Dissertation

# The $p$-Poisson Equation: Regularity Analysis and Adaptive Wavelet Frame Approximation

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# The $p$-Poisson Equation: Regularity Analysis and Adaptive Wavelet Frame Approximation

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# Contents

# Introduction

The mathematical modeling of many real life situations and phenomena leads to (systems of) *partial differential equations* (PDEs). The properties of linear PDEs and ways to numerically solve them have been subject to research for a long time with extensive results, see, e.g. [23, 74] for an overview. However, many real phenomena exhibit *nonlinear* characteristics and the description of the nonlinear situation under consideration often naturally leads to nonlinear PDEs of various types. The prominent class of *quasilinear* equations of the type

$$-\mathrm{div}(\alpha(\cdot, |\nabla u|)\nabla u) = F(u) \tag{QL}$$

appears in several problems in continuum mechanics, in particular in the mathematical description of non-Newtonian fluids [95].

This thesis is concerned with an important subclass of (QL): the *p-Poisson equations*

$$-\Delta_p(u) := -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = f \qquad \text{in} \quad \Omega, \tag{PP}$$

where $1 < p < \infty$, $f \in W_0^1(L_p(\Omega))'$ and $\Omega \subset \mathbb{R}^d$, $d \geq 2$, denotes some bounded Lipschitz domain. The differential operator $\Delta_p$ is called *p-Laplacian*, and the corresponding variational formulation takes the form

$$\int_\Omega \Big\langle |\nabla u|^{p-2}\, \nabla u, \nabla v \Big\rangle \,\mathrm{d}x = f(v) \qquad \text{for all} \quad v \in W_0^1(L_p(\Omega)).$$

A detailed description of the scope of problems treated in this work is contained in Chapter 3.

Problems of this latter type, i.e., with $p$-structure, arise in many applications, e.g., in non-Newtonian fluid theory [65, 95], non-Newtonian filtration [88, 109], turbulent flows of a gas in porous media [60], plasticity theory [8], bimaterial problems in elastic-plastic mechanics [101], and many others. Moreover, the $p$-Laplacian has a similar model character for nonlinear (quasilinear) problems as the ordinary Laplace equation for linear problems. We refer to [99] for an introduction. By now, many results concerning existence and uniqueness of solution are known, see Section 3.3 and [100], as well as [99] and the references therein. However, in many cases, the concrete shape of the solutions is unknown, so that numerical schemes for the constructive approximation are needed. Such schemes are generally based on a discretization of the problem, for instance, with respect to some finite grid or triangulation of the domain. The numerical computation of sufficiently accurate approximations to the exact solution of a (nonlinear) partial differential equation in practice typically leads to linear systems of many unknowns. Therefore, the efficiency of such schemes is of fundamental importance. Now, the performance of such methods is usually governed by the regularity of the exact solutions in specific scales of function spaces. In this context, the *adaptivity scale*

$$B^\sigma_\tau(L_\tau(\Omega), \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}, \quad \sigma > 0, \tag{$*$}$$

of Besov spaces is of particular importance. This thesis is mainly concerned with the two following subjects.

**(S1)** Regularity estimates for solutions to the $p$-Poisson equation (PP) in the scale $(*)$ of Besov spaces.

**(S2)** A linearization scheme for the numerical solution of the $p$-Poisson equation with a focus on implementation and numerical testing.

Before we concretize these topics and formulate more specific objectives (stated in (O1) and (O2) below), we shall first motivate the consideration of these issues.

## Motivation

In the sequel, the relation between the smoothness scale $(*)$ and the convergence properties of certain numerical methods is outlined. Hereby, the emphasis is put on the main principles and coherences. A more detailed presentation is contained in Chapter 4, for a rigorous treatment see [17, 26, 47, 76].

A typical characteristic of solutions to PDEs is that they exhibit *singularities* (i.e., discontinuities/unboundedness of the solution and/or its derivatives), which may be induced by a non-smooth domain boundary, the right-hand side or the operator itself. One prominent example are elliptic equations on domains with re-entrant corners, see [46, 70, 71, 89]. In such a situation, to obtain sufficiently accurate approximations while keeping the number of unknowns at a reasonable size, the usage of highly non-uniform spatial discretizations is often indispensable. In this context, *adaptive* numerical approximation schemes aim to efficiently resolve the singularities of the (unknown) solution.

Essentially, an adaptive algorithm is an updating strategy that iteratively generates a sequence of approximations, where additional degrees of freedom are only spent in regions where the numerical approximation is still “far away” from the exact solution. That is, given some Banach space $(X, \|\cdot\|_X)$ in which approximation takes place, as well as an initial finite dimensional subspace $X_0 \subset X$ (based, e.g., on a finite grid or triangulation of the domain), an adaptive method successively performs the following steps for $m \in \mathbb{N}_0$:

1. Compute approximation $u_m$ to $u$ in $X_m$.
2. Estimate the local error of $u - u_m$.
3. Refine (locally) the approximation space $X_m \to X_{m+1}$.

For instance, in case of the well-known Galerkin approach for linear elliptic problems, step 1 corresponds to the solution of a system of linear equations which originates from the projection of the continuous problem onto $X_m$. To perform step 2 - since the solution $u$ is generally unknown - an *a posteriori error estimator* is needed. Here, in order to provide information for the subsequent step on where refinement is necessary, such an error estimator should be based on local error indicators.

Nevertheless, although the idea of adaptivity is quite convincing, these schemes are in general harder to analyze and to implement, compared to more conventional uniform methods. In particular, the major difficulties concern the development of local a posteriori error estimators as needed for adaptive strategies, as well as the proof of convergence (rates) for adaptive schemes. Moreover, their implementation proves to be much more complex compared to the implementation of uniform methods.

Consequently, some theoretical foundations that justify the development, analysis and implementation of adaptive strategies are highly desirable. To this end, we need to check whether adaptive schemes admit the potential to realize a higher convergence rate than their uniform counterparts. Throughout this thesis, we say that a numerical method for a given problem has *convergence rate* $s > 0$ in the Banach space $(X, \|\cdot\|_X)$, if there exists a constant $C > 0$ such that

$$\|u - u_m\|_X \leq CN(m)^{-s} \qquad \text{for all} \quad m \in \mathbb{N}_0,$$

where $u \in X$ denotes the exact solution and $N(m) \in \mathbb{N}$ denotes for each $m \in \mathbb{N}_0$ the number of parameters needed to describe the approximant $u_m \in X_m$. Hence, we have to address the question if - for a given problem and a given class of numerical schemes (e.g., methods based on a specific type of discretization) - the best possible convergence rate which we can expect for an adaptive method is higher than the best possible rate of uniform schemes of this type.

The analysis in this thesis regarding (S1) is motivated by this problem, in particular in connection with *adaptive wavelet algorithms*. For schemes based on wavelet discretizations, there is a clear understanding of the maximal achievable convergence rates of adaptive and uniform methods, respectively. In summary, these optimal rates depend on the regularity of the true solution in specific scales of function spaces. Before we describe these coherences, we shall first introduce the concept of wavelets.

### Wavelet discretization

Wavelets are typically constructed to form special multiscale bases for the function space under consideration, where each basis element is obtained by a dyadic dilation and integer translation of one or more *mother wavelets*. In the classical one dimensional case, a typical wavelet basis for $L_2(\mathbb{R})$ takes the form

$$\Psi = \left\{ \psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k) \,\middle|\, j, k \in \mathbb{Z} \right\}.$$

The major benefits of wavelets rely on their strong analytical properties. In particular, wavelets can be designed to possess the following properties.

- Wavelet bases permit the *characterization of smoothness spaces* such as Sobolev and Besov spaces, in the sense that the corresponding norm is equivalent to some weighted sequence norm of the wavelet expansion coefficients.

- Wavelets of arbitrary *smoothness* and with *compact support* can be constructed.
- The $L_2$-inner product between a smooth function and a wavelet decays exponentially with increasing scale $j$ of the wavelet, i.e., wavelets satisfy certain *cancellation properties*.

Due to these features, wavelets admit the capability to resolve well local characteristics of a function such as singularities, while on the other hand smooth regions can be represented by comparatively few expansion coefficients. This renders wavelets suitable for numerous tasks. For instance, in the context of signal/image analysis and processing (e.g., compression), wavelets are by now widely and successfully used [103]. We refer to Chapter 1 and to the textbooks [45, 108, 135] for further information concerning the construction and the basic properties of wavelets.

Furthermore, many differential and integral operators admit a quasi-sparse representation with respect to a wavelet basis [123], i.e., the entries of the related system matrices decay exponentially in off-diagonal direction. By a simple diagonal preconditioning, the condition numbers of these matrices are usually uniformly bounded (see, e.g., [38, 39]). Due to these properties, wavelets have turned out to be a powerful tool also for the adaptive solution of a wide class of operator equations [6, 18, 19, 20, 27, 28, 34, 38, 66, 67, 84, 96, 111, 119, 134].

In the wavelet setting, there exists a natural *benchmark scheme* for adaptivity, referred to as *best $n$-term approximation*. In best $n$-term wavelet approximation, one does not approximate by linear spaces but by nonlinear manifolds $S_n$, consisting of functions of the form

$$S = \sum_{\lambda \in \overline{\Lambda}} c_\lambda \psi_\lambda,$$

where $\{\psi_\lambda \mid \lambda \in \Lambda\}$ denotes a given wavelet basis and $\overline{\Lambda} \subset \Lambda$ with $\#\overline{\Lambda} = n$. A best $n$-term wavelet approximation can be realized by extracting the $n$ biggest wavelet coefficients from the wavelet expansion of the (unknown) function one wants to approximate. Clearly, on the one hand, such a scheme can never be realized numerically, because this would require to compute all wavelet coefficients and to select the $n$ biggest. On the other hand, the best we can expect for an adaptive wavelet algorithm would be that it (asymptotically) realizes the approximation order of the best $n$-term approximation. In this sense, the use of adaptive wavelet methods is theoretically justified if the best $n$-term wavelet approximation realizes a significantly higher convergence order when compared to more conventional, uniform approximation schemes. In the wavelet setting, it is known that the convergence order of uniform schemes with respect to $L_p$ depends on the regularity of the object one wants to approximate in the scale $W^s(L_p(\Omega))$ of $L_p$-Sobolev spaces, whereas the rate of best $n$-term wavelet approximation in $L_p$ depends on the regularity in the *adaptivity scale* $B^\sigma_\tau(L_\tau(\Omega)), 1/\tau = \sigma/d + 1/p$, of Besov spaces. We again refer to Chapter 4 and [17, 26, 47, 76] for further information. Moreover, recently a similar relation was established in the context of finite element approximations [68], see also [10]. Therefore, the use of adaptive (wavelet) algorithms for (PP) would be justified if the Besov smoothness $\sigma$ of the solution in the adaptivity scale of Besov spaces is higher than its Sobolev regularity $s$.

## Main objectives

The first main objective of this thesis is motivated by the problem illustrated above, i.e., the question if adaptive schemes for the $p$-Poisson equation admit the potential to outperform uniform methods. To give a well-founded answer, we have to study if the Besov smoothness $\sigma$ of the solutions in the adaptivity scale of Besov spaces is higher than their Sobolev regularity $s$.

For linear second order elliptic equations, a lot of positive results in this direction already exist; see, e.g., [25, 29, 36]. In contrast, it seems that not too much is known for nonlinear equations. To the best of the author's knowledge, the only contribution is the paper [37] by Dahlke and Sickel which is concerned with semilinear equations (besides the recent results in [30, 78] by the author and collaborators, which are essential parts of this thesis). In the present work, we intend to show a first positive result for quasilinear elliptic equations, i.e., for the $p$-Poisson equation (PP). Results of Savaré [116] indicate that, on general Lipschitz domains, the Sobolev smoothness of the solutions to (PP) is given by $s^* = 1 + 1/p$ if $2 \leq p < \infty$, and by $s^* = 3/2$ if $1 < p < 2$. The first main objective of this thesis can thus be formulated as follows.

**(O1)** Regularity estimates for solutions of the $p$-Poisson equation (PP) in the adaptivity scale $(*)$ of Besov spaces shall be derived. Hereby we aim to clarify, whether $u \in B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$, for some $\sigma > s^*$. In order to cover a substantial class of problems, we shall consider the full range of the parameter $p$, i.e., $1 < p < \infty$, as well as the general class of bounded Lipschitz domains. Moreover, preferably also explicit assertions for the practically relevant class of polygonal domains shall be proved.

To derive Besov regularity estimates for solutions to the $p$-Poisson equation, we follow two approaches. The first one makes use of the fact that, under certain conditions, the solutions possess higher regularity away from the Lipschitz boundary, in the sense that they are locally Hölder continuous; see, e.g., [52, 61, 127, 132, 133]. The local Hölder semi-norms may explode as one approaches the boundary, but this singular behavior can be controlled by some power of the distance to the boundary as shown, e.g., in [54, 93, 97, 98]. Properties like this very often hold in the context of elliptic boundary problems on nonsmooth domains, we refer, e.g., to [105] and the references therein for details. It turns out that the combination of the global Sobolev smoothness and the local Hölder regularity can be used to establish Besov smoothness for the solutions to (PP). As we will see, in many cases the Besov smoothness $\sigma$ is much higher than the Sobolev smoothness $s^* = 1 + 1/p$ or $s^* = 3/2$ respectively, so that the development of adaptive schemes for the $p$-Poisson problem is completely justified.

On the one hand, this universal approach is applicable for the large class of general Lipschitz domains, but on the other hand the local Hölder regularity result which is used considers all boundary points as "equally bad". However, for solutions of PDEs on polygonal domains, it is known that the critical singularities typically occur only at the corners of the domain. Indeed, for nonnegative solutions of the $p$-Poisson equation on finite cones, there exist singular expansion results with respect to the vertex, see [57, 126]. Essentially, (the derivatives of) the solution can be estimated by some power of the distance to the vertex. Hence, by exploiting these stronger (local) results, one might expect better Besov smoothness estimates on polygonal domains.

The purpose of the second approach is to make a first step in improving some of the Besov regularity results derived with the first approach for polygonal domains. Therefore, as outlined above, the natural first step is to study the regularity of solutions in the vicinity of the corners $x_0 \in \partial\Omega$ of the domain. To this end, we will investigate the smoothness of solutions $u$ to (PP) in a small cone $\mathfrak{C} \subset \Omega$ with vertex $x_0$, measured in the adaptivity scale $B^\sigma_\tau(L_\tau(\mathfrak{C}))$, $1/\tau = \sigma/2 + 1/p$, of Besov spaces. As we shall see, in certain cases this ansatz indeed yields regularity assertions which are - in a local sense, i.e., when considering small neighborhoods of the vertices - stronger than those derived with the first approach in several aspects.

The findings of the first approach are stated in two steps. First of all, we prove a general embedding theorem which says that the intersection of a classical Sobolev space and a Hölder space with the properties outlined above can be embedded into Besov spaces in the adaptivity scale $1/\tau = \sigma/d + 1/p$. Essentially, we show that under suitable conditions on the parameters it holds that

$$C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \;\cap\; W^s(L_p(\Omega)) \;\hookrightarrow\; B^\sigma_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}. \tag{E1}$$

It turns out that for a large range of parameters, the Besov smoothness $\sigma$ is significantly higher compared to the Sobolev smoothness $s$. The proof of this embedding theorem is performed by exploiting the characterizations of Besov spaces by means of wavelet expansion coefficients. Then we verify that under certain natural conditions the solutions to (PP) indeed satisfy the assumptions of the embedding theorem, so that its application yields the desired result.

In regard of the second approach, the proofs are based on a singular expansion of the solution $u$ in the vicinity of a conical boundary point, as well as on embeddings of the type $\mathcal{K}^\ell_{p,a}(\Omega) \cap B^s_p(L_p(\Omega)) \hookrightarrow B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$, where $\mathcal{K}^\ell_{p,a}(\Omega)$ denote the weighted Sobolev spaces referred to as Babuska-Kondratiev spaces (see Section 1.3). As we will see in Chapter 7, in some cases the solutions to (PP) admit arbitrary high weighted Sobolev regularity in the vicinity of a corner. Due to this fact, we additionally examine the limit case $\ell \to \infty$ of the above embedding. In this context, we will consider the topological vector spaces $H^{\infty,s}_a(L_p(\Omega)) := \cap_{\ell=1}^\infty \mathcal{K}^\ell_{p,a}(\Omega) \cap B^s_p(L_p(\Omega))$ and $B^\infty_{\mathrm{NL}}(L_p(\Omega)) := \cap_{\sigma>0} B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$, study their topological properties (i.e., local convexity, metrizability and completeness), and show that the embeddings of the type

$$H^{\infty,s}_a(L_p(\Omega)) \hookrightarrow B^\infty_{\mathrm{NL}}(L_p(\Omega)), \tag{E2}$$

are continuous (in the sense of continuous mappings between topological vector spaces).

It is worth noting that by the above embeddings, we provide universal functional analytic tools, which allow to trace back the problem of deriving Besov regularity assertions in the scale (∗) to the analysis of suitably weighted Hölder or Sobolev regularity (cf. also Remark 5.10). Hence, besides our intention to use these embeddings in connection with the $p$-Poisson equation, they might prove beneficial for the regularity analysis of various further problems.

**Constructive approximation of the $p$-Poisson equation**

The second main subject of this thesis is the numerical solution of the $p$-Poisson equation (PP). Compared to the numerious and extensive results concerning schemes for linear problems, the study of methods for quasilinear equations is still in its infancy. To the author's best knowledge, by now there mainly exist two publications which are concerned with *numerically feasible* approximation schemes for the $p$-Poisson equation.

The work of Canuto and Urban [14] treats the fairly general framework of convex minimization in Banach spaces, where convergence of a steepest descent type method is established. This setting covers the $p$-Poisson problem with homogeneous Dirichlet boundary conditions for all $p > 2$, yet excluding the case $1 < p \leq 2$.

In [53] Diening et al. proposed an iterative linearization scheme for the $p$-Poisson equation with homogeneous Dirichlet boundary conditions. In particular, the case $1 < p \leq 2$ is treated. The main feature of this algorithm, which can be interpreted as a relaxed Kačanov iteration, is that in each iteration only a linear elliptic subproblem has to be solved, which is numerically accessible in a stable and approved way by, e.g., a finite element or wavelet method. In this thesis we want to implement and test this latter scheme, in connection with an appropriate adaptive wavelet frame method for the solution of the linear subproblems.

Roughly speaking, the classical Kačanov scheme is an iteration method for solving nonlinear problems via linearization. An early reference is [86]. For quasilinear elliptic equations of the type

$$-\operatorname{div}(\alpha\,(|\nabla u|)\,\nabla u) = f \qquad \text{in } \Omega,$$

the Kačanov iteration takes the following form. For a given function $u_0$, the new iterate $u_{n+1}$ is recursively defined as the solution of

$$-\operatorname{div}(\alpha\,(|\nabla u_n|)\,\nabla u_{n+1}) = f \qquad \text{in } \Omega, \quad n \geq 0.$$

Note that now at each iteration only a *linear* problem has to be solved. It is proved in [136] under certain assumptions on $\alpha$ that the Kačanov iteration converges to a fixed point $u$ which solves the original quasilinear problem. An a posteriori error estimate is derived in [75]. Unfortunately, the $p$-Poisson equation, i.e. $\alpha(\xi) = \xi^{p-2}$, does *not* satisfy these conditions. Moreover, the linear equations which have to be solved in the course of the Kačanov iteration are not numerically solvable in a stable way if $|\nabla u_n|$ vanishes or gets unbounded at certain points, since then the weight $|\nabla u_n|^{p-2}$ degenerates. One approach to overcome this problem is to truncate the weight function $\alpha$. With the notation $\varepsilon_- \vee x \wedge \varepsilon_+ := \max\{\varepsilon_-, \min\{x, \varepsilon_+\}\}$ for $0 < \varepsilon_- \leq \varepsilon_+ < \infty$ and $x \in \mathbb{R}$, the *relaxed Kačanov iteration* takes the form

$$-\operatorname{div}\Big((\varepsilon_- \vee |\nabla u_n| \wedge \varepsilon_+)^{p-2}\,\nabla u_{n+1}\Big) = f \qquad \text{in } \Omega, \quad n \geq 0. \tag{RKI}$$

To recover the $p$-Laplace operator, additionally the truncation interval $[\varepsilon_-, \varepsilon_+]$ has to be increased in the course of the iteration. This is shown in [53], where a convergence analysis for this scheme is performed under the assumption that the linear, uniformly elliptic PDEs occuring in (RKI) are solved *exactly* at each iteration. Clearly, to obtain a fully implementable algorithm, these subproblems have to be approximately solved by some numerical scheme. We shall pursue an adaptive approach based on wavelet discretizations here. The second main objective of this thesis can now be formulated as follows.

**(O2)** We want to develop and implement a new adaptive solver for the $p$-Poisson equation (PP) for all $1 < p < 2$. This method shall be based on the relaxed Kačanov iteration (RKI), where for the numerical solution of the arising linear elliptic subproblems an adaptive wavelet method shall be utilized. The practical properties of the new algorithm shall then be analyzed in a series of numerical tests.

In particular, for the numerical solution of the linear elliptic subproblems, we will apply the adaptive wavelet Galerkin method as proposed in [18], as well as the adaptive multiplicative Schwarz frame scheme introduced in [124] (see Section 8.4, and Section 1.5 for an introduction to wavelet frames). Both methods have been proved to be of asymptotically optimal complexity, in the sense that they indeed realize the same convergence rate as the best $n$-term approximation, while the number of floating point operations and storage locations needed to compute an approximant stays proportional to the respective number of degrees of freedom. Beyond that, numerical tests in [6, 134] revealed that these schemes may outperform a standard adaptive finite element solver with respect to the degrees of freedom spent.

This thesis provides a unified presentation of the Besov regularity results published in [30, 78], extended by some further results, additional proofs, as well as the results regarding (O2), i.e., the implementation and numerical testing of the Kačanov-type iteration method.

## Outline

This work is organized as follows: Part I contains several preliminaries required for our analysis. In **Chapter 1**, we introduce all function spaces that will be used in the thesis (Section 1.1 - Section 1.4) and summarize some notions and assertions concerning wavelet bases and frames (Section 1.5), including the wavelet characterization of Besov spaces (Section 1.6). In view of our analysis of (E2) - including topological properties of the involved spaces - in **Chapter 2** we first recapitulate some topological notions and facts (Section 2.1 & Section 2.2), after which we derive in Section 2.3 a quasi-/seminorm criterion for the continuity of a linear map from a locally convex topological vector space into a topological vector space equipped with a family of quasi-norms (Proposition 2.23).

Part II is devoted to the $p$-Poisson equation. In **Chapter 3**, after a short introduction to the $p$-Laplace operator, a detailed description of the scope of problems treated in this thesis is given in Section 3.2. Afterwards, a collection of some fundamental properties of the $p$-Poisson equation is contained in Section 3.3.

Part III of this thesis addresses the first main objective (O1), i.e., the Besov regularity analysis of solutions to the $p$-Poisson equation. In this context, in **Chapter 4** the well-known coherence between approximation rates of wavelet schemes and the smoothness of the solution in specific scales of Besov spaces is presented.

Afterwards, in **Chapter 5** - **Chapter 7** the main results are stated and proved: The general embeddings of function spaces can be found in **Chapter 5**. Here, at first in Section 5.1 an embedding of locally weighted Hölder spaces into the adaptivity scale of Besov spaces ($*$), i.e., an embedding of the type (E1), is presented and proved (Theorem 5.1). Then, in Section 5.2, after introducing the well-known embedding of Babuska-Kondratiev spaces into the scale ($*$), we treat the case $\ell \to \infty$, i.e. (E2). In Subsection 5.2.1, after showing that $H_a^{\infty,s}(L_p(\Omega))$ is a Fréchet space (Proposition 5.7) and $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$ an F-space (Proposition 5.8), we prove the continuity of the embedding (E2) (Theorem 5.9).

Then, in **Chapter 6** we are concerned with the Besov regularity of solutions to the $p$-Poisson equation on bounded Lipschitz domains. At first, in Section 6.1 we derive (generic) sufficient conditions on the parameters of $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ which ensure that the Besov regularity of solutions on general multidimensional domains exceeds their maximal Sobolev regularity (Theorem 6.5). Then, in Section 6.2 we prove explicit bounds on the Besov regularity of the unique solution to the $p$-Poisson equation with homogeneous Dirichlet boundary conditions in two dimensions - for Lipschitz domains (Theorem 6.14) as well as for polygonal domains (Theorem 6.17).

In **Chapter 7** we study the Besov regularity of solutions to (PP) in the vicinity of the vertices of a polygonal domain. Therefore, we first collect some results concerning the singular expansion of solutions to the $p$-Poisson equation on finite cones, see Section 7.1. Afterwards, our main (local) Besov regularity results are derived: those cases in which the right-hand side of (PP) vanishes in a (small) neighborhood of the corner are treated in Section 7.2 (Theorem 7.12), whereas in Section 7.3 local regularity assertions are proved under some local growth condition on the source term (Theorem 7.18).

Part IV of this thesis is concerned with the numerical solution of the $p$-Poisson equation, i.e., with (O2). **Chapter 8** contains a description as well as a collection of fundamental properties of the Kačanov-type iteration method under consideration. After the classical and the relaxed Kačanov iteration have been introduced in Section 8.1, some of the convergence results from [53] for the exact scheme are summarized in Section 8.2 and Section 8.3. Finally, the complete implementable algorithm is given in Section 8.4, including a short description of the adaptive multiplicative Schwarz frame scheme which we use for the solution of the linear subproblems. The results of a series of numerical tests are presented in **Chapter 9**, where we consider several non-trivial two-dimensional $p$-Poisson problems with homogeneous Dirichlet boundary conditions for $1 < p < 2$, on a convex as well as on a nonconvex polygonal domain with a re-entrant corner.

Finally, the thesis is concluded with an **Appendix** which contains a couple of auxiliary lemmata and propositions which are needed in our proofs, some additional results as well as a few alternative proofs.

# Part I

# Preliminaries

# Chapter 1

# Function Spaces and Wavelet Decompositions

In this chapter we first recall the definitions of several types of function spaces that will be needed in this thesis (Section 1.1 - 1.4). Afterwards, we recapitulate the multiscale decomposition of functions by means of a wavelet basis or frame (Section 1.5) and collect some assertions such as, e.g., the characterization of Besov spaces in terms of wavelet expansion coefficients (Section 1.6).

If not stated otherwise, in this chapter we assume $\Omega \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, to be either $\mathbb{R}^d$ itself, or some bounded domain, i.e., an open and connected set. Where necessary, the class of admissable domains may be further restricted, for instance to bounded Lipschitz domains (i.e., domains which possess a Lipschitz boundary; cf. [131, Definition 1.103]), which will be explicitly stated then.

### Notation

First of all, let us introduce some general notation we will use throughout this thesis. For families $\{a_{\mathcal{J}}\}_{\mathcal{J}}$ and $\{b_{\mathcal{J}}\}_{\mathcal{J}}$ of non-negative real numbers over a common index set we write $a_{\mathcal{J}} \lesssim b_{\mathcal{J}}$ if there exists a constant $C > 0$ (independent of the context-dependent parameters $\mathcal{J}$) such that

$$a_{\mathcal{J}} \leq C \cdot b_{\mathcal{J}}$$

holds uniformly in $\mathcal{J}$. Consequently, $a_{\mathcal{J}} \sim b_{\mathcal{J}}$ means $a_{\mathcal{J}} \lesssim b_{\mathcal{J}}$ and $b_{\mathcal{J}} \lesssim a_{\mathcal{J}}$. Next, for a countable (and hence totally ordered) index set $\mathcal{J}$ and $0 < p \leq \infty$, by $\ell_p(\mathcal{J})$ we denote the space of all real-valued sequences $c = \{c_j\}_{j\in\mathcal{J}}$, for which

$$\|c\|_{\ell_p(\mathcal{J})} := \begin{cases} \left(\sum_{j\in\mathcal{J}} |c_j|^p\right)^{1/p}, & 0 < p < \infty, \\ \sup_{j\in\mathcal{J}} |c_j|, & p = \infty, \end{cases}$$

is finite. Given a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$, we will commonly use the abbreviation $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_H$ if $H$ is implicitly given by the context. As well, for a normed space $V$ and its (topological) dual $V'$ we denote the dual pairing by $\langle v', v\rangle := v'(v)$ for $v' \in V'$, $v \in V$.

## 1.1 Strongly differentiable functions: (weighted) Hölder spaces

For $\ell \in \mathbb{N}_0$, by $C^\ell(\Omega)$ furnished with the norm

$$\left\| g \,\middle|\, C^\ell(\Omega) \right\| = \sum_{|\nu| \leq \ell} \sup_{x \in \Omega} |\partial^\nu g(x)|$$

we denote the space of all real-valued functions $g$ on $\Omega$ such that $\partial^\nu g$ is uniformly continuous and bounded on $\Omega$ for every multi-index $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ with $0 \leq |\nu| \leq \ell$. Therein $\partial^\nu = \partial^{|\nu|}/(\partial x_1^{\nu_1} \ldots \partial x_d^{\nu_d})$ denote the $\nu$-th order strong derivatives. If $\ell = 0$ we abbreviate the notation and write $C(\Omega)$. If $K$ is a compact subset of $\Omega$ (denoted by $K \subset\subset \Omega$), the spaces $C^\ell(K)$ are defined likewise. Unless otherwise stated we restrict ourselves to those $K \subset\subset \Omega$ which can be described as the closure of some open and simply connected set. Next let us recall that for $g \in C^\ell(\Omega)$ the $\ell$-th order Hölder semi-norm with exponent $0 < \alpha \leq 1$ is given by

$$|g|_{C^{\ell,\alpha}(\Omega)} = \sum_{|\nu|=\ell} \sup_{\substack{x,y \in \Omega,\\ x \neq y}} \frac{|\partial^\nu g(x) - \partial^\nu g(y)|}{|x-y|^\alpha}. \tag{1.1.1}$$

Consequently, for $\ell \in \mathbb{N}_0$ and $0 < \alpha \leq 1$,

$$C^{\ell,\alpha}(\Omega) = \left\{ g \in C^\ell(\Omega) \,\middle|\, \left\| g \,\middle|\, C^{\ell,\alpha}(\Omega) \right\| = \left\| g \,\middle|\, C^\ell(\Omega) \right\| + |g|_{C^{\ell,\alpha}(\Omega)} < \infty \right\},$$

denote the (classical) *Hölder spaces* on $\Omega$. Again we can replace $\Omega$ by $K$ at every occurrence to define the Hölder spaces also for compact subsets $K \subset\subset \Omega$. Standard proofs yield that all the spaces we defined so far are actually Banach spaces; see, e.g., [58, 90].

Furthermore, let us introduce the collection of all functions on $\Omega$ which are locally Hölder continuous of order $\ell \in \mathbb{N}_0$ with exponent $0 < \alpha \leq 1$. This set will be denoted by

$$C^{\ell,\alpha}_{\mathrm{loc}}(\Omega) = \left\{ g \colon \Omega \to \mathbb{R} \,\middle|\, g \in C^{\ell,\alpha}(K) \text{ for all } K \subset\subset \Omega \right\},$$

where we simplified the notation by denoting the restrictions $g\big|_K$ of functions $g$ from $\Omega$ to compact subsets $K$ by $g$ again. Since the latter collection of functions does not perfectly fit for our purposes, in the sequel the following closely related (non-standard) function spaces will be used instead. Therefore, let $\Omega$ be some bounded domain, and let $\mathcal{K}$ denote an arbitrary but non-trivial family of compact subsets $K \subset\subset \Omega$. Then for every $K \in \mathcal{K}$ the quantity

$$\delta_K = \operatorname{dist}(K, \partial\Omega), \tag{1.1.2}$$

i.e., the distance of $K$ to the boundary of $\Omega$, is strictly positive. Thus, for each $\ell \in \mathbb{N}_0$, all $0 < \alpha \leq 1$, and every $\gamma > 0$, the space

$$C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega; \mathcal{K})$$
$$= \left\{ g \colon \Omega \to \mathbb{R} \,\middle|\, g \in C^{\ell,\alpha}(K) \text{ for all } K \in \mathcal{K} \text{ and } |g|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} = \sup_{K \in \mathcal{K}} \delta_K^\gamma \, |g|_{C^{\ell,\alpha}(K)} < \infty \right\}$$

is well-defined and it is easily verified that $|\cdot|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}}$ provides a semi-norm for this space. In our applications (Section 5.1 and Chapter 6) $\mathcal{K}(c)$ will be the set of all closed balls $\overline{B_r} = \overline{B_r(x_0)} \subset \Omega$ (with center $x_0 \in \Omega$ and radius $r > 0$) such that the open ball $B_{cr} = B_{cr}(x_0)$ is still contained in $\Omega$. Here $c > 1$ denotes a constant which we assume to be given fixed in advance. Actually, it is not hard to see that the space $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega; \mathcal{K}(c))$ is independent of $c$. Consequently, we simply write $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) = C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega; \mathcal{K})$ for $\mathcal{K} = \mathcal{K}(c)$. Those spaces are then referred to as *locally weighted Hölder spaces*.

**Remark 1.1.** Obviously, for every choice of the parameters, $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ contains $C^{\ell,\alpha}(\Omega)$ as a linear subspace, but it also contains functions $g$ whose local Hölder semi-norms $|g|_{C^{\ell,\alpha}(K)}$ grow to infinity as the distance $\delta_K$ of $K \subset\subset \Omega$ to the boundary tends to zero. However, this possible blow-up is controlled by the parameter $\gamma$. Moreover, in the appendix we show that the intersection of $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ with some Besov space is a Banach space with respect to the canonical norm; see Proposition A.3. Finally, we want to mention that the spaces $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ are monotone in $\gamma$, meaning that $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \subseteq C^{\ell,\alpha}_{\mu,\mathrm{loc}}(\Omega)$ for $\gamma \leq \mu$. This can be seen by checking that $\delta_K^\mu = (\delta_K/C)^\mu C^\mu \leq (\delta_K/C)^\gamma C^\mu = \delta_K^\gamma C^{\mu-\gamma}$ for some universal constant $C \geq 1$ (e.g., $C = \max\{1, \mathrm{diam}(\Omega)\}$), thus $|\cdot|_{C^{\ell,\alpha}_{\mu,\mathrm{loc}}} \lesssim |\cdot|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}}$.

For the sake of completeness, we mention here that (as usual) the set of all infinitely often (strongly) differentiable functions with compact support in $\Omega$ will be denoted by $C_0^\infty(\Omega)$ or $\mathcal{D}(\Omega)$. For its dual space we write $\mathcal{D}'(\Omega)$. Once more, these definitions apply likewise when $\Omega$ is replaced by some compact set $K$.

## 1.2 Weakly differentiable functions: Sobolev spaces

Given $0 < p \leq \infty$ the *Lebesgue spaces* $L_p(\Omega)$ consist of all (equivalence classes of real-valued) measurable functions $g$ on $\Omega$ for which the (quasi-)norm

$$\|g \,|\, L_p(\Omega)\| = \begin{cases} \left( \displaystyle\int_\Omega |g(x)|^p \,\mathrm{d}x \right)^{1/p} & \text{if } p < \infty, \\ \operatorname*{ess-sup}_{x\in\Omega} |g(x)| & \text{if } p = \infty \end{cases}$$

is finite.

Moreover, for $1 \leq p < \infty$ and $\ell \in \mathbb{N}_0$, let

$$W^\ell(L_p(\Omega)) = \left\{ g \in L_p(\Omega) \,\middle|\, \left\|g \,\middle|\, W^\ell(L_p(\Omega))\right\| = \sum_{|\nu|\leq\ell} \|D^\nu g \,|\, L_p(\Omega)\| < \infty \right\}$$

denote the classical *Sobolev spaces* on $\Omega$, where $D^\nu$ are the weak partial derivatives of order $\nu \in \mathbb{N}_0^d$. For fractional smoothness parameters $s = \ell + \beta > 0$ (with $\ell \in \mathbb{N}_0$ and $0 < \beta < 1$) we extend the definition in the usual way by setting

$$W^s(L_p(\Omega)) = \left\{ g \in W^\ell(L_p(\Omega)) \,\middle|\, \|g \,|\, W^s(L_p(\Omega))\| < \infty \right\},$$

where here the norm is given by $\|g\,|\,W^s(L_p(\Omega))\| = \left\|g\,\middle|\,W^\ell(L_p(\Omega))\right\| + |g|_{W^s(L_p(\Omega))}$ and

$$|g|_{W^s(L_p(\Omega))} = \left(\sum_{|\nu|=\ell}\int_\Omega\int_\Omega \frac{|D^\nu g(x) - D^\nu g(y)|^p}{|x-y|^{d+\beta p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}$$

denotes the common *Sobolev-Slobodeckij* semi-norm on $\Omega$.

Furthermore, for $s > 0$ and $1 < p < \infty$, let us denote the closure of $C_0^\infty(\Omega)$ in the norm of $W^s(L_p(\Omega))$ by $W_0^s(L_p(\Omega))$. Then we define $W^{-s}(L_{p'}(\Omega))$ to be the dual space of $W_0^s(L_p(\Omega))$, where $p'$ is determined by the relation $1/p + 1/p' = 1$. By $\langle\cdot,\cdot\rangle$ we denote the canonical duality pairing on $W^{-s}(L_{p'}(\Omega)) \times W_0^s(L_p(\Omega))$, i.e., $\langle f, v\rangle = f(v)$ for $f \in W^{-s}(L_{p'}(\Omega))$, $v \in W_0^s(L_p(\Omega))$.

For a detailed discussion of the scale of Banach spaces $W^s(L_p(\Omega))$, $s \in \mathbb{R}$, we refer to standard textbooks such as [1, 129] and the references given therein.

Finally, for bounded Lipschitz domains $\Omega \subset \mathbb{R}^d$ and $1 \le p \le \infty$, the Lebesgue spaces $L_p(\partial\Omega)$ on the boundary $\partial\Omega$ consist of all measurable functions $g$ on $\partial\Omega$ for which the norm

$$\|g\,|\,L_p(\partial\Omega)\| = \begin{cases} \left(\int_{\partial\Omega} |g(x)|^p\,\mathrm{d}S\right)^{1/p} & \text{if } p < \infty,\\ \operatorname*{ess-sup}_{x\in\partial\Omega} |g(x)| & \text{if } p = \infty \end{cases}$$

is finite, where $\mathrm{d}S$ denotes the usual surface measure on $\partial\Omega$. Likewise, for $0 < s < 1$ and $1 \le p < \infty$, the Sobolev spaces on the boundary $\partial\Omega$ of some bounded Lipschitz domain $\Omega$ are defined as usual as

$$W^s(L_p(\partial\Omega)) = \{g \in L_p(\partial\Omega)\,|\,\|g\,|\,W^s(L_p(\partial\Omega))\| < \infty\},$$

where the norm is given by $\|g\,|\,W^s(L_p(\partial\Omega))\| = \|g\,|\,L_p(\partial\Omega)\| + |g|_{W^s(L_p(\partial\Omega))}$ and

$$|g|_{W^s(L_p(\partial\Omega))} = \left(\int_{\partial\Omega}\int_{\partial\Omega} \frac{|g(x)-g(y)|^p}{|x-y|^{d-1+sp}}\,\mathrm{d}S_x\,\mathrm{d}S_y\right)^{1/p}.$$

For further information on Sobolev spaces on surfaces or manifolds, see [70, Chapter 1.3.3] and [83].

The treatment of boundary value problems requires a proper definition of the restriction of a (Sobolev) function to the boundary. For a bounded Lipschitz domain $\Omega$ and $g \in C(\overline{\Omega})$, we define the *trace* of $g$ simply as its restriction to the boundary, i.e.,

$$tr\,g = g|_{\partial\Omega}. \tag{1.2.1}$$

Now, it turns out that this trace can be generalized to functions $g \in W^1(L_p(\Omega))$ in the following way. For a proof of this result see [70, Theorem 1.5.1.3].

**Proposition 1.2.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 < p < \infty$. Then, the trace operator* (1.2.1) *has a unique continuous extension to an operator from $W^1(L_p(\Omega))$ onto $W^{1-1/p}(L_p(\partial\Omega))$. This operator has a continuous right inverse $E$ independent of $p$, i.e.,*

$$E : W^{1-1/p}(L_p(\partial\Omega)) \to W^1(L_p(\Omega))$$

*with $tr \circ E = id$.*

With the help of this extended trace operator, which we denote as well by $tr$, the space $W_0^1(L_p(\Omega))$ admits the following characterization. A proof can be found in [70, Corollary 1.5.1.6].

**Proposition 1.3.** *Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $1 < p < \infty$. Then, $g \in W_0^1(L_p(\Omega))$ if and only if $g \in W^1(L_p(\Omega))$ and $tr\, g = 0$.*

# 1.3 Weighted Sobolev spaces

Besides the classical Sobolev spaces introduced above, we will make use of two classes of *weighted Sobolev spaces*. For our purposes it is most suitable to consider weight functions which essentially depend on the distance to a subset $S$ of the *boundary* of $\Omega$. Therefore, in case $\Omega$ is a bounded domain, let $S \subseteq \partial\Omega$. If $\Omega = \mathbb{R}^d$, we assume that $S \subseteq \partial\Omega'$ for some bounded domain $\Omega'$. In both cases we suppose that $S \neq \emptyset$. Then, let $\rho = \rho_S : \Omega \to [0,1]$ be the smooth distance to the set $S$, meaning that $\rho$ is smooth on $\Omega \setminus S$ and in the vicinity of $S$ it is equivalent to the distance to that set. Now, for $\ell \in \mathbb{N}_0$, $1 < p < \infty$ and $a \geq 0$, the *Babuska-Kondratiev spaces* $\mathcal{K}^\ell_{p,a}(\Omega)$ consist of all (equivalence classes of real-valued) measurable functions $g$ on $\Omega$ for which the norm

$$\left\| g \,\middle|\, \mathcal{K}^\ell_{p,a}(\Omega) \right\| = \left( \sum_{|\nu| \leq \ell} \int_\Omega \left| \rho(x)^{|\nu|-a} D^\nu g(x) \right|^p \, \mathrm{d}x \right)^{1/p} \tag{1.3.1}$$

is finite. For further reading, we refer to [5] and the references given therein.

**Remark 1.4.**

**(i)** For the range of parameters as stated above, the spaces $\mathcal{K}^\ell_{p,a}(\Omega)$ are Banach spaces, see [91, Theorem 1.11 & Remark 4.10].

**(ii)** Note that

$$\left\| \cdot \,\middle|\, \mathcal{K}^\ell_{p,a}(\Omega) \right\| \leq \left\| \cdot \,\middle|\, \mathcal{K}^{\ell+1}_{p,a}(\Omega) \right\|$$

and hence $\mathcal{K}^{\ell+1}_{p,a}(\Omega) \hookrightarrow \mathcal{K}^\ell_{p,a}(\Omega)$. That is, the Babuska-Kondratiev spaces $\mathcal{K}^\ell_{p,a}(\Omega)$ are ordered w.r.t. the smoothness index $\ell$. Moreover, if $a \geq \ell \in \mathbb{N}_0$, then $\mathcal{K}^\ell_{p,a}(\Omega) \hookrightarrow W^\ell(L_p(\Omega))$ with

$$\left\| \cdot \,\middle|\, W^\ell(L_p(\Omega)) \right\| \lesssim \left\| \cdot \,\middle|\, \mathcal{K}^\ell_{p,a}(\Omega) \right\|.$$

In particular, all $\mathcal{K}^\ell_{p,a}(\Omega)$ are continuously embedded into $L_p(\Omega)$ (with norm estimate $\| \cdot \mid L_p(\Omega) \| \leq \left\| \cdot \,\middle|\, \mathcal{K}^\ell_{p,a}(\Omega) \right\|$).

The second, closely related class of weighted Sobolev spaces is defined as follows. For $\ell \in \mathbb{N}$, $1 < p < \infty$ and some weight paramter $\beta \in \mathbb{R}$ we set $\mathcal{W}^\ell_{p,\beta}(\Omega) := \left\{ g : \Omega \to \mathbb{R} \,\middle|\, \left\| g \,\middle|\, \mathcal{W}^\ell_{p,\beta}(\Omega) \right\| < \infty \right\}$, where

$$\left\| g \,\middle|\, \mathcal{W}^\ell_{p,\beta}(\Omega) \right\| := \left( \sum_{|\nu| \leq \ell} \left\| D^\nu(\rho^\beta g) \,\middle|\, L_p(\Omega) \right\|^p \right)^{1/p} + \left\| \rho^{\beta-\ell} g \,\middle|\, L_p(\Omega) \right\|. \tag{1.3.2}$$

In regularity theory for elliptic PDEs, the set $S$ is typically chosen to be the so-called *singular set* of $\Omega$, i.e., the collection of non-smooth boundary points. E.g., for polyhedral domains, in case $d = 2$ this singular set consists exactly of the vertices of $\Omega$, whereas for $d = 3$ it consists of the vertices and edges of the domain.

**Remark 1.5.** In this work, the two introduced classes of weighted Sobolev spaces will occur only in connection with (two-dimensional, non-degenerated) finite cones or finite polygonal domains in $\mathbb{R}^2$. If not explicitly stated otherwise, we will at every occurance of these spaces implicitly assume that the set $S$ consists of the collection of vertices (of the polygonal domain or the cone, respectively).

## 1.4 Generalized smoothness: Besov spaces

A more advanced way to measure the smoothness of functions is provided by the framework of *Besov spaces* which essentially generalizes the concept of Sobolev spaces introduced above. Besov spaces can be defined in various ways which (for a large range of the parameters involved) lead to equivalent descriptions; cf. [9, 35, 129, 131]. For our purposes the following approach based on iterated differences seems to be the most reasonable one, since it provides an entirely *intrinsic* definition when dealing with Lipschitz domains. We refer, e.g., to [17, 47, 49, 50, 51].

In the following let $\Omega \subseteq \mathbb{R}^d$ be either $\mathbb{R}^d$ itself, or some bounded Lipschitz domain. Moreover, let $r \in \mathbb{N}$ and $h \in \mathbb{R}^d$. Then $\Omega_{r,h}$ denotes the set of all $x \in \Omega$ such that the line segment $[x, x + rh]$ belongs to $\Omega$. Moreover, for functions $g$ on $\Omega$ the *iterated difference* of order $r$ with step size $h$ is recursively given by

$$\Delta_h^1(g,x) = g(x+h) - g(x) \qquad \text{and} \qquad \Delta_h^r(g,x) = \Delta_h^1(\Delta_h^{r-1}(g,\cdot),x), \quad r \geq 2,$$

for every $x \in \Omega_{r,h}$. It is easily verified that

$$\Delta_h^r(g,x) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} g(x+kh) \qquad \text{for all} \quad r \in \mathbb{N},\ h \in \mathbb{R}^d,\ x \in \Omega_{r,h}.$$

Those differences can be used to quantify smoothness: For $0 < p \leq \infty$ and every $g \in L_p(\Omega)$ let

$$\omega_r(g,t,\Omega)_p = \sup_{h \in \mathbb{R}^d, |h| \leq t} \left\| \Delta_h^r(g,\cdot) \,|\, L_p(\Omega_{r,h}) \right\|, \qquad t > 0, \tag{1.4.1}$$

denote the *modulus of smoothness* of order $r$. It is well-known that $\omega_r(g,t,\Omega)_p \to 0$ monotonically as $t$ tends to zero and the faster this convergence the smoother is $g$.

Now let $s = \ell + \beta > 0$ with $\ell \in \mathbb{N}_0$ and $0 \leq \beta < 1$. Then, for $0 < p, q \leq \infty$, the *Besov space* $B_q^s(L_p(\Omega))$ is defined as the collection of all $g \in L_p(\Omega)$ for which the semi-norm

$$|g|_{B_q^s(L_p(\Omega))} = \begin{cases} \left( \displaystyle\int_0^\infty \left[ t^{-s}\, \omega_r(g,t,\Omega)_p \right]^q \frac{\mathrm{d}t}{t} \right)^{1/q} & \text{if } q < \infty, \\ \displaystyle\sup_{t>0} t^{-s}\, \omega_r(g,t,\Omega)_p & \text{if } q = \infty, \end{cases} \tag{1.4.2}$$

with $r \geq \ell + 1$ is finite. Endowed with the canonical (quasi-)norm

$$\left\| g \,\middle|\, B^s_q(L_p(\Omega)) \right\| = \| g \mid L_p(\Omega) \| + |g|_{B^s_q(L_p(\Omega))}$$

these spaces turn out to be quasi-Banach spaces (and Banach spaces if and only if $\min\{p, q\} \geq 1$). A proof of this fact (for the range of parameters which is relevant for our purposes) is given by [129, Theorem 2.3.3 & Proposition 3.2.3] in combination with the norm equivalences shown in [129, Section 2.5.12] and [131, Theorem 1.118 & Remark 1.119]. Roughly speaking, with $\left\| g \,\middle|\, B^s_q(L_p(\Omega)) \right\|$ we can control all (weak) partial derivatives $D^\nu g$ up to the order $s$, measured in $L_p(\Omega)$. The influence of the additional *fine index* $q$ is rather small compared to the *smoothness parameter* $s$ and the *integrability index* $p$.

**Remark 1.6.** Some comments are in order:

**(i)** We note that different choices of $r \geq \lfloor s \rfloor + 1$ in (1.4.2) lead to equivalent (quasi-)norms. The same is true when we restrict the range for $t$ in (1.4.2) to the interval $(0, 1)$. Both results are proved in [48, Chapter 2, Theorem 10.1]. By these two facts we conclude that for all $0 < s_0 < s_1$ and $0 < q, p \leq \infty$ it holds

$$B^{s_1}_q(L_p(\Omega)) \hookrightarrow B^{s_0}_q(L_p(\Omega)). \tag{1.4.3}$$

**(ii)** The scale of Besov spaces as defined above is well-studied. In particular, sharp assertions on embeddings, interpolation and duality properties, characterizations in terms of various building blocks (e.g., atoms, local means, quarks, or wavelets) and best $n$-term approximation results are known; see, e.g., [35, 47, 51, 77]. Many of them can also be shown using the Fourier analytic definition of $B^s_q(L_p(\Omega))$ as spaces of (restrictions of) tempered distributions [64, 129, 131]. It is known [55, 118, 131] that both definitions coincide in the sense of equivalent (quasi-)norms if

$$s > \sigma_p = d \cdot \max\left\{ \frac{1}{p} - 1, 0 \right\}. \tag{1.4.4}$$

**(iii)** Besov spaces are closely related to Sobolev spaces. Indeed, it has been shown that for $1 < p < \infty$ and $0 < s \notin \mathbb{N}$ the space $B^s_p(L_p(\Omega))$ coincides with $W^s(L_p(\Omega))$ in the sense of equivalent norms; see, e.g., [51, Theorem 6.7] in case $\Omega$ is a bounded Lipschitz domain and [130, Section 2.5.1] for $\Omega = \mathbb{R}^d$. Using the fact that $X^s(L_p(\Omega)) \hookrightarrow X^{s-\varepsilon}(L_p(\Omega))$ for $X \in \{B_p, W\}$ and arbitrary small $\varepsilon > 0$ we thus have

$$W^{s+\varepsilon}(L_p(\Omega)) \hookrightarrow B^s_p(L_p(\Omega)) \hookrightarrow W^{s-\varepsilon}(L_p(\Omega)) \tag{1.4.5}$$

for all $1 < p < \infty$ and every $s > \varepsilon > 0$.

**(iv)** For every bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$ there exists a linear extension operator

$$\mathcal{E}_\Omega \colon B^s_q(L_p(\Omega)) \to B^s_q(L_p(\mathbb{R}^d))$$

which is simultaneously bounded for all parameters that satisfy (1.4.4); cf. [115]. Moreover, $\mathcal{E}_\Omega$ is local in the sense that $\operatorname{supp}(\mathcal{E}_\Omega u)$ is contained in some bounded neighborhood of $\Omega$; see [35].

**Remark 1.7.** In addition to Remark 1.6(iv), let us also mention Stein's linear extension operator for Sobolev spaces on Lipschitz domains $\Omega \subset \mathbb{R}^d$ [121, Section VI.3.2], which we denote by $\mathcal{E}_S = \mathcal{E}_S(\Omega)$. We note that $\mathcal{E}_S : W^\ell(L_p(\Omega)) \to W^\ell(L_p(\mathbb{R}^d))$ is simultaneously bounded for all $\ell \in \mathbb{N}_0$ and $1 < p < \infty$, as well as local in the sense that $\operatorname{supp}(\mathcal{E}_S g)$ is contained in some bounded neighborhood of $\Omega$, see [121, Section VI.3.1, Theorem 5]. Hansen [76, Lemma 5.1] has shown for polyhedral Lipschitz domains that Stein's extension operator is also bounded with respect to the $\mathcal{K}^\ell_{p,a}$-norms, i.e., for every $a \geq 0$, $\mathcal{E}_S$ is bounded as a mapping $\mathcal{E}_S : \mathcal{K}^\ell_{p,a}(\Omega) \to \mathcal{K}^\ell_{p,a}(\mathbb{R}^d)$. Hence, it holds

$$\left\| \mathcal{E}_S g \,\middle|\, \mathcal{K}^\ell_{p,a}(\mathbb{R}^d) \right\| \lesssim \left\| g \,\middle|\, \mathcal{K}^\ell_{p,a}(\Omega) \right\|$$

for all $g \in \mathcal{K}^\ell_{p,a}(\Omega)$. Here, $\mathcal{K}^\ell_{p,a}(\Omega)$ and $\mathcal{K}^\ell_{p,a}(\mathbb{R}^d)$ are supposed to admit the same singular set $S$, consisting of the non-smooth boundary points of $\Omega$ (cf. Section 1.3). Furthermore, for every $s > 0$, $1 < p < \infty$ and $0 < q \leq \infty$, the operator $\mathcal{E}_S$ is bounded as a mapping $B^s_q(L_p(\Omega)) \to B^s_q(L_p(\mathbb{R}^d))$, i.e.,

$$\left\| \mathcal{E}_S g \,\middle|\, B^s_q(L_p(\mathbb{R}^d)) \right\| \lesssim \left\| g \,\middle|\, B^s_q(L_p(\Omega)) \right\|$$

for all $g \in B^s_q(L_p(\Omega))$, see [76, Lemma 5.2].

The relations of Besov spaces corresponding to different values of the integrability and fine index are summarized by the next lemma.

**Lemma 1.8.**

**(i)** *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be either $\mathbb{R}^d$ itself, or some bounded Lipschitz domain, as well as $0 < p < \infty$ and $s > \sigma_p$. Then, for all $0 < q_0 \leq q_1 \leq \infty$ we have the continuous embedding*

$$B^s_{q_0}(L_p(\Omega)) \hookrightarrow B^s_{q_1}(L_p(\Omega)), \tag{1.4.6}$$

*as well as*

$$B^{s+\varepsilon}_{q_1}(L_p(\Omega)) \hookrightarrow B^s_{q_0}(L_p(\Omega)) \tag{1.4.7}$$

*for any $\varepsilon > 0$.*

**(ii)** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be some bounded Lipschitz domain, as well as $s > 0$ and $0 < q \leq \infty$. Then, for all $0 < p_0 < p_1 \leq \infty$ the continuous embedding*

$$B^s_q(L_{p_1}(\Omega)) \hookrightarrow B^s_q(L_{p_0}(\Omega)) \tag{1.4.8}$$

*holds true.*

*Proof.* A proof of part (i) for $\Omega = \mathbb{R}^d$ can be found in [129, Proposition 2.3.2/2]. In case $\Omega$ is a bounded Lipschitz domain, see [131, (1.299)] for the first embedding and [131, Theorem 1.107] for the second. Let us remark that for the parameter constellations considered here, our characterization of Besov spaces (see (1.4.2)) coincides with the one used in the references given above in the sense of equivalent norms, see [129, Section 2.5.12] and [131, Theorem 1.118 & Remark 1.119].

To prove part (ii), note that from Hölder's inequality (applied for $p := p_1/p_0 > 1$, $\tilde{f} := |f|^{p_0} \in L_p(\Omega)$ and $g \equiv 1$) we know that $\|f \mid L_{p_0}(\Omega)\| \leq |\Omega|^{1/p_0 - 1/p_1} \|f \mid L_{p_1}(\Omega)\|$ for all $0 < p_0 < p_1 \leq \infty$. Now the assertion follows readily from (1.4.1) and (1.4.2). □

A result analog to the well-known Sobolev embedding theorem holds true for Besov spaces, as the following lemma states.

**Lemma 1.9.** *Let $\Omega \subseteq \mathbb{R}^d$, $d \geq 2$, be either $\mathbb{R}^d$ itself, or some bounded Lipschitz domain.*

**(i)** *Let $0 < p_0 < p_1 \leq \infty$ and $s_0 > s_1 > \sigma_{p_1}$ with $s_0 - d/p_0 = s_1 - d/p_1$. Then, the continuous embedding*

$$B^{s_0}_{q_0}(L_{p_0}(\Omega)) \hookrightarrow B^{s_1}_{q_1}(L_{p_1}(\Omega)) \tag{1.4.9}$$

*holds true if $0 < q_0 \leq q_1 \leq \infty$ and $q_0 < \infty$.*

**(ii)** *Let $1 < p < \infty$ and $\sigma_0 > \sigma_1 > 0$, as well as $1/\tau_i = \sigma_i/d + 1/p$ for $i \in \{0, 1\}$. Then, the continuous embeddings*

$$B^{\sigma_0}_{\tau_0}(L_{\tau_0}(\Omega)) \hookrightarrow B^{\sigma_1}_{\tau_1}(L_{\tau_1}(\Omega)) \hookrightarrow L_p(\Omega) \tag{1.4.10}$$

*hold true.*

*Proof.* Part (i). If $\Omega = \mathbb{R}^d$, the assertion follows from [129, Theorem 2.7.1/1] and [129, Proposition 2.3.2/2]. In case $\Omega$ is a bounded Lipschitz domain, see [131, (1.301)].

Part (ii). The first embedding follows from part (i) of this lemma. A proof of the second embedding for $\Omega = \mathbb{R}^d$ can be found in [131, Theorem 1.73(i)]. Finally, this latter result transfers to the case of bounded Lipschitz domains by an application of the bounded extension operator $\mathcal{E}_\Omega$, see Remark 1.6(iv). □

In Figure 1.1 some of the above embedding results for Besov spaces are visualized by means of a so-called DeVore-Triebel diagram. In this $(1/\tau, \sigma)$-diagram each point $(1/p, s) \in (0, \infty)^2$ represents a Besov space $B^s_p(L_p(\Omega))$. Moreover, the points $(1/p, 0)$, $0 < 1/p < \infty$, stand for the Lebesgue spaces $L_p(\Omega)$. The three arrows starting from $(1/p, s)$ and pointing to the north-west, north and east correspond to the embeddings (1.4.9) (with $q_i = p_i$ for $i \in \{0, 1\}$), (1.4.3) (with $q = p$), and (1.4.8) (with $q = p_1$), respectively. The last embedding clearly only holds true if $\Omega$ is bounded and the fine index $q$ of the target space is not decreased. However, if the smoothness of the target space is decreased by an arbitrary small $\varepsilon > 0$ (indicated by the $\varepsilon$ attached to the respective arrow), with the help of (1.4.7) we conclude that

$$B^s_{p_1}(L_{p_1}(\Omega)) \hookrightarrow B^{s-\varepsilon}_{p_0}(L_{p_0}(\Omega))$$

for $p_1 > p_0$.

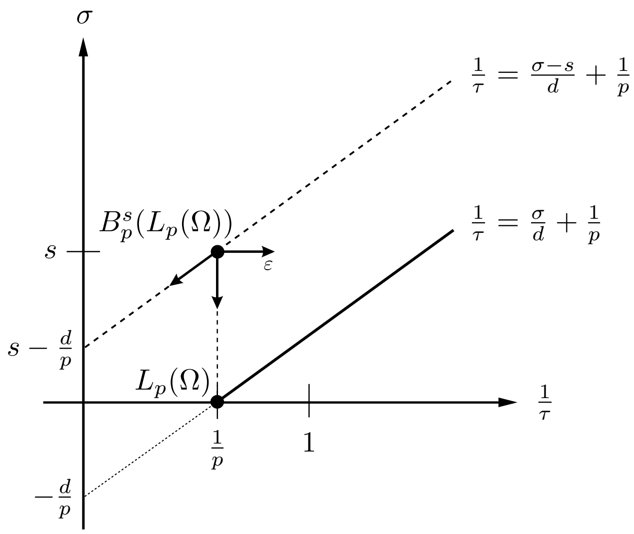



Figure 1.1: Embeddings of Besov spaces illustrated by a DeVore-Triebel diagram.

**Remark 1.10.** The demarcation line for embeddings of Besov spaces into $L_p(\Omega)$, $1 < p < \infty$, is given by

$$\frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}. \tag{1.4.11}$$

Every Besov space with smoothness and integrability indices corresponding to a point above that line is continuously embedded into $L_p(\Omega)$ (regardless of the fine index $q$). This follows from Lemma 1.8(i) and (1.4.10). The points below this line never embed into $L_p(\Omega)$. For spaces $B^\sigma_q(L_\tau(\Omega))$ with $(\sigma, \tau)$ that satisfy (1.4.11) some care is needed. However, if $q = \tau$, then the embedding still holds, see (1.4.10). Observe that (1.4.11) exactly coincides with the *adaptivity scale of Besov spaces* we are interested in.

# 1.5 Wavelet bases and frames

In numerous areas of applied mathematics one is interested in the decomposition of a general function $f$ into components corresponding to different scales of resolution. Such a *multiscale decomposition* can formally be written as

$$f = f_{j_0} + \sum_{j \geq j_0} g_j,$$

where $f_{j_0}$ denotes some coarsest approximation to $f$ and $g_j = f_{j+1} - f_j$ represents the fluctuation (additional detail) between the approximations $f_j$ and $f_{j+1}$ at resolution levels $j$ and $j+1$, respectively. Moreover, it is clearly desirable (in particular in the context of numerical applications) to further decompose each fluctuation $g_j$ into *local* contributions.

In many situations, *wavelet bases* constitute an elegant way to achieve this task. Moreover, they give rise to computationally efficient decomposition and reconstruction algorithms (see, e.g., [17, Section 2.6]), allow for the characterization of function spaces (see Section 1.6) and admit certain cancellation properties, in the sense that the $L_2$-inner product between a wavelet and a smooth function either vanishes or decays exponentially with increasing scale $j$ of the wavelet. Due to these properties, wavelets are widely-used in the context of signal analysis, image processing and numerical analysis. Here, let us in particular mention the wavelet discretization of operator equations [24].

In the following Subsection 1.5.1 we give a general introduction to wavelet bases including a short summary of a common construction method and typical properties of wavelets. To circumvent certain difficulties that appear in the construction of wavelet bases on general, more complex domains, the use of (wavelet) *frames* often constitutes a practical alternative to bases [31, 32, 33, 85, 111, 134]. In Subsection 1.5.2 we give a short description of the type of wavelet frames we will use for our numerical applications in Part IV.

