alpha_i + \ldots + \alpha_r$.

Example 4.19 (Normal form for C_r)**.** We assume that Δ is of type C_r, $r \geq 3$, and that the simple roots Π are enumerated as indicated in figure 7. The normal form tree for the long roots in Δ is given by $\mathcal{A}_i = \{A_{i+1}\}$ with

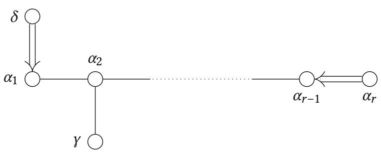

FIGURE 7. Modified Dynkin diagram for root systems of type C_r, $r \geq 3$. The highest root is $\delta = 2\alpha_1 + \ldots + 2\alpha_{r-1} + \alpha_r$, the highest short root is $\gamma = \alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{r-1} + \alpha_r$.

$A_i = \{\alpha_i, \ldots, \alpha_r\}$, because $A_{r-1} = \{\alpha_{r-1}, \alpha_r\}$ spans a root subsystem of type B_2 in Δ, with short root α_{r-1}. Hence, if Γ only contains long roots, then $\Gamma = \{\delta(A_1), \ldots, \delta(A_m)\}$ for some $m \leq r$ and to satisfy $p(\alpha_m) \in 2\mathbb{Z}$, we must have $m = r$. Next, suppose that Γ consists of short roots only and that $r = 2k+1$ or $r = 2k$. In this case the normal form tree for short roots is given by

$$\mathcal{B}_i = \Big\{ \{\alpha_{2i-1}\}, \{\alpha_{2i+1}, \ldots, \alpha_{2k+1}\} \Big\}, \text{ if } 1 \leq i \leq k-1, \text{ and } \mathcal{B}_k = \Big\{ \{\alpha_{2k-1}\} \Big\};$$

we carefully note that $\{\alpha_r\}$ is not contained in $\mathcal{B}_k$, because either $r = 2k+1$ and α_{2k+1} is a long root or $r = 2k$ and $\{\alpha_{2k-1}, \alpha_{2k}\} \in \mathcal{B}_{k-1}$ spans a root subsystem of type B_2, with short root α_{2k-1}. Put $B_i = \{\alpha_i, \ldots, \alpha_r\}$ for $i \geq 1$ and observe that r cannot be odd. In fact, if $r = 2k+1$, then the same reasoning as in example 4.12 shows the only way that the function p can be even valued is that there exists some $m \leq k$ such that $\Gamma^+ = \{\alpha_{2i-1}, \gamma(B_{2i-1}) \mid i \leq m\}$, and this contradicts our assumption that Γ consists of strongly orthogonal roots, because $\alpha_1 + \gamma(B_1) = \delta(A_1)$ is a root. Similarly, if $r = 2k$, then no root α_{2i-1} with $i \leq k$ can be contained in Γ, for then also $\gamma(B_1)$ and α_1 must be contained in Γ. This implies that $\Gamma^+ = \{\gamma(B_1), \gamma(B_3), \ldots, \gamma(B_{2k-1})\}$. Finally, suppose that Γ contains both long and short roots. Then $\Gamma_{\text{long}} = \{\delta(A_1), \ldots, \delta(A_m)\}$ for some $m < r$ and each root in Γ_{short} is the highest short root of some element in the normal form tree for the irreducible component Φ of $\delta(A_m)^\perp \cap A_m$ admitting short roots. Note, however, that $\delta(A_m)^\perp \cap A_m$ always consists of a single irreducible component. In particular, the case $m = r-1$ is excluded, because this component is equal to $\{\alpha_r\}$ and α_r is a long root. Therefore, $\Phi = \{\alpha_{m+1}, \ldots, \alpha_r\}$, whence if $m < r-2$, then $\Omega := \Delta_\Phi$ is a root subsystem of type C_{r-m} and $\langle \alpha, \delta(A_i) \rangle$ vanishes for all $\alpha \in \Phi$, $i \leq m$. Hence, $p(\Omega, \Gamma_{\text{short}}, \cdot)$ is even valued and Γ_{short} is one of the sets that we encountered above. Similarly, if $m = r-2$, then $\Phi = \{\alpha_{r-1}, \alpha_r\}$ and Ω is of type B_2, whence according to example 4.18 $\Gamma_{\text{short}} = \{\gamma(\Phi)\}$. In total, we have shown that Γ^+ equals

$$\{\delta_1, \ldots, \delta_{r-1}, \alpha_r\} \text{ or } \{\delta_1, \ldots, \delta_i, \gamma_{i+1}, \gamma_{i+3}, \ldots, \gamma_{r-1}\},$$

where $0 \leq i < r$, $r-i$ is even, $\delta_i = 2(\alpha_i + \ldots + \alpha_{r-1}) + \alpha_r$, and $\gamma_j = \alpha_j + 2(\alpha_{j+1} + \ldots + \alpha_{r-1}) + \alpha_r$ for $j < r-1$, while $\gamma_{r-1} = \alpha_{r-1} + \alpha_r$.

Example 4.20 (Normal form for F_4)**.** Suppose that Δ is of type F_4 and enumerate the simple roots as in figure 8. Let us first note that Γ cannot consist of short roots only: indeed, if γ is the highest short root of Δ and $\Gamma^+ = \{\gamma\}$,

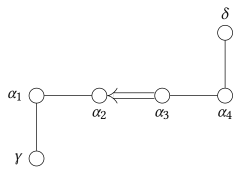

FIGURE 8. Modified Dynkin diagram for root systems of type F_4. The highest root is $\delta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$, the highest short root is $\gamma = 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4$.

then $p(\alpha_1) = 1$ is not even. Hence, there must be at least two short roots in Γ^+, and since $\gamma^\perp \cap \Pi = \{\alpha_2, \alpha_3, \alpha_4\}$ spans a root subsystem of type B_3, we must have $\Gamma^+ = \{\gamma, \gamma_2\}$, where $\gamma_2 = \alpha_2 + \alpha_3 + \alpha_4$. But then $\gamma + \gamma_2 = \delta$ is the highest long root of Δ, which is impossible, because the elements of Γ are supposed to be strongly orthogonal. Consequently, Γ contains the highest long root δ. Now $\delta^\perp \cap \Pi = \{\alpha_1, \alpha_2, \alpha_3\}$ spans a root subsystem of type C_3, so Γ^+ must be the union of $\{\delta\}$ and one of the normal forms given in example 4.19. It follows that Γ^+ is equal to $\{\delta, \delta_1, \delta_2, \alpha_3\}$ or $\{\delta, \delta_1, \gamma_2\}$, where $\delta_1 = 2(\alpha_1 + \alpha_2) + \alpha_3$, $\delta_2 = 2\alpha_2 + \alpha_3$, and $\gamma_2 = \alpha_2 + \alpha_3$ (note that $\alpha_4 + 2\gamma_2$ is a root, so $p(\alpha_4)$ is indeed even in the second case).

Example 4.21 (Normal form for G_2)**.** Suppose that Δ is of type G_2 and let the simple roots be enumerated as in figure 9. Note that the normal form tree for the long root length in Δ only is $\mathcal{A}_0 = \{\Pi\}$, because the only

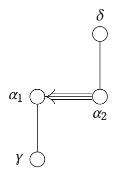

FIGURE 9. Modified Dynkin diagram for root systems of type G_2. The highest root is $\delta = 3\alpha_1 + 2\alpha_2$, the highest short root is $\gamma = 2\alpha_1 + \alpha_2$.

root which is non–perpendicular to the highest long root δ of Δ is α_1, which is a short root. Similarly, the normal form tree for the short root length in Δ is just $\mathcal{B}_0 = \{\Pi\}$, because only α_2 is non–perpendicular to the highest short root γ of Δ. Therefore, Γ must contain both a long and a short root, whence by our convention to enumerate the long roots first we must have $\Gamma^+ = \{\delta, \alpha_1\}$.

5. Cohomology of associated subalgebras

Let $\mathfrak{g}$ be a compact semisimple Lie algebra (equivalently: the Killing form on $\mathfrak{g}$ is negative–definite), $\mathfrak{t}$ a maximal torus, Δ the roots with respect to the Cartan subalgebra $\mathfrak{g}^{\mathbb{C}}$, and Δ^+ a notion of positivity. In the sequel, we frequently have to consider the *Lie subalgebra of* $\mathfrak{g}$ *associated with* Ω, where Ω is a non–empty subset of the roots Δ. By this we shall mean the smallest subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ containing the spaces $\mathfrak{g} \cap (\mathfrak{g}^{\mathbb{C}}_\alpha \oplus \mathfrak{g}^{\mathbb{C}}_{-\alpha})$ for each root $\alpha \in \Omega$. Explicitly, this subalgebra is given as the intersection of $\mathfrak{g}$ with the subalgebra

$$\sum_{\alpha\in\Delta^+\cap\mathrm{span}_{\mathbb{Z}}\Omega} \left[\mathfrak{g}^{\mathbb{C}}_\alpha, \mathfrak{g}^{\mathbb{C}}_{-\alpha}\right] \oplus \bigoplus_{\alpha\in\Delta\cap\mathrm{span}_{\mathbb{Z}}\Omega} \mathfrak{g}^{\mathbb{C}}_\alpha.$$

Note that if A is an automorphism of $\mathfrak{g}$ which leaves invariant $\mathfrak{t}$, then $A(\mathfrak{k})$ is the subalgebra associated with $(A^{-1})(\Omega) = \{\alpha\circ A^{-1} \mid \alpha \in \Omega\}$. We will almost exclusively be interested in the case that Ω is a subset of the set of all simple roots $\Pi \subseteq \Delta^+$, and we list some properties for such associated subalgebras in the following propositions. These are mostly straightforward to verify, but nonetheless, we decided to provide the proofs.

Proposition 5.1. Let $\Pi_0 \subseteq \Pi$ be a non–empty subset and $\mathfrak{k}$ the subalgebra associated with Π_0.

(1) $\mathfrak{k}$ is compact semisimple.

(2) A maximal torus for $\mathfrak{k}$ is given by

$$\mathfrak{s} = \mathfrak{g} \cap \bigoplus_{\alpha\in\Pi_0} \left[\mathfrak{g}^{\mathbb{C}}_\alpha, \mathfrak{g}^{\mathbb{C}}_{-\alpha}\right]$$

(3) Restriction to $\mathfrak{s}^{\mathbb{C}}$ induces a bijection Φ from $\Delta \cap \mathrm{span}_{\mathbb{Z}}\Pi_0$ onto the set of roots of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{s}^{\mathbb{C}}$. Moreover, $\Phi\left(\Delta^+ \cap \mathrm{span}_{\mathbb{Z}}\Pi_0\right)$ is a notion of positivity with simple roots $\Phi(\Pi_0)$.

Proof. The restrictions of the elements of Π_0 to $\mathfrak{s}^{\mathbb{C}}$ give a basis of $(\mathfrak{s}^{\mathbb{C}})^*$, so no non–trivial element of a root space $\mathfrak{g}^{\mathbb{C}}_\alpha$ with $\alpha \in \Delta \cap \mathrm{span}_{\mathbb{Z}}\,\Pi_0$ can simultaneously commute with all elements of $\mathfrak{s}$. This implies that $\mathfrak{s}$ is a maximal Abelian subspace of $\mathfrak{k}$. Consequently, the center of $\mathfrak{k}$ must be contained in $\mathfrak{s}$, and since the root space of every root in Π_0 is contained in $\mathfrak{k}^{\mathbb{C}}$, no non–trivial element of $\mathfrak{s}$ can be central. Now observe that the existence of an ad–invariant inner product on $\mathfrak{g}$ (hence $\mathfrak{k}$) implies that $\mathfrak{k}$ is semisimple and also compact.

Finally, let Δ' be the roots of $\mathfrak{k}^{\mathbb{C}}$ on $\mathfrak{s}^{\mathbb{C}}$ and $\Phi :\ \Delta \cap \mathrm{span}_{\mathbb{Z}}\,\Pi_0 \longrightarrow \Delta'$ be induced by restriction. As already noted, the elements $\Phi(\alpha)$, $\alpha \in \Pi_0$, constitute a basis of the dual of $\mathfrak{s}^{\mathbb{C}}$, so Φ is injective and, by construction of $\mathfrak{k}$, surjective. Thus $\Phi(\Pi_0)$ is a set of simple roots for the choice of positive roots $\Phi(\Delta^+ \cap \mathrm{span}_{\mathbb{Z}}\,\Pi_0)$. □

Note that the isomorphism type of $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k}$ is the subalgebra associated with a non–empty subset $\Pi_0 \subseteq \Pi$, only depends on $\mathfrak{g}$ and Π_0. In fact, suppose that $\mathfrak{h}$ is a compact semisimple Lie algebra with maximal torus $\mathfrak{s} \subseteq \mathfrak{h}$. Let Ω be the roots on $\mathfrak{s}^{\mathbb{C}}$, $\Omega^+ \subseteq \Omega$ a choice of positive roots, and $\Phi \subseteq \Omega^+$ the associated simple roots. Further suppose that $\mathfrak{m}$ is the subalgebra of $\mathfrak{h}$ associated with a non–empty subset $\Phi_0 \subseteq \Phi$ and that σ is an isomorphism between the Dynkin diagrams of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$ which satisfies $\sigma(\Pi_0) = \Phi_0$; here, we call a bijection $\sigma :\ \Pi \longrightarrow \Phi$ an *isomorphism between the Dynkin diagrams* of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{h}^{\mathbb{C}}$, or more precisely between $(\mathrm{span}_{\mathbb{R}}\Pi, \langle\cdot,\cdot\rangle)$ and $(\mathrm{span}_{\mathbb{R}}\Phi, \langle\!\langle\cdot,\cdot\rangle\!\rangle)$, if it satisfies

$$\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = \frac{2\langle\!\langle\sigma(\alpha),\sigma(\beta)\rangle\!\rangle}{\langle\!\langle\sigma(\alpha),\sigma(\alpha)\rangle\!\rangle}$$

for all simple roots $\alpha, \beta \in \Pi$ and negative–definite ad–invariant inner products $\langle\cdot,\cdot\rangle$, $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on $\mathfrak{g}$, $\mathfrak{h}$. Let $\{X_\alpha, X_{-\alpha}\}_{\alpha\in\Pi}$ be chosen such that X_α is a root vector for $\alpha \in \Pi$, $\overline{X_\alpha} = -X_{-\alpha}$, and $\langle X_\alpha, X_{-\alpha}\rangle = 1$ (see e.g. the proof of [7, Lemma 3.6]). Write $H_\alpha := [X_\alpha, X_{-\alpha}]$. Choose root vectors Y_γ, $Y_{-\gamma}$ for each $\gamma \in \Phi$ with the analogous properties and set $I_\gamma := [Y_\gamma, Y_{-\gamma}]$. According to the isomorphism theorem ([16, Theorem 2.108, sect. II.10]), the assignments $H_\alpha \mapsto I_{\sigma^{-1}\alpha}$ and $X_\alpha \mapsto Y_{\sigma-1\alpha}$, where $\alpha \in \Pi$, uniquely extend to an automorphism of complex Lie algebras $\sigma : \mathfrak{g}^{\mathbb{C}} \longrightarrow \mathfrak{h}^{\mathbb{C}}$. This automorphism necessarily maps $X_{-\alpha}$ to $Y_{-\sigma^{-1}\alpha}$, and hence satisfies

$$\sigma(\overline{X_\alpha}) = \sigma(-X_{-\alpha}) = -Y_{-\sigma^{-1}\alpha} = \overline{\sigma(X_\alpha)}.$$

Since the elements $(H_\alpha)_{\alpha\in\Pi}$ and $(I_\gamma)_{\gamma\in\Phi}$ span it and i$\mathfrak{s}$, σ maps $\mathfrak{t}$ onto $\mathfrak{s}$, and together with the relation above this implies $\sigma(\mathfrak{g}) = \mathfrak{h}$. But by construction, σ maps $\mathfrak{k}^{\mathbb{C}}$ isomorphically onto $\mathfrak{m}^{\mathbb{C}}$, and hence also has to map $\mathfrak{k}$ onto $\mathfrak{m}$. In summary, we have shown:

Proposition 5.2. The Lie algebra pairs $(\mathfrak{g}, \mathfrak{k})$ and $(\mathfrak{h}, \mathfrak{m})$ are isomorphic if and only if there is an automorphism of Dynkin diagrams that maps Π_0 onto Φ_0.

We remind the reader that a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is totally non–cohomologous to zero in $\mathfrak{g}$ if the canonical map $\mathrm{H}(\mathfrak{g}) \longrightarrow \mathrm{H}(\mathfrak{k})$ is surjective. If $\mathfrak{k}$ is compact, then, according to [11, Theorem X, sect. 10.19], the previous definition can be rephrased by saying that restriction of polynomials induces a surjection $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{k}}$ (recall that $\mathrm{A}_{\mathfrak{g}}$ and $\mathrm{A}_{\mathfrak{k}}$ are the spaces of invariant polynomials on $\mathfrak{g}$ and $\mathfrak{k}$, respectively).

Corollary 5.3. Suppose that $\mathfrak{g}$ is simple with $\mathfrak{g}^{\mathbb{C}}$ of type A_r and let $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be enumerated as in example 4.11. The subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ associated with $\alpha_i, \dots, \alpha_j$, where $1 \le i \le j \le r$, is totally non–cohomologous to zero in $\mathfrak{g}$.

Remark 5.4. The corresponding statement on the level of Lie groups, namely, that for $k \le r$ the subgroup $\mathrm{SU}(k) \subseteq \mathrm{SU}(r+1)$, embedded as a subblock, is totally non–cohomologous to zero, is well–known (see [11, Example 1, sect. 11.11], for example, and note that $\mathrm{U}(r+1)/\mathrm{U}(k) \cong \mathrm{SU}(r+1)/\mathrm{SU}(k)$ if $\mathrm{U}(k) \subseteq U(r+1)$ is embedded accordingly). The proof of corollary 5.3, which essentially establishes this correspondence, is merely included for the convenience of the reader.

Proof. For $1 \le m \le n \le r$ denote by $\mathfrak{k}_{m,n}$ the subalgebra of $\mathfrak{g}$ associated with $\{\alpha_m, \dots, \alpha_n\}$. We have a chain of inclusions

$$\mathfrak{k} = \mathfrak{k}_{i,j} \hookrightarrow \mathfrak{k}_{i,j+1} \hookrightarrow \dots \hookrightarrow \mathfrak{k}_{i,r} \hookrightarrow \mathfrak{k}_{i-1,r} \hookrightarrow \dots \hookrightarrow \mathfrak{k}_{1,r} = \mathfrak{g}$$

resulting in the chain of maps

$$\mathrm{A}_{\mathfrak{g}} = \mathrm{A}_{\mathfrak{k}_{1,r}} \longrightarrow \mathrm{A}_{\mathfrak{k}_{2,r}} \longrightarrow \dots \longrightarrow \mathrm{A}_{\mathfrak{k}_{i,r}} \longrightarrow \mathrm{A}_{\mathfrak{k}_{i,r-1}} \longrightarrow \dots \longrightarrow \mathrm{A}_{\mathfrak{k}_{i,j}} = \mathrm{A}_{\mathfrak{k}}.$$

The roots of $\mathfrak{k}_{m,n}$ with respect to the Cartan subalgebra $\bigoplus_{k=m}^{n}[\mathfrak{g}^{\mathbb{C}}_{\alpha_k}, \mathfrak{g}^{\mathbb{C}}_{-\alpha_k}]$ are exactly the restrictions of the roots $\Delta_{m,n} := \Delta \cap \mathrm{span}_{\mathbb{Z}}\{\alpha_m, \dots \alpha_n\}$, and with respect to the notion of positivity induced by $\Delta^+ \cap \Delta_{m,n}$, the simple roots are precisely the restrictions of $\alpha_m, \dots, \alpha_n$. Thus, $\mathfrak{k}_{m,n}$ is the Lie subalgebra of $\mathfrak{k}_{m,n+1}$ (and also $\mathfrak{k}_{m-1,n}$) associated with the simple roots $\{\alpha_m, \dots, \alpha_n\}$, whence in the statement of the corollary it suffices to consider the case that the difference $\mathrm{rank}(\mathfrak{g}) - \mathrm{rank}(\mathfrak{k})$ is 1, that is, the cases $i = 1, j = r - 1$ and $i = 2, j = r$. We shall treat the first case, the second case can be proven analogously.

Thus, we assume that $\mathfrak{k}$ is the subalgebra associated with the simple roots $\{\alpha_1, \dots, \alpha_{r-1}\}$. According to proposition 5.2, it will suffice to verify the statement of the corollary for a specific choice of Lie algebra of type A_r and a specific choice of Cartan subalgebra and (positive) roots. Consider the Lie algebra $\mathfrak{su}(r+1)$ and the set of all diagonal matrices $\mathfrak{s} \subseteq \mathfrak{su}(r+1)$. As is well–known, $\mathfrak{s}$ is a maximal torus, and we claim that

$$\mathfrak{h} := \left\{ \left(\begin{array}{cc} A & 0 \\ 0 & 0 \end{array} \right) \middle| \, A \in \mathfrak{su}(r) \right\},$$

is the Lie subalgebra associated with a suitable choice of simple roots of $\mathfrak{su}(r+1)$. In fact, consider the Lie algebra isomorphism

$$\Phi : \mathfrak{su}(r+1)^{\mathbb{C}} = \mathfrak{su}(r+1) \oplus \mathfrak{su}(r+1) \longrightarrow \mathfrak{sl}(r+1, \mathbb{C}), (A, B) \mapsto A + \mathrm{i}B,$$

and denote by $E_{i,j}$ the complex $(r+1)$–by–$(r+1)$ matrix with entries 1 in the (i,j)–th position and 0 everywhere else. Then Φ maps the Cartan subalgebra $\mathfrak{s}^{\mathbb{C}}$ onto the Cartan subalgebra of $\mathfrak{sl}(r+1,\mathbb{C})$ consisting of all diagonal matrices, and with respect to this choice, the roots of $\mathfrak{sl}(r+1,\mathbb{C})$ are the linear maps $\varepsilon_i - \varepsilon_j$, $i \neq j$, where $\varepsilon_1,\ldots,\varepsilon_{r+1}$ is the basis dual to the basis $E_{1,1},\ldots,E_{r+1,r+1}$, cf. [16, Example 1, sect. II.1]. If we declare the elements of the form $\varepsilon_i - \varepsilon_j$ with $i < j$ to be positive, then $\alpha_1,\ldots,\alpha_r$ with $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ are the corresponding simple roots. Moreover, since the root space of a root $\varepsilon_i - \varepsilon_j$ is exactly the subspace spanned by $E_{i,j}$, it follows that the subspace of $\mathfrak{sl}(r+1,\mathbb{C})$ associated with the simple roots $\alpha_1,\ldots,\alpha_{r-1}$ is precisely the Lie subalgebra $\mathfrak{sl}(r,\mathbb{C}) \subseteq \mathfrak{sl}(r+1,\mathbb{C})$ consisting of matrices whose last column and last row is identically zero. But Φ maps $\mathfrak{h}$ isomorphically onto $\mathfrak{sl}(r,\mathbb{C})$, and hence the subalgebra of $\mathfrak{su}(r+1)$ associated with the roots $\alpha_1\circ\Phi,\ldots,\alpha_{r-1}\circ\Phi$ is $\mathfrak{h}$. The claim now follows from the alternative characterization of surjectivity of the map $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{h}}$ given in [11, Theorem IX, sect. 10.18], because the cohomology algebra of the pair $(\mathfrak{su}(r+1),\mathfrak{h})$ is of dimension two: in fact, $\mathfrak{h}$ is the Lie algebra of the isotropy subgroup $\mathrm{SU}(r+1)_p$ of the standard action of $\mathrm{SU}(r+1)$ on the $(2r+1)$–sphere $S^{2r+1} \subseteq \mathbb{C}^{r+1}$, where $p = (0,\ldots,0,1)$ is an element of the standard basis of $\mathbb{C}^{r+1}$, so the cohomology of $(\mathfrak{su}(r+1),\mathfrak{h})$ is that of $\mathrm{SU}(r+1)/\mathrm{SU}(r+1)_p$. □

Let $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ be an automorphism of Lie algebras and $A : \Pi \longrightarrow \Pi$ an automorphism of the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$. We say that τ is *induced by* A, if τ leaves invariant $\mathfrak{t}$ and Π, and if there exists a collection of non–zero root vectors E_α for every simple root $\alpha \in \Pi$ such that $\tau(E_\alpha) = E_{A^{-1}(\alpha)}$. Note that in this case the map $\Pi \longrightarrow \Pi$, $\alpha \longmapsto \tau(\alpha)$, coincides with A and that τ is necessarily of finite order, since Π is a finite set and the root vectors $(E_\alpha)_{\alpha\in\Pi}$ together with their complex conjugates generate $\mathfrak{g}^{\mathbb{C}}$ as an algebra. In the language of [7], $\mathfrak{g}^\tau$ is a folded subalgebra, cf. [7, Proposition 3.7], and it was shown in [7, Proposition 3.5] that $\mathfrak{g}^\tau$ is compact semisimple with maximal torus $\mathfrak{s} = \mathfrak{t}^\tau$. Moreover, since τ fixes the Weyl chamber of $\mathfrak{g}^{\mathbb{C}}$ defined by the simple roots Π, a notion of positivity is obtained by declaring a root on $\mathfrak{s}^{\mathbb{C}}$ to be positive if it can be obtained by restricting a root in Δ^+. With respect to this choice of positivity, the restrictions of the roots in Π are the simple roots on $\mathfrak{s}^{\mathbb{C}}$.

Proposition 5.5. Let $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ be an automorphism induced by an automorphism of the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ and write $\mathfrak{h} := \mathfrak{g}^\tau$, $\mathfrak{s} := \mathfrak{t}^\tau$. Suppose that $\Pi_0 \subseteq \Pi$ is a non–empty subset satisfying $\tau(\Pi_0) = \Pi_0$ and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ associated with Π_0. Then $\mathfrak{k}$ is τ–invariant and with respect to the restricted roots

$$\Pi|_{\mathfrak{s}^{\mathbb{C}}} := \left\{ \alpha|_{\mathfrak{s}^{\mathbb{C}}} \mid \alpha \in \Pi \right\}$$

the subalgebra $\mathfrak{f}$ of $\mathfrak{h}$ associated with $\Pi_0|_{\mathfrak{s}^{\mathbb{C}}} \subseteq \Pi|_{\mathfrak{s}^{\mathbb{C}}}$ coincides with the fixed point subalgebra $\mathfrak{m} := \mathfrak{k}^\tau$.

Proof. Put $\Delta_0 := \Delta \cap \mathrm{span}_{\mathbb{Z}}\Pi_0$. In order to prove the statement, it suffices to consider the τ–invariant spaces

$$V_\alpha := \sum_{i\leq 0} \left[\mathfrak{g}^{\mathbb{C}}_{\tau^i(\alpha)}, \mathfrak{g}^{\mathbb{C}}_{-\tau^i(\alpha)}\right] \text{ and } W_\beta := \sum_{j\leq 0} \mathfrak{g}^{\mathbb{C}}_{\tau^j(\beta)}$$

for roots $\alpha \in \Pi_0$ and $\beta \in \Delta_0$, since $\mathfrak{k}^{\mathbb{C}}$ is a sum of such V_α and W_β.

We first show $\mathfrak{m} \subseteq \mathfrak{f}$. To this end, choose $\beta \in \Delta_0$ and suppose that a non–zero vector $X \in W_\beta$ is being fixed by τ. For any element $T \in \mathfrak{s}$ we then have $[T,X] = \beta(T)X$, whence $\tilde{\beta}$, the restriction of β to $\mathfrak{s}^{\mathbb{C}}$, is a root of $\mathfrak{h}^{\mathbb{C}}$. Because β is an element of Δ_0, $\tilde{\beta}$ must be contained in $\mathrm{span}_{\mathbb{Z}}\,\Pi_0|_{\mathfrak{s}^{\mathbb{C}}}$, and therefore

$$X \in \mathfrak{h}^{\mathbb{C}}_{\tilde{\beta}} \subseteq \mathfrak{f}^{\mathbb{C}}.$$

To conclude that $\mathfrak{m}$ is contained in $\mathfrak{f}$, recall that there exist root vectors $E_\alpha \in \mathfrak{g}^{\mathbb{C}}_\alpha$ for all $\alpha \in \Pi$ with the property that $\tau(E_\alpha) = E_{\tau^{-1}\alpha}$. If $k \geq 0$ is the smallest integer with $\tau^{k+1}(\alpha) = \alpha$, $\alpha \in \Pi_0$, then, since each root space is one–dimensional, the elements $[E_\alpha, \overline{E_\alpha}]$, $\tau[E_\alpha, \overline{E_\alpha}],\ldots,\tau^k[E_\alpha,\overline{E_\alpha}]$ hence constitute a basis of V_α; in particular, τ is of order k on V_α. It follows that the fixed point set of τ on V_α is one–dimensional, spanned by the non–zero vector $\sum_{i=0}^k \tau^i[E_\alpha,\overline{E_\alpha}]$. However, $\tilde{\alpha}$ is a root of $\mathfrak{h}^{\mathbb{C}}$ with non–zero root vector

$$X_{\tilde{\alpha}} := E_\alpha + \tau\left(E_\alpha\right) + \ldots + \tau^k\left(E_\alpha\right),$$

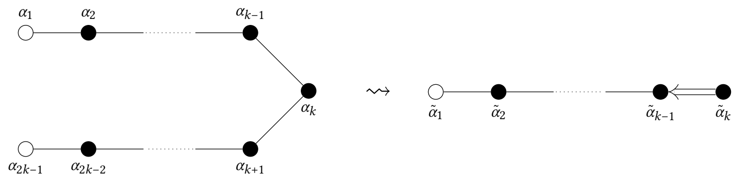

FIGURE 10. Folding a Lie algebra of type A_{2k-1}. Black nodes indicate the roots of associated subalgebras.

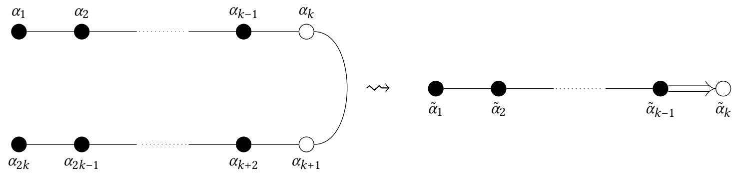

FIGURE 11. Folding a Lie algebra of type A_{2k}. Black nodes indicate the roots of associated subalgebras.

and the difference of the simple roots $\tau^i(\alpha) - \tau^j(\alpha)$ is never a root. Therefore,

$$[X_{\tilde\alpha}, \overline{X_{\tilde\alpha}}] = \sum_{i,j=0}^{k} [\tau^i(E_\alpha), \overline{\tau^j(E_\alpha)}] = \sum_{i=0}^{k} \tau^i[E_\alpha, \overline{E_\alpha}]$$

is an element of $\mathfrak{f}^{\mathbb{C}}$, and $\mathfrak{m} \subseteq \mathfrak{f}$. For the converse inclusion, note that $X_{\tilde\alpha}$ and hence $\mathfrak{h}^{\mathbb{C}}_{\tilde\alpha}$ is contained in $\mathfrak{m}^{\mathbb{C}}$. Therefore, $\mathfrak{m}$ is a subalgebra of $\mathfrak{h}$ that contains the spaces $\mathfrak{h} \cap (\mathfrak{h}^{\mathbb{C}}_{\alpha} \oplus \mathfrak{h}^{\mathbb{C}}_{-\alpha})$, so we have $\mathfrak{f} \subseteq \mathfrak{m}$ by definition. □

Example 5.6. Let us assume that $\mathfrak{g}$ is simple and that $\mathfrak{g}^{\mathbb{C}}$ is of type A_r, $r \geq 2$. Then there is only one non–trivial automorphism τ on the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$. Explicitly, if $\Pi = \{\alpha_1, \dots, \alpha_r\}$ is enumerated as in example 4.11, then this automorphism is given by $\tau(\alpha_i) = \alpha_{r-(i-1)}$, and arguing as in the proof of proposition 5.2, we may extend τ to an automorphism $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$. If r is odd, say $r = 2k-1$, then the complexfication of the fixed point set $\mathfrak{h}$ of $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ is of type C_k with simple roots $\tilde\alpha_1, \dots, \tilde\alpha_k$ and long root $\tilde\alpha_k$ (cf. [7, Lemma 5.2]), where we write $\tilde\alpha_i$ to denote the restriction of α_i to the complexification of the maximal torus $\mathfrak{s} = \mathfrak{t}^\sigma$ of $\mathfrak{h}$. If we let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ associated with the simple roots $\Pi_0 = \Pi \setminus \{\alpha_1, \alpha_r\}$, then according to proposition 5.5 the fixed point set of τ on $\mathfrak{k}$ is the Lie subalgebra $\mathfrak{f}$ associated with the simple roots $\tilde\alpha_2, \dots, \tilde\alpha_k$; its complexification is a Lie algebra of type C_{k-1}. The situation is visualized in figure 10.

If r is even, with $r = 2k$, then the fixed point set is of type B_k. The simple roots of $\mathfrak{h} = \mathfrak{g}^\tau$ are again given by $\tilde\alpha_1, \dots, \tilde\alpha_k$, and this time $\tilde\alpha_k$ is the short simple root. The fixed point set of τ on the subalgebra of $\mathfrak{g}$ associated with the simple roots $\Pi_0 = \Pi \setminus \{\alpha_k, \alpha_{k+1}\}$ is the subalgebra $\mathfrak{f}$ of $\mathfrak{h}$ associated with the simple roots $\tilde\alpha_1, \dots, \tilde\alpha_{k-1}$. Here, $\mathfrak{f}^{\mathbb{C}}$ is of type A_{k-1}, cf. figure 11.

Corollary 5.7. Suppose that $\mathfrak{g}^{\mathbb{C}}$ is of type B_r ($r \geq 2$) or C_r ($r \geq 3$) and let the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be enumerated as in example 4.18 or example 4.19, respectively. The subalgebra of $\mathfrak{g}$ associated with the simple roots $\alpha_i, \alpha_{i+1}, \dots, \alpha_r$, where $1 \leq i \leq r$, is totally non–cohomologous to zero in $\mathfrak{g}$.

PROOF. It will suffice to consider an arbitrary Lie algebra whose complexification is of type C_r or B_r, and it will also suffice to consider the case $i = 2$, cf. proposition 5.2 and the proof of corollary 5.3.

Let $\mathfrak{n} = \mathfrak{su}(2r)$ be the compact Lie algebra whose complexification is of type A_{2r-1} and choose a maximal torus $\mathfrak{s} \subseteq \mathfrak{n}$, a set of roots Ω, and positive roots Ω^+. Then the fixed point subalgebra $\mathfrak{h}$ of an automorphism $\tau : \mathfrak{n} \longrightarrow \mathfrak{n}$ induced by the non–trivial automorphism of the Dynkin diagram of $\mathfrak{n}^{\mathbb{C}}$ is of type C_r; if instead we start with the compact Lie algebra $\mathfrak{n}$ whose complexfication is of type A_{2r}, then $\mathfrak{h}^{\mathbb{C}}$ is of type B_r. Moreover, if we enumerate the simple roots $\beta_1, \dots, \beta_{2r-1}$ (respectively $\beta_1, \dots, \beta_{2r}$) as in example 5.6 and denote by $\tilde\beta_i$ the restriction of β_i to the complexfication of $\mathfrak{s}^\tau$, then $\{\tilde\beta_1, \dots, \tilde\beta_r\}$ is a set of simple roots for $\mathfrak{h}^{\mathbb{C}}$, enumerated as in example 4.19 (or example 4.18). The subalgebra $\mathfrak{m}$ of $\mathfrak{n}$ associated with the simple roots $\beta_2, \dots, \beta_{2r-2}$ (respectively with $\beta_2, \dots, \beta_{2r-1}$ in case that we are considering a Lie algebra of type A_{2r}) is τ−invariant and $\mathfrak{f} = \mathfrak{m}^\tau$ is the subalgebra of $\mathfrak{h}$ associated with the simple roots $\tilde{\beta}_2, \dots, \tilde{\beta}_r$. We obtain a commutative diagram

$$\begin{tikzcd} \mathrm{A}_{\mathfrak{n}} \arrow[r] \arrow[d] & \mathrm{A}_{\mathfrak{h}} \arrow[d] \\ \mathrm{A}_{\mathfrak{m}} \arrow[r] & \mathrm{A}_{\mathfrak{f}}, \end{tikzcd}$$

with all maps induced by canonical inclusions, so it remains to verify surjectivity of the right hand vertical map. But the left hand vertical map is surjective by corollary 5.3 and the lower horizontal map is surjective, because $\mathfrak{m}$ is compact semisimple and $\mathfrak{f}$ is the fixed point subalgebra of an automorphism induced by an automorphism of the Dynkin diagram of $\mathfrak{m}^{\mathbb{C}}$, cf. [7, Proposition 4.6]. Hence, $\mathrm{A}_{\mathfrak{h}} \longrightarrow \mathrm{A}_{\mathfrak{f}}$ is surjective too. □

Suppose that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$ is a decomposition of $\mathfrak{g}$ into two ideals. As was already noted in the proof of theorem 1.2, the diagonal embedding $\Delta(\mathfrak{k}) \subseteq \mathfrak{g}$ then is totally non−cohomologous to zero in $\mathfrak{g}$: indeed, there is a canonical isomorphism $\mathrm{A}_{\mathfrak{k}} \otimes \mathrm{A}_{\mathfrak{k}} \longrightarrow \mathrm{A}_{\mathfrak{k}\oplus\mathfrak{k}}$, induced by the projections $\mathfrak{k} \oplus \mathfrak{k} \longrightarrow \mathfrak{k} \oplus 0$ and $\mathfrak{k} \oplus \mathfrak{k} \longrightarrow \mathfrak{k} \oplus 0$, which takes $\mathrm{A}_{\Delta(\mathfrak{k})}$ onto $\Delta(\mathrm{A}_{\mathfrak{k}})$, and the restriction map $\mathrm{A}_{\mathfrak{k}} \otimes \mathrm{A}_{\mathfrak{k}} \longrightarrow \Delta(\mathrm{A}_{\mathfrak{k}})$ is surjective. The next proposition generalizes this observation to cases where we do not have a global decomposition of $\mathfrak{g}$ (note that $\Delta(\mathfrak{k})$ is the fixed point set of the involution exchanging the two summands of $\mathfrak{g}$).

Proposition 5.8. Let σ be an involutive Lie algebra automorphism of $\mathfrak{g}$ and suppose that $\mathfrak{k} \subseteq \mathfrak{g}$ is a Lie subalgebra, invariant under σ. Further suppose that $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ is a decomposition of $\mathfrak{k}$ into two ideals with $\sigma(\mathfrak{k}_1) = \mathfrak{k}_2$ and let $\mathfrak{h}$ be the fixed point subalgebra of σ on $\mathfrak{k}$.

(1) Let $I \subseteq \mathrm{A}_{\mathfrak{g}}$ be the graded subspace consisting of all polynomials with $\sigma^*(f) = f$. If the map $I \longrightarrow \mathrm{A}_{\mathfrak{k}_1}$, $f \mapsto f|_{\mathfrak{k}_1}$, is a surjection, then so is $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{h}}, f \mapsto f|_{\mathfrak{h}}$.

(2) Let $J_{\mathfrak{k}_1} \subseteq \mathrm{A}_{\mathfrak{k}_1}$ and $J_{\mathfrak{h}} \subseteq \mathrm{A}_{\mathfrak{h}}$ be the ideals generated by all polynomials of odd degree. If $I \longrightarrow \mathrm{A}_{\mathfrak{k}_1}/J_{\mathfrak{k}_1}$, $f \mapsto f|_{\mathfrak{k}_1} + J_{\mathfrak{k}_1}$, is a surjection, then so is $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{h}}/J_{\mathfrak{h}}, f \mapsto f|_{\mathfrak{h}} + J_{\mathfrak{h}}$.

Proof.

(1) For $j = 1, 2$ consider the linear isomorphisms

$$\Phi_j : \mathfrak{k}_j \longrightarrow \mathfrak{h}, X \mapsto X + \sigma(X).$$

Since $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$ as Lie algebras and σ maps $\mathfrak{k}_1$ onto $\mathfrak{k}_2$ and vice versa, these are actually homomorphisms of Lie algebras. Consequently, they induce isomorphisms $\Phi_j^* : A_{\mathfrak{h}} \longrightarrow A_{\mathfrak{k}_j}$. Now let $p_1 : \mathfrak{k} \longrightarrow \mathfrak{k}_1$ and $p_2 : \mathfrak{k} \longrightarrow \mathfrak{k}_2$ denote the projections with kernels $\mathfrak{k}_2$ and $\mathfrak{k}_1$, respectively. If $X \in \mathfrak{k}$ is fixed by σ, then $\sigma(p_1(X)) = p_2(X)$ and $\sigma(p_2(X)) = p_1(X)$. Hence, if $g \in \mathrm{A}_{\mathfrak{h}}$ is homogeneous, then

$$p_j^* \left(\Phi_j^*(g)\right)\Big|_{\mathfrak{h}}(X) = g\big(p_j(X) + \sigma(p_j(X))\big) = g\big(p_1(X) + p_2(X)\big) = g(X)$$

and $p_j^* \left(\Phi_j^*(g)\right)$ restricts to g. By assumption, we find a homogeneous polynomial $f \in I$ with the property that $f|_{\mathfrak{k}_1} = \Phi_1^*(g)$, and then, for all $X \in \mathfrak{k}_2$:

$$(\sigma^*(f))|_{\mathfrak{k}_2}(X) = f(\sigma(X)) = f|_{\mathfrak{k}_1}(\sigma(X)) = \Phi_1^*(g)(\sigma(X)) = g(X + \sigma(X)) = \Phi_2^*(g)(X).$$

But f is fixed by σ^*, and so $f|_{\mathfrak{k}_2} = \Phi_2^*(g)$. Since $\mathrm{A}_{\mathfrak{k}_1} \otimes \mathrm{A}_{\mathfrak{k}_2}$ is isomorphic to $\mathrm{A}_{\mathfrak{k}}$ via the map sending $f_1 \otimes f_2$ to $(p_1^*(f_1)) \cdot (p_2^*(f_2))$, we thus find that

$$f|_{\mathfrak{k}} = p_1^* \left(\Phi_1^*(g)\right) + q + p_2^* \left(\Phi_2^*(g)\right),$$

where $q \in A_{\mathfrak{k}}$ is a polynomial in the graded subspace generated by the set $p_1^*(A_{\mathfrak{k}_1}^+) \cdot p_2^*(A_{\mathfrak{k}_2}^+)$ consisting of products of polynomials without constant term. Thus, it follows that

$$f|_{\mathfrak{h}} = g + q|_{\mathfrak{h}} + g = 2g + q|_{\mathfrak{h}}\,.$$

In particular, if g is of degree 1, then necessarily $q = 0$ and g is in the image of the restriction map $A_{\mathfrak{g}} \longrightarrow A_{\mathfrak{h}}$. Proceeding by induction on the degree of g, we see that $A_{\mathfrak{g}} \longrightarrow A_{\mathfrak{h}}$ is surjective.

(2) We retain the notation of the previous item. Then, if $g \in A_{\mathfrak{h}}$ is a homogeneous polynomial, there exists $f \in I$ with $f|_{\mathfrak{k}_1} = \Phi_1^*(g) + p$ for some homogeneous polynomial $p \in J_{\mathfrak{k}_1}$. Arguing analogously as in the previous case, we find that

$$f|_{\mathfrak{k}_2} = (\sigma^*(f))\,|_{\mathfrak{k}_2} = \Phi_2^*(f) + \tilde{p},$$

where $\tilde{p} = (\sigma|_{\mathfrak{k}_2})^*(p)$ is a homogeneous polynomial in $J_{\mathfrak{k}_2}$, the ideal generated by all polnyomials of odd degree. Thus, we still have

$$f|_{\mathfrak{k}} + J_{\mathfrak{k}} = p_1^*(\Phi_1^*(g)) + q + p_2^*(\Phi_2^*(g) + J_{\mathfrak{k}},$$

where $q \in p_1^*(A_{\mathfrak{k}_1}^+) \cdot p_2^*(A_{\mathfrak{k}_2}^+)$ and $J_{\mathfrak{k}} \subseteq A_{\mathfrak{k}}$ again is the ideal generated by all odd degree polynomials. We conclude that

$$f|_{\mathfrak{h}} + J_{\mathfrak{h}} = 2g + q|_{\mathfrak{h}} + J_{\mathfrak{h}},$$

and, as in the proof of the previous item, that $A_{\mathfrak{g}} \longrightarrow A_{\mathfrak{h}} \longrightarrow A_{\mathfrak{h}}/J_{\mathfrak{h}}$ is surjective. □

Let us recall in passing some facts from [11]. If $\mathfrak{k} \subseteq \mathfrak{g}$ is a compact subalgebra, totally non–cohomologous to zero in $\mathfrak{g}$, then according to [11, Proposition VII, sect. 6.11] we have a commutative diagram

$$\begin{tikzcd} P_{\mathfrak{g}} \arrow[r, "\rho_{\mathfrak{g}}"] \arrow[d] & \mathrm{A}_{\mathfrak{g}} \arrow[d] \\ P_{\mathfrak{k}} \arrow[r, "\rho_{\mathfrak{k}}"] & \mathrm{A}_{\mathfrak{k}}; \end{tikzcd}$$

here, $P_{\mathfrak{g}} \subseteq \Omega(\mathfrak{g})^{\mathfrak{g}}$ and $P_{\mathfrak{k}} \subseteq \Omega(\mathfrak{k})^{\mathfrak{k}}$ are the primitive subspaces, the vertical maps are induced by the canonical inclusions, and the maps $\rho_{\mathfrak{g}}$, $\rho_{\mathfrak{k}}$ are the ("distinguished") transgressions, cf. [11, Definition, sect. 6.10]. By [11, Theorem X, sect. 10.19], the kernel of the left hand vertical map is exactly the Samelson subspace of the pair $(\mathfrak{g}, \mathfrak{k})$, so if $v_1, \dots, v_r$ is any homogeneous basis of $P_{\mathfrak{g}}$ such that $v_{s+1}, \dots, v_r$ is a basis of the kernel of $P_{\mathfrak{g}} \longrightarrow P_{\mathfrak{k}}$, then necessarily $s = \operatorname{rank}(\mathfrak{k})$ and the images of $w_1, \dots w_s$ of the elements $v_1, \dots, v_s$ form a homogeneous basis of $P_{\mathfrak{k}}$. Note that we are considering $\mathfrak{g}^* \subseteq \mathrm{S}(\mathfrak{g}^*)$ as concentrated in degree 1, so $\rho_{\mathfrak{g}}$ maps a homogeneous primitive element of degree k onto a homogeneous polynomial of degree $(k+1)/2$. Thus, if we put $x_i := \rho_{\mathfrak{g}}(v_i)$ and $y_j := \rho_{\mathfrak{k}}(w_j)$, then the canonical inclusions extend to isomorphisms of graded algebras $\mathbb{R}[x_1, \dots, x_r] \cong \mathrm{A}_{\mathfrak{g}}$ and $\mathbb{R}[y_1, \dots, y_s] \cong \mathrm{A}_{\mathfrak{k}}$ [11, Theorem I, sect. 6.13]; they fit into the commutative diagram

$$\begin{tikzcd} \mathrm{A}_{\mathfrak{g}} \arrow[r] & \mathrm{A}_{\mathfrak{k}} \\ \mathbb{R}[x_1, \dots, x_r] \arrow[r] \arrow[u, "\cong"] & \mathbb{R}[y_1, \dots, y_s], \arrow[u, "\cong"'] \end{tikzcd}$$

where the lower vertical map sends x_i to y_i if $i \leq s$ and $x_{s+1}, \dots, x_r$ to zero.

These observations in particular apply if $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an automorphism induced by an automorphism of the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k} = \mathfrak{g}^\tau$ is its fixed point set, because $\mathfrak{k}$ is a folded subalgebra and hence totally non–cohomologous to zero in $\mathfrak{g}$ by [7, Proposition 4.6]. Moreover, if τ is actually an involution, then we may choose $v_1, \dots, v_r$ to be a basis consisting of eigenvectors of $\tau^* : P_\mathfrak{g} \longrightarrow P_\mathfrak{g}$: in fact, given a form ω on $\mathfrak{g}$ the restriction of $\omega + \tau^*(\omega)$ to $\mathfrak{k}$ coincides with 2ω, so the kernel of $P_\mathfrak{g} \longrightarrow P_\mathfrak{k}$ is τ^* invariant and its image is spanned by all elements in the 1–eigenspace of τ^*. According to [11, Proposition VII, sect. 10.26] the elements $v_{s+1}, \dots, v_r$ constitute a basis of the (-1)–eigenspace of τ^* and hence $v_1, \dots, v_s$ must be a basis of its 1–eigenspace. Because of $\tau^* \circ \rho_\mathfrak{g} = \rho_\mathfrak{g} \circ \tau^*$ (see [11, Proposition VII, sect. 6.11]), the kernel of $\mathrm{A}_\mathfrak{g} \longrightarrow \mathrm{A}_\mathfrak{k}$ hence coincides with the ideal generated by the (-1)–eigenspace of $\tau^* : \mathrm{A}_\mathfrak{g} \longrightarrow \mathrm{A}_\mathfrak{g}$.

Specializing even further, suppose that $\mathfrak{g}^{\mathbb{C}}$ is of type A_r and that τ is induced by the non trivial automorphism of the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$. As is well–known (see e.g.[15, Proposition, sect. 3.7]), the degrees of any set of basic invariants of a simple Lie group, that is, the degrees of any set of algebraically independent generators of the invariant polynomials, are uniquely determined, up to permutation. For $\mathfrak{g}$ and $\mathfrak{k}$ the sets of degrees of basic invariants are given by $\{2, 3, \dots, r + 1\}$ and $\{2, 4, \dots, 2s\}$, respectively, see [15, Table 1, sect. 3.7]. In particular, the elements $x_{s+1}, \dots, x_r$ must be of odd degree and the kernel of the map $\mathrm{A}_\mathfrak{g} \longrightarrow \mathrm{A}_\mathfrak{k}$ is the ideal in $\mathrm{A}_\mathfrak{g}$ generated by all polynomials of odd degree (this is actually how surjectivity of the map $\mathrm{A}_\mathfrak{g} \longrightarrow \mathrm{A}_\mathfrak{k}$ was concluded in [7, Theorem 5.5]). Combined with proposition 5.8, this leads to the following

Corollary 5.9. Suppose that $\mathfrak{g}^{\mathbb{C}}$ is of type B_r ($r \geq 2$) or C_r ($r \geq 3$) and let the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be enumerated as in example 4.18 or example 4.19, respectively. If $\mathfrak{k}$ is the subalgebra of $\mathfrak{g}$ associated with the simple roots $\alpha_1, \dots, \alpha_{r-1}$, then the inclusion induced map $\mathrm{A}_\mathfrak{g} \longrightarrow \mathrm{A}_\mathfrak{k}/J$, where $J \subseteq \mathrm{A}_\mathfrak{k}$ is the ideal generated by all polynomials of odd degree, is surjective.

Proof. To clarify the exposition, we only consider the case that $\mathfrak{g}^{\mathbb{C}}$ is of type C_r, the proof in case that $\mathfrak{g}^{\mathbb{C}}$ is of type B_r only requires minor modifications. Recall from proposition 5.2 that it suffices to verify the statement for an arbitrary Lie algebra whose complexfication is of type C_r. We shall make use of this fact and proceed as in the proof of corollary 5.7: let $\mathfrak{n}$ be a Lie algebra such that $\mathfrak{n}^{\mathbb{C}}$ is of type A_{2r-1}, fix a maximal torus $\mathfrak{b}$ in $\mathfrak{n}$, a choice of positivity Ω^+ for the roots Ω on $\mathfrak{b}^{\mathbb{C}}$, and let Φ be the simple roots. Further suppose that $\mathfrak{h}$ is the fixed point set of an involution τ on $\mathfrak{n}$ which is induced by the non–trivial automorphism on the Dynkin diagram of $\mathfrak{n}^{\mathbb{C}}$. Then $\mathfrak{h}^{\mathbb{C}}$ is of type C_r, and if we enumerate the simple roots $\Phi = \{\beta_1, \dots, \beta_{2r-1}\}$ as in example 4.11 and denote the restriction of β_i to the complexfication of $\mathfrak{b}^\tau$ by $\tilde{\beta}_i$, then $\tilde{\beta}_1, \dots, \tilde{\beta}_r$ are simple roots for the notion of positivity induced by Ω^+, enumerated as in example 4.19.

Now consider the subalgebra $\mathfrak{m}$ of $\mathfrak{n}$ associated with $\Pi \setminus \{\beta_r\}$. It decomposes as $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where the ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$ of $\mathfrak{m}$ are the subalgebras of $\mathfrak{n}$ associated with the simple roots $\{\beta_1, \dots, \beta_{r-1}\}$ and $\{\beta_{r+1}, \dots, \beta_{2r-1}\}$, respectively. Note that the fixed point subalgebra $\mathfrak{f} = \mathfrak{m}^\tau$ is the subalgebra of $\mathfrak{h}$ associated with the simple roots $\tilde{\beta}_1, \dots, \tilde{\beta}_{r-1}$ and that τ maps $\mathfrak{m}_1$ onto $\mathfrak{m}_2$. The claim thus follows from proposition 5.8 once we show that the polynomials in the 1–eigenspace E of $\tau^* : \mathrm{A}_\mathfrak{n} \longrightarrow \mathrm{A}_\mathfrak{n}$ surject onto $\mathrm{A}_{\mathfrak{m}_1}/I_{\mathfrak{m}_1}$, where $I_{\mathfrak{m}_1} \subseteq \mathrm{A}_{\mathfrak{m}_1}$ denotes the ideal generated by all polynomials of odd degree, because the map $\mathrm{A}_\mathfrak{n} \longrightarrow \mathrm{A}_\mathfrak{f}$ factors through $\mathrm{A}_\mathfrak{h} \longrightarrow \mathrm{A}_\mathfrak{f}$. But in the paragraph preceding this corollary we have observed that E surjects onto $\mathrm{A}_\mathfrak{n}/I_\mathfrak{n}$, where $I_\mathfrak{n}$ is the ideal generated by all polynomials of odd degree, and by corollary 5.3 the canonical map $\mathrm{A}_\mathfrak{n} \longrightarrow \mathrm{A}_{\mathfrak{m}_1}$ is surjective. Thus, the map $E \longrightarrow \mathrm{A}_{\mathfrak{m}_1}/I_{\mathfrak{m}_1}$ must be surjective too. □

Another fact from [11], which enters the next corollary and also proposition 5.12, concerns the Samelson subspace $P \subseteq P_\mathfrak{g}$ of $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k}$ is a compact subalgebra of $\mathfrak{g}$. Denote by $\rho_\mathfrak{g} : P_\mathfrak{g} \longrightarrow \mathrm{A}_\mathfrak{g}$ the transgression. Then a primitive element $\omega \in P_\mathfrak{g}$ is contained in the Samelson subspace P if and only if $\rho_\mathfrak{g}(\omega)\big|_\mathfrak{k}$ is contained in the subspace generated by all elements of the form $\rho_\mathfrak{g}(\eta)\big|_\mathfrak{k} \cdot f$ with $f \in \mathrm{A}_\mathfrak{k}^+$ a non–constant polynomial and $\eta \in P_\mathfrak{g}$ arbitrary, see [11, Corollary II, sect. 10.8]. Thus, if $v_1, \dots, v_r$ is a homogeneous basis of $P_\mathfrak{g}$ and $x_1, \dots, x_r$ its image under ρ_g, so that $\mathrm{A}_\mathfrak{g} \cong \mathbb{R}[x_1, \dots, x_r]$, then v_i is contained in P if and only if x_i is contained in the subspace $x_1|_\mathfrak{k} \cdot \mathrm{A}_\mathfrak{k}^+ + \dots + x_r|_\mathfrak{k} \cdot \mathrm{A}_\mathfrak{k}^+$.

Corollary 5.10. Suppose that $\mathfrak{g}^{\mathbb{C}}$ is of type D_r ($r \geq 4$) and let the simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$ be enumerated as in example 4.12. If $\mathfrak{k}$ is the subalgebra of $\mathfrak{g}$ associated with the simple roots $\Pi \setminus \{\alpha_r\}$ or with the simple roots

$\Pi \setminus \{\alpha_{r-1}\}$, then the inclusion induced map $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{k}}/J$, where $J \subseteq \mathrm{A}_{\mathfrak{k}}$ is the ideal generated by all polynomials of odd degree, is surjective.

Proof. We shall only treat the case that $\mathfrak{k}$ is the subalgebra associated with the simple roots $\Pi_0 = \Pi \setminus \{\alpha_r\}$, the other case being similar. Recall from proposition 5.1 that a maximal torus for $\mathfrak{k}$ is given by

$$\mathfrak{s} = \mathfrak{g} \cap \bigoplus_{i=1}^{r-1} \left[\mathfrak{g}^{\mathbb{C}}_{\alpha_i}, \mathfrak{g}^{\mathbb{C}}_{-\alpha_i}\right],$$

that $\Delta \cap \mathrm{span}_{\mathbb{Z}}\Pi_0$ bijectively corresponds to the set of roots on $\mathfrak{s}^{\mathbb{C}}$ via restriction, and that Δ^+ induces a notion of positivity with simple roots the restrictions $\tilde{\alpha}_1, \dots, \tilde{\alpha}_{r-1}$ of the roots $\alpha_1, \dots, \alpha_{r-1}$. In particular, the subalgebras $\mathfrak{q}$ and $\mathfrak{h}$ of $\mathfrak{k}$ associated with the simple roots $\{\tilde{\alpha}_1, \dots, \tilde{\alpha}_{r-2}\}$ and $\{\tilde{\alpha}_2, \dots, \tilde{\alpha}_{r-2}\}$ are equal to the subalgebras of $\mathfrak{g}$ associated with the simple roots $\{\alpha_1, \dots, \alpha_{r-2}\}$ and $\{\alpha_2, \dots, \alpha_{r-2}\}$. We will show that the canonical inclusion induces a surjection $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{q}}/J_{\mathfrak{q}}$, where $J_{\mathfrak{q}}$ is the ideal generated by all polynomials of odd degree, but before doing so, let us see how surjectivity of the aforementioned map implies the statement of the corollary. For this, we will have to distinguish the cases r odd and even.

If $r = 2k+1$, then we note that the map $A_{\mathfrak{k}}/J_{\mathfrak{k}} \longrightarrow A_{\mathfrak{q}}/J_{\mathfrak{q}}$ is actually an isomorphism: indeed, $A_{\mathfrak{k}} \longrightarrow A_{\mathfrak{q}}$ is surjective by corollary 5.3, $A_{\mathfrak{k}}$ is a polynomial algebra on $r-1$ generators of degrees $2, 3, \dots, r$, and $A_{\mathfrak{q}}$ is polynomial algebra on $r-2$ generators of degrees $2, 3, \dots, r-1$. Since $r = 2k+1$ is odd, $A_{\mathfrak{k}}/J_{\mathfrak{k}}$ and $A_{\mathfrak{k}}/J_{\mathfrak{q}}$ hence are two polynomial algebras on k generators of degrees $2, 4, \dots, 2k$. For degree reasons the epimorphism $A_{\mathfrak{k}}/J_{\mathfrak{k}} \longrightarrow A_{\mathfrak{q}}/J_{\mathfrak{q}}$ then necessarily has to be an isomorphism.

The case $r = 2k$ is more involved. Let $\tau : \mathfrak{k} \longrightarrow \mathfrak{k}$ be an automorphism induced by the non–trivial automorphism of the Dynkin diagram of $\mathfrak{k}^{\mathbb{C}}$ and note that $\mathfrak{h}$ is τ–invariant; in fact, the restriction $\tau : \mathfrak{h} \longrightarrow \mathfrak{h}$ is an automorphism induced by the non–trivial Dynkin diagram automorphism of $\mathfrak{h}^{\mathbb{C}}$. Put $\mathfrak{m} = \mathfrak{h}^{\tau}$, $\mathfrak{n} = \mathfrak{k}^{\tau}$ and recall from our discussion before corollary 5.9 that in the commutative diagram

$$\begin{tikzcd}
\mathrm{A}_{\mathfrak{g}} \arrow[r] \arrow[d, equal] & \mathrm{A}_{\mathfrak{k}}/J_{\mathfrak{k}} \arrow[r] \arrow[d, "\cong"] & \mathrm{A}_{\mathfrak{h}}/J_{\mathfrak{h}} \arrow[d, "\cong"] \\
\mathrm{A}_{\mathfrak{g}} \arrow[r] & \mathrm{A}_{\mathfrak{n}} \arrow[r] & \mathrm{A}_{\mathfrak{m}},
\end{tikzcd}$$

in which all maps are induced by canonical inclusions and $J_{\mathfrak{h}}$ is the ideal generated by all polynomials of odd degree, the vertical maps are well–defined isomorphisms, because $\mathfrak{n}$, $\mathfrak{m}$ are folded subalgebras and $J_{\mathfrak{k}}$, $J_{\mathfrak{h}}$ are precisely the kernels of the restrictions $\mathrm{A}_{\mathfrak{k}} \longrightarrow \mathrm{A}_{\mathfrak{n}}$, $\mathrm{A}_{\mathfrak{h}} \longrightarrow \mathrm{A}_{\mathfrak{m}}$. Also note that $\mathrm{A}_{\mathfrak{q}}/J_{\mathfrak{q}} \longrightarrow \mathrm{A}_{\mathfrak{h}}/J_{\mathfrak{h}}$ is surjective by corollary 5.3, so if $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{q}}/J_{\mathfrak{q}}$ is a surjection, then $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{m}}$ is surjective as well. Now we use the transgression $P_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{g}}$ to identify a homogeneous basis $v_1, \dots, v_r$ of the primitive subspace $P_{\mathfrak{g}}$ with homogeneous polynomials $x_1, \dots, x_r \in \mathrm{A}_{\mathfrak{g}}$. Similarly, we may use the transgressions of $\mathfrak{n}$ and $\mathfrak{m}$ to choose homogeneous polynomials $p_1, \dots, p_k$ in $\mathrm{A}_{\mathfrak{n}}$, $q_1, \dots, q_{k-1}$ in $\mathrm{A}_{\mathfrak{m}}$, and since $\mathfrak{m}$ is totally non–cohomologous to zero in $\mathfrak{n}$, we may choose these polynomials in such a way that the diagram of graded algebras

$$\begin{tikzcd}
\mathrm{A}_{\mathfrak{g}} \arrow[r] & \mathrm{A}_{\mathfrak{n}} \arrow[r] & \mathrm{A}_{\mathfrak{m}} \\
\mathbb{R}[x_1, \dots, x_r] \arrow[r, "t"] \arrow[u, "\cong"] & \mathbb{R}[p_1, \dots, p_k] \arrow[r] \arrow[u, "\cong"] & \mathbb{R}[q_1, \dots, q_{k-1}], \arrow[u, "\cong"]
\end{tikzcd}$$

where the lower right horizontal map sends p_i to q_i if $i \leq k-1$ and to zero if $i = k$, commutes. Note that $\mathfrak{n}^{\mathbb{C}}$ and $\mathfrak{m}^{\mathbb{C}}$ are of type C_k and C_{k-1}, so the sets of degrees of the basic invariants of $\mathfrak{g}$, $\mathfrak{m}$, and $\mathfrak{n}$ are

$$\{2, 4, \dots, 2r-2\} \cup \{r\}, \{2, 4, \dots, 2k\}, \text{ and } \{2, 4, \dots, 2k-2\},$$

respectively (cf. [15, Table 1, sect. 3.7]), whence for degree reasons p_k must be of degree $2k$. Also, we may assume x_i to be of degree $2i$ if $i \leq r-1$ and x_r to be of degree r. With this arrangement the surjectivity of the map $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{m}}$ implies that the subalgebra generated by $t(x_1), \dots, t(x_{k-1})$ coincides with the subalgebra generated by $p_1, \dots, p_{k-1}$. Moreover, if $k+1 \leq i \leq r-1$, then the element $t(x_i)$, whose degree is at most $4k-2$, but not $2k$, is contained in the ideal $I = p_1 \cdot \mathrm{A}_{\mathfrak{n}}^+ + \dots + p_{k-1} \cdot \mathrm{A}_{\mathfrak{n}}^+$, whence the elements $v_{k+1}, \dots, v_{r-1}$ must be contained in the Samelson subspace $P \subseteq P_{\mathfrak{g}}$ of the pair $(\mathfrak{g}, \mathfrak{n})$. Since P is bounded in dimension by $\operatorname{rank}(\mathfrak{g}) - \operatorname{rank}(\mathfrak{n}) = k$, at most one of the elements v_k, v_r can hence be contained in P. In particular, if we write $t(x_k) = cp_k + u$ and $t(x_r) = dp_k + v$, with $u, v \in I$ and $c, d \in \mathbb{R}$, then one of c or d must be non–zero, for otherwise both v_k and v_r would be contained in P. Since $p_1, \dots, p_{k-1}$ are already part of the image of t, it follows that t is surjective.

Thus, it remains to verify that $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{q}}/J_{\mathfrak{q}}$ is an epimorphism, and we argue as follows. Let $\sigma : \Pi \longrightarrow \Pi$ be the involution exchanging α_{r-1} with α_r and fixing all other simple roots. It is an automorphism of the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ and hence extends to an involution $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$. Let $\mathfrak{f}$ be its fixed point set, put $\mathfrak{b} = \mathfrak{t}^{\sigma}$, and denote by $\beta_1, \dots, \beta_{r-1}$ the restrictions of the simple roots $\alpha_1, \dots, \alpha_{r-1}$ to the Cartan subalgebra $\mathfrak{b}^{\mathbb{C}}$ of $\mathfrak{f}^{\mathbb{C}}$. We already observed that $\beta_1, \dots, \beta_{r-1}$ is a set of simple roots for a suitable notion of positivity, and because $\alpha_1, \dots, \alpha_{r-2}$ are fixed by σ, proposition 5.5 implies that $\mathfrak{q} = \mathfrak{q}^{\sigma}$ also is the subalgebra of $\mathfrak{f}$ associated with the simple roots $\beta_1, \dots, \beta_{r-2}$. Since $\mathfrak{f}^{\mathbb{C}}$ is of type B_{r-1} (see [7, Lemma 5.2]), with short root β_{r-1}, corollary 5.9 thus implies that $\mathrm{A}_{\mathfrak{f}} \longrightarrow \mathrm{A}_{\mathfrak{q}}/J_{\mathfrak{q}}$ is surjective. But $\mathfrak{f}$ is also a folded subalgebra, and hence restriction gives a surjection $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{f}}$. In total, $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{q}}/J_{\mathfrak{q}}$ is surjective. ☐

For the proof of proposition 5.12 below we will have to collect some more results from [11]. Given a Lie subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ we shall use the symbol $\mathbb{A}_{\mathfrak{k}}$ to denote the set of invariant polynomials on $\mathfrak{k}$ with grading induced by viewing $\mathfrak{k}^*$ as a graded vector space concentrated in degree 2. More precisely, $\mathbb{A}_{\mathfrak{k}}$ is the graded algebra which is equal to $\mathrm{A}_{\mathfrak{k}}$ as an algebra, but whose k–th graded component $\mathbb{A}_{\mathfrak{k}}^k$ is zero, if k is odd, and $\mathrm{A}_{\mathfrak{k}}^j$, if $k = 2j$ is even. Thus, a homogeneous polynomial of degree k in $\mathrm{A}_{\mathfrak{k}}$ corresponds to a homogeneous element of degree $2k$ in $\mathbb{A}_{\mathfrak{k}}$ and the transgression $\rho : P_{\mathfrak{g}} \longrightarrow \mathbb{A}_{\mathfrak{g}}$ is homogeneous of degree 1. Now suppose that $\mathfrak{k}$ is compact and let d be the anti–derivation on $\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P_{\mathfrak{g}})$ sending $\mathbb{A}_{\mathfrak{k}} \otimes 1$ to zero and an element $1 \otimes w$ with $w \in P_{\mathfrak{g}}$ to $\rho(w)|_{\mathfrak{k}} \otimes 1$. The differential graded $\mathbb{R}$–algebra $(\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P_{\mathfrak{g}}), \mathrm{d})$ is called the *Koszul complex* for the pair $(\mathfrak{g}, \mathfrak{k})$, see [11, Section 10.8]; in the notation of [11, Section 2.17], the space $\mathbb{A}_{\mathfrak{k}}$ together with the restriction $\mathrm{d}|_{P_{\mathfrak{g}}} : P_{\mathfrak{g}} \longrightarrow \mathbb{A}_{\mathfrak{k}}$ is a symmetric $P_{\mathfrak{g}}$–algebra and the Koszul complex for $(\mathfrak{g}, \mathfrak{k})$ coincides with the Koszul complex for the symmetric $P_{\mathfrak{g}}$–algebra $(\mathbb{A}_{\mathfrak{k}}, \mathrm{d}|_{P_{\mathfrak{g}}})$. By [11, Theorem III, sect. 10.8] there is an isomorphism of graded algebras between $\mathrm{H}\big(\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P_{\mathfrak{g}})\big)$ and $\mathrm{H}(\mathfrak{g}, \mathfrak{k})$, so the graded algebra structure of $\mathrm{H}(\mathfrak{g}, \mathfrak{k})$ is determined by the one of $\mathrm{H}\big(\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P_{\mathfrak{g}})\big)$.