### 1.5.1 Wavelet bases

In order to keep the notation simple and to focus on the main features, let us first consider the Hilbert space $L_2(\Omega)$ with $\Omega = \mathbb{R}$. In this classical setting, the term *wavelet* usually denotes a univariate function $\psi : \mathbb{R} \to \mathbb{R}$ which, when subjected to shifts (i.e., translation by integers) and dyadic dilation, yields an orthogonal basis of $L_2(\mathbb{R})$. That is, the collection of functions

$$\Psi = \left\{ \psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k) \,\middle|\, j,k \in \mathbb{Z} \right\}$$

forms an orthonormal basis of $L_2(\mathbb{R})$, so that each $f \in L_2(\mathbb{R})$ admits a unique (wavelet) expansion

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k} \rangle \, \psi_{j,k}$$

with unconditional convergence in $L_2(\mathbb{R})$. The system $\Psi$ is called orthonormal *wavelet basis* of $L_2(\mathbb{R})$.

However, there are several generalizations of this definition which drop the requirement of orthogonality. In the following we outline a commonly used construction principle which leads to so-called *biorthogonal* wavelet bases. We confine ourselves hereby to the summary of those results which are relevant for our purposes. A rigorous treatment of this topic as well as proofs of all subsequent assertions can be found in [17, Chapter 2].

Before we describe this construction scheme, let us first recall the notion of a Riesz basis.

**Definition 1.11** (Riesz basis)**.** *Let $H$ be a separable Hilbert space with inner product $\langle \cdot,\cdot \rangle_H$ and induced norm $\|\cdot\|_H = \langle \cdot,\cdot \rangle_H^{1/2}$, and let $\mathcal{J}$ be a countable index set. Then, a sequence $\{e_n\}_{n\in\mathcal{J}}$ in $H$ is called* Riesz basis *for $H$, if the span of $\{e_n\}_{n\in\mathcal{J}}$ is dense in $H$ and if there exist constants $0 < A \le B < \infty$ (called Riesz basis bounds), such that*

$$A \, \|c\|_{\ell_2(\mathcal{J})}^2 \le \left\| \sum_{n\in\mathcal{J}} c_n e_n \right\|_H^2 \le B \, \|c\|_{\ell_2(\mathcal{J})}^2 \tag{1.5.1}$$

*for all* $c = \{c_n\}_{n\in\mathcal{J}} \in \ell_2(\mathcal{J})$.

Note that every Riesz basis is a Schauder basis, and that property (1.5.1) expresses the 'stability' of the expansion in this basis with respect to the coordinates.

**Common construction principle**

In practice, (biorthogonal) wavelet (Riesz) bases are typically constructed by means of a *multiresolution analysis* (MRA) [102, 107], i.e., a sequence $(V_j)_{j\in\mathbb{Z}}$ of closed linear subspaces of $L_2(\mathbb{R})$ with

$$V_j \subset V_{j+1} \qquad \text{and} \qquad f \in V_j \Leftrightarrow f(2\cdot) \in V_{j+1} \qquad \text{for all} \quad j \in \mathbb{Z}, \tag{1.5.2}$$

as well as

$$\overline{\left(\bigcup_{j\in\mathbb{Z}} V_j\right)} = L_2(\mathbb{R}) \qquad \text{and} \qquad \bigcap_{j\in\mathbb{Z}} V_j = 0.$$

Moreover, it is assumed that there exists a function $\phi \in V_0$ such that

$$\Phi_0 := \{\phi(\cdot - k) \,|\, k \in \mathbb{Z}\}$$

is a Riesz basis of $V_0$. The function $\phi$ is called *scaling function*, and from (1.5.2) it follows that the system $\Phi_j := \{\phi_{j,k} := 2^{j/2}\phi(2^j \cdot -k) \,\big|\, k \in \mathbb{Z}\}$ constitutes a Riesz basis for $V_j$, $j \in \mathbb{Z}$.

A pair of MRA's $V$ and $\tilde{V}$ with corresponding scaling fuctions $\phi$ and $\tilde{\phi}$ is called *biorthogonal*, if it holds

$$\langle \phi(\cdot - k), \tilde{\phi}(\cdot - \ell)\rangle = \delta_{k,\ell}, \qquad k, \ell \in \mathbb{Z}.$$

Now, given a pair of biorthogonal MRA $(V, \tilde{V})$ with compactly supported scaling functions $(\phi, \tilde{\phi})$, we can define (oblique) projectors $P_j : L_2(\mathbb{R}) \to V_j$ and $P_j^* : L_2(\mathbb{R}) \to \tilde{V}_j$ by

$$P_j f := \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}_{j,k}\rangle \phi_{j,k} \qquad \text{and} \qquad P_j^* f := \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle \tilde{\phi}_{j,k},$$

respectively. These projectors in turn induce (detail) operators $Q_j := P_{j+1} - P_j$ and $Q_j^* := P_{j+1}^* - P_j^*$. In fact, $Q_j$ is a projector onto a complementary space $W_j$ of $V_j$ in $V_{j+1}$, i.e.,

$$Q_j : L_2(\mathbb{R}) \to W_j \qquad \text{and} \qquad V_{j+1} = V_j \oplus W_j,$$

where $\oplus$ denotes the direct sum. Here, an analogous assertion holds true for $Q_j^*$. Now, it is possible to construct in a systematic way functions $\psi \in W_1$ and $\tilde{\psi} \in \tilde{W}_1$ (the biorthogonal wavelets) with

$$\langle \psi(\cdot - k), \tilde{\psi}(\cdot - \ell)\rangle = \delta_{k,\ell} \qquad \text{and} \qquad \langle \psi(\cdot - k), \tilde{\phi}(\cdot - \ell)\rangle = \langle \tilde{\psi}(\cdot - k), \phi(\cdot - \ell)\rangle = 0$$

for all $k, \ell \in \mathbb{Z}$, such that $\Psi_j := \{\psi_{j,k} := 2^{j/2}\psi(2^j \cdot -k) \,\big|\, k \in \mathbb{Z}\}$ and $\tilde{\Psi}_j := \{\tilde{\psi}_{j,k} := 2^{j/2}\tilde{\psi}(2^j \cdot -k) \,\big|\, k \in \mathbb{Z}\}$ constitute Riesz bases of $W_j$ and $\tilde{W}_j$, respectively. Altogether, the pairs of scaling functions and wavelets provide for each $f \in L_2(\mathbb{R})$ and $j_0 \in \mathbb{Z}$ the multiscale decomposition

$$f = \sum_{k\in\mathbb{Z}} \langle f, \tilde{\phi}_{j_0,k} \rangle \phi_{j_0,k} + \sum_{j\geq j_0} \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}_{j,k} \rangle \psi_{j,k} \tag{1.5.3}$$

and satisfy the biorthogonality relations

$$\langle \phi_{j_0,k}, \tilde{\psi}_{j,m} \rangle = \langle \tilde{\phi}_{j_0,k}, \psi_{j,m} \rangle = 0, \; j \geq j_0, \quad \text{and} \quad \langle \psi_{j,k}, \tilde{\psi}_{\ell,m} \rangle = \delta_{j,\ell}\delta_{k,m}$$

for all $j, k, \ell, m \in \mathbb{Z}$. Moreover, if $\phi, \tilde{\phi} \in W^\varepsilon(L_2(\mathbb{R}))$ for some $\varepsilon > 0$, the systems

$$\Psi := \Phi_{j_0} \cup \bigcup_{j\geq j_0} \Psi_j \qquad \text{and} \qquad \tilde{\Psi} := \tilde{\Phi}_{j_0} \cup \bigcup_{j\geq j_0} \tilde{\Psi}_j \tag{1.5.4}$$

constitute biorthogonal (wavelet) Riesz bases of $L_2(\mathbb{R})$ [17, Section 3.8]. If additionally $\phi, \tilde{\phi} \in C^{0,\varepsilon}(\mathbb{R})$ for some $\varepsilon > 0$, then the systems $\Psi$ and $\tilde{\Psi}$ are also unconditional bases for $L_p(\mathbb{R})$ for all $1 < p < \infty$ [17, Theorem 3.8.3]. Finally, we note that the expansion (1.5.3) can also be written as

$$f = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \tilde{\psi}_{j,k} \rangle \psi_{j,k},$$

and that the particular case of an orthonormal scaling function in the construction above, i.e., $\phi = \tilde{\phi}$, leads to an orthonormal wavelet basis, i.e., $\psi = \tilde{\psi}$.

### Typical properties

Typically wavelet bases are constructed in a way such that its elements have compact support, satisfy appropriate smoothness assumptions and that the spaces $V_j$ admit *polynomial exactness* of some order $m \in \mathbb{N}$, i.e, there exist constants $c_{k,q} \in \mathbb{R}$ such that

$$x^q = \sum_{k\in\mathbb{Z}} c_{k,q} \phi(x-k)$$

allmost everywhere for all $0 \leq q < m$. The latter property is closely related to the *vanishing moments* property of order $m$ of the dual wavelet, i.e.,

$$\int_{\mathbb{R}} x^q \, \tilde{\psi}(x) \, \mathrm{d}x = 0$$

for all $0 \leq q < m$.

As prominent instances of wavelet bases for the space $L_2(\mathbb{R})$ let us mention the compactly supported orthonormal wavelets of Daubechies [44, 45] and the compactly supported biorthogonal spline wavelets of Cohen, Daubechies and Feauveau [21].

**Remark 1.12.**

**(i)** By the commonly used tensor product strategy (see, e.g., [17, Section 2.12]), univariate wavelet bases can easily be extended to the multivariate case. We describe and apply this approach in Section 1.6 to construct wavelet bases for the spaces $L_p(\mathbb{R}^d)$, where $d \geq 2$ and $1 < p < \infty$, from univariate Daubechies wavelets.

**(ii)** The construction principle for wavelet bases via MRA we outlined above has been successfully adapted to the interval $\Omega = [0,1]$, see [2, 11, 16, 22, 40, 110]. An application of the tensor product approach from part (i) of this remark then readily yields wavelet bases of $L_2(\Omega)$ for the unit cube $\Omega = [0,1]^d$, $d \geq 2$. Clearly, by scaling, these instances generalize to rectangular domains $\Omega = I_1 \times \cdots \times I_d$, where $I_j$, $1 \leq j \leq d$, denote bounded intervals.

**(iii)** For bounded domains $\Omega \subset \mathbb{R}^d$ which can be represented as the disjoint union of smooth parametric images of the standard cube, i.e., $\overline{\Omega} = \cup_{i=1}^n \overline{\Omega_i}$, where $\Omega_i = T_i((0,1)^d)$ with smooth isoparametric maps $T_i$, $1 \leq i \leq n$, it is further possible to construct (composite) wavelet bases based on a basis on the reference cube $(0,1)^d$, see [12, 13, 41, 42, 43] for a detailed discussion.

Finally, we introduce the following notation which aims to simplify the representation of a function in a wavelet basis, in particular when working with bases on multivariate domains as described in Remark 1.12. We will use this notation whenever the concrete shape of the wavelet indices is not relevant (e.g., in Chapter 4).

By $\Phi_j := \{\phi_\gamma \mid \gamma \in \Gamma_j\}$ we denote the scaling function basis of $V_j$ and by $\Psi_j := \{\psi_\lambda \mid \lambda \in \Lambda_j\}$ the wavelet basis of $W_j$ for $j \geq j_0 \in \mathbb{Z}$ (with an analogous notation for the dual MRA). Moreover, we set $\Psi_{j_0-1} := \Phi_{j_0}$, i.e., at the coarsest level $j_0$ we denote $\{\phi_\gamma \mid \gamma \in \Gamma_{j_0}\}$ by $\{\psi_\lambda \mid \lambda \in \Lambda_{j_0-1}\}$ (analogously for the dual functions), so that the wavelet expansion takes the form

$$f = \sum_{j \geq j_0-1} \sum_{\lambda \in \Lambda_j} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda = \sum_{\lambda \in \Lambda} \langle f, \tilde{\psi}_\lambda \rangle \psi_\lambda,$$

where we have set $\Lambda := \cup_{j \geq j_0-1} \Lambda_j$. We also write $\Lambda^j := \cup_{j_0-1 \leq i < j} \Lambda_i$ for $j \geq j_0$. Further, we use the same notation as in (1.5.4) for the primal/dual wavelet basis. For an index $\lambda \in \Lambda_j \cup \Gamma_j$ we set $|\lambda| := j$. For notational convenience and w.l.o.g. we may assume that $j_0 = 0$.

### 1.5.2 Aggregated wavelet frames

In view of our applications in Part IV, in this subsection we introduce the concept of *aggregated wavelet frames*. This approach aims at a simplified construction principle, compared to the construction of wavelet bases on general bounded domains (cf. [41, 42, 43]). Therefore, one gives up the Riesz basis property and works in the context of frames, while retaining most of the beneficial properties of wavelets. In particular, aggregated wavelet frames are well suited for the discretization and adaptive approximation of linear operator equations ([31, 32, 122, 124, 134]). In Section 8.4 we will utilize such an adaptive solver based on aggregated wavelet frames for the numerical approximation of certain linear elliptic subproblems.

To begin with, let us shortly summarize some assertions regarding general frames in Hilbert spaces.

#### Frames in Hilbert spaces

In the following, let $H$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_H$ and induced norm $\|\cdot\|_H = \langle \cdot, \cdot \rangle_H^{1/2}$, and let $\mathcal{J}$ be a countable index set.

**Definition 1.13** (Hilbert frame). *A sequence* $\{e_n\}_{n\in\mathcal{J}}$ *in* $H$ *is called* (Hilbert) frame *for* $H$*, if there exist constants* $0 < A \leq B < \infty$ *(called frame bounds) such that*

$$A\,\|f\|_H^2 \leq \sum_{n\in\mathcal{J}} |\langle f, e_n\rangle_H|^2 \leq B\,\|f\|_H^2 \tag{1.5.5}$$

*for all* $f \in H$.

From the second inequality in (1.5.5) we conclude that the linear operator

$$F : H \to \ell_2(\mathcal{J}), \quad f \mapsto \{\langle f, e_n\rangle_H\}_{n\in\mathcal{J}}\,,$$

the so-called *analysis operator*, is bounded with $\|F\| \leq \sqrt{B}$ (here $\|\cdot\|$ denotes the usual operator norm). A short computation (cf. [45, (3.2.3)]) yields that the adjoint of $F$ is given by

$$F^* : \ell_2(\mathcal{J}) \to H, \quad \{c_n\}_{n\in\mathcal{J}} \mapsto \sum_{n\in\mathcal{J}} c_n e_n,$$

the so-called *synthesis operator* (with $\|F^*\| = \|F\|$). Now, for their composition

$$S := F^*F : H \to H, \quad f \mapsto \sum_{n\in\mathcal{J}} \langle f, e_n\rangle_H\, e_n,$$

the so-called *frame operator*, the following assertions are well-known. For a proof, see [15, Lemma 5.1.5].

**Lemma 1.14.** *Let* $\{e_n\}_{n\in\mathcal{J}}$ *be a frame for* $H$ *with frame operator* $S$ *and frame bounds* $0 < A \leq B < \infty$.

**(i)** *The frame operator* $S$ *is bounded, invertible, self-adjoint and positive definite. Moreover, it holds* $A\,\|f\|_H^2 \leq \langle Sf, f\rangle_H \leq B\,\|f\|_H^2$ *for all* $f \in H$.

**(ii)** *The sequence* $\{\tilde{e}_n\}_{n\in\mathcal{J}} := \{S^{-1}e_n\}_{n\in\mathcal{J}}$ *is as well a frame for* $H$ *with frame bounds* $B^{-1}$, $A^{-1}$. *The frame operator corresponding to* $\{\tilde{e}_n\}_{n\in\mathcal{J}}$ *is* $S^{-1}$.

Now, with the help of the frame $\{\tilde{e}_n\}_{n\in\mathcal{J}}$ from Lemma 1.14(ii), called *canonical dual frame*, each element in $H$ admits the following frame decomposition; cf. [15, Theorem 5.1.6].

**Proposition 1.15.** *Let* $\{e_n\}_{n\in\mathcal{J}}$ *be a frame for* $H$ *with frame operator* $S$ *and canonical dual frame* $\{\tilde{e}_n\}_{n\in\mathcal{J}}$. *Then, each* $f \in H$ *admits the representation*

$$f = S^{-1}Sf = \sum_{n\in\mathcal{J}} \langle f, e_n\rangle_H\, \tilde{e}_n = SS^{-1}f = \sum_{n\in\mathcal{J}} \langle f, \tilde{e}_n\rangle_H\, e_n \tag{1.5.6}$$

*with unconditional convergence in* $H$.

In addition, if we denote the analysis and synthesis operator corresponding to the canonical dual frame by $\tilde{F}$ and $\tilde{F}^*$, respectively, the frame decomposition (1.5.6) can also be written as $f = \tilde{F}^*Ff = F^*\tilde{F}f$. From this latter representation we infer that $F^*$ must be onto, and from the first inequality of (1.5.5) we conclude that $F$ is injective. However, $F^*$ is not necessarily injective, as the following remark points out.

**Remark 1.16.**

**(i)** Every Riesz basis for $H$ is in particular a frame for $H$, and the Riesz basis bounds coincide with the frame bounds. A proof of this fact can be found in [15, Theorem 5.4.1].

**(ii)** The opposite statement is not true, i.e., a frame is in general not a (Riesz or Schauder) basis; for an example see [15, Example 5.4.5]. If a frame $\{e_n\}_{n\in\mathcal{J}}$ for $H$ is not a (Schauder) basis, then there exists some $c = \{c_n\}_{n\in\mathcal{J}} \in \ell_2(\mathcal{J})$ with $c \neq 0$, such that $\sum_{n\in\mathcal{J}} c_n e_n = 0$ (cf. [15, Theorem 6.1.1]). Hence, frames generally allow for redundancies, in which case the frame representation (1.5.6) is not unique.

### Aggregated wavelet frames

As outlined at the beginning of this subsection, we now turn to the specific instance of *wavelet frames* which are well suited for the discretization of linear elliptic PDEs. In what follows we summarize a method (introduced in [122]) to construct wavelet frames for $W_0^s(L_2(\Omega))$, $s \geq 0$, over a bounded domain $\Omega \subset \mathbb{R}^d$. Therefore, let us assume that $\Omega$ is decomposed into $N_\Omega \in \mathbb{N}$ *overlapping* subdomains $\Omega_i \subset \Omega$, i.e.,

$$\Omega = \bigcup_{i=0}^{N_\Omega - 1} \Omega_i. \tag{1.5.7}$$

Now, the basic idea is to choose (local) wavelet bases for the subspaces $W_0^s(L_2(\Omega_i))$, $0 \leq i < N_\Omega$, and to aggregate them into a (global) frame for $W_0^s(L_2(\Omega))$. However, for this approach an appropriate partition of unity is required.

**Definition 1.17.** *A collection of functions* $\{\eta_i\}_{i=0}^{N_\Omega-1}$, $\eta_i : \Omega \to \mathbb{R}$ *is called a* partition of unity subordinate to the covering $\{\Omega_i\}_{i=0}^{N_\Omega-1}$, $\Omega = \cup_{i=0}^{N_\Omega-1}\Omega_i$, *if for a given* $s \geq 0$ *the following conditions hold true.*

*(i)* $\operatorname{supp}\eta_i \subset \overline{\Omega_i}$ *and* $\Sigma_{i=0}^{N_\Omega-1}\eta_i \equiv 1$.

*(ii)* $\eta_i v \in W_0^s(L_2(\Omega_i))$ *for all* $v \in W_0^s(L_2(\Omega))$.

*(iii)* $\|\eta_i v \,|\, W^s(L_2(\Omega_i))\| \lesssim \|v \,|\, W^s(L_2(\Omega))\|$ *for all* $v \in W_0^s(L_2(\Omega))$.

Further, by $E_i : W_0^s(L_2(\Omega_i)) \to W_0^s(L_2(\Omega))$, $0 \leq i < N_\Omega$, we denote the canonical extension by zero. Then, a frame for $W_0^s(L_2(\Omega))$ can be constructed as follows. For a proof, see [134, Proposition 2.8].

**Proposition 1.18.** *Let* $\mathcal{F}^{(i)}$, $0 \leq i < N_\Omega$, *be a Riesz basis or a frame for* $W_0^s(L_2(\Omega_i))$. *Furthermore, assume that a partition of unity subordinate to the covering* $\{\Omega_i\}_{i=0}^{N_\Omega-1}$ *from* (1.5.7) *exists. Then, the collection*

$$\mathcal{F} := \bigcup_{i=0}^{N_\Omega-1} E_i \mathcal{F}^{(i)}$$

*is a frame for* $W_0^s(L_2(\Omega))$.

**Remark 1.19.** For general domain decompositions, the existence of a partition of unity as required by Proposition 1.18 is not always obvious. However, for the two dimensional L-shaped domain which we will use for our numerical experiments in Chapter 9,

$$\Omega_L = (-1,1)^2 \setminus [0,1)^2 \tag{1.5.8}$$

with covering $\Omega_L = \Omega_0 \cup \Omega_1$, where $\Omega_0 = (-1,0) \times (-1,1)$ and $\Omega_1 = (-1,1) \times (-1,0)$, an appropriate partition of unity has been constructed in [134, Chapter 2.7] which satisfies the conditions of Definition 1.17 for all $s \in \mathbb{N}$.

Now, a common way to construct Riesz bases or frames for $W_0^s(L_2(\Omega_i))$ on the patches $\Omega_i$ is to *lift* a basis or frame for $W_0^s(L_2(\square))$ on a reference domain $\square$ to $\Omega_i$. Here, as the reference domain we choose the open unit cube, i.e., $\square := (0,1)^d$. In detail, for $0 \le i < N_\Omega$, let

$$\kappa_i : \square \to \Omega_i$$

be smooth parametrizations of $\Omega_i$ with respect to $\square$, where we additionally assume that the $\kappa_i$ are $C^k$-diffeomorphisms for some $k \ge s$ and $|\det D\kappa_i| \sim 1$ uniformly in $\square$. Now, for each $0 \le i < N_\Omega$, let $\mathcal{F}_i^\square$ be a Riesz basis or frame for $W_0^s(L_2(\square))$. Then, the collection of lifted elements

$$\mathcal{F}^{(i)} := \mathcal{F}_i^\square \circ \kappa_i^{-1} = \left\{ e^{(i)} \circ \kappa_i^{-1} \,\middle|\, e^{(i)} \in \mathcal{F}_i^\square \right\},$$

forms a Riesz basis or frame for $W_0^s(L_2(\Omega_i))$, respectively.

Finally, by choosing (e.g., tensor product type) wavelet Riesz bases $\Psi_i^\square$ for $W_0^s(L_2(\square))$ over the unit cube, i.e., setting $\mathcal{F}_i^\square = \Psi_i^\square$ in the construction above, by Proposition 1.18 we obtain a frame for $W_0^s(L_2(\Omega))$, which we refer to as an *aggregated wavelet frame*.

## 1.6 Wavelet characterization of Besov spaces

Under suitable conditions on the parameters involved it is possible to characterize Besov spaces by means of wavelet decompositions [45, 79, 108, 131]. These characterizations are one of the most important ingredients of wavelet analysis. In particular, they provide the basis for several numerical applications such as preconditioning and the design of adaptive algorithms. We refer to [17, 18, 26] for details. Moreover, the resulting (quasi-)norm equivalences provide a powerful tool which allows to prove continuous embeddings such as the one stated in Theorem 5.1 in Chapter 5 below.

To start with, we recall some basic assertions related to expansions with respect to Daubechies wavelets. We essentially follow the lines of [29]: Let $\{D_m \mid m \in \mathbb{N}\}$ denote the univariate family of compactly supported Daubechies wavelets [44, 45]. We remind the reader that $D_m$ has $m$ vanishing moments and the smoothness of these functions increases without bound as $m$ tends to infinity. So, let us fix an arbitrary value of $m$ and let $\psi^0 = \phi_m$ denote the univariate scaling function which generates the wavelet $\psi^1 = D_m$. Furthermore, by $E$ we denote the non-zero vertices of the unit cube $[0,1]^d$. Then, in dimension $d$, the set

$$\Psi^M = \Psi^M(d) = \left\{ \psi^e = \bigotimes_{n=1}^{d} \psi^{e_n} \,\middle|\, e = (e_1, \ldots, e_d) \in E \right\}$$

of $2^d - 1$ (tensor product) functions (so-called *mother wavelets*) generates (by shifts and dilates) an orthonormal wavelet basis for $L_2(\mathbb{R}^d)$ as follows: If

$$\mathcal{I} = \mathcal{I}(\mathbb{R}^d) = \left\{ I_{j,k} = 2^{-j}k + 2^{-j}[0,1]^d \,\middle|\, k \in \mathbb{Z}^d, j \in \mathbb{Z} \right\}$$

denotes the set of all dyadic intervals in $\mathbb{R}^d$, then the basis $\Psi$ consists of all functions of the form

$$\psi_I = \psi_{j,k} = 2^{j\,d/2}\, \psi(2^j \cdot -k) \quad \text{with} \quad I = I_{j,k} \in \mathcal{I}, \quad k \in \mathbb{Z}^d, \quad j \in \mathbb{Z}, \quad \text{and} \quad \psi \in \Psi^M. \tag{1.6.1}$$

In view of our application below, we remark that there exists some open cube $Q \subset \mathbb{R}^d$, centered at the origin with sides parallel to the coordinate axes, such that $\operatorname{supp}(\psi) \subset Q$ for all $\psi \in \Psi^M$. Accordingly, all basis functions (1.6.1) satisfy $\operatorname{supp}(\psi_I) \subset Q(I) = 2^{-j}k + 2^{-j}Q$, where

$$|Q(I)| \sim |I| = 2^{-j\,d} \quad \text{and} \quad Q(I) \subset B(I) = B_{2^{-(j+1)}\operatorname{diam}(Q)}(2^{-j}k), \quad I = I_{j,k} \in \mathcal{I}. \tag{1.6.2}$$

For every $1 < q < \infty$ the system defined in (1.6.1) also forms an unconditional basis for $L_q(\mathbb{R}^d)$. Hence, for those $q$ each $g \in L_q(\mathbb{R}^d)$ possesses a wavelet expansion

$$g = \sum_{I \in \mathcal{I}} \sum_{\psi \in \Psi^M} \langle g, \psi_I \rangle\, \psi_I \tag{1.6.3}$$

which converges in $L_q(\mathbb{R}^d)$.

For our purposes it is convenient to slightly modify this decomposition. Therefore let $S_0$ be the closure of all finite linear combinations of integer shifts of $\bigotimes_{n=1}^d \phi_m$ in $L_2(\mathbb{R}^d)$ and let $P_0$ denote the orthogonal projector which maps $L_2(\mathbb{R}^d)$ onto $S_0$. Then, for every $1 < q < \infty$, the operator $P_0$ can be extended to a projector on $L_q(\mathbb{R}^d)$ and in (1.6.3) we can restrict ourselves to those $\psi_I$ for which

$$I \in \mathcal{I}^+ = \mathcal{I}^+(\mathbb{R}^d) = \{ I \in \mathcal{I}(\mathbb{R}^d) \,|\, |I| \leq 1 \},$$

i.e., to wavelets corresponding to levels $j \in \mathbb{N}_0$. Moreover, we shall renormalize our wavelets and set

$$\psi_{I,p} = |I|^{1/2 - 1/p}\, \psi_I \qquad \text{for all} \quad I \in \mathcal{I}^+, \quad \psi \in \Psi^M, \quad \text{and} \quad 0 < p < \infty,$$

such that $\left\| \psi_{I,p} \,\middle|\, L_p(\mathbb{R}^d) \right\| = \left\| \psi \,\middle|\, L_p(\mathbb{R}^d) \right\|$ does not depend on $I$. Incorporating these conventions, from (1.6.3) we conclude that every $g \in L_q(\mathbb{R}^d)$, $1 < q < \infty$, can be expanded as

$$\begin{aligned} g &= P_0(g) + \sum_{I \in \mathcal{I}^+} \sum_{\psi \in \Psi^M} \langle g, \psi_I \rangle\, \psi_I \\ &= P_0(g) + \sum_{I \in \mathcal{I}^+} \sum_{\psi \in \Psi^M} \langle g, \psi_{I,p'} \rangle\, \psi_{I,p}, \end{aligned} \tag{1.6.4}$$

where $p'$ satisfies $1/p' = 1 - 1/p$.

**Lemma 1.20.** *Let $d \in \mathbb{N}$, $0 < p < \infty$, and $\sigma_p < s < r \in \mathbb{N}$. Moreover, choose $m \in \mathbb{N}$ such that $\phi_m, D_m \in C^r(\mathbb{R})$. Then a function $g$ belongs to the Besov space $B^s_p(L_p(\mathbb{R}^d))$ if and only if (1.6.4) holds with*

$$\left\| P_0(g) \,\middle|\, L_p(\mathbb{R}^d) \right\| + \left( \sum_{I \in \mathcal{I}^+} \sum_{\psi \in \Psi^M} |I|^{-sp/d} \, |\langle g, \psi_{I,p'} \rangle|^p \right)^{1/p} < \infty. \tag{1.6.5}$$

*Furthermore, (1.6.5) provides an equivalent (quasi-)norm for $B^s_p(L_p(\mathbb{R}^d))$.*

The proof of this assertion is quite standard. For the case of Banach spaces ($p \geq 1$) it can be found, e.g., in [108]. For the quasi-Banach case $0 < p < 1$ we refer to [94]. Similar assertions can also be found in [131].

**Remark 1.21.** We stress the point that due to $s > \sigma_p$ every $g \in B^s_p(L_p(\mathbb{R}^d))$ belongs to some $L_q(\mathbb{R}^d)$, $1 < q < \infty$, such that (1.6.4) is well-defined; see Lemma 1.9 and Remark 1.10. Moreover, we can use the extension operator $\mathcal{E}_\Omega$ described in Remark 1.6(iv) to obtain similar norm equivalences for functions in $B^s_p(L_p(\Omega))$, where $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain.

As mentioned already in the introduction, we are particularly interested in Besov spaces $B^\sigma_\tau(L_\tau(\Omega))$ within the adaptivity scale of $L_p(\Omega)$, $1 < p < \infty$, i.e., spaces with parameters that satisfy (1.4.11). Therefore, we specialize Lemma 1.20 for the corresponding spaces on $\mathbb{R}^d$:

**Proposition 1.22.** *Let $d \in \mathbb{N}$, $1 < p < \infty$, as well as $0 < \sigma < r \in \mathbb{N}$, and $\tau = (\sigma/d + 1/p)^{-1}$. Moreover, choose $m \in \mathbb{N}$ such that $\phi_m, D_m \in C^r(\mathbb{R})$. Then a function $g$ belongs to the Besov space $B^\sigma_\tau(L_\tau(\mathbb{R}^d))$ if and only if*

$$g = P_0(g) + \sum_{I \in \mathcal{I}^+} \sum_{\psi \in \Psi^M} \langle g, \psi_{I,p'} \rangle \, \psi_{I,p}$$

*with*

$$\left\| P_0(g) \,\middle|\, L_\tau(\mathbb{R}^d) \right\| + \left( \sum_{I \in \mathcal{I}^+} \sum_{\psi \in \Psi^M} |\langle g, \psi_{I,p'} \rangle|^\tau \right)^{1/\tau} < \infty \tag{1.6.6}$$

*and (1.6.6) provides an equivalent (quasi-)norm for $B^\sigma_\tau(L_\tau(\mathbb{R}^d))$.*

*Proof.* Observe that $\psi_{I,\tau'} = |I|^{1/p' - 1/\tau'} \, \psi_{I,p'}$ implies $|I|^{-\sigma\tau/d} \, |\langle g, \psi_{I,\tau'} \rangle|^\tau = |\langle g, \psi_{I,p'} \rangle|^\tau$. Then the proof easily follows from Lemma 1.20. $\square$

# Chapter 2

# Topological Vector Spaces and Continuity

In view of the embeddings of function spaces that will be presented and proved in Subsection 5.2.1, we introduce some topological notions and facts. In Section 2.1 some topological basics are recapitulated. After this the concept of locally convex topological vector spaces is outlined in Section 2.2. Quasi-normed spaces, and in particular topological vector spaces whose topology is induced by a family of quasi-norms, are treated in Section 2.3. There, we also study linear continuous mappings from a locally convex topological vector space into such a topological vector space, which is induced by a family of quasi-norms. Finally, in Section 2.4 the notions of metrizability and completeness of topological vector spaces are summarized. For additional reading on this topic we refer to the textbooks [80, 87, 117, 128].

## 2.1 Topological basics

We begin with a summary of some basic topological notions.

**Definition 2.1** (Topology)**.** *A* topology $\mathcal{O}$ *on a set* $X$ *is a collection of subsets* $\mathcal{O} \subset \mathcal{P}(X)$*, called* open sets*, with*

*(i)* $\emptyset, X \in \mathcal{O}$*;*

*(ii) for* $U_i \in \mathcal{O}$*,* $i \in I$*, where* $I$ *is an arbitrary index set, it holds* $\bigcup_{i\in I} U_i \in \mathcal{O}$*;*

*(iii) for* $U, V \in \mathcal{O}$ *it holds* $U \cap V \in \mathcal{O}$*.*

*Then,* $(X, \mathcal{O})$ *is called a* topological space*.*

**Definition 2.2** (Neighborhood)**.** *Let* $(X, \mathcal{O})$ *be a topological space and* $x \in X$*. A set* $V \subset X$ *is called* neighborhood *of* $x$*, if there exists an open set* $U \in \mathcal{O}$ *with* $x \in U$ *and* $U \subset V$*.*

A topological space $(X, \mathcal{O})$ is called *Hausdorff space* (or *separated*), if for all $x, y \in X$ with $x \neq y$, there exist neighborhoods $U(x), U(y)$ of $x$ and $y$, respectively, such that $U(x) \cap U(y) = \emptyset$. In other words, any two distinct points can be separated by neighborhoods. Furthermore, we will need the following notion of a basis of a topology.

**Definition 2.3** (Neighborhood basis)**.** *Let $(X, \mathcal{O})$ be a topological space and $x \in X$. A system of open subsets $\{U_i(x) \mid i \in I\}$ is called* neighborhood basis *of $x$, if for each neighborhood $V$ of $x$ there exists an Index $k \in I$ with $U_k(x) \subset V$.*

If a neighborhood basis is given for all $x \in X$, every open set $U \in \mathcal{O}$ can be expressed in terms of these basis elements, namely, as the union $U = \cup_{x \in U} U(x)$, where for each $x \in U$, $U(x) \subset U$ is an element of the neighborhood basis of $x$. In this case, the union of all neighborhood bases forms a *basis* of the topology $\mathcal{O}$.

**Definition 2.4** (Basis)**.** *Let $(X, \mathcal{O})$ be a topological space. A collection $\mathcal{B} \subset \mathcal{O}$ of open sets is called* basis *for $\mathcal{O}$, if every open set $U \in \mathcal{O}$ is a union of members of $\mathcal{B}$.*

Now, for the moment, let us consider an arbitrary set $X$ without a given topology. One way to define a topology on $X$ is described in the following.

**Definition 2.5** (Local basis)**.** *Let $X$ be an arbitrary set.*

**(i)** *A nonempty system of subsets $\{U_i(x) \mid i \in I\}$ is called* local basis *of $x \in X$, if $x \in U_i(x)$ for all $i \in I$, and if for all $i, j \in I$ there exists $k \in I$ with $U_k(x) \subset U_i(x) \cap U_j(x)$.*

**(ii)** *Assume that a local basis is given for each $x \in X$ and let $A$ be a subset of $X$. Then, a point $x \in A$ is called* inner point *of $A$, if there exists an element $U_i(x)$ of the local basis of $x$ with $U_i(x) \subset A$. The set $A$ is called* open*, if each point of $A$ is an inner point.*

If a local basis is given for each $x \in X$, the open sets as defined in Definition 2.5(ii) indeed form a topology on $X$, which immediately follows from Definition 2.1. We state this fact by the following lemma.

**Lemma 2.6.** *Let $X$ be an arbitrary set and assume a local basis is given for each $x \in X$. Then the open sets as defined in Definition 2.5(ii) form a topology on $X$.*

In case $X$ is a normed space, a natural choice for a local basis are the balls with radius $r > 0$ centered at $x \in X$, i.e., $B_r(x) = \{y \in X \mid \|x - y\| < r\}$. The topology which is defined via this local basis will be referred to as the *norm topology.* Note that in this case the local basis elements $B_r(x)$ are open by theirselves, and hence form a neighborhood basis as well. The closed balls $\overline{B_r(x)}$ also form a local basis and generate the same topology, but are clearly not open and hence no neighborhood basis anymore.

Next, we recall the notion of continuity for functions between general topological spaces.

**Definition 2.7** (Continuity)**.** *Let $X, Y$ be topological spaces and $f : X \to Y$ a function.*

**(i)** *$f$ is called* continuous in *$x \in X$, if for each (open) neighborhood $V$ of $y = f(x)$ there exists an (open) neighborhood $U$ of $x$ with $f(U) \subset V$.*

**(ii)** *$f$ is called* continuous*, if $f$ is continuous in $x$ for all $x \in X$.*

Sometimes it is more convenient to use the following, equivalent characterization of continuity. Its proof is straightforward, see e.g. [104, Theorem 3.28].

**Lemma 2.8.** *Let $X, Y$ be topological spaces and $f : X \to Y$ a function. Then the two following properties are equivalent:*

*(i) $f$ is continuous.*

*(ii) For each open set $V \subset Y$, the inverse image $f^{-1}(V)$ is an open subset of $X$.*

For *vector spaces* $X$, which are additionally equipped with a topology $\mathcal{O}$ on $X$, we recap the notion of topological vector spaces. Here and in the following, $\mathbb{K}$ may denote either $\mathbb{R}$ or $\mathbb{C}$.

**Definition 2.9** (Topological vector space)**.** *Let $X$ be a $\mathbb{K}$-vector space and $\mathcal{O}$ a topology on $X$. If the vector space addition, as well as the scalar multiplication, are continuous functions (with respect to $\mathcal{O}$), then $(X, \mathcal{O})$ is called a* topological vector space *(TVS).*

Note that in a *normed* vector space, the vector space operations are always continuous with respect to the norm topology, i.e., to the topology induced by the norm. Hence, normed vector spaces are always topological vector spaces. In the subsequent section we will be faced with (function-) spaces that are possibly not normable, but which are equipped with a family of seminorms.

## 2.2 Locally convex topological vector spaces

**Definition 2.10** (Seminorm)**.** *Let $X$ be a $\mathbb{K}$-vector space. A* seminorm *on $X$ is a mapping $s : X \to [0, \infty)$ with the following properties:*

*(i)* $s(\lambda x) = |\lambda| s(x)$ *for all* $x \in X, \lambda \in \mathbb{K}$.

*(ii)* $s(x + y) \leq s(x) + s(y)$ *for all* $x, y \in X$.

One can show completely analogously to normed spaces, that a seminorm on a vector space $X$ induces a topology $\mathcal{O}$ on $X$, and $(X, \mathcal{O})$ is a topological vector space. But, $(X, \mathcal{O})$ is in general *not* separated in this case, in contrast to a normed space equipped with the norm topology. Hence, the concept of locally convex vector space topologies is introduced.

**Definition 2.11** (Locally convex topological vector space)**.** *Let $X$ be a $\mathbb{K}$-vector space and $\{s_i \,|\, i \in I\}$ a family of seminorms on $X$, such that*

$$\text{for each} \quad x \in X \setminus \{0\} \quad \text{there exists} \quad i \in I \quad \text{with} \quad s_i(x) \neq 0. \tag{2.2.1}$$

*For $i \in I$, $r > 0$ and $x \in X$ we set*

$$V_{i,r}(x) := \{y \in X \,|\, s_i(y - x) < r\} = x + V_{i,r}(0).$$

*Then, the topology generated by the local bases*

$$U_{I_0,r}(x) := \bigcap_{i \in I_0} V_{i,r}(x), \qquad I_0 \subset I \text{ finite}, \quad r > 0, \quad x \in X,$$

*is called the* locally convex vector space topology *and $(X, \{s_i \mid i \in I\})$ is called a* locally convex topological vector space *(LCTVS).*

**Remark 2.12.**

**(i)** Note that the local basis elements $U_{I_0,r}(x)$ are open. Hence, the sets $U_{I_0,r}(x)$ build up a neighborhood basis of $x$.

**(ii)** Due to assumption (2.2.1) on the seminorms $s_i$, every LCTVS is a Hausdorff space.

For a proof of the following lemma we refer to [72, Theorem B.5].

**Lemma 2.13.** *Let $(X, \{s_i \mid i \in I\})$ be a locally convex topological vector space. Then, the seminorms $s_i$, the vector space addition and the scalar multiplication are continuous.*

Hence, a LCTVS is indeed a topological vector space in the sense of Definition 2.9. Instead of the definition via a family of seminorms, LCTVSs can equivalently be defined via convex sets in the following way. Let $U$ be a subset of $X$. Then, $U$ is called

- *convex*, if $\quad x, y \in U, 0 \leq t \leq 1 \implies tx + (1-t)y \in U,$
- *balanced*, if $\quad x \in U, |\lambda| \leq 1 \implies \lambda x \in U,$
- *absorbent*, if $\quad x \in X \implies tx \in U$ for some $t > 0$.

Now, a topological vector space is locally convex if and only if the origin has a neighborhood basis of convex, balanced and absorbent sets. We formulate this result in the following lemma, a proof can be found in [114, Theorem 1.36, Theorem 1.37, Remark 1.38].

**Lemma 2.14.** *Let $(X, \mathcal{O})$ be a topological vector space.*

**(i)** *If the topology $\mathcal{O}$ is generated by a family of seminorms as described in Definition 2.11, i.e., if $X$ is a LCTVS, then the sets $U_{I_0,r}(0)$, which constitute a neighborhood basis of the origin, are convex, balanced and absorbent.*

**(ii)** *If the origin admits a neighborhood basis of convex, balanced and absorbent sets, then $X$ is a LCTVS, i.e., there exists a family of seminorms, which generate the topology $\mathcal{O}$.*

Finally, let us consider linear functions between two LCTVSs. The following characterization of continuity in terms of the corresponding seminorms is well-known, for a proof see [72, Lemma B.7 & Remark B.8].

**Lemma 2.15.** *Let $(X, \{s_i \mid i \in I\})$, $(Y, \{\tilde{s}_j \mid j \in J\})$ be locally convex topological vector spaces and $T : X \to Y$ a linear map. Then, the following properties are equivalent:*

*(i) $T$ is continuous.*

*(ii) $T$ is continuous in 0.*

*(iii) For each $j \in J$ there exists a finite subset $I_0 \subset I$ and a constant $C$, such that*

$$\tilde{s}_j(Tx) \leq C \max_{i \in I_0} s_i(x) \qquad \text{for all} \qquad x \in X.$$

## 2.3 Quasi-normed spaces

In what follows we will show that, like in the case of semi-norms, any family of *quasi*-norms defined on a vector space turns this space into a Hausdorff TVS. Furthermore, we will derive an assertion similar to Lemma 2.15.

**Definition 2.16** (Quasi-norm, $p$-norm)**.** *Let $X$ be a $\mathbb{K}$-vector space.*

**(i)** *A* quasi-norm *on $X$ is a mapping $q : X \to [0, \infty)$ with the following properties:*

*(1) If $q(x) = 0$, then $x = 0$.*
*(2) $q(\lambda x) = |\lambda|\, q(x)$ for all $x \in X, \lambda \in \mathbb{K}$.*
*(3) There exists a constant $C > 0$, such that*

$$q(x + y) \le C\,(q(x) + q(y)) \qquad \text{for all} \qquad x, y \in X. \tag{2.3.1}$$

**(ii)** *Let $0 < p \le 1$. A $p$*-norm *on $X$ is a mapping $\|\cdot\|_p : X \to [0, \infty)$, which satisfies properties (1), (2) and*

*(4) $\|x + y\|_p^p \le \|x\|_p^p + \|y\|_p^p$ for all $x, y \in X$.*

Note that for a quasi-normed space $(X, q)$, the constant $C$ of inequality (2.3.1) always satisfies $C \ge 1$, since property (2) implies $q(0) = 0$ so that (3) gives $q(x) = q(x+0) \le Cq(x)$ for all $x \in X$. We set

$$C_X := \min\{C \ge 1 \,|\, C \text{ satisfies inequality (2.3.1) for all } x, y \in X\}.$$

If $C_X = 1$, then $(X, q)$ is just a normed space.

Of course, a $p$-norm with $p = 1$ is just a norm. We note that every $p$-norm is a quasi-norm. This can be seen with the help of Lemma A.13. On the other hand, for every quasi-norm there exists an equivalent $p$-norm, which has been proven by Aoki [3] and Rolewicz [112].

**Lemma 2.17** (Aoki-Rolewicz)**.** *Let $(X, q)$ be a quasi-normed vector space. Let $p \in (0, 1]$ be such that*

$$C = 2^{1/p-1},$$

*where $C$ is the constant of the quasi-triangle inequality* (2.3.1)*. Then there exists a $p$-norm $\|\cdot\|_p$ on $X$, which is equivalent to $q$. In detail it holds*

$$\|x\|_p \le q(x) \le 4^{1/p}\, \|x\|_p \qquad \text{for all} \qquad x \in X.$$

In the following, let us consider a vector space $Y$ equipped with a family of quasi-norms $\{q_j \,|\, j \in J\}$, where $J$ denotes an arbitrary index set. Like in the case of semi-norms, the family of quasi-norms defines a topology on $Y$.

**Proposition 2.18.** *Let $Y$ be a $\mathbb{K}$-vector space and $\{q_j \mid j \in J\}$ a family of quasi-norms on $Y$. For $j \in J$, $r > 0$ and $y \in Y$ we set*

$$V_{j,r}(y) := \{z \in Y \mid q_j(z-y) < r\} = y + V_{j,r}(0). \tag{2.3.2}$$

*Then, the sets*

$$U_{J_0,r}(y) := \bigcap_{j \in J_0} V_{j,r}(y), \qquad J_0 \subset J \text{ finite}, \quad r > 0, \quad y \in Y, \tag{2.3.3}$$

*form a local basis on $Y$, and hence generate a topology $\mathcal{O}$ on $Y$.*

*Proof.* Let $y \in Y$. Clearly, $y \in U_{J_0,r}(y)$ for all finite $J_0 \subset J$ and $r > 0$. Next, let $U_{J_0,r_0}(y)$ and $U_{J_1,r_1}(y)$ be two arbitrary sets as defined in (2.3.3). With $J_2 := J_0 \cup J_1$ and $r_2 := \min\{r_0, r_1\}$ we have

$$\begin{aligned}
U_{J_2,r_2}(y) &= \bigcap_{j \in J_2} V_{j,r_2}(y) \\
&= \left( \bigcap_{j \in J_0} V_{j,r_2}(y) \right) \cap \left( \bigcap_{j \in J_1} V_{j,r_2}(y) \right) \\
&\subseteq \left( \bigcap_{j \in J_0} V_{j,r_0}(y) \right) \cap \left( \bigcap_{j \in J_1} V_{j,r_1}(y) \right) \\
&= U_{J_0,r_0}(y) \cap U_{J_1,r_1}(y).
\end{aligned}$$

Thus, the sets as defined in (2.3.3) form a local basis on $Y$, and the assertion follows with Lemma 2.6. □

**Remark 2.19.**

**(i)** Note that the sets $U_{J_0,r}(y)$ as defined in (2.3.3) are not necessarily open in the sense that they are contained in $\mathcal{O}$. Furthermore, quasi-norms need not be continuous. An example of a discontinuous quasi-norm can be found in [3], where, as a consequence, the sets $U_{J_0,r}(y)$ are not open.

**(ii)** The situation is quite different for $p$-norms. In case all quasi-norms $q_j$ are $p$-norms, i.e., $q_j = \|\cdot\|_{p_j}$ with $0 < p_j \leq 1$ for all $j \in J$, then the sets $U_{J_0,r}(y)$ as defined in (2.3.3) are open. Furthermore, then all $q_j$ are continuous functions (with respect to the topology as defined in Proposition 2.18). A proof is given in the appendix, see Proposition A.18.

Although the sets $U_{J_0,r}(y)$ are not necessarily open, they nevertheless always contain an open neighborhood of $y$, as the following lemma states.

**Lemma 2.20.** *Let the assumptions of Proposition 2.18 hold. Then, for each of the sets $U_{J_0,r}(y)$ as defined in* (2.3.3)*, there exists an open set $U$, such that $y \in U$ and $U \subseteq U_{J_0,r}(y)$.*

*Proof.* It suffices to show that each of the sets $V_{j,r}(y)$ defined in (2.3.2) contains an open neighborhood $V$ of $y$. Thus, in the following we fix $j \in J$, $r > 0$ and $y \in Y$.

*Step 1.* From Lemma 2.17 we know that there exists a $p$-norm $\|\cdot\|_{p_j}$, which is equivalent to $q_j$. Based on this we define the balls

$$\tilde{V}_{j,r}(y) := \left\{ z \in Y \,\middle|\, \|z - y\|_{p_j} < r \right\} \tag{2.3.4}$$

with respect to this $p$-norm and show that

$$\tilde{V}_{j,\varepsilon}(y) \subseteq V_{j,r}(y) \subseteq \tilde{V}_{j,r}(y) \qquad \text{for all} \qquad 0 < \varepsilon < 4^{-1/p_j} r. \tag{2.3.5}$$

To do so, let $z \in \tilde{V}_{j,\varepsilon}(y)$, i.e., $\|z - y\|_{p_j} < \varepsilon$. Then, with Lemma 2.17 we get

$$q_j(z - y) \le 4^{1/p_j} \|z - y\|_{p_j} < 4^{1/p_j} \varepsilon < r.$$

Hence, $z \in V_{j,r}(y)$. Completely analogously, one proves the second inclusion of (2.3.5).

*Step 2.* We show that the balls $\tilde{V}_{j,r}(y)$ as defined in (2.3.4) are open. Therefore, for each $q_j$ we choose an equivalent $p$-norm $\|\cdot\|_{p_j}$ (which exists due to Lemma 2.17). Now, from (2.3.5) it follows that both families of quasi-norms, $\{q_j \mid j \in J\}$ and $\{\|\cdot\|_{p_j} \,\big|\, j \in J\}$, induce the same topology on $Y$. In this topology, the sets $\tilde{V}_{j,r}(y) = \{z \in Y \mid \|z - y\|_{p_j} < r\}$ are open, see Proposition A.18.

*Step 3.* We choose $0 < \varepsilon < 4^{-1/p_j} r$ and set $V := \tilde{V}_{j,\varepsilon}(y)$. Now, $V$ is open due to *Step 2*, and with (2.3.5) we conclude that

$$y \in V \subseteq V_{j,r}(y),$$

i.e., we have found an open neighborhood of $y$, which is contained in $V_{j,r}(y)$. □

We note that due to Lemma 2.20, the local basis elements $U_{J_0,r}(y)$ of $y$ as defined in (2.3.3) are indeed neighborhoods of $y$.

**Remark 2.21.** The topology we found in Proposition 2.18 is Hausdorff, since for $x \neq y \in Y$ and arbitrary $j \in J$, the neighborhoods $V_{j,r}(x)$ and $V_{j,r}(y)$ are disjoint for $0 < r < \delta_j/(2C_j)$, where $\delta_j := q_j(x - y)$ and $C_j$ denotes the constant of the quasi-triangle inequality of $q_j$.

Next we show that a vector space $Y$, together with the topology induced by a family of quais-norms as stated in Proposition 2.18, indeed forms a topological vector space.

**Proposition 2.22.** *Let $Y$ be a $\mathbb{K}$-vector space and $\{q_j \mid j \in J\}$ a family of quasi-norms on $Y$. Let $\mathcal{O}$ denote the topology induced by $\{q_j \mid j \in J\}$, as stated in Proposition 2.18. Then $(Y, \mathcal{O})$ is a topological vector space.*

*Proof.* We have to show that the vector space operations are continuous with respect to $\mathcal{O}$. Let $\alpha$ and $\beta$ denote the vector addition and scalar multiplication on $Y$, respectively.

*Step 1.* We consider $\alpha : Y \times Y \to Y$ and prove continuity in $(x, y) \in Y \times Y$. Thus, we have to show that for all open neighborhoods $W(z)$ of $z := \alpha(x, y) = x + y$

there exists an open neighborhood $W(x,y)$ of $(x,y)$, such that $\alpha(W(x,y)) \subseteq W(z)$. Now, fix $(x,y)$ and $W(z)$. Since $W(z)$ is open, there exists a local basis element $U_{J_0,r}(z)$ of $z$ with

$$U_{J_0,r}(z) \subseteq W(z).$$

For $j \in J_0$, let $C_j$ denote the constant from the quasi-triangle inequality of $q_j$ and set

$$\varepsilon_j := \frac{r}{2C_j}.$$

Then, for $(\tilde{x},\tilde{y}) \in V_{j,\varepsilon_j}(x) \times V_{j,\varepsilon_j}(y)$ we have

$$\begin{aligned} q_j(\alpha(\tilde{x},\tilde{y}) - z) &= q_j(\tilde{x} + \tilde{y} - (x+y)) \\ &\le C_j\left(q_j(\tilde{x} - x) + q_j(\tilde{y} - y)\right) \\ &< C_j\left(\varepsilon_j + \varepsilon_j\right) \\ &= r. \end{aligned}$$

Thus, $\alpha(\tilde{x},\tilde{y}) \in V_{j,r}(z)$ and therefore

$$\alpha\left(V_{j,\varepsilon_j}(x) \times V_{j,\varepsilon_j}(y)\right) \subseteq V_{j,r}(z).$$

Next, from Lemma 2.20 we know that there exist open neighborhoods $W_j(x)$ and $W_j(y)$ of $x$ and $y$, respectively, which are contained in $V_{j,\varepsilon_j}(x)$ and $V_{j,\varepsilon_j}(y)$, respectively. Hence, $W_j(x) \times W_j(y)$ is an open neighborhood of $(x,y)$ with

$$\alpha\left(W_j(x) \times W_j(y)\right) \subseteq V_{j,r}(z).$$

Setting

$$W(x,y) := \bigcap_{j \in J_0} W_j(x) \times W_j(y)$$

yields

$$\alpha\left(W(x,y)\right) \subseteq \bigcap_{j \in J_0} V_{j,r}(z) = U_{J_0,r}(z) \subseteq W(z).$$

*Step 2.* It remains to prove continuity of $\beta : \mathbb{K} \times Y \to Y$ in $(\lambda,y) \in \mathbb{K} \times Y$. Thus, we have to show that for all open neighborhoods $W(z)$ of $z := \beta(\lambda,y) = \lambda y$ there exists an open neighborhood $W(\lambda,y)$ of $(\lambda,y)$, such that $\alpha(W(\lambda,y)) \subseteq W(z)$. Now, fix $(\lambda,y)$ and $W(z)$. Since $W(z)$ is open, there exists a local basis element $U_{J_0,r}(z)$ of $z$ with

$$U_{J_0,r}(z) \subseteq W(z).$$

For $j \in J_0$, let $C_j$ denote the constant from the quasi-triangle inequality of $q_j$ and set

$$\varepsilon_j := \frac{r}{2C_j|\lambda|}$$

and

$$\delta_j := \frac{\varepsilon_j |\lambda|}{C_j \left(\varepsilon_j + q_j(y)\right)}.$$

Then, for $\tilde{y} \in V_{j,\varepsilon_j}(y)$ and $\tilde{\lambda} \in W_j(\lambda) := \{\xi \in \mathbb{K} \,\big|\, |\lambda - \xi| < \delta_j\}$ we have

$$\begin{aligned} q_j(\beta(\tilde{\lambda}, \tilde{y}) - z) &= q_j(\tilde{\lambda}\tilde{y} - \lambda y) \\ &= q_j(\lambda(\tilde{y} - y) + (\tilde{\lambda} - \lambda)\tilde{y}) \\ &\leq C_j \left( |\lambda| q_j(\tilde{y} - y) + |\tilde{\lambda} - \lambda| q_j(\tilde{y}) \right) \\ &< C_j \left( |\lambda| \varepsilon_j + \delta_j q_j(\tilde{y} - y + y) \right) \\ &\leq \frac{r}{2} + C_j^2 \delta_j \left( q_j(\tilde{y} - y) + q_j(y) \right) \\ &< \frac{r}{2} + C_j^2 \delta_j \left( \varepsilon_j + q_j(y) \right) \\ &= r. \end{aligned}$$

Thus, $\beta(\tilde{\lambda}, \tilde{y}) \in V_{j,r}(z)$ and therefore

$$\beta\left(W_j(\lambda) \times V_{j,\varepsilon_j}(y)\right) \subseteq V_{j,r}(z).$$

Next, from Lemma 2.20 we know that there exists an open neighborhood $W_j(y)$ of $y$, which is contained in $V_{j,\varepsilon_j}(y)$. Hence, $W_j(\lambda) \times W_j(y)$ is an open neighborhood of $(\lambda, y)$ with

$$\beta\left(W_j(\lambda) \times W_j(y)\right) \subseteq V_{j,r}(z).$$

Setting

$$W(\lambda, y) := \bigcap_{j \in J_0} W_j(\lambda) \times W_j(y)$$

then yields

$$\beta(W(\lambda, y)) \subseteq \bigcap_{j \in J_0} V_{j,r}(z) = U_{J_0,r}(z) \subseteq W(z)$$

and completes the proof. □

Hence, given a vector space $Y$ and a family of quasi-norms $\{q_j \,|\, j \in J\}$ on $Y$, we have seen that the $q_j$ induce a topology $\mathcal{O}$ on $Y$ which is Hausdorff, and that $(Y, \mathcal{O})$ is a topological vector space. But, in contrast to the case of seminorms, the resulting topology is not necessarily locally convex.
Now we have everything at hand to formulate and prove a variant of Lemma 2.15, where we consider functions mapping a LCTVS $X$ into a topological vector space $Y$ equipped with a family of quasi-norms.

**Proposition 2.23.** *Let $(X, \{s_i \,|\, i \in I\})$ be a locally convex topological vector space and $Y$ a topological vector space, where the topology of $Y$ is defined by a family of quasi-norms $\{q_j \,|\, j \in J\}$ as stated in Proposition 2.18. Furthermore, let $T : X \to Y$ be a linear map. Then, the following properties are equivalent:*

*(i) $T$ is continuous.*

*(ii) $T$ is continuous in 0.*

*(iii) For each $j \in J$ there exists a finite subset $I_0 \subset I$ and a constant $C$, such that*

$$q_j(Tx) \leq C \max_{i \in I_0} s_i(x) \qquad \text{for all} \qquad x \in X.$$

*Proof.* (i) $\Leftrightarrow$ (ii): The first implication is trivial. To show (ii) $\Rightarrow$ (i), we fix $x \in X$ and note that

$$T(\tilde{x}) = T(x) + T(\tilde{x} - x), \quad \tilde{x} \in X. \tag{2.3.6}$$

For $\xi \in X$, by $\Theta_\xi$ we denote the translation in $X$ by $\xi$, i.e., $\Theta_\xi(\tilde{x}) = \tilde{x}+\xi$. Analogously, by $\Upsilon_\eta$ we denote the translation by $\eta \in Y$. Then, (2.3.6) can be written as

$$T(\tilde{x}) = (\Upsilon_{T(x)} \circ T \circ \Theta_{-x})(\tilde{x}).$$

Next, note that from Lemma 2.13 and Proposition 2.22 we know that $X$ and $Y$ are topological vector spaces. By definition, translations are continuous in topological vector spaces, and we conclude that $\Theta_\xi$ and $\Upsilon_\eta$ are continuous. Now, $T$ is continuous in $0 = \Theta_{-x}(x)$, and therefore $T = \Upsilon_{T(x)} \circ T \circ \Theta_{-x}$ is continuous in $x$ as a concatenation of continuous functions.

(ii) $\Rightarrow$ (iii): Let $j \in J$. Due to Lemma 2.20, the local basis element $V_{j,1}(0_Y)$ contains an open neighborhood $V$ of $0_Y$. Since $T$ is continuous in $0_X$, there exists an open neighborhood $U$ of $0_X$, such that $T(U) \subseteq V$. Because $U$ is open, there exists an element $U_{I_0,r}(0_X)$ of the local basis of $0_X$ with

$$U_{I_0,r}(0_X) \subseteq U,$$

and we have

$$T\left(U_{I_0,r}(0_X)\right) \subseteq T(U) \subseteq V \subseteq V_{j,1}(0_Y).$$

Hence,

$$x \in U_{I_0,r}(0_X) \implies T(x) \in V_{j,1}(0_Y),$$

which is equivalent to

$$\max_{i \in I_0} s_i(x) < r \implies q_j(Tx) < 1. \tag{2.3.7}$$

Now, let $x \in X$ and set

$$C_x := \max_{i \in I_0} s_i(x).$$

If $C_x > 0$, we have

$$\max_{i \in I_0} s_i \left(\frac{r}{2C_x} x\right) = \frac{r}{2C_x} \max_{i \in I_0} s_i(x) = \frac{r}{2},$$

and (2.3.7) yields

$$q_j\left(T\left(\frac{r}{2C_x}x\right)\right) < 1,$$

thus

$$q_j(Tx) < \frac{2C_x}{r} = \frac{2}{r}\max_{i\in I_0} s_i(x),$$

which proves the assertion with $C := 2/r$. If $C_x = 0$, i.e., $s_i(x) = 0$ for all $i \in I_0$, it holds that $s_i(\lambda x) = 0$ for all $i \in I_0$ and all $\lambda \in \mathbb{K}$. Hence, from (2.3.7) we conclude

$$q_j(T(\lambda x)) < 1 \qquad \text{for all} \qquad \lambda \in \mathbb{K},$$

and therefore

$$q_j(Tx) < |\lambda|^{-1} \qquad \text{for all} \qquad \lambda \in \mathbb{K} \setminus \{0\},$$

which yields $q_j(Tx) = 0$.

(iii) $\Rightarrow$ (ii): From (iii) we know that for all finite $J_0 \subset J$ there exits a finite $I_0 \subset I$ and a constant $C > 0$, such that

$$\max_{j\in J_0} q_j(Tx) \le C \max_{i\in I_0} s_i(x) \qquad \text{for all} \qquad x \in X. \tag{2.3.8}$$

Now, let $V$ be an open neighborhood of $0_Y$. Since $V$ is open, there exists an element $U_{J_0,r}(0_Y)$ of the local basis of $0_Y$ with $U_{J_0,r}(0_Y) \subseteq V$. Next, from (iii) we know that for this index set $J_0$ there exists a finite $I_0 \subset I$ and a constant $C$, such that (2.3.8) holds true. Then, for $x \in U_{I_0,r/C}(0_X)$, i.e., for each $x \in X$ with

$$\max_{i\in I_0} s_i(x) < \frac{r}{C},$$

we know that

$$\max_{j\in J_0} q_j(Tx) < r,$$

i.e., $Tx \in U_{J_0,r}(0_Y)$. Hence, we have

$$T\left(U_{I_0,r/C}(0_X)\right) \subseteq U_{J_0,r}(0_Y) \subseteq V.$$

Finally, from Remark 2.12(i) we know that $U_{I_0,r/C}(0_X)$ is open, and we have shown that $T$ is continuous in $0_X$. □

## 2.4 Metrizability, normability and completeness

In this section we recall the notion of completeness for TVSs, and address the question whether a topological (vector) space is compatible with some metric or norm on that space (in the sense of Definition 2.24 below).