Let $P \subseteq P_{\mathfrak{g}}$ be the Samelson space and choose a graded vector space $P' \subseteq P_{\mathfrak{g}}$ complementary to P, that is, such that $P_{\mathfrak{g}} = P \oplus P'$. A well–known theorem (cf. [11, Theorem V and corollary I, sect. 2.15]) now states that there is an isomorphism of graded algebras between $\mathrm{H}\big(\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P_{\mathfrak{g}})\big)$ and $\mathrm{H}\big(\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P')\big) \otimes \Lambda(P)$, where we think of $\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P')$ as a differential graded subalgebra of $(\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P_{\mathfrak{g}}), \mathrm{d})$. In particular, if $(\mathfrak{g}, \mathfrak{k})$ is formal, then $P' = 0$ and $\mathrm{H}\big(\mathbb{A}_{\mathfrak{k}} \otimes \Lambda(P')\big)$ reduces to $\mathbb{A}_{\mathfrak{k}}/J$, where J is the ideal in $\mathbb{A}_{\mathfrak{k}}$ generated by the image of the inclusion induced map $\mathbb{A}_{\mathfrak{g}} \longrightarrow \mathbb{A}_{\mathfrak{k}}$, see [11, Theorem VII, sect. 2.19].

Lemma 5.11. Let $\mathfrak{k}$ be a compact Lie algebra and $\mathfrak{h} \subseteq \mathfrak{k}$ a simple subalgebra. Then the inclusion induced map $\mathrm{A}_{\mathfrak{k}} \longrightarrow \mathrm{A}_{\mathfrak{h}}$ is a surjection in degree 2.

Proof. For every vector space V there is a natural isomorphism of graded vector spaces between $\mathrm{S}(V^*)$ and the space of symmetric multilinear forms $\operatorname{Sym}(V)$; it induces the commutative diagram

$$\begin{tikzcd} \operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \arrow[r, "\cong"] \arrow[d] & \mathrm{A}_{\mathfrak{k}}^2 \arrow[d] \\ \operatorname{Sym}^2(\mathfrak{h})^{\mathfrak{h}} \arrow[r, "\cong"] & \mathrm{A}_{\mathfrak{h}}^2, \end{tikzcd}$$

where both vertical maps are induced by the canonical inclusion $\iota : \mathfrak{h} \longrightarrow \mathfrak{k}$. Hence, it will suffice to verify the surjectivity of the map $\iota^* : \operatorname{Sym}(\mathfrak{k})^{\mathfrak{k}} \longrightarrow \operatorname{Sym}(\mathfrak{h})^{\mathfrak{h}}$ in degree 2. Let $\langle\cdot,\cdot\rangle$ be an ad–invariant inner product on $\mathfrak{k}$, that is, such that ad_X is skew–symmetric for all $X \in \mathfrak{k}$; such an inner product exists, because we are assuming $\mathfrak{k}$ to be compact. Then $\iota^*\langle\cdot,\cdot\rangle$ is an ad–invariant inner product on $\mathfrak{h}$, so the image of $\iota^* : \operatorname{Sym}^2(\mathfrak{k})^{\mathfrak{k}} \longrightarrow \operatorname{Sym}^2(\mathfrak{h})^{\mathfrak{h}}$ is at least one–dimensional and $\mathfrak{h}$ is necessarily compact. However, it is a well–known consequence of Schur's Lemma (see [16, Proposition 5.1, sect. V.1]) that the space of ad–invariant symmetric bilinear forms on a compact simple Lie algebra is one–dimensional. In fact, fix some ad–invariant inner product $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on $\mathfrak{h}$. Choose $h \in \operatorname{Sym}^2(\mathfrak{h})^{\mathfrak{h}}$ arbitrarily and define $T : \mathfrak{h} \longrightarrow \mathfrak{h}$ by requiring that $\langle\!\langle T(X), Y\rangle\!\rangle = h(X,Y)$ holds for all $X, Y \in \mathfrak{h}$. The ad–invariance of $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and h implies that $\mathrm{ad}_X \circ T = T \circ \mathrm{ad}_X$ holds for all $X \in \mathfrak{h}$ and the symmetry of h forces T to be self–adjoint with respect to $\langle\!\langle\cdot,\cdot\rangle\!\rangle$. Therefore, T is diagonalizable, with real eigenvalues, and if λ is an eigenvalue of T, then the kernel of $T - \lambda\,\mathrm{id}$ is a non–trivial ideal of $\mathfrak{h}$, hence already equal to $\mathfrak{h}$. $\square$

Proposition 5.12. Suppose that $\mathfrak{g}^{\mathbb{C}}$ is of type E_7 and that the simple roots $\Pi = \{\alpha_1, \ldots, \alpha_7\}$ are enumerated as in example 4.14. Let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ associated with $\Pi \setminus \{\alpha_1\}$ and $J \subseteq \mathrm{A}_{\mathfrak{k}}$ the ideal generated by all polynomials of odd degree. Then the canonical restriction $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{k}}/J$ is surjective.

Proof. Note that $\mathfrak{k}^{\mathbb{C}}$ is a Lie algebra of type E_6, so the set of degrees of the basic invariants of $\mathfrak{k}$ is $\{2,5,6,8,9,12\}$; the set of degrees of the basic invariants of $\mathfrak{g}$ is $\{2,6,8,10,12,14,18\}$. Let v_2, v_6, v_8, v_{10}, v_{12}, v_{14}, v_{18} be a homogeneous basis of $P_{\mathfrak{g}}$, increasingly ordered by degree, and $x_2, \ldots, x_{18} \in \mathrm{A}_{\mathfrak{g}}$ the images under the transgression (note the grading). We claim that there is a dichotomy: either the element v_6 corresponding to the homogeneous polynomial x_6 of degree 6 is contained in the Samelson space $P \subseteq P_{\mathfrak{g}}$ of $(\mathfrak{g},\mathfrak{k})$ or the map $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{k}}/J$ is surjective. To see this, suppose that v_6 is not contained in P and let $\mathfrak{h}$ be the fixed point set of an automorphism on $\mathfrak{k}$ induced by the non–trivial automorphism of the Dynkin diagram of $\mathfrak{k}^{\mathbb{C}}$. Then $\mathfrak{h}$ is a folded subalgebra, with $\mathfrak{h}^{\mathbb{C}}$ of type F_4, and its set of degrees of basic invariants is given by $\{2,6,8,12\}$. Thus, as in the proof of corollary 5.10, the kernel of the inclusion induced map $\mathrm{A}_{\mathfrak{k}} \longrightarrow \mathrm{A}_{\mathfrak{h}}$ is precisely J and we may choose homogeneous polynomials p_2, $p_5, p_6, p_8, p_9, p_{12}$ in $\mathrm{A}_{\mathfrak{k}}$ and q_2, q_6, q_8, q_{12} in $\mathrm{A}_{\mathfrak{h}}$, enumerated in increasing order of degree, such that the diagram

$$\begin{tikzcd}
\mathrm{A}_{\mathfrak{g}} \arrow[r] & \mathrm{A}_{\mathfrak{k}} \arrow[r] & \mathrm{A}_{\mathfrak{h}} \\
\mathbb{R}[x_2,\ldots,x_{18}] \arrow[r] \arrow[u, "\cong"] & \mathbb{R}[p_2,\ldots,p_{12}] \arrow[r] \arrow[u, "\cong"] & \mathbb{R}[q_2,\ldots,q_{12}], \arrow[u, "\cong"]
\end{tikzcd}$$

where the lower right horizontal map sends (p_2,p_6,p_8,p_{12}) to (q_2,q_6,q_8,q_{12}) and p_5, p_9 to zero, commutes. Now recall that v_i is an elment of P if and only the restriction of x_i to $\mathfrak{k}$ is contained in $x_2|_{\mathfrak{k}} \cdot \mathrm{A}_{\mathfrak{k}}^+ + \ldots + x_{18}|_{\mathfrak{k}} \cdot \mathrm{A}_{\mathfrak{k}}^+$ and that x_2 restricts to a non–zero multiple of p_2 as well as q_2 by lemma 5.11. So if $v_6 \notin P$, then $x_6|_{\mathfrak{k}} = cp_6 + dp_2^3$ for some non–zero constant $c \in \mathbb{R}$ and some constant $d \in \mathbb{R}$, and $x_6|_{\mathfrak{h}} = cq_6 + dq_2^3$ as well. We conclude that $x_2|_{\mathfrak{h}}$ and $x_6|_{\mathfrak{h}}$ generate the same subalgebra as q_2 and q_6, and it follows for degree reasons that the restrictions of the elements x_{10}, x_{14}, and x_{18} are contained in the ideal $q_2 \cdot \mathrm{A}_{\mathfrak{h}}^+ + q_6 \cdot \mathrm{A}_{\mathfrak{h}}^+$. Hence, v_{10}, v_{14}, and v_{18} are contained in the Samelson space of the pair $(\mathfrak{g},\mathfrak{h})$, and since the latter space is at most three–dimensional, the element $x_8|_{\mathfrak{h}}$ cannot be contained in the ideal generated by $x_2|_{\mathfrak{h}}$ and $x_6|_{\mathfrak{h}}$. Similarly, $x_{12}|_{\mathfrak{h}}$ cannot be contained in the ideal generated by the restrictions of x_2, x_6, and x_8. But this means that $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{h}}$ is surjective, and since the kernel of $\mathrm{A}_{\mathfrak{k}} \longrightarrow \mathrm{A}_{\mathfrak{h}}$ is the ideal generated by p_5 and p_9, which is exactly J, the map $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{k}}/J$ is surjective as well.

Therefore, we only need to show that v_6 is not contained in P, and we assume $v_6 \in P$ to hold for a contradiction. Then $(\mathfrak{g},\mathfrak{k})$ is formal and v_8 cannot be contained in P. Thus, by the same reasoning as before, p_2 and p_8 must be contained in the ideal I generated by the image of the restriction map $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{k}}$. Once more it follows for degree reasons that modulo the ideal (p_2,p_8) generated by p_2 and p_8 we have the equalities

$$x_{10}|_{\mathfrak{k}} = ap_5^2,\ \ x_{12}|_{\mathfrak{k}} = bp_{12} + cp_6^2,\ \ x_{14}|_{\mathfrak{k}} = dp_9p_5, \text{ and }\ x_{18}|_{\mathfrak{k}} = ep_{12}p_6 + fp_9^2 + gp_6^3,$$

for certain real constants a, b, c, d, e, f, and g. As a consequence, the ideal I_0 generated by p_2, p_5, p_8, and the restrictions of x_{12} and x_{18} already contains I, because $x_{10}|_{\mathfrak{k}}$, $x_{14}|_{\mathfrak{k}} \in I_0$. By the discussion preceeding lemma 5.11

the quotient $\mathrm{A}_{\mathfrak{k}}/I$ is finite–dimensional, $\mathbb{A}_{\mathfrak{k}}/I \otimes \Lambda(P)$ being isomorphic to $\mathrm{H}(\mathfrak{g},\mathfrak{k})$, so the quotient $\mathrm{A}_{\mathfrak{k}}/I_0$ must be finite–dimensional too. This observation leads to the desired contradiction, because

$$\mathrm{A}_{\mathfrak{k}}/I_0 \cong \mathbb{R}[p_2,\dots,p_{12}]/\big(p_2,p_5,p_8,\, x_{12}|_{\mathfrak{k}}\,,\, x_{18}|_{\mathfrak{k}}\big) \cong \mathbb{R}[p_6,p_9,p_{12}]/\big(bp_{12}+cp_6^2, ep_{12}p_6+fp_9^2+gp_6^3\big)$$

is a quotient of a polynomial algebra in three variables by an ideal generated by two homogeneous, but non–constant polynomials, which is infinite–dimensional. Thus, v_6 is not an element of P. □

6. Equivariant and ordinary cohomology of simple $\mathbb{Z}_2\times\mathbb{Z}_2$–symmetric spaces

6.1. Inner automorphisms

Throughout this section, we fix a simple, compact connected Lie group G and two commuting involutive Lie group automorphisms σ_1, σ_2 on G. As in section 2, we denote by $K_i=(G^{\sigma_i})_0$ the fixed point set of σ_i, $i=1,2$, choose a maximal torus $S\subseteq(G^{\sigma_1}\cap G^{\sigma_2})_0$ and an Ad– as well as σ_1– and σ_2–invariant negative–definite inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Then $T_1 = Z_{K_1}(S)$ is a maximal torus in K_1, $T = Z_G(T_1)$ a maximal torus in G. Let Δ be the $\mathfrak{g}^{\mathbb{C}}$–roots with respect to the Cartan subalgebra $\mathfrak{t}^{\mathbb{C}}$, Δ^+ a choice of positive roots with simple roots Π, $\Gamma\subseteq\Delta$ the set of roots vanishing on $\mathfrak{s}$, and $\Gamma^+ := \Gamma\cap\Delta^+$. Once an enumeration $\Pi=\{\alpha_1,\dots,\alpha_r\}$ of the simple roots has been chosen, we shall also denote by $(u_1,\dots,u_r)$ the basis dual to the basis $(\alpha_1,\dots,\alpha_r)$ of $\mathrm{span}_{\mathbb{R}}\Pi$.

Our first goal is to show that the pair $(\mathfrak{g},\mathfrak{k}_1\cap\mathfrak{k}_2)$ is (equivariantly) formal if σ_1 and σ_2 are both inner automorphisms. If Γ is empty, then this is certainly the case, as then $\mathfrak{s}=\mathfrak{t}$ is a maximal torus (cf. proposition 2.1), whence $\mathfrak{k}_1\cap\mathfrak{k}_2$ is a Lie subalgebra of maximal rank. Thus, we may assume that $\Gamma\neq\emptyset$. Now we observe that Γ is a set of strongly orthogonal roots by proposition 2.2 and that $p(\beta)=\sum_{\alpha\in\Gamma^+}2\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle$ is an even number for all β, cf. proposition 3.5. Therefore, Γ possesses a normal form, that is, there exists a Weyl group element $w\in W(\Delta)$ such that $w(\Gamma)\cap\Delta^+$ is one of the sets specified in examples 4.11 to 4.15 or examples 4.18 to 4.21. It is a well–known fact (see [16, Theorem 4.54, sect. IV.6]) that the abstract Weyl group $W(\Delta)$ corresponds under the isomorphism $\mathrm{span}_{\mathbb{R}}\Delta\longrightarrow(i\mathfrak{t})^*$, $\alpha\mapsto\alpha|_{i\mathfrak{t}}$, to the action induced by the coadjoint action of the analytic Weyl group $\mathrm{N}_G(T)/T$ on $(i\mathfrak{t})^*$, so there exists an element $n\in \mathrm{N}_G(T)$ such that the dual map $(\mathrm{Ad}_n)^*$ coincides with w on $\mathrm{span}_{\mathbb{R}}\Delta$. Put $A := (c_n)^{-1}$ and consider the inner automorphisms $A\circ\sigma_1\circ A^{-1}$ and $A\circ\sigma_2\circ A^{-1}$. These are two commuting involutions and their fixed point subalgebra is $A(\mathfrak{k}_1\cap\mathfrak{k}_2)$ with maximal torus $A(\mathfrak{s})$. Moreover, the set of roots vanishing on $A(\mathfrak{s})$ is

$$(A^{-1})^*(\Gamma) = \left\{\alpha\circ A^{-1}\,|\,\alpha\in\Delta\right\} = \{\alpha\circ\mathrm{Ad}_n\,|\,\alpha\in\Delta\} = (\mathrm{Ad}_n)^*(\Gamma) = w(\Gamma),$$

and since A maps $[\mathfrak{k}_1\cap\mathfrak{k}_2]_{\mathfrak{f}}$ onto $[A(\mathfrak{k}_1\cap\mathfrak{k}_2)]_{\mathfrak{f}}$, there is no loss of generality if we assume that $w=\mathrm{id}$. Also note that by corollary 3.7 we may assume that

$$\sigma_1 = c_h \text{ with } h=\exp\left(\sum_{\alpha\in\Gamma^+}\frac{i\pi}{|\alpha|^2}H_\alpha\right),$$

as c_h commutes with σ_2 and $\mathfrak{k}_1\cap\mathfrak{k}_2$ and $\mathfrak{g}^{\mathrm{Ad}_h}\cap\mathfrak{k}_2$ share the same maximal torus $\mathfrak{s}$.

Now we check that $[\mathfrak{k}_1\cap\mathfrak{k}_2]_{\mathfrak{f}}$ contains a subalgebra that is totally non–cohomologous to zero in $\mathfrak{g}$ by considering the various Lie algebra isomorphism classes that $\mathfrak{g}^{\mathbb{C}}$ may assume.

Theorem 6.1. If $\mathfrak{g}^{\mathbb{C}}$ is of type A_r, $r\geq 1$, then some element in $[\mathfrak{k}_1\cap\mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. As noted before, we may assume that Γ is in normal form and that the simple roots $\Pi=\{\alpha_1,\dots,\alpha_r\}$ are enumerated as in example 4.11. Recall that the rank r of G necessarily is an odd number, say $r=2k+1$, and that $\Gamma\cap\Delta^+=\{\delta_1,\dots,\delta_{k+1}\}$, with $\delta_i=\alpha_i+\ldots+\alpha_{r-i+1}$. Since σ_1 is given by conjugation with elements in T, the fixed point set $\mathfrak{k}_1^{\mathbb{C}}$ of σ_1 is the direct sum of $\mathfrak{t}^{\mathbb{C}}$ and those root spaces $\mathfrak{g}_\alpha^{\mathbb{C}}$ on which σ_1 acts as the identity. Now if

$\alpha = m_1\alpha_1 + \ldots + m_r\alpha_r$ is a root, then

$$\sigma_1|_{\mathfrak{g}^{\mathbb{C}}_\alpha} = \exp\left(\frac{\mathrm{i}\pi}{2}\sum_{j=1}^{k+1}\frac{2\langle\delta_j,\alpha\rangle}{\langle\delta_j,\delta_j\rangle}\right)\cdot \mathrm{id} = (-1)^{m_{k+1}}\,\mathrm{id},$$

so σ_1 is the identity of the root space of α if and only if $m_{k+1} \in 2\mathbb{Z}$. However, every positive root of $\mathfrak{g}$ is of the form $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ for integers $1 \le i \le j \le r$, whence the fixed point set of σ_1 is

$$\mathfrak{k}_1^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta'}\mathfrak{g}^{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta''}\mathfrak{g}^{\mathbb{C}}_\alpha,$$

where $\Delta' = \Delta \cap \mathrm{span}_{\mathbb{Z}}\{\alpha_1,\ldots,\alpha_k\}$ and $\Delta'' = \Delta \cap \mathrm{span}_{\mathbb{Z}}\{\alpha_{k+2},\ldots,\alpha_{2k+1}\}$. Thus, we have

$$\begin{aligned}\mathfrak{k}_1^{\mathbb{C}} &= \Big(\bigcap_{i\neq k+1}\ker\alpha_i\Big) \oplus \Big(\bigoplus_{i=1}^{k}[\mathfrak{g}^{\mathbb{C}}_{\alpha_i},\mathfrak{g}^{\mathbb{C}}_{-\alpha_i}] \oplus \bigoplus_{\alpha\in\Delta'}\mathfrak{g}^{\mathbb{C}}\Big) \oplus \Big(\bigoplus_{i=k+2}^{2k+1}[\mathfrak{g}^{\mathbb{C}}_{\alpha_i},\mathfrak{g}^{\mathbb{C}}_{-\alpha_i}] \oplus \bigoplus_{\alpha\in\Delta''}\mathfrak{g}^{\mathbb{C}}\Big)\\ &= \Big(\bigcap_{i\neq k+1}\ker\alpha_i\Big) \oplus \mathfrak{i}_1^{\mathbb{C}} \oplus \mathfrak{i}_2^{\mathbb{C}},\end{aligned}$$

where $\mathfrak{i}_1$ is the subalgebra of $\mathfrak{g}$ associated with the simple roots $\alpha_1,\ldots,\alpha_k$ and $\mathfrak{i}_2$ is the subalgebra associated with $\alpha_{k+2},\ldots,\alpha_{2k+1}$. We claim that σ_2 maps $\mathfrak{i}_1$ onto $\mathfrak{i}_2$. Indeed, the common kernel of all simple roots different from α_{k+1} constitutes the one–dimensional center of $\mathfrak{k}_1^{\mathbb{C}}$ and hence is invariant under σ_2. The subalgebras $\mathfrak{i}_1^{\mathbb{C}}$ and $\mathfrak{i}_2^{\mathbb{C}}$ are simple ideals of $\mathfrak{k}_1^{\mathbb{C}}$, both of type A_k, and thus either interchanged by σ_2 or invariant subspaces. However, σ_2 maps the root $\alpha = \alpha_1 + \ldots + \alpha_k$ onto $\beta = -(\alpha_{k+2} + \ldots + \alpha_{2k+1})$ and hence also sends $\mathfrak{g}^{\mathbb{C}}_\alpha$ isomorphically onto $\mathfrak{g}^{\mathbb{C}}_\beta$. Therefore, $\sigma_2(\mathfrak{i}_1)$ intersects $\mathfrak{i}_2$ non–trivially, whence $\sigma_2(\mathfrak{i}_1) = \mathfrak{i}_2$.

Now it follows from proposition 5.1 and corollary 3.3 that $\mathfrak{k}_1 \cap \mathfrak{k}_2$ is of rank k, and we have just observed that the fixed point set of σ_2 on $\mathfrak{i}_1 \oplus \mathfrak{i}_2$ is isomorphic to the diagonal $\Delta(\mathfrak{i}_1) \subseteq \mathfrak{i}_1 \oplus \mathfrak{i}_1$, a Lie algebra of rank k. Therefore, σ_2 acts as $-\,\mathrm{id}$ on $\mathrm{Z}(\mathfrak{k}_1)$, and $\mathfrak{k}_1 \cap \mathfrak{k}_2$ is precisely the fixed point set of σ_2 on $\mathfrak{i}_1 \oplus \mathfrak{i}_2$. Since $\mathfrak{i}_1$ is totally non–cohomologous to zero in $\mathfrak{g}$ by corollary 5.3 and every invariant polynomial on $\mathfrak{g}$ is in particular a fixed point of σ_2^*, the theorem now follows from proposition 5.8. ☐

Theorem 6.2. If $\mathfrak{g}^{\mathbb{C}}$ is of type B_r, $r \ge 2$, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. Let $\alpha_1,\ldots,\alpha_r$ be the simple roots associated with Δ^+, enumerated as in example 4.18. This time, there are three normal forms to consider, and all of them can be treated simultaneously as follows. We know that $\mathfrak{s}$, the maximal torus of $\mathfrak{k}_1 \cap \mathfrak{k}_2$, is the fixed point set of σ_2 on $\mathfrak{t}$, so in the second normal form, there exists an odd number $k < r$ such that $\mathfrak{s}^{\mathbb{C}}$ is the common kernel of the roots $\{\alpha_1,\delta_1,\alpha_3,\delta_3,\ldots,\alpha_k,\delta_k\}$, with $\delta_i = \alpha_i + 2(\delta_{i+1} + \ldots + \delta_r)$ for $i < r$. The first and third normal forms state that $\mathfrak{s}^{\mathbb{C}}$ is the common fixed point set of the root reflections defined by the elements in one of the sets $\{\gamma_1\}$ or $\{\alpha_1,\delta_1,\alpha_3,\delta_3,\ldots,\alpha_k,\delta_k,\gamma_{k+2}\}$, where $\gamma_i = \alpha_i + \ldots + \alpha_r$ and $k < r$ is again odd. Thus, in all three cases, the maximal torus $\mathfrak{s}^{\mathbb{C}}$ is given by

$$\mathfrak{s}^{\mathbb{C}} = \bigoplus_{j=\ell+1}^{r}\left[\mathfrak{g}^{\mathbb{C}}_{\alpha_j},\mathfrak{g}^{\mathbb{C}}_{-\alpha_j}\right],$$

where $\ell = |\Gamma^+|$. But this means that $\mathfrak{s}$ is the maximal torus of the Lie subalgebra of $\mathfrak{g}$ associated with the simple roots $\alpha_{\ell+1},\ldots,\alpha_r$, which is non–cohomologous to zero in $\mathfrak{g}$ by corollary 5.7. ☐

Theorem 6.3. If $\mathfrak{g}^{\mathbb{C}}$ is of type C_r, $r \ge 3$, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. According to example 4.19, there are two normal forms to consider, one of which cannot apply: indeed, if Γ^+ would be equal to $\{\delta_1,\delta_2,\ldots,\delta_{r-1},\alpha_r\}$, where $\delta_i = 2(\alpha_i + \ldots + \alpha_{r-1}) + \alpha_r$, then by corollary 3.3 $\mathfrak{k}_1 \cap \mathfrak{k}_2$ would be of rank 0, which is impossible. Therefore, we only need to consider the normal form in which Γ^+ is equal to $\{\delta_1,\ldots,\delta_i,\gamma_{i+1},\gamma_{i+3},\ldots,\gamma_{r-1}\}$ for some $i < r$ such that $r - i$ is even, where $\gamma_j = \delta_j - \alpha_j$ is the highest short root in the root subsystem spanned by $\{\alpha_j,\ldots,\alpha_r\}$. To compute the fixed point set of σ_1, let $\alpha = m_1\alpha_1 + \ldots + m_r\alpha_r$ be an arbitrary root and set $m_0 := 0$. We have, for all $i < r-1$,

$$\frac{2\langle \delta_i, \alpha\rangle}{\langle \delta_i, \delta_i\rangle} = m_i - m_{i-1},\ \frac{2\langle \gamma_i, \alpha\rangle}{\langle \gamma_i, \gamma_i\rangle} = m_{i+1} - m_{i-1},\ \frac{2\langle \gamma_{r-1}, \alpha\rangle}{\langle \gamma_{r-1}, \gamma_{r-1}\rangle} = 2m_r - m_{r-2}, \text{ and } \ \sigma_1|_{\mathfrak{g}^{\mathbb{C}}_\alpha} = (-1)^{m_r}\,\mathrm{id},$$

so a root vector of α is fixed if and only if $m_r \in 2\mathbb{Z}$. However, there is no root with $m_r \notin \{0, -1, 1\}$, because the highest root of $\mathfrak{g}^{\mathbb{C}}$ is $\delta_1 = 2(\alpha_1 + \ldots + \alpha_{r-1}) + \alpha_r$. Thus, the fixed point set of σ_1 is $\mathfrak{k}_1 = \mathrm{Z}(\mathfrak{k}_1) \oplus \mathfrak{h}$, where the center $\mathrm{Z}(\mathfrak{k}_1) = \mathbb{R}Z$ is spanned by an element $Z \in \mathfrak{t}$ in the common kernel of the roots in $\Pi \setminus \{\alpha_r\}$ and $\mathfrak{h}$ is the rank $r-1$ subalgebra of $\mathfrak{g}$ associated with the simple roots $\Pi \setminus \{\alpha_r\}$; in fact, $\mathfrak{h}^{\mathbb{C}}$ is of type A_{r-1}. The automorphism σ_2 sends $\mathfrak{h}$ to $\mathfrak{h}$, so the restriction $\sigma_2 : \mathfrak{h} \longrightarrow \mathfrak{h}$ cannot be an inner automorphism, as otherwise $\mathfrak{k}_1 \cap \mathfrak{k}_2$ would be of rank at least $r-1$, and we claim that this implies that $\mathfrak{k}_1 \cap \mathfrak{k}_2$ is equal to $\mathfrak{h}^\sigma$. To see this, recall that $\mathfrak{h}$ is compact semisimple, whence there exists a maximal torus $\mathfrak{b} \subseteq \mathfrak{h}$ and a choice of positive roots Ω^+ for the roots Ω on $\mathfrak{b}^{\mathbb{C}}$, all of which are σ_2–invariant. Let $\Phi \subseteq \Omega^+$ be the corresponding simple roots and $\tau : \mathfrak{h} \longrightarrow \mathfrak{h}$ the automorphism induced by the Dynkin diagram automorphism $\sigma_2 : \Phi \longrightarrow \Phi$; the fixed point set $\mathfrak{f}$ of τ shares the maximal torus $\mathfrak{b}^\tau = \mathfrak{b}^{\sigma_2}$ with $\mathfrak{h}^{\sigma_2}$, so $\sigma_2 : \Phi \longrightarrow \Phi$ must be non–trivial. But there is only one non–trivial Dynkin diagram automorphism on a Lie algebra of type A_{r-1}, and thus $\mathfrak{f}^{\mathbb{C}}$ must be of type C_k, if $r = 2k$, or of type B_k, if $r = 2k+1$, cf. [7, Lemma 5.2]. In both cases $\mathfrak{h}^{\sigma_2}$ is of rank $k := \lfloor r/2 \rfloor$, so according to the decomposition $\mathfrak{k}_1 = \mathrm{Z}(\mathfrak{k}_1) \oplus \mathfrak{h}$ the rank of $\mathfrak{k}_1 \cap \mathfrak{k}_2$ must be at least k. On the other hand, Γ^+ consists of $i + (r-i)/2$ elements, whence $\mathfrak{k}_1 \cap \mathfrak{k}_2$ is of rank $(r-i)/2$ by corollary 3.3, and this is only possible if either r is even and $i = 0$ or r is odd and $i = 1$. In any case it follows for rank reasons that $\mathfrak{k}_1 \cap \mathfrak{k}_2 = \mathfrak{h}^{\sigma_2}$. In particular, $\mathfrak{k}_1 \cap \mathfrak{k}_2$ and $\mathfrak{f}$ share a maximal torus, so it will suffice to verify that $\mathfrak{f}$ is totally non–cohomologous to zero in $\mathfrak{g}$. However, the ideal $J_{\mathfrak{h}}$ in $\mathrm{A}_{\mathfrak{h}}$ generated by all polynomials of odd degree is exactly the kernel of the inclusion induced surjection $\mathrm{A}_{\mathfrak{h}} \longrightarrow \mathrm{A}_{\mathfrak{f}}$, and restriction induces a surjection $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{h}}/J_{\mathfrak{h}}$ by corollary 5.9. Hence, $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{f}}$ is surjective and $\mathfrak{f}$ is totally non–cohomologous to zero in $\mathfrak{g}$. □

Theorem 6.4. If $\mathfrak{g}^{\mathbb{C}}$ is of type D_r, $r \geq 4$,then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. We can immediately rule out one of the normal forms that may appear by example 4.12: if r is even and $\Gamma^+ = \{\alpha_1, \delta_1, \ldots, \alpha_{r-3}, \delta_{r-3}, \alpha_{r-1}, \alpha_r\}$, then $\mathfrak{k}_1 \cap \mathfrak{k}_2$ is a Lie algebra of rank $r - |\Gamma^+| = 0$, which is impossible. Thus, there are only two normal forms to consider. The second normal form that we shall treat is when $r = 2k$ is even and Γ^+ is equal to the set $\{\delta_1, \delta_3, \ldots, \delta_{r-3}, \gamma\}$, where $\delta_i = \alpha_i + 2(\alpha_{i+1} + \ldots + \alpha_{r-2}) + \alpha_{r-1} + \alpha_r$ and $\gamma \in \{\alpha_{r-1}, \alpha_r\}$. Both cases, $\gamma = \alpha_{r-1}$ and $\gamma = \alpha_r$, can be handled analogously, so let us assume that $\gamma = \alpha_r$. Then the proof of theorem 6.3, almost verbatimely carries over: in fact, if $\alpha = \sum_{j=1}^r m_j\alpha_j$ is a root and $i < r-2$, then

$$\frac{2\langle \delta_i, \alpha\rangle}{\langle \delta_i, \delta_i\rangle} = m_{i+1} - m_i \text{ and } \frac{2\langle \delta_{r-2}, \alpha\rangle}{\langle \delta_{r-2}, \delta_{r-2}\rangle} = 2m_{r-2} - m_{r-3} + m_{r-1} + m_r,$$

so σ_1 restricts to $(-1)^{m_r} \cdot \mathrm{id}$ on $\mathfrak{g}^{\mathbb{C}}_\alpha$. Hence $\mathfrak{k}_1$ is equal to $\mathrm{Z}(\mathfrak{k}_1) \oplus \mathfrak{h}$, where $\mathrm{Z}(\mathfrak{k}_1)$ is the one–dimensional center of $\mathfrak{k}_1$, spanned by an element in the joint kernel of all roots in Γ^+, and $\mathfrak{h}$ is the subalgebra associated with the simple roots $\alpha_1, \ldots, \alpha_{r-1}$, with $\mathfrak{h}^{\mathbb{C}}$ of type A_{2k-1}. Since $\mathrm{rank}(\mathfrak{k}_1 \cap \mathfrak{k}_2) \leq r-2$, the restriction $\sigma_2 : \mathfrak{h} \longrightarrow \mathfrak{h}$ cannot be inner, $\mathfrak{h}^{\sigma_2}$ must be equal to $\mathfrak{k}_1 \cap \mathfrak{k}_2$, and the fixed point set $\mathfrak{f}$ of an automorphism $\tau : \mathfrak{h} \longrightarrow \mathfrak{h}$ induced by the non–trivial Dynkin diagram automorphism of $\mathfrak{h}^{\mathbb{C}}$ with respect to a suitable Cartan subalgebra shares a maximal torus with $\mathfrak{h}^{\sigma_2}$. By corollary 5.9, $\mathfrak{f}$ is totally non–cohomologous to zero in $\mathfrak{g}$.

Thus, we consider the last normal form, according to which $\Gamma^+ = \{\alpha_1, \delta_1, \ldots, \alpha_k, \delta_k\}$ holds for some odd number $k \leq r-2$. The effect of σ_1 on the root space of a root $\alpha = m_1\alpha_1 + \ldots + m_r\alpha_r$ is

$$\sigma_1|_{\mathfrak{g}^{\mathbb{C}}_\alpha} = \exp\left(\frac{\mathrm{i}\pi}{2}\sum_{j=1}^{k} \frac{2\langle \alpha_j, \alpha\rangle}{\langle \alpha_j, \alpha_j\rangle} + \frac{2\langle \delta_j, \alpha\rangle}{\langle \delta_j, \delta_j\rangle}\right)\mathrm{id} = (-1)^{m_1+\ldots+m_k}\,\mathrm{id},$$

and we will further have to distinguish between the cases $k = 1$ and $k > 1$.

The case $k = 1$. Here $\mathfrak{k}_1$ is equal to $\mathrm{Z}(\mathfrak{k}_1) \oplus \mathfrak{h}$, where $\mathrm{Z}(\mathfrak{k}_1)$ is one–dimensional and $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ associated with the simple roots $\alpha_2, \ldots, \alpha_r$. Let $\tilde{\alpha}_2, \ldots, \tilde{\alpha}_r$ denote their restrictions to the complexification of the standard maximal torus $\mathfrak{b} = \mathfrak{g} \cap \bigoplus_{i=2}^r [\mathfrak{g}^{\mathbb{C}}_{\alpha_i}, \mathfrak{g}^{\mathbb{C}}_{-\alpha_i}]$ of $\mathfrak{h}$ and put $\Pi_0 = \{\tilde{\alpha}_2, \ldots, \tilde{\alpha}_r\}$. Further let $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ be an automorphism induced by the automorphism $\Pi \longrightarrow \Pi$ which exchanges α_{r-1} with α_r and fixes all other roots, and observe that τ leaves invariant $\mathfrak{b}$ and Π_0, interchanges $\tilde{\alpha}_{r-1}$ with $\tilde{\alpha}_r$, and fixes all other elements of Π_0. Moreover, the connected subgroup $H \subseteq G$ with Lie algebra $\mathfrak{h}$ is compact, because $\mathfrak{h}$ is compact semisimple, and we claim that there exists a Weyl group element $h \in H$ such that $\mathrm{Ad}_h \circ \sigma_2 \circ \mathrm{Ad}_{h^{-1}} : \Pi_0 \longrightarrow \Pi_0$ coincides with $\tau : \Pi_0 \longrightarrow \Pi_0$. In fact, σ_2 leaves invariant $\mathfrak{b}$ and the Weyl group acts transitively on the set of Weyl chambers, so we may choose h such that $\mathrm{Ad}_h \circ \sigma_2 \circ \mathrm{Ad}_{h^{-1}}$ leaves invariant Π_0. Then $\mathrm{Ad}_h \circ \sigma_2 \circ \mathrm{Ad}_{h^{-1}} : \Pi_0 \longrightarrow \Pi_0$ cannot be the identity map, as otherwise the $r-1$ dimensional torus $\mathfrak{b}$ would be fixed by $\mathrm{Ad}_h \circ \sigma_2 \circ \mathrm{Ad}_{h^{-1}}$, which is impossible, because $\mathfrak{h}^{\sigma_2}$ has rank $r-2$. Now if $r \neq 5$, then the Dynkin diagram of $\mathfrak{h}^{\mathbb{C}}$ is either of type A_3 or of type D_{r-1} with $r-1 \geq 5$, and since there is only one non–trivial automorphism on such diagrams, the existence of h is verified for $r \neq 5$. If $r = 5$ we observe that

$$\sigma_2(\tilde{\alpha}_2) = \sigma_2(\alpha_2)|_{\mathfrak{b}^{\mathbb{C}}} = s_{\alpha_1} s_{\delta_1}(\alpha_2)|_{\mathfrak{b}^{\mathbb{C}}} = -\tilde{\delta}_2,$$

where $\tilde{\delta}_2$ denotes the restriction of δ_2, and that σ_2 fixes the remaining roots $\tilde{\alpha}_3$, $\tilde{\alpha}_4$, and $\tilde{\alpha}_5$. Let $h \in H$ be an element such that $(\mathrm{Ad}_h)^*$ corresponds on $\mathrm{span}_{\mathbb{R}}\Pi_0$ to the Weyl group element $w := s_\beta s_{\beta'}$ with $\beta = \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{\alpha}_4$ and $\beta' = \tilde{\alpha}_3 + \tilde{\alpha}_4 + \tilde{\alpha}_5$. Recalling that $\mathfrak{h}^{\mathbb{C}}$ is of type D_4, with triple node $\tilde{\alpha}_3$, we compute

$$w(\tilde{\alpha}_2) = \tilde{\alpha}_5,\ w(\tilde{\alpha}_3) = \tilde{\alpha}_3,\ w(\tilde{\alpha}_4) = -\tilde{\delta}_2, \text{ and } w(\tilde{\alpha}_5) = \tilde{\alpha}_2,$$

so $\mathrm{Ad}_h \circ \sigma_2 \circ \mathrm{Ad}_{h^{-1}}$ is as desired. Now we already observed that for rank reasons $\mathfrak{h}^{\sigma_2}$ equals $\mathfrak{k}_1 \cap \mathfrak{k}_2$. Since $\mathrm{Ad}_h(\mathfrak{h}^{\sigma_2})$ and $\mathfrak{f} := \mathfrak{h}^\tau$ share the maximal torus $\mathrm{Ad}_h(\mathfrak{s})$ (indeed, $\mathfrak{f}^{\mathbb{C}}$ is of type B_{r-2}), it hence suffices to verify that $\mathfrak{f}$ is totally non–cohomologous to zero in $\mathfrak{g}$. But if $\beta_1, \dots, \beta_{r-1}$ denotes the restrictions of the simple roots $\alpha_1, \dots, \alpha_{r-1}$ to the complexification of the maximal torus $\mathfrak{t}^\tau$ of $\mathfrak{g}^\tau$, then according to proposition 5.5 $\mathfrak{f}$ is the subalgebra of $\mathfrak{g}^\tau$ associated with $\beta_2, \dots, \beta_{r-1}$. This subalgebra is totally non–cohomologous to zero in $\mathfrak{g}^\tau$ by corollary 5.7, and since $\mathfrak{g}^\tau$ is a folded subalgebra, also $\mathfrak{f}$ is totally non–cohomologous to zero in $\mathfrak{g}$.

The case $k \geq 2$. We shall see that this case cannot occur, the reason being as follows. Let Δ_0 be the roots of $\mathfrak{k}_1^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ and observe that $\Delta_0^+ := \Delta_0 \cap \Delta^+$ is a notion of positivity. We claim that the simple roots $\Pi_0 \subseteq \Delta_0^+$ decompose as a disjoint union $\Pi_0 = \Pi_0' \cup \Pi_0''$ with

$$\Pi_0' = \{\alpha_i + \alpha_{i+1} \mid i \text{ even}, i < k\} \cup \{\alpha_{k+1}, \dots, \alpha_r\} \text{ and } \Pi_0'' = \{\alpha_j + \alpha_{j+1} \mid j \text{ odd}, j < k\} \cup \{\kappa\},$$

where $\kappa = \delta_{k-2} - \alpha_{k-1}$. Indeed, a root α is contained in Δ_0 if and only if the integer $u_1(\alpha) + \dots + u_k(\alpha)$ is even, so Π_0' and Π_0'' are subsets of Δ_0^+. Since none of the simple roots $\alpha_1, \dots, \alpha_k$ is contained in Δ_0, we conclude that $\alpha_i + \alpha_{i+1}$ is a simple root for $i < k$. Now the roots of $\mathfrak{g}^{\mathbb{C}}$ are contained in one of the sets

$$\mathrm{span}_{\mathbb{Z}}\big(\Pi \setminus \{\alpha_{r-1}\}\big) \cap \Delta,\ \mathrm{span}_{\mathbb{Z}}\big(\Pi \setminus \{\alpha_r\}\big) \cap \Delta, \text{ or } \left\{ \sum_{s=i}^{j-1} \alpha_s + 2 \sum_{t=j}^{r-2} \alpha_t + \alpha_{r-1} + \alpha_r \,\middle|\, i < j \leq r-1 \right\},$$

and the former two sets are root subsystems of type A_{r-1}. Hence, if β and β' are two elements of Δ_0^+ with $\kappa = \beta + \beta'$, then either $u_{k-2}(\beta) = 1$ or $u_{k-2}(\beta') = 1$. Without loss of generality, assume that $u_{k-2}(\beta) = 1$ and $u_{k-2}(\beta') = 0$. Then also $u_{k-1}(\beta) = 1$, for otherwise $\beta \notin \Delta_0$, and this implies that $u_{k-1}(\beta')$ vanishes too. But then β' is only a root if $u_k(\beta') < 2$, whence in order for $\beta \in \Delta_0$ to hold we must have $u_k(\beta) = 2$ and already $\beta = \kappa$. Thus, κ is simple and $\Pi_0' \cup \Pi_0'' \subseteq \Pi_0$. Since k is odd, Π_0' and Π_0'' consist, respectively, of $m := (k-1)/2 + (r-k)$ and $n := (k-1)/2 + 1$ elements, and since $m + n = r$, we actually have an equality $\Pi_0 = \Pi_0' \cup \Pi_0''$; in particular, $\mathfrak{k}_1$ is semisimple.

By examining the various root strings of the elements of Π_0' and Π_0'', we find that any two elements $\alpha' \in \Pi_0'$ and $\alpha'' \in \Pi_0''$ are strongly orthogonal, and that $\Delta_0 \cap \mathrm{span}_{\mathbb{Z}}\Pi_0$, $\Delta_0 \cap \mathrm{span}_{\mathbb{Z}}\Pi_0''$ are two root systems of types D_m and D_n (or A_n, if $n < 4$), respectively; for example, if $n \geq 4$, then $\alpha_{k-4} + \alpha_{k-3}$ is the triple node. Hence, if we let $\mathfrak{i}'$, $\mathfrak{i}''$ be the subalgebras of $\mathfrak{k}_1$ associated with the simple roots Π_0' and Π_0'', then $\mathfrak{k}_1 = \mathfrak{i}' \oplus \mathfrak{i}''$ as Lie algebras and σ_2 either interchanges $\mathfrak{i}'$ with $\mathfrak{i}''$ or leaves both ideals invariant. However, $\sigma_2(\alpha_r) = \alpha_r$, because α_r is contained in the common kernel of all roots in Γ^+. So σ_2 has to leave Π_0' and $\mathfrak{i}'$ invariant, hence also $\mathfrak{i}''$. But the fixed point set of an involution on a Lie algebra of type D_ℓ (or A_ℓ with $\ell \leq 3$) is at least of rank $\ell - 1$, whence $\mathfrak{k}_1 \cap \mathfrak{k}_2$ must be of rank at least $r - 2$, contradicting the inequality $r - |\Gamma^+| \leq r - 4$. □

Theorem 6.5. If $\mathfrak{g}^{\mathbb{C}}$ is of type E_6, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. We shall see that in this case the subalgebra $\mathfrak{k}_1 \cap \mathfrak{k}_2$ is actually of full rank. Assuming the contrary, then, as in the proofs for the previously dealt Lie algebra types, we may assume Γ to be in normal form and the simple roots $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ to be enumerated as in example 4.13. Thus,

$$\Gamma^+ = \{\delta, \delta_1, \delta_2, \alpha_3\}, \text{ where } \delta = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 \text{ and } \delta_i = \alpha_i + \ldots + \alpha_{4-i}.$$

Given a root $\alpha = m_1\alpha_1 + \ldots + m_6\alpha_6$, we have

$$\frac{2\langle \alpha, \delta\rangle}{\langle \delta, \delta\rangle} = m_6, \ \frac{2\langle \alpha, \delta_1\rangle}{\langle \delta_1, \delta_1\rangle} = m_1 + m_5 - m_6, \text{ and } \frac{2\langle \alpha, \delta_2\rangle}{\langle \delta_2, \delta_2\rangle} = m_2 + m_4 - m_1 - m_5 - m_6,$$

so σ_1 restricts to $(-1)^{m_3+m_6}\,\mathrm{id}$ on $\mathfrak{g}_\alpha^{\mathbb{C}}$. Hence, if Δ_0 are the roots of $\mathfrak{k}_1^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ and $\Delta_0^+ = \Delta_0 \cap \Delta^+$ is the notion of positivity induced by Δ^+, with simple roots Π_0, then

$$\Pi_0' = \{\alpha_1, \alpha_2, \alpha_3 + \alpha_6, \alpha_4, \alpha_5\} \text{ and } \Pi_0'' = \{\delta - \alpha_6\}$$

are two sets of positive roots. We claim that $\Pi_0 = \Pi_0' \cup \Pi_0''$. In fact, the elements of Π_0' are simple in Δ_0^+, because α_3 and α_6 are not roots of $\mathfrak{k}_1^{\mathbb{C}}$. Thus, it only remains to verify the simplicity of $\kappa := \delta - \alpha_6$, for then $\Pi_0' \cup \Pi_0''$ consists of 6 elements and $\mathfrak{k}_1$ hence is a Lie subalgebra of rank 6. So suppose that $\kappa = \beta + \beta'$ for roots $\beta, \beta' \in \Delta_0^+$ and write $\beta = m_1\alpha_1 + \ldots + m_6\alpha_6$, $\beta' = n_1\alpha_1 + \ldots + n_6\alpha_6$. Since $u_6(\kappa)$ is equal to 1, we may assume that $m_6 = 1$ and $n_6 = 0$. Then β' is contained in $\mathrm{span}_{\mathbb{Z}}(\Pi \setminus \{\alpha_6\})$, a root system of type A_5, and hence there exist integers $s \leq t \leq 5$ such that $n_i = 1$ for all $s \leq i \leq t$ and $n_i = 0$ else. However, the only simple root different from α_6 that is not perpendicular to $\kappa = \delta - \alpha_6$ is α_3, and since $\beta = \kappa - \beta'$ is a root, whereas $\kappa + \beta'$ is not, because δ is the highest root, it follows that

$$0 \neq \frac{2\langle \beta', \kappa\rangle}{\langle \beta', \beta'\rangle} = n_3 \frac{2\langle \alpha_3, \kappa\rangle}{\langle \beta', \beta'\rangle},$$

so $n_3 = 1$. But then β' is not a root of $\mathfrak{k}_1^{\mathbb{C}}$, because $n_3 + n_6 = 1$ is not an even number.

Therefore, $\Pi_0 = \Pi_0' \cup \Pi_0''$ and $\mathfrak{k}_1$ is semisimple. Let $\mathfrak{i}'$ and $\mathfrak{i}''$ be the subalgebras of $\mathfrak{k}_1$ associated with Π_0' and Π_0'', respectively. By examining the roots strings, we find that $\langle \alpha_3 + \alpha_6, \kappa\rangle = 0$, so the roots in Π_0' are orthogonal to κ, and that Π_0' is of type A_5. Therefore, $\mathfrak{k}_1 = \mathfrak{i}' \oplus \mathfrak{i}''$ is a direct sum of Lie algebras and σ_2 necessarily has to leave $\mathfrak{i}'$ invariant. Since the fixed point set of an involution on $\mathfrak{i}'$ has rank at least 3 and $r - |\Gamma^+| = 2$, we obtain the desired contradiction. □

Theorem 6.6. If $\mathfrak{g}^{\mathbb{C}}$ is of type E_8, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. Note that one of the normal forms states that Γ^+ consists of 8 elements, and hence cannot apply. The other normal form can be treated similarly as in the proof of theorem 6.5. Here are the details. Recall that $\Gamma^+ = \{\delta, \delta', \alpha_1, \delta_1\}$, where δ is the highest root, δ' is the highest root of the root subsystem of type E_7 spanned by the simple roots $\{\alpha_1, \ldots, \alpha_7\}$, and δ_1 is the highest root of the root subsystem of type D_6 spanned by the simple roots $\{\alpha_1, \ldots, \alpha_6\}$. Hence, if $\alpha = m_1\alpha_1 + \ldots + m_8\alpha_8$ is a root, then

$$\frac{2\langle \delta, \alpha\rangle}{\langle \delta, \delta\rangle} = m_8, \ \frac{2\langle \delta', \alpha\rangle}{\langle \delta', \delta'\rangle} = m_7 - m_8, \ \frac{2\langle \delta_1, \alpha\rangle}{\langle \delta_1, \delta_1\rangle} = m_2 - m_7 - m_8, \text{ and } \ \sigma_1|_{\mathfrak{g}_\alpha^{\mathbb{C}}} = (-1)^{m_1+m_8}\,\mathrm{id}\,.$$

Let Δ_0 be the set of roots of $\mathfrak{k}_1^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$. Then $\Pi_0' = \Pi \setminus \{\alpha_1, \alpha_8\} \cup \{\alpha_1 + \alpha_8\}$ and $\Pi_0'' = \{\delta - \alpha_8\}$ are two sets of positive roots with respect to the notion of positivity Δ_0^+ induced by Δ^+. Moreover, Π_0' are simple roots in Δ_0^+, and we check that $\kappa := \delta - \alpha_8$ is simple too. So let $\beta, \beta' \in \Delta_0^+$ with $\kappa = \beta + \beta'$ be given. Since $u_8(\kappa) = 1$, we may assume that $u_8(\beta) = 1$ and $u_8(\beta') = 0$. In particular, β' is contained in the root subsystem of type E_7 spanned by $\Pi \setminus \{\alpha_8\}$, and since $u_1(\delta) = 1$, $u_1(\beta')$ can either be 0 or 1. The latter possibility is excluded, because β' would not be a root of $\mathfrak{k}_1^{\mathbb{C}}$ otherwise, and so $u_1(\beta') = 0$. But the only root different from α_8 that is not perpendicular to κ is α_1, and since $\kappa - \beta' = \beta$ is a root and $\kappa + \beta'$ is not, we must have

$$0 \neq \frac{2\langle \beta', \kappa \rangle}{\langle \beta', \beta' \rangle} = u_1(\beta') \frac{2\langle \alpha_1, \kappa \rangle}{\langle \beta', \beta' \rangle} = 0,$$

which is impossible. Therefore, κ is simple too. Now it follows that $\Pi_0' \cup \Pi_0''$ is a simple system for $\mathfrak{k}_1^{\mathbb{C}}$ and that $\mathfrak{k}_1$ is a sum of two simple ideals of types E_7 and A_1. Since any involution on a Lie algebra of type E_7 has full rank, but $r - |\Gamma^+| = 4$, Γ must be empty. □

Theorem 6.7. If $\mathfrak{g}^{\mathbb{C}}$ is of type F_4, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. The proof is very similar to the proofs of theorems 6.5 and 6.6. We only need to consider the normal form in which Γ^+ is equal to $\{\delta, \delta_1, \gamma_2\}$, with $\delta = 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4$ the highest long root of Δ^+, $\delta_1 = 2\alpha_1 + 2\alpha_2 + \alpha_3$ the highest long root of the root subsystem of type C_3 spanned by $\{\alpha_1, \alpha_2, \alpha_3\}$, and γ_2 the highest short root of the root subsystem of type B_2 spanned by $\{\alpha_2, \alpha_3\}$. Note that δ_1 and γ_2 are long and short roots in Δ, respectively, so if $\alpha \in \Delta$ is arbitrary, with $\alpha = m_1\alpha_1 + m_2\alpha_2 + m_3\alpha_3 + m_4\alpha_4$, then

$$\frac{2\langle \delta, \alpha \rangle}{\langle \delta, \delta \rangle} = m_4, \ \frac{2\langle \delta_1, \alpha \rangle}{\langle \delta_1, \delta_1 \rangle} = m_1 - m_4, \ \frac{2\langle \gamma_2, \alpha \rangle}{\langle \gamma_2, \gamma_2 \rangle} = 2m_3 - m_1 - 2m_4, \text{ and } \ \sigma_1|_{\mathfrak{g}_\alpha^{\mathbb{C}}} = (-1)^{m_3+m_4}.$$

Denote by Δ_0 the roots of $\mathfrak{k}_1^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ and put $\Delta_0^+ := \Delta_0 \cap \Delta^+$. It follows that $\Pi_0' = \Pi \setminus \{\alpha_3, \alpha_4\} \cup \{\alpha_3 + \alpha_4\}$ and $\Pi_0'' = \{\kappa\}$, with $\kappa = \delta - \alpha_4$, are sets of positive roots in Δ_0. The elements of Π_0' are simple in Δ_0^+ and we claim that κ is simple as well. To see this, assume that $\kappa = \beta + \beta'$ holds for roots $\beta, \beta' \in \Delta_0^+$. Since $u_4(\kappa) = 1$, we may assume that $u_4(\beta) = 1$ and $u_4(\beta') = 0$. Then β' is contained in the root subsystem spanned by $\{\alpha_1, \alpha_2, \alpha_3\}$, and since its highest long root is δ_1, it follows that $u_3(\beta')$ equals 0 or 1. Hence, since β' is supposed to be a root of $\mathfrak{k}_1^{\mathbb{C}}$, we must have $u_3(\beta') = 0$. But the only simple root different from α_4 not perpendicular to κ is α_3, and since $\kappa - \beta'$ is a root but $\kappa + \beta'$ is not, we have

$$0 \neq \frac{2\langle \beta', \kappa \rangle}{\langle \beta', \beta' \rangle} = u_3(\beta') \frac{2\langle \alpha_3, \kappa \rangle}{\langle \beta', \beta' \rangle} = 0,$$

a contradiction. Thus, $\Pi_0' \cup \Pi_0'$ is a simple system for $\mathfrak{k}_1^{\mathbb{C}}$ and the subalgebras $\mathfrak{i}'$, $\mathfrak{i}''$ associated with Π_0', Π_0'' are actually two ideals. Their complexifications are Lie algebras of types A_3 and A_1, respectively, whence the fixed point set of σ_2 on $\mathfrak{k}_1$ is a subalgebra of rank at least 2. But the normal form dictates that the rank of $\mathfrak{k}_1 \cap \mathfrak{k}_2$ be 1, which is impossible. Therefore, Γ must be empty and $\mathfrak{k}_1 \cap \mathfrak{k}_2$ has full rank. □

Theorem 6.8. If $\mathfrak{g}^{\mathbb{C}}$ is of type E_7, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. Of the three normal forms that may appear by example 4.14, one states that Γ^+ consists of 7 elements and hence cannot apply. If $\Gamma^+ = \{\delta, \delta_1, \delta_3, \alpha_6\}$, where δ is the highest root of Δ^+ and δ_1, δ_3 are the highest roots of the root subsystems of types D_6, D_4 spanned by $\Pi \setminus \{\alpha_7\}$ and $\Pi \setminus \{\alpha_1, \alpha_2, \alpha_7\}$, respectively, then for a root $\alpha = m_1\alpha_1 + \ldots + m_7\alpha_7$ we have

$$\frac{2\langle \delta, \alpha \rangle}{\langle \delta, \delta \rangle} = m_7, \ \frac{2\langle \delta_1, \alpha \rangle}{\langle \delta_1, \delta_1 \rangle} = m_2 - m_7, \ \frac{2\langle \delta_3, \alpha \rangle}{\langle \delta_3, \delta_3 \rangle} = m_4 - m_2 m - m_7, \text{ and } \ \sigma_1|_{\mathfrak{g}_\alpha^{\mathbb{C}}} = (-1)^{m_6+m_7} \,\mathrm{id}\,.$$

It follows, similarly as in theorems 6.5 and 6.6, that $\Pi' = \Pi \setminus \{\alpha_7\} \cup \{\alpha_6 + \alpha_7\}$ and $\Pi'' = \{\delta - \alpha_7\}$ are two mutually orthogonal sets whose union is the set of simple roots for $\mathfrak{k}_1$ with respect to the notion of positivity induced by Δ^+. The sets Π' and Π'' give rise to a decomposition of $\mathfrak{k}_1$ into two ideals whose complexifications are of types

Type of ambient Lie algebra	Type of fixed point set
A_3	$\mathrm{A}_1 \oplus \mathrm{A}_1$ or C_2
A_{2r} $(r \geq 1)$	B_r
A_{2r+1} $(r \geq 2)$	C_{r+1} or D_{r+1}
D_{r+1} $(r \geq 3)$	$\mathrm{B}_p \oplus \mathrm{B}_{r-p}$
E_6	C_4 or F_4

Table 1. Possible Lie algebra type of the fixed point set of an outer involution on a complex simple Lie algebra. The case A_3 is listed separately to clarify the meaning of D_2.

D_6 and A_1, respectively, and since the fixed point set of an involution on a Lie algebra of type D_6 has rank at least 5, we conclude that this normal form cannot occur either.

Finally, suppose that $\Gamma^+ = \{\delta, \delta_1, \alpha_1\}$. Then σ_1 is given by $(-1)^{m_1}$ id on the root space of a root $\alpha = m_1\alpha_1 + \ldots + m_7\alpha_7$ and the fixed point set of σ_1 has a one–dimensional center; it decomposes as $\mathfrak{k}_1 = Z(\mathfrak{k}_1) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the subalgebra of $\mathfrak{k}_1$ associated with the simple roots $\Pi \setminus \{\alpha_1\}$. The proof now proceeds along the same lines as the proof of theorem 6.3: using the standard maximal torus, the standard set of roots, positive roots, and simple roots introduced in proposition 5.1, we find that $\mathfrak{h}^{\mathbb{C}}$ is of type E_6 and that the fixed point set $\mathfrak{f}$ of an automorphism $\tau : \mathfrak{h} \longrightarrow \mathfrak{h}$ induced by the non–trivial automorphism of the Dynkin diagram of $\mathfrak{h}^{\mathbb{C}}$ shares a maximal torus with $\mathfrak{h}$, because $\mathfrak{k}_1 \cap \mathfrak{k}_2$ has rank 4 and $\sigma_2 : \mathfrak{h} \longrightarrow \mathfrak{h}$ hence cannot be an inner automorphism. Moreover, $\mathfrak{f}$ is a folded subalgebra in $\mathfrak{h}$, with $\mathfrak{f}^{\mathbb{C}}$ of type F_4, and the kernel of the surjective map $\mathrm{A}_{\mathfrak{h}} \longrightarrow \mathrm{A}_{\mathfrak{f}}$ is the ideal J generated by all polynomials of odd degree. Since $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{h}}/J$ is a surjection by proposition 5.12, $\mathfrak{f}$ is totally non–cohomologous to zero in $\mathfrak{g}$. □

Theorem 6.9. If $\mathfrak{g}^{\mathbb{C}}$ is of type G_2, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathrm{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. Indeed, $\mathfrak{k}_1 \cap \mathfrak{k}_2$ must be of full rank and Γ must be empty, for if Γ was non–empty, the only normal form for root systems of type G_2 would state that Γ^+ consists of two elements, whence $\mathfrak{s}$ would be trivial. □

6.2. Outer automorphisms

We continue to use the notation established in the previous section, but this time we assume that σ_1 is an outer automorphism; σ_2 might be inner or outer. For this case we will have to employ the classification of involutive automorphisms on complex simple Lie algebras given in [14, Theorem 5.15, chap. X], or more precisely the classification of the type of the fixed point set of such automorphisms presented in [14, Tables II and III, pp. 514 and 515]. For the convenience of the reader we have reproduced the classification results for the cases that arise from non–inner automorphisms in table 1.

As an immediate consequence of this classification we have

Theorem 6.10. If $\mathfrak{g}^{\mathbb{C}}$ is of type E_6, then some $\mathfrak{h} \in [\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathrm{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. According to table 1 $\mathfrak{k}_1^{\mathbb{C}}$ is either of type C_4 or F_4 and hence does not admit any outer automorphism. Therefore, $\sigma_2 : \mathfrak{k}_1 \longrightarrow \mathfrak{k}_1$ fixes a maximal torus of $\mathfrak{k}_1$, whence $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathrm{f}} = [\mathfrak{k}_1]_{\mathrm{f}}$, and we already know that there exists a subalgebra $\mathfrak{h} \in [\mathfrak{k}_1]_{\mathrm{f}}$ with the desired property. □

Proposition 6.11. Suppose that the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ only admits one non–trivial automorphism $A : \Pi \longrightarrow \Pi$. Then there exists an involution $\tau : G \longrightarrow G$ induced by A and elements $t \in T_1$, $n \in \mathrm{N}_H(T_1) \cdot T$ such that $\sigma_1 = c_t \circ \tau$ and $\sigma_2 = c_n \circ \nu$, where $\nu \in \{\tau, \mathrm{id}_G\}$ and $H = (G^\tau)_0$.

Remark 6.12. If one only requires that n be contained in $\mathrm{N}_G(T_1)$, then the statement of proposition 6.11 is known, cf. [23, Lemma 5.3]. The point of proposition 6.11 is that we can take $n \in \mathrm{N}_H(T_1) \cdot T$.

Proof. First of all note that $\sigma_1 : \Pi \longrightarrow \Pi$ and A must coincide, because σ_1 is not inner, whence both maps are non–trivial automorphisms on the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$. Now let $\Pi' \subseteq \Pi$ be the set of all simple roots that are fixed by σ_1 and choose a subset $\Pi'' \subseteq \Pi \setminus \Pi'$ with the property that $\Pi = \Pi' \cup \Pi'' \cup \sigma_1(\Pi'')$ is a disjoint union; such a decomposition exists, because $\sigma_1 : \Pi \longrightarrow \Pi$ is an automorphism of order 2. Next, pick a non–zero weight vector E_α for each simple root $\alpha \in \Pi' \cup \Pi''$ and put $E_\beta := \sigma_1(E_{\sigma_1(\beta)})$ for all $\beta \in \sigma_1(\Pi'')$. In this way we obtain a collection of root vectors $\{E_\alpha \,|\, \alpha \in \Pi\}$, one for each simple root. The elements of Π constitute a real basis of $(\mathrm{i}\mathfrak{t})^*$, because $\mathfrak{g}$ is (semi–)simple, and the only eigenvalues of σ_1 are 1 and (−1), so we may pick an element $X \in \mathfrak{t}$ such that $\alpha(\mathrm{i}X) = \pi$, if $\sigma_1(E_\alpha) = -E_\alpha$, and $\alpha(\mathrm{i}X) = 0$ else. In particular, X is contained in the joint kernel of all members of Π'' and $\sigma_1(\Pi'')$, and since $\alpha(X + \sigma_1(X)) = 2\alpha(X)$ holds for all $\alpha \in \Pi$, we even have $X \in \mathfrak{t}_1$. Now consider the automorphism $\tau := c_{\exp(X)} \circ \sigma_1$. It satisfies $\tau(E_\alpha) = E_{A^{-1}(\alpha)}$ for all $\alpha \in \Pi$, and since τ and σ_1 agree on $\mathfrak{t}$, τ is an automorphism induced by A.

For the second part of the statement, we first show that $\sigma_2 = c_n \circ \nu$ holds, at least with n an element of $\mathrm{N}_G(T_1)$. This is true if σ_2 is an inner automorphism (cf. proposition 3.2), so let us assume that σ_2 is outer. Note that σ_2 permutes the roots of $\mathfrak{t}^{\mathbb{C}}$, and so maps the Weyl chamber defined by Δ^+ (that is, the set of elements $Y \in \mathrm{i}\mathfrak{t}$ with $\alpha(Y) > 0$ if and only if $\alpha \in \Delta^+$) onto a different Weyl chamber. As is known, the Weyl group acts transitively on the set of Weyl chambers, hence we find $g \in \mathrm{N}_G(T)$ such that $\mathrm{Ad}_{g^{-1}} \circ \sigma_2$ preserves the Weyl chamber defined by Δ^+, and consequently also Δ^+ and Π. Also note: $c_{g^{-1}} \circ \sigma_2$ is outer, because an automorphism is inner if and only if it fixes a maximal torus. In particular, the non–trivial map $\mathrm{Ad}_{g^{-1}} \circ \sigma_2 : \Pi \longrightarrow \Pi$ must coincide with A, so we may use the weight vectors $\{E_\alpha \,|\, \alpha \in \Pi\}$ chosen earlier and proceed analogously as in the first part of the proof to find $t' \in T$ such that $c_{g^{-1}} \circ \sigma_2 = c_{t'} \circ \tau$, the only difference to the proof of the first part being that the root vector E_α of a root $\alpha \in \Pi''$ is not necessarily mapped onto $E_{A(\alpha)}$, but rather a scalar multiple of $E_{A(\alpha)}$; this is why t' can only be assumed to be an element of T. Then we have $\sigma_2 = c_{gt'} \circ \tau$ and $n := gt'$ must be contained in $\mathrm{N}_G(T_1)$ by proposition 3.2.