First recall that a *metric* on a set $X$ is a mapping $d : X \times X \to [0, \infty)$, such that for any $x, y, z \in X$ it holds

(i) $d(x, y) = 0 \Leftrightarrow x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$.

**Definition 2.24** (Metrizability, normability)**.**

**(i)** *A topological space $(X, \mathcal{O})$ is called* metrizable*, if there exists a metric $d$ on $X$ which generates the topology $\mathcal{O}$, i.e., the open balls $B_r(x) = \{y \in X \mid d(y, x) < r\}$, where $x \in X$ and $r > 0$, form a basis for $\mathcal{O}$.*

**(ii)** *A TVS $(X, \mathcal{O})$ is called* normable*, if there exists a norm $\|\cdot\|$ on $X$ which generates the topology $\mathcal{O}$, i.e., the open balls $B_r(x) = \{y \in X \mid \|y - x\| < r\}$, where $x \in X$ and $r > 0$, form a basis for $\mathcal{O}$.*

Now, for TVSs, the following characterization of metrizability holds true [117, Chapter I, §6].

**Proposition 2.25.** *Let $(X, \mathcal{O})$ be a TVS. Then, the following properties are equivalent:*

*(i) $(X, \mathcal{O})$ is metrizable.*

*(ii) $(X, \mathcal{O})$ is a Hausdorff space and has a countable neighborhood basis of the origin.*

*Furthermore, if $(X, \mathcal{O})$ is metrizable, the metric $d$ which generates $\mathcal{O}$ can be chosen to be* translationally invariant*, i.e., it holds $d(x, y) = d(x + z, y + z)$ for all $x, y, z \in X$.*

In order to characterize the normable TVSs, we first need to introduce the notion of *bounded sets*.

**Definition 2.26** (Bounded set)**.** *Let $(X, \mathcal{O})$ be a TVS. A subset $U \subset X$ is called* bounded*, if for every neighborhood $V(0)$ of the origin there exists $t_0 \in \mathbb{R}$ such that*

$$U \subset tV(0)$$

*for all $t > t_0$.*

For a proof of the following proposition see [117, Chapter II, §2].

**Proposition 2.27.** *Let $(X, \mathcal{O})$ be a TVS. Then, the following properties are equivalent:*

*(i) $(X, \mathcal{O})$ is normable.*

*(ii) $(X, \mathcal{O})$ is a Hausdorff space and has a bounded convex neighborhood of the origin.*

In analogy to metric spaces, we define (sequential) completeness of general TVSs via the convergence of Cauchy sequences. Recall that in a metric space $(X, d)$ a sequence $(x_n)_{n\in\mathbb{N}}$ is called Cauchy sequence, if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$, such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq N$.

**Definition 2.28** (Cauchy sequence)**.** *Let* $(X, \mathcal{O})$ *be a TVS. Then, a sequence* $(x_n)_{n\in\mathbb{N}} \subset X$ *is called* Cauchy sequence*, if for all neighborhoods* $V(0)$ *of the origin there exists* $N \in \mathbb{N}$*, such that*

$$x_m - x_n \in V(0) \qquad \textit{for all} \qquad m, n \geq N.$$

**Remark 2.29.** If a TVS $(X, \mathcal{O})$ is metrizable, where we denote the translationally invariant metric by $d$, then a sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the sense of Definition 2.28, if and only if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the metric $d$.

Now, a TVS $(X, \mathcal{O})$ is called *sequentially complete*, if every Cauchy sequence $(x_n)_{n\in\mathbb{N}} \subset X$ converges to a point $x \in X$. A metrizable and sequentially complete TVS is just called *complete*.

**Definition 2.30** (F-space, Fréchet space)**.** *A metrizable and complete TVS is called* F-space*. A locally convex F-space is called* Fréchet space*.*

Note that every Banach space is a Fréchet space, and every Fréchet space is an F-space.

# Part II

# The $p$-Poisson Equation

# Chapter 3

# Scope of Problems and Basic Properties

In this chapter, after a short introduction to the $p$-Laplace operator, we specify the class of $p$-Poisson problems considered in this thesis. Afterwards, a summary of some basic properties of (solutions to) the $p$-Poisson equation concludes this chapter.

## 3.1 The $p$-Laplacian

*Quasilinear* equations of the type

$$-\mathrm{div}(\alpha(\cdot, |\nabla u|)\nabla u) = F(u) \tag{3.1.1}$$

appear in several problems in continuum mechanics, in particular in the mathematical description of non-Newtonian fluids [95]. Other relevant instances of (3.1.1) come from classical problems, e.g., the mean curvature equation [63, 69] or modern applications in image processing, with the minimal total variation equation [113]. In this thesis we will focus on a prominent subclass of equations of the type (3.1.1), which we introduce in the following.

For $1 < p < \infty$, the *$p$-Poisson equations* are given by

$$-\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = f \qquad \text{in} \quad \Omega. \tag{3.1.2}$$

Here, the differential operator

$$-\Delta_p(u) := -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big)$$

is called the *$p$-Laplacian*. There are various reasons for the importance of this class of equations. On the one hand, problems of this type arise in many applications, as outlined in the introduction of this thesis. On the other hand the $p$-Laplacian has a similar prototype character for more general quasilinear problems as the ordinary Laplace equation for linear problems.

At the critical points, i.e., where $\nabla u = 0$, the equation is *degenerate* for $p > 2$ and *singular* for $p < 2$. Moreover, note that for $p = 2$ the equation (3.1.2) corresponds to the classical Poisson equation and for $p = 1$ it formally yields the equation of total variation minimization [113]. In analogy to the linear case, equation (3.1.2) with $f = 0$ is called $p$-Laplace equation and the corresponding solutions are called $p$-harmonic functions. Clearly, the $p$-Poisson equations (3.1.2) are quasilinear PDEs of order two. For suitable right-hand side $f$, the variational formulation corresponding to (3.1.2) is given by

$$\int_\Omega \left\langle |\nabla u|^{p-2}\,\nabla u, \nabla v \right\rangle \mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \qquad \text{for all} \quad v \in W_0^1(L_p(\Omega)). \tag{3.1.3}$$

In this weak setting, the $p$-Laplacian can be considered as a mapping from $W^1(L_p(\Omega))$ into its (topological) dual $W^1(L_p(\Omega))'$. Indeed, for any $u, v \in W^1(L_p(\Omega))$, with Hölder's inequality and Lemma A.13 we estimate

$$\left| \int_\Omega \left\langle |\nabla u|^{p-2}\,\nabla u, \nabla v \right\rangle \mathrm{d}x \right| \lesssim \left\| u \,\middle|\, W^1(L_p(\Omega)) \right\|^{p-1} \left\| v \,\middle|\, W^1(L_p(\Omega)) \right\| \tag{3.1.4}$$

with a constant depending on $d$ and $p$.

## 3.2 Scope of $p$-Poisson problems

Let us briefly describe the scope of problems treated in this work. Throughout this thesis, we consider the $p$-Poisson equation (3.1.2) with Dirichlet boundary conditions, i.e.,

$$\begin{aligned} -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned} \tag{3.2.1}$$

In the most general case we assume that $\Omega \subset \mathbb{R}^d$, $d \geq 2$, denotes some bounded domain, $1 < p < \infty$, as well as $f \in W^{-1}(L_{p'}(\Omega))$ and $g \in W^1(L_p(\Omega))$. Then, we consider the problem of finding a *weak solution* to (3.2.1), i.e., we are searching for a solution $u \in W^1(L_p(\Omega))$ to the variational problem

$$\int_\Omega \left\langle |\nabla u|^{p-2}\,\nabla u, \nabla v \right\rangle \mathrm{d}x = f(v) \qquad \text{for all} \quad v \in W_0^1(L_p(\Omega)), \tag{3.2.2}$$

which satisfies

$$u - g \in W_0^1(L_p(\Omega)).$$

Occasionally, the main focus will be put on the $p$-Poisson equation with *homogeneous* Dirichlet boundary conditions,

$$\begin{aligned} -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{3.2.3}$$

in which case we are searching for a weak solution $u \in W_0^1(L_p(\Omega))$ which satisfies (3.2.2).

**Remark 3.1.**

**(i)** For most of the results in this work, some (minimum) regularity of the boundary of the domain is required. We will restrict the class of admissible domains to Lipschitz or polyhedral Lipschitz domains then (e.g., for the numerical discretization in Part IV).

**(ii)** Furthermore, for some results a certain (minimum) integrability of the right-hand side is needed. We will assume that $f \in L_q(\Omega)$ for appropriate values of $q$ then. However, in all considered cases Lemma A.8 ensures that $f$ is also contained in $W^{-1}(L_{p'}(\Omega))$. Note that the variational formulation (3.2.2) then takes the form (3.1.3).

## 3.3 Basic properties

Existence and uniqueness of weak solutions to all problems we are going to consider is guaranteed by the following fairly general result which is well-known in the literature. Its proof can be found, e.g., in Lions [100, Chapter 2].

**Proposition 3.2** (Existence and uniqueness)**.** *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ denote a bounded domain and let $1 < p < \infty$. Moreover, assume $f \in W^{-1}(L_{p'}(\Omega))$, as well as $g \in W^1(L_p(\Omega))$. Then the p-Poisson problem* (3.2.1)*, i.e.,*

$$\begin{aligned} -\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = f \qquad &\textit{in } \Omega, \\ u - g \ \in \ &W_0^1(L_p(\Omega)), \end{aligned}$$

*admits a unique weak solution $u \in W^1(L_p(\Omega))$.*

In regard of classical regularity, the local Hölder regularity of solutions to the $p$-Poisson equation (3.1.2), as well as to more general quasi-linear elliptic problems, was studied in several papers. We refer, e.g., to Ural'ceva [133], Uhlenbeck [132], Evans [61], Lewis [97], DiBenedetto [52], Tolksdorf [127], Diening, Kaplický and Schwarzacher [54], Kuusi and Mingione [93], as well as to Teixeira [125]. The subsequent proposition can be derived as a special case from [54, Corollary 5.5] (see also [54, Remark 5.7]).

**Proposition 3.3** ($C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ regularity)**.** *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ denote any bounded domain, let $1 < p < \infty$, and $q > d$. Then there exists $\alpha \in (0,1)$ such that all $u \in W^1(L_p(\Omega))$ which are weak solutions to* (3.1.2) *with $f \in L_q(\Omega)$ belong to $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.*

**Remark 3.4.** It is well-known that, for $p > 2$, solutions to (3.1.2) do not possess continuous second derivatives in general, even if $f$ is smooth. For instance, a weak solution to the equation

$$\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = 1 \quad \text{on} \quad B_1(0)$$

is given by

$$u(x_1, \ldots, x_d) = \frac{p-1}{p}\, |x_1|^{p/(p-1)}\,,$$

see [120, Proposition 5.4] and [98]. Hence, in this respect $\ell = 1$ in Proposition 3.3 is sharp at least for $p > 2$.

Here and in what follows we shall say a given problem is of *sharp regularity* $\alpha$ if $\alpha$ is a lower bound for the smoothness (measured in a certain scale) of *all* solutions to *any* problem instances (e.g., for all Lipschitz domains $\Omega$ and each $f \in L_{p'}(\Omega)$), but for every $\varepsilon > 0$ there exists a problem instance such that its corresponding solution has a regularity strictly less than $\alpha + \varepsilon$.

We conclude this section by noting that the weak comparison principle is well-known to hold true for the $p$-Laplace operator. For a proof see, e.g., [126, Lemma 3.1] or [56, Section 1].

**Proposition 3.5** (Weak comparison principle). *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be some bounded Lipschitz domain, $1 < p < \infty$ and assume that for $u_1, u_2 \in W^1(L_p(\Omega))$ it holds $-\Delta_p(u_1) \leq -\Delta_p(u_2)$ in the weak sense, i.e.,*

$$\int_\Omega \left\langle |\nabla u_1|^{p-2} \nabla u_1, \nabla v \right\rangle \mathrm{d}x \leq \int_\Omega \left\langle |\nabla u_2|^{p-2} \nabla u_2, \nabla v \right\rangle \mathrm{d}x$$

*for all nonnegative $v \in W_0^1(L_p(\Omega))$. Then, the inequality*

$$u_1(x) \leq u_2(x) \qquad \textit{for a.e.} \qquad x \in \partial\Omega$$

*implies that*

$$u_1(x) \leq u_2(x) \qquad \textit{for a.e.} \qquad x \in \Omega.$$

Further results in regard of the $p$-Poisson equation - in particular Sobolev regularity assertions or the singular expansion of the solution near conical boundary points - will be presented further below at the places where we need them.

# Part III

# Besov Regularity

# Chapter 4

# Approximation rates and smoothness of the solution

The *efficiency* of a numerical method for the approximate solution of a PDE such as (3.1.2) is doubtlessly of vital importance. Here, by efficiency we mean the following: assume that a numerical scheme generates for every error tolerance $\varepsilon > 0$ an approximation $u_{N(\varepsilon)}$ to the exact solution $u$ of the PDE with $\left\| u - u_{N(\varepsilon)} \right\| < \varepsilon$ for some norm $\|\cdot\|$ we want to measure the error with, and that $N(\varepsilon) \in \mathbb{N}$ represents the number of parameters wich are needed to describe $u_{N(\varepsilon)}$. Typically, $N(\varepsilon)$ goes to infinity as $\varepsilon$ tends to zero. In many cases it can be shown that the computational cost - i.e., the number of arithmetic operations - that is required to compute $u_{N(\varepsilon)}$ stays proportional to $N(\varepsilon)$ as $\varepsilon$ decreases. Now, we are interested in the efficiency of this trade-off, i.e., the relation between the number of degrees of freedom $N(\varepsilon)$ and the error of approximation. If $N(\varepsilon) \lesssim \varepsilon^{-1/s}$ or equivalently $\varepsilon \lesssim N(\varepsilon)^{-s}$ for some $s > 0$, we say that $u$ is approximated at rate $s$.

In this chapter we are concerned with the *maximal* approximation rate that can be realized by a certain class of numerical schemes. Since in this thesis we are in particular interested in methods based on wavelet discretizations, we will consider the two prominent types of approximation referred to as *uniform* and *n-term wavelet approximation*. As we will see, in this setting there exist natural benchmark schemes, namely best uniform and best $n$-term wavelet approximation, constituting the optimal approximation scheme of the respective type. The convergence rate of these benchmark schemes in turn depends on the smoothness of the function one wants to approximate in certain scales of Besov spaces.

This chapter is organized as follows. After introducing the general concepts of uniform (linear) and $n$-term (nonlinear) wavelet approximation in Section 4.1, we describe the connection between the best possible approximation rate of such schemes and the smoothness of the target function one wants to approximate in Section 4.2.

## 4.1 Uniform and $n$-term wavelet approximation

### Linear and nonlinear approximation

The central subject in approximation theory is the problem to approximate a (possibly complicated) function $f$ of a normed linear space $(X, \|\cdot\,|\,X\|)$ by (in general simpler) functions $f_n \in Y_n$, where $(Y_n)_{n\in\mathbb{N}} =: \mathcal{Y}$ denotes a sequence of subspaces of $X$. Hereby one generally distinguishes two types of approximation: In case the subsets $Y_n \subset X$ constitute *linear* subspaces of $X$ we speak of *linear approximation*, whereas in *nonlinear approximation* the subsets $Y_n$ can be chosen to be *nonlinear manifolds*. In either case, for $f \in X$ and $n \in \mathbb{N}$ we denote the *best approximation error* in $Y_n$ by

$$E_{n,X,\mathcal{Y}}(f) := \operatorname{dist}_X(f, Y_n) := \inf_{f_n \in Y_n} \|f - f_n\,|\,X\|\,.$$

Note that if the subsets $Y_n$ are nested and asymptotically dense in $X$, it follows that $E_{n,X,\mathcal{Y}}(f)$ monotonically tends to zero as $n \to \infty$. Moreover, if for some $s > 0$ it holds that

$$E_{n,X,\mathcal{Y}}(f) \lesssim n^{-s} \qquad \text{for all} \quad n \in \mathbb{N},$$

we say that $f$ can be approximated at *rate* $s$.

### Approximation by wavelets

For the rest of this chapter we will be concerned with the following setting of wavelet approximation. Let be given:

- $\Omega = \mathbb{R}^d$ or $\Omega \subset \mathbb{R}^d$ a bounded Lipschitz domain of the type as described in Remark 1.12,
- a *target function* $u \in L_p(\Omega)$, where $1 < p < \infty$,
- a unconditional wavelet basis $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$ of $L_p(\Omega)$, constructed by means of a pair of biorthogonal MRA $(V_j)_{j\geq 0}$ and $(\tilde{V}_j)_{j\geq 0}$ with compactly supported tensor product type scaling functions $\phi$ and $\tilde{\phi}$, see Subsection 1.5.1. For simplicity, throughout this chapter we assume that $\phi$ and $\tilde{\phi}$ possess some minimal smoothness in the sense that $\phi, \tilde{\phi} \in C^{0,\varepsilon}(\Omega)$ for some $\varepsilon > 0$.

In regard of the wavelet basis $\Psi$ we use the notation as introduced at the end of Subsection 1.5.1. Then, clearly $u$ admits a unique wavelet expansion

$$u = \sum_{\lambda\in\Lambda} c_\lambda \psi_\lambda,$$

with $c_\lambda = \langle u, \tilde{\psi}_\lambda\rangle$. Now, a wavelet approximation method is defined by the choice of the sequence of subsets $Y_n \subset L_p(\Omega)$, $n \in \mathbb{N}$, and a rule how to select approximants $f_n$ to $f$ from $Y_n$. In the sequel we will describe two fundamental classes of wavelet approximation schemes, corresponding to linear and nonlinear approximation, respectively.

### Uniform approximation

As an instance of linear approximation we will consider *uniform* wavelet approximation in the following. Therefore one sets

$$Y_n := V_n = \operatorname{clos}_{L_p(\Omega)} \operatorname{span} \{\psi_\lambda \mid \lambda \in \Lambda, |\lambda| < n\}, \qquad n \in \mathbb{N}_0,$$

i.e., the approximants come from the linear subspaces $V_n$ of $L_p(\Omega)$ which are spanned by all wavelets up to some given fixed refinement level $n$. Note that in case $\Omega$ is a bounded domain, the spaces $V_n$ are finite dimensional with

$$N_n := \dim V_n = \#\Lambda^n \sim 2^{dn}. \tag{4.1.1}$$

We set $\mathcal{V} := (V_n)_{n\in\mathbb{N}_0}$. Now, if we measure the approximation error in some subspace $X \subset L_p(\Omega)$ with (quasi-)norm $\|\cdot \mid X\|$ and $u \in X$ as well as $V_n \subset X$, $n \in \mathbb{N}_0$, the best approximation error writes as

$$E_{n,X,\mathcal{V}}(u) = \inf_{u_n \in V_n} \|u - u_n \mid X\| .$$

Clearly, the best one can hope to expect for any uniform wavelet scheme is that its error propagates like $E_{n,X,\mathcal{V}}(u)$.

### $n$-term approximation

A prominent instance of nonlinear approximation is *n-term* wavelet approximation. There, the approximants are chosen from the sets

$$S_n := \left\{ \sum_{\lambda\in\overline{\Lambda}} c_\lambda \psi_\lambda \;\middle|\; \overline{\Lambda} \subset \Lambda,\ \#\overline{\Lambda} \leq n, c_\lambda \in \mathbb{R} \right\},$$

i.e., the nonlinear manifold of all functions from $L_p(\Omega)$ that can be represented by a linear combination of at most $n$ *arbitrary* wavelet basis elements. Clearly these subspaces are not linear, since in general the sum of two functions from $S_n$ is only contained in $S_{2n}$ (it holds $S_n + S_n = S_{2n}$). We set $\mathcal{S} := (S_n)_{n\in\mathbb{N}_0}$. Further note that on bounded domains $n$-term approximation covers uniform approximation since $V_n \subset S_{N_n}$.

Moreover, let us stress the fact that every *adaptive* wavelet method is a form of $n$-term approximation. Hence, the best we can expect for *any* adaptive wavelet method (corresponding to the basis $\Psi$) is to realize the convergence rate of *best n-term approximation*, i.e., that its approximation error decays with the same rate as the error of best approximation,

$$E_{n,X,\mathcal{S}}(u) = \inf_{u_n \in S_n} \|u - u_n \mid X\| .$$

In this sense, the approximation rate of best $n$-term approximation serves as a *benchmark* for adaptive methods.

## 4.2 Approximation rates

For the remainder of this chapter we are concerned with the decay rate of the best approximation error for uniform and $n$-term wavelet approximation, respectively. It is well-known that these rates are governed by the smoothness of the target function in certain scales of Besov spaces. First of all, to subsume all functions permitting a certain rate of approximation, let us introduce the following definition.

**Definition 4.1** (Approximation space). *Let $X$ be a Banach space and $\mathcal{Y} = (Y_n)_{n\in\mathbb{N}_0}$ a nested sequence of (not necessarily linear) subspaces of $X$ such that $\cup_{n\in\mathbb{N}_0} Y_n$ is dense in $X$. Moreover, we assume that $aY_n = Y_n$ for all $a \in \mathbb{R}$, $a \neq 0$, as well as $0 \in Y_0$ and $Y_n + Y_n \subset Y_{n+b}$ for some $b \in \mathbb{N}_0$ independent of $n \in \mathbb{N}_0$. Then, for $s > 0$ and $0 < q \leq \infty$ we define the approximation space $\mathcal{A}^s_q(X, \mathcal{Y})$ by*

$$\mathcal{A}^s_q(X, \mathcal{Y}) := \left\{ f \in X \,\middle|\, |f|_{\mathcal{A}^s_q(X,\mathcal{Y})} < \infty \right\}, \tag{4.2.1}$$

*where*

$$|f|_{\mathcal{A}^s_q(X,\mathcal{Y})} := \left\| (2^{sn} \operatorname{dist}_X(f, Y_n))_{n\in\mathbb{N}_0} \,\middle|\, \ell_q(\mathbb{N}_0) \right\|. \tag{4.2.2}$$

*We further set*

$$\left\| f \,\middle|\, \mathcal{A}^s_q(X, \mathcal{Y}) \right\| := \| f \mid X \| + |f|_{\mathcal{A}^s_q(X,\mathcal{Y})}. \tag{4.2.3}$$

**Remark 4.2.**

**(i)** The approximation space (4.2.1) is a linear subspace of $X$ with quasi-norm (4.2.3). This can readily be seen as follows. Since $0 \in Y_n$ for all $n \in \mathbb{N}_0$, it clearly holds that $\left\| f \,\middle|\, \mathcal{A}^s_q(X, \mathcal{Y}) \right\| = 0$ if and only if $f = 0$. The homogeneity follows from the assumption that $aY_n = Y_n$ for all $a \in \mathbb{R}$, $a \neq 0$. Finally, since $Y_n + Y_n \subset Y_{n+b}$ for some $b \in \mathbb{N}_0$ independent of $n$, it holds that $E_{n+b,X,\mathcal{Y}}(f+g) \leq E_{n,X,\mathcal{Y}}(f) + E_{n,X,\mathcal{Y}}(g)$, from which we derive the validity of the quasi-triangle inequality.

**(ii)** In case the subsets $Y_n$, $n \in \mathbb{N}_0$, in Definition 4.1 are *linear subspaces* of $X$ and if $q \geq 1$, then (4.2.3) defines a *norm* on $\mathcal{A}^s_q(X, \mathcal{Y})$. This follows from part (i) with $b = 0$.

**(iii)** From (4.2.2) we immediately deduce the continuous embeddings

$$\mathcal{A}^{s_1}_{q_1}(X, \mathcal{Y}) \hookrightarrow \mathcal{A}^{s_2}_{q_2}(X, \mathcal{Y}) \qquad \text{if } s_1 > s_2, \quad \text{or if } s_1 = s_2 \text{ and } q_1 \leq q_2.$$

Hence, each space $\mathcal{A}^s_q(X, \mathcal{Y})$ is contained in $\mathcal{A}^s_\infty(X, \mathcal{Y})$, which consists exactly of those functions $f \in X$ that satisfy $E_{n,X,\mathcal{Y}}(f) \lesssim 2^{-sn}$.

Now, it is well-known that the approximation spaces corresponding to uniform wavelet approximation in $L_p(\Omega)$ can be exactly characterized by means of classical smoothness spaces. For a proof of the following result see [17, Theorem 3.6.1 & Section 3.9].

**Proposition 4.3.** *Let $1 \leq p, q \leq \infty$. Moreover, let $m$ be the order of polynomial reproduction of the spaces $(V_j)_{j\in\mathbb{N}_0}$ and $\phi \in B^{\bar{s}}_{q_0}(L_p(\Omega))$ for some $\bar{s} > 0$ and $q_0 > 0$. Then it holds*

$$\left\| f \,\middle|\, \mathcal{A}^s_q(L_p(\Omega), \mathcal{V}) \right\| \sim \left\| f \,\middle|\, B^s_q(L_p(\Omega)) \right\| \tag{4.2.4}$$

*for all* $0 < s < \min\{m, \bar{s}\}$.

If we measure the approximation error in a Besov norm, an analog result holds true (cf. [17, Corollary 3.6.1 & Section 3.9]).

**Proposition 4.4.** *Under the same assumptions as in Proposition 4.3 it holds*

$$\left\| f \,\middle|\, \mathcal{A}^{s-t}_q(B^t_p(L_p(\Omega)), \mathcal{V}) \right\| \sim \left\| f \,\middle|\, B^s_q(L_p(\Omega)) \right\|$$

*for all* $0 < t < s < \min\{m, \bar{s}\}$.

Next, let us consider approximation rates of best $n$-term wavelet approximation. Therefore, let $\Sigma$ denote the subsequence of $\mathcal{S}$ defined by

$$\Sigma_j := S_{2^{dj}}, \quad j \in \mathbb{N}_0,$$

and note that $\Sigma_j + \Sigma_j \subset \Sigma_{j+1}$.

**Remark 4.5.** From the monotonicity of the sequence $E_{n,X,\mathcal{S}}(f)$, it follows that for $s > 0$ and $0 < q < \infty$ it holds

$$|f|^q_{\mathcal{A}^s_q(X,\Sigma)} = \sum_{j\geq 0} \left[ 2^{js} E_{2^{dj},X,\mathcal{S}}(f) \right]^q \sim \sum_{n\geq 1} n^{-1} \left[ n^{s/d} E_{n,X,\mathcal{S}}(f) \right]^q, \tag{4.2.5}$$

where the constants of equivalence depend on $d, s$, and $q$. In case $q = \infty$, we have

$$|f|_{\mathcal{A}^s_\infty(X,\Sigma)} = \sup_{j\geq 0} \left( 2^{js} E_{2^{dj},X,\mathcal{S}}(f) \right) \sim \sup_{n\geq 1} \left( n^{s/d} E_{n,X,\mathcal{S}}(f) \right) \tag{4.2.6}$$

with constants depending only on $s$.

If the approximation error is measured in the $L_p(\Omega)$-norm, the following result holds true [17, Theorem 4.3.3, Theorem 3.7.7 & Section 3.9].

**Proposition 4.6.** *Let $1 < p < \infty$. Moreover, let $m \in \mathbb{N}$ be the order of polynomial reproduction of the spaces $(V_j)_{j\in\mathbb{N}_0}$ and $\phi \in B^{s'}_q(L_q(\Omega))$ for some $s' > \bar{s} > 0$ and $1/q = \bar{s}/d + 1/p$. Then, the (quasi-)norm equivalence*

$$\| f \,|\, \mathcal{A}^\sigma_\tau(L_p(\Omega), \Sigma) \| \sim \| f \,|\, B^\sigma_\tau(L_\tau(\Omega)) \|$$

*holds true for all* $0 < \sigma < \min\{m, \bar{s}\}$ *and* $1/\tau = \sigma/d + 1/p$.

For a proof of the following result, see [17, Theorem 4.2.2, Theorem 3.7.7 & Section 3.9].

**Proposition 4.7.** *Let $1 < p < \infty$ and $t > 0$. Moreover, let $m > t$ be the order of polynomial reproduction of the spaces $(V_j)_{j\in\mathbb{N}_0}$ and $\phi \in B_q^{s'}(L_q(\Omega))$ for some $s' > \bar{s} > t$ and $1/q = (\bar{s} - t)/d + 1/p$. Then, the (quasi-)norm equivalence*

$$\left\| f \,\middle|\, \mathcal{A}_\tau^{\sigma-t}(B_p^t(L_p(\Omega)), \Sigma)\right\| \sim \|f \mid B_\tau^\sigma(L_\tau(\Omega))\|$$

*holds true for all $t < \sigma < \min\{m, \bar{s}\}$ and $1/\tau = (\sigma - t)/d + 1/p$.*

**Remark 4.8.** An analogous result holds true if the approximation error is measured in a Sobolev norm with integral smoothness parameter. Under the assumptions of Proposition 4.7 with $t = \ell \in \mathbb{N}$, the (quasi-)norm equivalence

$$\left\| f \,\middle|\, \mathcal{A}_\tau^{\sigma-\ell}(W^\ell(L_p(\Omega)), \Sigma)\right\| \sim \|f \mid B_\tau^\sigma(L_\tau(\Omega))\|$$

holds true for all $\ell < \sigma < \min\{m, \bar{s}\}$ and $1/\tau = (\sigma - \ell)/d + 1/p$. For a proof of this result see [17, Remark 4.3.3].

Note that Proposition 4.3 and Proposition 4.6 (Proposition 4.4 and Proposition 4.7) provide an exact characterization of the approximation spaces corresponding to uniform and $n$-term wavelet approximation when the error is measured in the $L_p(\Omega)$-norm ($B_p^t(L_p(\Omega))$-norm). The relevant scales of function spaces are visualized by a $(1/\tau, \sigma)$-DeVore-Triebel diagram in Figure 4.1 (Figure 4.2).

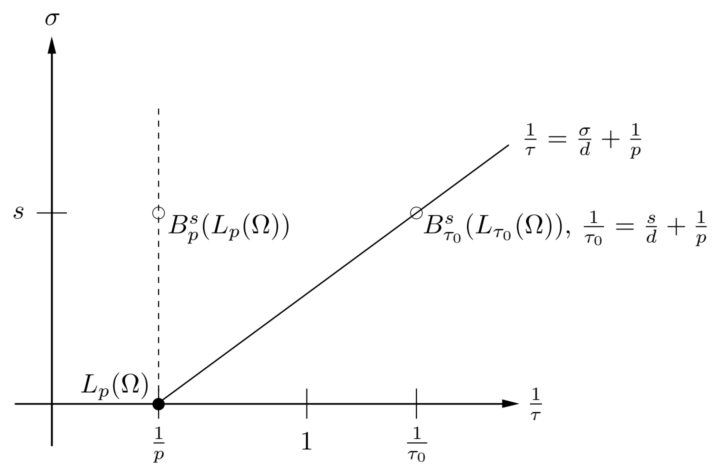



Figure 4.1: Scales of function spaces governing the convergence rates of uniform (linear) and $n$-term (nonlinear) wavelet approximation in $L_p(\Omega)$.

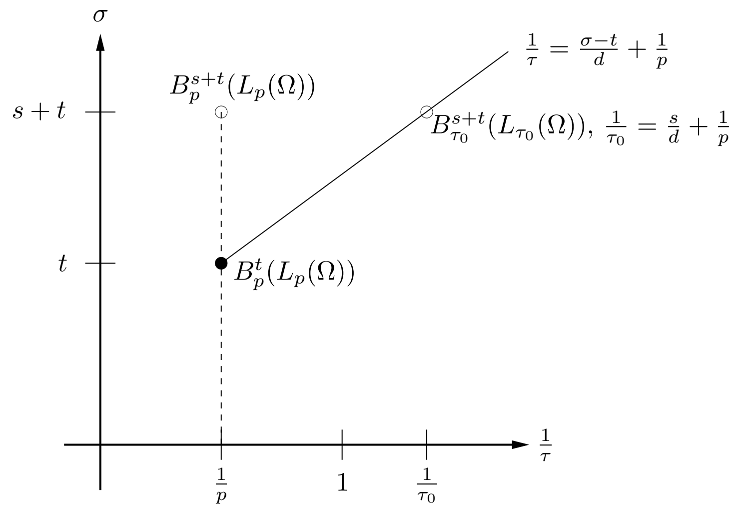



Figure 4.2: Scales of function spaces governing the convergence rates of uniform (linear) and $n$-term (nonlinear) wavelet approximation in $B_p^t(L_p(\Omega))$.

Now, let us compare the above results for uniform and $n$-term wavelet approximation, respectively, when measuring the approximation error in the $L_p(\Omega)$-norm on a bounded domain. Therefore, from Proposition 4.3 and Proposition 4.6 we deduce the following corollaries.

**Corollary 4.9.** *Let $1 < p < \infty$ and $\Omega$ be bounded, and let $(V_j)_{j\in\mathbb{N}_0}$ and $\phi$ satisfy the assumptions of Proposition 4.3. Then, for $0 < s < \min\{m, \bar{s}\}$, it holds*

$$\sum_{n=0}^{\infty} \Big[N_n^{s/d} E_{n,L_p(\Omega),\mathcal{V}}(u)\Big]^p \;<\; \infty \qquad \Longleftrightarrow \qquad u \in B_p^s(L_p(\Omega)). \tag{4.2.7}$$

*Consequently, for arbitrarily small $0 < \varepsilon < s$, the implications*

$$f \in W^s(L_p(\Omega)) \qquad \Longrightarrow \qquad E_{n,L_p(\Omega),\mathcal{V}}(f) \lesssim N_n^{-(s-\varepsilon)/d} \quad \textit{for all} \quad n \in \mathbb{N}_0, \tag{4.2.8}$$

*and*

$$E_{n,L_p(\Omega),\mathcal{V}}(f) \lesssim N_n^{-s/d} \quad \textit{for all} \quad n \in \mathbb{N}_0 \qquad \Longrightarrow \qquad f \in W^{s-\varepsilon}(L_p(\Omega)), \tag{4.2.9}$$

*hold true.*

*Proof.* The equivalence (4.2.7) readily follows from Proposition 4.3 by setting $q = p$ and using that $N_n \sim 2^{dn}$, see (4.1.1). Next, from (4.2.7) and (1.4.5) we derive (4.2.8). To prove (4.2.9), note that from $E_{n,L_p(\Omega),\mathcal{V}}(f) \lesssim N_n^{-s/d}$, i.e., $N_n^{(s-\varepsilon)/d} E_{n,L_p(\Omega),\mathcal{V}}(f) \lesssim N_n^{-\varepsilon/d}$, it follows that $\sum_{n=0}^{\infty} \Big[N_n^{(s-\varepsilon)/d} E_{n,L_p(\Omega),\mathcal{V}}(u)\Big]^p \lesssim \sum_{n=0}^{\infty} 2^{-\varepsilon n p} < \infty$, and apply (4.2.7) and (1.4.5). □

**Corollary 4.10.** *Let $1 < p < \infty$, and let $(V_j)_{j\in\mathbb{N}_0}$ and $\phi$ satisfy the assumptions of Proposition 4.6. Then, for $0 < \sigma < \min\{m, \bar{s}\}$ and $1/\tau = \sigma/d + 1/p$, it holds*

$$\sum_{n=1}^{\infty} n^{-1} \Big[n^{\sigma/d} E_{n,L_p(\Omega),\mathcal{S}}(u)\Big]^\tau \;<\; \infty \qquad \Longleftrightarrow \qquad u \in B_\tau^\sigma(L_\tau(\Omega)). \tag{4.2.10}$$

*Consequently, the implications*

$$f \in B^{\sigma}_{\tau}(L_{\tau}(\Omega)) \qquad \Longrightarrow \qquad E_{n,L_p(\Omega),\mathcal{S}}(f) \lesssim n^{-\sigma/d} \quad \textit{for all} \quad n \in \mathbb{N}, \qquad (4.2.11)$$

*and*

$$E_{n,L_p(\Omega),\mathcal{S}}(f) \lesssim n^{-\sigma/d} \quad \textit{for all} \quad n \in \mathbb{N} \qquad \Longrightarrow \qquad f \in B^{\tilde{\sigma}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega)) \qquad (4.2.12)$$

*hold true for all* $0 < \tilde{\sigma} < \sigma$ *and* $1/\tilde{\tau} = \tilde{\sigma}/d + 1/p$.

*Proof.* The equivalence (4.2.10) readily follows from Proposition 4.6 and (4.2.5), and (4.2.10) clearly implies (4.2.11). To prove (4.2.12), choose $0 < \tilde{\sigma} < \sigma$ and set $1/\tilde{\tau} = \tilde{\sigma}/d + 1/p$ and $\varepsilon = \sigma - \tilde{\sigma} > 0$. Then, from $E_{n,L_p(\Omega),\mathcal{S}}(f) \lesssim n^{-\sigma/d}$, i.e., $n^{\tilde{\sigma}/d} E_{n,L_p(\Omega),\mathcal{S}}(f) \lesssim n^{-\varepsilon/d}$, we conclude that $\sum_{n=1}^{\infty} n^{-1} \left[ n^{\tilde{\sigma}/d} E_{n,L_p(\Omega),\mathcal{S}}(u) \right]^{\tilde{\tau}} \lesssim \sum_{n=1}^{\infty} n^{-(1+\varepsilon\tilde{\tau}/d)} < \infty$, and hence $f \in B^{\tilde{\sigma}}_{\tilde{\tau}}(L_{\tilde{\tau}}(\Omega))$ due to (4.2.10). □

Hence, under appropriate assumptions on the wavelet basis, the rate of best uniform wavelet approximation in $L_p(\Omega)$ is essentially governed by the smoothness $s$ of the target function in the scale of Sobolev spaces $W^s(L_p(\Omega))$, whereas the rate of best $n$-term wavelet approximation is determined by the smoothness $\sigma$ of the target function in the scale of Besov spaces $B^{\sigma}_{\tau}(L_{\tau}(\Omega))$, $1/\tau = \sigma/d + 1/p$. Consequently, adaptive wavelet schemes potentially pay off, if

$$f \in B^{\sigma}_{\tau}(L_{\tau}(\Omega)), \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}, \quad \text{for some} \quad \sigma > s^* := \sup \{ s > 0 \,|\, f \in W^s(L_p(\Omega)) \} .$$

We finish this chapter with some additional notes on the corresponding scales of function spaces.

**Remark 4.11.**

**(i)** Note that with increasing parameter $s$ the functions in $B^s_p(L_p(\Omega))$ become smooth in the classical sense due to Sobolev's embedding theorem, whereas functions from $B^{\sigma}_{\tau}(L_{\tau}(\Omega))$, $1/\tau = \sigma/d + 1/p$, may still be discontinuous even for large values of $\sigma$. This can be seen by the following example on the unit interval $\Omega = (0, 1)$. Let $f = \chi_{[1/2,1)}$. A short computation yields that $|f|_{B^s_q(L_p(\Omega))}$ is finite if and only if $\int_0^1 t^{q(1/p-s)} t^{-1}\, \mathrm{d}t$ is finite. Thus, we conclude that $f \in B^s_p(L_p(\Omega))$ if and only if $0 < s < 1/p$, whereas $f \in B^{\sigma}_{\tau}(L_{\tau}(\Omega))$ for all $\sigma > 0$ and $1/\tau = \sigma + 1/p$.

**(ii)** For the solution $u$ to a linear elliptic boundary value problem on a polygonal domain $\Omega$ with right-hand side $f \in W^{\overline{s}}(L_2(\Omega))$ for some $\overline{s} \geq -1$, it holds that $u \notin W^s(L_2(\Omega))$ for $s > 3/2$ in general (cf. [82]), but $u \in B^{\sigma}_{\tau}(L_{\tau}(\Omega))$ for all $0 < \sigma < \overline{s} + 2$, where $1/\tau = \sigma/2 + 1/2$, see [25].

# Chapter 5

# General Embeddings

One of the main goals of this thesis is the study if adaptive wavelet schemes for the numerical solution of the $p$-Poisson equation (3.1.2) admit the potential to outperform uniform methods, i.e., to explore if the development (and use) of adaptive methods is theoretically justified for this class of problems. According to the results of Chapter 4, to this end we have to compare the convergence rates of best $n$-term and best uniform approximation, which in turn are governed by the regularity of the solution in the adaptivity scale of Besov spaces and the Sobolev smoothness, respectively; cf. Corollary 4.10 and Corollary 4.9 for the case of $L_p(\Omega)$-error measurement we are interested in.

To derive nontrivial Besov regularity assertions for solutions of the $p$-Poisson equation, we follow two approaches. Firstly, for general Lipschitz domains it is well-known that the solution possesses certain local Hölder regularity (see Proposition 3.3). Hereby, the Hölder seminorm over compact subsets of the domain usually tends to infinity as these subsets approach the boundary of the domain, but this growth can be controlled by some power of the distance to the boundary. In summary, as we will see in Chapter 6, one can show the containedness of the solution in certain locally weighted Hölder spaces $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$. Secondly, on polygonal domains it is known that the solution admits a singular expansion in the vicinity of a corner of the domain, which implies the membership of the solution in certain Babuska-Kondratiev spaces $\mathcal{K}^{\ell}_{p,a}$ as will be shown in Chapter 7. Here, it turns out that in some cases the solution even possesses arbitrary high Babuska-Kondratiev regularity $\ell \in \mathbb{N}$ (in a neighborhood of the corner).

In order to exploit the just mentioned additional information about the solution, we derive appropriate general *embeddings of function spaces* in this chapter. In Section 5.1 we prove that

$$C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \cap B^s_p(L_p(\Omega)) \hookrightarrow B^\sigma_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p},$$

under certain conditions on the parameters involved. Afterwards, we introduce the well-known embeddings of the type

$$\mathcal{K}^{\ell}_{p,a}(\Omega) \cap B^s_p(L_p(\Omega)) \hookrightarrow B^\sigma_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}, \tag{5.0.1}$$

at the beginning of Section 5.2. As noted above, some solutions are contained in $\mathcal{K}^{\ell}_{p,a}$ for all $\ell \in \mathbb{N}$. For that reason, in Subsection 5.2.1 we then consider (5.0.1) as $\ell \to \infty$, i.e., we examine an embedding of the type

$$\bigcap_{\ell=1}^{\infty} \mathcal{K}^{\ell}_{p,a}(\Omega) \cap B^s_p(L_p(\Omega)) \hookrightarrow \bigcap_{\sigma>0} B^{\sigma}_{\tau}(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}.$$

Besides the continuity of this embedding, we will furthermore prove some topological properties of the involved function spaces.

We stress the fact that all subsequent embeddings are independent of the $p$-Poisson problem and therefore universally applicable - for instance to derive Besov regularity assertions for solutions to some other PDE. In this sense, the results of this chapter are of certain interest on their own.

## 5.1 Embeddings of locally weighted Hölder spaces

In this section we prove that, under some growth condition on the local Hölder semi-norm, the intersection $B^s_p(L_p(\Omega)) \cap C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ is continuously embedded into certain Besov spaces $B^{\sigma}_{\tau}(L_\tau(\Omega))$ in the adaptivity scale. The results of this section have been primarily published in [30]. Now, the assertion reads as follows.

**Theorem 5.1.** *For $d \in \mathbb{N}$ with $d \geq 2$, let $\Omega \subset \mathbb{R}^d$ denote some bounded Lipschitz domain. Moreover, let $s > 0$ and $1 < p < \infty$, as well as $\ell \in \mathbb{N}_0$, $0 < \alpha \leq 1$, and $0 < \gamma < \ell + \alpha + 1/p$. If we define*

$$\sigma^* = \begin{cases} \ell + \alpha & \text{if} \quad 0 < \gamma < \dfrac{\ell+\alpha}{d} + \dfrac{1}{p}, \\ \dfrac{d}{d-1}\left(\ell + \alpha + \dfrac{1}{p} - \gamma\right) & \text{if} \quad \dfrac{\ell+\alpha}{d} + \dfrac{1}{p} \leq \gamma < \ell + \alpha + \dfrac{1}{p}, \end{cases} \tag{5.1.1}$$

*then for all*

$$0 < \sigma < \min\left\{\sigma^*, \frac{d}{d-1}\, s\right\} \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p} \tag{5.1.2}$$

*we have the continuous embedding*

$$B^s_p(L_p(\Omega)) \;\cap\; C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \;\hookrightarrow\; B^{\sigma}_{\tau}(L_\tau(\Omega)),$$

*i.e., for all $u \in B^s_p(L_p(\Omega)) \cap C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ it holds*

$$\|u \mid B^{\sigma}_{\tau}(L_\tau(\Omega))\| \lesssim \max\left\{ \left\| u \,\middle|\, B^s_p(L_p(\Omega)) \right\|, |u|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \right\}. \tag{5.1.3}$$

Let us briefly comment on Theorem 5.1 before we give its proof: From the theory of function spaces it is well-known that (standard) embeddings between Besov spaces, e.g.,

$$B^s_p(L_p(\Omega)) \hookrightarrow B^{\sigma}_{\tau}(L_\tau(\Omega)),$$

are valid *only if* the regularity of the target space is at most as large as the smoothness of the space we start from, i.e., only if $\sigma \leq s$. Theorem 5.1 now states that, under suitable assumptions on the parameters involved, exploiting the additional information on locally weighted Hölder regularity (encoded by the membership of $u$ in $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$) enables us to prove that functions from $B^s_p(L_p(\Omega))$ indeed possess a higher-order Besov regularity $\sigma > s$ measured in the adaptivity scale corresponding to $L_p(\Omega)$. Since $B^s_p(L_p(\Omega))$ almost equals the Sobolev space $W^s(L_p(\Omega))$ (cf. Remark 1.6(iii)) this shows that approximating $u \in W^s(L_p(\Omega)) \cap C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ in an adaptive way is justified whenever $\sigma^*$ defined by (5.1.1) is larger than $s$. At this point we remark that $\sigma^*$ is a continuous piecewise linear function of $\gamma \in (0, \ell + \alpha + 1/p)$ which decreases to zero when $\gamma$ approaches its upper bound. Hence, in any case $0 < \sigma^* \leq \ell + \alpha$. Thus, for a fixed value of $s$, the maximal regularity $d/(d-1) \cdot s$ is achieved if $\ell + \alpha$ is sufficiently large and $\gamma$ is small enough.

The proof of Theorem 5.1 given below is inspired by ideas first given in [29]. For the rest of this section we adopt the notation of Section 1.6. Due to extension arguments in conjunction with the wavelet characterization of Besov spaces on $\mathbb{R}^d$ (see Remark 1.21) it suffices to find suitable estimates for the wavelet coefficients $\langle u, \psi_{I,p'} \rangle$, $I \in \mathcal{I}^+$, $\psi \in \Psi^M$, which then imply (5.1.3). The contribution of (the relatively small number of) wavelets supported in the vicinity of the boundary of $\Omega$ (*boundary wavelets*) can be bounded in terms of the norm of $u$ in $B^s_p(L_p(\Omega))$. Here the restriction $\sigma < s \cdot d/(d-1)$ comes in. The coefficients corresponding to the remaining *interior wavelets* can be upper bounded by the semi-norm of $u$ in $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}$ using a Whitney-type argument which then gives rise to the restriction $\sigma < \sigma^*$. The detailed proof reads as follows:

*Proof (of Theorem 5.1). Step 1.* Let $u \in B^s_p(L_p(\Omega)) \cap C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$. Since for $1 < p < \infty$ it is $\sigma_p = 0$ and $s > 0$, every such $u$ can be extended to some $\mathcal{E}_\Omega u \in B^s_p(L_p(\mathbb{R}^d))$; see Remark 1.6(iv). In particular, $\mathcal{E}_\Omega u \in L_p(\mathbb{R}^d)$ such that it can be written as

$$\mathcal{E}_\Omega u = P_0(\mathcal{E}_\Omega u) + \sum_{(I,\psi) \in \mathcal{I}^+ \times \Psi^M} \langle \mathcal{E}_\Omega u, \psi_{I,p'} \rangle \, \psi_{I,p}.$$

Here the $\psi_I$ form a system of Daubechies wavelets (1.6.1), where $m \in \mathbb{N}$ is chosen such that $m > \ell$ and $\phi_m, D_m \in C^r(\mathbb{R})$ for some $r \in \mathbb{N}$ with $r > \max\{\sigma, s\}$; see Section 1.6 for details. We restrict the latter expansion and consider only those wavelets for which $(I, \psi)$ belongs to

$$\Lambda = \bigcup_{j \in \mathbb{N}_0} \Lambda_j,$$

where we set

$$\Lambda_j = \left\{ (I, \psi) \in \mathcal{I}^+ \times \Psi^M \,\middle|\, \overline{B_c(I)} \cap \Omega \neq \emptyset \;\text{ and }\; |I| = 2^{-jd} \right\}.$$

Therein $B_c(I)$ denotes the ball $B(I)$ (see (1.6.2)) concentrically expanded by the factor $c > 1$ which we used to define the class $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$; cf. Section 1.1. Note that thus $\operatorname{supp}(\psi_I) \subset \overline{B_c(I)}$ for all $I$ and $\psi$. Next we split up the index sets $\Lambda_j$ once more and write

$$\Lambda_j = \bigcup_{n \in \mathbb{N}_0} \Lambda_{j,n}$$

with

$$\Lambda_{j,n} = \Big\{ (I_{j,k}, \psi) \in \Lambda_j \,\Big|\, n\, 2^{-j} \leq \operatorname{dist}\Big(2^{-j}k, \partial\Omega\Big) < (n+1)\, 2^{-j} \Big\},$$

for every dyadic level $j \in \mathbb{N}_0$. Note that, due to the boundedness of $\Omega$, there exists an absolute constant $C_1$ such that $\Lambda_{j,n} = \emptyset$ for all $j \in \mathbb{N}_0$ and $n > C_1\, 2^j$. For example, we may take $C_1 = \max\{\operatorname{diam}(\Omega), c\, \operatorname{diam}(Q)\}$. Moreover, our assumption that $\Omega$ is a bounded Lipschitz domain ensures that all remaining index sets satisfy at least $|\Lambda_{j,n}| \lesssim 2^{-j(d+1)}$. Finally, we note that all balls $\overline{B_c(I)}$ corresponding to $(I, \psi) \in \Lambda_{j,n}$ with $j \in \mathbb{N}_0$ and $n$ strictly larger than $C_0 = \lceil c\, \operatorname{diam}(Q)/2 \rceil$ are completely contained in $\Omega$. These considerations justify the disjoint splitting $\Lambda = \left(\bigcup_{j\in\mathbb{N}_0} \Lambda_j^{\mathrm{bnd}}\right) \cup \left(\bigcup_{j\in\mathbb{N}_0} \Lambda_j^{\mathrm{int}}\right)$, where

$$\Lambda_j^{\mathrm{bnd}} = \bigcup_{n=0}^{C_0} \Lambda_{j,n} \qquad \text{and} \qquad \Lambda_j^{\mathrm{int}} = \bigcup_{n=C_0+1}^{C_1\, 2^j} \Lambda_{j,n}$$

correspond to the sets of boundary and interior wavelets at level $j \in \mathbb{N}_0$, respectively. Observe that then $\tilde{u} = u_0 + u_1 + u_2$, defined by

$$u_0 = P_0(\mathcal{E}_\Omega u), \quad u_1 = \sum_{j\in\mathbb{N}_0} \sum_{(I,\psi)\in\Lambda_j^{\mathrm{bnd}}} \langle \mathcal{E}_\Omega u, \psi_{I,p'} \rangle\, \psi_{I,p}, \quad u_2 = \sum_{j\in\mathbb{N}_0} \sum_{(I,\psi)\in\Lambda_j^{\mathrm{int}}} \langle u, \psi_{I,p'} \rangle\, \psi_{I,p},$$

is an extension of $u$ as well, i.e., it satisfies $\tilde{u}\big|_\Omega = u$. In Step 2–4 below we will show that for the adaptivity scale $\tau = (\sigma/d + 1/p)^{-1}$ it holds

$$\Big\| u_0 \,\Big|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \Big\| \lesssim \Big\| P_0(\mathcal{E}_\Omega u) \,\Big|\, L_p(\mathbb{R}^d) \Big\| \qquad \text{if } 0 < \sigma, \tag{5.1.4}$$

$$\Big\| u_1 \,\Big|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \Big\| \lesssim \left[ \sum_{j\in\mathbb{N}_0} \sum_{(I,\psi)\in\Lambda_j^{\mathrm{bnd}}} |I|^{-s\,p/d}\, |\langle \mathcal{E}_\Omega u, \psi_{I,p'} \rangle|^p \right]^{1/p} \qquad \text{if } 0 < \sigma < \frac{d}{d-1}\, s, \tag{5.1.5}$$

$$\Big\| u_2 \,\Big|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \Big\| \lesssim |u|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \qquad \text{if } 0 < \sigma < \sigma^*. \tag{5.1.6}$$

Suppose we already know that those relations hold for all $\sigma$ and $\tau$ that satisfy (5.1.2). Then we can extend the index set in (5.1.5) from $\bigcup_{j\in\mathbb{N}_0} \Lambda_j^{\mathrm{bnd}}$ to $\mathcal{I}^+ \times \Psi^M$ and the wavelet characterization of $\mathcal{E}_\Omega u \in B^s_p(L_p(\mathbb{R}^d))$ (cf. Lemma 1.20) together with the continuity of $\mathcal{E}_\Omega$ implies

$$\Big\| u_0 + u_1 \,\Big|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \Big\| \lesssim \Big\| \mathcal{E}_\Omega u \,\Big|\, B^s_p(L_p(\mathbb{R}^d)) \Big\| \sim \Big\| u \,\Big|\, B^s_p(L_p(\Omega)) \Big\| \tag{5.1.7}$$

which is finite due to our assumptions. Therefore, the special choice $g = \tilde{u} = (u_0 + u_1) + u_2$, in conjunction with (5.1.6) and (5.1.7), yields the desired estimate

$$\begin{aligned} \| u \,|\, B^\sigma_\tau(L_\tau(\Omega)) \| &\sim \inf\Big\{ \Big\| g \,\Big|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \Big\| \,\Big|\, g \in B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \text{ with } g\big|_\Omega = u \Big\} \\ &\lesssim \Big\| u_0 + u_1 \,\Big|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \Big\| + \Big\| u_2 \,\Big|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \Big\| \\ &\lesssim \max\left\{ \Big\| u \,\Big|\, B^s_p(L_p(\Omega)) \Big\|, |u|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \right\}. \end{aligned}$$

This proves Theorem 5.1 since $u \in B^s_p(L_p(\Omega))$ with $s > 0 = \sigma_p$ particularly implies that $u \in L_p(\Omega) \hookrightarrow L_\tau(\Omega)$, due to $\tau < p$ and the boundedness of $\Omega$. Hence, $u \in B^\sigma_\tau(L_\tau(\Omega))$.

*Step 2 (Estimate for $u_0$).* To show the bound on the projection onto the coarse levels let $\tau = (\sigma/d + 1/p)^{-1}$ and $\sigma > 0$. We note that $u_0 \perp \psi_{I,p'}$ for all $I \in \mathcal{I}^+$ and $\psi \in \Psi^M$, i.e., $u_0 = P_0(u_0)$. Moreover, by definition, this equals $P_0(\mathcal{E}_\Omega u)$ which has compact support in $\mathbb{R}^d$ since $\mathcal{E}_\Omega$ is local; see Remark 1.6(iv). Proposition 1.22, i.e., the wavelet characterization of $B^\sigma_\tau(L_\tau(\mathbb{R}^d))$, therefore gives

$$\left\| u_0 \,\middle|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \right\| \sim \left\| P_0(\mathcal{E}_\Omega u) \,\middle|\, L_\tau(\mathbb{R}^d) \right\| \lesssim \left\| P_0(\mathcal{E}_\Omega u) \,\middle|\, L_p(\mathbb{R}^d) \right\|,$$

due to $\tau < p$. That is, we have shown (5.1.4).

*Step 3 (Estimate for $u_1$).* Here we establish the bound on the contribution of all wavelets near $\partial\Omega$. To this end, assume again that $\tau = (\sigma/d + 1/p)^{-1}$ with $\sigma > 0$. We fix $j \in \mathbb{N}_0$ for a moment and apply Hölder's inequality (with $q = p/\tau > 1$) to estimate

$$\begin{aligned}\sum_{(I,\psi)\in\Lambda_j^{\mathrm{bnd}}} |\langle \mathcal{E}_\Omega u, \psi_{I,p'}\rangle|^\tau &\leq \left|\Lambda_j^{\mathrm{bnd}}\right|^{1-\tau/p} \left( \sum_{(I,\psi)\in\Lambda_j^{\mathrm{bnd}}} |\langle \mathcal{E}_\Omega u, \psi_{I,p'}\rangle|^p \right)^{\tau/p} \\ &\lesssim 2^{j(d-1)(1-\tau/p)}\, 2^{-j\,s\,\tau} \left( \sum_{(I,\psi)\in\Lambda_j^{\mathrm{bnd}}} |I|^{-s\,p/d}\, |\langle \mathcal{E}_\Omega u, \psi_{I,p'}\rangle|^p \right)^{\tau/p}.\end{aligned}$$

Taking the sum over all levels $j$ and using Hölder's inequality once more (with the same $q$), we find

$$\begin{aligned}&\sum_{j\in\mathbb{N}_0} \sum_{(I,\psi)\in\Lambda_j^{\mathrm{bnd}}} |\langle \mathcal{E}_\Omega u, \psi_{I,p'}\rangle|^\tau \qquad (5.1.8)\\ &\quad\lesssim \left( \sum_{j\in\mathbb{N}_0} \left[ 2^{(d-1)-s\,\tau/(1-\tau/p)} \right]^j \right)^{1-\tau/p} \left( \sum_{j\in\mathbb{N}_0} \sum_{(I,\psi)\in\Lambda_j^{\mathrm{bnd}}} |I|^{-s\,p/d}\, |\langle \mathcal{E}_\Omega u, \psi_{I,p'}\rangle|^p \right)^{\tau/p} \\ &\quad\lesssim \left( \sum_{j\in\mathbb{N}_0} \sum_{(I,\psi)\in\Lambda_j^{\mathrm{bnd}}} |I|^{-s\,p/d}\, |\langle \mathcal{E}_\Omega u, \psi_{I,p'}\rangle|^p \right)^{\tau/p},\end{aligned}$$

provided that we additionally assume

$$\sigma < \frac{d}{d-1}\, s,$$

since this condition is equivalent to $1/\tau < s/(d-1) + 1/p$ which in turn holds if and only if $(d-1) - s\,\tau/(1-\tau/p) < 0$. Finally, the structure of $u_1$ together with Proposition 1.22 shows that the quantity (5.1.8) is equivalent to $\left\| u_1 \,\middle|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d)) \right\|^\tau$ such that (5.1.5) follows.

*Step 4 (Estimate for $u_2$).* We are left with the proof of (5.1.6), i.e., the bound for the interior wavelets indexed by $(I,\psi) \in \bigcup_{j\in\mathbb{N}_0} \Lambda_j^{\mathrm{int}}$. Recall that $\psi_{I,p'}$ is orthogonal to every polynomial $\mathcal{P}$ of total degree strictly less than $m$. Therefore, for all $(I,\psi)$ under consideration,

$$\begin{aligned}|\langle u, \psi_{I,p'}\rangle| = |\langle u-\mathcal{P}, \psi_{I,p'}\rangle| &\leq \|u-\mathcal{P} \,|\, L_p(Q(I))\| \cdot \|\psi_{I,p'} \,|\, L_{p'}(Q(I))\| \\ &\lesssim \|u-\mathcal{P} \,|\, L_p(Q(I))\|.\end{aligned}$$

Consequently, a Whitney-type argument (i.e., the application of Proposition A.1 stated in the Appendix with $t=\ell+\alpha$ and $q=\infty$) shows that

$$|\langle u, \psi_{I,p'}\rangle| \lesssim \inf_{\mathcal{P}\in\Pi_\ell} \|u-\mathcal{P} \,|\, L_p(Q(I))\| \lesssim |Q(I)|^{(\ell+\alpha)/d+1/p} \, |u|_{B^{\ell+\alpha}_\infty(L_\infty(Q(I)))},$$

since we assumed $m>\ell$. Next we use (1.6.2) and estimate the Besov semi-norm by the Hölder semi-norm (see Proposition A.2) to obtain

$$\begin{aligned}|\langle u, \psi_{I,p'}\rangle| &\lesssim 2^{-j(\ell+\alpha+d/p)} \, |u|_{C^{\ell,\alpha}(Q(I))} \\ &\lesssim 2^{-j(\ell+\alpha+d/p)} \delta_{\overline{B(I)}}^{-\gamma} \, |u|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \qquad \text{for all} \quad (I,\psi)\in \bigcup_{j\in\mathbb{N}_0} \Lambda_j^{\mathrm{int}} = \bigcup_{j\in\mathbb{N}_0} \bigcup_{n=C_0+1}^{C_1\,2^j} \Lambda_{j,n},\end{aligned} \tag{5.1.9}$$

because the open cubes $Q(I)$ are contained in the closed balls $\overline{B(I)}$ by definition. For fixed $j\in\mathbb{N}_0$, $n\in\{C_0+1, C_0+2, \ldots, C_1\,2^j\}$, and $(I,\psi)\in\Lambda_{j,n}$, we have

$$\delta_{\overline{B(I)}} \geq \delta_{\overline{B_c(I)}} \geq \operatorname{dist}\!\left(2^{-j}k, \partial\Omega\right) - \frac{c\,\operatorname{diam}(Q)}{2}\, 2^{-j} \geq (n-C_0)\, 2^{-j}. \tag{5.1.10}$$

Now let $\tau>0$ and recall the estimate $|\Lambda_{j,n}| \lesssim 2^{j(d-1)}$ which we found in Step 1. Combining this with (5.1.9) and (5.1.10) thus yields

$$\begin{aligned}\sum_{(I,\psi)\in\Lambda_j^{\mathrm{int}}} |\langle u, \psi_{I,p'}\rangle|^\tau &\lesssim \sum_{n=C_0+1}^{C_1\,2^j} \sum_{(I,\psi)\in\Lambda_{j,n}} 2^{-j(\ell+\alpha+d/p)\tau} \, (n-C_0)^{-\gamma\tau} \, 2^{j\gamma\tau} \, |u|^\tau_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \\ &\lesssim |u|^\tau_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \, 2^{-j(\ell+\alpha+d/p-\gamma)\tau+j(d-1)} \sum_{t=1}^{C_1\,2^j} t^{-\gamma\tau}, \qquad j\in\mathbb{N}_0.\end{aligned} \tag{5.1.11}$$

Note that, due to the assumption $\gamma>0$, the quantity $\gamma\tau$ is always positive. Then straightforward calculations show that for all $j\in\mathbb{N}_0$

$$1 \leq \sum_{t=1}^{C_1\,2^j} t^{-\gamma\tau} \lesssim \begin{cases} 2^{j(1-\gamma\tau)} & \text{if} \quad \gamma\tau\in(0,1), \\ 1+j & \text{if} \quad \gamma\tau=1, \\ 1 & \text{if} \quad \gamma\tau>1, \end{cases}$$

such that we have to distinguish several cases for $\gamma$ in what follows:

*Substep 4.1 (Small $\gamma$).* Let us consider the case $0<\gamma<(\ell+\alpha)/d+1/p$ first. Then obviously $d(\gamma-1/p)<\ell+\alpha$, such that we can set

$$\tau = \left(\frac{\sigma}{d}+\frac{1}{p}\right)^{-1} \qquad \text{with} \qquad \max\!\left\{0, d\left(\gamma-\frac{1}{p}\right)\right\} < \sigma < \ell+\alpha. \tag{5.1.12}$$

From $d(\gamma - 1/p) < \sigma$ we particularly infer that $\gamma < \tau^{-1}$, i.e., $\gamma\tau < 1$, for this choice of $\tau$. Therefore, from the considerations stated above we conclude that

$$\begin{aligned}\sum_{j\in\mathbb{N}_0}\sum_{(I,\psi)\in\Lambda_j^{\mathrm{int}}} |\langle u, \psi_{I,p'}\rangle|^\tau &\lesssim |u|^\tau_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \sum_{j\in\mathbb{N}_0} 2^{-j(\ell+\alpha+d/p-\gamma)\tau+j(d-1)+j(1-\gamma\tau)} \\ &= |u|^\tau_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \sum_{j\in\mathbb{N}_0} \left(2^{d-(\ell+\alpha+d/p)\tau}\right)^j \\ &\lesssim |u|^\tau_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}},\end{aligned}$$

because the sum in the second line converges for $d - (\ell + \alpha + d/p)\tau < 0$ which is equivalent to $\sigma < \ell + \alpha = \sigma^*$. Similar to the end of Step 3, we note that the double sum on the left-hand side is equivalent to $\left\| u_2 \,\middle|\, B^\sigma_\tau(L_\tau(\mathbb{R}^d))\right\|^\tau$ such that (5.1.6) follows (in the case of small $\gamma$) for all $\sigma$ that satisfy (5.1.12). Note that if $\gamma > 1/p$, then the maximum in (5.1.12) is strictly positive. The result (5.1.6) for $\sigma > 0$ below this value can be deduced from the assertion we just proved by means of the embedding along the adaptivity scale:

$$B^{\sigma_2}_{\tau_2}(L_{\tau_2}(\mathbb{R}^d)) \hookrightarrow B^{\sigma_1}_{\tau_1}(L_{\tau_1}(\mathbb{R}^d)) \qquad \text{for all} \qquad \sigma_2 \geq \sigma_1 > 0,$$

where $1/\tau_i = \sigma_i/d + 1/p$ for each $i \in \{1, 2\}$, see Lemma 1.9(ii).

*Substep 4.2 (Large $\gamma$).* We turn to the case

$$\frac{\ell+\alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p}.$$

As mentioned right after the statement of Theorem 5.1, for $\gamma$ in this range we have that

$$\sigma^* = \frac{d}{d-1}\left(\ell + \alpha + \frac{1}{p} - \gamma\right) \leq \ell + \alpha.$$

The lower bound for $\gamma$ thus implies that $\sigma^* \leq d\gamma - d/p$. Therefore, for every $0 < \sigma < \sigma^*$ the corresponding $\tau$ in the adaptivity scale satisfies

$$\frac{1}{p} < \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p} < \gamma,$$

i.e., $\gamma\tau > 1$. Hence, proceeding as in the previous substep yields

$$\begin{aligned}\sum_{j\in\mathbb{N}_0}\sum_{(I,\psi)\in\Lambda_j^{\mathrm{int}}} |\langle u, \psi_{I,p'}\rangle|^\tau &\lesssim |u|^\tau_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \sum_{j\in\mathbb{N}_0} 2^{-j(\ell+\alpha+d/p-\gamma)\tau+j(d-1)} \\ &= |u|^\tau_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \sum_{j\in\mathbb{N}_0} \left(2^{d-1-\tau(\ell+\alpha+d/p-\gamma)}\right)^j \lesssim |u|^\tau_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}},\end{aligned}$$

where this time the sum over $j$ converges if $d - 1 - \tau(\ell + \alpha + d/p - \gamma) < 0$ which is (for the assumed range of $\gamma$) equivalent to $\sigma < \sigma^*$. Since this implies the desired estimate (5.1.6), finally, the proof is complete. □

**Remark 5.2.**

**(i)** The interested reader might ask what happens if $\gamma \geq \ell + \alpha + 1/p$. For $\gamma \geq \ell + \alpha + d/p$ the sum over (5.1.11) w.r.t. $j \in \mathbb{N}_0$ can never be convergent, because due to $\tau > 0$ the exponent $-j(\ell + \alpha + d/p - \gamma)\tau + j(d-1)$ would be non-negative for all $j$ and the sum over $t$ is bounded from below by 1. Hence, we are left with $\ell + \alpha + 1/p \leq \gamma < \ell + \alpha + d/p$. Choosing $\tau > 0$ such that $\gamma \leq 1/\tau$ then implies $\sigma \geq d(\ell + \alpha)$ for $\sigma$ in the adaptivity scale. On the other hand, $\sigma < \ell + \alpha$ would be necessary for the geometric series to converge; see Substep 4.1. In contrast, if we choose $\tau > 0$ such that $\gamma > 1/\tau$, then convergence is equivalent to $\sigma < \frac{d}{d-1}(\ell + \alpha + 1/p - \gamma)$ which contradicts $\sigma > 0$ for the range of $\gamma$ under consideration.

**(ii)** The assertion of Theorem 5.1 clearly holds true analogously when $B^s_p(L_p(\Omega))$ is replaced by $W^s(L_p(\Omega))$. This follows immediately from (1.4.5).

## 5.2 Embeddings of Babuska-Kondratiev spaces

First, let us mention that all results of this section stem from [78].

The derivation of our Besov regularity results for solutions to the $p$-Poisson equation in Chapter 7 is based on embeddings from Babuska-Kondratiev spaces $\mathcal{K}^\ell_{p,a}(\Omega)$, intersected with some Besov space $B^s_p(L_p(\Omega))$, into Besov spaces in the adaptivity scale $B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$. Such kind of embeddings - for slightly different scales of Besov spaces - have been proved to hold true by Hansen [76] for polyhedral domains in $\mathbb{R}^d$, $d \geq 2$, under certain conditions on the parameters involved. One of his results ([76, Theorem 3]), formulated for $d = 2$, reads as follows.

**Proposition 5.3.** *Let $\Omega \subset \mathbb{R}^2$ be some bounded polygonal domain and $1 < p < \infty$. Furthermore, let $\ell \in \mathbb{N}$, $a > 0$ and $s > 0$. We set $1/\tau_* := \ell/2 + 1/p$. Then there exists some $\tau_0 \in (\tau_*, p]$, such that the chain of continuous embeddings*

$$\mathcal{K}^\ell_{p,a}(\Omega) \cap B^s_\infty(L_p(\Omega)) \hookrightarrow B^\ell_\infty(L_\tau(\Omega)) \hookrightarrow L_p(\Omega)$$

*holds true for all $\tau \in (\tau_*, \tau_0)$.*

We recall in passing that here (and in the following) the singular set $S \subseteq \partial\Omega$ corresponding to the spaces $\mathcal{K}^\ell_{p,a}(\Omega)$ is implicitly assumed to consist of the collection of vertices of the domain $\Omega$, cf. Remark 1.5.

Note that the smoothness $\ell$ of the Babuska-Kondratiev space leads to Besov smoothness $\ell$ measured in the scale $B^\ell_\infty(L_\tau(\Omega))$, but excluding the case $1/\tau = \ell/2 + 1/p$ we are interested in. However, if we relax the Besov smoothness parameter of the target space slightly to $\sigma < \ell$, the above embedding still holds true with the right-hand side replaced by $B^\sigma_\tau(L_\tau(\Omega))$, where $1/\tau = \sigma/2 + 1/p$. Furthermore, we may certainly impose the slightly more restrictive condition $q = p$ for the fine index parameter on the left-hand side. We state the resulting embedding by the following Corollary 5.4, and carry out in detail the arguments just given. For the interested reader, an alternative proof based on wavelet characterizations of Besov spaces can be found in Appendix A, see Subsection A.4.1.

**Corollary 5.4.** *Let $\Omega \subset \mathbb{R}^2$ denote some bounded polygonal domain and let $1 < p < \infty$. Furthermore, let $\ell \in \mathbb{N}$, $a > 0$ and $s > 0$. Then for all*

$$0 < \sigma < \ell \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p} \tag{5.2.1}$$

*we have the continuous embedding*

$$\mathcal{K}^{\ell}_{p,a}(\Omega) \cap B^s_p(L_p(\Omega)) \hookrightarrow B^{\sigma}_{\tau}(L_{\tau}(\Omega)), \tag{5.2.2}$$

*i.e., for all $u \in \mathcal{K}^{\ell}_{p,a}(\Omega) \cap B^s_p(L_p(\Omega))$ it holds*

$$\|u \,|\, B^{\sigma}_{\tau}(L_{\tau}(\Omega))\| \lesssim \left\|u \,\middle|\, \mathcal{K}^{\ell}_{p,a}(\Omega)\right\| + \left\|u \,\middle|\, B^s_p(L_p(\Omega))\right\|. \tag{5.2.3}$$

*Proof.* Let $\tau_*$ and $\tau_0$ denote the bounds from Proposition 5.3. At first, we consider the case $\sigma_0 < \sigma < \ell$, where $\sigma_0 := 2(1/\tau_0 - 1/p)$. We set $1/\tau = \sigma/2 + 1/p$. Since $\sigma_0 < \sigma < \ell$ is equivalent to $\tau_* < \tau < \tau_0$, from Proposition 5.3 we know that

$$\mathcal{K}^{\ell}_{p,a}(\Omega) \cap B^s_{\infty}(L_p(\Omega)) \hookrightarrow B^{\ell}_{\infty}(L_{\tau}(\Omega)). \tag{5.2.4}$$

Next, from the embedding (1.4.7), see Lemma 1.8, we know that

$$B^{\ell}_{\infty}(L_{\tau}(\Omega)) \hookrightarrow B^{\sigma}_{\tau}(L_{\tau}(\Omega)) \tag{5.2.5}$$

for all $0 < \sigma < \ell$. With the help of the embedding

$$B^s_p(L_p(\Omega)) \hookrightarrow B^s_{\infty}(L_p(\Omega)),$$

see (1.4.6), from (5.2.4) and (5.2.5) we conclude that

$$\mathcal{K}^{\ell}_{p,a}(\Omega) \cap B^s_p(L_p(\Omega)) \hookrightarrow B^{\sigma}_{\tau}(L_{\tau}(\Omega))$$

holds true for all $\sigma_0 < \sigma < \ell$ and $1/\tau = \sigma/2 + 1/p$. Finally, to prove the case $0 < \sigma \leq \sigma_0$, note that from Lemma 1.9 we know that for all $\sigma_1 > \sigma_2 > 0$ it holds

$$B^{\sigma_1}_{\tau_1}(L_{\tau_1}(\Omega)) \hookrightarrow B^{\sigma_2}_{\tau_2}(L_{\tau_2}(\Omega)),$$

where $1/\tau_i = \sigma_i/2 + 1/p$ for $i \in \{1, 2\}$. This completes the proof. □

**Remark 5.5.**

**(i)** In view of Remark 1.6(iii) the space $B^s_p(L_p(\Omega))$ in (5.2.2) and (5.2.3) can be replaced by $W^s(L_p(\Omega))$. Moreover, Remark 1.4(ii) shows that under the additional assumption $a \geq 1$ we can even drop it completely, because then $\mathcal{K}^{\ell}_{p,a}(\Omega) \hookrightarrow \mathcal{K}^{1}_{p,a}(\Omega) \hookrightarrow W^1(L_p(\Omega))$.

**(ii)** The assertion of Proposition 5.3 - and consequently Corollary 5.4 - also holds true for the more general setting of Lipschitz domains with polyhedral structure, see [76, Chapter 2.3]. For a precise definition of such domains we refer to [46, 106]. The finite cones we will be concerned with in Chapter 7 (defined by (7.0.1)) are in particular included by this class of domains.

Due to the ordering of the Kondratiev spaces (see Remark 1.4(ii)) it is obvious that also the spaces $\mathcal{K}^{\ell}_{p,a}(\Omega) \cap B^s_p(L_p(\Omega))$ get smaller with increasing $\ell$. In turn, by Corollary 5.4 we then gain more and more smoothness in the adaptivity scale of Besov spaces. Therefore, the question arises what happens to the embedding (5.2.2) if we let $\ell \to \infty$.

### 5.2.1 The limit case $\ell \to \infty$

For the rest of this section, we will examine the borderline case $\ell = \infty$ of the embeddings (5.2.2) stated in Corollary 5.4. Therefore, let us introduce the abbreviation

$$H_a^{\ell,s}(L_p(\Omega)) := \mathcal{K}_{p,a}^{\ell}(\Omega) \cap B_p^s(L_p(\Omega)).$$

The canonical norm on $H_a^{\ell,s}(L_p(\Omega))$ is given by

$$\left\| \cdot \,\middle|\, H_a^{\ell,s}(L_p(\Omega)) \right\| := \left\| \cdot \,\middle|\, \mathcal{K}_{p,a}^{\ell}(\Omega) \right\| + \left\| \cdot \,\middle|\, B_p^s(L_p(\Omega)) \right\|. \tag{5.2.6}$$

Then, for some fixed bounded polygonal domain $\Omega \subset \mathbb{R}^2$ and $s > 0$, as well as $a \geq 0$ and $1 < p < \infty$, we consider the vector space

$$H_a^{\infty,s}(L_p(\Omega)) := \bigcap_{\ell=1}^{\infty} H_a^{\ell,s}(L_p(\Omega))$$

endowed with the family of norms

$$\mathcal{N} := \{ n_\ell \,|\, \ell \in \mathbb{N} \}, \qquad \text{where} \qquad n_\ell := \left\| \cdot \,\middle|\, H_a^{\ell,s}(L_p(\Omega)) \right\|$$

as defined in (5.2.6). Analogously, for $1 < p < \infty$, we define the vector space $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$ as the intersection of all Besov spaces $B_\tau^\sigma(L_\tau(\Omega))$, $\sigma > 0$, in the adaptivity scale, thus

$$B_{\mathrm{NL}}^{\infty}(L_p(\Omega)) := \bigcap_{\sigma>0} B_\tau^\sigma(L_\tau(\Omega)), \qquad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}, \tag{5.2.7}$$

equipped with the family of quasi-norms

$$\mathcal{Q} := \{ q_\sigma \,|\, \sigma > 0 \}, \qquad \text{where} \qquad q_\sigma := \| \cdot \,|\, B_\tau^\sigma(L_\tau(\Omega)) \| \quad \text{with} \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}.$$

**Remark 5.6.** Recall that a family of (semi-, quasi-, $p$-)norms induces a topology on the corresponding vector space, cf. Chapter 2. In the particular case of $H_a^{\infty,s}(L_p(\Omega))$ equipped with $\mathcal{N}$, for each $g \in H_a^{\infty,s}(L_p(\Omega))$ a local basis is given by the sets

$$U_{L_0,r}(g) = \bigcap_{\ell \in L_0} V_{\ell,r}(g) \qquad \text{with} \qquad L_0 \subset \mathbb{N} \text{ finite}, \ r > 0, \tag{5.2.8}$$

where

$$V_{\ell,r}(g) = \{ f \in H_a^{\infty,s}(L_p(\Omega)) \,|\, n_\ell(f - g) < r \} = g + V_{\ell,r}(0), \tag{5.2.9}$$

see Definition 2.5(i). These local bases in turn allow to define a topology on $H_a^{\infty,s}(L_p(\Omega))$, see Definition 2.5(ii) and Lemma 2.6. The topology on $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$ induced by $\mathcal{Q}$ is defined likewise, cf. Proposition 2.18.

In a first step, we investigate properties of the topologies of the spaces $H_a^{\infty,s}(L_p(\Omega))$ and $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$ generated by $\mathcal{N}$ and $\mathcal{Q}$, respectively.

**Proposition 5.7.** *Let $\Omega \subset \mathbb{R}^2$ denote some bounded polygonal domain and $1 < p < \infty$. Furthermore, let $s > 0$ and $a \geq 0$. Then, the space $H_a^{\infty,s}(L_p(\Omega))$, equipped with the topology induced by the family of norms $\mathcal{N}$, is a LCTVS. Furthermore, it is metrizable and complete, i.e., $H_a^{\infty,s}(L_p(\Omega))$ is a Fréchet space.*

*Proof. Step 1.* At first, according to Remark 5.6 and Definition 2.11, the space $(H_a^{\infty,s}(L_p(\Omega)), \mathcal{N})$ clearly is a LCTVS.

*Step 2.* We prove metrizability. From Remark 2.12 we know that $H_a^{\infty,s}(L_p(\Omega))$ is a Hausdorff space and that the sets

$$U_{L_0,r}(0) \qquad \text{with} \qquad L_0 \subset \mathbb{N} \text{ finite, } r > 0,$$

defined by (5.2.8), (5.2.9) constitute a neighborhood basis of the origin. Next, from (5.2.6) and Remark 1.4(ii) it follows that

$$n_i(\cdot) \leq n_\ell(\cdot) \qquad \text{for all} \qquad i \leq \ell,$$

where $i, \ell \in \mathbb{N}$. Hence,

$$V_{\ell,r}(0) \subset V_{i,r}(0) \qquad \text{for all} \qquad i \leq \ell,$$

and therefore

$$U_{L_0,r}(0) = V_{\ell_0,r}(0), \qquad \text{where} \quad \ell_0 = \max\{\ell \mid \ell \in L_0\}.$$

Thus, the sets $V_{\ell,r}(0)$, $\ell \in \mathbb{N}$, $r > 0$, form a neighborhood basis of the origin, and one easily checks that countable many of them, namely $V_{\ell,1/m}(0)$, $\ell \in \mathbb{N}$, $m \in \mathbb{N}$, are a neighborhood basis of the origin as well. Now, from Proposition 2.25 we conclude that $H_a^{\infty,s}(L_p(\Omega))$ is metrizable.

*Step 3.* The proof of completeness is subdivided into two substeps.

*Substep 3.1.* We first show that for all $\ell \in \mathbb{N}$ the spaces $\left(H_a^{\ell,s}(L_p(\Omega)), n_\ell\right)$ are Banach spaces. This can either be seen by the use of standard arguments [9], or be proved directly. For the sake of completeness, a straightforward, direct proof is presented in the following. Therefore, let $(g_j)_{j\in\mathbb{N}} \subset H_a^{\ell,s}(L_p(\Omega))$ denote some arbitrary Cauchy sequence. Note that $\left\| g \,\middle|\, \mathcal{K}_{p,a}^\ell(\Omega) \right\| \leq n_\ell(g)$ for all $g \in H_a^{\ell,s}(L_p(\Omega))$, and hence $(g_j)_{j\in\mathbb{N}}$ is also a Cauchy sequence in $\mathcal{K}_{p,a}^\ell(\Omega)$. Now, from Remark 1.4(i) we know that $\mathcal{K}_{p,a}^\ell(\Omega)$ is a Banach space, and therefore

$$g_j \to g^{\mathcal{K}} \in \mathcal{K}_{p,a}^\ell(\Omega) \tag{5.2.10}$$

for $j \to \infty$. The same holds true with respect to $B_p^s(L_p(\Omega))$, i.e.,

$$g_j \to g^B \in B_p^s(L_p(\Omega)) \tag{5.2.11}$$

for $j \to \infty$, since $\left\| \cdot \,\middle|\, B_p^s(L_p(\Omega)) \right\| \leq n_\ell(\cdot)$ and $B_p^s(L_p(\Omega))$ is a Banach space, see Section 1.4. Next, from Remark 1.4(ii) we know that $\|\cdot \mid L_p(\Omega)\| \leq \left\| \cdot \,\middle|\, \mathcal{K}_{p,a}^\ell(\Omega) \right\|$, and the definition of the Besov norm immediately yields $\|\cdot \,|\, L_p(\Omega)\| \le \left\|\cdot \,\middle|\, B^s_p(L_p(\Omega))\right\|$. With these inequalities we estimate for $j \in \mathbb{N}$,

$$\begin{aligned}
0 &\le \left\| g^{\mathcal{K}} - g^B \,\middle|\, L_p(\Omega) \right\| \\
&\le \left\| g^{\mathcal{K}} - g_j \,\middle|\, L_p(\Omega) \right\| + \left\| g^B - g_j \,\middle|\, L_p(\Omega) \right\| \\
&\le \left\| g^{\mathcal{K}} - g_j \,\middle|\, \mathcal{K}^\ell_{p,a}(\Omega) \right\| + \left\| g^B - g_j \,\middle|\, B^s_p(L_p(\Omega)) \right\|,
\end{aligned}$$

and with (5.2.10) and (5.2.11) we conclude that $\left\| g^{\mathcal{K}} - g^B \,\middle|\, L_p(\Omega) \right\| = 0$, i.e., $g^{\mathcal{K}} = g^B$ in $L_p(\Omega)$. Thus, setting $\hat{g} := g^{\mathcal{K}} = g^B$, we have shown that $\hat{g} \in H^{\ell,s}_a(\Omega)$ with

$$n_\ell(g_j - \hat{g}) = \left\| g_j - \hat{g} \,\middle|\, \mathcal{K}^\ell_{p,a}(\Omega) \right\| + \left\| g_j - \hat{g} \,\middle|\, B^s_p(L_p(\Omega)) \right\| \underset{j\to\infty}{\longrightarrow} 0,$$

i.e., $g_j \to \hat{g}$ in $\left(H^{\ell,s}_a(L_p(\Omega)), n_\ell\right)$.

*Substep 3.2.* Finally, we prove completeness of $H^{\infty,s}_a(L_p(\Omega))$. Hence, let $(g_j)_{j\in\mathbb{N}} \subset H^{\infty,s}_a(L_p(\Omega))$ denote some arbitrary Cauchy sequence. At first, we show that $(g_j)_{j\in\mathbb{N}}$ is also a Cauchy sequence in $\left(H^{\ell,s}_a(L_p(\Omega)), n_\ell\right)$ for all $\ell \in \mathbb{N}$. Therefore, fix $\ell \in \mathbb{N}$ and note that $H^{\infty,s}_a(L_p(\Omega)) \subset H^{\ell,s}_a(L_p(\Omega))$. Let $U_{0,\ell}$ denote some arbitrary neighborhood of the origin in $\left(H^{\ell,s}_a(L_p(\Omega)), n_\ell\right)$. Consequently, there exists an open ball $\tilde{V}_{\ell,r}(0) = \{g \in H^{\ell,s}_a(L_p(\Omega)) \,|\, n_\ell(g) < r\}$ with $\tilde{V}_{\ell,r}(0) \subset U_{0,\ell}$. Now, consider the element $V_{\ell,r}(0)$ of the neighborhood basis of the origin in the LCTVS $H^{\infty,s}_a(L_p(\Omega))$, i.e.,

$$V_{\ell,r}(0) = \{g \in H^{\infty,s}_a(L_p(\Omega)) \,|\, n_\ell(g) < r\}.$$

Clearly, $V_{\ell,r}(0) \subset \tilde{V}_{\ell,r}(0)$, and since $(g_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $H^{\infty,s}_a(L_p(\Omega))$, there exists $N \in \mathbb{N}$ such that

$$g_m - g_j \in V_{\ell,r}(0) \subset \tilde{V}_{\ell,r}(0) \subset U_{0,\ell} \qquad \text{for all} \qquad m, j \ge N.$$

Thus, we have shown that $(g_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in the space $\left(H^{\ell,s}_a(L_p(\Omega)), n_\ell\right)$, which is complete according to Substep 3.1. Hence, there exists $g^\ell \in H^{\ell,s}_a(L_p(\Omega))$ with

$$g_j \underset{j\to\infty}{\longrightarrow} g^\ell, \tag{5.2.12}$$

i.e., for all neighborhoods $U_{0,\ell}$ of 0 in $H^{\ell,s}_a(L_p(\Omega))$ there exists $N(U_{0,\ell}) \in \mathbb{N}$ such that $g_m - g^\ell \in U_{0,\ell}$ for all $m \ge N(U_{0,\ell})$. Equivalently, for every $\varepsilon > 0$ there exists $N(\varepsilon, \ell) \in \mathbb{N}$ such that

$$n_\ell(g_m - g^\ell) < \varepsilon \qquad \text{for all} \qquad m \ge N(\varepsilon, \ell). \tag{5.2.13}$$

Now, since $\|\cdot \,|\, L_p(\Omega)\| \le n_1(\cdot) \le n_\ell(\cdot)$, for sufficiently large $m$ it holds

$$0 \le \left\| g^1 - g^\ell \,\middle|\, L_p(\Omega) \right\| \le n_1(g^1 - g^\ell) \le n_1(g^1 - g_m) + n_\ell(g^\ell - g_m) \le 2\varepsilon,$$

and we conclude that $\left\| g^1 - g^\ell \,\middle|\, L_p(\Omega) \right\| = n_1(g^1 - g^\ell) = 0$. As $\ell$ was arbitrary, setting $\hat{g} := g^1$ we have shown that $\hat{g} = g^\ell$ in $L_p(\Omega)$ for all $\ell \in \mathbb{N}$, and since $g^\ell \in H^{\ell,s}_a(L_p(\Omega))$, this shows $\hat{g} \in H^{\ell,s}_a(L_p(\Omega))$ for all $\ell \in \mathbb{N}$, i.e., $\hat{g} \in H^{\infty,s}_a(L_p(\Omega))$.

Finally, we show that $g_j \to \hat{g}$ in $H_a^{\infty,s}(L_p(\Omega))$ as $j \to \infty$. Therefore, it suffices to consider an arbitrary element $V_{\ell,r}(0)$ of the neighborhood basis of the origin in $H_a^{\infty,s}(L_p(\Omega))$. With (5.2.13) and the fact that $n_\ell(\hat{g} - g^\ell) = 0$, for sufficiently large $j$ we estimate

$$n_\ell(\hat{g} - g_j) \leq n_\ell(\hat{g} - g^\ell) + n_\ell(g^\ell - g_j) = n_\ell(g^\ell - g_j) < r,$$

i.e., $\hat{g} - g_j \in V_{\ell,r}(0)$. This shows that $g_j \to \hat{g}$ in $H_a^{\infty,s}(L_p(\Omega))$. □

Let us turn to the space $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$ as defined in (5.2.7). Note that for $\sigma > 2(p-1)/p$ it holds $\tau < 1$ such that then $q_\sigma = \|\cdot \,|\, B_\tau^\sigma(L_\tau(\Omega))\| \in \mathcal{Q}$ is only a *quasi*-norm. However, it turns out that $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$, equipped with the topology induced by $\mathcal{Q}$, is a metrizable and complete TVS, as stated by the following Proposition 5.8. The corresponding proof is quite similar to the one of Proposition 5.7; nevertheless, for the readers convenience it is presented below.

**Proposition 5.8.** *Let $\Omega \subset \mathbb{R}^2$ denote some bounded polygonal domain and $1 < p < \infty$. Then, the space $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$, equipped with the topology induced by the family of quasi-norms $\mathcal{Q}$, is a TVS. Furthermore, it is metrizable and complete, i.e., $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$ is an F-space.*

*Proof. Step 1.* From Proposition 2.22 it follows that $(B_{\mathrm{NL}}^{\infty}(L_p(\Omega)), \mathcal{Q})$ is a TVS.

*Step 2.* We prove metrizability, i.e., due to Proposition 2.25, we have to show that $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$ is a Hausdorff space and admits a countable neighborhood basis of the origin. The Hausdorff property follows from Remark 2.21. Recall that according to (2.3.3) the local basis of the origin is given by

$$U_{\Sigma_0,r}(0) = \bigcap_{\sigma \in \Sigma_0} V_{\sigma,r}(0), \qquad \Sigma_0 \subset \{\sigma \in \mathbb{R} \,|\, \sigma > 0\} \text{ finite, } r > 0,$$

where

$$V_{\sigma,r}(0) = \{g \in B_{\mathrm{NL}}^{\infty}(L_p(\Omega)) \,|\, q_\sigma(g) < r\}.$$

Now, consider the countable collection of sets

$$V_{m,1/j}(0), \quad m, j \in \mathbb{N}.$$

From Lemma 2.20 we know that for each of the local basis elements $V_{m,1/j}(0)$ there exists an open neighborhood $\tilde{V}_{m,1/j}(0)$ of the origin in $B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$ with $\tilde{V}_{m,1/j}(0) \subset V_{m,1/j}(0)$. We will show that the collection of open sets

$$\tilde{V}_{m,1/j}(0), \quad m, j \in \mathbb{N}, \tag{5.2.14}$$

indeed forms a neighborhood basis of the origin. Hence, let $U$ denote some arbitrary open neighborhood of 0. Then, since 0 is an inner point of $U$, due to Definition 2.5(ii) there exists a local basis element $U_{\Sigma_0,r}(0)$ such that

$$U_{\Sigma_0,r}(0) \subset U. \tag{5.2.15}$$

Now, from Lemma 1.9 we know that for $\sigma_1 > \sigma_0 > 0$ we have

$$B^{\sigma_1}_{\tau_1}(L_{\tau_1}(\Omega)) \hookrightarrow B^{\sigma_0}_{\tau_0}(L_{\tau_0}(\Omega)), \tag{5.2.16}$$

where $1/\tau_i = \sigma_i/2 + 1/p$ for $i \in \{0,1\}$, i.e.,

$$q_{\sigma_0}(g) = \left\| g \,\middle|\, B^{\sigma_0}_{\tau_0}(L_{\tau_0}(\Omega)) \right\| \leq C \left\| g \,\middle|\, B^{\sigma_1}_{\tau_1}(L_{\tau_1}(\Omega)) \right\| = C q_{\sigma_1}(g) \tag{5.2.17}$$

for some $C > 0$ and all $g \in B^{\sigma_1}_{\tau_1}(L_{\tau_1}(\Omega))$. Hence, fixing $m \in \mathbb{N}$ with $m \geq \max\{\sigma \,|\, \sigma \in \Sigma_0\}$, it follows from (5.2.17) that we can always find some $j > 0$ such that

$$V_{m,1/j}(0) \subset U_{\Sigma_0,r}(0).$$

Finally, with (5.2.15) and the fact that $\tilde{V}_{m,1/j}(0) \subset V_{m,1/j}(0)$, we conclude that $\tilde{V}_{m,1/j}(0) \subset U$. Thus, the sets (5.2.14) indeed form a countable neighborhood basis of the origin.

*Step 3.* We prove completeness of $B^{\infty}_{\mathrm{NL}}(L_p(\Omega))$. Hence, let $(g_j)_{j\in\mathbb{N}} \subset B^{\infty}_{\mathrm{NL}}(L_p(\Omega))$ denote some arbitrary Cauchy sequence. At first, we show that $(g_j)_{j\in\mathbb{N}}$ is also a Cauchy sequence in $(B^{\sigma}_{\tau}(L_\tau(\Omega)), q_\sigma)$ for all $\sigma > 0$. Therefore, fix $\sigma > 0$ and note that $B^{\infty}_{\mathrm{NL}}(L_p(\Omega)) \subset B^{\sigma}_{\tau}(L_\tau(\Omega))$. Let $U_{0,\sigma}$ denote some arbitrary neighborhood of the origin in $(B^{\sigma}_{\tau}(L_\tau(\Omega)), q_\sigma)$. Consequently, there exists a local basis element $\tilde{V}_{\sigma,r}(0) = \{g \in B^{\sigma}_{\tau}(L_\tau(\Omega)) \,\big|\, q_\sigma(g) < r\}$ with $\tilde{V}_{\sigma,r}(0) \subset U_{0,\sigma}$. Now, consider the element $V_{\sigma,r}(0)$ of the local basis of the origin in the TVS $B^{\infty}_{\mathrm{NL}}(L_p(\Omega))$, i.e.,

$$V_{\sigma,r}(0) = \{g \in B^{\infty}_{\mathrm{NL}}(L_p(\Omega)) \,|\, q_\sigma(g) < r\}.$$

Clearly, $V_{\sigma,r}(0) \subset \tilde{V}_{\sigma,r}(0)$, and from Lemma 2.20 we know that there exists an open set $W(0)$ in $B^{\infty}_{\mathrm{NL}}(L_p(\Omega))$ with $0 \in W(0) \subset V_{\sigma,r}(0)$. Hence,

$$W(0) \subset V_{\sigma,r}(0) \subset \tilde{V}_{\sigma,r}(0) \subset U_{0,\sigma},$$

and since $(g_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $B^{\infty}_{\mathrm{NL}}(L_p(\Omega))$, there exists $N \in \mathbb{N}$ such that

$$g_m - g_j \in W(0) \subset U_{0,\sigma} \qquad \text{for all} \qquad m, j \geq N.$$

Thus, we have shown that $(g_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in the space $(B^{\sigma}_{\tau}(L_\tau(\Omega)), q_\sigma)$. Since $B^{\sigma}_{\tau}(L_\tau(\Omega))$ is complete (see Section 1.4), there exists $g^\sigma \in B^{\sigma}_{\tau}(L_\tau(\Omega))$ with

$$g_j \underset{j\to\infty}{\longrightarrow} g^\sigma, \tag{5.2.18}$$

i.e., for all neighborhoods $U_{0,\sigma}$ of 0 in $B^{\sigma}_{\tau}(L_\tau(\Omega))$ there exists $N(U_{0,\sigma}) \in \mathbb{N}$ such that $g_m - g^\sigma \in U_{0,\sigma}$ for all $m \geq N(U_{0,\sigma})$. Equivalently, for every $\varepsilon > 0$ there exists $N(\varepsilon, \sigma) \in \mathbb{N}$ such that

$$q_\sigma(g_m - g^\sigma) < \varepsilon \tag{5.2.19}$$

for all $m \geq N(\varepsilon, \sigma)$. Now, from (5.2.17) we know that $q_1(\cdot) \leq C_\sigma q_\sigma(\cdot)$ for some $C_\sigma > 0$ and $\sigma > 1$. For the moment, assume that $\sigma > 1$. Then, for sufficiently large $m$ it holds

$$0 \leq q_1(g^1 - g^\sigma) \leq C\left(q_1(g^1 - g_m) + C_\sigma q_\sigma(g^\sigma - g_m)\right) \leq 2\varepsilon,$$

where $C \geq 1$ denotes the constant from the (quasi-)triangle inequality of $q_1$. We conclude that $q_1(g^1 - g^\sigma) = 0$. Since $\|\cdot \,|\, L_p(\Omega)\| \lesssim q_1(\cdot)$ (see Lemma 1.9), setting $\hat{g} := g^1$, we have shown that $\hat{g} = g^\sigma$ in $L_p(\Omega)$ for all $\sigma \geq 1$, and analogously one proves that $\hat{g} = g^\sigma$ in $L_p(\Omega)$ for all $0 < \sigma < 1$. Since $g^\sigma \in B^\sigma_\tau(L_\tau(\Omega)) \hookrightarrow L_p(\Omega)$, it holds that $\hat{g} \in B^\sigma_\tau(L_\tau(\Omega))$ for all $\sigma > 0$, i.e., $\hat{g} \in B^\infty_{\mathrm{NL}}(L_p(\Omega))$.

Finally, we show that $g_j \to \hat{g}$ in $B^\infty_{\mathrm{NL}}(L_p(\Omega))$ as $j \to \infty$. Since every neighborhood of the origin contains a local basis element of 0, it suffices to consider such an element $U_{\Sigma_0,r}(0)$. Let $\sigma \in \Sigma_0$. With (5.2.19) and the fact that $q_\sigma(\hat{g} - g^\sigma) = 0$, for sufficiently large $j$ we estimate

$$q_\sigma(\hat{g} - g_j) \leq C\left(q_\sigma(\hat{g} - g^\sigma) + q_\sigma(g^\sigma - g_j)\right) = C q_\sigma(g^\sigma - g_j) < r, \tag{5.2.20}$$

i.e., $\hat{g} - g_j \in V_{\sigma,r}(0)$. Since $\Sigma_0$ is finite, it holds that $\hat{g} - g_j \in U_{\Sigma_0,r}(0)$ for sufficiently large $j$. This shows that $g_j \to \hat{g}$ in $B^\infty_{\mathrm{NL}}(L_p(\Omega))$. □

From Corollary 5.4 we immediately conclude that $H^{\infty,s}_a(L_p(\Omega)) \subset B^\infty_{\mathrm{NL}}(L_p(\Omega))$. Now, with respect to the topologies on these spaces induced by the corresponding families of (quasi-)norms, this embedding is in fact continuous.

**Theorem 5.9.** *Let $\Omega \subset \mathbb{R}^2$ denote some bounded polygonal domain and $1 < p < \infty$. Furthermore, let $s > 0$ and $a > 0$. Then, the Fréchet space $H^{\infty,s}_a(L_p(\Omega))$ is continuously embedded into the F-space $B^\infty_{\mathrm{NL}}(L_p(\Omega))$, i.e.,*

$$H^{\infty,s}_a(L_p(\Omega)) \hookrightarrow B^\infty_{\mathrm{NL}}(L_p(\Omega)).$$

*Here, the (locally convex) topology of $H^{\infty,s}_a(L_p(\Omega))$ is induced by the family of norms $\mathcal{N}$, whereas the topology of $B^\infty_{\mathrm{NL}}(L_p(\Omega))$ is induced by the family of quasi-norms $\mathcal{Q}$.*

*Proof.* First note that $H^{\infty,s}_a(L_p(\Omega)) \subset B^\infty_{\mathrm{NL}}(L_p(\Omega))$ due to Corollary 5.4. To prove continuity of the identity map $\mathrm{id} : H^{\infty,s}_a(L_p(\Omega)) \to B^\infty_{\mathrm{NL}}(L_p(\Omega))$, from Proposition 2.23 we know that it suffices to show that for each $\sigma > 0$ there exists a finite subset $L_0 \subset \mathbb{N}$ and a constant $C > 0$, such that

$$q_\sigma(u) \leq C \max_{\ell \in L_0} n_\ell(u) \qquad \text{for all} \qquad u \in H^{\infty,s}_a(L_p(\Omega)),$$

i.e.,

$$\|u \,|\, B^\sigma_\tau(L_\tau(\Omega))\| \leq C \max_{\ell \in L_0} \left\| u \,\middle|\, H^{\ell,s}_a(L_p(\Omega)) \right\| \qquad \text{for all} \qquad u \in H^{\infty,s}_a(L_p(\Omega)),$$

where $1/\tau = \sigma/2 + 1/p$. Now, for given $\sigma > 0$, choose $\ell_0 \in \mathbb{N}$ with $\ell_0 > \sigma$ and set $L_0 := \{\ell_0\}$. Then the assertion follows from Corollary 5.4. □

**Remark 5.10.** Note that by continuous embeddings of TVSs, convergence of a sequence in the source space implies convergence in the target space. E.g., each Cauchy sequence in $H^{\infty,s}_a(L_p(\Omega))$ is also a Cauchy sequence in $B^\infty_{\mathrm{NL}}(L_p(\Omega))$, and since $H^{\infty,s}_a(L_p(\Omega))$ is complete (Proposition 5.7), it converges to an element in $B^\infty_{\mathrm{NL}}(L_p(\Omega))$. This observation leads to the following approach to derive regularity assertions for solutions to general (nonlinear) PDEs: Assume that little is known about the exact solution $u$ to some specific PDE in terms of classical (weighted) Hölder or

Sobolev regularity, except that it is contained in $L_p(\Omega)$. Now, if an $L_p$-convergent approximation scheme is known, where the approximants $u_n$ form a Cauchy sequence in some complete subspace $X \subset L_p$ (e.g. $X = H_a^{\infty,s}(L_p(\Omega))$) which is continuously embedded into some other function space $Y$ of interest (e.g. $Y = B_{\mathrm{NL}}^{\infty}(L_p(\Omega))$), then we know that the exact solution, i.e. the limit of $u_n$, is contained in $Y$. The advantage of this concept - compared to directly proving that $u \in Y$ - is that often the approximants $u_n$ are solutions to linear (sub-) problems, for which in general much more is known in regard of regularity compared to the solution of the nonlinear problem itself. In this way, the embedding Theorem 5.9 provides a tool for such kind of regularity proofs.

However, the derivation of our Besov regularity results in the next two chapters is not based on this approach. Instead, we will be able to apply the corresponding embeddings to solutions of the $p$-Poisson equation directly.

# Chapter 6

# Besov Regularity on Lipschitz Domains

This chapter is concerned with the regularity of solutions to the $p$-Poisson equation (3.1.2), $1 < p < \infty$, in the adaptivity scale of Besov spaces $B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$. In Section 6.1 we deal with the general case of multidimensional, bounded Lipschitz domains. The main result of this part, Theorem 6.5, describes (generic) sufficient conditions on the parameters of locally weighted Hölder spaces which ensure that the Besov regularity of all solutions $u$ to (3.1.2) that are contained in such spaces exceeds the Sobolev smoothness of $u$. Section 6.2 then is devoted to problems on two-dimensional domains, since there many more results concerning local Hölder regularity are available in the literature. Among other things, in this section, we state and prove explicit Besov regularity assertions for the unique solution to the $p$-Poisson equation (3.1.2), with a right-hand in $L_q(\Omega)$, $q \geq p'$, which satisfies a homogeneous Dirichlet boundary condition. These statements constitute the main results of the present chapter. In Theorem 6.14 we deal with general bounded Lipschitz domains $\Omega \subset \mathbb{R}^2$, whereas Theorem 6.17 contains the results for the special case of bounded polygonal domains. All results of this chapter have been primarily published in [30].

**Remark 6.1.** Note that, since we like to deal with bounded Lipschitz domains $\Omega$ and $q \geq p'$, the chain of embeddings

$$L_q(\Omega) \hookrightarrow L_{p'}(\Omega) \hookrightarrow W^{-1}(L_{p'}(\Omega))$$

together with Proposition 3.2 (applied for $g \equiv 0$) guarantees that there is at least one $u \in W^1(L_p(\Omega))$ that solves the $p$-Poisson equation (3.1.2) with $f \in L_q(\Omega)$.

In order to prove non-trivial Besov regularity results, we will make use of the general embedding Theorem 5.1. For that reason, we need to determine preferably small spaces $B^s_p(L_p(\Omega))$ and $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ which still contain the solution $u$ to the respective problem under consideration. Clearly, smoothness results w.r.t. the Besov scale $B^s_p(L_p(\Omega))$ can be derived easily from corresponding Sobolev regularity assertions using the intimate relation of Sobolev and Besov spaces described in Remark 1.6(iii).

## 6.1 The $p$-Poisson equation in arbitrary dimensions

Regularity results for partial differential equations are usually stated in terms of shift theorems. Concerning the $p$-Poisson equation with homogeneous Dirichlet boundary conditions (3.2.3), i.e.,

$$\begin{aligned} -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

and the scale of Sobolev spaces $W^s(L_p(\Omega))$ one such result is due to Savaré [116, Theorems 2 and 2']:

**Proposition 6.2** (Sobolev regularity on Lipschitz domains)**.** *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Given $1 < p < \infty$ and $f \in W^{-1}(L_{p'}(\Omega))$ let $u \in W_0^1(L_p(\Omega))$ denote the unique solution to (3.2.3). Then, for $\theta \in [0,1)$,*

$$f \in W^{t_\theta}(L_{p'}(\Omega)) \quad \textit{with} \quad t_\theta = \begin{cases} -1+\theta/2 & \textit{if} \quad 1 < p \leq 2, \\ -1+\theta/p' & \textit{if} \quad 2 < p < \infty \end{cases} \tag{6.1.1}$$

*implies that*

$$u \in W^{s_\theta}(L_p(\Omega)) \quad \textit{with} \quad s_\theta = \begin{cases} 1+\theta/2 & \textit{if} \quad 1 < p \leq 2, \\ 1+\theta/p & \textit{if} \quad 2 < p < \infty. \end{cases}$$

**Remark 6.3.** In [116, Remark 4.3] Savaré states that the regularity results given in Proposition 6.2 are sharp (in the sense defined in Section 3.3), even for the class of smooth domains.

Observe that $L_{p'}(\Omega) \hookrightarrow W^{t_\theta}(L_{p'}(\Omega))$ for all $\theta$ under consideration. Hence, provided that $f \in L_{p'}(\Omega)$, the preceding Proposition 6.2 shows that the unique solution to (3.2.3) is contained in $W^s(L_p(\Omega))$ and in $B^s_p(L_p(\Omega))$, respectively, for all $s < s^*$, where we set

$$s^* = \begin{cases} 3/2 & \text{if} \quad 1 < p \leq 2, \\ 1+1/p & \text{if} \quad 2 < p < \infty. \end{cases} \tag{6.1.2}$$

Moreover, let us mention that Savaré actually proved (for an even larger class of equations and slightly weaker assumptions on $f$) that we may replace $B^s_p(L_p(\Omega))$, $s < s^*$, by $B^{s^*}_\infty(L_p(\Omega))$. However, this slightly stronger assertion would not provide any gain in what follows.

**Remark 6.4.** In addition to Remark 6.3 we state that there are good reasons to assume that $s^*$ given in (6.1.2) defines a sharp bound for the Sobolev regularity of solutions $u$ to (3.2.3), even for much smoother right-hand sides $f$. First of all, this conjecture is supported by the well-known fact that there exist Lipschitz domains $\Omega$ such that the solution for $p = 2$ and some $f \in C^\infty(\overline{\Omega})$ does not belong to any $W^{3/2+\varepsilon}(L_2(\Omega))$, $\varepsilon > 0$; see, e.g., Jerison and Kenig [82, Theorem A]. Moreover, for

$d = 2$ and $p > 2$ it can be seen easily that $s^* = 1 + 1/p$ can not be improved for general Lipschitz domains, as the following example shows: Given $\omega \in (0, 2\pi)$ let

$$\mathfrak{C}(\omega) = \{(r, \theta) \in [0, \infty) \times [0, 2\pi] \,|\, 0 < r < 1 \text{ and } 0 < \theta < \omega\}$$

denote an (open) circular sector of radius 1 which is centered at the origin and possesses a central angle $\omega$. Then, by [57, Theorem 3] (see also Section 7.1 and [4]), there exist $\alpha(\omega) > 0$ which can be computed explicitly and some function $t$ such that, under quite mild conditions on the right-hand side $f$, for every solution $u$ to (3.2.3) in $\Omega = \mathfrak{C}(\omega)$ there exist a positive constant $k$ and a function $v$ such that

$$u(r, \theta) = k \cdot r^{\alpha(\omega)} t(\theta) + v(r, \theta), \qquad (r, \theta) \in \mathfrak{C}(\omega), \tag{6.1.3}$$

where $v$ fulfills

$$|v(r, \theta)| \lesssim r^{\alpha(\omega)+\eta} \qquad \text{and} \qquad |\nabla v(r, \theta)| \lesssim r^{\alpha(\omega)-1+\eta} \tag{6.1.4}$$

for some absolute constant $\eta > 0$. It follows from (6.1.3), (6.1.4), and the special structure of $t(\theta)$, cf. [57, Theorem 1], that $|\nabla u(r, \theta)| \sim r^{\alpha(\omega)-1}$ near the origin. Therefore $|\nabla u| \in L_\mu(\mathfrak{C}(\omega))$ can hold true only if $\mu \cdot (\alpha(\omega) - 1) > -2$. On the other hand, the behavior of $\alpha(\omega)$ for large central angles $\omega$, is known: It has been shown that

$$\lim_{\omega \to 2\pi} \alpha(\omega) = \frac{p-1}{p}. \tag{6.1.5}$$

Hence, by (6.1.5), for every $\mu > 2p$ there exists a two-dimensional Lipschitz domain $\Omega = \mathfrak{C}(\omega)$ and a solution $u$ to (3.2.3) such that $|\nabla u|$ does not belong to $L_\mu(\Omega)$. Consequently, for this solution Sobolev's embedding yields that $|\nabla u|$ is not contained in $W^{1/p+\varepsilon}(L_p(\Omega))$ for any $\varepsilon > 0$ and thus

$$u \notin W^{1+1/p+\varepsilon}(L_p(\Omega)). \tag{6.1.6}$$

Finally, let us remark that for the open circular sector with $\omega = 2\pi$ the same arguments yield (6.1.6) with $\varepsilon = 0$. However, note that then $\Omega = \mathfrak{C}(2\pi)$ is not a Lipschitz domain anymore.

Unfortunately, if $d \geq 3$, then (to the author's best knowledge) finding the sharp local Hölder regularity $\alpha$ of solutions to (3.1.2), (3.2.1), or (3.2.3), respectively, still is an open problem. Moreover, in the articles mentioned before the statement of Proposition 3.3, there appear too many unspecified constants that do not seem to allow estimates for the local Hölder semi-norms which are sufficient for our purposes, i.e., to obtain a satisfactory bound for the parameter $\gamma$. In contrast, for the case $d = 2$ much more explicit results are available such that these two drawbacks can be resolved. Consequently, we present a detailed discussion of the two-dimensional case in Section 6.2. To conclude the current subsection, at least we want to determine the *range* of the parameters $\alpha$ and $\gamma$ for which the Besov regularity of the solution $u$ (in the general multidimensional setting) *would* exceed its Sobolev regularity.

**Theorem 6.5.** *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ denote a bounded Lipschitz domain. Moreover, for $1 < p < \infty$ and $f \in W^{-1}(L_{p'}(\Omega))$ let $u$ be a weak solution to (3.1.2) which satisfies $u \in W^s(L_p(\Omega))$ for all $s < \overline{s} \in [\ell, \ell+1)$ with some $\ell \in \mathbb{N}$. If, additionally, $u$ is contained in $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ with*

$$\overline{s} - \ell < \alpha \leq 1 \qquad \text{and} \qquad 0 < \gamma < \ell + \alpha + \frac{1}{p} - \frac{d-1}{d}\,\overline{s}, \tag{6.1.7}$$

*then there exists $\overline{\sigma} > \overline{s}$ such that*

$$u \in B^\sigma_\tau(L_\tau(\Omega)) \qquad \text{for all} \qquad 0 < \sigma < \overline{\sigma} \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}.$$

Before proving Theorem 6.5 we stress that, according to Proposition 3.2, we know that there indeed exists $\overline{s} \geq 1$ such that all solutions $u$ to the $p$-Poisson equation (3.1.2) are contained in $W^s(L_p(\Omega))$ for all $s < \overline{s}$. Moreover, at least when dealing with homogeneous boundary conditions (i.e., solutions of (3.2.3)), it is reasonable to assume that $s \in [\ell, \ell+1)$ and that $u \in C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ with $\ell = 1$; see Remark 6.4 and Remark 3.4, respectively. Hence, Theorem 6.5 particularly describes a wide range of sufficient conditions which ensure that the Besov regularity $\sigma$ (measured in the adaptivity scale w.r.t. $L_p(\Omega)$) of solutions $u$ to (3.2.3) on bounded Lipschitz domains is strictly larger than its maximal Sobolev regularity $\overline{s}$. Moreover, we note that the upper bound $\overline{\sigma}$ can be calculated (from $p$, the regularity parameters $\ell$, $\alpha$, and $\gamma$, as well as the dimension $d$), as the following proof shows.

*Proof (of Theorem 6.5).* Since we assume that $u \in W^s(L_p(\Omega))$, $s < \overline{s}$, standard embeddings (cf. Remark 1.6(iii)) imply that $u \in B^s_p(L_p(\Omega))$ for all $s \in (0, \overline{s})$. Then, for general $0 < \alpha \leq 1$ and $0 < \gamma < \ell+\alpha+1/p$, our embedding result (Theorem 5.1) states that the additional assumption $u \in C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ yields $u \in B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$, for all

$$0 < \sigma < \min\left\{\sigma^*, \frac{d}{d-1}\,(\overline{s} - \varepsilon)\right\} =: \overline{\sigma},$$

where $\varepsilon > 0$ can be chosen arbitrarily small and $\sigma^*$ depends on $d$, $p$, $\ell$, $\alpha$, and $\gamma$, as described in (5.1.1). Thus, the maximal Besov regularity (w.r.t. the adaptivity scale) $\overline{\sigma}$ of the solution $u$ exceeds its maximal Sobolev regularity $\overline{s}$ provided that $\sigma^* > \overline{s}$. Due to (5.1.1), this is the case if $\alpha$ and $\gamma$ satisfy

$$\ell + \alpha > \overline{s} \quad \text{and} \quad 0 < \gamma < \frac{\ell+\alpha}{d} + \frac{1}{p},$$

or if

$$\frac{d}{d-1}\left(\ell + \alpha + \frac{1}{p} - \gamma\right) > \overline{s} \quad \text{and} \quad \frac{\ell+\alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p}. \tag{6.1.8}$$

Now the first inequality in (6.1.8) is equivalent to $\gamma < \ell + \alpha + 1/p - \overline{s}\,(d-1)/d$ such that (6.1.8) reduces to

$$\frac{\ell+\alpha}{d} + \frac{1}{p} \leq \gamma < \ell + \alpha + \frac{1}{p} - \frac{d-1}{d}\,\overline{s}.$$

This range for $\gamma$ is non-empty if and only if $\ell + \alpha > \overline{s}$. In summary, the condition $\ell + \alpha > \overline{s}$ is necessary in both cases and the union of the two ranges for $\gamma$ yields that $\sigma^* > \overline{s}$ for all values of $\alpha$ and $\gamma$ satisfying (6.1.7), as claimed. □

## 6.2 The $p$-Poisson equation in two dimensions

As mentioned earlier, in order to derive non-trivial Besov regularity results by means of Theorem 5.1, we need to determine (preferably small) spaces $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ which contain the solutions $u$ to the $p$-Poisson equation (3.1.2); see Section 1.1 for the definition of these spaces. For this purpose we proceed as follows. Starting from a known local Hölder regularity result, we estimate the Hölder semi-norms $|u|_{C^{\ell,\alpha}(K)}$ on compact subsets $K \subset\subset \Omega$ in terms of $\delta_K$, in order to conclude estimates on the parameter $\gamma$. In what follows we restrict ourselves to the situation $d = 2$, because in this case explicit bounds on the (local) Hölder regularity are available in the literature. In particular, quite recently Kuusi and Mingione [92] and Lindgren and Lindqvist [98] have proven a lower bound for the Hölder exponent of solutions to (3.1.2) with right-hand side $f \in L_q(\Omega)$, $q > 2$; see Proposition 6.8 below.

**Remark 6.6.** We note in passing that we have $L_q(\Omega) \hookrightarrow W^{-1}(L_{p'}(\Omega))$ in dimension two, provided that $2/q < 1 + 2/p'$. Hence, Proposition 3.2 guarantees that the problem (3.1.2) is solvable for all $1 < p < \infty$ and $q > 2$.

The subsequent definition is inspired by [98].

**Definition 6.7.** *Let us define the local Hölder exponent $\alpha^*_q = \alpha^*_q(p)$ for $2 < q \leq \infty$ by*

- *$1 < p \leq 2$: If $q = \infty$, let $\alpha^*_q$ be any number less than 1, and if $q < \infty$, let*

$$\alpha^*_q = 1 - \frac{2}{q}.$$

- *$2 < p < \infty$: If $q = \infty$, let $\alpha^*_q = 1/(p-1)$, and if $q < \infty$, let*

$$\alpha^*_q = \frac{1 - 2/q}{p - 1}.$$

The results of Kuusi and Mingione [92] and Lindgren and Lindqvist [98] can be summarized as follows.

**Proposition 6.8.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and let $1 < p < \infty$. For $2 < q \leq \infty$, let $f \in L_q(\Omega)$ and set $\alpha = \alpha^*_q$ as specified in Definition 6.7. Moreover, let $u \in W^1(L_p(\Omega))$ be a solution to (3.1.2). Then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ and for any compact set $K \subset \Omega$, it holds*

$$|u|_{C^{1,\alpha}(K)} \leq C(q, p, \alpha, K) \max\Big\{ \|f \mid L_q(\Omega)\|^{1/(p-1)}, \|u \mid L_\infty(\Omega)\| \Big\}. \tag{6.2.1}$$

*Proof.* Everything except the case $p > 2$ and $q = \infty$ is contained in [98]. In the remaining case Lindgren and Lindqvist only show membership in $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for any $\alpha < 1/(1-p)$. The limiting case $\alpha = 1/(1-p)$ however follows from the sharper pointwise estimates [92, (1.38)] and the characterization of Hölder spaces by Campanato spaces. □

**Remark 6.9.** It is known that the Hölder exponent $\alpha_q^*$ defined above is sharp, at least for $p > 2$ and $2 < q \leq \infty$. If $q = \infty$, then this follows from the example given in Remark 3.4. Corresponding examples for finite $q$ can be found in [98].

Based on the local Hölder regularity result given in Proposition 6.8, we are able to show that, for $\alpha = \alpha_q^*$ and certain values of $\gamma$, solutions to the $p$-Poisson equation (3.1.2) are contained in locally *weighted* Hölder spaces $C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$, too; see Proposition 6.11 below. To do so, we have to examine the dependence of the constant $C(q, p, \alpha, K)$ in (6.2.1) on $K \subset\subset \Omega$. This is performed in the subsequent lemma.

**Lemma 6.10.** *Let the assumptions of Proposition 6.8 be satisfied. Then, for every closed disc $\overline{B_{r/4}} \subset \Omega$ of radius $r/4 > 0$ such that $B_{2r}$ is contained in $\Omega$ as well, we have*

$$|u|_{C^{1,\alpha}(\overline{B_{r/4}})} \leq C(q,p,\alpha,\Omega)\, r^{-\alpha-1} \max\Big\{\|f \,|\, L_q(B_r)\|^{1/(p-1)}, \|u \,|\, L_\infty(B_r)\|\Big\} \quad (6.2.2)$$

*and, for $t > 2$,*

$$|u|_{C^{1,\alpha}(\overline{B_{r/4}})} \leq \hat{C}(q,p,\alpha,\Omega,t)\, r^{-\alpha-2/t} \max\Big\{\|f \,|\, L_q(B_{2r})\|^{1/(p-1)}, \|\nabla u \,|\, L_t(B_{2r})\|\Big\}. \quad (6.2.3)$$

*Proof.* To show the claim, assume that $u$ solves (3.1.2) on the whole domain $\Omega$ and let $\overline{B_r(x_0)} \subset \Omega$ denote a disc of radius $r > 0$ around an arbitrary point $x_0$. Then, certainly, $u$ is a solution of the restricted problem $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f$ in $B_r(x_0)$, as well. Moreover, from Proposition 6.8 we infer that $u$ belongs to $C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ with $\alpha = \alpha_q^*$ given in Definition 6.7. Hence, in particular $u \in L_\infty(B_r(x_0))$.

Now let us perform a translation to the origin. One checks easily that then $\tilde{u} = u(\cdot + x_0)$ solves

$$-\mathrm{div}\Big(|\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\Big) = \tilde{f} \quad \text{in} \quad B_r(0),$$

where $\tilde{f} = f(\cdot + x_0)$. Thus, it suffices to prove (6.2.2) and (6.2.3) only for solutions to the $p$-Poisson equation (3.1.2) in $B_r(0)$, $r > 0$.

To do so, we use a result for the unit disc $B_1(0)$. By Proposition 6.8, with $\Omega = B_1(0)$ and $K = \overline{B_{1/4}(0)}$, we know that if $u$ solves $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f$ in $B_1(0)$ with $u \in L_\infty(B_1(0))$ and $f \in L_q(B_1(0))$, then there exists a constant $C = C(q,p,\alpha) > 0$, such that for all $x, y \in \overline{B_{1/4}(0)}$ it holds

$$|\nabla u(x) - \nabla u(y)| \leq C|x-y|^\alpha \max\Big\{\|f \,|\, L_q(B_1(0))\|^{1/(p-1)}, \|u \,|\, L_\infty(B_1(0))\|\Big\}. \quad (6.2.4)$$

Now suppose that $u$ solves $-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f$ in some dilated disc $B_r(0)$ and let $F = r^p f(r\cdot)$. Then it is easy to see that $U = u(r\cdot)$ solves

$$-\mathrm{div}\Big(|\nabla U|^{p-2}\nabla U\Big) = F \quad \text{in} \quad B_1(0).$$

Clearly, it holds that $\|F \,|\, L_q(B_1(0))\| = r^{p-2/q} \|f \,|\, L_q(B_r(0))\|$ and $\|U \,|\, L_\infty(B_1(0))\| = \|u \,|\, L_\infty(B_r(0))\|$. Next, we apply the estimate (6.2.4) to $U$ which yields that for all

$x, y \in \overline{B_{1/4}(0)}$

$$\begin{aligned}&|\nabla u(rx) - \nabla u(ry)|\\&\quad = r^{-1}\,|\nabla U(x) - \nabla U(y)|\\&\quad \le C\,r^{-1}\,|x-y|^{\alpha}\,\max\Big\{\|F\,|\,L_q(B_1(0))\|^{1/(p-1)}\,,\|U\,|\,L_\infty(B_1(0))\|\Big\}\\&\quad \le C\,r^{-1-\alpha}\,|rx-ry|^{\alpha}\,\max\Big\{r^{(p-2/q)/(p-1)}\,\|f\,|\,L_q(B_r(0))\|^{1/(p-1)}\,,\|u\,|\,L_\infty(B_r(0))\|\Big\}\,.\end{aligned}$$

Hence, for all $x \neq y$ in $\overline{B_{r/4}(0)}$ it holds

$$\begin{aligned}&\frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\alpha}}\\&\qquad \le C\,r^{-1-\alpha}\,\max\Big\{r^{(p-2/q)/(p-1)}\,\|f\,|\,L_q(B_r(0))\|^{1/(p-1)}\,,\|u\,|\,L_\infty(B_r(0))\|\Big\}\\&\qquad \le \tilde{C}\,r^{-1-\alpha}\,\max\Big\{\|f\,|\,L_q(B_r(0))\|^{1/(p-1)}\,,\|u\,|\,L_\infty(B_r(0))\|\Big\}\,,\end{aligned} \tag{6.2.5}$$

where $\tilde{C} = C\cdot\max\Big\{1, \mathrm{diam}(\Omega)^{(p-2/q)/(p-1)}\Big\}$ and $(p-2/q)/(p-1) > 0$, since $2/q < 1 < p$. This shows (6.2.2) for all discs $\overline{B_{r/4}(0)}$ under consideration.