To conclude the proof, it thus will suffice to show that $\mathrm{N}_G(T_1) = \mathrm{N}_H(T_1) \cdot T$. To this end, recall that T is the unique maximal torus of G containing T_1, so any element of $X := \mathrm{N}_G(T_1)$ also normalizes T, whence $P := \mathrm{N}_H(T_1) \cdot T$ actually is a (closed) subgroup of X. Consider the inclusion induced diagram

$$X/P \longrightarrow \mathrm{N}_G(T)/P \longleftarrow \mathrm{N}_G(T)/T.$$

The left hand map is injective and the right hand map is a surjection originating from the Weyl group of G, so X/P is a finite set. To compute its number of elements, we use [8, Proposition 2.3], according to which

$$\dim \mathrm{H}^0(X/P) = \frac{\dim \mathrm{H}^0(X)}{\dim \mathrm{H}^0(P)} \cdot \dim \mathrm{H}^0(P \cap X_0).$$

Now $P = \mathrm{N}_H(T_1) \cdot T$ is a space having as many components as $\mathrm{N}_H(T_1)$ does, because T is path–connected, and the identity component of $X = \mathrm{N}_G(T_1)$ is $\mathrm{Z}_G(T_1) = T$. Moreover, H is a folded subgroup with maximal torus T_1, and it was shown in [7, Proposition 4.4] that the number of connected components of $\mathrm{N}_G(T_1)$ equals the number of connected components of $\mathrm{N}_H(T_1)$. Therefore, X/P is connected, and so $\mathrm{N}_G(T_1) = \mathrm{N}_H(T_1) \cdot T$. □

Now suppose that the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ only admits one non–trivial automorphism. Then by propositions 3.4 and 6.11 we may assume that $\sigma_1 = c_t \circ \tau$ and that $\sigma_2 = c_n \circ \nu$, where $t \in T$, $n \in \mathrm{N}_H(T_1)$, $\nu \in \{\tau, \mathrm{id}_G\}$, and H is the identity component of the fixed point set of an automorphism $\tau : G \longrightarrow G$ induced by the non–trivial Dynkin diagram automorphism $\Pi \longrightarrow \Pi$. Moreover, if $w \in W(\Delta)$ is a Weyl group element that is represented by $(\mathrm{Ad}_h)^*$ for some element $h \in H$, then we may assume that Γ is equal to $w(\Gamma)$: in fact, in this case c_h and τ commute, and since $\mathfrak{t}_1 = \mathfrak{t}^\tau$, h must be an element of $\mathrm{N}_H(T_1)$. Thus, $c_{h^{-1}} \circ \sigma_1 \circ c_h = c_{(c_h)^{-1}(t)} \circ \tau$ and $c_{h^{-1}} \circ \sigma_2 \circ c_h = c_{(c_h)^{-1}(n)} \circ \tau$ are two commuting involutions with common fixed subalgebra $\mathrm{Ad}_h(\mathfrak{k}_1 \cap \mathfrak{k}_2)$ and for which $w(\Gamma)$ is the set of roots vanishing on the maximal torus $\mathrm{Ad}_h(\mathfrak{s})$.

Having applied all desired transformations to Γ, we define $\Pi_{\mathrm{odd}} \subseteq \Pi$ to be the set of simple roots β for which $p(\beta) := \sum_{\alpha \in \Gamma^+} 2\langle \alpha, \beta \rangle / \langle \alpha, \alpha \rangle$ is odd and put $\Pi_{\mathrm{even}} := \Pi \setminus \Pi_{\mathrm{odd}}$. We have shown in corollary 3.7 and proposition 3.5 that no root in Π_{odd} is fixed by τ and that for any choice of integers $\{\epsilon_\alpha \in \{\pm 1\} \,|\, \alpha \in \Pi_{\mathrm{odd}}\}$ such that $\epsilon_\alpha = -\epsilon_{\tau(\alpha)}$ there exists an element $p \in T$ with the property that for all simple roots α the map Ad_p is multiplication with $\epsilon_\alpha \mathrm{i}$ on $\mathfrak{g}_\alpha^{\mathbb{C}}$, if $\alpha \in \Pi_{\mathrm{odd}}$, and equal to id else. We also showed that there exists $s \in T$ such that Ad_s restricts to multiplication with $\exp(\mathrm{i}\pi \sum_{\alpha\in\Gamma^+}\langle\alpha,\beta\rangle/\langle\alpha,\alpha\rangle)$ on $\mathfrak{g}_\alpha^{\mathbb{C}}$, that $\nu_1 := c_{ps}$ is an involution which commutes with $\nu_2 := \sigma_2$, and that $\mathfrak{s}$ is a maximal torus for the common fixed point set of ν_1 and ν_2.

Theorem 6.13. If $\mathfrak{g}^{\mathbb{C}}$ is of type A_r, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. Note that if $r = 2k$ is even, then $\mathfrak{k}_1^{\mathbb{C}}$ is of type B_k by table 1 and hence only admits inner automorphisms. Thus, $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_{\mathfrak{f}} = [\mathfrak{k}_1]_{\mathfrak{f}}$ and the claim follows in this case, cf. also theorem 6.10.

Henceforth, we assume that $r = 2k - 1$ is odd. Then $k \geq 2$, necessarily, and we further suppose that $\sigma_2 : \mathfrak{k}_1 \longrightarrow \mathfrak{k}_1$ is not an inner automorphism. By table 1 $\mathfrak{k}_1^{\mathbb{C}}$ must be either of type $\mathrm{A}_1 \oplus \mathrm{A}_1$ or of type D_k and $\sigma_2 : \mathfrak{k}_1 \longrightarrow \mathfrak{k}_1$ either interchanges the two simple summands of $\mathfrak{k}_1$ or is an outer automorphism of order two. In both cases we have $\operatorname{rank}(\mathfrak{k}_1 \cap \mathfrak{k}_2) = \operatorname{rank}(\mathfrak{k}_1) - 1$ and $\operatorname{rank}\mathfrak{k}_1 = k$. Now note that the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$ only admits one non–trivial automorphism, so the considerations preceeding this theorem apply. In particular, we know from corollary 3.3 that $\operatorname{rank}(\mathfrak{k}_1 \cap \mathfrak{k}_2) = \operatorname{rank}(\mathfrak{k}_1) - |\Gamma^+|$, so Γ^+ consists of a single element γ, and this element must be fixed by τ because of lemma 3.6. If we enumerate the simple roots Π as in example 4.11, then τ maps a root α_{k-i} to α_{k+i}, and since the elements of Δ^+ are of the form $\alpha_i + \alpha_{i+1} + \ldots + \alpha_j$ for integers i, j with $i \leq j$, we hence have $\gamma = \alpha_{k-\ell} + \ldots + \alpha_{k+\ell}$ for some $\ell = 0, \ldots, k-1$.

To conclude the theorem, let us consider the involutions ν_1 and ν_2 constructed earlier. Observe that

$$\frac{2\langle\gamma,\alpha_i\rangle}{\langle\gamma,\gamma\rangle} = \begin{cases} 1, & i = k-\ell \text{ or } i = k+\ell, \\ -1, & i = k-\ell-1 \text{ or } i = k+\ell+1, \\ 2, & i = k \text{ and } \ell = 0, \\ 0, & \text{else,} \end{cases}$$

and recall that we need to choose integers $\{\epsilon_\alpha \in \{\pm 1\} \mid \alpha \in \Pi_{\mathrm{odd}}\}$ satisfying $\epsilon_\alpha = -\varepsilon_{\tau(\alpha)}$ in order to define ν_1. We define ϵ_α to be equal to 1 if and only if $\alpha = \alpha_i$ for some $i \leq k$. With this choice it follows that for any root $\alpha \in \Delta$ with $\alpha = m_1\alpha_1 + \ldots + m_{2k-1}\alpha_{2k-1}$ we have $\nu_1|_{\mathfrak{g}_\alpha^{\mathbb{C}}} = (-1)^m$, where

$$m = \begin{cases} m_1, & \ell = k-1, \\ m_{k-\ell} + m_{k+\ell+1}, & \text{else.} \end{cases}$$

Now we are ready to determine the fixed point subalgebra $\mathfrak{f}_1$ of ν_1. In fact, if $\Delta_0 \subseteq \Delta$ are the roots of $\mathfrak{f}_1^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ and Δ_0^+ is the notion of positivity induced by Δ^+, with resulting simple roots Π_0, then

$$\Pi_0 = \begin{cases} \Pi \setminus \{\alpha_1\}, & \ell = k-1, \\ \Pi \setminus \{\alpha_{k-\ell}, \alpha_{k+\ell+1}\} \cup \{\gamma + \alpha_{k+\ell+1}\}, & \text{else,} \end{cases}$$

because a root $\alpha \in \Delta_0^+$ with $\alpha = m_1\alpha_1 + \ldots + m_{2k-1}\alpha_{2k-1}$ either has $m_1 = 0$, if $\ell = k-1$, or $m_{k-\ell} = m_{k+\ell+1}$, if $\ell \neq k-1$. Thus, $\mathfrak{f}_1 = \mathrm{Z}(\mathfrak{f}_1) \oplus \mathfrak{m}$ has a one–dimensional center $\mathrm{Z}(\mathfrak{f}_1)$ given by the common kernel of the $2k-2$ elements in Π_0. Moreover, the roots of $\mathfrak{f}_1$ correspond bijectively to the roots of $\mathfrak{m}^{\mathbb{C}}$ with respect to $\mathfrak{m}^{\mathbb{C}} \cap \mathfrak{t}^{\mathbb{C}}$ via restriction and the positive roots induce a notion of positivity whose simple roots are the restrictions of the elements in Π_0. In particular, if $\ell = k-1$, then $\mathfrak{m}$ is the subalgebra of $\mathfrak{g}$ associated with $\alpha_2, \ldots, \alpha_{2k-1}$ and hence is of type $\mathrm{A}_{2(k-1)}$. Since we already know that $\operatorname{rank}(\mathfrak{k}_1 \cap \mathfrak{k}_2) = k-1$, $\nu_2 : \mathfrak{m} \longrightarrow \mathfrak{m}$ hence must be an outer automorphism and $\mathfrak{f}_1 \cap \mathfrak{f}_2 = \mathfrak{m}^{\nu_2}$, where $\mathfrak{f}_2 = \mathfrak{g}^{\nu_2}$. Since $[\mathfrak{f}_1 \cap \mathfrak{f}_2]_{\mathfrak{f}}$ contains a subalgebra which is totally non–cohomologous to zero in $\mathfrak{m}$ and $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{m}}$ is surjective, the claim follows if $\ell = k-1$.

An analogous argument shows that the claim also holds if $\ell = 0$: for $i = 1, \ldots, k$ we define $\mathfrak{m}_i$ to be the subalgebra of $\mathfrak{g}$ associated with $\Pi \setminus \{\alpha_i, \alpha_{i+1}\} \cup \{\alpha_i + \alpha_{i+1}\}$. Then $\mathfrak{m}_k = \mathfrak{m}$ and $\mathrm{Ad}_g : \mathfrak{m}_i \longrightarrow \mathfrak{m}_{i+1}$ is an isomorphism, provided that $g \in \mathrm{N}_G(T)$ is such that $(\mathrm{Ad}_{g^{-1}})^* = s_{\alpha_{i+1}}$. Since the inner automorphism corresponding to s_{α_1} maps the Lie subalgebra of $\mathfrak{g}$ associated with $\{\alpha_2, \ldots, \alpha_r\}$ onto $\mathfrak{m}_1$, it hence follows that $\mathfrak{m}$ is totally non–cohomologous to zero in $\mathfrak{g}$, so the equivalence class $[\mathfrak{f}_1 \cap \mathfrak{f}_2]_\mathfrak{f}$ of $\mathfrak{f}_1 \cap \mathfrak{f}_2 = \mathfrak{m}^{\nu_2}$ will contain a respresentative which is totally non–cohomologous to zero in $\mathfrak{g}$ as well.

Finally, we consider the case $\ell \neq 0, k-1$. One can show, but we will not, that $\mathfrak{m}$ is a sum of two simple ideals whose complexifications are of types $A_{2\ell}$ and $A_{2(k-\ell-1)}$, and that these ideals are ν_2–invariant. Since the arguments of the previous cases cannot be adapted to this situation, we compute $\mathfrak{s}$ instead and explicitly construct a subalgebra which is totally non–cohomologous to zero in $\mathfrak{g}$.

To this end, recall that by corollary 3.3 $\mathfrak{s}$ is the fixed point set of s_{H_γ} on $\mathfrak{t}_1 = \mathfrak{t}^\tau$ and that $H_\gamma = H_{\alpha_{k-\ell}} + \ldots + H_{\alpha_{k+\ell}}$. We also know that $\mathfrak{t}_1$ is a maximal torus of $\mathfrak{h} = \mathfrak{g}^\tau$, that the restrictions $\beta_1, \ldots, \beta_k$ of $\alpha_1, \ldots, \alpha_k$ to $\mathfrak{t}_1^\mathbb{C}$ form a set of simple roots for the notion of positivity induced by Δ^+, and that $\mathfrak{h}^\mathbb{C}$ is of type C_k, with long root β_k. In particular, since the elements $L_i = 1/2(H_{\alpha_i} + H_{\alpha_{2k-i}})$ are fixed by τ and satisfy $\langle L_i, \cdot \rangle = \beta_i$ on $\mathfrak{t}_1$, we must have $\mathbb{C}L_i = [\mathfrak{h}^\mathbb{C}_{\beta_i}, \mathfrak{h}^\mathbb{C}_{-\beta_i}]$ for all $i = 1, \ldots, k$. Moreover, the elements L_i with $i < k-\ell-1$ or $i > k-\ell$ are fixed by s_{H_γ}, as is $L_{k-\ell-1} + L_{k-\ell}$, and since $\mathfrak{s}$ is of rank $k-1$, they must comprise a basis of i$\mathfrak{s}$. So, if we write $\Phi = \{\beta_1, \ldots, \beta_k\}$ and define $\mathfrak{m}_i$ to be the subalgebra of $\mathfrak{h}$ associated with $\Phi \setminus \{\beta_i, \beta_{i+1}\} \cup \{\beta_i + \beta_{i+1}\}$, then $\mathfrak{m}_{k-\ell-1}$ shares the maximal torus $\mathfrak{s}$ with $\mathfrak{f}_1 \cap \mathfrak{f}_2$. But the Weyl group element Ad_x of $\mathfrak{h}$ with $(\mathrm{Ad}_{x^{-1}})^* = s_{\beta_i}$ and $x \in \mathrm{N}_H(T_1)$ maps $\mathfrak{m}_i$ isomorphically onto $\mathfrak{m}_{i+1}$, and the Weyl group element of $\mathfrak{h}$ representing s_{β_1} maps $\mathfrak{m}_1$ isomorphically onto the subalgebra of $\mathfrak{h}$ associated with $\{\beta_2, \ldots, \beta_k\}$. Since the latter is totally non–cohomologous to zero in $\mathfrak{h}$ by corollary 5.7 and $\mathfrak{h}$ is totally non–cohomologous to zero in $\mathfrak{g}$, it follows that $\mathfrak{m}_{k-\ell-1}$ must be totally non–cohomologous to zero in $\mathfrak{g}$ as well. Hence, $\mathfrak{m}_{k-\ell-1} \in [\mathfrak{f}_1 \cap \mathfrak{f}_2]_\mathfrak{f}$ is the desired subalgebra. □

Theorem 6.14. If $\mathfrak{g}^\mathbb{C}$ is of type D_r, $r \geq 4$, then some element in $[\mathfrak{k}_1 \cap \mathfrak{k}_2]_\mathfrak{f}$ is non–cohomologous to zero in $\mathfrak{g}$.

Proof. We first note that we may assume $r = 2k+1$: from the classification we know that $\mathfrak{k}_1 = \mathfrak{i}' \oplus \mathfrak{i}''$ is a sum of two simple ideals whose complexfications are of types B_k and B_{r-k-1}, respectively. Since Lie algebras of type B_m only admit inner automorphisms (even in the case $m = 1$), it follows that σ_2 either fixes a maximal torus of $\mathfrak{k}_1$ or that σ_2 interchanges $\mathfrak{i}'$ and $\mathfrak{i}''$. In the former case $\mathfrak{k}_1 \in [\mathfrak{k}_1 \cap \mathfrak{k}_2]_\mathfrak{f}$ is totally non–cohomologous to zero in $\mathfrak{g}$, and in the latter case $\operatorname{rank} \mathfrak{i}' = \operatorname{rank} \mathfrak{i}''$, so $r = 2k+1$.

In particular, $r \neq 4$, so the Dynkin diagram of $\mathfrak{g}^\mathbb{C}$ only admits one non–trivial automorphism and the maps ν_1, ν_2 are defined. Enumerate the simple roots Π as in example 4.12. We show that the root reflections $s_{\alpha_1}, \ldots, s_{\alpha_{2k-1}}$, and $s_{\delta_{2k-1}}$, where $\delta_{2k-1} = \alpha_{2k-1} + \alpha_{2k} + \alpha_{2k+1}$, can be represented by $(\mathrm{Ad}_h)^*$ with $h \in \mathrm{N}_H(T_1)$. Indeed, it is a well–known fact (cf. the proof of [16, Theorem 4.54, sect. IV.6]) that the root reflection s_α of a root α is represented by $\mathrm{Ad}_{\exp(X)}$ for some element $X \in \mathfrak{g}^\mathbb{C}_\alpha \oplus \mathfrak{g}^\mathbb{C}_{-\alpha}$, so if τ is the identity on $\mathfrak{g}^\mathbb{C}_\alpha$, then s_α can be represented by some element in $\mathrm{N}_H(T_1)$. This is definitely the case for the simple roots $\alpha_1, \ldots, \alpha_{2k-1}$, just by definition of an automorphism induced by a Dynkin diagram automorphism; and if X_{2k-1}, X_{2k}, and X_{2k+1} are non–zero weight vectors for the roots α_{2k-1}, α_{2k}, and α_{2k+1}, respectively, such that $\tau(X_{2k}) = X_{2k+1}$, then $[X_{2k+1}, [X_{2k-1}, X_{2k}]]$ is a non–zero root vector for δ_{2k-1} and

$$\tau\big([X_{2k+1}, [X_{2k-1}, X_{2k}]]\big) = [X_{2k}, [X_{2k-1}, X_{2k+1}]] = [[X_{2k}, X_{2k-1}], X_{2k+1}] = [X_{2k+1}, [X_{2k-1}, X_{2k}]],$$

because $\mathrm{ad}_{X_{2k}}$ is a derivation and α_{2k}, α_{2k+1} are perpendicular.

Now observe that the root subsystem Ω of Δ spanned by the roots $\{\alpha_1, \ldots, \alpha_{2k-1}, \delta_{2k-1}\}$ is of type D_{2k}, with triple node α_{2k-2} connected to the mutually perpendicular roots α_{2k-3}, α_{2k-1}, and δ_{2k-1}. Moreover, we deduce from our explicit description of the roots Δ given in theorem 6.4 that Ω is equal to the set of all roots in Δ which are fixed by τ, whence Γ is a set of strongly orthogonal roots in Ω. Thus, if we can show that p is even valued on Ω, then it follows from theorem 4.9 together with our discussion before theorem 6.13 that we may assume Γ^+ to be in one of the normal forms obtained in example 4.12, because the Weyl group $W(\Omega)$ is generated by the root reflections $s_{\alpha_1}, \ldots, s_{\alpha_{2k-1}}$, and $s_{\delta_{2k}}$. However, $p(\alpha_i)$ is even for roots $i \leq 2k-1$, because such roots are fixed by τ and hence contained in Π_{even}. Moreover, the roots in Γ are fixed by τ as well, whence $p \circ \tau = p$, and since $\tau(\alpha_{2k}) = \alpha_{2k+1}$, it hence follows that $p(\delta_{2k-1}) = p(\alpha_{2k-1}) + 2p(\alpha_{2k})$ is even. Combined with the facts $\operatorname{rank} \mathfrak{k}_1 = 2k$, $\operatorname{rank}(\mathfrak{k}_1 \cap \mathfrak{k}_2) = k$, and $|\Gamma^+| = \operatorname{rank} \mathfrak{k}_1 - \operatorname{rank}(\mathfrak{k}_1 \cap \mathfrak{k}_2)$, we henceforth assume Γ^+ to be equal to one of the sets $\{\alpha_1, \delta_1, \alpha_3, \delta_3, \ldots, \alpha_{k-1}, \delta_{k-1}\}$, $\{\delta_1, \delta_3, \ldots, \delta_{2k-1}\}$, or $\{\delta_1, \delta_3, \ldots, \delta_{2k-3}, \alpha_{2k-1}\}$, where

$\delta_i = \alpha_i + 2(\alpha_{i+1} + \ldots + \alpha_{2k-1}) + \alpha_{2k} + \alpha_{2k+1}$. Suppose that $\delta_{2k-1} \in \Gamma^+$, so Γ^+ is the second of the three sets in question. Then we have

$$p_j := p(\alpha_j) = \sum_{\alpha\in\Gamma^+} \frac{2\langle\alpha,\alpha_j\rangle}{\langle\alpha,\alpha\rangle} = \begin{cases} 1, & j = 2k, 2k+1, \\ 0, & \text{else}, \end{cases}$$

so $\Pi_{\mathrm{odd}} = \{\alpha_{2k}, \alpha_{2k+1}\}$ consists of two elements. Hence, if we choose $\epsilon_{\alpha_{2k}} = -\mathrm{i}$, $\epsilon_{\alpha_{2k+1}} = \mathrm{i}$, then ν_1 is $(-\mathrm{id})$ on $\mathfrak{g}^{\mathbb{C}}_{\alpha_{2k+1}}$ and the identity on the root space of all other simple roots cf. corollary 3.7. Thus, the fixed point set $\mathfrak{k}_1$ of ν_1 is $\mathfrak{k}_1 = \mathrm{Z}(\mathfrak{k}_1) \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the subalgebra of $\mathfrak{g}$ associated with the simple roots $\{\alpha_1, \ldots, \alpha_{2k}\}$. Its complexification is of type A_{2k} and since the inclusion $\mathfrak{m} \longrightarrow \mathfrak{g}$ induces a surjection $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{m}}$ modulo the ideal in $\mathrm{A}_{\mathfrak{m}}$ generated by all polynomials of odd degree, it follows that $\mathfrak{m}^{\nu_2} = (\mathfrak{k}_1)^{\nu_2}$ shares a maximal torus with a subalgebra that is totally non−cohomologous to zero in $\mathfrak{g}$, cf. also the proof of theorem 6.4.

The proof is similar in case α_{2k-1} is contained in Γ^+, the difference being that $p_{2k} = p_{2k+1} = -1$ and that $p_{2k-1} = 2$. Hence, if we let $\epsilon_{\alpha_{2k}} = \mathrm{i}$, $\epsilon_{\alpha_{2k+1}} = -\mathrm{i}$, then ν_1 is $(-\mathrm{id})$ on the root space of the roots α_{2k-1}, α_{2k+1} and the identity on the weight spaces of the remaining simple roots. Thus, $\mathfrak{k}_1 = \mathrm{Z}(\mathfrak{k}_1) \oplus \mathfrak{m}$, and this time $\mathfrak{m}$ is the subalgebra of $\mathfrak{g}$ associated with $\{\alpha_1, \ldots, \alpha_{2k-2}, \alpha_{2k-1} + \alpha_{2k+1}, \alpha_{2k}\}$. But for any element $g \in \mathrm{N}_G(T)$ such that $(\mathrm{Ad}_{g^{-1}})^* = s_{\alpha_{2k+1}}$ the automorphism Ad_g sends the subalgebra of $\mathfrak{g}$ associated with $\{\alpha_1, \ldots, \alpha_{2k}\}$ isomorphically onto $\mathfrak{m}$, so $\mathrm{A}_{\mathfrak{g}} \longrightarrow \mathrm{A}_{\mathfrak{m}}$ is surjective modulo the ideal $J \subseteq \mathrm{A}_{\mathfrak{m}}$ generated by all polynomials of odd degree too. Since J is the kernel of any involution on $\mathfrak{m}$ that is induced by the non−trivial Dynkin diagram automorphism on some Cartan subalgebra of $\mathfrak{m}^{\mathbb{C}}$, the claim follows.

Now suppose that Γ^+ is neither of the two previous sets. Then Π_{odd} is empty and $\nu_1 = (-1)^{m_1+\ldots+m_{k-1}}$. The proof given in theorem 6.4 carries over almost verbatimely if $k-1 > 1$, the only difference being that ν_2 does not fix α_{2k+1} but α_{2k-1}. Hence, the same rank considerations show that this case cannot occur. If $k-1 = 1$, we note that ν_2 fixes $L_3 = H_{\alpha_3}$ and $L_4 = 1/2(H_{\alpha_4} + H_{\alpha_5})$, so $\mathrm{i}L_3$ and $\mathrm{i}L_4$ are basis vectors for $\mathfrak{s}$. Moreover, if we write $\tilde{\alpha}_i$ for the restriction of α_i to the complexification of $\mathfrak{t}_1 = \mathfrak{t}^\tau$, then $\langle L_3, \cdot\rangle = \tilde{\alpha}_3$ and $\langle L_4, \cdot\rangle = \tilde{\alpha}_4$, $\mathfrak{h}^{\mathbb{C}}$ is of type B_4, and $\{\tilde{\alpha}_1, \ldots, \tilde{\alpha}_4\}$ is a set of simple roots on $\mathfrak{t}_1^{\mathbb{C}}$ with respect to the notion of positivity induced by Δ^+, with short root $\tilde{\alpha}_4$. Therefore, $\mathfrak{s}$ is the maximal torus of the subalgebra $\mathfrak{m}$ of $\mathfrak{h}$ associated with $\{\tilde{\alpha}_3, \tilde{\alpha}_4\}$, which is totally non−cohomologous to zero in $\mathfrak{h}$ by corollary 5.7, and since $\mathfrak{h}$ is a folded subalgebra, $\mathfrak{m}$ is also totally non−cohomologous to zero in $\mathfrak{g}$. □

Chapter III.

An algebraic model for the equivariant cohomology of isotropy actions

1. $\mathfrak{g}$–actions

Let $\mathfrak{g}$ be a (finite–dimensional real) Lie algebra, $\Lambda(\mathfrak{g})$ the exterior algebra of $\mathfrak{g}$, and $\Omega(\mathfrak{g})$ the space of alternating forms on $\mathfrak{g}$. Then $\Omega(\mathfrak{g})$ is a differential graded $\mathbb{R}$–algebra with respect to the exterior derivative d, which is the unique anti–derivation, homogeneous of degree 1, such that $d\omega(X, Y) = -\omega([X, Y])$ holds for all $\omega \in \Omega^1(\mathfrak{g})$ and all $X, Y \in \mathfrak{g}$. We can use d to introduce a differential on $\Lambda(\mathfrak{g})$: for each integer $p \geq 0$ there is a canonical and non–degenerate pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle :\ & \Omega^p(\mathfrak{g}) \otimes \Lambda^p(\mathfrak{g}) \longrightarrow \mathbb{R}, \\ & \omega \otimes (X_1 \wedge \ldots \wedge X_p) \longmapsto \omega(X_1, \ldots, X_p), \end{aligned}$$

so we may uniquely define a linear map $\partial : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$, homogeneous of degree −1, which is dual to (−d) with respect to this pairing, that is, such that we have $\langle \omega, \partial\lambda \rangle = -\langle \mathrm{d}\omega, \lambda \rangle$ for all $\omega \in \Omega^p(\mathfrak{g})$ and all $\lambda \in \Lambda^{p+1}(\mathfrak{g})$. The non–degeneracy of the pairing above readily shows that ∂ is a differential, but, unfortunately, it is not an anti–derivation on $\Lambda(\mathfrak{g})$ with respect to the canonical ring structure on $\Lambda(\mathfrak{g})$, unless d is trivial. In fact, ∂ vanishes on $\Lambda^1(\mathfrak{g})$, because d is zero on $\Omega^0(\mathfrak{g})$, while we have $\partial(X \wedge Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

Moreover, just as the adjoint map induces a representation $\mathfrak{g} \longrightarrow \mathrm{End}(\Omega(\mathfrak{g}))$, $X \longmapsto -(\mathrm{ad}_X)^*$, where $(\mathrm{ad}_X)^* : \Omega(\mathfrak{g}) \longrightarrow \Omega(\mathfrak{g})$ is the unique extension of $\mathrm{ad}_X : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ to a derivation in $\Omega(\mathfrak{g})$, we obtain a representation $\mathfrak{g} \longrightarrow \mathrm{End}(\Lambda(\mathfrak{g}))$ by extending each of the maps $\mathrm{ad}_X : \mathfrak{g} \longrightarrow \mathfrak{g}$ to a derivation $\mathrm{ad}_X : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$. The maps ad_X and $(\mathrm{ad}_X)^*$ then are dual to each other with respect to the canonical pairing between $\Omega(\mathfrak{g})$ and $\Lambda(\mathfrak{g})$, and we shall denote the subalgebra of all invariant elements in $\Lambda(\mathfrak{g})$ by $\Lambda(\mathfrak{g})^{\mathfrak{g}}$ or simply Λ, if there is no source for confusion.

Definition 1.1. Let (M, d) be a differential graded $\mathbb{R}$–module ($\mathbb{R}$–dgm for short), that is, a $\mathbb{Z}$–graded vector space M over $\mathbb{R}$ together with a differential $d : M \longrightarrow M$, homogeneous of degree 1. An *action of* $\mathfrak{g}$ in (M, d) is a tuple $(\mathrm{i}, \mathcal{L})$ consisting of $\mathbb{R}$–linear maps $\mathrm{i} : \mathfrak{g} \longrightarrow \mathrm{End}(M)$ and $\mathcal{L} : \mathfrak{g} \longrightarrow \mathrm{End}(M)$, subject to the following conditions, for all $X, Y \in \mathfrak{g}$:

(1) i_X is homogeneous of degree −1 and $\mathcal{L}_X$ is homogeneous of degree 0,

(2) we have $(\mathrm{i}_X)^2 = 0$ and $\mathrm{i}_{[X,Y]} = \mathcal{L}_X \circ \mathrm{i}_Y - \mathrm{i}_Y \circ \mathcal{L}_X$,

(3) $\mathcal{L}_{[X,Y]} = \mathcal{L}_X \circ \mathcal{L}_Y - \mathcal{L}_Y \circ \mathcal{L}_X$,

(4) $\mathcal{L}_X = d \circ \mathrm{i}_X + \mathrm{i}_X \circ d$.

We remark that if $\mathfrak{g}$ is the Lie algebra of a Lie group G, then [13] refers to (M, d) and $(\mathrm{i}, \mathcal{L})$ as a $G^\star$–module, cf. [13, Definition 2.3.1, sect. 2.3]. If M actually is a differential graded $\mathbb{R}$–*algebra*, $\mathcal{L}_X$ is a derivation, and i_X is an anti–derivation for all $X \in \mathfrak{g}$, then the data of the definition above is also known as a differential graded $\mathfrak{g}$–algebra (cf. [9, Definition 3.1]) or operation of $\mathfrak{g}$ (see [11, Definition, sect. 7.1], although there it additionally is required that M is non–negatively graded).

Example 1.2.

(1) If we consider the contraction operator $\mathrm{i}_X : \Omega(\mathfrak{g}) \longrightarrow \Omega(\mathfrak{g})$, $X \in \mathfrak{g}$, as a map $\mathrm{i} : \mathfrak{g} \longrightarrow \mathrm{End}(\Omega(\mathfrak{g}))$, $X \longmapsto \mathrm{i}_X$, and the contragredient representation as a map $-\mathrm{ad}^* : \mathfrak{g} \longrightarrow \mathrm{End}(\Omega(\mathfrak{g}))$, $X \longmapsto -(\mathrm{ad}_X)^*$, then the pair $(\mathrm{i}, -\mathrm{ad}^*)$ is a $\mathfrak{g}$–action in the differential graded $\mathbb{R}$–module $(\Omega(\mathfrak{g}), \mathrm{d})$.

(2) Let $L_X : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$ denote multiplication from the left with $X \in \mathfrak{g}$ in the algebra $\Lambda(\mathfrak{g})$. Then L_X is dual to $\mathrm{i}_X : \Omega(\mathfrak{g}) \longrightarrow \Omega(\mathfrak{g})$ with respect to the canonical pairing introduced earlier, and since ad_X and $(\mathrm{ad}_X)^*$ are also dual to each other, it follows that $(L_{(\cdot)}, \mathrm{ad})$ is a $\mathfrak{g}$–action in $(\Lambda^{-\bullet}(\mathfrak{g}), \partial)$, where $\Lambda^{-\bullet}(\mathfrak{g})$ coincides with $\Lambda(\mathfrak{g})$ as a vector space, but its p–th graded component is given by $(\Lambda^{-\bullet}(\mathfrak{g}))^p = \Lambda^{-p}(\mathfrak{g})$ for all integers p. For example, if X, Y are arbitrary elements of $\mathfrak{g}$ and $\omega \in \Omega(\mathfrak{g})$ and $\lambda \in \Lambda(\mathfrak{g})$ are arbitrary homogeneous elements of (ordinary) degree $p > 0$, then to verify the equation $L_{[X,Y]} = \mathrm{ad}_X \circ L_Y - L_Y \circ \mathrm{ad}_X$, we observe that

$$\langle \omega, L_{[X,Y]}(\lambda)\rangle = \langle \mathrm{i}_{[X,Y]}\omega, \lambda\rangle = \langle (-\mathrm{ad}_X)^* \mathrm{i}_Y \omega + \mathrm{i}_Y (\mathrm{ad}_X)^* \omega, \lambda\rangle = \langle \omega, -L_Y \,\mathrm{ad}_X(\lambda) + \mathrm{ad}_X\, L_Y(\lambda)\rangle.$$

(3) If $\mathfrak{g}$ is the Lie algebra of a Lie group G and M is a (smooth) G–manifold, then $\mathfrak{g}$ acts on the $\mathbb{R}$–dgm $(\Omega(M), \mathrm{d})$ of forms on M together with the exterior derivative $\mathrm{d} : \Omega(M) \longrightarrow \Omega(M)$. In fact, if given a vector field X on M we let i_X denote contraction of a form with X and write $\mathcal{L}_X$ for the Lie derivative in the direction of X, then the defining equations for an action are satisfied by all pairs of vector fields X, Y on M. In particular, if $X \in \mathfrak{g}$, then the assignments $X \longmapsto \mathrm{i}_{\overline{X}}$ and $X \longmapsto \mathcal{L}_{\overline{X}}$ define an action of $\mathfrak{g}$ in $\Omega(M)$, where $\overline{X}$ is the vector field *induced by the G–action*, that is, the complete vector field with flow $M \times \mathbb{R} \longrightarrow M$, $(t, p) \longmapsto \exp(-tX).p$. Note that some authors declare $-\overline{X}$ to be the induced vector field, however we shall see later that the choice of sign that we make is dictated if we require that all maps between differential graded $\mathbb{R}$–modules be chain maps.

(4) Actions can be pulled back along Lie algebra homomorphisms: if $(\mathrm{i}, \mathcal{L})$ is a $\mathfrak{g}$–action in a differential graded $\mathbb{R}$–module M, $\mathfrak{h}$ is a Lie algebra, and $F : \mathfrak{h} \longrightarrow \mathfrak{g}$ is a homomorphism of Lie algebras, then $(\mathrm{i}\circ F, \mathcal{L}\circ F)$ is an $\mathfrak{h}$–action in M.

Now suppose that $\mathfrak{g}$–acts on an $\mathbb{R}$–dgm (M, d) and extend the representation of $\mathfrak{g}$ in $\mathrm{S}(\mathfrak{g}^*)$ and M to the tensor product of vector spaces $\mathrm{S}(\mathfrak{g}^*) \otimes M$ via the assignment $X \longmapsto \mathcal{L}_X$, with $\mathcal{L}_X := (-\mathrm{ad}_X)^* \otimes \mathrm{id} + \mathrm{id} \otimes \mathcal{L}_X$ for all $X \in \mathfrak{g}$. We denote by $C_{\mathfrak{g}}(M)$ the space of all invariant elements in $\mathrm{S}(\mathfrak{g}^*) \otimes M$ and endow $C_{\mathfrak{g}}(M)$ with the $\mathbb{Z} \times \mathbb{Z}$–bigrading

$$C_{\mathfrak{g}}^{p,q}(M) := \left(\mathrm{S}^p(\mathfrak{g}^*) \otimes M^{q-p}\right)^{\mathfrak{g}}.$$

As is well known, $C_{\mathfrak{g}}(M)$ is a double complex, called *Cartan complex*, whose total cohomology is commonly referred to as the equivariant cohomology of the $\mathfrak{g}$–action on (M, d). The details are collected in

Proposition 1.3. Let $X_1, \dots, X_n$ be a basis of $\mathfrak{g}$ with dual basis $\varepsilon_1, \dots, \varepsilon_n \in \mathfrak{g}^*$. Let further $\mathrm{id} \otimes d$ and $M_{\varepsilon_j} \otimes \mathrm{i}_{X_j}$ be the $\mathbb{R}$–linear maps on $\mathrm{S}(\mathfrak{g}^*) \otimes M$ induced by the $\mathbb{R}$–bilinear assignments $(f, m) \longmapsto f \otimes d(m)$ and $(f, m) \longmapsto f\varepsilon_j \otimes \mathrm{i}_{X_j} m$, respectively.

(1) The maps d and $\mathcal{L}_X$ commute for all $X \in \mathfrak{g}$, and

(2) i extends to a homomorphism of $\mathbb{R}$–algebras $\mathrm{i} : \Lambda(\mathfrak{g}) \longrightarrow \mathrm{End}(M)$.

(3) The maps $\mathrm{id} \otimes d$ and $\iota := \sum_{j=1}^n M_{\varepsilon_j} \otimes \mathrm{i}_{X_j}$ restrict to endomorphisms on $C_{\mathfrak{g}}(M)$. As such they are differentials and homogeneous of bidegrees $(0, 1)$ and $(1, 0)$, respectively.

(4) the maps $\mathrm{id} \otimes d$ and ι anti–commute on $C_{\mathfrak{g}}(M)$.

Proof. To prove the first item, note that

$$\mathcal{L}_X \circ d = (d \circ \mathrm{i}_X + \mathrm{i}_X \circ d) \circ d = d \circ (d \circ \mathrm{i}_X + \mathrm{i}_X \circ d) = d \circ \mathcal{L}_X.$$

The second statement is a consequence of the fact that, for all $X, Y \in \mathfrak{g}$, we have $0 = \mathrm{i}_{X+Y} \circ \mathrm{i}_{X+Y}$ and that the right hand side of this equation is equal to $\mathrm{i}_X \circ \mathrm{i}_Y + \mathrm{i}_Y \circ \mathrm{i}_X$, by linearity of the map i.

To prove the third item, note that $\mathrm{id} \otimes d$ is even a differential on $\mathrm{S}(\mathfrak{g}^*) \otimes M$, and we claim that so is ι. In fact, $\mathrm{S}(\mathfrak{g}^*)$ is a commutative ring, so $M_{\varepsilon_j} \circ M_{\varepsilon_i} = M_{\varepsilon_i} \circ M_{\varepsilon_j}$ for all i, j and

$$\iota \circ \iota = \sum_{i,j=1}^{n} (M_{\varepsilon_i} \circ M_{\varepsilon_j}) \otimes (\mathrm{i}_{X_i} \circ \mathrm{i}_{X_j}) = \sum_{i<j}^{n} (M_{\varepsilon_i} \circ M_{\varepsilon_j}) \otimes (\mathrm{i}_{X_i} \circ \mathrm{i}_{X_j}) - \sum_{i>j} (M_{\varepsilon_i} \circ M_{\varepsilon_j}) \otimes (\mathrm{i}_{X_j} \circ \mathrm{i}_{X_i}) = 0.$$

Moreover, since d commutes with $\mathcal{L}_X$ for all $X \in \mathfrak{g}$, also $\mathrm{id} \otimes d$ commutes with the representation of $\mathfrak{g}$ in $\mathrm{S}(\mathfrak{g}^*) \otimes M$. Hence, to finish the proof of the third statement it suffices to show that ι commutes with $\mathcal{L}_X$ too. To this end, we compute, using that $\mathcal{L}_X$ is a derivation on $\mathrm{S}(\mathfrak{g}^*)$, for all pure tensors $f \otimes m$

$$\begin{aligned}
(\mathcal{L}_X \circ \iota)(f \otimes m) &= \sum_{j=1}^{n} \mathcal{L}_X(f\varepsilon_j) \otimes \mathrm{i}_{X_j} m + (f\varepsilon_j) \otimes \mathcal{L}_X \mathrm{i}_{X_j} m \\
&= (\iota \circ \mathcal{L}_X)(f \otimes m) + \sum_{j=1}^{n} \big(f \mathcal{L}_X(\varepsilon_j)\big) \otimes \mathrm{i}_{X_j} m + (f\varepsilon_j) \otimes \mathrm{i}_{[X, X_j]} m,
\end{aligned}$$

and observe that $\mathcal{L}_X(\varepsilon_j) = -\sum_{i=1}^{n} \varepsilon_j([X, X_i])\varepsilon_i$, whence

$$\sum_{j=1}^{n} \big(f \mathcal{L}_X(\varepsilon_j)\big) \otimes \mathrm{i}_{X_j} m = -\sum_{i=1}^{n} \sum_{j=1}^{n} (f\varepsilon_i) \otimes \varepsilon_j([X, X_i]) \mathrm{i}_{X_j} m = -\sum_{i=1}^{n} (f\varepsilon_i) \otimes \mathrm{i}_{[X, X_i]} m.$$

To verify the last item, we first observe that the operator $\sum_{j=1}^{n} M_{\varepsilon_j} \circ \mathcal{L}_{X_j}$ vanishes identically on $\mathrm{S}(\mathfrak{g}^*)$. In fact, since each of the maps $\mathcal{L}_{X_j}$ is a derivation on $\mathrm{S}(\mathfrak{g}^*)$, it will suffice to check this for $h \in \mathfrak{g}^*$, and for such an element we compute

$$\sum_{j=1}^{n} \varepsilon_j \mathcal{L}_{X_j}(h) = -\sum_{i,j=1}^{n} h([X_j, X_i]) \varepsilon_i \varepsilon_j = 0,$$

the last equation being true due to the skew–symmetry of $[\cdot, \cdot]$. Now observe that on $C_{\mathfrak{g}}(M)$ we have

$$(\mathrm{id} \otimes d) \circ \iota + \iota \circ (\mathrm{id} \otimes d) = \sum_{j=1}^{n} M_{\varepsilon_j} \otimes \mathcal{L}_{X_j} = -\sum_{j=1}^{n} (M_{\varepsilon_j} \circ \mathcal{L}_{X_j}) \otimes \mathrm{id}. \qquad \square$$

It should be noted that the definition of the differentials does not depend on the actual choice of basis and dual basis. Indeed, there is a canonical homomorphism of $\mathbb{R}$–algebras from $\mathrm{S}(\mathfrak{g}^*)$ into the space of all maps $\mathfrak{g}^* \longrightarrow \mathbb{R}$ given by interpreting a tensor $f \in \mathrm{S}^1(\mathfrak{g}^*)$ as the form $X \longmapsto f(X)$, and this homomorphism is injective in each degree. Hence, for each degree p one obtains an identification of $\mathrm{S}^p(\mathfrak{g}^*) \otimes M$ with a certain subspace of all maps $\mathfrak{g}^* \longrightarrow M$, usually referred to as the space of M–valued polynomials on $\mathfrak{g}$. Under this identification the sum $\mathrm{id} \otimes d - \iota$ becomes the map sending an M–valued polynomial f to the map $X \longmapsto d(f(X)) - \mathrm{i}_X(f(X))$.

We use the symbol $\mathrm{H}_{\mathfrak{g}}(M)$ to denote the cohomology of the Cartan complex $(C_{\mathfrak{g}}(M), \mathrm{id} \otimes d - \iota)$. If $M = \Omega(X)$ is the space of smooth forms on a smooth manifold X and G is compact connected, then $\mathrm{H}_{\mathfrak{g}}(M)$ is isomorphic, as an $\mathrm{A}_{\mathfrak{g}}$–algebra, to the topological model $\mathrm{H}_G(X)$ of equivariant cohomology introduced in section I.1, cf. [12, Theorem C.4]. Next, suppose that $\mathfrak{g}$ also acts on an $\mathbb{R}$–dgm $(\mathcal{N}, d')$ and let $\Phi : M \longrightarrow \mathcal{N}$ be a chain map, i. e. a map with $\Phi \circ d = d' \circ \Phi$. Motivated by the next result, we call Φ a *morphism of* $\mathfrak{g}$*–actions* if Φ additionally is a morphism of representations and if $\Phi \circ \mathrm{i} - \mathrm{i} \circ \Phi$ or $\Phi \circ \mathrm{i} + \mathrm{i} \circ \Phi$ is the zero map $\mathfrak{g} \longrightarrow \mathrm{End}(M, \mathcal{N})$; thus either for all $X \in \mathfrak{g}$ we have $\Phi \circ \mathrm{i}_X = \mathrm{i}_X \circ \Phi$ or for all $X \in \mathfrak{g}$ we have $\Phi \circ \mathrm{i}_X = -\mathrm{i}_X \circ \Phi$.

Proposition 1.4. Suppose that $\mathfrak{g}$ is compact and let M, $\mathcal{N}$ be two differential graded $\mathbb{R}$–modules which are acted on by $\mathfrak{g}$. Further suppose that $\Phi : M \longrightarrow \mathcal{N}$ is a morphism of $\mathfrak{g}$–actions, homogeneous of degree 0, which induces an isomorphism on cohomology, that the inclusions $M^{\mathfrak{g}} \longrightarrow M$ and $\mathcal{N}^{\mathfrak{g}} \longrightarrow \mathcal{N}$ are quasi–isomorphisms, and that there exists an integer $q_0 \in \mathbb{Z}$ with $M^q = 0$, $\mathcal{N}^q = 0$ for all $q < q_0$. Then the map $\epsilon \otimes \Phi : C_{\mathfrak{g}}(M) \longrightarrow C_{\mathfrak{g}}(\mathcal{N})$ is a quasi–isomorphism, where ϵ is the linear map sending a homogeneous polynomial $f \in \mathrm{S}^p(\mathfrak{g}^*)$ to $\sigma^p \cdot f$ and $\sigma \in \{\pm 1\}$ is chosen in such a way that $\Phi \circ \mathrm{i}_X = \sigma \cdot \mathrm{i}_X \circ \Phi$ for all $X \in \mathfrak{g}$.

Proof. Note that $\epsilon \otimes \Phi$ is a map of double complexes: if we denote the differentials on M and N by d_M and d_N, respectively, then certainly $(\mathrm{id} \otimes d_N)\circ(\epsilon \otimes \Phi) = (\epsilon \otimes \Phi)\circ(\mathrm{id} \otimes d_M)$, since Φ is assumed to be a chain map. If $f \otimes m$ is a pure tensor with f homogeneous of degree p and $X_1, \dots, X_n$ is a basis of $\mathfrak{g}$ with dual basis $\varepsilon_1, \dots, \varepsilon_n$, then

$$\begin{aligned}(M_{\varepsilon_j} \otimes \mathrm{i}_{X_j})\circ(\epsilon \otimes \Phi)(f \otimes m) &= \sigma^p \cdot (f\varepsilon_j) \otimes \mathrm{i}_{X_j}\Phi(m) \\ &= \sigma^{p+1} \cdot (f\varepsilon_j) \otimes \Phi(\mathrm{i}_{X_j} m) \\ &= (\epsilon \otimes \Phi)\circ(M_{\varepsilon_j} \otimes \mathrm{i}_{X_j})(f \otimes m),\end{aligned}$$

and this implies that $\iota_N\circ(\epsilon \otimes \Phi) = (\epsilon \otimes \Phi)\circ\iota_M$. Consequently, $\epsilon \otimes \Phi$ induces a map between the vertical filtrations on $C_{\mathfrak{g}}(M)$ and $C_{\mathfrak{g}}(N)$ (called "first filtration" in [3, Section A.2]), hence also a map $\epsilon \otimes \Phi : E_{1,M} \longrightarrow E_{1,N}$ between the first pages of the associated spectral sequences. This map fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^q\big(C_{\mathfrak{g}}^{p,\bullet}(M), \mathrm{id} \otimes d_M\big) \arrow[r, "\epsilon\otimes\Phi"] \arrow[d] & \mathrm{H}^q\big(C_{\mathfrak{g}}^{p,\bullet}(N), \mathrm{id} \otimes d_N\big) \arrow[d] \\
E_{1,M}^{p,q} \arrow[r, "\epsilon\otimes\Phi"] & E_{1,N}^{p,q}
\end{tikzcd}$$

for all integers p, q, where the vertical maps are isomorphisms, and since we are assuming that M, N are concentrated in positive degrees with the exception of finitely many negative degrees, it will suffice to show that the upper horizontal map is an isomorphism for all p, q to conclude that $\epsilon \otimes \Phi : E_{1,M} \longrightarrow E_{1,N}$ and hence also $\epsilon \otimes \Phi : \mathrm{H}_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}_{\mathfrak{g}}(N)$ is an isomorphism, see [3, Section A.4].

However, since $\mathfrak{g}$ is compact, the canonical inclusion $\mathrm{A}_{\mathfrak{g}} \otimes M^{\mathfrak{g}} \longrightarrow (\mathrm{S}(\mathfrak{g}^*) \otimes M)^{\mathfrak{g}} = C_{\mathfrak{g}}(M)$ is a quasi–isomorphism, cf. [11, Proposition IV, sect. 7.6]. Similarly, $\mathrm{A}_{\mathfrak{g}} \otimes N^{\mathfrak{g}} \longrightarrow C_{\mathfrak{g}}(N)$ is a quasi–isomorphism, so we have, for all integers p and q, a commutative diagram

$$\begin{tikzcd}
\mathrm{A}_{\mathfrak{g}}^p \otimes \mathrm{H}^{q-p}(M^{\mathfrak{g}}) \arrow[r, "\epsilon\otimes\Phi"] \arrow[d] & \mathrm{A}_{\mathfrak{g}}^p \otimes \mathrm{H}^{q-p}(N^{\mathfrak{g}}) \arrow[d] \\
\mathrm{H}^q\big(C_{\mathfrak{g}}^{p,\bullet}(M)\big) \arrow[r, "\epsilon\otimes\Phi"] & \mathrm{H}^q\big(C_{\mathfrak{g}}^{p,\bullet}(N)\big)
\end{tikzcd}$$

in which the vertical maps are again isomorphisms. Moreover, Φ is a morphism of the representations of $\mathfrak{g}$ in M and N, hence restricts to a map $\Phi : M^{\mathfrak{g}} \longrightarrow N^{\mathfrak{g}}$. This restriction of the quasi–isomorphism Φ must again be a quasi–isomorphism, because $M^{\mathfrak{g}} \longrightarrow M$ as well as $N^{\mathfrak{g}} \longrightarrow N$ are so. Therefore, the upper horizontal map in the diagram above is an isomorphism, as is $\epsilon \otimes \Phi : \mathrm{H}_{\mathfrak{g}}(M) \longrightarrow \mathrm{H}_{\mathfrak{g}}(N)$. ☐

2. Constructing $\mathfrak{g}$–actions

Throughout this section we fix a compact connected Lie group G and a differential graded $\mathbb{R}$–module (M, d). Recall that we set $\Lambda = \Lambda(\mathfrak{g})^{\mathfrak{g}}$ and suppose that we are given an $\mathbb{R}$–algebra homomorphism $\mathrm{i} : \Lambda \longrightarrow \mathrm{End}(M)$ with the following property: whenever $\nu \in \Lambda$ is homogeneous of degree p, then

(1) i_ν is homogeneous of degree $-p$ and

(2) $\mathrm{i}_\nu\circ d = (-1)^p \cdot d\circ\mathrm{i}_\nu$.

Note that i turns M into a left Λ–module, and if given $\nu \in \Lambda$ we define $R_\nu : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$ via $R_\nu(\lambda) = \lambda \wedge \nu$ for $\lambda \in \Lambda(\mathfrak{g})$, then $\Lambda(\mathfrak{g})$ becomes a right Λ–module. Hence, we may form the tensor product of Λ–modules

$$\mathcal{E} := \Lambda(\mathfrak{g}) \otimes_\Lambda M.$$

Observe that $\mathcal{E}$ does not canonically inherit a bigrading from $\Lambda(\mathfrak{g})$ and M, however, if we let $\mathcal{E}^k = \sum_{j-i=k} \Lambda^i(\mathfrak{g}) \otimes_\Lambda M^j$, then $\mathcal{E} = \bigoplus_{k\in\mathbb{Z}} \mathcal{E}^k$ is a $\mathbb{Z}$–grading.

Proposition 2.1. Let $f : M \longrightarrow M$ be an $\mathbb{R}$–linear map with $\mathrm{i}_\nu \circ f = (-1)^p \cdot f \circ \mathrm{i}_\nu$ for all homogeneous elements $\nu \in \Lambda$ of degree p and let $\epsilon : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$ be the degree involution, that is, the linear map taking a homogeneous element $\lambda \in \Lambda^p(\mathfrak{g})$ to $\epsilon(\lambda) = (-1)^p \cdot \lambda$. Then the assignment $\Lambda(\mathfrak{g}) \times M \longrightarrow \mathcal{E}$, $(\lambda, m) \longmapsto \epsilon(\lambda) \otimes_\Lambda f(m)$, is Λ–balanced and hence descends to an $\mathbb{R}$–linear map $\epsilon \otimes_\Lambda f : \mathcal{E} \longrightarrow \mathcal{E}$.

Proof. Let $\nu \in \Lambda$ and $\lambda \in \Lambda(\mathfrak{g})$ be homogeneous elements of respective degrees p and q, and choose $m \in M$ arbitrarily. Balancedness of the map in question is implied by the chain of equations

$$\epsilon\big(R_\nu(\lambda)\big) \otimes_\Lambda f(m) = (-1)^{p+q} \cdot \lambda \otimes_\Lambda \mathrm{i}_\nu f(m) = (-1)^q \cdot \lambda \otimes_\Lambda f(\mathrm{i}_\nu m) = \epsilon(\lambda) \otimes_\Lambda f(\mathrm{i}_\nu(m)). \qquad \square$$

Remark 2.2. Note that ϵ and f are not maps of right– and left–Λ–modules, though, so the notation $\epsilon \otimes_\Lambda f$ is not customary. However, if $g : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$ is an endomorphism of the right Λ–module $\Lambda(\mathfrak{g})$, not necessarily homogeneous, then we still have $(g \otimes_\Lambda \mathrm{id}) \circ (\epsilon \otimes_\Lambda f) = (g \circ \epsilon) \otimes_\Lambda f$ and $(\epsilon \otimes_\Lambda f) \circ (g \otimes_\Lambda \mathrm{id}) = (\epsilon \circ g) \otimes_\Lambda f$, because the maps on the right hand sides of the previous two equations are induced by Λ–balanced maps $\Lambda(\mathfrak{g}) \times M \longrightarrow \mathcal{E}$. For example, if $\lambda \in \Lambda(\mathfrak{g})$, $\nu \in \Lambda$ is homogeneous of degree p, and $m \in M$, then we have

$$(\epsilon \circ g)\big(R_\nu(\lambda)\big) \otimes_\Lambda f(m) = (-1)^p \cdot R_\nu\big((\epsilon \circ g)(\lambda)\big) \otimes_\Lambda f(m) = (\epsilon \circ g)(\lambda) \otimes_\Lambda f(\mathrm{i}_\nu m).$$

Similarly, if $h : M \longrightarrow M$ is a map of the left Λ–module M, then $(\epsilon \otimes_\Lambda f) \circ (\mathrm{id} \otimes_\Lambda h) = \epsilon \otimes_\Lambda (f \circ h)$ and $(\mathrm{id} \otimes_\Lambda h) \circ (\epsilon \otimes_\Lambda f) = \epsilon \otimes_\Lambda (h \circ f)$.

Proposition 2.3. The map ∂ is a morphism of the right Λ–module $\Lambda(\mathfrak{g})$.

Proof. It is a well known fact (cf. [11, Lemma I, sect. 5.12]) that each element in Λ is closed with respect to ∂. Now suppose that we have shown that $\partial(R_\nu(\lambda)) = R_\nu(\partial(\lambda))$ for all homogeneous elements λ of degree at most p and all elements $\nu \in \Lambda$. Let $X, X_1, \ldots, X_p \in \mathfrak{g}$ and put $\lambda = X_1 \wedge \ldots \wedge X_p$. By the Cartan formula and because R_ν commutes with L_X and ad_X, it follows that

$$(\partial \circ R_\nu)\big(L_X(\lambda)\big) = (\partial \circ L_X)\big(R_\nu(\lambda)\big) = R_\nu\big(\mathrm{ad}_X(\lambda) - L_X \circ \partial(\lambda)\big) = (R_\nu \circ \partial)\big(L_X(\lambda)\big).$$

Since the elements of the form $L_X(\lambda)$ span $\Lambda^{p+1}(\mathfrak{g})$, we inductively conclude that $R_\nu \circ \partial = \partial \circ R_\nu$. $\square$

In a similar fashion, one shows that each element $X \in \mathfrak{g}$ gives rise to maps $L_X \otimes_\Lambda \mathrm{id} : \mathcal{E} \longrightarrow \mathcal{E}$ and $\mathrm{ad}_X \otimes_\Lambda \mathrm{id} : \mathcal{E} \longrightarrow \mathcal{E}$, uniquely determined by the condition that a pure tensor $\lambda \otimes_\Lambda m$ be mapped to $(L_X \otimes_\Lambda \mathrm{id})(\lambda \otimes m) = L_X(\lambda) \otimes_\Lambda m$ and $(\mathrm{ad}_X \otimes_\Lambda \mathrm{id})(\lambda \otimes m) = \mathrm{ad}_X(\lambda) \otimes m$, respectively.

Proposition 2.4. The map $\delta := \partial \otimes_\Lambda \mathrm{id} + \epsilon \otimes_\Lambda d$ is a differential on $\mathcal{E}$, homogeneous of degree 1, and the tuple $(L_{(.)} \otimes_\Lambda \mathrm{id}, \mathrm{ad} \otimes_\Lambda \mathrm{id})$ is a $\mathfrak{g}$–action in $(\mathcal{E}, \delta)$.

Proof. The maps ∂ and ϵ anti–commute, hence so do $\partial \otimes_\Lambda \mathrm{id}$ and $\epsilon \otimes_\Lambda d$, which is why δ is a differential on $\mathcal{E}$. It is homogeneous of degree 1, because so are $\epsilon \otimes_\Lambda d$ and $\partial \otimes_\Lambda \mathrm{id}$, by our choice of grading. Next, recall that $(L_{(.)}, \mathrm{ad})$ already is a $\mathfrak{g}$–action in $(\Lambda^{-\bullet}(\mathfrak{g}), \partial)$ by example 1.2, so of all the properties that need to be verified in order for the specified tuple to define a $\mathfrak{g}$–action in $(\mathcal{E}, \delta)$, those not involving the differential δ are already satisfied. Hence, it only remains to verify the Cartan formula. The latter indeed holds for all $X \in \mathfrak{g}$, because L_X and ϵ anti–commute, whence

$$\delta \circ L_X \otimes_\Lambda \mathrm{id} + L_X \otimes_\Lambda \mathrm{id} \circ \delta = (\partial \circ L_X + L_X \circ \partial) \otimes_\Lambda \mathrm{id} + (\epsilon \circ L_X + L_X \circ \epsilon) \otimes_\Lambda d = \mathrm{ad}_X \otimes_\Lambda \mathrm{id}. \qquad \square$$

Our next goal is to show that the natural inclusion $M \longrightarrow \mathcal{E}$, $m \longmapsto 1 \otimes_\Lambda m$, is a quasi–isomorphism between (M, d) and $(\mathcal{E}, \delta)$ by providing an explicit quasi–inverse map $\mathcal{E} \longrightarrow M$. To construct this map, we need to make a general observation. So suppose that X is a topological space and that $I : \mathrm{C}(X) \longrightarrow \mathbb{R}$ is an $\mathbb{R}$–linear map on the space of continuous real valued functions on X. Given a finite–dimensional $\mathbb{R}$–vector space V, equipped with its canonical smooth structure, we can extend I to an operator $I : \mathrm{C}(X, V) \longrightarrow V$ by requiring that the following universal property be satisfied: for all forms $\alpha \in V^*$ and all continuous functions $f : X \longrightarrow V$ we have $(\alpha \circ I)(f) = I(\alpha \circ f)$. Indeed, if $v_1, \dots, v_n$ is any basis of V with dual basis $\varepsilon_1, \dots, \varepsilon_n$, then the operator $\mathrm{C}(X, V) \longrightarrow V$ mapping f to $\sum_{i=1}^n I(\varepsilon_i \circ f) v_i$ satisfies the universal property for each ε_j, whence by linearity of I it must be satisfied for arbitrary forms $\alpha \in V^*$.

We apply this reasoning in case that X is a compact oriented (smooth) manifold, with or without boundary, and $I = \fint_X \, dx$ is a notion of integration of continous functions on X. More precisely, $\fint_X f(x)\, dx = \int_X f \mathrm{V}$ for some fixed volume form V on X, where the right hand side is the ordinary integral of forms on oriented manifolds. Extend $\fint_X \, dx$ to an operator $\mathrm{C}(X, \Lambda(\mathfrak{g})) \longrightarrow \mathbb{R}$ and suppose that $f : X \longrightarrow G$ is a continuous function. Then for all $\lambda \in \Lambda(\mathfrak{g})$ the assignment $X \longrightarrow \Lambda(\mathfrak{g})$, $x \mapsto \mathrm{Ad}_{f(x)}(\lambda)$, defines another continuous function, where we have extended each $\mathrm{Ad}_g : \mathfrak{g} \longrightarrow \mathfrak{g}$ to a homomorphism of $\mathbb{R}$–algebras $\mathrm{Ad}_g : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$. Consequently, we obtain an operator

$$\mu_{X,f} : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g}), \lambda \mapsto \fint_X \mathrm{Ad}_{f(x)}(\lambda)\, dx,$$

which is homogeneous of degree 0. Also note that if an $\mathbb{R}$–linear map $F : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$ commutes with Ad_g for all $g \in G$, then it also commutes with $\mu_{X,f}$: in fact, if $\alpha \in (\Lambda(\mathfrak{g}))^*$ and $\lambda \in \Lambda(\mathfrak{g})$ are arbitrary, then by the universal property

$$(\alpha \circ \mu_{X,f} \circ F)(\lambda) = \fint_X \alpha\big(\mathrm{Ad}_{f(x)}(F(\lambda))\big)\, dx = \fint_X (\alpha \circ F)\big(\mathrm{Ad}_{f(x)}(\lambda)\big)\, dx = (\alpha \circ F \circ \mu_{X,f})(\lambda),$$

whence $\mu_{X,f} \circ F = F \circ \mu_{X,f}$. In particular, ∂ and R_ν commute with $\mu_{X,f}$: the former because the exterior derivative d on $\Omega(\mathfrak{g})$ commutes with $(\mathrm{Ad}_g)^* \in \mathrm{End}(\Omega(\mathfrak{g}))$ and $(\mathrm{Ad}_g)^*$ is dual to $\mathrm{Ad}_g \in \mathrm{End}(\Lambda(\mathfrak{g}))$ with respect to the canonical pairing between $\Lambda(\mathfrak{g})$ and $\Omega(\mathfrak{g})$; and the latter because G is connected, so that $\Lambda = \Lambda(\mathfrak{g})^{\mathfrak{g}}$ is precisely the space of elements which are invariant with respect to the representation $\mathrm{Ad} : G \longrightarrow \mathrm{End}(\Lambda(\mathfrak{g}))$. Therefore, $\mu_{X,f}$ descends to a well–defined chain map $\mu_{X,f} \otimes_\Lambda \mathrm{id} : \mathcal{E} \longrightarrow \mathcal{E}$.

Let us be more specific about the choices that we make if $X = [0,1]$ or $X = G$, since these are the only cases of interest to us. If $X = [0,1]$, we take the volume form used to define $\mu_{[0,1],f}$ to be the standard volume form on $[0,1]$, and then $\mu_{[0,1],f}(\lambda)$ is just the ordinary integral of the path $t \mapsto \mathrm{Ad}_{f(t)}(\lambda)$ in $\Lambda(\mathfrak{g})$. For $X = G$ we choose V to be a biinvariant volume form, so $\mu_{G,\mathrm{id}}(\lambda)$ will be Ad– and hence ad–invariant. Given that we will make frequent use of $\mu_{G,\mathrm{id}}$, let us also write $\mu := \mu_{G,\mathrm{id}}$. Now the promised quasi–inverse map $M \longrightarrow A$ is introduced in the following

Proposition 2.5. There is a unique $\mathbb{R}$–linear map $\pi : \mathcal{E} \longrightarrow M$ taking a pure tensor $\lambda \otimes m$ to $\pi(\lambda \otimes m) = \mathrm{i}_{\mu(\lambda)} m$, and this map is a chain map.

Proof. We just argued that μ commutes with R_ν for all $\nu \in \Lambda$, so the assignment $\Lambda(\mathfrak{g}) \times M \longrightarrow M$, $(\lambda, m) \mapsto \mathrm{i}_{\mu(\lambda)} m$, is Λ–balanced and induces a map $\pi : \mathcal{E} \longrightarrow M$. Moreover, since the elements of Λ are ∂–closed and μ commutes with ∂, we have $\mu \circ \partial = 0$. This implies that π is a chain map, for if $\lambda \in \Lambda(\mathfrak{g})$ is homogeneous of degree p and $m \in M$ is arbitrary, then

$$(\pi \circ \delta)(\lambda \otimes_\Lambda m) = \mathrm{i}_{\mu(\partial(\lambda))} m + \mathrm{i}_{\mu(\epsilon(\lambda))} dm = (-1)^p \cdot \mathrm{i}_{\mu(\lambda)} dm = d\mathrm{i}_{\mu(\lambda)} m = (d \circ \pi)(\lambda \otimes_\Lambda m). \qquad \square$$

Theorem 2.6. There exists a chain homotopy $H : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$, homogeneous of degree 1, between μ and id which commutes with R_ν for all $\nu \in \Lambda$.

Proof. In fact, a fairly standard chain homotopy will do. Here are the details. First note that since G is compact and connected we find a finite open cover $\mathcal{U}$ of G such that each set $U \in \mathcal{U}$ admits a smooth map $F_U : U \times$

$[0,1] \longrightarrow G$ connecting the identity map on U to the constant map $U \longrightarrow \{e\}$, that is, such that $F_U(g,1) = g$ and $F_U(g,0) = e$ for all $g \in U$ and the neutral element $e \in G$. We further find a partition of unity $(\xi_U)_{U\in\mathcal{U}}$ subordinate to $\mathcal{U}$. For $\lambda \in \Lambda(\mathfrak{g})$ we define

$$\mu_U(\lambda) := \int_G \xi_U(g)\,\mathrm{Ad}_g(\lambda)\,dg \text{ and } \lambda_U := \int_G \xi_U(g)\lambda\,dg.$$

By linearity of the (extended) integral we then have $\mu(\lambda) - \lambda = \sum_{U\in\mathcal{U}} \mu_U(\lambda) - \lambda_U$. Next, fix $U \in \mathcal{U}$, $\lambda \in \Lambda(\mathfrak{g})$, a point $g \in G$, and $t \in [0,1]$. If we let $k := \mathrm{Ad}_{F_U(g,t)}$, then

$$\frac{d}{ds}\Big|_{s=t} \mathrm{Ad}_{F_U(g,s)}(\lambda) = \mathrm{Ad}_k \frac{d}{ds}\Big|_{s=t} \mathrm{Ad}_{k^{-1}F_U(g,s)}(\lambda) = (\mathrm{Ad}_k \circ \mathrm{ad}_{W_U(g,t)})(\lambda),$$

where $W_U(g,t) \in \mathfrak{g}$ is the vector field which at e evaluates to $\gamma'(t)$ and $\gamma(s) = k^{-1}F_U(g,s)$; to see this, recall that there is an isomorphism $T_{\mathrm{id}}\mathrm{End}(\mathfrak{g}) \longrightarrow \mathrm{End}(\mathfrak{g})$ taking a tangent vector $\alpha'(0)$, α a smooth curve in $\mathrm{End}(\mathfrak{g})$, to the map $X \longmapsto \frac{d}{ds}\big|_{s=0} \alpha(s)(X)$ and use that $(\mathrm{Ad}\circ\gamma)'(t) = (d\,\mathrm{Ad})_e(\gamma'(t))$. We also observe that $W_U(g,t)$ depends smoothly on t since $\gamma(t) = e$ and the exponential map of G is a diffeomorphism onto some open neighborhood of e. Combined with the universal property of the extended integral and the fundamental theorem of calculus we conclude that

$$\mathrm{Ad}_g(\lambda) - \lambda = \int_0^1 \frac{d}{ds}\Big|_{s=t} \mathrm{Ad}_{F_U(g,s)}(\lambda)\,dt = \int_0^1 (\mathrm{Ad}_{F_U(g,t)} \circ \mathrm{ad}_{W_U(g,t)})(\lambda)\,dt.$$

Now put $T_U(g,\lambda) := \xi_U(g) \cdot \int_0^1 (\mathrm{Ad}_{F_U(g,t)} \circ L_{W_U(g,t)})(\lambda)\,dt$ and note that $T_U(g,\cdot)$ commutes with R_ν for all $\nu \in \Lambda$, because L_X and Ad_k do so for all $X \in \mathfrak{g}$, $k \in G$. Hence, if we use the generalized Cartan formula to replace $\mathrm{ad}_{W_U(g,t)}$ by $\partial\circ L_{W_U(g,t)} + L_{W_U(g,t)}\circ\partial$ in the displayed formula above, multiply the result with $\xi_U(g)$, and integrate over G afterwards, then we obtain

$$\mu_U(\lambda) - \lambda_U = \int_G \partial\big(T_U(g,\lambda)\big) + T_U\big(g,\partial(\lambda)\big)\,dg = (\partial\circ H_U + H_U\circ\partial)(\lambda),$$

where we have set $H_U(\lambda) = \int_G T_U(g,\lambda)\,dg$. Again, observe that H_U commutes with R_ν for all $\nu \in \Lambda$, because $T_U(g,\cdot)$ already does. Thus, if we write $H := \sum_{U\in\mathcal{U}} H_U$, then $\mu - \mathrm{id} = \partial\circ H + H\circ\partial$ and H is as claimed. ☐

Corollary 2.7. H induces a chain homotopy $H \otimes_\Lambda \mathrm{id}$, homogeneous of degree -1, between $\mu \otimes_\Lambda \mathrm{id}$ and $\mathrm{id}_{\mathcal{E}}$.