We are left with the proof of (6.2.3) for these discs. Note that if $u$ solves (3.1.2), so does $u - c$ for every constant $c$. Hence, from (6.2.5) we infer

$$\begin{aligned}&\frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\alpha}}\\&\qquad \le C\,r^{-1-\alpha}\,\max\Big\{r^{(p-2/q)/(p-1)}\,\|f\,|\,L_q(B_r(0))\|^{1/(p-1)}\,,\|u-c\,|\,L_\infty(B_r(0))\|\Big\}\,,\end{aligned} \tag{6.2.6}$$

whenever $x \neq y$ belong to $\overline{B_{r/4}(0)}$. Next we apply Whitney's estimate (see Proposition A.1) with $k = 1$, $d = 2$, $p = \infty$, and $q = t$. Thus, for every $t > d = 2$ and every square $Q \subset \Omega$, there exist constants $c$ and $C'$, such that

$$\|u - c\,|\,L_\infty(Q)\| \le C'\;|Q|^{1/2-1/t}\;|u|_{W^1(L_t(Q))}\,. \tag{6.2.7}$$

Let $Q_r$ denote the square in $\mathbb{R}^2$ with sides parallel to the coordinate axes and side length $2r$ that contains $B_r(0)$. Using the fact that $|Q_r|^{1/2-1/t} = (2r)^{1-2/t}$, from (6.2.7) we conclude

$$\|u - c\,|\,L_\infty(B_r(0))\| \le C'\;|Q_r|^{1/2-1/t}\;|u|_{W^1(L_t(Q_r))} \le C''\,r^{1-2/t}\;\|\nabla u\,|\,L_t(B_{2r}(0))\| \tag{6.2.8}$$

Now, (6.2.6) and (6.2.8) together yield the upper bound

$$\begin{aligned}&\frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\alpha}}\\&\le CC''\,r^{-2/t-\alpha}\,\max\Big\{r^{-1+2/t+(p-2/q)/(p-1)}\,\|f\,|\,L_q(B_r(0))\|^{1/(p-1)}\,,\|\nabla u\,|\,L_t(B_{2r}(0))\|\Big\}\,.\end{aligned}$$

Since, clearly,

$$-1+\frac{2}{t}+\frac{p-2/q}{p-1}=\frac{2}{t}+\frac{1-2/q}{p-1}>0,$$

by setting $\hat{C}=C\cdot C''\cdot\max\Big\{1,\operatorname{diam}(\Omega)^{2/t+(1-2/q)/(p-1)}\Big\}$ we finally arrive at

$$\frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^{\alpha}}\leq \hat{C}\,r^{-2/t-\alpha}\,\max\Big\{\|f\,|\,L_q(B_{2r}(0))\|^{1/(p-1)},\|\nabla u\,|\,L_t(B_{2r}(0))\|\Big\}$$

for all $x\neq y$ in $\overline{B_{r/4}(0)}$. This shows (6.2.3) for all discs of interest. □

The locally weighted Hölder regularity result which forms the basis for our further analysis now can be derived from (6.2.3):

**Proposition 6.11** ($C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ regularity)**.** *Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain and assume $1<p<\infty$. Furthermore, for $2<q\leq\infty$ and $f\in L_q(\Omega)$, let $u\in W^1(L_p(\Omega))$ be some solution to the p-Poisson equation* (3.1.2) *and set $\alpha=\alpha^*_q$ as in Definition 6.7.*

**(i)** *If $|\nabla u|\in L_t(\Omega)$ for some $t>2$, then we have*

$$u\in C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega)\qquad \text{for}\qquad \alpha=\alpha^*_q, \tag{6.2.9}$$

*as well as every weight parameter $\gamma\geq\alpha+2/t$.*

**(ii)** *If $u\in W^s(L_p(\Omega))$ for all $s<\overline{s}$ with some $\overline{s}>\max\{2/p,1\}$, then (6.2.9) holds true for all*

$$\gamma>\alpha+\max\left\{0,1-\overline{s}+\frac{2}{p}\right\}.$$

*Proof.* Let us first prove part (i). Since the locally weighted Hölder spaces $C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega)=C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega;\mathcal{K}(c))$ are monotone in $\gamma$ (see Remark 1.1), we may restrict ourselves to the limiting case $\gamma=\alpha+2/t$. Moreover, without loss of generality, we can assume $c>8$; cf. Section 1.1. Then let us consider a compact disc $\overline{B_r}\in\mathcal{K}(c)$, i.e., $\overline{B_r}=\overline{B_r(x_0)}$ with $x_0\in\Omega$ and $r>0$ such that the (open) disc $B_{cr}(x_0)$ still is contained in $\Omega$. Clearly, $r<\operatorname{dist}(x_0,\partial\Omega)/8$, so that we can choose $R\geq r$ with

$$\frac{\operatorname{dist}(x_0,\partial\Omega)}{16}<R<\frac{\operatorname{dist}(x_0,\partial\Omega)}{8}.$$

Consequently, $\overline{B_R}=\overline{B_R(x_0)}$ is a compact disc with $\overline{B_r}\subseteq\overline{B_R}\subset B_{8R}\subset\Omega$. Therefore, (6.2.3) applied for $\overline{B_R}$ yields

$$|u|_{C^{1,\alpha}(\overline{B_r})}\leq|u|_{C^{1,\alpha}(\overline{B_R})}\leq C\,R^{-\alpha-2/t}\,\max\left\{\|f\,|\,L_q(B_{8R})\|^{1/(p-1)},\|\nabla u\,|\,L_t(B_{8R})\|\right\},$$

where $C=C(q,p,\alpha,\Omega)$ does not depend on $r$. Since $\delta_{\overline{B_r}}<\operatorname{dist}(x_0,\partial\Omega)<16\,R$ and $\gamma=\alpha+2/t$, setting $C'=C\cdot 16^{\gamma}$ we may estimate further

$$|u|_{C^{1,\alpha}(\overline{B_r})}\leq C'\,\delta_{\overline{B_r}}^{-\gamma}\max\left\{\|f\,|\,L_q(\Omega)\|^{1/(p-1)},\|\nabla u\,|\,L_t(\Omega)\|\right\}.$$

Observe that the latter maximum is finite due to the additional assumption that $|\nabla u|$ belongs to $L_t(\Omega)$. Multiplying by $\delta_{\overline{B_r}}^{\gamma}$ and taking the supremum over all $\overline{B_r} \in \mathcal{K}(c)$ thus proves the claim stated in (i).

The proof of (ii) follows from Sobolev's embedding: At first, note that $\overline{s} > 2/p$ yields that $1 > \max\{0, 1 - \overline{s} + 2/p\}$. Therefore, we can choose $s < \overline{s}$ and $t > 2$ such that $2/t > \max\{0, 1 - s + 2/p\}$ is arbitrary close to $\max\{0, 1 - \overline{s} + 2/p\}$. Thus, in view of (6.2.9), it remains to show that $|\nabla u| \in L_t(\Omega)$ for this choice of $s$ and $t$. To do so, observe that $s - 1 > 2/p - 2/t$. Since we imposed the additional condition that $\overline{s} > 1$, we may assume that $s - 1 > 0$. Hence, it follows

$$s - 1 > 2 \cdot \max\left\{0, \frac{1}{p} - \frac{1}{t}\right\}$$

which particularly implies the embedding $W^{s-1}(L_p(\Omega)) \hookrightarrow L_t(\Omega)$. Finally, the fact that $u \in W^s(L_p(\Omega))$ yields $|\nabla u| \in W^{s-1}(L_p(\Omega))$ completes the proof. □

Next let us combine the locally weighted Hölder regularity result obtained in Proposition 6.11 above with the generic Besov regularity result stated in Theorem 6.5. This leads to conditions on the Sobolev smoothness of solutions $u$ to the $p$-Poisson equation (3.1.2) which imply (non-trivial) Besov regularity assertions for these $u$.

**Theorem 6.12.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume $1 < p < \infty$. Moreover, for $2 < q \leq \infty$, as well as $f \in L_q(\Omega)$, let $u$ be some solution to the $p$-Poisson equation (3.1.2) which satisfies $u \in W^s(L_p(\Omega))$ for all $s < \overline{s}$. Then the conditions*

- •*)* $1 < p \leq 2$ *and* $\frac{2}{p} < \overline{s} < 2 - \frac{2}{q}$,
- •*)* $2 < p < \infty$ *and* $1 < \overline{s} < 1 + \frac{1-2/q}{p-1}$

*imply that there exists $\overline{\sigma} > \overline{s}$ such that*

$$u \in B^{\sigma}_{\tau}(L_\tau(\Omega)) \qquad \text{for all} \qquad 0 < \sigma < \overline{\sigma} \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}. \tag{6.2.10}$$

*Proof.* Note that our assumptions particularly imply

$$\max\left\{1, \frac{2}{p}\right\} < \overline{s} < 2. \tag{6.2.11}$$

Therefore, in view of Theorem 6.5 (applied with $d = 2$ and $\ell = 1$), it suffices to find parameters $\alpha$ and $\gamma$ with $\overline{s} - 1 < \alpha \leq 1$ and

$$0 < \gamma < 1 + \alpha + \frac{1}{p} - \frac{\overline{s}}{2} \tag{6.2.12}$$

such that $u \in C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$. Observe that from (6.2.11) it follows

$$\alpha + \max\left\{0, 1 - \overline{s} + \frac{2}{p}\right\} < 1 + \alpha + \frac{1}{p} - \frac{\overline{s}}{2} \qquad \text{for all} \qquad 0 < \alpha \leq 1.$$

Thus, due to Proposition 6.11(ii), choosing $\alpha = \alpha_q^*$ (as given in Definition 6.7), there exists $\gamma$ which satisfies (6.2.12) such that $u \in C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$. To complete the proof, it remains to check that this choice of $\alpha$ belongs to the interval $(\overline{s}-1, 1]$ which is obvious in view of Definition 6.7, as well as our restrictions on $\overline{s}$. □

**Remark 6.13.** Note that the bound $\overline{\sigma}$ in Theorem 6.12 can be calculated explicitly, provided that the maximal Sobolev regularity $\overline{s}$ is known; see, e.g., the proof of Theorem 6.14 below.

Now we are well-prepared to state and prove one of the main results of this thesis. It shows that for a large range of parameters $p$ and $q$ the (unique) solution to (3.2.3), i.e., to the $p$-Poisson with homogeneous Dirichlet boundary conditions, has a significantly higher Besov regularity compared to its Sobolev smoothness. Indeed, as we shall see, on bounded Lipschitz domains $\Omega \subset \mathbb{R}^2$ this happens whenever $4/3 < p < \infty$ and $\max\{4, 2p\} < q \leq \infty$. Therefore, for the same range of parameters, the application of adaptive (wavelet) algorithms for the numerical treatment of (3.2.3) is completely justified. Recall that from Proposition 6.2 (and the subsequent remarks) it follows that the solution $u$ to this problem is contained in $W^s(L_p(\Omega))$ for all $s < s^*$ given in (6.1.2). Consequently, the proof of the subsequent result is obtained by applying Theorem 6.12 with $\overline{s} = s^*$ together with some straightforward calculations.

**Theorem 6.14** (Besov regularity on Lipschitz domains in 2D)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $1 < p < \infty$, as well as $f \in L_q(\Omega)$ with $2 < q \leq \infty$ and $q \geq p'$. Then the unique solution $u$ to the $p$-Poisson equation with homogeneous Dirichlet boundary conditions* (3.2.3) *satisfies*

$$u \in B^\sigma_\tau(L_\tau(\Omega)) \qquad \textit{for all} \qquad 0 < \sigma < \overline{\sigma} \qquad \textit{and} \qquad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p},$$

*where*

$$\overline{\sigma} = \begin{cases} \frac{3}{2} & \textit{if} \quad 1 < p < 4/3 \textit{ and } p' \leq q \leq \infty, \\ \frac{3}{2} & \textit{if} \quad p = 4/3 \textit{ and } 4 < q \leq \infty, \\ 3 - \frac{2}{p} & \textit{if} \quad 4/3 < p \leq 2 \textit{ and } (\frac{1}{p} - \frac{1}{2})^{-1} \leq q \leq \infty, \\ 2 - \frac{2}{q} & \textit{if} \quad 4/3 < p \leq 2 \textit{ and } 4 < q < (\frac{1}{p} - \frac{1}{2})^{-1}, \\ \frac{3}{2} & \textit{if} \quad 4/3 \leq p < 2 \textit{ and } p' \leq q \leq 4, \\ \frac{3}{2} & \textit{if} \quad p = 2 \textit{ and } 2 < q \leq 4, \\ 1 + \frac{1-2/q}{p-1} & \textit{if} \quad 2 < p < \infty \textit{ and } 2p < q \leq \infty, \\ 1 + \frac{1}{p} & \textit{if} \quad 2 < p < \infty \textit{ and } 2 < q \leq 2p. \end{cases}$$

*Proof. Step 1.* Let us start with the cases where $\overline{\sigma} = s^*$, i.e., where $\overline{\sigma}$ equals $3/2$ or $1 + 1/p$. Then from classical embeddings of Besov spaces it follows that $u \in W^s(L_p(\Omega))$ for all $0 < s < \overline{s}$ implies that $u$ also belongs to $B^s_p(L_p(\Omega))$ for all these $s$ which in turn yields the claim; cf. Remark 1.6(iii).

*Step 2.* We are left with proving the assertion for the third, fourth, and seventh line in the definition of $\overline{\sigma}$. According to (the proof of) Theorem 6.12 we know that in all these remaining cases Proposition 6.11(ii) ensures the existence of some reasonably small $\gamma$ such that $u \in C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$, where $\alpha = \alpha_q^*$ (as given in Definition 6.7) and $\ell = 1$. In fact, it can be checked that we can use

$$\gamma = \alpha + \varepsilon + \begin{cases} 2/p - 1/2 & \text{if} \quad p < 2, \\ 1/p, & \text{if} \quad p \geq 2 \end{cases}$$

with arbitrarily small $\varepsilon > 0$. As shown in the proof of Theorem 6.5 (which we used to derive Theorem 6.12), the desired quantity $\overline{\sigma}$ then is given by $\sigma^*$ defined in (5.1.1) in Theorem 5.1. Thus, we need to determine whether our choice of $\gamma$ is smaller or larger than $(1+\alpha)/2 + 1/p$. Note that, according to Theorem 6.5, we already know that for all cases of interest it is smaller than $1 + \alpha + 1/p$. It turns out that for $4/3 < p \leq 2$ and $(1/p - 1/2)^{-1} \leq q \leq \infty$, i.e., for the constellation described in the third line, the second case in (5.1.1) applies, i.e., then

$$\frac{1+\alpha}{2} + \frac{1}{p} \leq \gamma < 1 + \alpha + \frac{1}{p}.$$

Consequently, for these $p$ and $q$, the quantity $\overline{\sigma} = \sigma^*$ is given by $2(1 + \alpha + 1/p - \gamma) = 3 - 2/p - \varepsilon$, where $\varepsilon$ can be neglected since it can be chosen arbitrarily small.

For the remaining two ranges for $p$ and $q$ the chosen weight $\gamma$ is small enough such that the first case in (5.1.1) applies. Thus, for $p$ and $q$ as described in the fourth and seventh line, we obtain $\overline{\sigma} = \sigma^* = \ell + \alpha$ with $\ell = 1$ and $\alpha = \alpha_q^*$. This finishes the proof. □

In the more restrictive (but practically more important) setting of polygonal domains slightly better Besov regularity assertions for the unique solutions to (3.2.3) with $f \in L_q(\Omega)$ can be deduced using our method, at least for some cases. For this purpose, we will employ a further Sobolev regularity result which was shown by Ebmeyer [59, Corollary 2.3] for polyhedral Lipschitz domains in arbitrary dimensions:

**Proposition 6.15.** *For $d \geq 2$ let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral Lipschitz domain and for $1 < p < \infty$ let $f \in L_{p'}(\Omega)$. Then the unique solution $u \in W^1(L_p(\Omega))$ to (3.2.3) satisfies*

$$|\nabla u| \in L_t(\Omega) \qquad \textit{for all} \qquad t < \frac{d}{d-1}\, p.$$

**Remark 6.16.** The example described in Remark 6.4 shows that, for $d = 2$, Ebmeyer's result (Proposition 6.15) is sharp, meaning that there are cases in which

$$|\nabla u| \notin L_t(\Omega) \qquad \text{if} \qquad t > 2p = \frac{d}{d-1}\, p.$$

Our improved Besov regularity result for solutions to $p$-Poisson equations with homogeneous boundary conditions (3.2.3) on bounded polygonal domains then reads as follows.

**Theorem 6.17** (Besov regularity on polygonal domains)**.** *Let $\Omega \subset \mathbb{R}^2$ denote a bounded polygonal domain and let $1 < p < \infty$, as well as $f \in L_q(\Omega)$ with $2 < q \leq \infty$*

*and $q \geq p'$. Then the unique solution $u$ to the $p$-Poisson equation with homogeneous Dirichlet boundary conditions* (3.2.3) *satisfies*

$$u \in B^\sigma_\tau(L_\tau(\Omega)) \qquad \text{for all} \qquad 0 < \sigma < \overline{\sigma} \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p},$$

*where*

$$\overline{\sigma} = \begin{cases} 2 - \frac{2}{q} & \text{if} \quad 1 < p < 4/3 \text{ and } p' \leq q \leq \infty, \\ 2 - \frac{2}{q} & \text{if} \quad p = 4/3 \text{ and } 4 < q \leq \infty, \\ 2 - \frac{2}{q} & \text{if} \quad 4/3 < p \leq 2 \text{ and } 4 < q \leq \infty, \\ \frac{3}{2} & \text{if} \quad 4/3 \leq p < 2 \text{ and } p' \leq q \leq 4, \\ \frac{3}{2} & \text{if} \quad p = 2 \text{ and } 2 < q \leq 4, \\ 1 + \frac{1-2/q}{p-1} & \text{if} \quad 2 < p < \infty \text{ and } 2p < q \leq \infty, \\ 1 + \frac{1}{p} & \text{if} \quad 2 < p < \infty \text{ and } 2 < q \leq 2p. \end{cases}$$

Before giving the proof of this assertion we want to stress that in the first three cases, as well as in the sixth one, the upper bound $\overline{\sigma}$ for the regularity of the solution $u$ in the adaptivity scale of Besov spaces is strictly larger than $\overline{s} = s^*$ as defined in (6.1.2) which is considered to be a sharp bound for the regularity in the Sobolev scale; see Remark 6.4. Hence, in contrast to Theorem 6.14 (which deals with general bounded Lipschitz domains in $\mathbb{R}^2$), on polygonal domains $u$ gains some additional regularity also in the range $1 < p \leq 4/3$ (except for the case $p = 4/3$ and $q = 4$). Furthermore, observe that for the case of $p \in (4/3, 2)$ and large $q$ the value $3 - 2/p$ for Lipschitz domains is strictly worse than $2 - 2/q$ obtained in Theorem 6.17 for polygonal domains. Finally we note that, given some fixed $p$, in all cases in which $\overline{\sigma} > \overline{s}$ this quantity grows with increasing integrability $q$ of the right-hand side $f$. This is not the case for $s^*$. Accordingly, the largest gain $\overline{\sigma} - \overline{s}$ is obtained for $f \in L_\infty(\Omega)$. This situation is illustrated in Figure 6.1 below.

*Proof (of Theorem 6.17). Step 1.* Since $q \geq p'$, we have that $L_q(\Omega) \hookrightarrow L_{p'}(\Omega) \hookrightarrow W^{-1}(L_{p'}(\Omega))$, see Lemma A.8. Consequently, Proposition 3.2 assures a unique solution $u \in W^1(L_p(\Omega))$. Then Remark 1.6(iii) implies $u \in B^{1-\varepsilon}_p(L_p(\Omega))$ for all $\varepsilon \in (0,1)$. Moreover, by Proposition 6.11(i) we know that $u \in C^{1,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ for all $\gamma \geq \alpha + 2/t$, with $\alpha = \alpha^*_q$ given in Definition 6.7 and $t > 2$ such that $|\nabla u| \in L_t(\Omega)$. Proposition 6.15 shows that the latter condition is fulfilled for all $t < 2p$, i.e., for all $2/t$ strictly larger (but arbitrary close to) $1/p$. Thus, since $\alpha \in (0,1)$, we can choose $\gamma$ such that

$$\alpha + \frac{1}{p} < \gamma < \frac{1+\alpha}{2} + \frac{1}{p}.$$

Then, for this choice of $\alpha$ and $\gamma$, as well as $d = 2$, $s = 1 - \varepsilon$, and $\ell = 1$, we apply Theorem 5.1 (note that every polygonal domain $\Omega \subset \mathbb{R}^2$ is Lipschitz!) and conclude that $u$ belongs to $B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$, for all

$$0 < \sigma < \min\left\{1 + \alpha, \frac{2}{2-1}\,(1-\varepsilon)\right\} = 1 + \alpha,$$

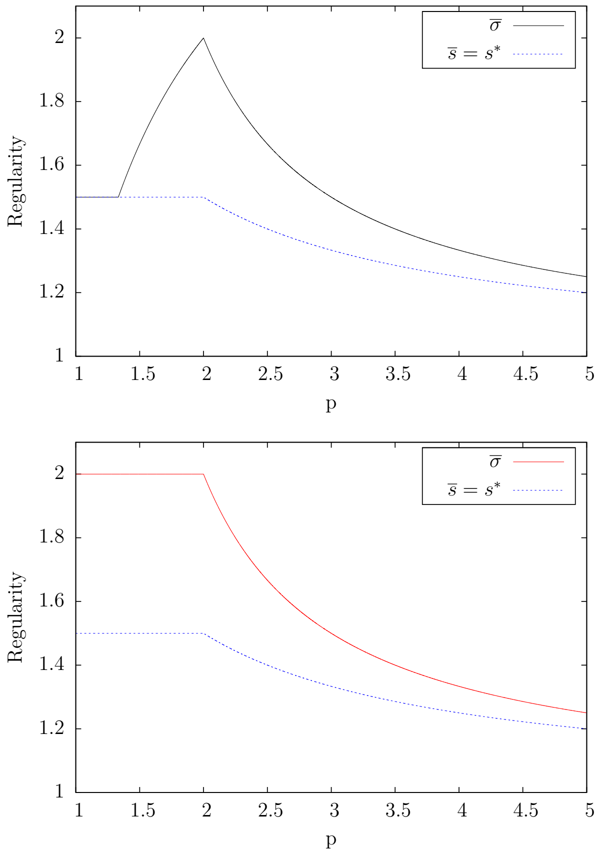



Figure 6.1: Bounds $\overline{\sigma}$ and $s^*$ for the regularity of solutions $u$ to (3.2.3) with $f \in L_\infty(\Omega)$ on bounded 2D Lipschitz domains (top) and bounded polygonal domains (bottom), measured in $B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$, and in $W^s(L_p(\Omega))$, respectively.

where the last equality holds provided that $\varepsilon > 0$ is chosen sufficiently small.

*Step 2.* Since $f \in L_{p'}(\Omega)$, we furthermore can employ Proposition 6.2 (as well as the subsequent remarks) to see that $u \in W^s(L_p(\Omega))$ for all $s < s^*$. This implies that $u$ belongs to $B^s_p(L_p(\Omega))$ and $B^\sigma_\tau(L_\tau(\Omega))$ for all $s$ and $\sigma$ less than $s^*$, respectively.

In conclusion, combining both steps yields

$$u \in B^\sigma_\tau(L_\tau(\Omega)) \qquad \text{for all} \qquad 0 < \sigma < \max\{1+\alpha, s^*\} \qquad \text{and} \qquad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}.$$

Now the claim directly follows from the definitions of $\alpha = \alpha^*_q$ and $s^*$. □

**Remark 6.18.** We add some comments on our main results in Theorem 6.14 and 6.17, respectively:

**(i)** The restriction $q \geq p'$ in Theorem 6.14 can be weakened. Anyhow, note that for $p$ in the vicinity of 1 and $q$ close to 2, Proposition 6.2 only guarantees that the unique solution $u$ to (3.2.3) satisfies $u \in W^s(L_p(\Omega))$ for all $s < \overline{s}$ with some $1 \leq \overline{s} < s^*$.

**(ii)** According to [59, Section 5.3] Proposition 6.15 remains valid for special classes of bounded Lipschitz domains with polyhedral structure. Hence, also Theorem 6.17 applies to this slightly generalized situation.

**(iii)** Observe that for large $q$ our bound $\overline{\sigma}$ in Theorem 6.17 always equals $1+\alpha$, where $\alpha = \alpha_q^*$ is the local Hölder exponent given in Definition 6.7 which is known to be optimal at least for $p > 2$; see Remark 6.9. Thus, by (5.1.1), as well as the subsequent statements, we see that the results stated in Theorem 6.17 are the best possible we can achieve by our method (i.e., by Theorem 5.1). On the other hand, we do not know whether they are sharp, as (for general $p$) in the current literature there seem to exist no results at all which address comparable regularity questions. However, for example in the case of the classical Laplacian ($p = 2$) Besov regularity larger than two cannot be expected for general right-hand sides of smoothness zero, since then we deal with a linear operator of order two.

Finally, let us briefly consider *p-harmonic* functions, i.e., solutions to the *p-Laplace equation*

$$\operatorname{div}\Big(|\nabla u|^{p-2}\, \nabla u\Big) = 0 \qquad \text{in} \quad \Omega, \tag{6.2.13}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain and $1 < p < \infty$. In [59, Remark 2.5(iv)] Ebmeyer states that if $\Omega$ is a bounded polyhedral Lipschitz domain (of arbitrary dimension $d \geq 2$), then all solutions to (6.2.13) with boundary data $g \in W^1(L_p(\partial\Omega))$ are as well contained in $W^s(L_p(\Omega))$ for all $s < s^*$ defined by (6.1.2). However, he does not provide a proof of this statement. Using this claim, the arguments in Step 1 of the proof of Theorem 6.17 would imply that all $p$-harmonic functions $u$ on bounded polygonal domains $\Omega$ satisfy

$$u \in B^\sigma_\tau(L_\tau(\Omega)) \quad \text{for all} \quad 0 < \sigma < \begin{cases} 2 & \text{if} \quad 1 < p \leq 2, \\ 1 + \frac{1}{p-1} & \text{if} \quad 2 < p < \infty \end{cases} \quad \text{and} \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}. \tag{6.2.14}$$

In addition, we remark that the local Hölder regularity of two-dimensional $p$-harmonic functions is known to be higher than for general solutions to the $p$-Poisson equation (3.1.2): In fact, Iwaniec and Manfredi [81] showed that in the case $d = 2$ all $p$-harmonic functions are contained in $C^{\ell,\alpha}_{\mathrm{loc}}(\Omega)$, where $\ell \in \mathbb{N}$ and $0 < \alpha \leq 1$ are determined by the formula

$$\ell + \alpha = 1 + \frac{1}{6}\left(1 + \frac{1}{p-1} + \sqrt{1 + \frac{14}{p-1} + \frac{1}{(p-1)^2}}\right). \tag{6.2.15}$$

Furthermore, for $p \neq 2$ this result is known to be sharp; see [81]. Note that for all $1 < p < \infty$ the right-hand side of (6.2.15) indeed is larger than $1 + \alpha^*_\infty$. In conclusion, one might expect to achieve even higher Besov regularity for $p$-harmonic functions than stated in (6.2.14). To prove this conjecture (by means of our embedding result Theorem 5.1), we would need to exploit the sharp Hölder regularity (6.2.15) instead of Proposition 6.8; provided we could show that $p$-harmonic functions belong to $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ for these $\ell$ and $\alpha$, as well as for sufficiently small values of $\gamma$, and provided that Ebmeyer's claim holds true. Unfortunately, sufficient estimates for the parameter $\gamma$ do not seem to exist, yet.

# Chapter 7

# Besov Regularity in the Vicinity of a Corner

In this chapter we study the Besov smoothness of solutions to the $p$-Poisson equation (3.1.2), measured in the adaptivity scale of Besov spaces, in the vicinity of a corner of a polygonal domain. The results of this chapter stem from [78].

To describe the scope of problems considered here, let us introduce the following notation first. For a function $f$ (in Cartesian coordinates $x$), by $\tilde{f}$ we denote its representation in polar coordinates, i.e., $\tilde{f}(r, \phi) := f(\Xi^{-1}(r, \phi))$, where $\Xi$ denotes the corresponding transformation of coordinates, see (A.2.1). When appropriate, we will omit the transformation $\Xi$ or $\Xi^{-1}$, and just write $f$ or $\tilde{f}$ for the representation in Cartesian or polar coordinates, respectively. Analogously, the same applies to domains, i.e., $\tilde{\Omega} := \Xi(\Omega)$. Recall that $B_R(x) \subset \mathbb{R}^2$ denotes the open Euclidean ball with radius $R > 0$ centered at $x \in \mathbb{R}^2$.

Now, let $\Omega \subset \mathbb{R}^2$ be some bounded polygonal domain and let $x_0 \in \partial\Omega$ denote some arbitrary but fixed corner of $\Omega$. Furthermore, let $R_0 > 0$ be sufficiently small such that

$$\mathfrak{C}_{R_0}(x_0) := B_{R_0}(x_0) \cap \Omega$$

is congruent to some cone

$$\tilde{C}(R_0, \omega) = \{(r, \phi) \in (0, R_0) \times (0, \omega)\} \tag{7.0.1}$$

of radius $R_0$ and inner angle $0 < \omega < 2\pi$. As part of the boundary, we denote the straight sides of $\partial\mathfrak{C}_{R_0}(x_0)$ by $S_{R_0}(x_0)$, i.e.,

$$S_{R_0}(x_0) := B_{R_0}(x_0) \cap \partial\Omega,$$

see Figure 7.1.

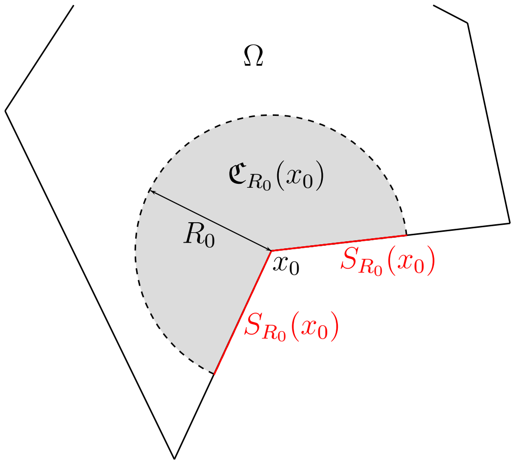



Figure 7.1: Polygonal domain $\Omega$ and cone $\mathfrak{C}_{R_0}(x_0)$ with sides $S_{R_0}(x_0)$.

Then, for $1 < p < \infty$, we consider problem (3.2.1), i.e., the problem of finding $u \in W^1(L_p(\Omega))$ with

$$\begin{aligned} -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

We assume that

$$g \in W^{1-1/p}(L_p(\partial\Omega)) \quad \text{and} \quad f \in L_q(\Omega) \quad \text{with} \quad \begin{cases} \frac{2p}{3p-2} \le q \le \infty, & \text{if } 1 < p < 2, \\ 1 < q \le \infty, & \text{if } \ p = 2, \\ 1 \le q \le \infty, & \text{if } 2 < p < \infty, \end{cases} \tag{7.0.2}$$

satisfy

$$f \ge 0 \quad \text{and} \quad g \ge 0 \quad \text{pointwise almost everywhere.} \tag{7.0.3}$$

**Remark 7.1.**

**(i)** Problem (3.2.1), (7.0.2) ,(7.0.3) admits a uniquely determined weak solution $u \in W^1(L_p(\Omega))$. To see this, first note that (7.0.2) implies $L_q(\Omega) \hookrightarrow W^{-1}(L_{p'}(\Omega))$, see Lemma A.8. Furthermore, by Proposition 1.2 there exists a continuous extension operator $E : W^{1-1/p}(L_p(\partial\Omega)) \to W^1(L_p(\Omega))$. Now, for $f \in W^{-1}(L_{p'}(\Omega))$ and Dirichlet boundary conditions given by $g \in W^1(L_p(\Omega))$ in the sense that $u - g \in W_0^1(L_p(\Omega))$, we know from Proposition 3.2 that the $p$-Poisson equation admits a unique weak solution.

**(ii)** Clearly, $q = 2$ would be a sufficient condition for (7.0.2) to hold true for all $1 < p < \infty$. Hence, for simplicity, one may impose the (less general) condition $f \in L_2(\Omega)$.

**(iii)** Using the weak comparison principle (Proposition 3.5) the assumption (7.0.3) assures that $u \ge 0$ in $\Omega$. We treat only nonnegative solutions because the type of results - concerning the (growth-) behavior in the vicinity of a conical boundary point - needed for our proofs below are known only for solutions which are nonnegative in $\mathfrak{C}_{R_0}(x_0)$. Hence, all results remain valid if we drop assumption (7.0.3) and instead require the solution $u$ to be nonnegative a.e. in $\mathfrak{C}_{R_0}(x_0)$.

This chapter is organized as follows. In Section 7.1 we collect some known facts about the singular expansion of solutions to the $p$-Poisson equation in a cone. Afterwards, in Section 7.2 we treat problem (3.2.1), (7.0.2), (7.0.3) under the additional assumption that $f = 0$ in $\mathfrak{C}_{R_0}(x_0)$ and $g = 0$ on $S_{R_0}(x_0)$. Our main result here (Theorem 7.12) will show that then the solution $u$ possesses *arbitrary high* Besov regularity in $\mathfrak{C}_R(x_0)$ for some $0 < R < R_0$. Finally, in Section 7.3 we prove that for the more general case of local growth conditions on $f$ in $\mathfrak{C}_{R_0}(x_0)$ it holds $u \in B^\sigma_\tau(L_\tau(\mathfrak{C}_R(x_0)))$, $1/\tau = \sigma/2 + 1/p$, at least for all $0 < \sigma < 2$ (see Theorem 7.18).

## 7.1 Singular expansions

In this section we recall some known facts about the singular expansion of solutions to the $p$-Poisson equation in a cone. In the following, for notational convenience, we consider the unit cone $C(1,\omega) \subset \mathbb{R}^2$ with opening angle $0 < \omega < 2\pi$, see (7.0.1). However, we remark that all results are valid as well for cones $C(R_0,\omega)$ of arbitrary radius $R_0 > 0$. In order to describe these expansion results, we first consider the $p$-Laplace equation

$$\begin{aligned} -\operatorname{div}\Big(|\nabla u|^{p-2}\nabla u\Big) &= 0 \qquad \text{in } C(1,\omega),\\ u &= 0 \qquad \text{on } S(1,\omega), \end{aligned} \tag{7.1.1}$$

where $S(1,\omega)$ in polar coordinates is given by $\tilde{S}(1,\omega) = [0,1) \times \{0,\omega\}$. First of all, let us look for (strong) solutions of the form $u = s$ with

$$\tilde{s}(r,\phi) := r^\alpha t(\phi), \qquad (r,\phi) \in \tilde{C}(1,\omega), \tag{7.1.2}$$

where $\alpha \in \mathbb{R}$ and $t \in C^2((0,\omega))$. From Proposition A.17 we know that these solutions satisfy the nonlinear eigenvalue problem

$$\begin{aligned} &\frac{\partial}{\partial\phi}\left\{\left[\alpha^2 t(\phi)^2 + \left(\frac{\partial t}{\partial\phi}(\phi)\right)^2\right]^{(p-2)/2} \frac{\partial t}{\partial\phi}(\phi)\right\}\\ &\quad + \alpha\left(\alpha\left(p-1\right)+2-p\right)\left[\alpha^2 t(\phi)^2 + \left(\frac{\partial t}{\partial\phi}(\phi)\right)^2\right]^{(p-2)/2} t(\phi) \;=\; 0 \quad \text{for all } \phi \in (0,\omega),\\ &t(0) = t(\omega) = 0, \end{aligned} \tag{7.1.3}$$

and vice versa, for solutions $(\alpha, t(\cdot))$ of (7.1.3), the function $u = s$ given by (7.1.2) solves the primary problem (7.1.1). The first positive eigenvalue $\alpha$ of (7.1.3) is given by the following lemma, see [57, Theorem 1].

**Lemma 7.2.** *Let $1 < p < \infty$ and $\omega \in (0, 2\pi]$. Moreover, for $\omega \neq 2\pi$ let*

$$\Theta(\Gamma, p) := \frac{(\Gamma - 1)p - 2\Gamma}{2\Gamma(p-1)}, \qquad \textit{where} \qquad \Gamma(\omega) := \left(\frac{\omega}{\pi} - 1\right)^2 - 1.$$

*Then, for*

$$\alpha := \alpha(\omega, p) := \begin{cases} \Theta(\Gamma, p) + \sqrt{\Theta(\Gamma, p)^2 + \Gamma^{-1}} & \textit{if} \quad \omega \leq \pi, \\ \Theta(\Gamma, p) - \sqrt{\Theta(\Gamma, p)^2 + \Gamma^{-1}} & \textit{if} \quad \pi \leq \omega < 2\pi, \\ (p-1)/p & \textit{if} \quad \omega = 2\pi, \end{cases} \tag{7.1.4}$$

*there exists a solution $t \in C^2([0, \omega])$ of* (7.1.3) *with $t > 0$ in $(0, \omega)$. Further, it holds*

$$\alpha > \max\{0, (p-2)/(p-1)\} \tag{7.1.5}$$

*and*

$$t(\phi)^2 + t'(\phi)^2 > 0 \qquad \textit{for all} \qquad \phi \in [0, \omega]. \tag{7.1.6}$$

*Any two positive solutions $t_1, t_2$ corresponding to $\alpha$ which satisfy* (7.1.6) *are scalar multiples of each other.*

**Remark 7.3.** A short computation yields that for all $1 < p < \infty$ it holds

$$\begin{aligned} \alpha(\omega, p) > 1 \quad &\Longleftrightarrow \quad 0 < \omega < \pi, \\ 0 < \alpha(\omega, p) < 1 \quad &\Longleftrightarrow \quad \pi < \omega \leq 2\pi, \end{aligned}$$

and hence $\alpha(\omega, p) = 1$ if and only if $\omega = \pi$. Moreover, for $\omega \neq \pi$ it holds

$$\lim_{p \to 1} \alpha(\omega, p) = \begin{cases} \infty & \text{if} \quad 0 < \omega < \pi, \\ 0 & \text{if} \quad \pi < \omega \leq 2\pi, \end{cases} \qquad \lim_{p \to \infty} \alpha(\omega, p) = \begin{cases} -1/\Gamma & \text{if} \quad 0 < \omega < \pi, \\ 1 & \text{if} \quad \pi < \omega \leq 2\pi. \end{cases}$$

The positive solution $t(\cdot)$ of problem (7.1.3) from Lemma 7.2 (corresponding to $\alpha$ given by (7.1.4)) turns out to be arbitrarily smooth. For a proof of this result, see [126, Theorem 2.1.1 & Corollary 2.1]. An alternative, more elementary proof based on a construction of $t(\cdot)$ is presented in Appendix A, see Subsection A.4.2.

**Lemma 7.4.** *Let $(\alpha, t(\cdot))$ be the solution of the nonlinear eigenvalue problem* (7.1.3) *according to Lemma 7.2. It holds $t \in C^\infty((0, \omega))$, as well as $t \in C^i([0, \omega])$ for all $i \in \mathbb{N}$.*

Now we have everything at hand to describe the above mentioned singular expansion results for solutions to the $p$-Poisson equation in a cone. We will consider both the homogeneous and the inhomogeneous case. In regard of the homogeneous problem (7.1.1), the following expansion result by Tolksdorf ([126, Theorem 1.3]) holds true.

**Proposition 7.5.** *For $1 < p < \infty$ and $\omega \in (0, 2\pi)$ let $s$ be given by* (7.1.2) *with $(\alpha, t(\cdot))$ according to Lemma 7.2. Then for every nonnegative (pointwise a.e.) solution $u \in W^1(L_p(C(1,\omega)))$ to* (7.1.1) *there exists some $0 < R < 1$, such that either*

$$u = 0 \qquad \text{in} \qquad C(R,\omega),$$

*or there exists $\kappa > 0$ such that for all $\nu \in \mathbb{N}_0^2$ it holds*

$$D^\nu u(x) = \kappa D^\nu s(x) + w_\nu(x) \qquad \text{for a.e.} \qquad x \in C(R,\omega),$$

*with*

$$|w_\nu(x)| < c_\nu \, |x|^{\alpha-|\nu|} \qquad \text{for some} \quad c_\nu > 0 \quad \text{and all} \quad x \in C(R,\omega). \tag{7.1.7}$$

*Moreover, for the functions $w_\nu$ it holds*

$$\lim_{|x|\to 0} \frac{|w_\nu(x)|}{|x|^{\alpha-|\nu|}} = 0$$

*in $C(R,\omega)$.*

Next, let us consider the inhomogeneous problem

$$\begin{aligned} -\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) &= f \qquad \text{in } C(1,\omega), \\ u &= 0 \qquad \text{on } S(1,\omega). \end{aligned} \tag{7.1.8}$$

We will use an expansion result due to Dobrowolski, see [57, Theorem 3], originally formulated for the case of a general domain containing a conical boundary point. However, the corresponding proof is based solely on considerations in a neighborhood of that point, by reducing the global problem to a related one on the cone at the beginning. Hence, we may reformulate the assertion [57, Theorem 3] for $C(1,\omega)$ as follows.

**Proposition 7.6.** *For $1 < p < \infty$ and $\omega \in (0, 2\pi)$ let $s$ be given by* (7.1.2) *with $(\alpha, t(\cdot))$ according to Lemma 7.2. Assume that for some $c > 0$ and $\gamma > \gamma_0 := (\alpha - 1)(p-1) - 1$ it holds*

$$0 \le f(x) \le c\,|x|^\gamma \qquad \text{for a.e.} \qquad x \in C(1,\omega). \tag{7.1.9}$$

*Then for every nonnegative (pointwise a.e.) solution $u \in W^1(L_p(C(1,\omega)))$ to* (7.1.8) *there exists some $0 < R < 1$ such that either*

$$u = 0 \qquad \text{in} \qquad C(R,\omega),$$

*or there exist constants $\kappa, c', \eta > 0$ such that $u$ admits the singular expansion*

$$u(x) = \kappa s(x) + w(x) \qquad \text{for a.e.} \qquad x \in C(R,\omega) \tag{7.1.10}$$

*with a remainder $w$ that satisfies*

$$|w(x)| \le c'\,|x|^{\alpha+\eta} \quad \text{and} \quad |\nabla w(x)| \le c'\,|x|^{\alpha-1+\eta} \quad \text{for all} \quad x \in C(R,\omega).$$

*The maximum $\eta > 0$ depends on $\gamma$ and the eigenvalue problem* (7.1.3).

**Remark 7.7.**

**(i)** For re-entrant corners, i.e. $\omega > \pi$, it holds $\gamma_0 < -1$, since $0 < \alpha < 1$ in this case, see Remark 7.3.

**(ii)** Note that the local growth condition (7.1.9) implies $f \in L_q(C(1,\omega))$ for all

$$\begin{cases} 1 \leq q < -2/\gamma, & \text{if } \gamma < 0, \\ 1 \leq q \leq \infty, & \text{if } \gamma \geq 0. \end{cases} \tag{7.1.11}$$

In case $\gamma < 0$, we can use $\alpha > \max\{(p-2)/(p-1), 0\}$, see (7.1.5), to estimate

$$-\frac{2}{\gamma} > \frac{-2}{(\alpha-1)(p-1)-1} > \begin{cases} 2/p, & \text{if } 1 < p \leq 2, \\ 1, & \text{if } 2 < p < \infty. \end{cases}$$

Since $2/p \geq 2p/(3p-2)$ for all $1 < p \leq 2$, we see that in all cases we can find some $q$ with (7.1.11) which satisfies condition (7.0.2). Hence, by Remark 7.1(i) it particularly follows that problem (7.1.8), (7.1.9) admits at least one solution $u \in W^1(L_p(C(1,\omega)))$, e.g., for the boundary condition $u = 0$ on the whole boundary $\partial C(1,\omega)$.

**(iii)** Retracing Dobrowolski's proof of Proposition 7.6, see [57], one checks that for the remainder $w$ in (7.1.10) it even holds

$$w \in \mathcal{W}^2_{q,\beta}(C(R,\omega)) \qquad \text{for all} \qquad 2 < q < \infty \tag{7.1.12}$$

and all $\beta$ with

$$-\min\left\{\alpha+\delta, \mathrm{Re}\beta^{(2)}\right\} + 2 - \frac{2}{q} < \beta < -\alpha + 2 - \frac{2}{q},$$

where $\delta := \gamma - (\alpha-1)(p-1)+1 > 0$. Here, $\beta^{(2)}$ denotes the second eigenvalue of the eigenvalue problem (7.1.3), where all eigenvalues are ordered by increasing real parts. It holds $0 < \mathrm{Re}\beta^{(1)} = \beta^{(1)} = \alpha < \mathrm{Re}\beta^{(2)}$, see [57, Lemma 4]. Recall that for the definition of the weighted Sobolev spaces $\mathcal{W}^2_{q,\beta}(C(R,\omega))$ in (7.1.12), the set $S \subset \partial C(R,\omega)$ corresponding to the smooth distance function $\rho : C(R,\omega) \to [0,1]$ has to be fixed, see Section 1.3. Here, as an exception (cf. Remark 1.5), $S$ consists only of the apex of the cone, i.e., $S = \{0\}$. Since $0 < R < 1$ we may then simply set

$$\rho(x) := |x|, \qquad x \in C(R,\omega), \tag{7.1.13}$$

for the spaces $\mathcal{W}^2_{q,\beta}(C(R,\omega))$ in (7.1.12).

## 7.2 Besov regularity: Locally vanishing source term

In order to discuss regularity properties of solutions to the $p$-Poisson equation (3.2.1), (7.0.2), (7.0.3) with right-hand sides $f, g$ that vanish in the vicinity of a corner, we first consider the corresponding problem (7.1.1) on the unit cone. With the help of the singular expansion results of the last section we are able to prove the following Babuska-Kondratiev regularity assertion.

**Proposition 7.8.** *Let* $1 < p < \infty$ *and* $\omega \in (0, 2\pi)$, *and let* $u \in W^1(L_p(C(1,\omega)))$ *be a nonnegative solution of* (7.1.1). *Then there exists some* $0 < R < 1$, *such that*

$$u \in \mathcal{K}^{\ell}_{p,a}(C(R,\omega)) \qquad \textit{for all} \qquad \ell \in \mathbb{N} \quad \textit{and} \quad 0 \leq a < \frac{2}{p}. \tag{7.2.1}$$

*Proof.* Clearly, the assertion holds true if $u = 0$ in $C(R,\omega)$ for some $0 < R < 1$. Hence, let us assume that $u \neq 0$ in the vicinity of the origin, and let $(\alpha, t(\cdot))$ be the solution of the eigenvalue problem (7.1.3) according to Lemma 7.2. For the rest of this proof let $\ell \in \mathbb{N}$ be arbitrary but fixed.

*Step 1.* From Proposition 7.5 we know that there exists some $0 < R < 1$ and $\kappa > 0$, such that for all $\nu \in \mathbb{N}_0^2$ it holds

$$D^\nu u(x) = \kappa D^\nu s(x) + w_\nu(x) \qquad \text{for a.e.} \qquad x \in C(R,\omega),$$

where $w_\nu$ satisfies (7.1.7). Since $s(x) = \tilde{s}(\Xi(x))$ with $\tilde{s}(r,\phi) = r^\alpha t(\phi)$, this means

$$D^\nu u(x) = \kappa D^\nu \left(|x|^\alpha t\left(\Xi_\phi(x)\right)\right) + w_\nu(x). \tag{7.2.2}$$

At first, with Leibniz' rule for higher order (weak) partial derivatives we compute

$$D^\nu \left(|x|^\alpha t\left(\Xi_\phi(x)\right)\right) = \sum_{\beta \leq \nu} \binom{\nu}{\beta} D^\beta \left(|x|^\alpha\right) D^{\nu-\beta} t\left(\Xi_\phi(x)\right),$$

where

$$\binom{\nu}{\beta} = \binom{\nu_1}{\beta_1} \cdot \binom{\nu_2}{\beta_2} \qquad \text{for} \qquad \beta \leq \nu,$$

and with the help of Lemma A.11, for arbitrary $\nu \in \mathbb{N}_0^2$ with $|\nu| \leq \ell$ we estimate

$$\left|D^\nu \left(|x|^\alpha t\left(\Xi_\phi(x)\right)\right)\right| \lesssim \sum_{\beta \leq \nu} |x|^{\alpha - |\beta|} \left|D^{\nu-\beta} t(\Xi_\phi(x))\right|. \tag{7.2.3}$$

Furthermore, with Lemma A.12 and Lemma 7.4 we find

$$\begin{aligned}
\left|D^{\nu-\beta} t(\Xi_\phi(x))\right| &\leq \sum_{k=1}^{|\nu-\beta|} \sum_{j_1+j_2=|\nu-\beta|} \left|c_{\nu-\beta,k,j_1,j_2} t^{(k)}(\Xi_\phi(x))\right| \frac{|x|^{j_1+j_2}}{|x|^{2|\nu-\beta|}} \\
&\lesssim \sum_{k=1}^{|\nu-\beta|} \sum_{j_1+j_2=|\nu-\beta|} \left\|t \,\middle|\, C^k([0,\omega])\right\| |x|^{-|\nu-\beta|} \\
&\lesssim \left\|t \,\middle|\, C^{|\nu-\beta|}([0,\omega])\right\| |x|^{-|\nu|+|\beta|}. 
\end{aligned} \tag{7.2.4}$$

Now, the estimates (7.2.3) and (7.2.4) yield

$$\left|D^\nu(|x|^\alpha t(\Xi_\phi(x)))\right| \lesssim \left\|t \,\middle|\, C^{|\nu|}([0,\omega])\right\| \sum_{\beta \leq \nu} |x|^{\alpha-|\nu|} \lesssim |x|^{\alpha-|\nu|},$$

so that from (7.1.7) and (7.2.2) we conclude

$$|D^\nu u(x)| \lesssim |x|^{\alpha-|\nu|} \qquad \text{for a.e.} \qquad x \in C(R,\omega).$$

Hence, for the $\mathcal{K}^\ell_{p,a}$-norm of $u$, see Section 1.3, it holds

$$\begin{aligned} \left\| u \,\middle|\, \mathcal{K}^\ell_{p,a}(C(R,\omega)) \right\|^p &= \sum_{|\nu|\le \ell} \int_{C(R,\omega)} \left| \rho(x)^{|\nu|-a} D^\nu u(x) \right|^p \,\mathrm{d}x \\ &\lesssim \sum_{|\nu|\le \ell} \int_{C(R,\omega)} \left( \rho(x)^{|\nu|-a} \, |x|^{\alpha-|\nu|} \right)^p \,\mathrm{d}x. \end{aligned} \tag{7.2.5}$$

*Step 2.* To estimate the terms occuring in (7.2.5), we will use the following partition of $C(R,\omega)$. By $x_i, i \in \{0,1,2\}$, we denote the vertices of $C(R,\omega)$, where $x_0 = (0,0)$. Then we set $U(x_i) := C(R,\omega) \cap B_\varepsilon(x_i), i \in \{0,1,2\}$, and choose $\varepsilon \in (0,R)$ sufficiently small such that $\rho(x) \sim |x - x_i|$ in $U(x_i)$ for all $i \in \{0,1,2\}$. Furthermore, we set $U^c := C(R,\omega) \setminus \{\cup_{i=0}^2 U(x_i)\}$ and $r_{\min} := \inf_{x\in U^c} \rho(x) > 0$. Now, since $r_{\min} \le \rho(x) \le 1$ for all $x \in U^c$, for arbitrary $\nu \in \mathbb{N}_0^2$ and $a \in \mathbb{R}$ it clearly holds

$$\rho(x)^{|\nu|-a} \le \max\left\{1, r_{\min}^{|\nu|-a}\right\} \qquad \text{for all} \qquad x \in U^c. \tag{7.2.6}$$

Similarly, since $\varepsilon \le |x| < 1$ for all $x \in C(R,\omega) \setminus U(x_0)$, it holds

$$|x|^{\alpha-|\nu|} \le \max\left\{1, \varepsilon^{\alpha-|\nu|}\right\} \qquad \text{for all} \qquad x \in C(R,\omega) \setminus U(x_0). \tag{7.2.7}$$

With the help of (7.2.6) and (7.2.7), for arbitrary $\nu \in \mathbb{N}_0^2$ we estimate

$$\begin{aligned} \int_{C(R,\omega)} \left( \rho(x)^{|\nu|-a} \, |x|^{\alpha-|\nu|} \right)^p \mathrm{d}x &\lesssim \int_{U^c} |x|^{(\alpha-|\nu|)p} \,\mathrm{d}x + \sum_{i=0}^2 \int_{U(x_i)} \left( \rho(x)^{|\nu|-a} \, |x|^{\alpha-|\nu|} \right)^p \mathrm{d}x \\ &\lesssim 1 + \int_{U(x_0)} \left( \rho(x)^{|\nu|-a} \, |x|^{\alpha-|\nu|} \right)^p \mathrm{d}x + \sum_{i=1}^2 \int_{U(x_i)} \left( \rho(x)^{|\nu|-a} \right)^p \mathrm{d}x. \end{aligned}$$

Next, using that $\rho(x) \sim |x - x_i|$ in $U(x_i)$ for all $i \in \{0,1,2\}$, we further get

$$\begin{aligned} \int_{C(R,\omega)} \left( \rho(x)^{|\nu|-a} \, |x|^{\alpha-|\nu|} \right)^p \mathrm{d}x &\lesssim 1 + \int_{U(x_0)} |x|^{(\alpha-a)p} \,\mathrm{d}x + \sum_{i=1}^2 \int_{U(x_i)} \left( |x - x_i|^{|\nu|-a} \right)^p \mathrm{d}x \\ &\lesssim 1 + \int_{U(x_0)} |x|^{-ap} \,\mathrm{d}x + \sum_{i=1}^2 \int_{U(x_i)} |x - x_i|^{-ap} \,\mathrm{d}x, \end{aligned} \tag{7.2.8}$$

where we used that $\alpha > 0$ and $|\nu| \ge 0$ in the last step. Finally, since for all $i \in \{0,1,2\}$ it holds

$$\int_{U(x_i)} |x - x_i|^{-ap} \,\mathrm{d}x < \int_{B_\varepsilon(0)} |x|^{-ap} \,\mathrm{d}x \, < \infty$$

for $-ap > -2$, i.e., $a < 2/p$, from (7.2.8) we conclude

$$\int_{C(R,\omega)} \left( \rho(x)^{|\nu|-a} \, |x|^{\alpha-|\nu|} \right)^p \mathrm{d}x \, < \infty \qquad \text{for all} \qquad a < \frac{2}{p}, \; \nu \in \mathbb{N}_0^2. \tag{7.2.9}$$

Now, this last estimate together with (7.2.5) proves the assertion. □

**Remark 7.9.**

**(i)** In regard of the Babuska-Kondratiev spaces $\mathcal{K}^{\ell}_{p,a}(C(R,\omega))$ in (7.2.1) we implicitly assumed that the corresponding singular set $S$ consists of the three vertices of $C(R,\omega)$, cf. Remark 1.5. However, if we choose $S$ to consist only of the apex of the cone, i.e., $S=\{0\}$, the assertion of Proposition 7.8 holds true analogously with an improved upper bound for $a$, i.e., then there exists some $0<R<1$ such that

$$u \in \mathcal{K}^{\ell}_{p,a}(C(R,\omega)) \qquad \text{for all} \qquad \ell \in \mathbb{N} \quad \text{and} \quad 0 \le a < \alpha + \frac{2}{p}.$$

To see this, only minor modifications in *Step 2* of the above proof of Proposition 7.8 are necessary (using exactly the same arguments).

**(ii)** Under the assumptions of Proposition 7.8, part (i) of this remark and Remark 1.4(ii) imply $u \in W^{\ell}(L_p(C(R,\omega)))$, provided that $\alpha = \alpha(\omega,p) > \ell - 2/p = \ell - 2 + 2/p'$. This happens, e.g., for small values of $p$ and small angles $\omega$, see Remark 7.3.

Now we are in the position to apply the embedding Corollary 5.4 of Section 5.2. The resulting Besov regularity assertion for solutions to (7.1.1) on the unit cone reads as follows.

**Proposition 7.10.** *Let* $1<p<\infty$ *and* $\omega \in (0,2\pi)$*, and let* $u \in W^1(L_p(C(1,\omega)))$ *be a nonnegative solution of* (7.1.1)*. Then there exists some* $0<R<1$*, such that*

$$u \in B^{\sigma}_{\tau}(L_\tau(C(R,\omega))) \qquad \textit{for all} \qquad \sigma > 0, \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p},$$

*i.e.,* $u \in B^{\infty}_{\mathrm{NL}}(L_p(C(R,\omega)))$*.*

*Proof.* First note that $W^1(L_p(C(R,\omega))) \hookrightarrow B^{1-\varepsilon}_p(L_p(C(R,\omega)))$ for all $0<\varepsilon<1$, see Remark 1.6(iii). Hence, with Proposition 7.8 we conclude that

$$u \in \mathcal{K}^{\ell}_{p,a}(C(R,\omega)) \cap B^{s}_{p}(L_p(C(R,\omega))) \qquad \text{for all} \qquad \ell \in \mathbb{N},\ 0 \le a < \frac{2}{p},\ 0<s<1. \tag{7.2.10}$$

Now, by Remark 5.5(ii) the continuous embedding stated in Corollary 5.4 can be applied which proves the assertion. □

**Remark 7.11.**

**(i)** Under the assumptions of Proposition 7.10, for arbitrary $\ell \in \mathbb{N}$, $0<a<2/p$ and $0<\sigma<\ell$ the (quasi-) norm estimate

$$\|u \,|\, B^{\sigma}_{\tau}(L_\tau(C(R,\omega)))\| \lesssim \left\|u \,\middle|\, \mathcal{K}^{\ell}_{p,a}(C(R,\omega))\right\| + \left\|u \,\middle|\, W^1(L_p(C(R,\omega)))\right\|,$$

$1/\tau = \sigma/2 + 1/p$, holds true. This follows from (7.2.10), Remark 5.5 and Corollary 5.4.

**(ii)** Note that all assertions stated in this section so far in particular hold for the singular function $u = s$, since $s$ is a (special) positive solution of (7.1.1).

Finally, let us turn to our primary global problem (3.2.1), (7.0.2), (7.0.3). By means of a transformation of coordinates, Proposition 7.10 implies the following Besov regularity result.

**Theorem 7.12.** *Let $\Omega \subset \mathbb{R}^2$ be some bounded polygonal domain, $1 < p < \infty$, and let $u \in W^1(L_p(\Omega))$ be the unique solution to problem* (3.2.1)*,* (7.0.2)*,* (7.0.3)*. Let $x_0 \in \partial\Omega$ denote an arbitrary vertex of $\Omega$. If, for some $R_0 > 0$, it holds $f = 0$ in $\mathfrak{C}_{R_0}(x_0) = B_{R_0}(x_0) \cap \Omega$ and $g = 0$ on $S_{R_0}(x_0) = B_{R_0}(x_0) \cap \partial\Omega$, then there exists some $0 < R < R_0$ such that*

$$u|_{\mathfrak{C}_R(x_0)} \in B^\sigma_\tau(L_\tau(\mathfrak{C}_R(x_0))) \qquad \text{for all} \qquad \sigma > 0, \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p},$$

*i.e.,*

$$u|_{\mathfrak{C}_R(x_0)} \in B^\infty_{\mathrm{NL}}(L_p(\mathfrak{C}_R(x_0))).$$

*Proof. Step 1.* Let us first verify that problem (3.2.1) is invariant with respect to translation and rotation in the following sense. Translating the whole problem to the origin, i.e., $\hat{\Omega} := \Omega - x_0$, $\hat{f}(\cdot) := f(\cdot + x_0)$ and $\hat{g}(\cdot) := g(\cdot + x_0)$, one immediatelly checks that the corresponding solution is given by $\hat{u}(\cdot) := u(\cdot + x_0)$. Similarly, for a rotation $\Upsilon$ by a fixed angle $\theta$ about the origin, with $\check{\Omega} := \Upsilon^{-1}(\hat{\Omega})$, $\check{f} := \hat{f} \circ \Upsilon$ and $\check{u} := \hat{u} \circ \Upsilon$, by a change of variables and the orthogonality of the corresponding rotation matrix $M_\theta$, a short computation yields that for all $\check{v} \in C_0^\infty(\check{\Omega})$, where we set $\hat{v} = \check{v} \circ \Upsilon^{-1} \in C_0^\infty(\hat{\Omega})$, it holds

$$\begin{aligned}
\int_{\check{\Omega}} \left\langle |\nabla\check{u}|^{p-2}\, \nabla\check{u}, \nabla\check{v} \right\rangle \mathrm{d}x &= \int_{\check{\Omega}} \left\langle \left| M_\theta^{-1}(\nabla\hat{u}) \circ \Upsilon \right|^{p-2} M_\theta^{-1}(\nabla\hat{u}) \circ \Upsilon, M_\theta^{-1}(\nabla\hat{v}) \circ \Upsilon \right\rangle \mathrm{d}x \\
&= \int_{\Upsilon^{-1}(\hat{\Omega})} \left\langle |(\nabla\hat{u}) \circ \Upsilon|^{p-2}\, (\nabla\hat{u}) \circ \Upsilon, (\nabla\hat{v}) \circ \Upsilon \right\rangle \mathrm{d}x \\
&= \int_{\hat{\Omega}} \left\langle |\nabla\hat{u}|^{p-2}\, \nabla\hat{u}, \nabla\hat{v} \right\rangle \mathrm{d}x,
\end{aligned}$$

where we have used that $\det D\Upsilon = 1$. Since $\hat{u}$ is the weak solution of the translated problem, we further get

$$\int_{\check{\Omega}} \left\langle |\nabla\check{u}|^{p-2}\, \nabla\check{u}, \nabla\check{v} \right\rangle \mathrm{d}x = \int_{\hat{\Omega}} \hat{f}\hat{v}\, \mathrm{d}x = \int_{\check{\Omega}} \check{f}\check{v}\, \mathrm{d}x,$$

i.e., $\check{u}$ solves the translated and rotated problem $-\mathrm{div}\Big(|\nabla\check{u}|^{p-2}\, \nabla\check{u}\Big) = \check{f}$ in $\check{\Omega}$ with $\check{u} = \check{g} := \hat{g} \circ \Upsilon$ on $\partial\check{\Omega}$.

*Step 2.* Now, let $u \in W^1(L_p(\Omega))$ denote the unique solution to problem (3.2.1), (7.0.2), (7.0.3), and let $\check{u}(x) = u(\Upsilon(x) + x_0)$ be the solution to the appropriately translated and rotated problem according to *Step 1*, such that the setting of equation (7.1.1) applies (i.e., with radius $R_0$ instead of 1). Note that the restriction $\check{u}|_{C(R_0,\omega)}$ of the solution $\check{u}$ on the whole polygonal domain $\check{\Omega}$ is in particular a solution of the local problem $-\mathrm{div}(|\nabla \check{u}|^{p-2}\nabla \check{u}) = \check{f}|_{C(R_0,\omega)} = 0$ in $C(R_0,\omega)$, $\check{u} = 0$ on $S(R_0,\omega)$. Since $f, g \geq 0$, from the weak comparison principle, see Proposition 3.5, we know that $u \geq 0$ in $\Omega$, which implies $\check{u} \geq 0$ in $\check{\Omega}$. Hence, $\check{u}|_{C(R_0,\omega)} \in W^1(L_p(C(R_0,\omega)))$ is a nonnegative solution of (7.1.1), and from Proposition 7.10 we know that there exists some $0 < R < R_0$, such that

$$\check{u}|_{C(R,\omega)} \in B^\sigma_\tau(L_\tau(C(R,\omega))) \qquad \text{for all} \qquad \sigma > 0, \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}.$$

Finally, from the definition of the Besov spaces in Section 1.4, we see that $\|\check{u} \,|\, B^\sigma_\tau(L_\tau(C(R,\omega)))\| = \|u \,|\, B^\sigma_\tau(L_\tau(C(x_0,R)))\|$, which proves the assertion. □

**Remark 7.13.**

**(i)** Note that for the (global) solution $u$ in Theorem 7.12 it also holds

$$u|_{\mathfrak{C}_R(x_0)} \in \mathcal{K}^\ell_{p,a}(\mathfrak{C}_R(x_0)) \qquad \text{for all} \qquad \ell \in \mathbb{N} \quad \text{and} \quad 0 \leq a < \frac{2}{p}. \tag{7.2.11}$$

This follows from Proposition 7.8 by using the same transformation arguments as in the proof of Theorem 7.12.

**(ii)** The assertions of Remark 7.9 hold true analogously for Theorem 7.12. I.e., if the set $S$ corresponding to the spaces $\mathcal{K}^\ell_{p,a}(\mathfrak{C}_R(x_0))$ is chosen as $S = \{x_0\}$, the upper bound in (7.2.11) improves to $0 \leq a < \alpha + 2/p$. In particular, $u|_{\mathfrak{C}_R(x_0)} \in W^\ell(L_p(\mathfrak{C}_R(x_0)))$, provided that $\alpha = \alpha(\omega, p) > \ell - 2/p$.

Now, by (7.2.11) and with the help of the embedding Theorem 5.9, we may formulate the following corollary.

**Corollary 7.14.** *Let the assumptions of Theorem 7.12 hold. Then there exists some* $0 < R < R_0$*, such that for all* $0 < a < 2/p$ *and* $0 < s < 1$ *it holds*

$$u|_{\mathfrak{C}_R(x_0)} \in H^{\infty,s}_a(L_p(\mathfrak{C}_R(x_0))) \hookrightarrow B^\infty_{\mathrm{NL}}(L_p(\mathfrak{C}_R(x_0))).$$

## 7.3 Besov regularity: The inhomogeneous equation

In the sequel we treat the class of $p$-Poisson problems (3.2.1), (7.0.2), (7.0.3), where $g$ locally vanishes and the right-hand side $f$ satisfies a local growth condition in the vicinity of a vertex of the polygonal domain. To prove our second main Besov smoothness result, similar to the previous section at first we derive a weighted Sobolev regulartiy result for problem (7.1.8) on the unit cone.

**Proposition 7.15.** *For* $1 < p < \infty$ *and* $\omega \in (0, 2\pi)$ *let* $\alpha > \max\{0, (p-2)/(p-1)\}$ *be given by* (7.1.4)*. Assume that for some* $c > 0$ *and* $\gamma > \gamma_0 := (\alpha - 1)(p-1) - 1$ *it holds* $0 \leq f(x) \leq c\,|x|^\gamma$ *for a.e.* $x \in C(1,\omega)$*. Let* $u \in W^1(L_p(C(1,\omega)))$ *be a nonnegative solution of* (7.1.8)*. Then there exists some* $0 < R < 1$ *such that*

$$u \in \mathcal{K}^2_{p,a}(C(R,\omega)) \qquad \textit{for all} \qquad 0 \leq a < \frac{2}{p}. \tag{7.3.1}$$

*Proof.* Again, w.l.o.g. we assume that $u \neq 0$ in the vicinity of the origin. First, from Proposition 7.6 we know that $u$ can be expanded into

$$u(x) = \kappa s(x) + w(x) \qquad \text{for a.e.} \qquad x \in C(R,\omega),$$

with $\kappa > 0$ and

$$|w(x)| \leq c' \, |x|^{\alpha+\eta}, \qquad |\nabla w(x)| \leq c' \, |x|^{\alpha-1+\eta} \qquad \text{for all} \qquad x \in C(R,\omega), \quad (7.3.2)$$

where $c', \eta > 0$. In regard of the singular part $s$, from Proposition 7.8 and Remark 7.11(ii) we know that

$$s \in \mathcal{K}^{\ell}_{p,a}(C(R,\omega)) \qquad \text{for all} \qquad \ell \in \mathbb{N} \quad \text{and} \quad 0 \leq a < \frac{2}{p}.$$

Hence, it remains to show that $w \in \mathcal{K}^2_{p,a}(C(R,\omega))$.

*Step 1.* Therefore, as a first step, we will show that

$$w \in \mathcal{K}^2_{q,a}(C(R,\omega)) \qquad \text{for all} \qquad q > 2, \quad 0 \leq a < \frac{2}{q}. \tag{7.3.3}$$

Thus, from now on let $q > 2$ and $0 \leq a < 2/q$ be arbitrary, but fixed. Note that $a < 1$ in this case and therefore $|\nu| - a > 0$ for all $\nu \in \mathbb{N}_0^2$ with $|\nu| \geq 1$. Since $\rho(x) \lesssim |x|$ for all $x \in C(R,\omega)$, using (7.3.2) we get

$$\begin{aligned} \left\| w \,\middle|\, \mathcal{K}^2_{q,a}(C(R,\omega)) \right\|^q &= \sum_{|\nu| \leq 2} \int_{C(R,\omega)} \left| \rho(x)^{|\nu|-a} D^\nu w(x) \right|^q \mathrm{d}x \\ &\lesssim \sum_{|\nu| \leq 1} \int_{C(R,\omega)} \left| \rho(x)^{|\nu|-a} \, |x|^{\alpha+\eta-|\nu|} \right|^q \mathrm{d}x + \sum_{|\nu|=2} \int_{C(R,\omega)} \left| |x|^{2-a} \, D^\nu w(x) \right|^q \mathrm{d}x. \end{aligned} \tag{7.3.4}$$

The first sum in (7.3.4) can be treated exactly like in *Step 2* of the proof of Proposition 7.8, thus from (7.2.9) we conclude

$$\sum_{0 \leq |\nu| \leq 1} \int_{C(R,\omega)} \left| \rho(x)^{|\nu|-a} \, |x|^{\alpha+\eta-|\nu|} \right|^q \mathrm{d}x \, < \infty. \tag{7.3.5}$$

To estimate the terms involving second order derivatives of $w$, we will make use of the fact that $w \in \mathcal{W}^2_{q,\beta}(C(R,\omega))$ for all $q > 2$ and $-\min\{\alpha + \delta, \mathrm{Re}\beta^{(2)}\} + 2 - 2/q < \beta < -\alpha + 2 - 2/q$, where $\delta > 0$ and $\mathrm{Re}\beta^{(2)} > \alpha$, see Remark 7.7(iii). Note that for those specific spaces $\mathcal{W}^2_{q,\beta}$ it holds $S = \{0\}$ and the corresponding smooth distance function is given by $\rho(x) = |x|$, see (7.1.13) in Remark 7.7(iii). Hence, with the help of Lemma A.13 we conclude that for the $\mathcal{W}^2_{q,\beta}$-norm (cf. (1.3.2)) it holds

$$\left\| w \,\middle|\, \mathcal{W}^2_{q,\beta}(C(R,\omega)) \right\|^q \sim \int_{C(R,\omega)} \left| |x|^{\beta-2} \, w(x) \right|^q \mathrm{d}x + \sum_{|\nu| \leq 2} \int_{C(R,\omega)} \left| D^\nu \left( |x|^{\beta} \, w(x) \right) \right|^q \mathrm{d}x. \tag{7.3.6}$$

Now, set $\varepsilon := 2 - a - \beta > 0$. With the help of Leibniz' rule and Lemma A.13, for $\nu \in \mathbb{N}_0^2$ with $|\nu| = 2$ we compute

$$\begin{aligned}
\int_{C(R,\omega)} \left| |x|^{2-a} D^\nu w(x) \right|^q \mathrm{d}x &= \int_{C(R,\omega)} |x|^{\varepsilon q} \left| |x|^\beta D^\nu w(x) \right|^q \mathrm{d}x \\
&= \int_{C(R,\omega)} |x|^{\varepsilon q} \left| D^\nu \left( |x|^\beta w(x) \right) - \sum_{0 \neq \mu \leq \nu} \binom{\nu}{\mu} \left( D^\mu |x|^\beta \right) \left( D^{\nu-\mu} w(x) \right) \right|^q \mathrm{d}x \\
&\lesssim \int_{C(R,\omega)} \left| D^\nu \left( |x|^\beta w(x) \right) \right|^q \mathrm{d}x + \int_{C(R,\omega)} \sum_{0 \neq \mu \leq \nu} \left| |x|^\varepsilon \left( D^\mu |x|^\beta \right) \left( D^{\nu-\mu} w(x) \right) \right|^q \mathrm{d}x.
\end{aligned}$$

With (7.3.2) and (7.3.6), as well as Lemma A.11 we further estimate

$$\begin{aligned}
\int_{C(R,\omega)} \left| |x|^{2-a} D^\nu w \right|^q \mathrm{d}x &\lesssim \left\| w \,\middle|\, \mathcal{W}^2_{q,\beta}(C(R,\omega)) \right\|^q + \int_{C(R,\omega)} \sum_{0 \neq \mu \leq \nu} \left| |x|^{\varepsilon+\beta+\alpha+\eta-2} \right|^q \mathrm{d}x \\
&\lesssim \left\| w \,\middle|\, \mathcal{W}^2_{q,\beta}(C(R,\omega)) \right\|^q + 1, \qquad (7.3.7)
\end{aligned}$$

since

$$(\varepsilon + \beta + \alpha + \eta - 2)\, q = (\alpha + \eta - a)\, q > -aq > -2.$$

Finally, using the estimates (7.3.5) and (7.3.7), from (7.3.4) we derive

$$\left\| w \,\middle|\, \mathcal{K}^2_{q,a}(C(R,\omega)) \right\| < \infty \qquad \text{for all} \qquad q > 2, \quad 0 \leq a < \frac{2}{q},$$

which proves (7.3.3). Hence, we have shown that the assertion (7.3.1) holds true for $2 < p < \infty$.

*Step 2.* To conclude the proof, it remains to consider the case $1 < p \leq 2$. Therefore, we will show that for $1 \leq p < q < \infty$ and $\ell \in \mathbb{N}_0$ the embedding

$$\mathcal{K}^\ell_{q,\tilde{a}}(C(R,\omega)) \hookrightarrow \mathcal{K}^\ell_{p,a}(C(R,\omega)) \qquad (7.3.8)$$

holds true for all $a, \tilde{a} \geq 0$ with $a < \tilde{a} - 2/q + 2/p$. With the help of Hölder's inequality we get

$$\begin{aligned}
\left\| w \,\middle|\, \mathcal{K}^\ell_{p,a}(C(R,\omega)) \right\|^p &= \sum_{|\nu| \leq \ell} \int_{C(R,\omega)} \left| \rho(x)^{|\nu|-a} D^\nu w \right|^p \mathrm{d}x \\
&= \sum_{|\nu| \leq \ell} \int_{C(R,\omega)} \rho(x)^{(\tilde{a}-a)p} \left| \rho(x)^{|\nu|-\tilde{a}} D^\nu w \right|^p \mathrm{d}x \\
&\leq \sum_{|\nu| \leq \ell} \left( \int_{C(R,\omega)} \left( \rho(x)^{(\tilde{a}-a)p} \right)^{q/(q-p)} \mathrm{d}x \right)^{1-p/q} \left( \int_{C(R,\omega)} \left| \rho(x)^{|\nu|-\tilde{a}} D^\nu w \right|^q \mathrm{d}x \right)^{p/q},
\end{aligned}$$

and to estimate the above integral not depending on $w$, note that $0 < \rho \leq 1$ only approaches zero in the vicinity of the corners of $C(R,\omega)$. Hence, we conclude that for $(\tilde{a} - a)pq/(q-p) > -2$, i.e., $a < \tilde{a} - 2/q + 2/p$ it holds

$$\left\| w \,\middle|\, \mathcal{K}^\ell_{p,a}(C(R,\omega)) \right\|^p \lesssim \sum_{|\nu| \leq \ell} \left( \int_{C(R,\omega)} \left| \rho(x)^{|\nu|-\tilde{a}} D^\nu w \right|^q \mathrm{d}x \right)^{p/q} \lesssim \left\| w \,\middle|\, \mathcal{K}^\ell_{q,\tilde{a}}(C(R,\omega)) \right\|^p,$$

which proves (7.3.8). Now, let $1 < p \leq 2$ and $0 \leq a < 2/p$. Choose any $q > 2$ and $\tilde{a} \geq 0$ such that $a - 2/p + 2/q < \tilde{a} < 2/q$. Then, from (7.3.3) and (7.3.8) we conclude that

$$w \in \mathcal{K}^2_{q,\tilde{a}}(C(R,\omega)) \hookrightarrow \mathcal{K}^2_{p,a}(C(R,\omega)).$$

□

**Remark 7.16.** The assertions of Remark 7.9 hold true analogously for Proposition 7.15 with $\ell = 2$. In particular, $u \in W^2(L_p(C(R,\omega)))$ provided that $\alpha(\omega,p) > 2 - 2/p$.

With the help of this weighted Sobolev regularity result, we can derive the following Besov smoothness estimate for solutions on the unit cone.

**Proposition 7.17.** *For $1 < p < \infty$ and $\omega \in (0, 2\pi)$ let $\alpha > \max\{0, (p-2)/(p-1)\}$ be given by (7.1.4). Assume that for some $c > 0$ and $\gamma > \gamma_0 := (\alpha - 1)(p-1) - 1$ it holds $0 \leq f(x) \leq c\,|x|^\gamma$ for a.e. $x \in C(1,\omega)$. Let $u \in W^1(L_p(C(1,\omega)))$ be a nonnegative solution of (7.1.8). Then there exists some $0 < R < 1$ such that*

$$u \in B^\sigma_\tau(L_\tau(C(R,\omega))) \qquad \text{for all} \qquad 0 < \sigma < 2, \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}.$$

*Proof.* The proof is completely analogous to the one of Proposition 7.10, where here we apply Proposition 7.15 instead of Proposition 7.8. □

By the same arguments as before, we can transfer the local smoothness estimate from the unit cone to our primary global problem (3.2.1), (7.0.2), (7.0.3) to obtain our second main Besov regularity result.

**Theorem 7.18.** *Let $\Omega \subset \mathbb{R}^2$ be some bounded polygonal domain, $1 < p < \infty$, and let $u \in W^1(L_p(\Omega))$ be the unique solution to problem (3.2.1), (7.0.2), (7.0.3). Let $x_0 \in \partial\Omega$ be a vertex of $\Omega$ with interior angle $\omega \in (0, 2\pi)$. Moreover, assume that for some $R_0 > 0$ there exists $c > 0$ and $\gamma > (\alpha - 1)(p-1) - 1$ with $\alpha = \alpha(\omega, p) > 0$ given by (7.1.4) such that $0 \leq f(x) \leq c\,|x - x_0|^\gamma$ for a.e. $x \in \mathfrak{C}_{R_0}(x_0) = B_{R_0}(x_0) \cap \Omega$ and $g = 0$ on $S_{R_0}(x_0) = B_{R_0}(x_0) \cap \partial\Omega$. Then there exists some $0 < R < R_0$ such that*

$$u|_{\mathfrak{C}_R(x_0)} \in B^\sigma_\tau(L_\tau(\mathfrak{C}_R(x_0))) \qquad \text{for all} \qquad 0 < \sigma < 2, \quad \frac{1}{\tau} = \frac{\sigma}{2} + \frac{1}{p}.$$

*Proof.* The proof follows almost exactly the lines of the proof of Theorem 7.12, where here we apply Proposition 7.17 instead of Proposition 7.10. □

**Remark 7.19.**

**(i)** In [57, p. 188] Dobrowolski notes that for the regular part $w$ of the expansion (7.1.10) it holds $|\nabla^2 w(x)| \leq c\,|x - x_0|^{\alpha-2+\eta}$ etc. in a neighborhood of $x_0$ if the right-hand side $f$ and $\partial\Omega \setminus \{x_0\}$ are sufficiently smooth. In this case, as can be seen from the proofs of Proposition 7.15 and Proposition 7.17, also the Besov regularity of the solution, measured in the adaptivity scale, would improve correspondingly.

**(ii)** By the same arguments as in Section 7.2 we conclude that for the solution $u$ in Theorem 7.18 it holds

$$u|_{\mathfrak{C}_R(x_0)} \in \mathcal{K}^2_{p,a}(\mathfrak{C}_R(x_0)) \qquad \text{for all} \qquad 0 \leq a < \frac{2}{p}.$$

Moreover, if $\alpha(\omega, p) > 2 - 2/p$ then $u \in W^2(L_p(\mathfrak{C}_R(x_0)))$, cf. Remark 7.13.

# Part IV

# Discretization of the $p$-Poisson Equation

# Chapter 8

# A Kačanov-type Iteration Method

In this chapter the *Kačanov-type iteration scheme* as proposed in [53] for the numerical treatment of the $p$-Poisson equation is introduced. Due to the nonlinear structure of the $p$-Laplacian, the numerical approximation is a nontrivial, rather unexplored subject of research. To the author's best knowledge, by now there mainly exist two publications which are concerned with numerically feasible approximation schemes for the $p$-Poisson equation.

The work of Canuto and Urban [14] treats the fairly general framework of convex minimization in Banach spaces, where convergence of a steepest descent type method is established. This setting covers the $p$-Poisson problem with homogeneous Dirichlet boundary conditions for all $p > 2$, yet excluding the case $1 < p \leq 2$.

In [53] Diening et al. proposed an iterative linearization scheme for the $p$-Poisson equation which can be interpreted as a relaxed Kačanov iteration. In particular, the case $1 < p \leq 2$ is treated. The main feature of this algorithm is that in each iteration only a linear elliptic subproblem has to be solved, which is numerically accessible in a stable and approved way by, e.g., a finite element or wavelet method. In this chapter we will focus on this algorithm, in connection with an appropriate adaptive wavelet frame method for the solution of the linear subproblems.

Throughout this chapter, we consider the $p$-Poisson problem with homogeneous Dirichlet boundary data (3.2.3), i.e.,

$$\begin{aligned} -\operatorname{div}\Big(|\nabla u|^{p-2}\,\nabla u\Big) = f \qquad &\text{in } \Omega, \\ u = 0 \qquad &\text{on } \partial\Omega, \end{aligned}$$

for $1 < p \leq 2$ on a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with $f \in W^{-1}(L_{p'}(\Omega))$. Where necessary (i.e., for some assertions in Section 8.3, as well as in Section 8.4), we may restrict the class of admissible domains to polyhedral Lipschitz domains, which will be explicitly stated then.

This chapter is organized as follows. First, in Section 8.1 the relaxed Kačanov iteration method is introduced. The error analysis done in [53] for this scheme - which we summarize in Section 8.3 - is based on a characterization of the $p$-Poisson problem and of the subproblems occuring in this Kačanov-type iteration, as certain energy minimization problems. Moreover, all approximation error assertions are formulated in terms of (differences of) those energies. Therefore, this correspondence between weak solutions and minimizers of specific energy functionals is outlined in Section 8.2. Finally, in Section 8.4, we will describe the discretization of the linear subproblems which have to be solved at each relaxed Kačanov iteration, by means of an adaptive wavelet (frame) algorithm.

## 8.1 A Kačanov-type iteration method

### 8.1.1 The classical Kačanov iteration

Roughly speaking, the classical Kačanov scheme is an iteration method for solving nonlinear problems via linearization. An early reference is [86]. For quasilinear elliptic equations of the type

$$-\mathrm{div}(\alpha\,(|\nabla u|)\,\nabla u) = f \qquad \text{in } \Omega, \tag{8.1.1}$$

the Kačanov iteration takes the following form. For a given function $u_0$, the new iterate $u_{n+1}$ is recursively defined as the solution of

$$-\mathrm{div}(\alpha\,(|\nabla u_n|)\,\nabla u_{n+1}) = f \qquad \text{in } \Omega, \quad n \geq 0. \tag{8.1.2}$$

Note that now at each iteration only a *linear* problem has to be solved.

If $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable, under certain monotonicity properties on $\alpha$, i.e., if

$$c_1 \leq \alpha(\xi) \leq c_2, \qquad \alpha'(\xi) \leq 0, \qquad 2\alpha'(\xi)\,\xi + \alpha(\xi) \geq c_3 \tag{8.1.3}$$

for suitable positive constants $c_1, c_2, c_3$ and all $\xi \geq 0$, it is proved in [136] that the Kačanov iteration (8.1.2) converges to a fixed point $u$ which solves (8.1.1). An a posteriori error estimate is derived in [75].

Unfortunately, the $p$-Poisson equation, i.e. $\alpha(\xi) = \xi^{p-2}$, does *not* satisfy the conditions in (8.1.3). Moreover, equation (8.1.2) is not numerically solvable in a stable way if $|\nabla u_n|$ vanishes or gets unbounded at certain points, since then the weight $|\nabla u_n|^{p-2}$ degenerates. A natural approach is to modify the weight function $\alpha(\xi) = \xi^{p-2}$ appropriately, as described in the next subsection.

### 8.1.2 A relaxed Kačanov iteration method

To overcome the above mentioned problem of a possibly degenerate weight $|\nabla u_n|^{p-2}$ in the course of the Kačanov iteration (8.1.2), a self-evident approach is to simply truncate the weight function. Therefore, for $0 < \varepsilon_- \leq \varepsilon_+ < \infty$ and $x \in \mathbb{R}$ we define

$$\varepsilon_- \vee x \wedge \varepsilon_+ := \begin{cases} \varepsilon_-, & \text{if } x \leq \varepsilon_-, \\ x, & \text{if } \varepsilon_- < x < \varepsilon_+, \\ \varepsilon_+, & \text{if } x \geq \varepsilon_+. \end{cases}$$

Similarly, for a function $a : \Omega \mapsto \overline{\mathbb{R}}$ we define the corresponding *truncated function*

$$\varepsilon_- \vee a \wedge \varepsilon_+ : \ \Omega \to [\varepsilon_-, \varepsilon_+]$$

by

$$x \mapsto \varepsilon_- \vee a(x) \wedge \varepsilon_+.$$

Note that $\varepsilon_- \vee a \wedge \varepsilon_+ \in L_\infty(\Omega)$ for any measurable function $a$. We will write $\varepsilon := [\varepsilon_-, \varepsilon_+]$ for the truncation interval.

Now, modifying the classical Kačanov iteration (8.1.2) by truncating the (argument of the) weight function $\alpha(\xi) = \xi^{p-2}$, the new iteration scheme takes the form

$$-\mathrm{div}\Big((\varepsilon_- \vee |\nabla u_n| \wedge \varepsilon_+)^{p-2}\, \nabla u_{n+1}\Big) = f \qquad \text{in } \Omega, \quad n \geq 0. \tag{8.1.4}$$

Moreover, to recover the $p$-Laplace operator, we will additionally have to increase the truncation interval during the iteration. Then, the resulting *relaxed* Kačanov algorithm for the $p$-Poisson problem, as proposed in [53], reads as follows.

**Algorithm 1** (relaxed Kačanov algorithm)**.**

*Data:* Given $f \in W^{-1}(L_{p'}(\Omega))$, $u_0 \in W_0^1(L_2(\Omega))$;
*Result:* Approximate solution of the $p$-Poisson problem (3.2.3);
*Initialize:* $\varepsilon_0 = [\varepsilon_{0,-}, \varepsilon_{0,+}] \subset (0, \infty)$, $n = 0$;

`while` desired accuracy is not achieved yet `do`

Define $u_{n+1} \in W_0^1(L_2(\Omega))$ as the solution of

$$\int_\Omega (\varepsilon_{n,-} \vee |\nabla u_n| \wedge \varepsilon_{n,+})^{p-2}\, \nabla u_{n+1} \cdot \nabla v \,\mathrm{d}x = \langle f, v\rangle \quad \text{for all} \quad v \in W_0^1(L_2(\Omega)); \tag{8.1.5}$$

Choose new relaxation interval $\varepsilon_{n+1} \supset \varepsilon_n$;

Increase $n$ by 1;

`end`

Some comments are in order. First note that the matrix of coefficients corresponding to (8.1.5), which is given by $(a_{i,j}(x))_{i,j=1}^d := (\varepsilon_{n,-} \vee |\nabla u_n(x)| \wedge \varepsilon_{n,+})^{p-2}\, I$, where $I \in \mathbb{R}^{d\times d}$ denotes the unit matrix, is uniformly elliptic because

$$\sum_{i,j=1}^d a_{i,j}(x)\xi_i\xi_j = (\varepsilon_{n,-} \vee |\nabla u_n(x)| \wedge \varepsilon_{n,+})^{p-2}\, |\xi|^2 \geq |\xi|^2 \cdot \begin{cases} \varepsilon_{n,+}^{p-2}, & \text{if } 1 < p < 2, \\ \varepsilon_{n,-}^{p-2}, & \text{if } p \geq 2, \end{cases}$$

for all $\xi \in \mathbb{R}^d$ and $x \in \Omega$. Hence, the subproblem (8.1.5) which has to be solved at each iteration is a linear, uniformly elliptic PDE. The unique solvability of these equations is guaranteed by the following well-known result, formulated for our particular instance (8.1.5). A proof can be found, e.g., in [73].

**Proposition 8.1.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain and $1 < p < \infty$, as well as $[\varepsilon_-, \varepsilon_+] \subset (0,\infty)$ and $a \in L_1(\Omega)$. Furthermore, let $f \in W^{-1}(L_2(\Omega))$ and*

$g \in W^1(L_2(\Omega))$. *Then, there exists a unique* $u \in W^1(L_2(\Omega))$ *with* $(u-g) \in W_0^1(L_2(\Omega))$ *such that*

$$\int_\Omega (\varepsilon_- \vee |a| \wedge \varepsilon_+)^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v \rangle \qquad \textit{for all} \quad v \in W_0^1(L_2(\Omega)).$$

**Remark 8.2.**

**(i)** Note that in **Algorithm 1** it is assumed $f \in W^{-1}(L_{p'}(\Omega))$ to assure the solvability of the $p$-Poisson problem (3.2.3), see Proposition 3.2, whereas for the solvability of the subproblems (8.1.5) it is required $f \in W^{-1}(L_2(\Omega))$, see Proposition 8.1. But, since $W_0^1(L_2(\Omega))$ is continuously embedded into $W_0^1(L_p(\Omega))$ for all $1 < p < 2$, we see that $W^{-1}(L_{p'}(\Omega)) \subset W^{-1}(L_2(\Omega))$. Hence, under the assumptions of **Algorithm 1** the linear subproblems (8.1.5) are uniquely solvable.

**(ii)** For the reader's convenience, we shortly sketch the proof of Proposition 8.1 given in [73]. Setting

$$b(u,v) := \int_\Omega (\varepsilon_- \vee |a| \wedge \varepsilon_+)^{p-2} \nabla u \cdot \nabla v \, \mathrm{d}x, \tag{8.1.6}$$

where $a \in L_1(\Omega)$, we estimate for $v \in W_0^1(L_2(\Omega))$

$$b(v,v) \geq C|v|^2_{W^1(L_2(\Omega))} \geq C_\Omega \|v\|^2_{W^1(L_2(\Omega))},$$

where the last estimate holds due to Poincaré-Friedrich's inequality (see, e.g., [73, Lemma 6.2.11]). Hence, $b(\cdot,\cdot)$ is $W_0^1(L_2(\Omega))$-elliptic, and it is not difficult to see that $b(\cdot,\cdot)$ is bounded on $W^1(L_2(\Omega)) \times W^1(L_2(\Omega))$. Consequently, $(W_0^1(L_2(\Omega)), b(\cdot,\cdot))$ is a Hilbert space on its own right, and the Riesz representation theorem yields the existence of a unique solution $\tilde{u} \in W_0^1(L_2(\Omega))$ of the problem

$$b(\tilde{u}, v) = \langle f, v \rangle - b(g, v) \qquad \text{for all} \quad v \in W_0^1(L_2(\Omega)),$$

since the above right-hand side is a linear bounded functional on $W_0^1(L_2(\Omega))$. Setting $u := \tilde{u} + g$ finally yields

$$b(u,v) = b(\tilde{u}, v) + b(g, v) = \langle f, v \rangle \qquad \text{for all} \quad v \in W_0^1(L_2(\Omega))$$

and $(u-g) \in W_0^1(L_2(\Omega))$.

In summary, we have seen that **Algorithm 1** is well-defined. Moreover, we remark that the algorithm is fully implementable, since clearly the linear, uniformly elliptic equations (8.1.5) can be approximately solved by a finite element or wavelet (frame) method in a stable way. Before we address this topic in Section 8.4, prior to that let us consider the question whether the relaxation and linearization performed above is consistent with the original $p$-Poisson equation. Therefore, we assume that the linear subproblems (8.1.5) are solved exactly, and summarize some convergence results for **Algorithm 1** in Section 8.3. To describe these results, we first need to introduce an equivalent characterization of the $p$-Poisson problem, as well as of the subproblems (8.1.5), as an energy minimization problem in the next section.

## 8.2 Energy minimizer and weak solutions

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain, $1 < p < \infty$ and $f \in W^{-1}(L_{p'}(\Omega))$. Then, the *energy functional* related to the $p$-Poisson problem (3.2.3) is defined as

$$\begin{aligned} &\mathcal{J} : W_0^1(L_p(\Omega)) \to \mathbb{R}, \\ &\mathcal{J}(u) := \int_\Omega \frac{1}{p} \, |\nabla u|^p \, \mathrm{d}x - \langle f, u \rangle. \end{aligned} \tag{8.2.1}$$

Now, the Dirichlet $p$-Poisson problem can be interpreted as an energy minimization problem as follows. A proof of this assertion for $f = 0$ can be found in [99]. For the readers convenience, we present the adapted proof below.

**Proposition 8.3.** *The following conditions are equivalent for $u \in W_0^1(L_p(\Omega))$.*

*(i) $u$ is a minimizer of $\mathcal{J}$:*

$$\mathcal{J}(u) \leq \mathcal{J}(v) \qquad \textit{for all} \qquad v \in W_0^1(L_p(\Omega)). \tag{8.2.2}$$

*(ii) The first variation vanishes:*

$$\int_\Omega |\nabla u|^{p-2} \, \nabla u \cdot \nabla v \, \mathrm{d}x = \langle f, v \rangle \qquad \textit{for all} \qquad v \in W_0^1(L_p(\Omega)). \tag{8.2.3}$$

*Proof.* (i) $\Rightarrow$ (ii). Let $u$ be a minimizer of $\mathcal{J}$ and $v \in W_0^1(L_p(\Omega))$ be arbitrary but fixed. For real $t$ we set $v_t := u + tv$. Note that $v_t \in W_0^1(L_p(\Omega))$ and hence $\mathcal{J}(u) \leq \mathcal{J}(v_t)$ for all $t \in \mathbb{R}$. Next we set

$$I(t) := \mathcal{J}(v_t) = \mathcal{J}(u + tv)$$

and note that $I$ has a minimum in $t = 0$. In the following, w.l.o.g. we restrict $t$ to the intervall $(-1, 1)$ and set

$$\begin{aligned} G(t) &:= \int_\Omega g(t,x) \, \mathrm{d}x \qquad &&\text{with} \qquad g(t,x) := \frac{1}{p} |\nabla u(x) + t\nabla v(x))|^p, \\ H(t) &:= (F \circ h)(t) &&\text{with} \qquad F(v) := \langle f, v \rangle \text{ and } h(t) = u + tv, \end{aligned}$$

i.e., $I = G - H$. Clearly, $g(t, \cdot) \in L_1(\Omega)$ for all $t \in (-1, 1)$ and $|\cdot|^p$ is continuously differentiable on $\mathbb{R}^d$, see Lemma A.10. It follows that $g(\cdot, x)$ is continuously differentiable for a.e. $x \in \Omega$ and we compute

$$\frac{\partial}{\partial t} g(t,x) = |\nabla u(x) + t\nabla v(x)|^{p-2} \, (\nabla u(x) + t\nabla v(x)) \cdot \nabla v(x).$$

We estimate

$$\left| \frac{\partial}{\partial t} g(t,x) \right| \leq (|\nabla u(x)| + |t| \, |\nabla v(x)|)^{p-1} \, |\nabla v(x)| \, ,$$

and since $|t| < 1$, it holds

$$\left|\frac{\partial}{\partial t}g(t,x)\right| \leq (|\nabla u(x)| + |\nabla v(x)|)^p \leq 2^{p-1}\left(|\nabla u(x)|^p + |\nabla v(x)|^p\right) \quad \in L_1(\Omega),$$

where for the last estimate we have used Lemma A.13. Now we have shown that $g$ satisfies all conditions of Lemma A.9 and hence $G$ is continuously differentiable. Clearly, $f$ is Fréchet differentiable and thus it holds $H'(t) = (F' \circ h)(t)h'(t) = F(h'(t)) = F(v)$. We derive

$$\begin{aligned} I'(t) &= G'(t) - H'(t) \\ &= \int_\Omega \frac{\partial}{\partial t} g(t,x)\,\mathrm{d}x - F(v) \\ &= \int_\Omega |\nabla u(x) + t\nabla v(x)|^{p-2}\left(\nabla u(x) + t\nabla v(x)\right)\cdot \nabla v(x)\,\mathrm{d}x - \langle f, v\rangle. \end{aligned}$$

Since $I$ has a minimum in $t = 0$, it must hold

$$0 = I'(0) = \int_\Omega |\nabla u(x)|^{p-2}\,\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x - \langle f, v\rangle.$$

This is (ii).

(ii) $\Rightarrow$ (i). Since $p > 1$, for vectors $\vartheta_1, \vartheta_2 \in \mathbb{R}^d$ the inequality

$$|\vartheta_2|^p \geq |\vartheta_1|^p + p\langle|\vartheta_1|^{p-2}\,\vartheta_1, \vartheta_2 - \vartheta_1\rangle$$

holds by convexity, see Lemma A.14, such that for $v \in W_0^1(L_p(\Omega))$ we estimate

$$\begin{aligned} \mathcal{J}(v) &= \int_\Omega \frac{1}{p}\,|\nabla v|^p\,\mathrm{d}x - \langle f, v\rangle \\ &\geq \int_\Omega \frac{1}{p}\,|\nabla u|^p\,\mathrm{d}x + \int_\Omega |\nabla u|^{p-2}\,\nabla u\cdot(\nabla v - \nabla u)\,\mathrm{d}x - \langle f, v\rangle \\ &= \int_\Omega \frac{1}{p}\,|\nabla u|^p\,\mathrm{d}x + \langle f, v-u\rangle - \langle f, v\rangle \\ &= \mathcal{J}(u). \end{aligned}$$

□

Hence, the $p$-Poisson equation with homogeneous Dirichlet boundary conditions can be equivalently described as an energy minimization problem with the same boundary constraint. The rest of this section will be devoted to the derivation of a characterization of the linear subproblems (8.1.5) as a (two-step) energy minimization problem.

### 8.2.1 Relaxed energy

If the energy functional $\mathcal{J}$ is extended by an additional parameter $a$ in the following way,

$$\mathcal{J}_s(u,a) := \int_\Omega \frac{1}{2}a^{p-2}\,|\nabla u|^2 + \left(\frac{1}{p} - \frac{1}{2}\right)a^p\,\mathrm{d}x - \langle f, u\rangle, \tag{8.2.4}$$

we see that $\mathcal{J}_s$ is now quadratic with respect to $u$. Note that for $a = |\nabla u|$ it holds $\mathcal{J}_s(u, |\nabla u|) = \mathcal{J}(u)$. Moreover, this functional is well-defined for $u \in W_0^1(L_p(\Omega))$ and measurable $a : \Omega \to [0, \infty)$, where $\mathcal{J}_s$ might take the value $\infty$. Furthermore, $\mathcal{J}_s$ is convex with respect to $(u, a)$.

Now, the main idea is to iteratively minimize $\mathcal{J}_s$ with respect to $u$ and $a$. A minimization with respect to $u$ (for fixed $a$) leads formally to the elliptic equation $-\mathrm{div}(a^{p-2}\nabla u) = f$, see [53, Section 2]. However, the same problem as in Subsection 8.1.1 of a degenerating ellipticity occurs at points where $a$ vanishes or gets unbounded. Therefore, like in Subsection 8.1.2, we will follow the approach to appropriately truncate the weight function $a$. Hence, for fixed $u$, let us confine the minimization of $\mathcal{J}_s(u, \cdot)$ to those functions which satisfy $\varepsilon_- \le a \le \varepsilon_+$. It turns out that this restricted minimization has a simple solution. A proof of the following statement can be found in [53, (2.1)].

**Proposition 8.4.** *Let $1 < p < 2$ and $\varepsilon = [\varepsilon_-, \varepsilon_+] \subset (0, \infty)$, as well as $u \in W_0^1(L_2(\Omega))$ be fixed. Then, when restricting the minimization of $\mathcal{J}_s(u, \cdot)$ to those functions $a \in L_1(\Omega)$ which satisfy $a = \varepsilon_- \vee a \wedge \varepsilon_+$, there exists a unique minimizer. This minimizer $m$ admits the representation*

$$m = \operatorname*{argmin}_{a\,:\,\varepsilon_- \le a \le \varepsilon_+} \mathcal{J}_s(u, a) = \varepsilon_- \vee |\nabla u| \wedge \varepsilon_+.$$

Hence, for the restricted minimization of $\mathcal{J}_s$ with respect to $a$ (for fixed $u$) it holds

$$\min_{a\,:\,\varepsilon_- \le a \le \varepsilon_+} \mathcal{J}_s(u, a) = \mathcal{J}_s(u, \varepsilon_- \vee |\nabla u| \wedge \varepsilon_+). \tag{8.2.5}$$

Next, for fixed measurable $a$, let us consider the minimization of $\mathcal{J}_s(\cdot, \varepsilon_- \vee a \wedge \varepsilon_+)$ with respect to $u$. To stress the fact that this step indeed corresponds to the solution of the linear elliptic problems (8.1.5), for the readers convenience we state this equivalence by the following proposition and give a short proof.

**Proposition 8.5.** *Let $a \in L_1(\Omega)$ be fixed. Then, the following conditions are equivalent for $u \in W_0^1(L_2(\Omega))$.*

*(i) $u$ is a minimizer of $\mathcal{J}_s(\cdot, \varepsilon_- \vee a \wedge \varepsilon_+)$:*

$$\mathcal{J}_s(u, \varepsilon_- \vee a \wedge \varepsilon_+) \le \mathcal{J}_s(v, \varepsilon_- \vee a \wedge \varepsilon_+) \qquad \text{for all} \quad v \in W_0^1(L_2(\Omega)).$$

*(ii) The first variation vanishes:*

$$\int_\Omega (\varepsilon_- \vee |a| \wedge \varepsilon_+)^{p-2}\, \nabla u \cdot \nabla v \,\mathrm{d}x = \langle f, v\rangle \qquad \text{for all} \quad v \in W_0^1(L_2(\Omega)).$$

*Proof.* First note that $u$ minimizes $\mathcal{J}_s(\cdot, \varepsilon_- \vee a \wedge \varepsilon_+)$ among all $v \in W_0^1(L_2(\Omega))$ if and only if $u$ minimizes $\mathcal{I}(v) := 1/2\, b(v, v) - F(v)$ among all such $v$, where

$$b(u, v) := \int_\Omega (\varepsilon_- \vee |a| \wedge \varepsilon_+)^{p-2}\, \nabla u \cdot \nabla v \,\mathrm{d}x \qquad \text{and} \qquad F(v) := \langle f, v\rangle.$$

Since $b(\cdot,\cdot)$ is symmetric and $W_0^1(L_2(\Omega))$-elliptic, we know from [73, Theorem 7.2.9] that for $u \in W_0^1(L_2(\Omega))$ the condition

$$\mathcal{I}(u) \leq \mathcal{I}(v) \quad \text{for all} \quad v \in W_0^1(L_2(\Omega)) \tag{8.2.6}$$

is equivalent to

$$b(u,v) = F(v) \quad \text{for all} \quad v \in W_0^1(L_2(\Omega)). \tag{8.2.7}$$

This finishes the proof. □

The full two-step energy minimization algorithm reads as follows.

**Algorithm 2** (relaxed Kačanov algorithm, energy version)**.**

*Data:* Given $f \in W^{-1}(L_{p'}(\Omega)$, $u_0 \in W_0^1(L_2(\Omega))$;
*Result:* Approximate solution of the $p$-Poisson problem (3.2.3);
*Initialize:* $\varepsilon_0 = [\varepsilon_{0,-}, \varepsilon_{0,+}] \subset (0,\infty)$, $n = 0$;

`while` desired accuracy is not achieved yet `do`

Define $a_n$ as the solution of

$$a_n := \operatorname*{argmin}_{a\,:\,\varepsilon_- \leq a \leq \varepsilon_+} \mathcal{J}_s(u_n, a); \tag{8.2.8}$$

Define $u_{n+1} \in W_0^1(L_2(\Omega))$ as the solution of

$$u_{n+1} := \operatorname*{argmin}_{u \in W_0^1(L_2(\Omega))} \mathcal{J}_s(u, a_n); \tag{8.2.9}$$

Choose new relaxation interval $\varepsilon_{n+1} \supset \varepsilon_n$;

Increase $n$ by 1;

`end`

Note that this is indeed **Algorithm 1** formulated as an energy minimization scheme, producing exactly the same iterates $u_n$.

To conclude this section, let us take a closer look at the correspondence of **Algorithm 1** and **2**. Recall that the relaxed Kačanov iteration can be interpreted as a relaxation and subsequent linearization of the $p$-Poisson equation, where the *relaxed* $p$-Poisson equation

$$-\operatorname{div}\Big((\varepsilon_- \vee |\nabla u| \wedge \varepsilon_+)^{p-2}\, \nabla u\Big) = f, \tag{8.2.10}$$

is approximated by the iterative linearization scheme (8.1.4), i.e.,

$$-\operatorname{div}\Big((\varepsilon_- \vee \nabla u_n \wedge \varepsilon_+)^{p-2}\, \nabla u_{n+1}\Big) = f.$$

With other words, the solution $u_\varepsilon \in W_0^1(L_2(\Omega))$ of (8.2.10) is approximated by the solutions $u_n \in W_0^1(L_2(\Omega))$ of the linear equations (8.1.4). Now, similar to the

$p$-Poisson problem and (8.1.5), also problem (8.2.10) admits a characterization as a certain energy minimization problem.

To introduce this energy, recall that from Proposition 8.4 we know that, for fixed $u \in W_0^1(L_2(\Omega))$, the minimizer of $\mathcal{J}_s(u,\cdot)$ among all measurable $a$ with $\varepsilon_- \leq a \leq \varepsilon_+$ is explicitly given by $\varepsilon_- \vee |\nabla u| \wedge \varepsilon_+$, see also (8.2.5). Hence, this restricted minimization with respect to the second variable only depends on the fixed and known first variable $u$. This suggests the following definition. For $1 < p < 2$ and $\varepsilon = [\varepsilon_-, \varepsilon_+] \subset (0,\infty)$ we define the *relaxed energy* $\mathcal{J}_\varepsilon : W_0^1(L_p(\Omega)) \to \mathbb{R} \cup \{\infty\}$ as

$$\mathcal{J}_\varepsilon(u) := \mathcal{J}_s(u, \varepsilon_- \vee |\nabla u| \wedge \varepsilon_+). \tag{8.2.11}$$

From (8.2.4) we see that $\mathcal{J}_\varepsilon$ may be written as

$$\mathcal{J}_\varepsilon(u) = \int_\Omega \frac{1}{2}\left(\varepsilon_- \vee |\nabla u| \wedge \varepsilon_+\right)^{p-2} |\nabla u|^2 + \left(\frac{1}{p} - \frac{1}{2}\right)\left(\varepsilon_- \vee |\nabla u| \wedge \varepsilon_+\right)^p \,\mathrm{d}x - \langle f, u\rangle.$$

Now, for this relaxed energy $\mathcal{J}_\varepsilon$, one can proof analogously as in Proposition 8.3 the following result.

**Proposition 8.6.** *The following conditions are equivalent for $u_\varepsilon \in W_0^1(L_2(\Omega))$.*

*(i) $u_\varepsilon$ is a minimizer of $\mathcal{J}_\varepsilon$:*

$$\mathcal{J}_\varepsilon(u_\varepsilon) \leq \mathcal{J}_\varepsilon(v) \qquad \text{for all} \qquad v \in W_0^1(L_2(\Omega)).$$

*(ii) The first variation vanishes:*

$$\int_\Omega \left(\varepsilon_- \vee |\nabla u_\varepsilon| \wedge \varepsilon_+\right)^{p-2} \nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x = \langle f, v\rangle \qquad \text{for all} \quad v \in W_0^1(L_2(\Omega)). \tag{8.2.12}$$

Hence, the above result allows to formulate the relaxed $p$-Poisson equation (8.2.12) as an energy minimization problem as well. Now we have everything at hand to summarize the convergence results mentioned at the beginning of this chapter.

## 8.3 Convergence results

In this section we outline the convergence analysis done in [53] for **Algorithm 1**. Therefore, in the following let $1 < p \leq 2$ and $f \in W^{-1}(L_{p'}(\Omega))$, where $\Omega \subset \mathbb{R}^d$ denotes some bounded domain.

Let us settle the notation for this section first. By $u \in W_0^1(L_p(\Omega))$ we denote the (unique) exact solution to the $p$-Poisson equation with homogeneous Dirichlet boundary conditions (3.2.3). For $\varepsilon \subset (0,\infty)$, by $u_\varepsilon \in W_0^1(L_2(\Omega))$ we denote the unique minimizer with vanishing trace of the relaxed energy functional $\mathcal{J}_\varepsilon$, see (8.2.11), i.e.,

$$u_\varepsilon := \operatorname*{argmin}_{u \in W_0^1(L_2(\Omega))} \mathcal{J}_\varepsilon(u).$$

Recall that $u_\varepsilon$ is also the solution to the relaxed $p$-Poisson equation (8.2.12)), see Proposition 8.6. Finally, by $u_n \in W_0^1(L_2(\Omega))$, $n \in \mathbb{N}$, we denote the iterates generated by **Algorithm 1**.

For the subsequent results to hold true, we assume that all iterates $u_n$ - i.e., the solutions to the linear elliptic subproblems (8.1.5) - are computed *exactly*. Hence, the discretization error which in practice is introduced by the numerical solution of the linear elliptic subproblems shall be neglected here.

At first, let us consider the approximation error with respect to the minimizer $u_\varepsilon$ of the relaxed energy functional $\mathcal{J}_\varepsilon$. The iterates $u_n$ of the regularized Kačanov iteration converge to $u_\varepsilon$ as follows. For a proof see [53, Corollary 4.2].

**Proposition 8.7.** *Let $u_n$, $n \geq \mathbb{N}$, denote the iterates generated by the regularized Kačanov iteration and let $u_\varepsilon$ denote the minimizer of $\mathcal{J}_\varepsilon$. Then there exists a constant $c > 1$, such that*

$$\mathcal{J}_\varepsilon(u_n) - \mathcal{J}_\varepsilon(u_\varepsilon) \leq (1-\delta)^n \left(\mathcal{J}_\varepsilon(u_0) - \mathcal{J}_\varepsilon(u_\varepsilon)\right),$$

*where $\delta = c^{-1}\,(\varepsilon_-/\varepsilon_+)^{2-p}$.*

Next we will see that the relaxation performed so far is indeed consistent with the original $p$-Poisson problem (3.2.3), in the sense that $u_\varepsilon \to u$ for $\varepsilon_- \to 0$, $\varepsilon_+ \to \infty$. For a proof of the following result, see [53, Corollary 3.4 & Corollary 3.8].

**Proposition 8.8.** *It holds $\mathcal{J}_\varepsilon(u_\varepsilon) \to \mathcal{J}(u)$ and $\mathcal{J}(u_\varepsilon) \to \mathcal{J}(u)$ as $\varepsilon \to [0,\infty]$. Furthermore, it holds $u_\varepsilon \to u$ in $W_0^1(L_p(\Omega))$ for $\varepsilon \to [0,\infty]$.*

The next result is proved in [53, Corollary 3.14].

**Proposition 8.9.** *Let $\Omega$ be a polyhedral domain and $f \in L_{p'}(\Omega)$, $1/p + 1/p' = 1$. Then*

$$\mathcal{J}_\varepsilon(u_\varepsilon) - \mathcal{J}(u) \lesssim \varepsilon_-^p + \varepsilon_+^{-p/(d-1)}.$$

Hence, from Proposition 8.7 and Proposition 8.9 we conclude convergence of the full error $\mathcal{J}_\varepsilon(u_n) - \mathcal{J}(u) \to 0$ for $\varepsilon \to [0,\infty]$, $n \to \infty$.

Finally, if $\varepsilon_{n,-} \to 0$ and $\varepsilon_{n,+} \to \infty$ for $n \to \infty$ are steered adequately, an overall algebraic convergence rate in the following sense can be guaranteed. For a proof see [53, Theorem 5.3].

**Theorem 8.10.** *Let $\Omega$ be a polyhedral domain, $f \in L_{p'}(\Omega)$, where $1/p + 1/p' = 1$, and let $\alpha, \beta > 0$ such that $\alpha + \beta < 1/(2-p)$. Moreover, let $u_n$ denote the iterates generated by **Algorithm 1**, where the truncation intervals are chosen as*

$$\varepsilon_n := \left[(n+1)^{-\alpha}, (n+1)^{\beta}\right], \qquad n \in \mathbb{N}.$$

*Then there exists a constant $c \geq 1$ such that*

$$\mathcal{J}_{\varepsilon_n}(u_n) - \mathcal{J}(u) \lesssim n^{-1/c}$$

*for all $n \in \mathbb{N}$. In particular, the energy error decreases at least algebraically.*

## 8.4 An adaptive wavelet frame Kačanov-type algorithm

In this section a fully implementable algorithm for problem (3.2.3) is presented. Therefore, it remains to treat the numerical approximation of the infinite dimensional linear elliptic problems (8.1.5) which have to be solved at each iteration of **Algorithm 1**. We want to pursue an adaptive approach based on wavelets here.

Given the case that we have an appropriate wavelet basis for $W_0^1(L_2(\Omega))$ over the bounded (polyhedral) domain $\Omega \subset \mathbb{R}^d$ at our disposal, then we shall utilize the adaptive wavelet Galerkin scheme introduced in [18]; there, besides convergence also quasi-optimality of this residual-based approach has been proved.

As a practical alternative to methods based on wavelet bases (in particular when dealing with more complicated domain geometries) we additionally want to consider an approach based on overlapping domain decompositions. In particular, we will employ the multiplicative Schwarz adaptive wavelet frame method as introduced in [124], see also [134, Chapter 6]. In the following description of this scheme we will put the focus on the main principle of the multiplicative Schwarz method. For the complete algorithm as well as further details we refer to [134, Algorithm 6, Chapter 6.1].

Therefore, we first refer to the exact multiplicative Schwarz method for the approximation of a linear elliptic equation of the form

$$a(u,v) = f(v) \quad \text{for all} \quad v \in W_0^s(L_2(\Omega)), \tag{8.4.1}$$

where $f \in W^{-s}(L_2(\Omega))$, $s \in \mathbb{N}$, and the symmetric bilinear form $a(\cdot,\cdot)$ is bounded and elliptic on $W_0^s(L_2(\Omega))$. Here, we assume that $\Omega$ is decomposed by an overlapping covering of $N_\Omega$ patches $\Omega_i \subset \Omega$, i.e., $\Omega = \cup_{i=0}^{N_\Omega-1}\Omega_i$. The exact multiplicative Schwarz method is presented in **Algorithm 3**.

**Algorithm 3** (Multiplicative Schwarz method).

*Data:* Given $f \in W^{-s}(L_2(\Omega))$, $u_0 \in W_0^s(L_2(\Omega))$;
*Result:* Approximate solution of the elliptic problem (8.4.1);
*Initialize:* $u_0 := 0$, $n = 0$;

`for` $k = 1, 2, \ldots,$
  $i := (k-1) \bmod m$
  Determine $e_{k-1} \in W_0^s(L_2(\Omega_i))$ as the solution of the problem

$$a(e_{k-1}, v) = f(v) - a(u_{k-1}, v) \quad \text{for all} \quad v \in W_0^s(L_2(\Omega_i)). \tag{8.4.2}$$

  $u_k := u_{k-1} + e_{k-1}$
`endfor`

The convergence of **Algorithm 3** is guaranteed under certain assumptions on the domain decomposition (i.e., on the subspaces $W_0^s(L_2(\Omega_i))$), see [134, Theorem

6.1]. However, for the L-shaped domain $\Omega_L$ as defined in (1.5.8), which will be used for our numerical tests in the next chapter, these conditions are satisfied (cf. [134, Remark 6.1 & Chapter 2]).

Next, to obtain an implementable version of the exact multiplicative Schwarz method, clearly the problems (8.4.2) on the subdomains $\Omega_i$ have to be numerically approximated. To this end, let $\Psi = \cup_{i=0}^{N_\Omega-1}\Psi^{(i)}$ be an aggregated wavelet frame composed of biorthogonal wavelet Riesz bases $\Psi^{(i)}$ for $W_0^s(L_2(\Omega_i))$, cf. Subsection 1.5.2. Then, based on a discretization of the elliptic problem (8.4.2) by means of the basis $\Psi^{(i)}$, an adaptive wavelet method shall be applied. To formulate the resulting (inexact) multiplicative Schwarz algorithm, we denote this adaptive wavelet scheme by **AdaptWav**.

**Algorithm 4 (MultSchw).**

*Data:* Given $f \in W^{-s}(L_2(\Omega))$, $u_0 \in W_0^s(L_2(\Omega))$;
*Result:* Approximate solution of the elliptic problem (8.4.1);
*Initialize:* $u_0 := 0$, $n = 0$;

`for` $k = 1, 2, \ldots,$

$i := (k-1) \bmod m$

Apply **AdaptWav** to determine approximate solution $\tilde{e}_{k-1} \in W_0^s(L_2(\Omega_i))$ to the problem

$$a(e_{k-1}, v) = f(v) - a(u_{k-1}, v) \quad \text{for all} \quad v \in W_0^s(L_2(\Omega_i)).$$

$u_k := u_{k-1} + \tilde{e}_{k-1}$

`endfor`

With the help of **Algorithm 4** we can now formulate a fully implementable variant of the relaxed Kačanov algorithm.

**Algorithm 5 (rKačanov).**

*Data:* Given $f \in W^{-1}(L_{p'}(\Omega))$, $u_0 \in W_0^1(L_2(\Omega))$;
*Result:* Approximate solution of the $p$-Poisson problem (3.2.3);
*Initialize:* $\varepsilon_0 = [\varepsilon_{0,-}, \varepsilon_{0,+}] \subset (0, \infty)$, $n = 0$;

`while` desired accuracy is not achieved yet `do`

Apply **MultSchw** to determine approximate solution $\tilde{u}_{n+1} \in W_0^1(L_2(\Omega))$ to the problem

$$\int_\Omega (\varepsilon_{n,-} \vee |\nabla \tilde{u}_n| \wedge \varepsilon_{n,+})^{p-2} \nabla u_{n+1} \cdot \nabla v \, \mathrm{d}x = \langle f, v \rangle \quad \text{for all} \quad v \in W_0^1(L_2(\Omega));$$

Choose new relaxation interval $\varepsilon_{n+1} \supset \varepsilon_n$;

Increase $n$ by 1;

`end`

For the numerical experiments in the next chapter, as **AdaptWav** we will always choose the adaptive wavelet method from [18].

# Chapter 9

# Numerical Tests

For all numerical computations in this chapter, we consider the $p$-Poisson problem (3.2.3) for $1 < p < 2$, i.e.,

$$-\operatorname{div}\Big(|\nabla u|^{p-2}\,\nabla u\Big) = f \qquad \text{in } \Omega,$$
$$u = 0 \qquad \text{on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^2$ denotes a polygonal domain. Furthermore, all examples are constructed in such a way that we start with a given solution $u \in W_0^1(L_p(\Omega))$, and set the right-hand side to

$$f(v) := \int_\Omega |\nabla u|^{p-2}\,\nabla u \cdot \nabla v \,\mathrm{d}x \tag{9.0.1}$$

for $v \in W_0^1(L_p(\Omega))$. From (3.1.4) we know that $f$ is bounded on $W_0^1(L_p(\Omega))$ with operator norm $\|f\| \lesssim \|u\,|\,W^1(L_p(\Omega))\|^{p-1}$, i.e., $f \in W^{-1}(L_{p'}(\Omega))$. Hence, the unique solvability of all subsequent problems is guaranteed (cf. Proposition 3.2) and algorithm **rKačanov** is well-defined. In case $u$ is sufficiently smooth, we may as well define the right-hand side $f$ pointwise as

$$f := -\Delta_p u = -(p-2)|\nabla u|^{p-4}\left(u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}\right) - |\nabla u|^{p-2}\Delta u. \tag{9.0.2}$$

The initial guess $u_0 \in W_0^1(L_2(\Omega))$ required by **rKačanov** is always chosen as $u_0 \equiv 0$.

## 9.1 Tests on the unit square

In this section we consider the unit square $\Omega^\square := (0,1)^2$. For discretization, a biorthogonal tensor product wavelet basis constructed from univariate spline wavelet bases on the unit interval as introduced in [110] is used. In particular we utilize linear spline wavelets having $m = 2$ vanishing moments. As already noted at the beginning of Section 8.4, for the solution of the linear subproblems (8.1.5) on $\Omega^\square$ the adaptive wavelet Galerkin method from [18] is applied in this case. That is, the frame method **MultSchw** in **rKačanov** is replaced by this wavelet method.

### 9.1.1 Example 1: smooth solution

For our first numerical experiment we choose a smooth solution $u \in C^\infty(\Omega^\square) \subset W^1(L_p(\Omega^\square))$, namely

$$u : [0,1]^2 \to \mathbb{R},$$
$$u(x,y) = x\,(x-1)\,y\,(y-1)\,.$$

The function $u$ is depicted in Figure 9.2(d). Here, the functional $f \in W^{-1}(L_{p'}(\Omega^\square))$ as defined in (9.0.1) also admits the following pointwise representation. With

$$\nabla u(x,y) = \begin{pmatrix} (2x-1)\,y\,(y-1) \\ (2y-1)\,x\,(x-1) \end{pmatrix}, \quad D^2 u(x,y) = \begin{pmatrix} 2y\,(y-1) & (2x-1)\,(2y-1) \\ (2y-1)\,(2x-1) & 2x\,(x-1) \end{pmatrix},$$

and (9.0.2), by setting $G(x,y) := (2x-1)^2\,(y^2-y)^2 + (2y-1)^2\,(x^2-x)^2$ and $H(x,y) := (2x-1)^2(y^2-y)^3 + (2y-1)^2(x^2-x)^3 + (2x-1)^2(x^2-x)(2y-1)^2(y^2-y)$, the right-hand side takes the form

$$f(x,y) = (4-2p)\,G(x,y)^{(p-4)/2} H(x,y) - 2G(x,y)^{(p-2)/2}\left(x^2+y^2-x-y\right).$$

A short computation shows that in the vicinity of each root $(x_0,y_0) \in \mathbb{R}^2$ of $G$, the growth of $f$ can be estimated as $|f(x,y)| \lesssim |(x,y)-(x_0,y_0)|^{p-2}$. Hence, we conclude that $f \in L_q(\Omega^\square)$ for all $1 \le q < 2/(2-p)$.

For our first numerical test we choose $p = 1.5$. The right-hand side corresponding to this value of $p$ is illustrated in Figure 9.1. Note that since $p' = 3 < 2/(2-p) = 4$, in this case in particular it holds $f \in L_{p'}(\Omega^\square)$. Moreover, we shall keep the relaxation interval constant in the course of the Kačanov iteration. We choose $\varepsilon_n = [\varepsilon_{n,-}, \varepsilon_{n,+}] = [10^{-3}, 10^3]$ for all $n \in \mathbb{N}_0$.

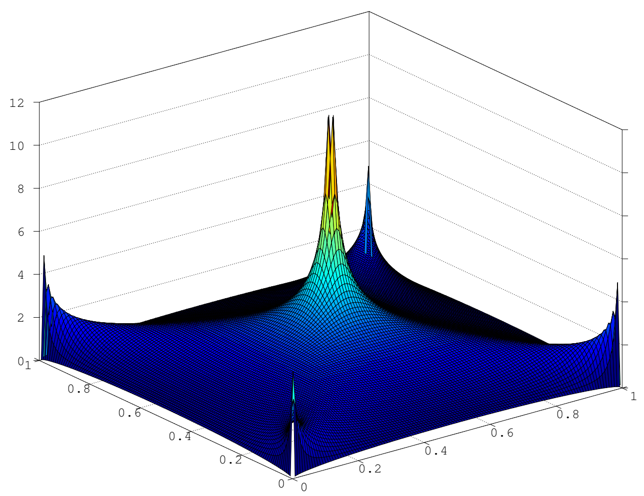
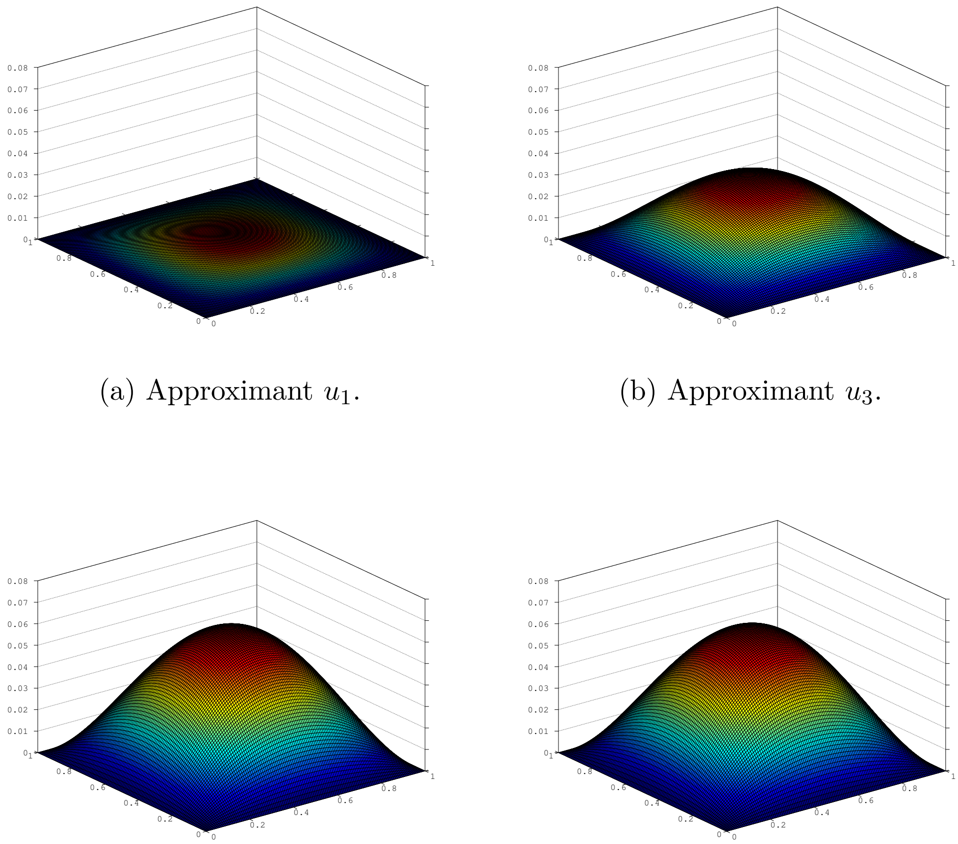



Figure 9.1: Right-hand side $f$ of example 1 for $p = 1.5$.

The approximations $u_1, u_3$ and $u_{10}$ generated by **rKačanov** after one, three and ten iterations, respectively, are depicted in Figure 9.2(a)-(c).