Proof. Part of the statement of theorem 2.6 was that H commutes with R_ν for all Λ, so we obtain a well defined map $H \otimes_\Lambda \mathrm{id} : \mathcal{E} \longrightarrow \mathcal{E}$. Moreover, ϵ and H anti-commute, because H is homogeneous of degree 1. Therefore,

$$H \otimes_\Lambda \mathrm{id} \circ \delta + \delta\circ H \otimes_\Lambda \mathrm{id} = (H\circ\partial + H\circ\partial) \otimes_\Lambda \mathrm{id} + (H\circ\epsilon + \epsilon\circ H) \otimes_\Lambda d = \mu \otimes_\Lambda \mathrm{id} - \mathrm{id}_{\mathcal{E}}\,.$$ ☐

Corollary 2.8. The natural inclusion $M \longrightarrow \mathcal{E}$, $m \longmapsto 1 \otimes_\Lambda m$, is a quasi-isomorphism with quasi-inverse π.

3. Compatibility with existing actions

We continue to use the notation of the previous section and additionally assume that $(j,\mathcal{L})$ is an action of a Lie algebra $\mathfrak{m}$ in (M,d) satisfying the following property: if $\nu \in \Lambda$ is homogeneous of degree p, then

(1) $j_A\circ\mathrm{i}_\nu = (-1)^p \cdot \mathrm{i}_\nu\circ j_A$ and

(2) $\mathcal{L}_A\circ\mathrm{i}_\nu = \mathrm{i}_\nu\circ\mathcal{L}_A$

for all $A \in \mathfrak{m}$. The second condition says that $\mathcal{L}_A$ is a homomorphism of the Λ-module M, hence induces a well-defined map $\mathrm{id} \otimes_\Lambda \mathcal{L}_A : \mathcal{E} \longrightarrow \mathcal{E}$ for all $A \in \mathfrak{m}$, and by proposition 2.1 we obtain a linear map $\epsilon \otimes_\Lambda j_A : \mathcal{E} \longrightarrow \mathcal{E}$ sending a pure tensor $\lambda \otimes_\Lambda m$ to $\epsilon(\lambda) \otimes_\Lambda j_A(m)$.

Proposition 3.1. We set $\mathrm{i}_{(X,A)} := L_X \otimes_\Lambda \mathrm{id} + \epsilon \otimes_\Lambda j_A$ and $\mathcal{L}_{(X,A)} := \mathrm{ad}_X \otimes_\Lambda \mathrm{id} + \mathrm{id} \otimes_\Lambda \mathcal{L}_A$ for all $X \in \mathfrak{g}$, $A \in \mathfrak{m}$. Then the assignments $(X, A) \mapsto \mathrm{i}_{(X,A)}$ and $(X, A) \mapsto \mathcal{L}_{(X,A)}$ define a $\mathfrak{g} \oplus \mathfrak{m}$–action in $(\mathcal{E}, \delta)$.

Proof. Ultimately, this is a consequence of the fact that the degree involution $\epsilon : M \longrightarrow M$ commutes with all homogeneous maps of even degree and anti–commutes with all homogeneous maps of odd degree. In more detail, let $[\cdot,\cdot] : \mathrm{End}(\mathcal{E}) \times \mathrm{End}(\mathcal{E}) \longrightarrow \mathrm{End}(\mathcal{E})$ also denote the commutator of endomorphisms. Then for all $X, Y \in \mathfrak{g}$ and $A, B \in \mathfrak{m}$ we have

$$\begin{aligned}[\mathcal{L}_{(X,A)}, \mathrm{i}_{(Y,B)}] &= [\mathrm{ad}_X \otimes_\Lambda \mathrm{id}, L_Y \otimes_\Lambda \mathrm{id}] + [\mathrm{ad}_X \otimes_\Lambda \mathrm{id}, \epsilon \otimes_\Lambda j_B] + [\mathrm{id} \otimes_\Lambda \mathcal{L}_A, L_Y \otimes_\Lambda \mathrm{id}] + [\mathrm{id} \otimes_\Lambda \mathcal{L}_A, \epsilon \otimes_\Lambda j_B] \\ &= \mathrm{i}_{[X,Y]} \otimes_\Lambda \mathrm{id} + \epsilon \otimes_\Lambda j_{[A,B]}.\end{aligned}$$

In a similar fashion one verifies the equation $[\mathcal{L}_{(X,A)}, \mathcal{L}_{(Y,B)}] = \mathcal{L}_{([X,Y],[A,B])}$, the right hand side of which, by definition of the bracket on the sum $\mathfrak{g} \oplus \mathfrak{m}$, is equal to $\mathcal{L}_{[(X,A),(Y,B)]}$. To validate the Cartan formula recall that $\delta = \partial \otimes_\Lambda \mathrm{id} + \epsilon \otimes_\Lambda d$ and that both $\epsilon \circ \partial + \partial \circ \epsilon$ and $L_X \circ \epsilon + \epsilon \circ L_X$ vanish, so we compute

$$\mathrm{i}_{(X,A)}\delta + \delta \mathrm{i}_{(X,A)} = (L_X \otimes_\Lambda \mathrm{id})\delta + \delta(L_X \otimes_\Lambda \mathrm{id}) + (\epsilon \otimes_\Lambda j_A)\delta + \delta(\epsilon \otimes_\Lambda j_A) = \mathrm{ad}_X \otimes_\Lambda \mathrm{id} + \mathrm{id} \otimes_\Lambda \mathcal{L}_A. \qquad \square$$

4. An exact sequence

Let G be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, and put $\Lambda = \Lambda(\mathfrak{g})^{\mathfrak{g}}$. If M and N are left– and right–Λ–modules, respectively, then restriction of scalars turns M and N into $\mathbb{R}$–vector spaces, so the tensor product (of $\mathbb{R}$–modules) $N \otimes M = N \otimes_{\mathbb{R}} M$ is declared. It is a real vector space and contains $N \otimes_\Lambda M$ as a quotient. In fact, we have a short exact sequence of real vector spaces

$$0 \longrightarrow I \longrightarrow N \otimes M \longrightarrow N \otimes_\Lambda M \longrightarrow 0,$$

where $I \subseteq N \otimes M$ is the subspace spanned by all elements of the form $(n\nu) \otimes m - n \otimes (\nu m)$, with $n \in N$, $m \in M$, and $\nu \in \Lambda$; the map $N \otimes M \longrightarrow N \otimes_\Lambda M$ is the natural map sending a pure tensor $n \otimes m$ to $n \otimes_\Lambda m$. Moreover, if $f : N \longrightarrow N$ and $g : M \longrightarrow M$ are $\mathbb{R}$–linear maps, then the assignment $N \otimes M \longrightarrow N \otimes_\Lambda M$, $(n, m) \mapsto f(n) \otimes_\Lambda g(m)$, is Λ–balanced if and only if I is an invariant subspace of $f \otimes g : N \otimes M \longrightarrow N \otimes M$.

Specifically, if $N = \Lambda^{-\bullet}(\mathfrak{g})$, M is a differential graded $\mathbb{R}$–module with differential d, and $\mathrm{i} : \Lambda \longrightarrow \mathrm{End}(M)$ is as in section 2, making M a left–Λ–module, then the balancedness of the maps $\mathrm{ad}_X \otimes_\Lambda \mathrm{id}$, $L_X \otimes_\Lambda \mathrm{id}$, $\partial \otimes_\Lambda \mathrm{id}$, and $\epsilon \otimes_\Lambda d$ implies that the maps $\mathrm{ad}_X \otimes \mathrm{id}$, $L_X \otimes \mathrm{id}$, $\partial \otimes \mathrm{id}$, and $\epsilon \otimes d$ restrict to endomorphisms of I for all $X \in \mathfrak{g}$. Hence, $(L_{(\cdot)} \otimes \mathrm{id}_M, \mathrm{ad} \otimes \mathrm{id}_M)$ is a $\mathfrak{g}$–action in $\Lambda^{-\bullet}(\mathfrak{g}) \otimes M$ which restricts to a $\mathfrak{g}$–action in I, and if $\mathfrak{u}$ is another compact Lie algebra and $F : \mathfrak{u} \longrightarrow \mathfrak{g}$ is a Lie algebra homomorphism, then also $\mathfrak{u}$ acts on (I, δ_0), $(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M, \delta_0)$, and $\mathcal{E} = \Lambda(\mathfrak{g}) \otimes_\Lambda M$ via the pullback of the respective $\mathfrak{g}$–action along F, where $\delta_0 := \partial \otimes \mathrm{id} + \epsilon \otimes d$. We claim that we obtain an exact sequence of differential graded A_u–modules

$$0 \longrightarrow C_{\mathfrak{u}}(I) \longrightarrow C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M) \longrightarrow C_{\mathfrak{u}}(\mathcal{E}) \longrightarrow 0.$$

Indeed, since tensoring with a fixed vector space preserves exact sequences, this is immediate for the left portion of the sequence above. To see that the map $\mathrm{S}(\mathfrak{u}^*) \otimes \Lambda^{-\bullet}(\mathfrak{g}) \otimes M \longrightarrow \mathrm{S}(\mathfrak{u}^*) \otimes \mathcal{E}$ is still surjective after passing to the subspaces of $\mathfrak{u}$–invariant elements, note that due to the compactness of $\mathfrak{u}$ we have, by [11, Lemma I, sect. 4.3, and theorem III, sect. 4.4], for each $p \geq 0$ a projection $\mu_p \in \mathrm{End}(\mathrm{S}^{\leq p}(\mathfrak{u}^*) \otimes \Lambda^{-\bullet}(\mathfrak{g}))$ onto the space of $\mathfrak{u}$–invariant elements $\left(\mathrm{S}^{\leq p}(\mathfrak{u}^*) \otimes \Lambda^{-\bullet}(\mathfrak{g})\right)^{\mathfrak{u}}$ whose kernel is spanned by all elements of the form $\mathcal{L}_X(f \otimes \lambda)$, where $X \in \mathfrak{g}$, $f \in \mathrm{S}^{\leq p}(\mathfrak{u}^*)$, and $\lambda \in \Lambda(\mathfrak{g})$. They assemble to a projection μ onto the $\mathfrak{u}$–invariants in $\mathrm{S}(\mathfrak{u}^*) \otimes \Lambda^{-\bullet}(\mathfrak{g})$, and μ commutes with $\mathrm{id} \otimes R_\nu$ for all $\nu \in \Lambda$. Thus, the map $\mu \otimes \mathrm{id}_M$, which — by definition of the $\mathfrak{u}$–action in $\mathrm{S}(\mathfrak{u}^*) \otimes \Lambda^{-\bullet}(\mathfrak{g}) \otimes M$ — is the projection onto the $\mathfrak{u}$–invariants, induces a map $\mu \otimes_\Lambda \mathrm{id}_M$. Hence, if $x \in \mathrm{S}(\mathfrak{u}^*) \otimes \Lambda^{-\bullet}(\mathfrak{g}) \otimes M$ is a preimage of $y \in C_{\mathfrak{u}}(\mathcal{E})$, then so is $(\mu \otimes \mathrm{id})(x) \in C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M)$.

Now the short exact sequence of differential graded A_u–modules induces a long exact cohomology sequence of A_u–modules, which may be rewritten as the exact sequence

$$\begin{tikzcd} \mathrm{H}_{\mathfrak{u}}(I) \arrow[r] & \mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M) \arrow[r] & \mathrm{H}_{\mathfrak{u}}(\mathcal{E}) \arrow[ll, bend left, "\chi"] \end{tikzcd}$$

with χ the "connecting homomorphism". Explicity, χ is the map, homogeneoeus of degree 1, sending $x \in \mathrm{H}_{\mathfrak{u}}(\mathcal{E})$, say with x represented by $\sum_j f_j \otimes \lambda_j \otimes_\Lambda m_j$, to the class of $\sum_j \delta_0(f_j \otimes \lambda_j \otimes m_j)$ in $\mathrm{H}_{\mathfrak{u}}(I)$.

Proposition 4.1. Suppose that $M^q = 0$ for all but finitely many $q < 0$ and that $F : \mathfrak{u} \longrightarrow \mathfrak{g}$ is not the trivial map. Then $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M)$ is a torsion A_u–module.

Lemma 4.2. Suppose that $M^q = 0$ for all but finitely many $q < 0$. Then the A_u–module $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M)$ is isomorphic to $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g})) \otimes \mathrm{H}(M)$, the $\mathfrak{u}$–action in $\Lambda^{-\bullet}(\mathfrak{g})$ being the pullback of the $\mathfrak{g}$–action along $F : \mathfrak{u} \longrightarrow \mathfrak{g}$.

Proof. Recall that $\mathfrak{g}$ acts on $\Lambda^{-\bullet}(\mathfrak{g}) \otimes M$ via $(L_{(\cdot)} \otimes \mathrm{id}_M, \mathrm{ad} \otimes \mathrm{id}_M)$ and that the $\mathfrak{u}$–action is the pullback of this $\mathfrak{g}$–action along the map F. On the other hand, a $\mathfrak{g}$–action in the $\mathbb{R}$–dgm $(\Lambda^{-\bullet}(\mathfrak{g}) \otimes \mathrm{H}(M), \partial \otimes \mathrm{id})$ is declared by $(L_{(\cdot)} \otimes \mathrm{id}, \mathrm{ad} \otimes \mathrm{id})$, and if $s : \mathrm{H}(M) \longrightarrow \ker(d)$ is a section, that is, an $\mathbb{R}$–linear map, homogeneous of degree 0, such that $s(x)$ represents the cohomology class $x \in \mathrm{H}(M)$, then $\mathrm{id} \otimes s : \Lambda^{-\bullet}(\mathfrak{g}) \otimes \mathrm{H}(M) \longrightarrow \Lambda^{-\bullet}(\mathfrak{g}) \otimes M$ is a map of $\mathfrak{g}$–actions. In fact, we have $\delta_0 \circ (\mathrm{id} \otimes s) = (\mathrm{id} \otimes s) \circ (\partial \otimes \mathrm{id})$, because s maps into the kernel of d and $\delta_0 = \partial \otimes \mathrm{id} + \epsilon \otimes d$. Thus, if $\mathrm{id} \otimes s$ is a quasi–isomorphism, then it also induces an isomorphism of A_u–modules $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes \mathrm{H}(M)) \longrightarrow \mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M)$, as according to [11, Theorem III, sect. 4.4] the inclusion of the $\mathfrak{u}$–invariants in $\Lambda^{-\bullet}(\mathfrak{g})$ is a quasi–isomorphism, so that proposition 1.4 applies. Since we have a canonical isomorphism of A_u–modules $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g})) \otimes \mathrm{H}(M) \cong \mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes \mathrm{H}(M))$, the claim will then follow.

Therefore, it only remains to show that $\mathrm{id} \otimes s$ is a quasi–isomorphism. But $(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M, \delta_0)$ is just the tensor product of two differential graded $\mathbb{R}$–modules, so, by the Künneth formula (cf. [20, Theorem 10.1, chap. V]), the map $p : \mathrm{H}(\Lambda^{-\bullet}(\mathfrak{g})) \otimes \mathrm{H}(M) \longrightarrow \mathrm{H}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M)$ sending $[\lambda] \otimes [m]$ to $[\lambda \otimes m]$ is an isomorphism, where square brackets indicate equivalence classes. Since the kernel of $\partial \otimes \mathrm{id}$ is spanned by all elements $\lambda \otimes x$ with $\lambda \in \ker \partial$, the restriction of $\mathrm{id} \otimes s$ to $\ker(\partial \otimes \mathrm{id})$ factors through p, and it follows that $\mathrm{id} \otimes s$ must be a quasi–isomorphism. □

Proof of proposition 4.1. By lemma 4.2 it will suffice to show that $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}))$ is a torsion $\mathrm{A}_{\mathfrak{u}}$–module, and we first assume that $F : \mathfrak{u} \longrightarrow \mathfrak{g}$ is injective. Put $\mathfrak{k} := F(\mathfrak{u})$. If $Y_1, \dots, Y_n$ is a basis of $\mathfrak{k}$ with dual basis $\varepsilon_1, \dots, \varepsilon_n$, then $F^{-1}(Y_1), \dots, F^{-1}(Y_n)$ is a basis of $\mathfrak{u}$ with dual basis $\varepsilon_1 \circ F, \dots, \varepsilon_n \circ F$; this observation shows that $(F^*) \otimes \mathrm{id}$ induces an isomorphism of differential graded $\mathbb{R}$–modules $C_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g})) \longrightarrow C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}))$, and under this isomorphism multiplication in $\mathrm{H}_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g}))$ with a polynomial $f \in \mathrm{A}_{\mathfrak{k}}$ corresponds to multiplication with $F^*(f)$ in $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}))$. Therefore, it suffices to show that $\mathrm{H}_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g}))$ is a torsion $\mathrm{A}_{\mathfrak{k}}$–module.

But, neglecting gradings, $\mathrm{H}_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g}))$ is just $\mathrm{H}(\mathfrak{g}, \mathfrak{k})$: to see this, choose a non–zero element $\mathrm{V} \in \Lambda^m(\mathfrak{g})$, where $m = \dim \mathfrak{g}$, and define the Lie algebraic Poincaré duality isomorphism $D : \Lambda^k(\mathfrak{g}) \longrightarrow \Omega^{m-k}(\mathfrak{g})$, $\lambda \longmapsto \mathrm{i}_\lambda \mathrm{V}$, for all k. It is a map of representations, because any non–zero element in $\Lambda^m(\mathfrak{g})$ is invariant, $\mathfrak{g}$ being compact, and it satisfies $D \circ L_X = \mathrm{i}_X \circ D$ for all $X \in \mathfrak{g}$. Using the Cartan formula and that V is ∂–closed, we conclude by induction on k that $\mathrm{d} \circ D = D \circ \partial$ on $\Lambda^{m-k}(\mathfrak{g})$, so $\mathrm{id} \otimes D$ induces an isomorphism of (ungraded) vector spaces $\mathrm{H}_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g})) \longrightarrow \mathrm{H}_{\mathfrak{k}}(\Omega(\mathfrak{g}))$. As is well known (see e.g. [11, Section 10.9]), $\mathrm{H}_{\mathfrak{k}}(\Omega(\mathfrak{g})) \cong \mathrm{H}(\mathfrak{g}, \mathfrak{k})$, by compactness of $\mathfrak{k}$. In particular, $\mathrm{H}_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g}))$ is finite–dimensional, and since multiplication in $\mathrm{H}_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g}))$ with a homogeneous element $f \in \mathrm{A}_{\mathfrak{k}}$ of degree $k > 0$ is a homogeneous endomorphism of $\mathrm{H}_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g}))$ of degree $2k$ and $\mathrm{A}_{\mathfrak{k}}$ is non–trivial, it follows that $\mathrm{H}_{\mathfrak{k}}(\Lambda^{-\bullet}(\mathfrak{g}))$ is a torsion $\mathrm{A}_{\mathfrak{k}}$–module.

Now let F be arbitrary, but non–trivial, and put $\mathfrak{u}_1 := \ker F$. Then $\mathfrak{u}_1$ is an ideal in $\mathfrak{u}$, and since $\mathfrak{u}$ is compact, we find a compact ideal $\mathfrak{u}_2 \subseteq \mathfrak{u}$ complementary to $\mathfrak{u}_1$, that is, such that $\mathfrak{u} = \mathfrak{u}_1 \oplus \mathfrak{u}_2$ as Lie algebras. The canonical isomorpism $\mathrm{S}((\mathfrak{u}_1)^*) \otimes \mathrm{S}((\mathfrak{u}_2)^*) \longrightarrow \mathrm{S}(\mathfrak{u}^*)$ induced by the projections $\mathfrak{u} \longrightarrow \mathfrak{u}_1$ and $\mathfrak{u} \longrightarrow \mathfrak{u}_2$ restricts to an isomorphism $\mathrm{A}_{\mathfrak{u}_1} \otimes \mathrm{A}_{\mathfrak{u}_2} \longrightarrow \mathrm{A}_{\mathfrak{u}}$ and induces, since $\mathfrak{u}_1$ acts trivially on $\Lambda^{-\bullet}(\mathfrak{g})$, an isomorphism of differential graded $\mathbb{R}$–modules $\mathrm{A}_{\mathfrak{u}_1} \otimes C_{\mathfrak{u}_2}(\Lambda^{-\bullet}(\mathfrak{g})) \longrightarrow C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}))$, where we consider the left hand side as the tensor product of the trivial $\mathbb{R}$–dgm $(\mathrm{A}_{\mathfrak{u}_1}, 0)$ with $C_{\mathfrak{u}_2}(\Lambda^{-\bullet}(\mathfrak{g}))$. In particular, we have an isomorphism $\mathrm{A}_{\mathfrak{u}_1} \otimes \mathrm{H}_{\mathfrak{u}_2}(\Lambda^{-\bullet}(\mathfrak{g})) \longrightarrow \mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}))$ under which multiplication with a polynomial $f \in \mathrm{A}_{\mathfrak{u}_2}$ corresponds to multiplication with the pullback of f along the projection $\mathfrak{u} \longrightarrow \mathfrak{u}_2$. Since $\mathfrak{u}_2$ is non–trivial by assumption, our earlier considerations show that $\mathrm{H}_{\mathfrak{u}_2}(\Lambda^{-\bullet}(\mathfrak{g}))$ is a torsion $\mathrm{A}_{\mathfrak{u}_2}$–module, whence also $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}))$ is a torsion $\mathrm{A}_{\mathfrak{u}}$–module. □

Corollary 4.3. Under the hypothesis of proposition 4.1 the connecting homomorphism induces an isomorphism of $K(\mathrm{A}_{\mathfrak{u}})$–vector spaces $\mathrm{H}_{\mathfrak{u}}(\mathcal{E})[S^{-1}] \longrightarrow \mathrm{H}_{\mathfrak{u}}(I)[S^{-1}]$, where $S = \mathrm{A}_{\mathfrak{u}} \setminus \{0\}$ and $K(\mathrm{A}_{\mathfrak{u}}) = \mathrm{A}_{\mathfrak{u}}[S^{-1}]$ is the quotient field of $\mathrm{A}_{\mathfrak{u}}$.

Proof. We just checked that $\mathrm{H}_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}) \otimes M)$ is a torsion $\mathrm{A}_{\mathfrak{u}}$–module, that is, a trivial $K(\mathrm{A}_{\mathfrak{u}})$–vector space. Since localization preserves exact sequences [5, Proposition 2.5, sect. 2.2], the localization at S of the long exact cohomology sequence for the triple $(I, \Lambda^{-\bullet}(\otimes)M, \mathcal{E})$ hence reduces to an isomorphism $\mathrm{H}_{\mathfrak{u}}(\mathcal{E})[S^{-1}] \longrightarrow \mathrm{H}_{\mathfrak{u}}(I)[S^{-1}]$. □

5. Applications to smooth manifolds

5.1. General results

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ and M a manifold which is acted on by G (from the left). We already have seen in example 1.2 that the G–action induces a $\mathfrak{g}$–action on $(\Omega(M), \mathrm{d})$ via the assignments $X \mapsto \mathrm{i}_{\overline{X}}$ and $X \mapsto \mathcal{L}_{\overline{X}}$, where $\overline{X}$ is the vector field induced by $X \in \mathfrak{g}$ and $\mathrm{i}_{\overline{X}}$, $\mathcal{L}_{\overline{X}}$ denotes contraction with, respectively Lie derivative in direction of X. Our goal in this section is to introduce on $\Omega(M)^{\mathfrak{g}}$, the differential graded submodule of $\Omega(M)$ consisting of $\mathfrak{g}$–invariant elements, a right–Λ–module structure and to show that $\Omega(M)$ is quasi–isomorphic, through a morphism of $\mathfrak{g}$–actions, to $\Lambda(\mathfrak{g}) \otimes_{\Lambda} \left(\Omega(M)^{\mathfrak{g}}\right)$.

We extend the assignment $\mathfrak{g} \longrightarrow \mathrm{End}(\Omega(M))$, $X \mapsto \mathrm{i}_{\overline{X}}$, to a morphism of $\mathbb{R}$–algebras $\Lambda(\mathfrak{g}) \longrightarrow \mathrm{End}(\Omega(M))$, $\lambda \mapsto \mathrm{i}_{\overline{\lambda}}$, and set $T_g : M \longrightarrow M$, $p \mapsto g.p$, for all $g \in G$. Also note that $\Omega(M)^{\mathfrak{g}} = \Omega(M)^G$, because G is connected.

Proposition 5.1. Choose an invariant form $\omega \in \Omega(M)^{\mathfrak{g}}$ and $\lambda \in \Lambda(\mathfrak{g})$. For all $g \in G$ we have

$$(T_g)^* \left(\mathrm{i}_{\overline{\lambda}}\, \omega\right) = \mathrm{i}_{\overline{\mathrm{Ad}_{g^{-1}}(\lambda)}}\, \omega.$$

Proof. Observe that if $X \in \mathfrak{g}$ is arbitrary, then $\overline{\mathrm{Ad}_{g^{-1}} X}$ is T_g–related to $\overline{X}$. Indeed, the integral curve of the former vector field emanating at p is given by the curve $t \mapsto \exp(\mathrm{Ad}_{g^{-1}} X).p = g^{-1}.\big(\exp(X).gp\big)$, so after composing with T_g we obtain the integral curve $t \mapsto \exp(tX).gp$ of $\overline{X}$ starting at $g.p = T_g(p)$. Hence, by invariance of ω we have

$$(T_g)^* \left(\mathrm{i}_{\overline{\lambda}}\, \omega\right) = \mathrm{i}_{\overline{\mathrm{Ad}_{g^{-1}}(\lambda)}}\, (T_g)^* \omega = \mathrm{i}_{\overline{\mathrm{Ad}_{g^{-1}}(\lambda)}}\, \omega. \qquad \square$$

Corollary 5.2. For all $\nu \in \Lambda$ the map $\mathrm{i}_{\overline{\nu}}$ restricts to an endomorphism $\Omega(M)^{\mathfrak{g}} \longrightarrow \Omega(M)^{\mathfrak{g}}$. Moreover, if $\lambda \in \Lambda(\mathfrak{g})$ is homogeneous of degree p and $\omega \in \Omega(M)^{\mathfrak{g}}$, then $\mathrm{d}\mathrm{i}_{\overline{\lambda}} \omega = \mathrm{i}_{\overline{\partial\lambda}}\, \omega + (-1)^p \cdot \mathrm{i}_{\overline{\lambda}}\, \mathrm{d}\omega$.

Remark 5.3. We stress that the formula above holds even in case that p is equal to or exceeds the degree of (any homogeneous component of) ω. In this case the right hand side of the formula vanishes identically.

Proof. The first statement is an immediate consequence of proposition 5.1. To prove the second assertion, we proceed by induction on the degree of λ, that is, we show that for all elements $\lambda \in \Lambda(\mathfrak{g})$ of degree at most p the claimed formula holds. If $p = 0$, then λ is a scalar and the operator $\mathrm{i}_{\overline{\lambda}}$ is just multiplication by λ. For the induction step, note that it will suffice to consider elements of the form $L_X(\lambda)$ with $\lambda \in \Lambda(\mathfrak{g})$ of degree p, since $\Lambda^{p+1}(\mathfrak{g})$ is spanned by such elements. Then we have $\mathrm{i}_{\overline{L_X(\lambda)}} = \mathrm{i}_{\overline{X}} \circ \mathrm{i}_{\overline{\lambda}}$, whence for invariant forms ω on M the Cartan formula implies

$$\mathrm{d}\mathrm{i}_{\overline{L_X(\lambda)}}\, \omega = \mathcal{L}_{\overline{X}} \mathrm{i}_{\overline{\lambda}}\, \omega - \mathrm{i}_{\overline{X}}\, \mathrm{d}\mathrm{i}_{\overline{\lambda}}\, \omega;$$

note that this formula in particular holds for 0–forms (i.e. smooth functions), so the equation remains true in case that ω is homogeneous and of degree at most p. On the other hand, if $q \in M$, then by proposition 5.1

$$\left(\mathcal{L}_{\overline{X}} \mathrm{i}_{\overline{\lambda}}\, \omega\right)_q = \left.\frac{d}{dt}\right|_{t=0} \left(\left(T_{\exp(-tX)}\right)^* \left(\mathrm{i}_{\overline{\lambda}}\, \omega\right)\right)_q = \left.\frac{d}{dt}\right|_{t=0} \left(\mathrm{i}_{\overline{\mathrm{Ad}_{\exp(tX)}(\lambda)}}\, \omega\right)_q = \left(\mathrm{i}_{\overline{\mathrm{ad}_X(\lambda)}}\, \omega\right)_q,$$

because contraction of forms on a fixed tangent space is a linear endomorphism of a finite–dimensional vector space and hence commutes with taking differential. Using the Cartan formula in $\Lambda(\mathfrak{g})$ as well as the induction hypotheses, we thus find

$$\mathrm{di}_{\overline{L_X(\lambda)}}\, \omega = \left(\mathrm{i}_{\overline{\partial(L_X(\lambda))}} + \mathrm{i}_{\overline{L_X(\partial(\lambda))}}\right) \omega - \mathrm{i}_{\overline{X}} \left(\mathrm{i}_{\overline{\partial\lambda}} + (-1)^p \cdot \mathrm{i}_{\overline{\lambda}} \mathrm{d}\right) \omega = \mathrm{i}_{\overline{\partial(L_X(\lambda))}}\, \omega + (-1)^{p+1} \cdot \mathrm{i}_{\overline{L_X(\lambda)}}\, \mathrm{d}\omega. \qquad \square$$

In particular, if $\nu \in \Lambda$ is homogeneous of degree p, then $d \circ \mathrm{i}_{\overline{\nu}} = (-1)^p \cdot \mathrm{i}_{\overline{\nu}} \circ d$ on $\Omega(M)^{\mathfrak{g}}$. Hence, if we consider $\Omega(M)^{\mathfrak{g}}$ a left–Λ–module via the maps $\mathrm{i}_{\overline{\nu}}$ and let $\mathcal{E} := \Lambda(\mathfrak{g}) \otimes_\Lambda \left(\Omega(M)^{\mathfrak{g}}\right)$, then the results of the previous sections apply to $\mathcal{E}$.

Theorem 5.4. The map $\Lambda(\mathfrak{g}) \times \Omega(M)^{\mathfrak{g}} \longrightarrow \Omega(M)$, $(\lambda, \omega) \mapsto \mathrm{i}_{\overline{\lambda}}\omega$, is Λ–balanced and descends to a quasi–isomorphism of $\mathfrak{g}$–actions $\Phi : \mathcal{E} \longrightarrow \Omega(M)$.

Proof. We extended $\mathrm{i} : \mathfrak{g} \longrightarrow \mathrm{End}(\Omega(M))$ to a homomorphism of $\mathbb{R}$–algebras, so we have $\mathrm{i}_{\overline{R_\nu(\lambda)}}\omega = \mathrm{i}_{\overline{\lambda}} \mathrm{i}_{\overline{\nu}}\omega$ for all $\lambda \in \Lambda(\mathfrak{g})$ and $\nu \in \Lambda$, and this proves that we indeed have a well–defined map Φ on $\mathcal{E}$. For the same reason we have $\Phi \circ (L_X \otimes_\Lambda \mathrm{id}) = \mathrm{i}_{\overline{X}} \circ \Phi$ for all $X \in \mathfrak{g}$. Checking that Φ is a chain map amounts to verifying that the diagram

$$\begin{tikzcd}
\mathcal{E}^k \arrow[r, "\delta"] \arrow[d, "\Phi"'] & \mathcal{E}^{k+1} \arrow[d, "\Phi"] \\
\Omega(M)^k \arrow[r, "\mathrm{d}"] & \Omega(M)^{k+1}
\end{tikzcd}$$

is commutative for all integers k, which it is by definition of δ and corollary 5.2; carefully note that the diagram in particular commutes when k is negative and so one or both spaces in the bottom row of the diagram are trivial, whereas the spaces in the top row might be non–zero, cf. remark 5.3. Next, note that in the proof of corollary 5.2 we also showed that $\mathcal{L}_{\overline{X}} \mathrm{i}_{\overline{\lambda}}\omega = \mathrm{i}_{\overline{\mathrm{ad}_X(\lambda)}}\omega$ for all $X \in \mathfrak{g}$ and all $\lambda \in \Lambda(\mathfrak{g})$, $\omega \in \Omega(M)^{\mathfrak{g}}$, proving that $\mathcal{L}_{\overline{X}} \circ \Phi = \Phi \circ (\mathrm{ad}_X \otimes_\Lambda \mathrm{id})$.