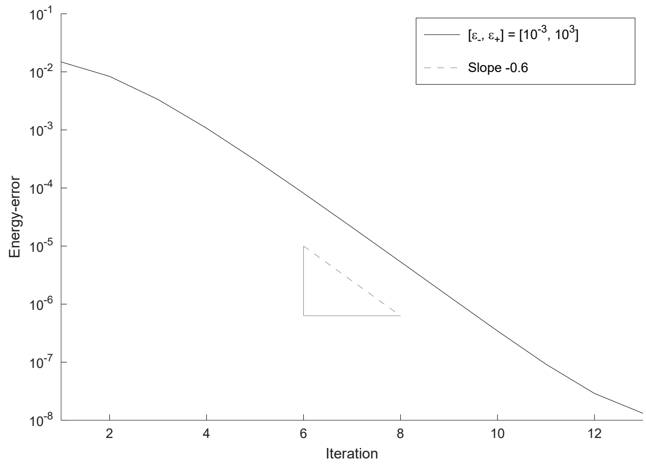
In Figure 9.3 the error decay of **rKačanov** for example 1 with $p = 1.5$ is illustrated for various error measures. By the pictured slope of $-0.6$ in Figure 9.3(a), we observe that within a limited range for $n$ the energy error is reduced by a constant factor of approximately 1/4 at each iteration, i.e., $\mathcal{J}(u_{n+1}) - \mathcal{J}(u) \sim 1/4\,(\mathcal{J}(u_n) - \mathcal{J}(u))$. Measured in the $W^1(L_p(\Omega^\square))$- and $(L_p(\Omega^\square)$-norm, the error is reduced by a factor of about 1/2 per iteration. Hence, we can observe linear convergence with respect to all error measures.



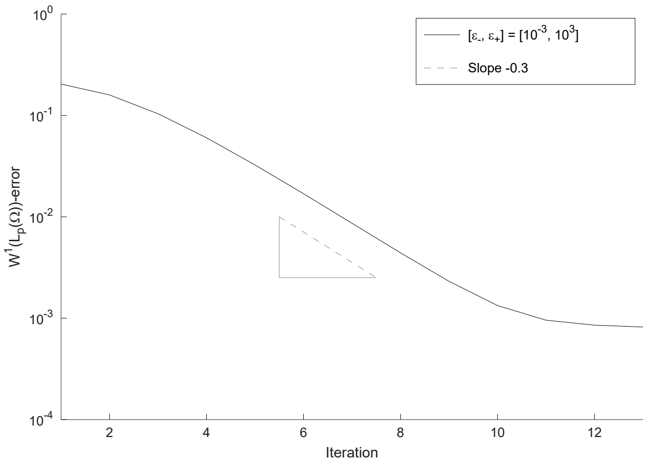
(a) Approximant $u_1$. (b) Approximant $u_3$.

(c) Approximant $u_{10}$. (d) Exact solution $u$.

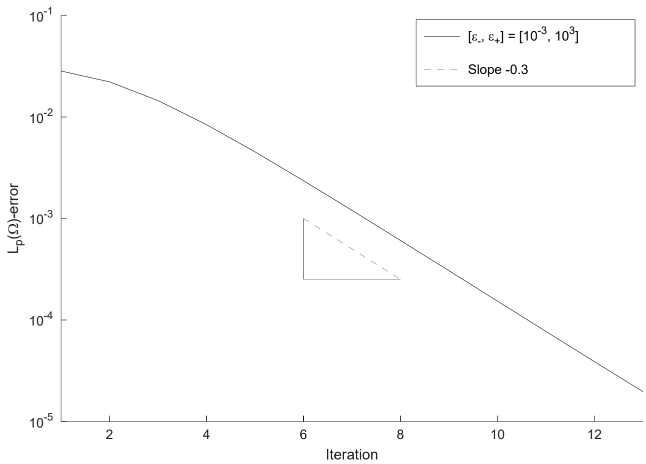
Figure 9.2: From top left to bottom right: approximations $u_1$, $u_3$, $u_{10}$ and the exact solution $u$ of example 1.




(a) Energy error.




(b) $W^1(L_p)$-error.




(c) $L_p$-error.

Figure 9.3: Error plots for example 1 with $p = 1.5$ on $\Omega^{\square}$: Energy-, $W^1(L_p)$- and $L_p$-error.

### 9.1.2 Example 2: solution of absolute value-type

For our next practical test we consider the function

$$u : [0,1]^2 \to \mathbb{R},$$
$$u(x,y) = \max\left\{\frac{1}{2} - \left|(x,y) - \left(\frac{1}{2},\frac{1}{2}\right)\right|,\ 0\right\}.$$

A graphical representation of $u$ is given in Figure 9.5(d). With $B := B_{1/2}((1/2,1/2))$ (note that $\operatorname{supp} u = \overline{B}$), the gradient of $u$ for $(x,y) \in \mathcal{G} := \Omega^{\square} \setminus \{\partial B \cup (1/2,1/2)\}$ takes the form

$$\nabla u(x,y) = \frac{\chi_B(x,y)}{\left|(x,y) - (\frac{1}{2},\frac{1}{2})\right|} \cdot \begin{pmatrix} \frac{1}{2} - x \\ \frac{1}{2} - y \end{pmatrix}.$$

Since $|\nabla u| = \chi_B$ on $\mathcal{G}$, we conclude that $u \in W^1(L_\infty(\Omega^{\square}))$. We remark in passing that hence the functional $f$ defined by (9.0.1) is contained in $W^{-1}(L_{p'}(\Omega^{\square}))$. The second partial derivatives of $u$ on $\mathcal{G}$,

$$\frac{\partial^2 u}{\partial x^2}(x,y) = \frac{-(y-1/2)^2}{|(x,y)-(1/2,1/2)|^3} \cdot \chi_B, \qquad \frac{\partial^2 u}{\partial y^2}(x,y) = \frac{-(x-1/2)^2}{|(x,y)-(1/2,1/2)|^3} \cdot \chi_B,$$

and

$$\frac{\partial^2 u}{\partial x \partial y}(x,y) = \frac{(x-1/2)(y-1/2)}{|(x,y)-(1/2,1/2)|^3} \cdot \chi_B = \frac{\partial^2 u}{\partial y \partial x}(x,y),$$

are bounded by $|(x,y)-(1/2,1/2)|^{-1}$, and therefore $u \in W^2(L_q(\Omega^{\square}))$ for all $1 \leq q < 2$. Moreover, since

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = -\frac{1}{|(x,y)-(1/2,1/2)|} \notin L_2(\Omega^{\square}),$$

we conclude that $u \notin W^2(L_2(\Omega^{\square}))$.

According to (9.0.2), the pointwise representation of $f$ on $\mathcal{G}$ computes to

$$f(x,y) = \frac{\chi_B}{\left|(x,y) - \left(\frac{1}{2},\frac{1}{2}\right)\right|}.$$

We note that in this example the right-hand side does not depend on the parameter $p$, and that $f \in L_q(\Omega^{\square})$ for all $1 \leq q < 2$, but $f \notin L_2(\Omega^{\square})$. In particular, $f \notin L_{p'}(\Omega^{\square})$ for all $1 < p < 2$. A pictorial representation of $f$ is given in Figure 9.4.

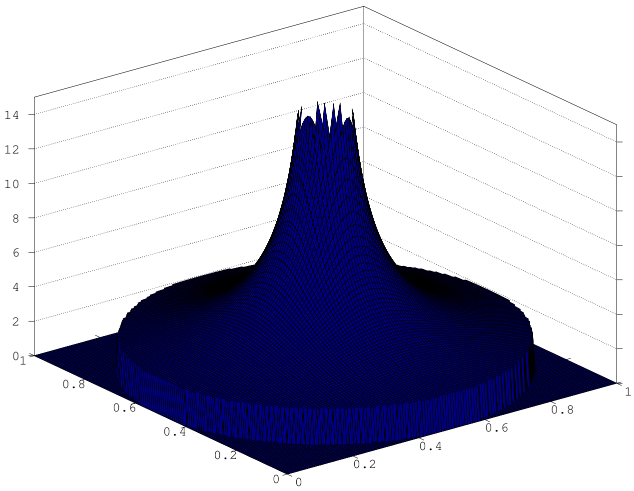
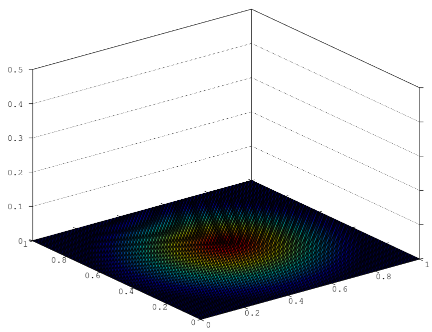
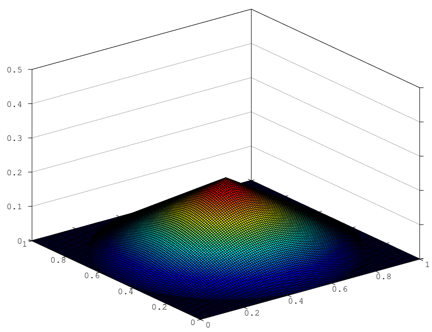


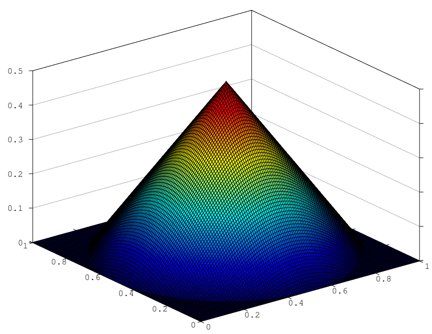

Figure 9.4: Right-hand side $f$ of example 2.



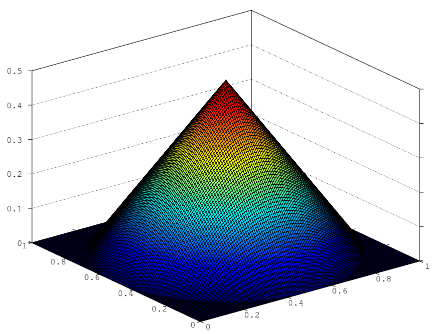
(a) Approximant $u_1$.



(b) Approximant $u_3$.



(c) Approximant $u_{10}$.



(d) Exact solution $u$.

Figure 9.5: From top left to bottom right: approximations $u_1$, $u_3$, $u_{10}$ and the exact solution $u$ of example 2.

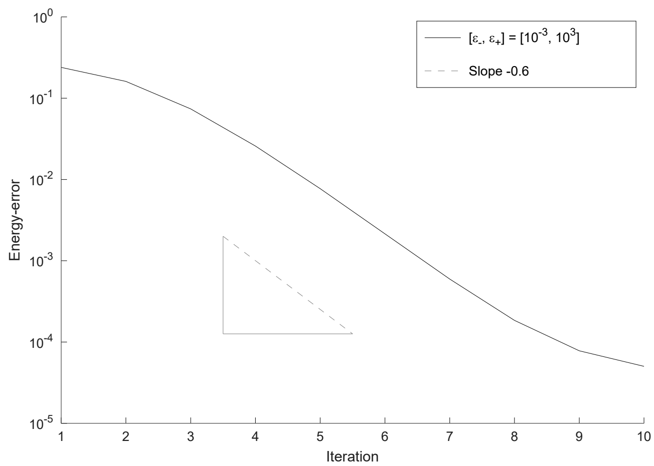
The energy of $u$, see (8.2.1), can be determined explicitly as

$$J(u) = \int_{\Omega^\square} \frac{1}{p} |\nabla u|^p \, \mathrm{d}x - f(u) = \left(\frac{1}{p} - 1\right) \int_{\Omega^\square} |\nabla u|^p \, \mathrm{d}x = \left(\frac{1}{p} - 1\right) \frac{\pi}{4},$$

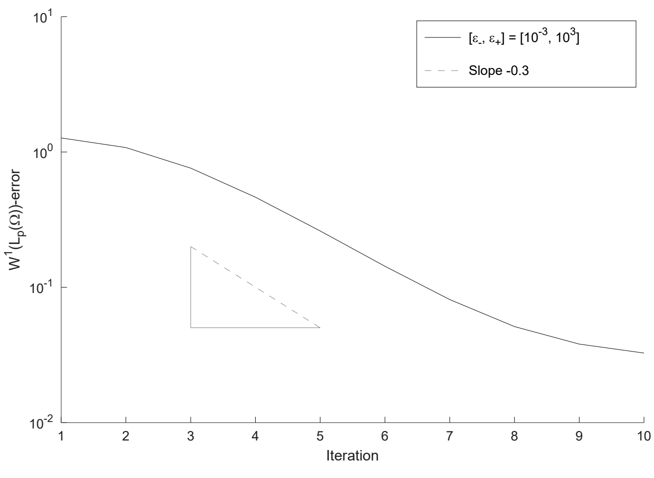
where we used that $|\nabla u| = \chi_B$.

For our second test run of **rKačanov** we choose the same parameters as in example 1, i.e., $p = 1.5$ and $\varepsilon_n = [\varepsilon_{n,-}, \varepsilon_{n,+}] = [10^{-3}, 10^3]$ for all $n \in \mathbb{N}_0$. Analogous to the first test run, the approximations $u_1, u_3$ and $u_{10}$ generated by **rKačanov** after one, three and ten iterations, respectively, are depicted (Figure 9.5(a)-(c)) and error plots are presented with respect to various error measures (Figure 9.6).

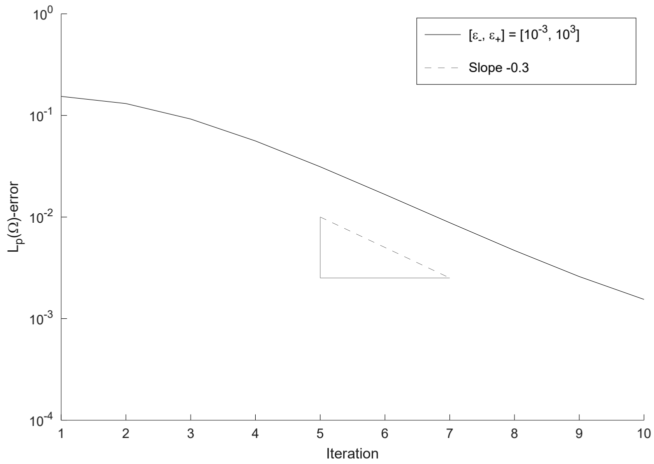
The observed error decay is of approximately the same order as in example 1, although slightly worse. For instance, in regard of the $W^1(L_p)$-error, the smallest ratio $\left\|u - u_{n+1} \,\middle|\, W^1(L_p(\Omega^\square))\right\| / \left\|u - u_n \,\middle|\, W^1(L_p(\Omega^\square))\right\|$ was achieved for $n = 5$ with a value of about 0.55, compared to a value of 0.48 for $n = 7$ in the first test run.




(a) Energy error.




(b) $W^1(L_p)$-error.




(c) $L_p$-error.

Figure 9.6: Error plots for example 2 with $p = 1.5$ on $\Omega^{\square}$: Energy-, $W^1(L_p)$- and $L_p$-error.

## 9.2 Tests on the L-shaped domain

For all subsequent numerical tests we consider the L-shaped domain as defined in (1.5.8), i.e.,

$$\Omega_L = (-1,1)^2 \setminus [0,1)^2. \tag{9.2.1}$$

For discretization, we use an aggregated wavelet frame constructed from tensor product type wavelets bases, where again the univariate wavelet basis from [110] is used. Here, we utilize quadratic spline wavelets having order $m = 3$ of polynomial reproduction. As noted above, for the solution of the linear subproblems (8.1.5) on $\Omega_L$, the adaptive multiplicative Schwarz frame method as introduced in [124] is applied in this case, i.e., algorithm **MultSchw** of Section 8.4.

### 9.2.1 Example 3: smooth solution

We start with a smooth solution $u \in C^\infty(\Omega_L)$ to the $p$-Poisson problem (3.2.3) on $\Omega_L$, namely

$$\begin{aligned} &u : \Omega_L \to \mathbb{R}, \\ &u(x,y) = x\left(1-x^2\right) y \left(1-y^2\right). \end{aligned} \tag{9.2.2}$$

A graphical representation of the function $u$ is given in Figure 9.7.

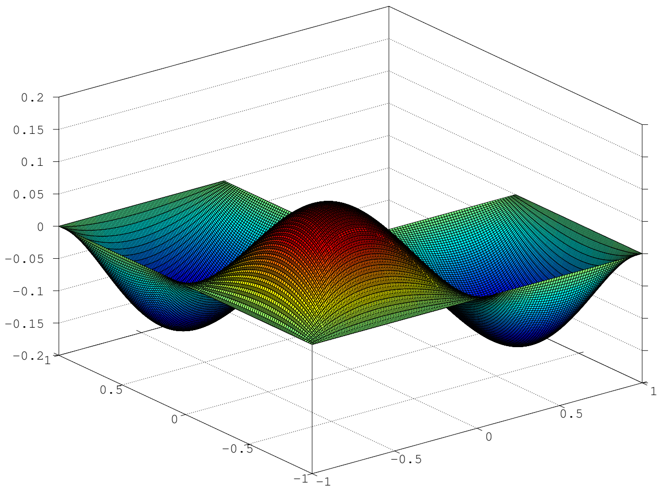
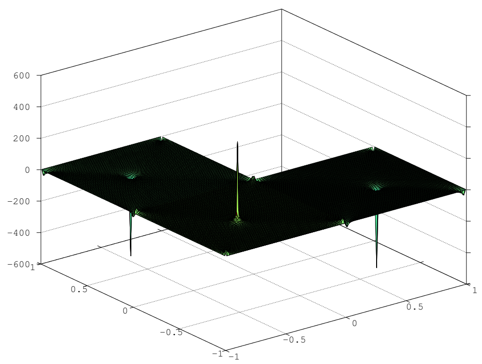
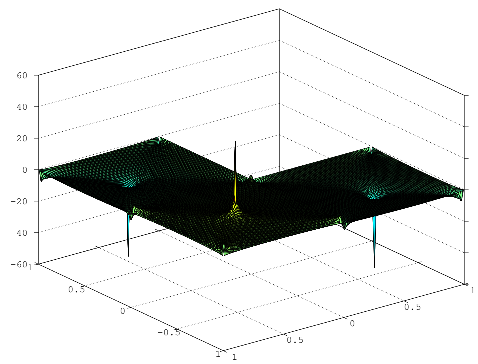


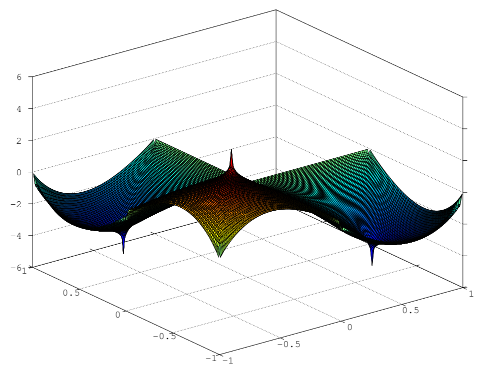
Figure 9.7: Exact solution of example 3.



(a) $p = 1.1$.



(b) $p = 1.5$.



(c) $p = 1.9$.

Figure 9.8: Right-hand side of example 3 for different values of $p$.

With

$$\nabla u(x,y) = \begin{pmatrix} (1-3x^2)(y-y^3) \\ (x-x^3)(1-3y^2) \end{pmatrix}, D^2u(x,y) = \begin{pmatrix} -6x(y-y^3) & (1-3x^2)(1-3y^2) \\ (1-3x^2)(1-3y^2) & -6y(x-x^3) \end{pmatrix},$$

and by setting $G(x,y) := (1-3x^2)^2(y-y^3)^2 + (x-x^3)^2(1-3y^2)^2$ and $H(x,y) := 2(1-3x^2)^2(y-y^3)(x-x^3)(1-3y^2)^2 - (1-3x^2)^2(y-y^3)^3 6x - (x-x^3)^3(1-3y^2)^2 6y$, according to (9.0.2) the right-hand side $f$ computes to

$$f(x,y) = (2-p)G(x,y)^{(p-4)/2}H(x,y) + 6G(x,y)^{(p-2)/2}\left[x\left(y-y^3\right) + y\left(x-x^3\right)\right].$$

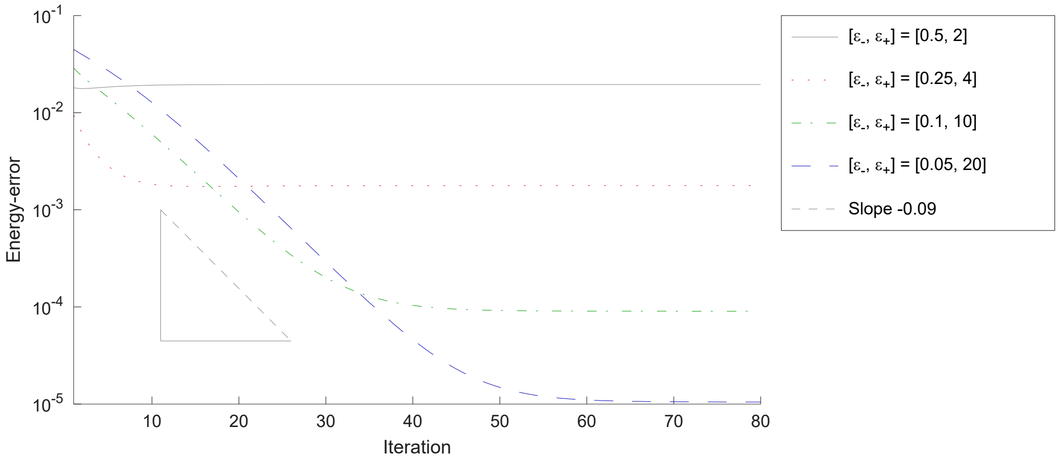
Similar to Subsection 9.1.1, one estimates $|f(x,y)| \lesssim |(x-x_0, y-y_0)|^{p-2}$ in the vicinity of the roots $(x_0, y_0)$ of $G$, such that $f \in L_q(\Omega_L)$ for all $1 \leq q < 2/(2-p)$.

We shall perform several test runs for $p = 1.1$, $p = 1.5$ and $p = 1.9$. In Figure 9.8 the right-hand side $f$ of the $p$-Poisson equation, corresponding to the solution (9.2.2), is depicted for these values of $p$. Note that $f \notin L_{p'}(\Omega_L)$ for $p = 1.1$.

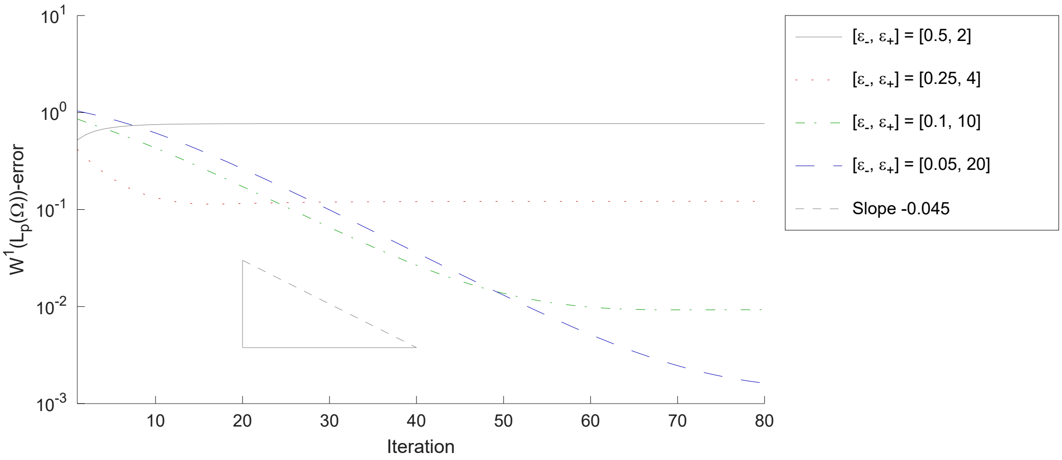
### Tests for p=1.1

For our first numerical tests on $\Omega_L$, we choose the relaxation interval to be constant in the course of the Kačanov iteration, i.e., $\varepsilon_n = [\varepsilon_{n,-}, \varepsilon_{n,+}] := [\varepsilon_-, \varepsilon_+]$ for all $n \in \mathbb{N}_0$. The results, in terms of error plots with respect to various error measures, are shown in Figure 9.9 for several different values of $[\varepsilon_-, \varepsilon_+]$.

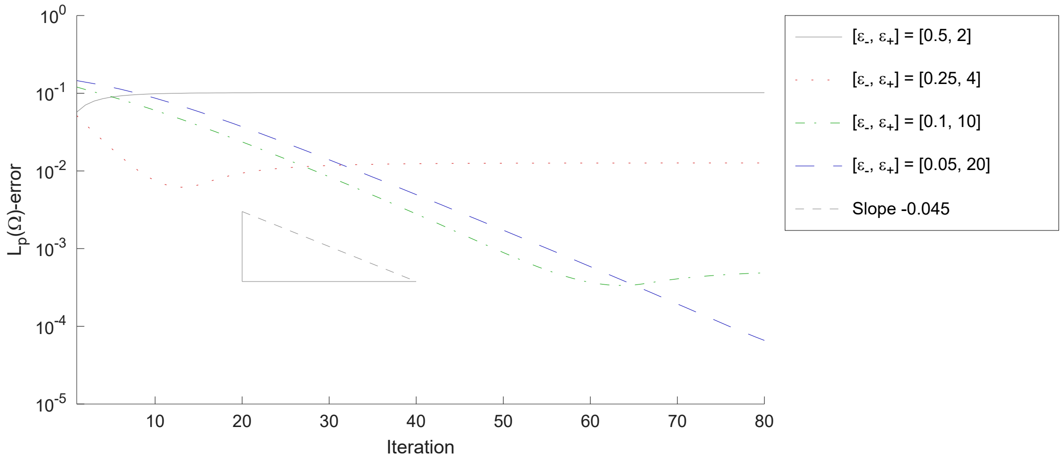
We can observe that with respect to all error measures, no error reduction is obtained for the relaxation interval $[\varepsilon_-, \varepsilon_+] = [0.5, 2]$, and that the approximation accuracy which is achieved after 80 iterations increases as the relaxation interval gets larger. For the two largest intervals, within a limited range for $n$ the energy error is reduced by a constant factor of approximately 0.8 at each iteration, see Figure 9.9(a), whereas the $L_p$- and $W^1(L_p)$-error are reduced by a factor of approximately 0.9 per iteration.




(a) Energy error.




(b) $W^1(L_p)$-error.




(c) $L_p$-error.

Figure 9.9: Error plots for example 3 and $p = 1.1$ for different constant values of the relaxation parameter $\varepsilon$: Energy-, $W^1(L_p)$- and $L_p$-error.

### Tests for p=1.5

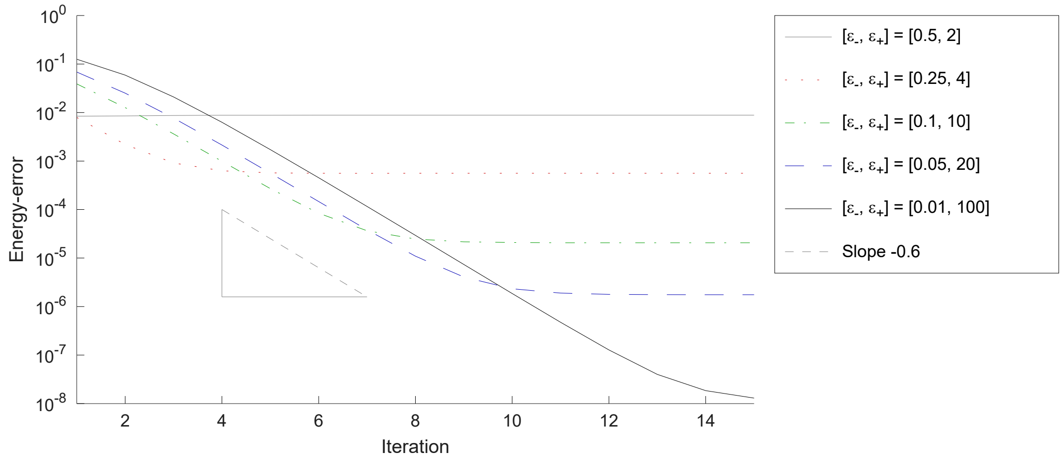
At first, we again choose the relaxation interval $\varepsilon = [\varepsilon_-, \varepsilon_+]$ to be constant in the course of the Kačanov iteration. The results for $p = 1.5$ and various values of $\varepsilon$ can be seen in Figure 9.10(a)-(c) for the energy error, $W^1(L_p(\Omega_L))$-error and $L_p(\Omega_L)$-error, respectively.

Again, the maximal approximation accuracy increases (at least with respect to the energy- and $W^1(L_p)$-error) with growing relaxation interval. However, the error decay is much faster than for $p = 1.1$, with similar decrease as in example 1.

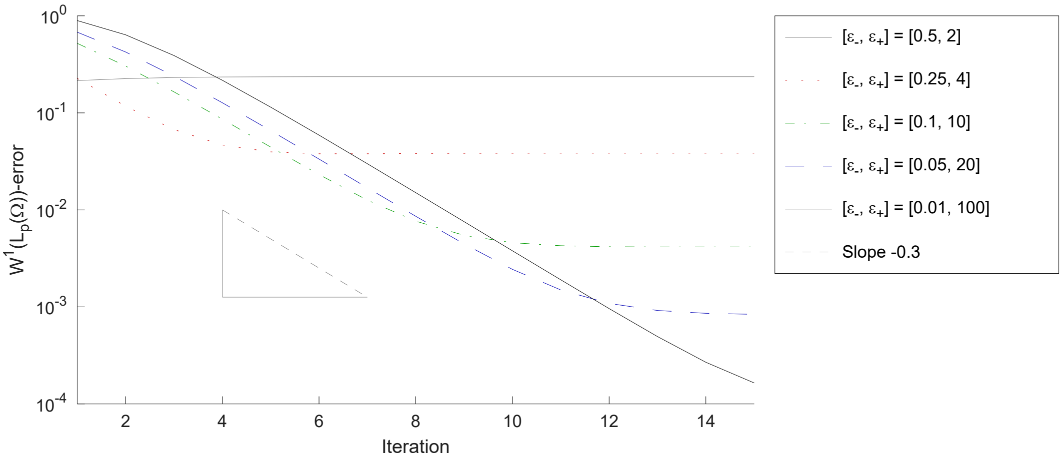
In our next test runs, we want to pursue several different strategies regarding the way how to decrease $\varepsilon_-$ and increase $\varepsilon_+$, i.e., how to enlarge the size of the relaxation interval $\varepsilon$, in the course of the Kačanov iteration. In particular, we choose

$$\varepsilon_n = \left[(n+1)^{-\alpha}, (n+1)^{\alpha}\right], \qquad n \in \mathbb{N}_0, \tag{9.2.3}$$

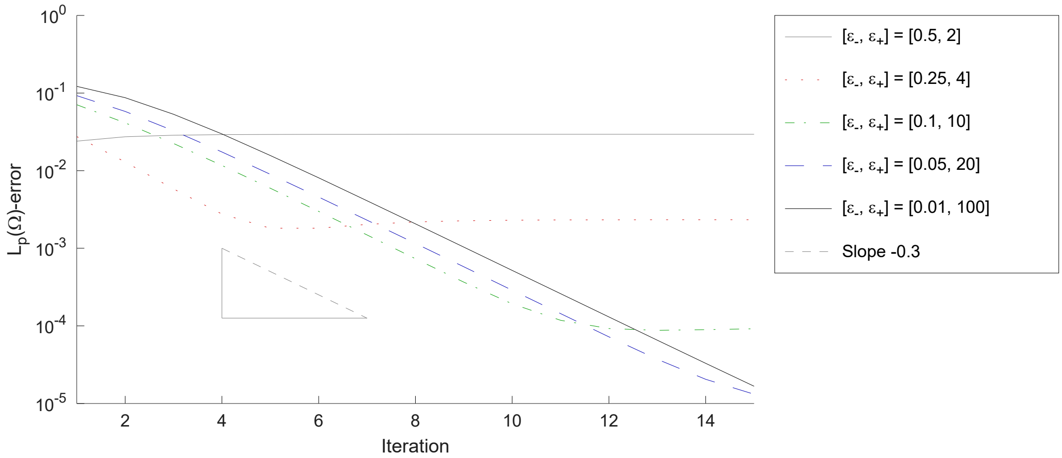
and perform tests for $\alpha = 0.9$, $\alpha = 1$ and $\alpha = 2$.




(a) Energy error.




(b) $W^1(L_p)$-error.

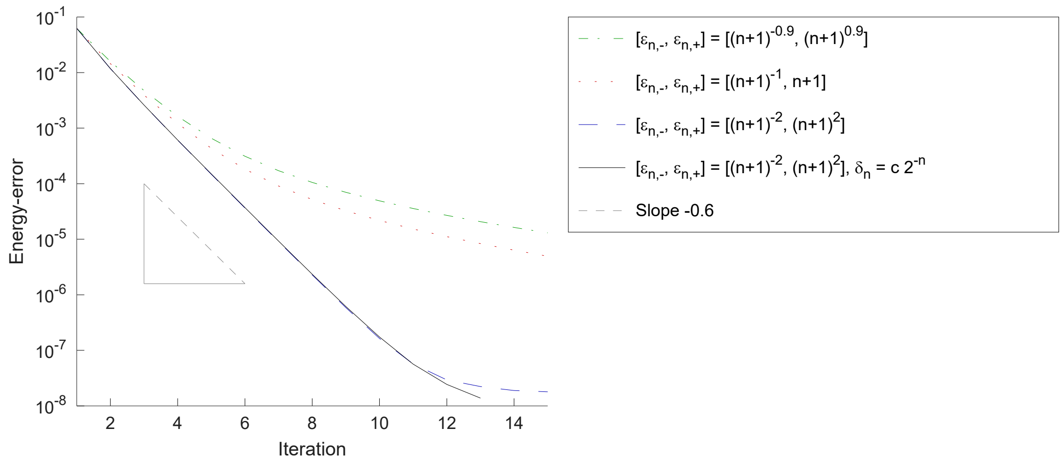



(c) $L_p$-error.

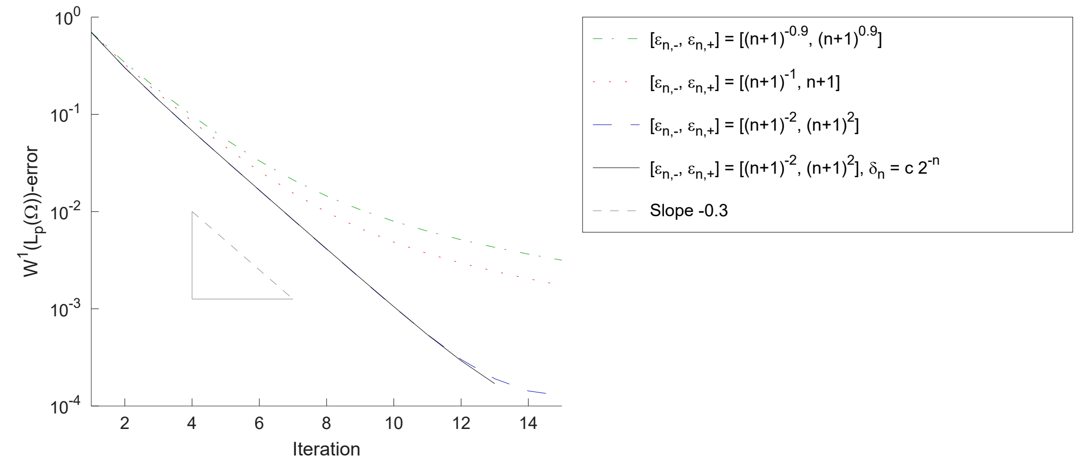
Figure 9.10: Error plots for example 3 and $p = 1.5$ for different constant values of the relaxation parameter $\varepsilon$: Energy-, $W^1(L_p)$- and $L_p$-error.

Hereby, the prescribed error tolerance for the solution of the linear subproblems (8.1.5), handed to **MultSchw** in terms of an $\ell_2$-tolerance $\delta$, is set to a rather small value, being constant for all $n \in \mathbb{N}_0$. Clearly, the accurate approximation of the problems (8.1.5) helps to highlight the influence of the relaxation parameter $\varepsilon$. However, a natural approach is to gradually decrease the error tolerance $\delta$ as $n$ increases. Therefore, we perform another test run with $\alpha = 2$ and $\delta_n = c\,2^{-n}$. Here, $c > 0$ denotes a suitably chosen constant. The respective results are given in Figure 9.11.

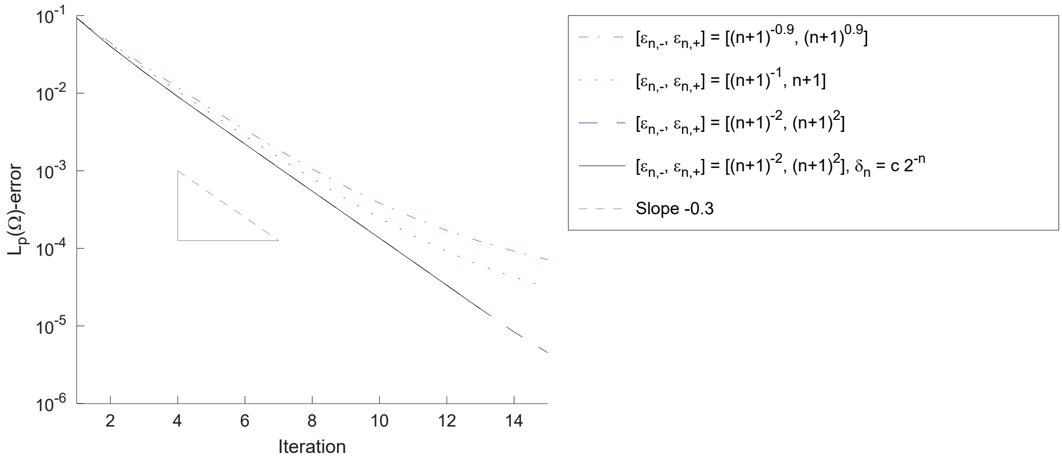
It can be seen that the fastest convergence is obtained for the choice of $\alpha = 2$ in (9.2.3). Moreover, the corresponding test run with decreasing error tolerances $\delta_n$ for the solution of the linear subproblems practically shows the same error decay as the one with a constant, small tolerance $\delta$. For the former test run, in Figure 9.12 an additional error plot is given, where the error in relation to the degrees of freedom is depicted. In regard to the energy- and $W^1(L_p)$-error, a convergence rate of approximately 2 and 1 can be observed, respectively.




(a) Energy error.

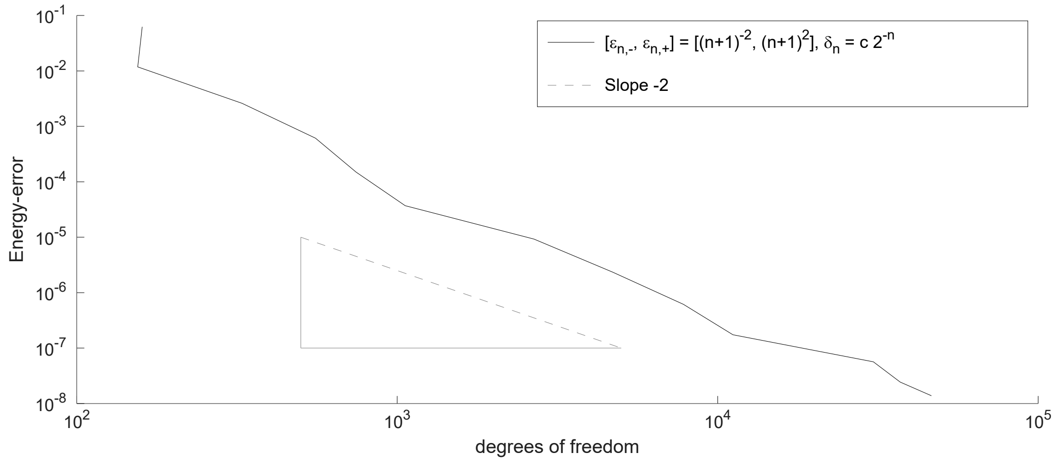



(b) $W^1(L_p)$-error.

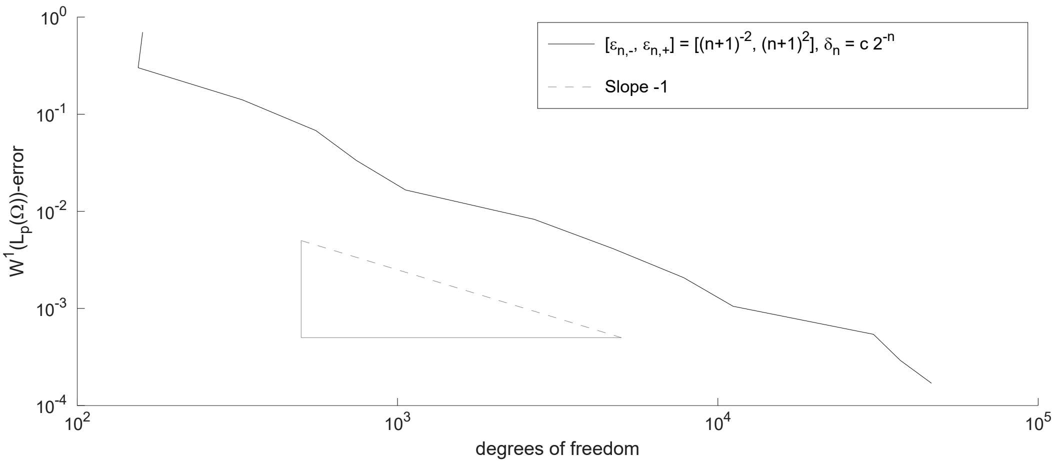



(c) $L_p$-error.

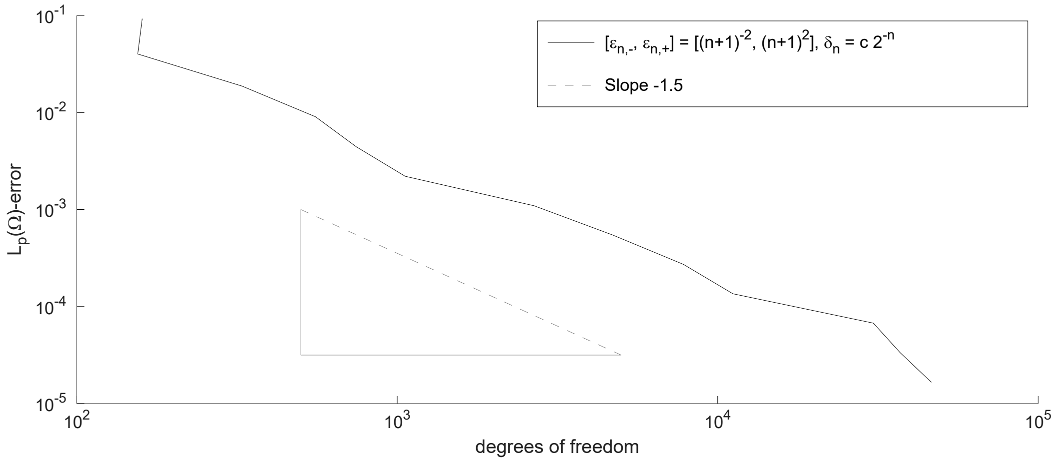
Figure 9.11: Error plots for example 3 and $p = 1.5$ for different strategies for decreasing the relaxation parameter $\varepsilon$: Energy-, $W^1(L_p)$- and $L_p$-error.




(a) Energy error.




(b) $W^1(L_p)$-error.

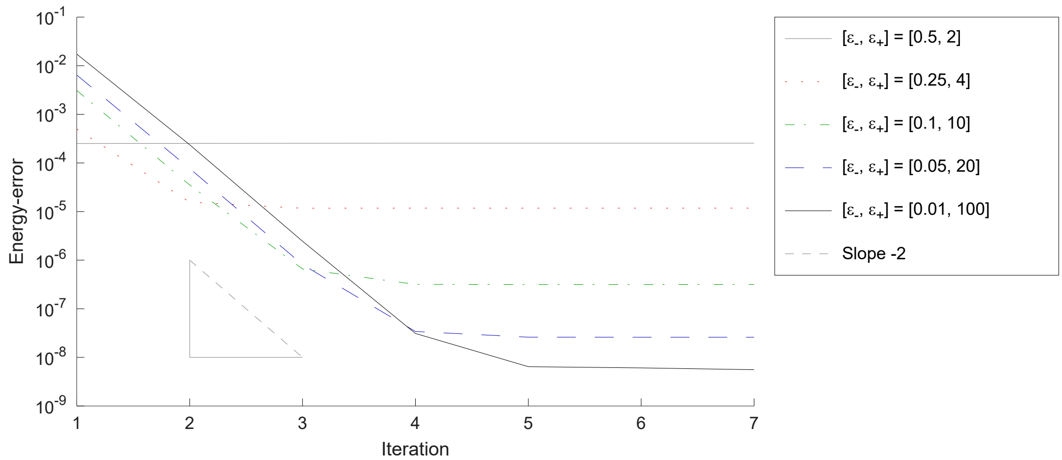



(c) $L_p$-error.

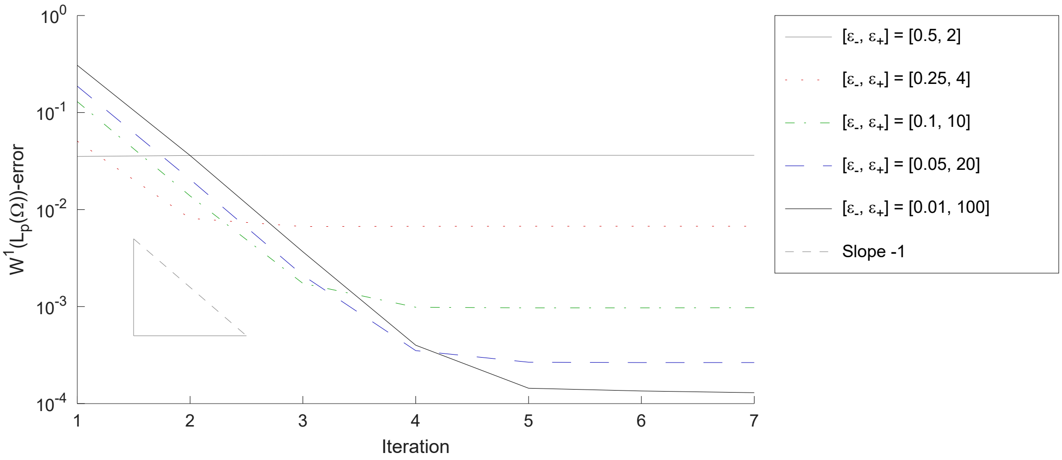
Figure 9.12: Error plots for example 3 and $p = 1.5$ with respect to degrees of freedom: Energy-, $W^1(L_p)$- and $L_p$-error.

### Tests for p=1.9

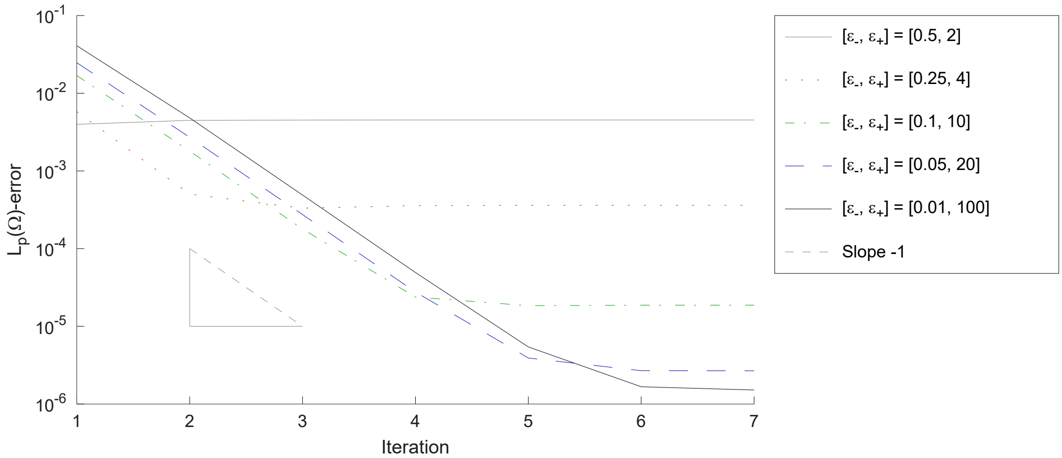
We adopt the setting from the tests for $p = 1.1$, i.e., we again choose constant relaxation intervals and set $\delta$ to a fixed, small value. The results are given in Figure 9.13. Compared to the tests for $p = 1.1$ and $p = 1.5$, here the fastest convergence can be observed. The energy error is reduced by a constant factor of approximately 0.01 at each iteration, whereas the $W^1(L_p)$- and $L_p$-error are decreased by a factor of about 0.1 per iteration.




(a) Energy error.




(b) $W^1(L_p)$-error.




(c) $L_p$-error.

Figure 9.13: Error plots for example 3 and $p = 1.9$ for different constant values of the relaxation parameter $\varepsilon$: Energy-, $W^1(L_p)$- and $L_p$-error.

## 9.2.2 Example 4: singularity function

For our last numerical tests we consider the singularity function

$$\begin{aligned} &u : \Omega_L \to \mathbb{R}, \\ &\tilde{u}(r, \phi) = \zeta(r) r^{2/3} \sin(\frac{2}{3}\phi), \end{aligned} \tag{9.2.4}$$

where $\tilde{u}$ denotes the representation of $u$ in polar coordinates $(r, \phi) = \Xi(x, y)$ with respect to the re-entrant corner at the origin (cf. (A.2.1)). Therein, $\zeta : [0, 1] \to [0, 1]$ is a smooth (cutoff) function which is identically 1 on $[0, r_0]$ and vanishes on $[r_1, 1]$ for some $0 < r_0 < r_1 < 1$. A graphical representation of the function $u$ is given in Figure 9.14.

We perform two test runs, for which we choose the relaxation intervals as in (9.2.3), with $\alpha = 1$ and $\alpha = 2$, respectively. In addition, another one is performed for $[\varepsilon_{n,-}, \varepsilon_{n,+}] = \left[2^{-(n+1)}, 2^{n+1}\right]$. In all cases, the $\ell_2$-tolerance for **MultSchw** is chosen as $c2^{-n}$. The corresponding error plots are presented in Figure 9.15.

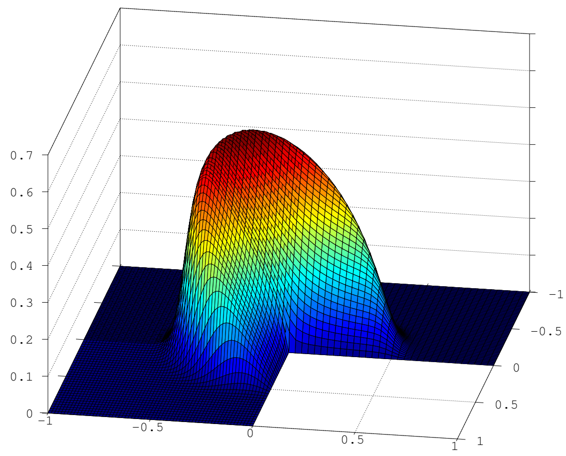
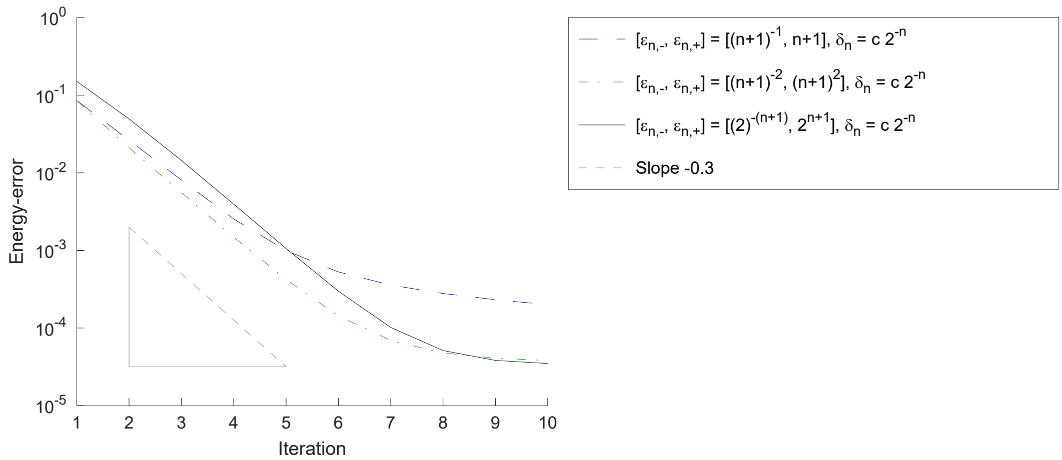
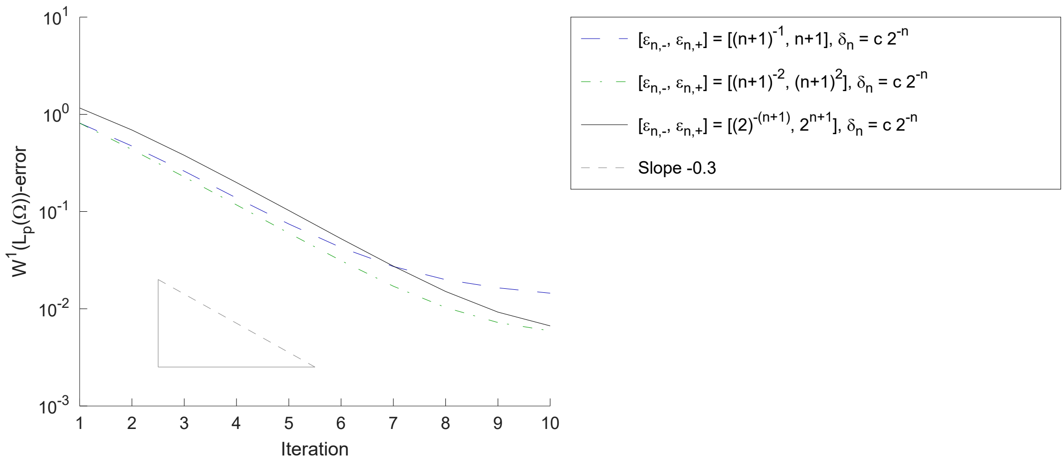


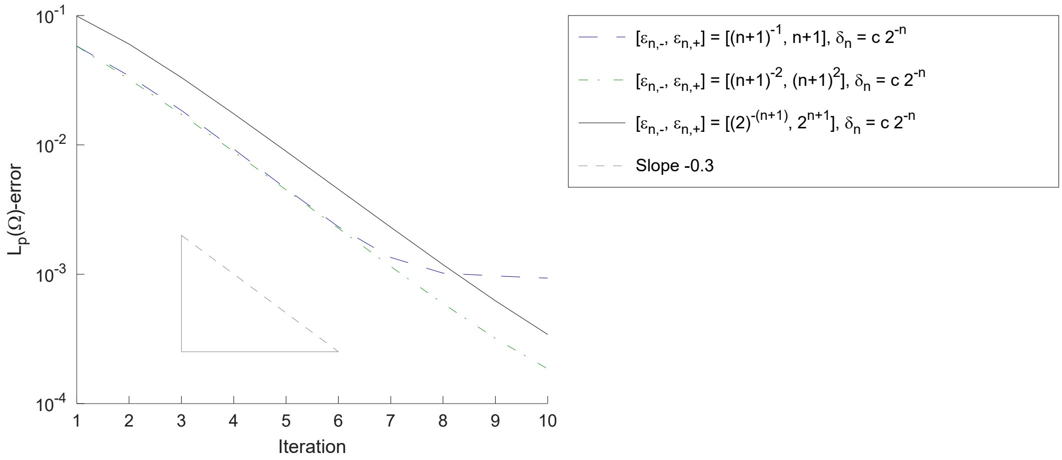
Figure 9.14: Exact solution of example 4.




(a) Energy error.




(b) $W^1(L_p)$-error.




(c) $L_p$-error.

Figure 9.15: Error plots for example 4 and $p = 1.5$ for different strategies for decreasing the relaxation parameter $\varepsilon$: Energy-, $W^1(L_p)$- and $L_p$-error.

# Concluding Remarks

The first main objective of this thesis - the derivation of regularity estimates for solutions of the $p$-Poisson equation in the adaptivity scale

$$B^\sigma_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}, \quad \sigma > 0, \tag{$*$}$$

of Besov spaces - was motivated by the question whether adaptive schemes for the numerical solution of the $p$-Poisson equation pay off, in comparison to more conventional, uniform methods. In summary, the theoretical superiority of adaptive wavelet methods for a large class of $p$-Poisson problems could be shown. In particular for the practically relevant cases of polygonal domains we could prove that, for a wide range of parameters, a significantly higher optimal convergence rate can be achieved by adaptive wavelet methods, compared to the best possible rate of uniform schemes. In this sense, it could be shown that the development of adaptive methods for the numerical solution of the $p$-Poisson equation is completely justified.

The second main objective - the implementation and extensive testing of the regularized Kačanov-type iteration method for the numerical solution of the $p$-Poisson equation - was accomplished in Part IV of this thesis. A series of numerical experiments revealed a promising stability and convergence behavior of this approach. In summary, existing theoretical results on the convergence of the exact scheme could be practically confirmed for the fully implemented algorithm.

### Discussion of the results

For the derivation of our Besov regularity results for the $p$-Poisson equation, we made use of the fact that in many cases the solutions possess a certain higher regularity in appropriately weighted function spaces. In order to provide a more general framework for the usage of this sort of information - independent of the $p$-Poisson equation - we proved two general embeddings of function spaces. At first, in Section 5.1 we showed that the intersection of a locally weighted Hölder space with an $L_p$-Besov space is continuously embedded into certain Besov spaces in the adaptivity scale, i.e., we derived an embedding of the type

$$C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \;\cap\; B^s_p(L_p(\Omega)) \;\hookrightarrow\; B^\sigma_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p},$$

under suitable assumptions on the involved parameters, see Theorem 5.1. The proof of this embedding theorem was performed using extension arguments in conjunction with characterizations of Besov spaces by means of wavelet expansion coefficients. The second embedding that we considered - allowing to infer Besov regularity assertions from information on weighted Sobolev regularity, encoded by the membership in certain Babuska-Kondratiev spaces $\mathcal{K}^{\ell}_{p,a}(\Omega)$ - is well-known for finite $\ell \in \mathbb{N}$. Inspired by singular expansion results for the $p$-Poisson equation, we extended this embedding to infinite intersections, i.e., we proved that

$$H^{\infty,s}_a(L_p(\Omega)) \hookrightarrow B^{\infty}_{\mathrm{NL}}(L_p(\Omega)),$$

see Theorem 5.9. To verify the continuity of this embedding, we made use of a quasi-norm criterion that we derived in Proposition 2.23. Moreover, we showed that $H^{\infty,s}_a(L_p(\Omega))$ is a Fréchet space (Proposition 5.7) and $B^{\infty}_{\mathrm{NL}}(L_p(\Omega))$ is an F-space (Proposition 5.8) in the sense as described in Subsection 5.2.1.

Let us stress the fact that with the help of these functional analytic tools, the knowledge of a certain regularity in locally weighted Hölder or Kondratiev spaces for the solution of any PDE can thus universally be employed to derive Besov regularity assertions in the scale $(*)$.

Then, for the case of general multidimensional Lipschitz domains, we first derived generic sufficient conditions on the parameters of the spaces $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ which would ensure that the Besov regularity of all solutions $u$ to (3.1.2) that are contained in such spaces exceeds the Sobolev smoothness (Theorem 6.5). Afterwards, in Section 6.2 we proved explicit Besov regularity results for the case $d = 2$ and homogeneous Dirichlet boundary conditions. We derived smoothness estimates for the $p$-Poisson equation (3.2.3) on bounded Lipschitz domains (Theorem 6.14). This was accomplished by establishing sufficient regularity of the solutions in locally weighted Hölder spaces (Proposition 6.11), combined with the embedding Theorem 5.1. A comparison of these results with existing Sobolev smoothness assertions (cf. Proposition 6.2 and the subsequent remarks), showed that the Besov regularity in the scale $(*)$ is significantly higher whenever

$$\frac{4}{3} < p < \infty \quad \text{and} \quad f \in L_q(\Omega) \quad \text{with} \quad \max\{4, 2p\} < q \leq \infty. \tag{$**$}$$

Let us recall that this assertion is based on the Sobolev regularity bounds $s^* = 3/2$ for $1 < p \leq 2$ and $s^* = 1 + 1/p$ for $2 < p < \infty$. While for the latter case it is known that $s^*$ indeed constitutes a sharp bound even for problems with smooth right-hand side $f$ (cf. Remark 6.4 and Proposition 7.6), this is not rigorously assured for $1 < p < 2$. However, there are good reasons to assume that this conjecture is true, as outlined at the beginning of Section 6.1.

The derived Besov regularity estimates could be improved for the case of polygonal domains and small values of $p$ (Theorem 6.17). In particular, for the parameter constellations $1 < p < 4/3$ and $p' \leq q \leq \infty$, as well as for $4/3 \leq p < 2$ and $(1/p - 1/2)^{-1} < q \leq \infty$, the bounds for the Besov regularity on polygonal domains are significantly larger than those for Lipschitz domains. This extends the cases $(**)$, for which the Besov regularity $\sigma$ of solutions to (3.2.3) is significantly higher than their Sobolev regularity $s^*$, to

$$1 < p < \frac{4}{3} \quad \text{and} \quad p' \leq q \leq \infty,$$

as well as

$$\frac{4}{3} \leq p < \infty \quad \text{and} \quad \max\{4, 2p\} < q \leq \infty.$$

Therefore, for the same range of parameters, the development and application of adaptive (wavelet) algorithms for the numerical treatment of (3.2.3) on polygonal domains is completely justified.

Moreover, for the $p$-Poisson problem (3.2.1) on polygonal domains we could show - under the additional assumption that both the right-hand side and Dirichlet boundary data vanish in a small neighborhood of a corner - that nonnegative solutions admit arbitrary high Besov regularity in the vicinity of that corner (Theorem 7.12). When the condition on $f$ is weakened to a rather mild local growth condition, then the solutions still possess Besov regularity $\sigma$ for all $\sigma < 2$ (Theorem 7.18). We remark that for certain cases and in a local sense, i.e., when considering solely a small neighborhood of a corner, these results are indeed stronger than those of Theorem 6.17. E.g., for nonnegative solutions of problem (3.2.3) with a right-hand side $f \in L_\infty(\Omega)$ that vanishes in the vicinity of a corner, from Theorem 6.17 we obtain the global Besov regularity estimate of 2 for $1 < p \leq 2$ and $1 + 1/(p-1)$ for $2 < p < \infty$, whereas Theorem 7.18 assures arbitrary high local Besov smoothness for all $p$. Also note that in case of a re-entrant corner, each $f \in L_\infty(\Omega)$ satisfies a local growth condition as required by Theorem 7.18, since $\gamma_0 < 0$ in this case (cf. Remark 7.7).