Finally, recall from corollary 2.8 that the canonical inclusion $\Omega(M)^{\mathfrak{g}} \longrightarrow \mathcal{E}$ is a quasi–isomorphism. Since by compactness of G also the canonical inclusion $\Omega(M)^{\mathfrak{g}} \hookrightarrow \Omega(M)$ is a quasi–isomorphism [11, Proposition XIII, sect. 7.20] and the latter map factors through Φ, so must be Φ. $\square$

Corollary 5.5. Let U be a compact connected Lie group, $F : U \longrightarrow G$ a homomorphism of Lie groups, and consider the pulled back action of $\mathfrak{u}$ in $\mathcal{E}$, $\Omega(M)$ along $F : \mathfrak{u} \longrightarrow \mathfrak{g}$. Then $\mathrm{id} \otimes_\Lambda \Phi : \mathrm{H}_{\mathfrak{u}}(\mathcal{E}) \longrightarrow \mathrm{H}_{\mathfrak{u}}(\Omega(M))$ is an isomorphism of $\mathrm{A}_{\mathfrak{u}}$–modules.

Example 5.6 (Actions by multiplication, biquotients)**.** Let H and K be two compact and connected Lie groups, $\sigma : H \longrightarrow G$ and $\tau : K \longrightarrow G$ two Lie group homomorphisms, and consider the action of a closed subgroup $U \subseteq H \times K$ on $M = G$ given by $(h, k).g = \sigma(h) g \tau(k^{-1})$ for all $(h, k) \in U$ and $g \in G$. We can consider the induced $\mathfrak{u}$–action as the pullback of a $\mathfrak{g} \oplus \mathfrak{g}$–action on $\Omega(G)$: indeed, $G \times G$ acts on G by the rule $(g_1, g_2).g = g_1 g (g_2)^{-1}$ for all $g_1, g_2, g \in G$, and then the $\mathfrak{u}$–action is the pullback of the induced $\mathfrak{g} \oplus \mathfrak{g}$–action along the map $\sigma \oplus \tau : \mathfrak{u} \longrightarrow \mathfrak{g} \oplus \mathfrak{g}$. Thus, according to corollary 5.5 the $\mathfrak{u}$–equivariant cohomology can be modeled on the $\mathbb{R}$–dgm $\mathcal{E} = \Lambda(\mathfrak{g} \oplus \mathfrak{g}) \otimes_\Lambda \left(\Omega(G)^{\mathfrak{g} \oplus \mathfrak{g}}\right)$, where now $\Lambda = \Lambda(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{g} \oplus \mathfrak{g}}$.

Let us be explicit about the Λ–module structure on $\Omega(G)^{\mathfrak{g} \oplus \mathfrak{g}}$. If we denote by ι_1 and ι_2 the inclusions of $\mathfrak{g} = \mathfrak{g} \oplus 0$ and $\mathfrak{g} = 0 \oplus \mathfrak{g}$ into $\mathfrak{g} \oplus \mathfrak{g}$, respectively, and extend both maps to $\mathbb{R}$–algebra homomorphisms $\Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g} \oplus \mathfrak{g})$, then they induce an isomorphism $\Lambda(\mathfrak{g}) \otimes \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g} \oplus \mathfrak{g})$ sending $\lambda_1 \otimes \lambda_2$ to $\iota_1(\lambda_1)\iota_2(\lambda_2)$. This isomorphism restricts to an isomorphism $\Lambda(\mathfrak{g})^{\mathfrak{g}} \otimes \Lambda(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \Lambda(\mathfrak{g} \oplus \mathfrak{g})^{\mathfrak{g} \oplus \mathfrak{g}}$, so it will suffice to examine the effect of each of the factors separately. To this end, note that $\Omega(G)^{\mathfrak{g}\oplus\mathfrak{g}}$ is just the space of biinvariant forms on G, so the map $\Psi : \Omega(\mathfrak{g}) \longrightarrow \Omega(G)$ extending a form $\omega \in \Omega(\mathfrak{g})$ to a left–invariant form on G restricts to an isomorphism $\Omega(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \Omega(G)^{\mathfrak{g}\oplus\mathfrak{g}}$. Moreover, if $X \in \mathfrak{g}$, then

$$\overline{\iota_1(X)}(e) = -X(e) \text{ and } \overline{\iota_2(X)}(e) = X(e),$$

because the integral curves of these vectors fields starting at the identity element $e \in G$ are given by $t \mapsto \exp(-tX) \cdot e$ and $t \mapsto e \cdot \exp(-tX)^{-1}$, respectively. Since contracting a biinvariant form on G with an element of Λ gives a biinvariant form again (corollary 5.2) and a left–invariant form is determined by the value it takes at e, it follows that for $\omega \in \Omega(\mathfrak{g})^{\mathfrak{g}}$ and homogeneous $\nu \in \Lambda(\mathfrak{g})^{\mathfrak{g}}$, say of degree p, we have

$$\mathrm{i}_{\overline{\iota_1(\nu)}}\Psi(\omega) = (-1)^p \cdot \Psi(\mathrm{i}_\nu\omega) \text{ and } \mathrm{i}_{\overline{\iota_2(\nu)}}\Psi(\omega) = \Psi(\mathrm{i}_\nu\omega).$$

5.2. Commuting actions

Let G and K be two compact connected Lie groups, both acting on a smooth manifold M. Suppose that the actions commute and that we are interested in computing the equivariant cohomology of the induced $\mathfrak{g} \oplus \mathfrak{k}$–action on $\Omega(M)$. One way to do so is to consider $\Omega(M)$ as a $\Lambda(\mathfrak{g} \oplus \mathfrak{k})^{\mathfrak{g}\oplus\mathfrak{k}}$–module and to apply the previously established results, but in the present situation it actually suffices to regard $\Omega(M)$ as a module over $\Lambda = \Lambda(\mathfrak{g})^{\mathfrak{g}}$.

More precisely, let $\overline{X}$ and $Y^\#$ denote the vector fields on M induced by the G– and K–action, respectively, where $X \in \mathfrak{g}$ and $Y \in \mathfrak{k}$. Denote by $T_g : M \longrightarrow M$, $p \mapsto g.p$, and $S_k : M \longrightarrow M$, $p \mapsto k.p$, translation by $g \in G$ and $k \in K$ and note that T_g and S_k commute by assumption; in particular, $(S_k)^*$ and consequently $\mathcal{L}_{Y^\#}$ restricts to an endomorphism on $\Omega(M)^{\mathfrak{g}}$ for all $Y \in \mathfrak{k}$. Moreover, if $X \in \mathfrak{g}$, then $\overline{X}$ is S_k–related to itself for every $k \in K$, so for any form ω on G and all $Y \in \mathfrak{k}$, $t \in \mathbb{R}$ we have

$$(S_{\exp(-tY)})^*\left(\mathrm{i}_{\overline{X}}\omega\right) = \mathrm{i}_{\overline{X}}(S_{\exp(-tY)})^*\omega.$$

Differentiating this equality, we hence find that $\mathcal{L}_{Y^\#}\circ\mathrm{i}_{\overline{X}} = \mathrm{i}_{\overline{X}}\circ\mathcal{L}_{Y^\#}$. By the same reasoning $\mathrm{i}_{Y^\#}$ restricts to a map $\Omega(M)^{\mathfrak{g}} \longrightarrow \Omega(M)^{\mathfrak{g}}$, so we are in the situation of proposition 3.1, with $j_Y = \mathrm{i}_{Y^\#}$. Thus, we have a $\mathfrak{g} \oplus \mathfrak{k}$–action in $\mathcal{E} = \Lambda(\mathfrak{g}) \otimes_\Lambda \left(\Omega(M)\right)^{\mathfrak{g}}$.

Proposition 5.7. The map $\Phi : \mathcal{E} \longrightarrow \Omega(M)$ introduced in theorem 5.4 is a morphism of $\mathfrak{g} \oplus \mathfrak{k}$–actions.

Proof. In fact, given $Y \in \mathfrak{k}$ it is immediate that $\Phi\circ\mathcal{L}_{(0,Y)} = \mathcal{L}_{Y^\#}\circ\Phi$. Moreover, if $\lambda \otimes_\Lambda \omega$ is a pure tensor, with $\lambda \in \Lambda(\mathfrak{g})$ homogeneous of degree p, then

$$(\Phi\circ\mathrm{i}_{(0,Y)})(\lambda \otimes_\Lambda \omega) = (\Phi\circ\epsilon \otimes_\Lambda \mathrm{i}_{Y^\#})(\lambda \otimes_\Lambda \omega) = (-1)^p \cdot \mathrm{i}_{\overline{\lambda}}\mathrm{i}_{Y^\#}\omega = \mathrm{i}_{Y^\#}\mathrm{i}_{\overline{\lambda}}\omega.$$

These observations, together with the fact that Φ already is a morphism of $\mathfrak{g}$–actions, imply that Φ is a morphism of $\mathfrak{g} \oplus \mathfrak{k}$–actions. ☐

Since Φ is a quasi–isomorphism, corollary 5.5 generalizes accordingly and we have

Corollary 5.8. Let U be a compact connected Lie group, $F : U \longrightarrow G \times K$ a homomorphism of Lie groups, and consider the pulled back action of $\mathfrak{u}$ in $\mathcal{E}$, $\Omega(M)$ along $F : \mathfrak{u} \longrightarrow \mathfrak{g} \oplus \mathfrak{k}$. Then $\mathrm{id} \otimes_\Lambda \Phi : \mathrm{H}_{\mathfrak{u}}(\mathcal{E}) \longrightarrow \mathrm{H}_{\mathfrak{u}}(\Omega(M))$ is an isomorphism of $\mathrm{A}_{\mathfrak{u}}$–modules.

Example 5.9 (Homogeneous spaces)**.** Let $H \subseteq G$ be a closed connected subgroup, suppose that $M = G/H$, and that G acts on G/H by multiplication from the left; no additional assumptions are made about the action of K. If U is another compact connected Lie group and $F : U \longrightarrow G \times K$ is a homomorphism, then we can pull back the action of $\mathfrak{g} \oplus \mathfrak{k}$ along $F : \mathfrak{u} \longrightarrow \mathfrak{g} \oplus \mathfrak{k}$. According to corollary 5.8 the equivariant cohomology of this $\mathfrak{u}$–action on $\Omega(G/H)$ then is computed by the $\mathfrak{u}$–action on $\mathcal{E} = \Lambda(\mathfrak{g}) \otimes_\Lambda \left(\Omega(G/H)^{\mathfrak{g}}\right)$, and $\Omega(G/H)^{\mathfrak{g}}$ is just the space of left–invariant forms on G/H.

Now the map $\Psi : \Omega(\mathfrak{g},\mathfrak{h}) \longrightarrow \Omega(G/H)^{\mathfrak{g}}$ extending an $\mathfrak{h}$–basic form on $\mathfrak{g}$ to a left–invariant form on G/H is an isomorphism of differential graded $\mathbb{R}$–modules, so we can also pullback the $\mathfrak{k}$–action on G/H along Ψ to a $\mathfrak{k}$–action $(j,\mathcal{L})$ on the differential graded $\mathbb{R}$–submodule $\Omega(\mathfrak{g},\mathfrak{h})$ of $\Omega(\mathfrak{g})$. Under this isomorphism Ψ the map $\mathrm{i}_{\overline{\nu}}$ corresponds, just as in example 5.6, to the map $(-1)^p \cdot \mathrm{i}_\nu$ on $\Omega(\mathfrak{g},\mathfrak{h})$, whenever $\nu \in \Lambda$ is homogeneous of degree p. Also observe that we still have $j_Y \circ \mathrm{i}_\nu = (-1)^p \cdot \mathrm{i}_\nu \circ j_Y$ and $\mathcal{L}_Y \circ \mathrm{i}_\nu = \mathrm{i}_\nu \circ \mathcal{L}_Y$ for all $Y \in \mathfrak{k}$, so $\mathfrak{g} \oplus \mathfrak{k}$ acts on $\mathcal{B} = \Lambda(\mathfrak{g}) \otimes_\Lambda \Omega(\mathfrak{g},\mathfrak{h})$ as well. Moreover, if we pull back this $\mathfrak{g} \oplus \mathfrak{k}$ action on $\mathcal{B}$ along F, then the induced map $\mathrm{id} \otimes \Psi : \mathrm{H}_u(\mathcal{B}) \longrightarrow \mathrm{H}_u(\mathcal{E})$ is an isomorphism of A_u–modules too, because $\Psi : \Omega(\mathfrak{g},\mathfrak{h}) \longrightarrow \Omega(G/H)^{\mathfrak{g}}$ is an isomorphism. Hence, the equivariant cohomology of the $\mathfrak{u}$–action on $\Omega(G/H)$ is also isomorphic, as an A_u–module, to $\mathrm{H}_u(\mathcal{B})$.

As an explicit example, let $K = \{e\}$ be the trivial subgroup, $U \subseteq G$ any closed connected subgroup, and F the inclusion. Then $\mathrm{H}_u(\mathcal{B})$ is the equivariant cohomology of the action of U on G/H given by left–multiplication, and the action of $\mathfrak{u}$ in $\mathcal{B}$ is just the restriction of the $\mathfrak{g}$–action $(L_{(\cdot)} \otimes_\Lambda \mathrm{id}, \mathrm{ad} \otimes_\Lambda \mathrm{id})$.

5.3. Symmetric spaces

Let G be a compact connected Lie group, $\sigma : G \longrightarrow G$ an involution and $U := (G^\sigma)_0$ the identity component of the fixed point set of σ. If $H \subseteq G$ is a closed connected subgroup, then we have shown in example 5.9 that $\mathfrak{u}$ acts on $\mathcal{B} = \Lambda(\mathfrak{g}) \otimes_\Lambda \Omega(\mathfrak{g},\mathfrak{h})$ via the pullback of the $\mathfrak{g}$–action $(L_{(\cdot)} \otimes_\Lambda \mathrm{id}, \mathrm{ad} \otimes_\Lambda \mathrm{id})$ along the inclusion, and that $\mathrm{H}_\mathfrak{u}(\mathcal{B})$ is isomorphic, as an $\mathrm{A}_\mathfrak{u}$–module, to the equivariant cohomology of the U–action on G/H by left–multiplication. As an application of this result we shall show

Theorem 5.10. Suppose that H contains U. Then the action of U on G/H by left–multiplication is equivariantly formal.

Lemma 5.11. Let $\omega \in \Omega(\mathfrak{g},\mathfrak{h})$ be a closed form, homogeneous of degree p. The map $\Lambda^{-\bullet}(\mathfrak{g}) \longrightarrow \mathcal{B}$, $\lambda \longmapsto \lambda \otimes_\Lambda \omega$, is homogeneous of degree p and a map of $\mathfrak{u}$–actions. Hence, it induces a map of double complexes $j : C_\mathfrak{u}(\Lambda^{-\bullet}(\mathfrak{g})) \longrightarrow C_\mathfrak{u}(\mathcal{B})$.

Proof. Just observe that $\delta(\lambda \otimes_\Lambda \omega) = (\partial \otimes_\Lambda \mathrm{id})(\lambda \otimes_\Lambda \omega)$ for all $\lambda \in \Lambda(\mathfrak{g})$, because ω is closed. Now it is immediate from the $\mathfrak{g}$–actions in $\Lambda^{-\bullet}(\mathfrak{g})$ and $\mathcal{B}$ that the assignment $\Lambda^{-\bullet}(\mathfrak{g}) \longrightarrow \mathcal{B}$, $\lambda \longmapsto \lambda \otimes_\Lambda \omega$, is a map of $\mathfrak{g}$–actions, and by definition of the grading in $\mathcal{B}$ this assignment is homogeneous of degree p. ☐

Lemma 5.12. The map $\Lambda(\mathfrak{g}) \times \Omega(\mathfrak{g},\mathfrak{h}) \longrightarrow \mathcal{B}$, $(\lambda,\omega) \longmapsto \sigma(\lambda) \otimes_\Lambda (\sigma^*\omega)$, descends to a map $\sigma \otimes_\Lambda (\sigma^*) : \mathcal{B} \longrightarrow \mathcal{B}$, and $\sigma \otimes_\Lambda (\sigma^*)$ is a morphism of $\mathfrak{u}$–actions.

Proof. Given $\lambda \in \Lambda(\mathfrak{g})$ and $\omega \in \Omega(\mathfrak{g},\mathfrak{h})$ we have $\sigma^*(\mathrm{i}_\lambda \omega) = \mathrm{i}_{\sigma(\lambda)} \sigma^*(\omega)$, so the map $\Lambda(\mathfrak{g}) \times \Omega(\mathfrak{g},\mathfrak{h}) \longrightarrow \mathcal{B}$ sending a pair (λ,ω) to $\sigma(\lambda) \otimes_\Lambda (\sigma^*\omega)$ is Λ–balanced. The resulting map $\sigma \otimes_\Lambda (\sigma^*)$ is morphism of $\mathfrak{u}$–actions, because $\mathfrak{u}$ is the fixed point set of σ, whence $L_X \circ \sigma = \sigma \circ L_X$ and $\mathrm{ad}_X \circ \sigma = \sigma \circ \mathrm{ad}_X$ for all $X \in \mathfrak{u}$. ☐

Lemma 5.13. Let $\mu : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$ be the projection onto Λ (cf. theorem 2.6) and write E_+ for the 1–eigenspace of $\sigma : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$. Then E_+ is μ–invariant.

Proof. By the universal property of the extended integral we have $(\alpha \circ \mu)(\lambda) = \int_G (\alpha \circ \mathrm{Ad}_g)(\lambda)\, dg$ for all elements α in $(\Lambda(\mathfrak{g}))^*$ and $\lambda \in \Lambda(\mathfrak{g})$, and if $f : G \longrightarrow \mathbb{R}$ is continuous, then the right hand side is defined as $\int_G f(g)\, dg = \int_G f\mathrm{V}$ for some biinvariant volume form V on G. Observe that since $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{u} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the (-1)–eigenspace of σ on $\mathfrak{g}$, and since the map $\Omega(\mathfrak{g}) \longrightarrow \Omega(G)^G$ extending a form on $\Omega(\mathfrak{g})$ to a left–invariant form is an isomorphism, we have $\sigma^*(\mathrm{V}) = \epsilon \cdot \mathrm{V}$ for some $\epsilon \in \{\pm 1\}$. In particular, if $\lambda \in E_+$ and we let $f(g) := \mathrm{Ad}_g(\lambda)$, then $\sigma \circ f = f \circ \sigma$, because $\sigma \circ \mathrm{Ad}_g = \mathrm{Ad}_{\sigma(g)} \circ \sigma$, and

$$(\alpha \circ \sigma \circ \mu)(\lambda) = \int_G (\alpha \circ \sigma \circ f)(g)\, dg = \int_G (\alpha \circ f \circ \sigma)\mathrm{V} = \epsilon \cdot \int_G \sigma^*\big((\alpha \circ f)\mathrm{V}\big) = \int_G (\alpha \circ f)\mathrm{V} = (\alpha \circ \mu)(\lambda),$$

because σ^* is orientation preserving if and only if $\epsilon = 1$. ☐

Proof of theorem 5.10. We need to show that the spectral sequence associated with the vertical filtration on $C_{\mathfrak{u}}(\mathcal{B})$ collapses on the first page, see [13, Section 6.9]. Equivalently, we need to prove that the map $\mathrm{H}_{\mathfrak{u}}(\mathcal{B}) \longrightarrow \mathrm{H}(\mathcal{B})$ induced by the map $\mathrm{S}(\mathfrak{u}^*) \otimes \mathcal{B} \longrightarrow \mathcal{B}$ sending a pure tensor $f \otimes \omega$ to $f(0) \cdot \omega$ is surjective. Thus, we fix $x \in \mathrm{H}(\mathcal{B})$, and since the natural inclusion $\Omega(\mathfrak{g}, \mathfrak{h}) \longrightarrow \mathcal{B}$ is a quasi–isomorphism, we may assume that x is represented by an element of the form $1 \otimes_\Lambda \omega$, with $\omega \in \Omega(\mathfrak{g}, \mathfrak{h})$ homogeneous of degree $p \geq 0$.

Let E_+, E_- be the 1– and (−1)–eigenspaces of $\sigma : \Lambda(\mathfrak{g}) \longrightarrow \Lambda(\mathfrak{g})$ and denote by $j : C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g})) \longrightarrow C_{\mathfrak{u}}(\mathcal{B})$ the map constructed in lemma 5.11. We shall prove by induction that for each $r \geq 0$ there exist elements $c_0, \dots, c_r \in C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}))$ with the following properties:

(1) $c_0 = 1 \otimes 1$,

(2) c_i is contained in $C_{\mathfrak{u}}^{i,-i}(\Lambda^{-\bullet}(\mathfrak{g})) \cap \mathrm{S}(\mathfrak{u}^*) \otimes E_+$, and

(3) $d_C(j(c_0 + \dots + c_r)) \in C_{\mathfrak{u}}^{r+1,p-r}(\mathcal{B})$,

where $d_C = \mathrm{id} \otimes \delta - \iota$ is the differential on $C_{\mathfrak{u}}(\mathcal{B})$, δ is the differential on $\mathcal{B}$, and $\iota = \sum_t M_{\epsilon_t} \otimes L_{X_t} \otimes_\Lambda \mathrm{id}$ for some basis $X_1, \dots, X_k$ of $\mathfrak{u}$ with dual basis $\epsilon_1, \dots, \epsilon_k$. Note that existence of such elements implies surjectivity of the map in question, because $C_{\mathfrak{u}}(\mathcal{B})$ vanishes in bidegrees $(*, -i)$ for sufficiently large $i \geq 0$. Also note that the statement is true for $r = 0$ and $c_0 = 1 \otimes 1$, because ω is closed and $j(c_0) = 1 \otimes 1 \otimes_\Lambda \omega$ is an element of bidegree $(0, p)$, whence $d_C(j(c_0)) = -\iota(j(c_0))$ is an element of bidegree $(1, p)$. Therefore, we may assume that the induction hypothesis holds up to some natural number $r \geq 0$, and choose $c_0, \dots, c_r$ satisfying the induction statement.

Our first claim is that $c' := \iota(j(c_r))$ is closed with respect to $\mathrm{id} \otimes \delta$. We have just checked this in case that $r = 0$, so assume that $r > 0$. Since by induction hypothesis the element $d_C(j(c_0 + \dots + c_r))$ is of bidegree $(r + 1, p - r)$, all homogeneous components of $d_C(j(c_0 + \dots + c_r))$ of bidegree different from $(r + 1, p - r)$ must vanish individually. However, the map j is homogeneous of bidegree $(0, p)$, so the element $(\mathrm{id} \otimes \delta)(j(c_i))$ has bidegree $(i, p - i + 1)$ and $\iota(j(c_i))$ is of bidegree $(i + 1, p - i)$. Therefore, we already must have $(\mathrm{id} \otimes \delta)(j(c_r)) = \iota(j(c_{r-1}))$, and so

$$(\mathrm{id} \otimes \delta)\big((\iota \circ j)(c_r)\big) = -\big(\iota \circ (\mathrm{id} \otimes \delta)\big)(j(c_r)) = -(\iota)^2(j(c_{r-1})) = 0,$$

because $(\mathrm{id} \otimes \delta)$ and ι anti–commute and ι already is a differential.

Now consider the projection μ from $\Lambda(\mathfrak{g})$ onto Λ. We have already seen in section 2 that μ descends to a well–defined map $\mu \otimes_\Lambda \mathrm{id}$ on $\mathcal{B}$, that $\mu \otimes_\Lambda \mathrm{id}$ commutes with the $\mathfrak{g}$–representation in $\mathcal{B}$, and that there exists a chain homotopy $H : \Lambda^{-\bullet}(\mathfrak{g}) \longrightarrow \Lambda^{-\bullet}(\mathfrak{g})$, homogeneous of degree −1, between μ and id which also induces a chain homotopy $H \otimes_\Lambda \mathrm{id}$ between $\mu \otimes_\Lambda \mathrm{id}$ and the identity map on $\mathcal{B}$. Since c' is closed with respect to $\mathrm{id} \otimes \delta$, we hence have

$$c' - (\mathrm{id} \otimes \mu \otimes_\Lambda \mathrm{id})(c') = \big((\mathrm{id} \otimes \delta) \circ (\mathrm{id} \otimes H \otimes_\Lambda \mathrm{id})\big)(c') = \big((\mathrm{id} \otimes \partial \otimes_\Lambda \mathrm{id}) \circ (\mathrm{id} \otimes H \otimes_\Lambda \mathrm{id})\big)(c'),$$

because any element in the image of j is already closed with respect to $\mathrm{id} \otimes \epsilon \otimes_\Lambda \mathrm{d}$.

Observe that the expression $(\mathrm{id} \otimes \mu \otimes_\Lambda \mathrm{id})(c')$ is identically zero: in fact, it follows from the very definition of the map j that we have $(\mathrm{id} \otimes \mu \otimes_\Lambda \mathrm{id}) \circ j = j \circ (\mathrm{id} \otimes \mu)$. Moreover, if $\lambda \in E_+$, then also $L_X(\lambda) \in E_+$ for all $X \in \mathfrak{u}$, and since μ leaves the space E_+ invariant as well, we have

$$(\mathrm{id} \otimes \mu \otimes_\Lambda \mathrm{id})(c') = \sum_{t=1}^{k} \big(j \circ (\mathrm{id} \otimes \mu) \circ (M_{\varepsilon_t} \otimes L_{X_t})\big)(c_r) \in j\left(\mathrm{S}(\mathfrak{u}^*) \otimes \left(E_+ \cap \Lambda^{2r+1}\right)\right).$$

However, if $\lambda \in E_+$ is homogeneous of degree $2r + 1$, then necessarily $\lambda \in \Lambda^+(\mathfrak{u}) \otimes \Lambda(\mathfrak{p})$, where $\Lambda^+(\mathfrak{u})$ now is the space generated by all homogeneous elements of non–zero degree. Since $\omega \in \Omega(\mathfrak{g}, \mathfrak{h})$ is $\mathfrak{h}$–basic and $\mathfrak{u} \subseteq \mathfrak{h}$, we hence have $\mathrm{i}_\lambda \omega = 0$ for all such λ. As $\nu \otimes_\Lambda \omega = \pm 1 \otimes_\Lambda \mathrm{i}_\nu \omega$ for all homogeneous $\nu \in \Lambda$, it follows that $(\mathrm{id} \otimes \mu \otimes_\Lambda \mathrm{id})(c') = 0$.

It remains to note that, due to the compactness of $\mathfrak{u}$ and [11, Theorem III, sect. 4.4], both $\mathrm{S}(\mathfrak{u}^*) \otimes \Lambda^{-\bullet}(\mathfrak{g})$ and $\mathrm{S}(\mathfrak{u}^*) \otimes \mathcal{B}$ can be exhausted as a union of finite–dimensional $\mathfrak{u}$–invariant subspaces and hence admit decompositions $C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g})) \oplus W$ and $C_{\mathfrak{u}}(\mathcal{B}) \oplus W'$, respectively, where W is the subspace spanned by all elements of the form $\mathcal{L}_X(a)$ with $a \in S(\mathfrak{u}^*) \otimes \Lambda^{-\bullet}(\mathfrak{g})$, $X \in \mathfrak{g}$, and W' is the subspace spanned by all elements of the form $\mathcal{L}_X(b)$ with $b \in S(\mathfrak{u}^*) \otimes \mathcal{B}$; that is, W and W' are the kernels of the projections onto the $\mathfrak{u}$–invariants. Since $C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g}))$ further decomposes as a sum of the 1– and (-1)–eigenspaces of $\mathrm{id} \otimes \sigma$, we may decompose $(\mathrm{id} \otimes H)(\iota(c_r))$ accordingly as $(\mathrm{id} \otimes H)(\iota(c_r)) = c_{r+1} + n + w$, where

$$c_{r+1} \in C_{\mathfrak{u}}^{r+1,-r-1}(\Lambda^{-\bullet}(\mathfrak{g})) \cap S(\mathfrak{u}^*) \otimes E_+,\ n \in C_{\mathfrak{u}}(\Lambda^{-\bullet}(\mathfrak{g})) \cap S(\mathfrak{u}^*) \otimes E_-, \text{ and } w \in W.$$

Since j sends W into W', $\mathrm{id} \otimes \partial \otimes_\Lambda \mathrm{id}$ leaves W' invariant, and c' is an element of $C_{\mathfrak{u}}(\mathcal{B})$, it follows that $c' = (\mathrm{id} \otimes \partial \otimes_\Lambda \mathrm{id})(j(c_{r+1} + n))$. Moreover, since every homogeneous element in $\Omega(\mathfrak{g}, \mathfrak{h})$ of degree q is an eigenvector of σ^* to the value $(-1)^q$, it follows that $j(n)$ is an eigenvector of $\mathrm{id} \otimes \sigma \otimes_\Lambda (\sigma^*)$ for the eigenvalue $(-1)^{p+1}$, whereas c' and $j(c_{r+1})$ are eigenvectors for the eigenvalue $(-1)^p$. Therefore, we already must have

$$c' = (\mathrm{id} \otimes \partial \otimes_\Lambda \mathrm{id})(j(c_{r+1})) = (\mathrm{id} \otimes \delta)(j(c_{r+1})),$$

whence the elements $c_0, \dots, c_{r+1}$ are as required by the induction claim. □

Remark 5.14. As already pointed out earlier, theorem 5.10 above can actually be deduced from the main theorem of [6], which treats the case $H = U$. To see this, note that we may assume that $\mathfrak{u}$ contains no non–zero ideal of $\mathfrak{g}$. In fact, if $\mathfrak{g}'' \subseteq \mathfrak{u}$ is an ideal of $\mathfrak{g}$, then $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{g}''$, $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{g}''$, and $\mathfrak{u} = \mathfrak{u}' \oplus \mathfrak{g}''$ for some ideal $\mathfrak{g}'$ in $\mathfrak{g}$, where $\mathfrak{h}', \mathfrak{u}' \subseteq \mathfrak{g}'$. Write G', G'', H', and U' for the corresponding Lie groups. Since the action of $\mathfrak{u}' \oplus \mathfrak{g}''$ on $\Omega\big(\mathfrak{g}' \oplus \mathfrak{g}'', \mathfrak{h}' \oplus \mathfrak{g}''\big)$ defined in example 5.9 is isomorphic to that of $\mathfrak{u}$ on $\Omega(\mathfrak{g}, \mathfrak{h})$, the action of U on G/H is equivariantly formal if and only if so is the action of $U' \times G''$ on $(G' \times G'')/(H' \times G'')$, which, in turn, is the case if and only if U' acts in an equivariantly formal fashion on G'/H'. Hence, we may assume $G = G'$, $H = H'$, and $U = U'$ right away.

We claim that then necessarily $\mathfrak{h} = (\mathfrak{u} \cap [\mathfrak{g}, \mathfrak{g}]) \oplus \mathfrak{a}$ for some subspace $\mathfrak{a} \subseteq Z(\mathfrak{g})$. Indeed, it follows from [14, Proposition 5.2, sect. VIII.5], that $[\mathfrak{g}, \mathfrak{g}]$ decomposes as $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{l}_1 \oplus \dots \oplus \mathfrak{l}_n$ in such a way that each $\mathfrak{l}_i$ is an ideal in $[\mathfrak{g}, \mathfrak{g}]$ and such that $\mathfrak{u} \cap [\mathfrak{g}, \mathfrak{g}] = \mathfrak{u}_1 \oplus \dots \oplus \mathfrak{u}_n$, where $\mathfrak{u}_i = \mathfrak{u} \cap \mathfrak{l}_i$. Moreover, $\mathfrak{l}_i$ is an invariant subspace of the involution defining $\mathfrak{u}$, and if $\mathfrak{p}_i$ is the (-1)–eigenspace of this involution on $\mathfrak{l}_i$, then the representation $\mathfrak{u}_i \longrightarrow \mathrm{End}(\mathfrak{p}_i)$, $X \longmapsto \mathrm{ad}_X$, is irreducible. Now fix an index $j > 0$ and let $\mathfrak{h}_j$ denote the image of $\mathfrak{h}$ under the projection $Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}] \longrightarrow \mathfrak{l}_j$. Note that this projection is a Lie algebra homomorphism, whence $\mathfrak{h}_j$ is a Lie subalgebra, and that $\mathfrak{u}_j \subseteq \mathfrak{h}_j$, because $\mathfrak{u}_j \subseteq \mathfrak{h}$. If $\mathfrak{u}_j$ was a proper subspace of $\mathfrak{h}_j$ there would be an element $X \in \mathfrak{h}$ whose component $X_j \in \mathfrak{h}_j$ under the aforementioned projection would not be contained in $\mathfrak{u}_j$, and since $\mathfrak{u}_j \subseteq \mathfrak{h}$ and $\mathfrak{l}_j = \mathfrak{u}_j \oplus \mathfrak{p}_j$, we could assume $X_j \in \mathfrak{p}_j$. Thus, $\mathfrak{h}_j \cap \mathfrak{p}_j$ would be a non–trivial $\mathfrak{u}_j$–invariant subspace, whence by irreducibility of the representation of $\mathfrak{u}_j$ in $\mathfrak{p}_j$ necessarily $\mathfrak{h}_j \cap \mathfrak{p}_j = \mathfrak{p}_j$ would have to hold. But then we would have $\mathfrak{l}_j \subseteq \mathfrak{h}$, because

$$\mathfrak{h} \supseteq [\mathfrak{u}_j, \mathfrak{h}] = [\mathfrak{u}_j, \mathfrak{h}_j] \supseteq [\mathfrak{u}_j, \mathfrak{p}_j] = \mathfrak{p}_j,$$

contradicting our assumption that $\mathfrak{h}$ does not contain any ideal of $\mathfrak{g}$. Therefore, $\mathfrak{u}_j = \mathfrak{h}_j$. So if $X \in \mathfrak{h}$ is arbitrary and we write $X = X' + X''$ with $X' \in Z(\mathfrak{g})$ and $X'' \in [\mathfrak{g}, \mathfrak{g}]$, then $X'' \in \mathfrak{u}$ and hence already $X' \in \mathfrak{h}$, proving that $\mathfrak{h} = (\mathfrak{u} \cap [\mathfrak{g}, \mathfrak{g}]) \oplus (\mathfrak{h} \cap Z(\mathfrak{g}))$. It remains to note that also $\mathfrak{u} = (\mathfrak{u} \cap [\mathfrak{g}, \mathfrak{g}]) \oplus (\mathfrak{u} \cap Z(\mathfrak{g}))$ and that the action of U on G/H is equivariantly formal if G is Abelian.

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