The numerical results in Chapter 9 demonstrated the stability and convergence of the Kačanov-type iteration method in practice. After a functional demonstration of the implemented algorithm by means of some tests on the unit square, in a series of numerical experiments on the L-shaped domain, the algorithm **rKačanov** could be practically verified to work stable for a wide range of the parameter $1 < p < 2$. In particular, the existing convergence result for the Kačanov scheme with *exact* subproblem solves (cf. Theorem 8.10) could be practically confirmed also for the case of an inexact solution of the linear subproblems. Moreover, several test runs indicated that the condition on the relaxation intervals in Theorem 8.10 may be weakend.

# Part V

# Appendix

# Appendix

The final part of this work is concerned with estimates needed in our proofs, as well as with auxiliary assertions and additional results that are of interest on their own. Finally, some alternative proofs are presented.

## A.1 Auxiliary lemmata and propositions

To begin with, we state the following well-known Whitney-type estimates which can be found, e.g., in DeVore [47, Subsection 6.1]. Here and in what follows we let $\Pi_k(S)$ denote the set of all polynomials $\mathcal{P}$ on some bounded and simply connected set $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, which possess a total degree $\deg \mathcal{P}$ not larger than $k \in \mathbb{N}_0$. As usual, $\lceil x \rceil$ (and $\lfloor x \rfloor$, respectively) means the smallest (largest) integer larger (smaller) or equal to $x \in \mathbb{R}$.

**Proposition A.1** (Whitney's estimate)**.** *For $d \in \mathbb{N}$ let $Q$ denote an arbitrary cube in $\mathbb{R}^d$ with sides parallel to the coordinate axes.*

**(i)** *Let $1 \leq p, q \leq \infty$ and $k \in \mathbb{N}$ with $k > d \max\{0, 1/q - 1/p\}$. Then it holds*

$$\inf_{\mathcal{P} \in \Pi_{k-1}(Q)} \|f - \mathcal{P} \mid L_p(Q)\| \leq C \, |Q|^{k/d + 1/p - 1/q} \, |f|_{W^k(L_q(Q))} ,$$

*whenever the right-hand side is finite. Therein the constant $C$ depends only on $k$.*

**(ii)** *Let $1 \leq p \leq \infty$ and $0 < q \leq \infty$. Furthermore, assume that $0 < t < \infty$ satisfies $t \geq d \max\{0, 1/q - 1/p\}$. Then we have*

$$\inf_{\mathcal{P} \in \Pi_{\lceil t \rceil - 1}(Q)} \|f - \mathcal{P} \mid L_p(Q)\| \leq C \, |Q|^{t/d + 1/p - 1/q} \, |f|_{B^t_q(L_q(Q))} ,$$

*whenever the right-hand side is finite. Here the constant $C$ depends only on $t$.*

In the proof of our general embedding result (Theorem 5.1) the subsequent bound is used. As no explicit derivation of this quite natural assertion seems to be available in the literature, a detailed proof is added here for the reader's convenience.

**Proposition A.2.** *For $d \in \mathbb{N}$ let $Q$ denote some open cube in $\mathbb{R}^d$ with sides parallel to the coordinate axes. Then for all $\ell \in \mathbb{N}_0$ and $0 < \alpha \leq 1$ it holds*

$$|g|_{B^{\ell+\alpha}_\infty(L_\infty(Q))} \lesssim |g|_{C^{\ell,\alpha}(Q)} ,$$

*whenever the right-hand side is finite.*

*Proof. Step 1.* Assume that $\ell = 0$. Then, for $0 < \alpha < 1$, the assertion follows from the definition of the involved semi-norms; see (1.1.1) and (1.4.2) in Chapter 1. If $\alpha = 1$, then we use the triangle inequality to see that for all $h \in \mathbb{R}^d$ it holds

$$\left\| \Delta_h^2(g,\cdot) \,\middle|\, L_\infty(Q_{2,h}) \right\| = \left\| \Delta_h^1(g,\cdot+h) - \Delta_h^1(g,\cdot) \,\middle|\, L_\infty(Q_{2,h}) \right\| \lesssim \left\| \Delta_h^1(g,\cdot) \,\middle|\, L_\infty(Q_{1,h}) \right\|, \tag{A.1.1}$$

where we recall that for $r \in \mathbb{N}$ the set $Q_{r,h}$ denotes the collection of all $x \in Q$ such that $[x, x + rh] \subset Q$. Then, as before, the claim directly follows from the definitions of the semi-norms.

*Step 2.* Now let $\ell \in \mathbb{N}$. Given $t > 0$, as well as $h \in \mathbb{R}^d$ with $0 < |h| \leq t$, and any function $f$ on some domain $\Omega \subset \mathbb{R}^d$, the mean value theorem ensures that for all $x \in \Omega_{1,h}$ there exists some $\xi_x \in [x, x + h] \subset \Omega$ with

$$\left| \Delta_h^1(f,x) \right| = |h \cdot \nabla f(\xi_x)| \leq |h| \, |\nabla f(\xi_x)| \lesssim t \sum_{|\nu|=1} |\partial^\nu f(\xi_x)| \,,$$

whenever the right-hand side is finite. Obviously, the same is true also for $h = 0$. Thus, we conclude that for every such $f$ and all $|h| \leq t$

$$\left\| \Delta_h^1(f,\cdot) \,\middle|\, L_\infty(\Omega_{1,h}) \right\| \lesssim t \sup_{x\in\Omega_{1,h}} \sum_{|\nu|=1} |\partial^\nu f(\xi_x)| \leq t \sum_{|\nu|=1} \|\partial^\nu f \mid L_\infty(\Omega)\| \,. \tag{A.1.2}$$

Observe that $r := \lfloor \ell + \alpha \rfloor + 1 \geq 2$ for all $0 < \alpha \leq 1$. Therefore, if we use (A.1.2) for $f := \Delta_h^{r-1}(g, \star)$ together with the linearity of $\partial^\nu$ and $\Delta_h^{r-1}$,

$$\begin{aligned}
|g|_{B_\infty^{\ell+\alpha}(L_\infty(Q))} &= \sup_{t>0} t^{-(\ell+\alpha)} \sup_{h\in\mathbb{R}^d, |h|\leq t} \left\| \Delta_h^1(\Delta_h^{r-1}(g,\star),\cdot) \,\middle|\, L_\infty(Q_{r,h}) \right\| \\
&\lesssim \sup_{t>0} t^{-(\ell+\alpha)} \sup_{h\in\mathbb{R}^d, |h|\leq t} t \sum_{|\nu|=1} \left\| \partial^\nu \Delta_h^{r-1}(g,\star) \,\middle|\, L_\infty(\Omega_{r-1,h}) \right\| \\
&\leq \sum_{|\nu|=1} \sup_{t>0} t^{-(\ell+\alpha)+1} \sup_{h\in\mathbb{R}^d, |h|\leq t} \left\| \Delta_h^{r-1}(\partial^\nu g,\cdot) \,\middle|\, L_\infty(\Omega_{r-1,h}) \right\|.
\end{aligned}$$

If necessary, we can iterate this argument and deduce

$$|g|_{B_\infty^{\ell+\alpha}(L_\infty(Q))} \lesssim \sum_{|\nu|=r-1} \sup_{t>0} t^{-(\ell+\alpha)+r-1} \sup_{h\in\mathbb{R}^d, |h|\leq t} \left\| \Delta_h^1(\partial^\nu g,\cdot) \,\middle|\, L_\infty(\Omega_{1,h}) \right\|. \tag{A.1.3}$$

For $0 < \alpha < 1$ it is $r - 1 = \ell$. Consequently, in this case we obtain

$$\begin{aligned}
|g|_{B_\infty^{\ell+\alpha}(L_\infty(Q))} &\lesssim \sum_{|\nu|=\ell} \sup_{t>0} \sup_{h\in\mathbb{R}^d, |h|\leq t} \frac{\|\partial^\nu g(\cdot+h) - \partial^\nu g(\cdot) \mid L_\infty(\Omega_{1,h})\|}{t^\alpha} \qquad \text{(A.1.4)} \\
&= \sum_{|\nu|=\ell} \sup_{\substack{x,y\in Q,\\ x\neq y}} \frac{|\partial^\nu g(x) - \partial^\nu g(y)|}{|x-y|^\alpha}.
\end{aligned}$$

Since the last term equals $|g|_{C^{\ell,\alpha}(Q)}$, this shows the claim in the case $\alpha < 1$.

Finally, we note that if $\alpha = 1$, then $r \geq 3$. Thus, by means of the same (iterative) argument as above, this time we derive

$$|g|_{B_\infty^{\ell+\alpha}(L_\infty(Q))} \lesssim \sum_{|\nu|=r-2} \sup_{t>0} t^{-(\ell+\alpha)+r-2} \sup_{h\in\mathbb{R}^d, |h|\leq t} \left\| \Delta_h^2(\partial^\nu g,\cdot) \,\middle|\, L_\infty(\Omega_{2,h}) \right\|$$

instead of (A.1.3). Using $r-2=\ell$ in conjunction with an estimate similar to (A.1.1) from Step 1 this allows to conclude (A.1.4) also for this case. Hence, the proof is complete. $\square$

In Remark 1.1, among other things, we stated that intersections of locally weighted Hölder spaces (as introduced in Section 1.1) with certain Besov spaces form Banach spaces w.r.t. the canonical maximum norm. Proposition A.3 below is devoted to this claim. The subsequent three lemmata are used to derive a sound mathematical proof.

**Proposition A.3.** *For $d \in \mathbb{N}$ let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and for $\ell \in \mathbb{N}_0$, $0<\alpha\leq 1$, as well as $\gamma>0$, let $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ denote a locally weighted Hölder space. Then for all $s>0$ and $1\leq p,q\leq\infty$ the space*

$$B^s_q(L_p(\Omega)) \cap C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \tag{A.1.5}$$

*endowed with the norm*

$$\|\cdot\| = \max\left\{ \left\| \cdot \,\middle|\, B^s_q(L_p(\Omega)\right\|, |\cdot|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}} \right\} \tag{A.1.6}$$

*is a Banach space.*

*Proof.* Since $\left\| \cdot \,\middle|\, B^s_q(L_p(\Omega)\right\|$ is a norm on $B^s_q(L_p(\Omega))$ and $|\cdot|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}}$ defines a semi-norm for $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$, it obviously holds that $\|\cdot\|$ is a norm for the space (A.1.5). To show completeness, let $\{f_j\}_{j\in\mathbb{N}_0}$ be a Cauchy sequence in (A.1.5) with respect to $\|\cdot\|$. Then, by completeness of the Besov space, there exists some $f \in B^s_q(L_p(\Omega))$ such that

$$f_j \to f \quad \text{in} \quad B^s_q(L_p(\Omega)), \quad \text{as } j\to\infty. \tag{A.1.7}$$

This clearly remains true for all restrictions of $f_j$ and $f$, respectively, e.g., when $\Omega$ is replaced by an open ball $B \subset \Omega$.

In the following, we will show that $f_j$ converges to $f$ with respect to $|\cdot|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}}$, too. Let $\overline{B}=\overline{B_r(x_0)} \subset \Omega$ be a non-empty closed ball such that $B_{cr}(x_0)$ is still contained in $\Omega$ for some $c>1$. Given some function $g\in C^\ell(\overline{B})$ we denote by $T^{\ell,x_0}[g]$ its Taylor polynomial of degree $\ell$ at $x_0$, i.e.,

$$T^{\ell,x_0}[g](x) = \sum_{|\nu|\leq\ell} \frac{\partial^\nu g(x_0)}{\nu!}(x-x_0)^\nu, \quad x\in\overline{B}.$$

*Step 1.* Here we prove that, if $\{f_j\}_{j\in\mathbb{N}_0}$ is a Cauchy sequence w.r.t. $|\cdot|_{C^{\ell,\alpha}(\overline{B})}$, then

$$\{f_j - T^{\ell,x_0}[f_j]\}_{j\in\mathbb{N}_0} \tag{A.1.8}$$

forms a Cauchy sequence with respect to the norm in the Hölder space $C^{\ell,\alpha}(\overline{B})$,

$$\left\|\cdot \,\middle|\, C^{\ell,\alpha}(\overline{B})\right\| = \left\|\cdot \,\middle|\, C^{\ell}(\overline{B})\right\| + |\cdot|_{C^{\ell,\alpha}(\overline{B})}\,.$$

Since the definition of the semi-norm $|\cdot|_{C^{\ell,\alpha}(\overline{B})}$ given in (1.1.1) is based on derivatives of degree $\ell$, we have

$$\left|f_j - T^{\ell,x_0}[f_j]\right|_{C^{\ell,\alpha}(\overline{B})} = |f_j|_{C^{\ell,\alpha}(\overline{B})}\,. \tag{A.1.9}$$

Therefore it remains to show that (A.1.8) is a Cauchy sequence with respect to the norm $\left\|\cdot \,\middle|\, C^\ell(\overline{B})\right\|$. For $j,k \in \mathbb{N}_0$ let $g_{j,k} = f_j - f_k$ and choose $\nu \in \mathbb{N}_0^d$ with $|\nu| \leq \ell$. Then, by linearity of the Taylor polynomial, for all $x \in \overline{B}$ it holds

$$\begin{aligned}\partial^\nu\Big(\big(f_j - T^{\ell,x_0}[f_j]\big) - \big(f_k - T^{\ell,x_0}[f_k]\big)\Big)(x) &= \partial^\nu\big(g_{j,k} - T^{\ell,x_0}[g_{j,k}]\big)(x)\\ &= \partial^\nu g_{j,k}(x) - T^{\ell-|\nu|,x_0}[\partial^\nu g_{j,k}](x).\end{aligned} \tag{A.1.10}$$

According to Lemma A.5 below, we thus have

$$\sup_{x\in\overline{B}}\left|\partial^\nu g_{j,k}(x) - T^{\ell-|\nu|,x_0}[\partial^\nu g_{j,k}](x)\right| \lesssim |\partial^\nu g_{j,k}|_{C^{\ell-|\nu|,\alpha}(\overline{B})} \leq |f_j - f_k|_{C^{\ell,\alpha}(\overline{B})}$$

for all $|\nu| \leq \ell$. Together with (A.1.10) this shows that

$$\left\|\big(f_j - T^{\ell,x_0}[f_j]\big) - \big(f_k - T^{\ell,x_0}[f_k]\big) \,\middle|\, C^\ell(\overline{B})\right\| \lesssim |f_j - f_k|_{C^{\ell,\alpha}(\overline{B})}\,,$$

i.e., (A.1.8) forms a Cauchy sequence w.r.t. $\left\|\cdot \,\middle|\, C^\ell(\overline{B})\right\|$. This observation in conjunction with (A.1.9) finally proves that $\{f_j - T^{\ell,x_0}[f_j]\}_{j\in\mathbb{N}_0}$ is a Cauchy sequence in the norm of the Hölder space $C^{\ell,\alpha}(\overline{B})$, too.

*Step 2.* Of course, the space $C^{\ell,\alpha}(\overline{B})$ endowed with the norm $\left\|\cdot \,\middle|\, C^{\ell,\alpha}(\overline{B})\right\|$ is complete. Since we have shown that $\{f_j - T^{\ell,x_0}[f_j]\}_{j\in\mathbb{N}_0}$ is a Cauchy sequence with respect to this norm, there exists some $f_{\overline{B}} \in C^{\ell,\alpha}(\overline{B})$ such that

$$\big(f_j - T^{\ell,x_0}[f_j]\big) \to f_{\overline{B}} \quad \text{in} \quad \left\|\cdot \,\middle|\, C^{\ell,\alpha}(\overline{B})\right\|, \quad \text{as } j\to\infty.$$

*Step 3.* In the previous steps it was proven that every Cauchy sequence $\{f_j\}_{j\in\mathbb{N}_0}$ in $B^s_q(L_p(\Omega)) \cap C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ (w.r.t. $\|\cdot\|$) converges to some $f$ in $B^s_q(L_p(\Omega))$ and that for every non-empty closed ball $\overline{B} = \overline{B_r(x_0)} \subset \mathbb{R}^d$ for which $B_{c\,r}(x_0)$ is still contained in $\Omega$ the sequence $\{f_j - T^{\ell,x_0}[f_j]\}_{j\in\mathbb{N}_0}$ restricted to $\overline{B}$ converges to some $f_{\overline{B}}$ with respect to $\left\|\cdot \,\middle|\, C^{\ell,\alpha}(\overline{B})\right\|$. It remains to show that $f_j \to f$ in the semi-norm of $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$. Let $B$ be the interior of $\overline{B}$. Lemma A.6, applied to $X = B^s_q(L_p(B))$, implies that the restriction of $f$ to $\overline{B}$ belongs to $C^{\ell,\alpha}(\overline{B})$ and that

$$f_j \to f \quad \text{with respect to} \quad |\cdot|_{C^{\ell,\alpha}(\overline{B})}\,, \quad \text{as } j\to\infty.$$

Since clearly, for all $j \in \mathbb{N}_0$ and every $\overline{B}$, it holds

$$|f_j - f|_{C^{\ell,\alpha}(\overline{B})} \leq \lim_{k\to\infty} |f_j - f_k|_{C^{\ell,\alpha}(\overline{B})}\,,$$

the definition of $|\cdot|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)}$ as a weighted supremum of $C^{\ell,\alpha}(\overline{B})$-semi-norms yields

$$|f_j - f|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)} \leq \lim_{k\to\infty} |f_j - f_k|_{C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)}\,.$$

Hence, from the assumption that $\{f_j\}_{j\in\mathbb{N}_0}$ is a Cauchy sequence in $C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega)$ and by (A.1.7) it follows that

$$f_j \to f \quad \text{in} \quad B^s_q(L_p(\Omega)) \cap C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega), \quad \text{as } j \to \infty,$$

and thus the proof is finished. □

**Remark A.4.** Let $s > 0$. If $0 < p < 1$ or $0 < q < 1$, then $B^s_q(L_p(\Omega))$ is only a quasi-Banach space, i.e., it is complete with respect to the *quasi*-norm $\left\|\cdot \,\middle|\, B^s_q(L_p(\Omega))\right\|$. However, in the same way as in Proposition A.3, one can show that in this case the intersection (A.1.5) endowed with the quasi-norm $\|\cdot\|$ given by (A.1.6) defines a quasi-Banach space.

**Lemma A.5.** *Let $\overline{B} \subset \mathbb{R}^d$, $d \in \mathbb{N}$, denote a non-trivial closed ball with center $x_0$ and let $\ell \in \mathbb{N}_0$. For $g \in C^{\ell,\alpha}(\overline{B})$ let $T^{\ell,x_0}[g]$ be the Taylor polynomial of degree $\ell$ at $x_0$. Then there exists a constant $C_{\ell,\alpha,\overline{B}} > 0$ such that*

$$\sup_{x\in\overline{B}} \left| g(x) - T^{\ell,x_0}[g](x) \right| \leq C_{\ell,\alpha,\overline{B}} \, |g|_{C^{\ell,\alpha}(\overline{B})} \qquad \textit{for all} \qquad g \in C^{\ell,\alpha}(\overline{B}).$$

*Proof.* Let $\ell \in \mathbb{N}$. Then, by Taylor's theorem for order $\ell - 1$, for all $x \in \overline{B}$ there exists a $\theta \in (0,1)$ such that

$$\begin{aligned} g(x) - T^{\ell,x_0}[g](x) &= g(x) - T^{\ell-1,x_0}[g](x) - \sum_{|\nu|=\ell} \frac{\partial^\nu g(x_0)}{\nu!}(x-x_0)^\nu \\ &= \sum_{|\nu|=\ell} \frac{\partial^\nu g(x_0 + \theta(x-x_0))}{\nu!}(x-x_0)^\nu - \sum_{|\nu|=\ell} \frac{\partial^\nu g(x_0)}{\nu!}(x-x_0)^\nu. \end{aligned}$$

Now, estimating the right-hand side with the help of $|g|_{C^{\ell,\alpha}(\overline{B})}$ results in

$$\begin{aligned} \left| g(x) - T^{\ell,x_0}[g](x) \right| &\leq \sum_{|\nu|=\ell} \frac{|\partial^\nu g(x_0 + \theta(x-x_0)) - \partial^\nu g(x_0)|}{|(x_0 + \theta(x-x_0)) - x_0|^\alpha} \, \frac{\theta^\alpha \, |x-x_0|^{|\nu|+\alpha}}{\nu!} \\ &\leq C_{\ell,\alpha,\overline{B}} \, |g|_{C^{\ell,\alpha}(\overline{B})} \end{aligned}$$

for all $x \in \overline{B} \setminus \{x_0\}$ and $\ell \in \mathbb{N}$. Since this bound obviously holds for $x = x_0$ and for $\ell = 0$ as well, the claim is proven. □

**Lemma A.6.** *Let $\overline{B} \subset \mathbb{R}^d$ be a non-trivial closed ball and denote its interior by $B$. Moreover, for $k, \ell \in \mathbb{N}_0$ with $k \leq \ell$, let $\{\mathcal{P}^k_j\}_{j\in\mathbb{N}_0} \subset \Pi_k(\overline{B})$ be a sequence of polynomials and suppose that $X(B)$ denotes a quasi-Banach space of functions on $B$, which is continuously embedded into $\mathcal{D}'(B)$. Finally, assume that*

$$(f_j - \mathcal{P}^k_j) \to f^1 \quad \textit{with respect to} \quad \left\|\cdot \,\middle|\, C^{\ell,\alpha}(\overline{B})\right\| \qquad \textit{and} \qquad f_j \to f \quad \textit{in} \quad X(B),$$

*as $j$ approaches infinity. Then $f \in C^{\ell,\alpha}(\overline{B})$ and*

$$f_j \to f \quad \textit{with respect to} \quad |\cdot|_{C^{\ell,\alpha}(\overline{B})}, \quad \textit{as } j \to \infty.$$

*Proof.* Since both the spaces $C^{\ell,\alpha}(B)$ and $X(B)$ are continuously embedded into $\mathcal{D}'(B)$, the convergence

$$(f_j - \mathcal{P}_j^k) \to f^1 \quad \text{and} \quad f_j \to f$$

takes place in $\mathcal{D}'(B)$. Hence, $\mathcal{P}_j^k \to (f - f^1) \in \mathcal{D}'(B)$, as $j \to \infty$.

On the other hand, the linear space $\Pi_k(B)$ of polynomials of degree not larger than $k$ is closed with respect to the convergence (cf. Lemma A.7 below) in $\mathcal{D}'(B)$. Consequently, $f - f^1 =: \mathcal{P}^k \in \Pi_k(\overline{B})$ and

$$f = f^1 + \mathcal{P}^k \in C^{\ell,\alpha}(\overline{B}).$$

Finally, as $|\cdot|_{C^{\ell,\alpha}(\overline{B})}$ can not distinguish polynomials of degree less or equal to $\ell$,

$$\begin{aligned} |f_j - f|_{C^{\ell,\alpha}(\overline{B})} &= \left|\left(f_j - \mathcal{P}_j^k\right) - \left(f - \mathcal{P}^k\right)\right|_{C^{\ell,\alpha}(\overline{B})} \\ &= \left|\left(f_j - \mathcal{P}_j^k\right) - f^1\right|_{C^{\ell,\alpha}(\overline{B})} \to 0, \quad \text{as } j \to \infty, \end{aligned}$$

due to our assumption. □

**Lemma A.7.** *Let $B$ denote an open ball in $\mathbb{R}^d$, $d \in \mathbb{N}$. Then the set of polynomials $\Pi_k(B)$ of degree at most $k \in \mathbb{N}_0$ on $B$ is closed with respect to convergence in $\mathcal{D}'(B)$.*

*Proof.* For all $\{\mathcal{P}_j^k\}_{j\in\mathbb{N}_0} \subset \Pi_k(B)$ with

$$\mathcal{P}_j^k \to \mathcal{P} \in \mathcal{D}'(B), \quad \text{as } j \to \infty,$$

we have to show that $\mathcal{P} \in \Pi_k(B)$. We shall prove this statement by induction on $k \in \mathbb{N}_0$. Let $k = 0$. Then $\mathcal{P}_j^0 \equiv a_j \in \mathbb{R}$ is a sequence of constants converging to $\mathcal{P} \in \mathcal{D}'(B)$, i.e.,

$$a_j \int_B \varphi(x)\,\mathrm{d}x = \int_B \mathcal{P}_j^0(x)\,\varphi(x)\,\mathrm{d}x \to \mathcal{P}(\varphi) \quad \text{for all} \quad \varphi \in \mathcal{D}(B), \quad \text{as } j \to \infty.$$

Obviously, the sequence $\{a_j\}_{j\in\mathbb{N}_0}$ has to be bounded in $\mathbb{R}$ and hence there is a subsequence $\{a_{j_\ell}\}_{\ell\in\mathbb{N}_0}$ with $a_{j_\ell} \to a \in \mathbb{R}$, as $\ell \to \infty$. By uniqueness of convergence of this subsequence it holds

$$\mathcal{P}(\varphi) = a \int_B \varphi(x)\,\mathrm{d}x$$

and thus $\mathcal{P} \equiv a \in \Pi_0(B)$.

Let us now assume that $k \in \mathbb{N}$ and that the statement of the lemma is already shown for $0 \leq \ell \leq k-1$. In addition, let $\nu \in \mathbb{N}_0^d$ with $|\nu| = k$ be a given multi-index. If $\mathcal{P}_j^k \to \mathcal{P}$ in $\mathcal{D}'(B)$, then also $\partial^\nu \mathcal{P}_j^k \to \partial^\nu \mathcal{P}$ in $\mathcal{D}'(B)$, as $j \to \infty$. But, for all $j \in \mathbb{N}_0$, $\partial^\nu \mathcal{P}_j^k \equiv a_j^\nu \in \mathbb{R}$ is a polynomial of degree 0. Hence, by the base step of the induction, the sequence $\{\partial^\nu \mathcal{P}_j^k\}_{j\in\mathbb{N}_0}$ converges to some constant $a^\nu$ in $\mathcal{D}'(B)$. This shows that

$$\mathcal{P}_j^{k-1} := \mathcal{P}_j^k - \sum_{|\nu|=k} \frac{\partial^\nu \mathcal{P}_j^k}{\nu!} x^\nu \quad \text{tends to} \quad \tilde{\mathcal{P}} := \mathcal{P} - \sum_{|\nu|=k} \frac{a^\nu}{\nu!} x^\nu \quad \text{in} \quad \mathcal{D}'(B), \quad \text{as } j \to \infty.$$

Since $\mathcal{P}_j^{k-1}$ belongs to $\Pi_{k-1}(B)$, by induction it follows that $\tilde{\mathcal{P}} \in \Pi_{k-1}(B)$, too. Therefore, $\mathcal{P}$ belongs to $\Pi_k(B)$ and the proof is complete. □

The next embedding result concerning dual Sobolev spaces readily follows from the classical embedding of Sobolev spaces and Hölder's inequality. We additionally refer to [62, Theorem 1.7] for a proof (actually compact embeddings are proven in this reference - however the modifications which yield the assertion below are straightforward).

**Lemma A.8.** *Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain, $k \in \mathbb{N}$, $1 < p < \infty$ and $p^* := dp/(d-kp)$. Then, for $q \leq \infty$ with*

$$\begin{cases} q \geq \frac{p^*}{p^*-1}, & \text{if } kp < d, \\ q > 1, & \text{if } kp = d, \\ q \geq 1, & \text{if } kp > d, \end{cases}$$

*there exists a continuous embedding*

$$L_q(\Omega) \hookrightarrow W^{-k}(L_{p'}(\Omega)), \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

For a proof of the following well-known differentiability result concerning integrals involving parameters, see [7, Corollary 16.3].

**Lemma A.9.** *Let $U \subset \mathbb{R}^d$ be open, $j \in \{1, \dots, d\}$ and $(\Omega, \mathcal{A}, \mu)$ a measure space. Assume that for a mapping $f : U \times \Omega \to \mathbb{R}$ it holds*

*(i) $f(x, \cdot) \in \mathcal{L}_1(\Omega, \mu, \mathbb{R})$ for all $x \in U$;*

*(ii) $f(\cdot, \omega)$ has an $j$-th partial derivative at each $x \in U$ for $\mu -$ a.e. $\omega \in \Omega$;*

*(iii) there exists $g \in \mathcal{L}_1(\Omega, \mu, \mathbb{R})$ with $|(\partial f/\partial x_j)(x, \omega)| \leq g(\omega)$ for all $(x, \omega) \in U \times \Omega$.*

*Then $F : U \to \mathbb{R},\ x \mapsto \int_\Omega f(x, \omega)\, \mu(\mathrm{d}\omega)$ has an $j$-th partial derivative at every $x \in U$ with*

$$\frac{\partial F}{\partial x_j}(x) = \int_\Omega \frac{\partial f}{\partial x_j}(x, \omega)\, \mu(\mathrm{d}\omega).$$

**Lemma A.10.** *For $p > 1$, $|\cdot|^p : \mathbb{R}^d \to \mathbb{R}$ is continuously differentiable on $\mathbb{R}^d$.*

*Proof.* Clearly, $|\cdot|^p$ is continuously differentiable on $\mathbb{R}^d \setminus \{0\}$. For $x = 0$ and $1 \leq i \leq d$ one computes

$$\frac{\partial}{\partial x_i}|0|^p = \lim_{h \to 0} \frac{|he_i|^p}{h} = \lim_{h \to 0} \frac{|h|^p}{h} = \lim_{h \to 0} sgn(h)|h|^{p-1} = 0,$$

thus all partial derivatives exist in $x = 0$. For $x \neq 0$, with $(\partial/\partial x_i)\, |x|^p = p\, |x|^{p-2}\, x_i$ we estimate

$$\left| \frac{\partial}{\partial x_i}\, |x|^p \right| =\leq p\, |x|^{p-1},$$

hence $\lim_{x \to 0}(\partial/\partial x_i)\, |x|^p = 0$, i.e., all partial derivatives are continuous in 0. □

**Lemma A.11.** *Let $r : \mathbb{R}^2 \to \mathbb{R}$, $r(x) = |x|$ and $\beta \in \mathbb{R}$. Then for all $\nu \in \mathbb{N}_0^2$ it holds*

$$\left| \left( \partial^\nu r^\beta \right)(x) \right| \lesssim r(x)^{\beta - |\nu|} \qquad \textit{for all} \qquad x \in \mathbb{R}^2 \setminus \{0\}.$$

*Proof. Step 1.* We first prove that for $m \in \mathbb{N}_0$, where $k \in \mathbb{N}_0$ is chosen such that $m = 2k$ or $m = 2k+1$, it holds that

$$\frac{\partial^m}{\partial x_i^m} r^\beta = \sum_{j=0}^{k} c_{j,m} r^{\beta - 2(m-k+j)} x_i^{m-2k+2j}, \qquad i \in \{1,2\}. \tag{A.1.11}$$

Therefore, note that the equality surely holds for $m = 0$ and $m = 1$. Assuming that (A.1.11) holds for $m = 2k$, we derive

$$\begin{aligned}
&\frac{\partial^{m+1}}{\partial x_i^{m+1}} r^\beta \\
&\quad = c_{0,m} \frac{\partial}{\partial x_i} r^{\beta - 2(m-k)} + \sum_{j=1}^{k} c_{j,m} \frac{\partial}{\partial x_i} \left( r^{\beta-2(m-k+j)} x_i^{m-2k+2j} \right) \\
&\quad = c_{0,m} \left( \beta - 2(m-k) \right) r^{\beta - 2(m-k) - 2} x_i \\
&\qquad + \sum_{j=1}^{k} c_{j,m} \Big[ (\beta - 2(m-k+j)) r^{\beta - 2(m-k+j) - 2} x_i^{m-2k+2j+1} \\
&\qquad\qquad + r^{\beta - 2(m-k+j)} (m - 2k + 2j) x_i^{m-2k+2j-1} \Big] \\
&\quad =: \tilde{c}_{0,m} r^{\beta - 2((m+1)-k)} x_i^{(m+1)-2k} \\
&\qquad + \sum_{j=1}^{k} \tilde{c}_{j,m} r^{\beta - 2((m+1)-k+j)} x_i^{(m+1)-2k+2j} + \hat{c}_{j,m} r^{\beta - 2((m+1)-k+(j-1))} x_i^{(m+1)-2k+2(j-1)} \\
&\quad =: \sum_{j=0}^{k} c_{j,m+1} r^{\beta - 2((m+1)-k+j)} x_i^{(m+1)-2k+2j},
\end{aligned}$$

this is (A.1.11) for $m+1$. For $m = 2k+1$, the induction step $m \to m+1$ is proved analogously.

*Step 2.* Now, let $\nu = (\nu_1, \nu_2) \in \mathbb{N}_0^2$ and choose $k_i$, $i = 1,2$, such that $\nu_i = 2k_i$ or $\nu_i = 2k_i + 1$. With (A.1.11) we get

$$\begin{aligned}
\partial^\nu r^\beta &= \frac{\partial^{\nu_2}}{\partial x_2^{\nu_2}} \left( \frac{\partial^{\nu_1}}{\partial x_1^{\nu_1}} r^\beta \right) \\
&= \frac{\partial^{\nu_2}}{\partial x_2^{\nu_2}} \left( \sum_{i=0}^{k_1} c_{i,\nu_1} r^{\beta - 2(\nu_1 - k_1 + i)} x_1^{\nu_1 - 2k_1 + 2i} \right) \\
&= \sum_{i=0}^{k_1} c_{i,\nu_1} x_1^{\nu_1 - 2k_1 + 2i} \left( \frac{\partial^{\nu_2}}{\partial x_2^{\nu_2}} r^{\beta - 2(\nu_1 - k_1 + i)} \right) \\
&= \sum_{i=0}^{k_1} c_{i,\nu_1} x_1^{\nu_1 - 2k_1 + 2i} \left( \sum_{j=0}^{k_2} c_{j,\nu_2,i} r^{\beta - 2(\nu_1 - k_1 + i) - 2(\nu_2 - k_2 + j)} x_2^{\nu_2 - 2k_2 + 2j} \right) \\
&= \sum_{i=0}^{k_1} \sum_{j=0}^{k_2} c_{i,j,\nu} r^{\beta - 2(\nu_1 + \nu_2 - k_1 - k_2 + i + j)} x_1^{\nu_1 - 2k_1 + 2i} x_2^{\nu_2 - 2k_2 + 2j}.
\end{aligned}$$

Now, since $(\nu_1 - 2k_1 + 2i)$ and $(\nu_2 - 2k_2 + 2j)$ are nonnegative and $|x_1|, |x_2| \le r$, we finally estimate

$$\begin{aligned}\left|\partial^\nu r^\beta\right| &\le \sum_{i=0}^{k_1}\sum_{j=0}^{k_2} |c_{i,j,\nu}|\, r^{\beta-2(\nu_1+\nu_2-k_1-k_2+i+j)} r^{\nu_1+\nu_2-2k_1-2k_2+2i+2j} \\ &= \sum_{i=0}^{k_1}\sum_{j=0}^{k_2} |c_{i,j,\nu}|\, r^{\beta-|\nu|} \\ &\lesssim r^{\beta-|\nu|}.\end{aligned}$$

□

**Lemma A.12.** *For $1 < p < \infty$ and $\omega \in (0, 2\pi)$ let $(\alpha, t(\cdot))$ be the solution of the eigenvalue problem* (7.1.3) *according to Lemma 7.2. We set $T(x, y) := t((\Xi_\phi(x, y))$ for $(x, y) \in C(1, \omega)$, where $\Xi$ denotes the transformation of coordinates as defined in* (A.2.1)*. Then, for all $\nu \in \mathbb{N}_0^2$ there exist constants $c_{\nu,k,j_1,j_2} \in \mathbb{R}$, such that*

$$\partial^\nu\, T(x, y) = \sum_{k=1}^{|\nu|}\sum_{j_1+j_2=|\nu|} c_{\nu,k,j_1,j_2} t^{(k)}\left(\Xi_\phi(x, y)\right) \frac{x^{j_1} y^{j_2}}{r^{2|\nu|}}, \qquad (x, y) \in C(1, \omega), \tag{A.1.12}$$

*where $r = |(x, y)|$.*

*Proof.* At first, since

$$\begin{aligned}\frac{\partial}{\partial x}\left(t(\Xi_\phi(x, y))\right) &= -t'(\Xi_\phi(x, y))\frac{y}{r^2}, \\ \frac{\partial}{\partial y}\left(t(\Xi_\phi(x, y))\right) &= t'(\Xi_\phi(x, y))\frac{x}{r^2},\end{aligned}$$

we see that (A.1.12) holds true for $|\nu| = 1$. Now, let $\ell \in \mathbb{N}$ and assume that (A.1.12) holds for all $\nu \in \mathbb{N}_0^2$ with $|\nu| \le \ell$. Let $\tilde\nu \in \mathbb{N}_0^2$ be arbitrary with $|\tilde\nu| = \ell + 1$. W.l.o.g. we assume that $\partial^{\tilde\nu} = \partial/\partial x \circ \partial^\nu$ for some $\nu \in \mathbb{N}_0^2$ with $|\nu| = \ell$. Then

$$\begin{aligned}\partial^{\tilde\nu}\, T(x, y) &= \frac{\partial}{\partial x}\left(\sum_{k=1}^{|\nu|}\sum_{j_1+j_2=|\nu|} c_{\nu,k,j_1,j_2} t^{(k)}(\Xi_\phi(x, y))\frac{x^{j_1}y^{j_2}}{r^{2|\nu|}}\right) \\ &= \sum_{k=1}^{|\nu|}\sum_{j_1+j_2=|\nu|} c_{\nu,k,j_1,j_2}\Big( - t^{(k+1)}(\Xi_\phi(x, y))\frac{y}{r^2}\frac{x^{j_1}y^{j_2}}{r^{2|\nu|}} \\ &\qquad + t^{(k)}(\Xi_\phi(x, y))\frac{j_1 x^{j_1-1}y^{j_2}r^{2|\nu|} - 2\,|\nu|\, x^{j_1}y^{j_2}r^{2|\nu|-2}x}{r^{4|\nu|}}\Big).\end{aligned}$$

With $j_1 x^{j_1-1} y^{j_2} r^{2|\nu|} = j_1\left(x^{j_1+1}y^{j_2} + x^{j_1-1}y^{j_2+2}\right) r^{2|\nu|-2}$ we arrive at

$$\begin{aligned}\partial^{\tilde\nu}\, T(x, y) &= \sum_{k=1}^{|\nu|}\sum_{j_1+j_2=|\nu|} c_{\nu,k,j_1,j_2}\Big( - t^{(k+1)}(\Xi_\phi(x, y))\frac{x^{j_1}y^{j_2+1}}{r^{2(|\nu|+1)}} \\ &\qquad + t^{(k)}(\Xi_\phi(x, y))\frac{j_1\left(x^{j_1+1}y^{j_2} + x^{j_1-1}y^{j_2+2}\right) - 2\,|\nu|\, x^{j_1+1}y^{j_2}}{r^{2(|\nu|+1)}}\Big) \\ &= \sum_{k=1}^{|\tilde\nu|}\sum_{j_1+j_2=|\tilde\nu|} c_{\tilde\nu,k,j_1,j_2} t^{(k)}(\Xi_\phi(x, y))\frac{x^{j_1}y^{j_2}}{r^{2(|\tilde\nu|)}}\end{aligned}$$

as claimed. □

**Lemma A.13.** *Let* $p > 0$ *and* $a, b \geq 0$.

**(i)** *If* $0 < p \leq 1$*, then*

$$2^{p-1}(a^p + b^p) \leq (a+b)^p \leq a^p + b^p.$$

**(ii)** *If* $p > 1$*, then*

$$a^p + b^p \leq (a+b)^p \leq 2^{p-1}(a^p + b^p).$$

*Proof.* Part (i). Since $(\cdot)^p$ is concave for $0 < p \leq 1$, it holds

$$\left(\frac{a+b}{2}\right)^p \geq \frac{1}{2}(a^p + b^p)$$

and the first inequality of (i) follows directly. To prove the second inequality, w.l.o.g. let $b \geq a > 0$. Then there exists $0 < \lambda \leq 1$ such that $a = \lambda b$. We estimate

$$(a+b)^p = (\lambda b + b)^p = (1+\lambda)^p\, b^p \leq (1+\lambda)\, b^p = \lambda \frac{a^p}{\lambda^p} + b^p = \lambda^{1-p} a^p + b^p \leq a^p + b^p.$$

Part (ii). Let $p > 1$. To prove the first inequality, we write

$$a^p + b^p = \left((a^p + b^p)^{1/p}\right)^p,$$

and since $1/p < 1$, we can apply the second inequality of part (i) to estimate

$$a^p + b^p \leq \left((a^p)^{1/p} + (b^p)^{1/p}\right)^p = (a+b)^p.$$

The last inequality follows from the convexity of $(\cdot)^p$, analogously to the case $0 < p \leq 1$ above. □

**Lemma A.14.** *For* $p \geq 1$ *and* $a, b \in \mathbb{R}^d$ *it holds*

$$|b|^p \geq |a|^p + p\langle |a|^{p-2} a, b - a\rangle.$$

*Proof.* First note that for a convex and continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ it holds that

$$f(x) + f'(x)(y - x) \leq f(y) \qquad \text{for all } x, y \in \mathbb{R},$$

i.e., for $|\cdot|^p : \mathbb{R} \to \mathbb{R}$ we get

$$|x|^p + p|x|^{p-2} x (y - x) \leq |y|^p.$$

With the choice $x = |a|$ and $y = |b|$ for $a, b \in \mathbb{R}^d$ it follows that

$$|a|^p + p|a|^{p-2}|a|\,(|b| - |a|) \leq |b|^p,$$

and we finally estimate

$$\begin{aligned} |a|^p + p\langle |a|^{p-2} a, b - a\rangle &= |a|^p + p\langle |a|^{p-2} a, b\rangle - p|a|^p \\ &\leq |a|^p + p|a|^{p-1}(|b| - |a|) \\ &\leq |b|^p. \end{aligned}$$

(One can moreover show: a differentiable function $f : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^d$ convex, is convex if and only if $f(y) \geq f(x) + \nabla f(x) \cdot (y - x)$ for all $x, y \in \Omega$.) □

**Lemma A.15.** *Let $1 < p < \infty$ and $0 < \omega < 2\pi$. Moreover, let $(\alpha, t)$ be the solution of the eigenvalue problem (7.1.3) from Lemma 7.2. It holds $r^\alpha t \in W^1(L_p(C(1,\omega)))$.*

*Proof.* First, with $z := (x, y)$, Lemma 7.4 and the fact that $\alpha > 0$ we estimate

$$\|r^\alpha t \,|\, L_p(C(1,\omega))\|^p = \int_{C(1,\omega)} r^{\alpha p} \, |t(\Xi_\phi(z)|^p \, \mathrm{d}z \le \|t \,|\, C([0,\omega])\|^p \int_{C(1,\omega)} r^{\alpha p} \, \mathrm{d}z \quad < \infty,$$

where $\Xi$ denotes the transformation of coordinates as defined in (A.2.1). Next, with

$$\frac{\partial}{\partial x} (r^\alpha t(\phi)) = \alpha x r^{\alpha-2} t(\phi) - y r^{\alpha-2} t'(\phi),$$

and Lemma 7.4 we estimate

$$\begin{aligned}\left\| \frac{\partial}{\partial x} (r^\alpha t) \,\middle|\, L_p(C(1,\omega)) \right\|^p &\le \int_{C(1,\omega)} \left( \alpha r^{\alpha-1} \, |t(\Xi_\phi(z))| + r^{\alpha-1} \, |t'(\Xi_\phi(z))| \right)^p \, \mathrm{d}z \\ &\lesssim \int_{C(1,\omega)} r^{(\alpha-1)p} \left( \|t \,|\, C([0,\omega])\| + \left\| t \,\middle|\, C^1([0,\omega]) \right\| \right)^p \, \mathrm{d}z \\ &\lesssim \int_{C(1,\omega)} r^{(\alpha-1)p} \, \mathrm{d}z. \end{aligned} \tag{A.1.13}$$

Finally, since $\alpha > \max\{0, (p-2)/(p-1)\}$,see (7.1.5), for $p \ge 2$ it holds

$$\alpha > \frac{p-2}{p-1} > \frac{p-2}{p} = 1 - \frac{2}{p},$$

i.e., $(\alpha - 1)p > -2$. For $1 < p < 2$, it holds

$$1 - \frac{2}{p} < 0 < \alpha,$$

i.e., $(\alpha - 1)p > -2$ also in this case. Hence, the integral in (A.1.13) is finite. The $L_p$-norm of the partial derivative of $r^\alpha t$ with respect to $y$ is estimated completely analogously. □

## A.2 The $p$-Laplace equation on a cone

In this section we consider the $p$-Laplace equation on finite cones $C(R, \omega) \subset \mathbb{R}^2$ of radius $R > 0$ and inner angle $0 < \omega < 2\pi$, see (7.0.1). Therefore, we introduce the following notation first. For $d = 2$, let $\Xi$ denote the transformation of Cartesian coordinates to polar coordinates, i.e.,

$$\begin{aligned} \Xi : \quad \mathbb{R}^2 &\to [0, \infty) \times [0, 2\pi), \\ (x, y) &\mapsto (r, \phi), \end{aligned} \tag{A.2.1}$$

where

$$r := \Xi_r(x,y) := \sqrt{x^2+y^2},$$

$$\phi := \Xi_\phi(x,y) := \begin{cases} \arctan\left(\frac{y}{x}\right), & \text{if } x>0,\ y\geq 0,\\ \arctan\left(\frac{y}{x}\right)+2\pi, & \text{if } x>0,\ y<0,\\ \arctan\left(\frac{y}{x}\right)+\pi, & \text{if } x<0,\\ \frac{\pi}{2}, & \text{if } x=0,\ y>0,\\ \frac{3\pi}{2}, & \text{if } x=0,\ y<0,\\ 0, & \text{if } x=0,\ y=0. \end{cases}$$

Furthermore, we will need the derivative of $\Xi$,

$$D\Xi(x,y) = \begin{pmatrix} \frac{\partial \Xi_r}{\partial x}(x,y) & \frac{\partial \Xi_r}{\partial y}(x,y) \\ \frac{\partial \Xi_\phi}{\partial x}(x,y) & \frac{\partial \Xi_\phi}{\partial y}(x,y) \end{pmatrix} = \begin{pmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix} = \begin{pmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{pmatrix},$$

where $(x,y)\in\mathbb{R}^2\setminus\{0\}$. Now, for a function $u:\mathbb{R}^2\to\mathbb{R}$, we denote its representation in polar coordinates by $\tilde{u}$, i.e.,

$$\tilde{u}:[0,\infty)\times[0,2\pi)\to\mathbb{R}, \qquad \tilde{u}(r,\phi) := u\left(\Xi^{-1}(r,\phi)\right).$$

Hence, $\tilde{u}\circ\Xi = u$.

### A.2.1 The $p$-Laplace operator in polar coordinates

**Lemma A.16.** *With the notation of this section, the identity*

$$\begin{aligned} &-\operatorname{div}\Big(|\nabla u(x,y)|^{p-2}\nabla u(x,y)\Big) \\ &\quad = -\frac{1}{r}\frac{\partial}{\partial r}\left\{\left[\left(\frac{\partial \tilde{u}}{\partial r}(r,\phi)\right)^2 + \left(\frac{1}{r}\frac{\partial \tilde{u}}{\partial \phi}(r,\phi)\right)^2\right]^{(p-2)/2} r\frac{\partial \tilde{u}}{\partial r}(r,\phi)\right\} \\ &\quad\quad - \frac{1}{r^2}\frac{\partial}{\partial \phi}\left\{\left[\left(\frac{\partial \tilde{u}}{\partial r}(r,\phi)\right)^2 + \left(\frac{1}{r}\frac{\partial \tilde{u}}{\partial \phi}(r,\phi)\right)^2\right]^{(p-2)/2} \frac{\partial \tilde{u}}{\partial \phi}(r,\phi)\right\} \end{aligned}$$

*holds true for all $(x,y)\in\mathbb{R}^2\setminus\{0\}$.*

*Proof.* Let $(x,y)\in\mathbb{R}^2\setminus\{0\}$. First, we compute

$$\begin{aligned} \nabla u(x,y) &= \nabla\tilde{u}\left(\Xi(x,y)\right)\cdot D\Xi(x,y) \\ &= \left(\frac{\partial \tilde{u}}{\partial r}(\Xi(x,y)) \quad , \quad \frac{\partial \tilde{u}}{\partial \phi}(\Xi(x,y))\right)\cdot\begin{pmatrix} \frac{x}{r} & \frac{y}{r} \\ \frac{-y}{r^2} & \frac{x}{r^2} \end{pmatrix} \\ &= \left(\frac{x}{r}\frac{\partial \tilde{u}}{\partial r}(\Xi(x,y)) - \frac{y}{r^2}\frac{\partial \tilde{u}}{\partial \phi}(\Xi(x,y)) \quad , \quad \frac{y}{r}\frac{\partial \tilde{u}}{\partial r}(\Xi(x,y)) + \frac{x}{r^2}\frac{\partial \tilde{u}}{\partial \phi}(\Xi(x,y))\right) \end{aligned}$$

and

$$|\nabla u(x,y)|^2 = \left(\frac{\partial \tilde{u}}{\partial r}(\Xi(x,y))\right)^2 + \frac{1}{r^2}\left(\frac{\partial \tilde{u}}{\partial \phi}(\Xi(x,y))\right)^2.$$

Hence,

$$
\begin{aligned}
&|\nabla u(x,y)|^{p-2}\nabla u(x,y)\\
&\quad=\left[\left(\frac{\partial\tilde u}{\partial r}(\Xi(x,y))\right)^2+\left(\frac{1}{r}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\right)^2\right]^{(p-2)/2}\\
&\qquad\cdot\left(\frac{x}{r}\frac{\partial\tilde u}{\partial r}(\Xi(x,y))-\frac{y}{r^2}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\quad,\quad\frac{y}{r}\frac{\partial\tilde u}{\partial r}(\Xi(x,y))+\frac{x}{r^2}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\right).
\end{aligned}
\tag{A.2.2}
$$

Now, with

$$
G(\Xi(x,y)):=\left[\left(\frac{\partial\tilde u}{\partial r}(\Xi(x,y))\right)^2+\left(\frac{1}{r}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\right)^2\right]^{(p-2)/2} r\frac{\partial\tilde u}{\partial r}(\Xi(x,y))
$$

and

$$
H(\Xi(x,y)):=\left[\left(\frac{\partial\tilde u}{\partial r}(\Xi(x,y))\right)^2+\left(\frac{1}{r}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\right)^2\right]^{(p-2)/2} \frac{\partial\tilde u}{\partial\phi}(\Xi(x,y)),
$$

the partial derivative of the first component of $|\nabla u(x,y)|^{p-2}\,\nabla u(x,y)$ with respect to $x$ computes to

$$
\begin{aligned}
&\frac{\partial}{\partial x}\left\{\left[\left(\frac{\partial\tilde u}{\partial r}(\Xi(x,y))\right)^2+\left(\frac{1}{r}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\right)^2\right]^{(p-2)/2}\left(\frac{x}{r}\frac{\partial\tilde u}{\partial r}(\Xi(x,y))-\frac{y}{r^2}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\right)\right\}\\
&\quad=\frac{\partial}{\partial x}\left\{G(\Xi(x,y))\frac{x}{r^2}-H(\Xi(x,y))\frac{y}{r^2}\right\}\\
&\quad=\frac{\partial G(\Xi(x,y))}{\partial x}\,\frac{x}{r^2}+G(\Xi(x,y))\frac{y^2-x^2}{r^4}-\frac{\partial H(\Xi(x,y))}{\partial x}\,\frac{y}{r^2}+H(\Xi(x,y))\frac{2xy}{r^4}\\
&\quad=\left(\frac{\partial G}{\partial r}(\Xi(x,y))\frac{x}{r}-\frac{\partial G}{\partial\phi}(\Xi(x,y))\frac{y}{r^2}\right)\frac{x}{r^2}+G(\Xi(x,y))\frac{y^2-x^2}{r^4}\\
&\qquad-\left(\frac{\partial H}{\partial r}(\Xi(x,y))\frac{x}{r}-\frac{\partial H}{\partial\phi}(\Xi(x,y))\frac{y}{r^2}\right)\frac{y}{r^2}+H(\Xi(x,y))\frac{2xy}{r^4},
\end{aligned}
\tag{A.2.3}
$$

and analogously taking the partial derivative of the second component with respect to $y$ yields

$$
\begin{aligned}
&\frac{\partial}{\partial y}\left\{\left[\left(\frac{\partial\tilde u}{\partial r}(\Xi(x,y))\right)^2+\left(\frac{1}{r}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\right)^2\right]^{(p-2)/2}\left(\frac{y}{r}\frac{\partial\tilde u}{\partial r}(\Xi(x,y))+\frac{x}{r^2}\frac{\partial\tilde u}{\partial\phi}(\Xi(x,y))\right)\right\}\\
&\quad=\frac{\partial}{\partial y}\left\{G(\Xi(x,y))\frac{y}{r^2}+H(\Xi(x,y))\frac{x}{r^2}\right\}\\
&\quad=\frac{\partial G(\Xi(x,y))}{\partial y}\,\frac{y}{r^2}+G(\Xi(x,y))\frac{x^2-y^2}{r^4}+\frac{\partial H(\Xi(x,y))}{\partial y}\,\frac{x}{r^2}-H(\Xi(x,y))\frac{2xy}{r^4}\\
&\quad=\left(\frac{\partial G}{\partial r}(\Xi(x,y))\frac{y}{r}+\frac{\partial G}{\partial\phi}(\Xi(x,y))\frac{x}{r^2}\right)\frac{y}{r^2}+G(\Xi(x,y))\frac{x^2-y^2}{r^4}\\
&\qquad+\left(\frac{\partial H}{\partial r}(\Xi(x,y))\frac{y}{r}+\frac{\partial H}{\partial\phi}(\Xi(x,y))\frac{x}{r^2}\right)\frac{x}{r^2}-H(\Xi(x,y))\frac{2xy}{r^4}.
\end{aligned}
\tag{A.2.4}
$$

Then, from (A.2.2), (A.2.3) and (A.2.4) it follows

$$
\begin{aligned}
-\mathrm{div}\Big(|\nabla u(x,y)|^{p-2}\nabla u(x,y)\Big) &= -\left(\frac{\partial G}{\partial r}(\Xi(x,y))\frac{x^2+y^2}{r^3} + \frac{\partial G}{\partial \phi}(\Xi(x,y))\frac{xy-xy}{r^4}\right) \\
&\quad - \left(\frac{\partial H}{\partial r}(\Xi(x,y))\frac{xy-xy}{r^3} + \frac{\partial H}{\partial \phi}(\Xi(x,y))\frac{x^2+y^2}{r^4}\right) \\
&= -\frac{1}{r}\frac{\partial G}{\partial r}(\Xi(x,y)) - \frac{1}{r^2}\frac{\partial H}{\partial \phi}(\Xi(x,y)).
\end{aligned}
$$

□

### A.2.2 $p$-harmonic functions of the form $r^\alpha t(\phi)$

Now, it is possible to characterize all strong solutions to the $p$-Laplace equation on the cone which take the form $\tilde{u}(r,\phi) = r^\alpha t(\phi)$.

**Proposition A.17.** *Let $1 < p < \infty$ and let $C(R,\omega)$ with $R > 0$ and $0 < \omega < 2\pi$ denote some cone as defined by (7.0.1). Then, for $\tilde{u}(r,\phi) = r^\alpha t(\phi)$ with $\alpha \in \mathbb{R}$ and $t \in C^2((0,\omega))$, the following two properties are equivalent:*

*(i) $\tilde{u}$ is $p$-harmonic in $C(R,\omega)$.*

*(ii) $\alpha$ and $t(\cdot)$ satisfy*

$$
\begin{aligned}
&\frac{\partial}{\partial \phi}\left\{\left[\alpha^2 t(\phi)^2 + \left(\frac{\partial t}{\partial \phi}(\phi)\right)^2\right]^{(p-2)/2}\frac{\partial t}{\partial \phi}(\phi)\right\} \\
&\quad + \alpha\Big(\alpha\,(p-1)+2-p\Big)\left[\alpha^2 t(\phi)^2 + \left(\frac{\partial t}{\partial \phi}(\phi)\right)^2\right]^{(p-2)/2} t(\phi) \;=\; 0 \quad \text{for all} \quad \phi \in (0,\omega).
\end{aligned}
$$

*Proof.* We apply Lemma A.16 for $\tilde{u}(r,\phi) = r^\alpha t(\phi)$. Hence, for $(x,y) \in C(R,\omega)$ and $(r,\phi) = \Xi(x,y)$ we get

$$
\begin{aligned}
&-\mathrm{div}\Big(|\nabla(\tilde{u}\circ\Xi)(x,y)|^{p-2}\,\nabla(\tilde{u}\circ\Xi)(x,y)\Big) \\
&\qquad = -\frac{1}{r}\frac{\partial}{\partial r}\left\{\left[\Big(\alpha r^{\alpha-1}t(\phi)\Big)^2 + \frac{1}{r^2}\left(r^\alpha\frac{\partial t}{\partial \phi}(\phi)\right)^2\right]^{(p-2)/2} r\alpha r^{\alpha-1}t(\phi)\right\} \\
&\qquad\quad - \frac{1}{r^2}\frac{\partial}{\partial \phi}\left\{\left[\Big(\alpha r^{\alpha-1}t(\phi)\Big)^2 + \frac{1}{r^2}\left(r^\alpha\frac{\partial t}{\partial \phi}(\phi)\right)^2\right]^{(p-2)/2} r^\alpha\frac{\partial t}{\partial \phi}(\phi)\right\} \\
&\qquad = -\frac{1}{r}\frac{\partial}{\partial r}\left\{r^{(\alpha-1)(p-2)}\left[\alpha^2 t(\phi)^2 + \left(\frac{\partial t}{\partial \phi}(\phi)\right)^2\right]^{(p-2)/2}\alpha r^\alpha t(\phi)\right\} \\
&\qquad\quad - \frac{1}{r^2}\frac{\partial}{\partial \phi}\left\{r^{(\alpha-1)(p-2)}\left[\alpha^2 t(\phi)^2 + \left(\frac{\partial t}{\partial \phi}(\phi)\right)^2\right]^{(p-2)/2} r^\alpha\frac{\partial t}{\partial \phi}(\phi)\right\}.
\end{aligned}
$$

Next, calculating the derivatives further yields

$$
\begin{aligned}
&-\operatorname{div}\Big(|\nabla(\tilde{u}\circ\Xi)(x,y)|^{p-2}\,\nabla(\tilde{u}\circ\Xi)(x,y)\Big)\\
&\quad= -\frac{1}{r}\left((\alpha-1)(p-2)+\alpha\right) r^{(\alpha-1)(p-2)+\alpha-1}\alpha\left[\alpha^2 t(\phi)^2+\left(\frac{\partial t}{\partial\phi}(\phi)\right)^2\right]^{(p-2)/2} t(\phi)\\
&\qquad\quad -\frac{1}{r^2} r^{(\alpha-1)(p-2)+\alpha}\frac{\partial}{\partial\phi}\left\{\left[\alpha^2 t(\phi)^2+\left(\frac{\partial t}{\partial\phi}(\phi)\right)^2\right]^{(p-2)/2}\frac{\partial t}{\partial\phi}(\phi)\right\}\\
&\quad= -\alpha\left(\alpha(p-1)+2-p\right) r^{(\alpha-1)(p-1)-1}\left[\alpha^2 t(\phi)^2+\left(\frac{\partial t}{\partial\phi}(\phi)\right)^2\right]^{(p-2)/2} t(\phi)\\
&\qquad\quad - r^{(\alpha-1)(p-1)-1}\frac{\partial}{\partial\phi}\left\{\left[\alpha^2 t(\phi)^2+\left(\frac{\partial t}{\partial\phi}(\phi)\right)^2\right]^{(p-2)/2}\frac{\partial t}{\partial\phi}(\phi)\right\}.
\end{aligned}
$$

Thus, $\tilde{u}$ is $p$-harmonic in $C(R,\omega)$ if and only if

$$
\begin{aligned}
0 = &-\alpha\left(\alpha(p-1)+2-p\right) r^{(\alpha-1)(p-1)-1}\left[\alpha^2 t(\phi)^2+\left(\frac{\partial t}{\partial\phi}(\phi)\right)^2\right]^{(p-2)/2} t(\phi)\\
&- r^{(\alpha-1)(p-1)-1}\frac{\partial}{\partial\phi}\left\{\left[\alpha^2 t(\phi)^2+\left(\frac{\partial t}{\partial\phi}(\phi)\right)^2\right]^{(p-2)/2}\frac{\partial t}{\partial\phi}(\phi)\right\}
\end{aligned}
$$

for all $0 < r < R$ and all $0 < \phi < \omega$. Now, dividing through $-r^{(\alpha-1)(p-1)-1}$ proves the assertion. □

## A.3 Further results: Quasi-normed spaces

**Proposition A.18.** *Let $Y$ be a $\mathbb{K}$-vector space and $\{\|\cdot\|_{p(j),j} \mid j \in J\}$ a family of $p$-norms on $Y$, where $0 < p(j) \leq 1$ for all $j \in J$. For $j \in J$, $r > 0$ and $y \in Y$ we set*

$$V_{j,r}(y) = \Big\{z \,\Big|\, \|z-y\|_{p(j),j} < r\Big\} = y + V_{j,r}(0).$$

*Furthermore, we define the sets*

$$U_{J_0,r}(y) = \bigcap_{j\in J_0} V_{j,r}(y), \qquad J_0 \subset J \text{ finite}, \quad r > 0, \quad y \in Y.$$

*With $\mathcal{O}$ we denote the topology generated by the sets $U_{J_0,r}(y)$, as stated in Proposition 2.18. Then, with respect to this topology $\mathcal{O}$, the sets $U_{J_0,r}(y)$ are open and all $p$-norms $\|\cdot\|_{p(j),j}$ are continuous.*

*Proof. Step 1.* We first show that the sets $U_{J_0,r}(y)$ are open. Since finite intersections of open sets are always open, it suffices to show that the sets $V_{j,r}(y)$ are open. Therefore, let $y \in Y$, $j \in J$ and $r > 0$ be arbitrary, but fixed. To simplify notation, we set $p = p(j)$. Due to Definition 2.5(ii), we have to show that each $x \in V_{j,r}(y)$ is an inner point of $V_{j,r}(y)$, i.e., that there exists an element of the local basis of $x$ which is contained in $V_{j,r}(y)$. Hence, fix $x \in V_{j,r}(y)$ and set $\|x-y\|_{p,j} =: \delta < r$. Since $r^p > \delta^p$, it holds that $\varepsilon^p := r^p - \delta^p > 0$. Now, for arbitrary $z \in V_{j,\varepsilon}(x)$ we estimate

$$\begin{aligned} \|z-y\|_{p,j}^p &= \|z-x+x-y\|_{p,j}^p \\ &\leq \|z-x\|_{p,j}^p + \|x-y\|_{p,j}^p \\ &< \varepsilon^p + \delta^p \\ &= r^p, \end{aligned}$$

thus $z \in V_{j,r}(y)$ and therefore $V_{j,\varepsilon}(x) \subset V_{j,r}(y)$. Hence, $x$ is an inner point of $V_{j,r}(y)$, and since $x \in V_{j,r}(y)$ was arbitrary, we proved that $V_{j,r}(y)$ is open.

*Step 2.* Let $j \in J$ and $y \in Y$ be arbitrary, but fixed. Again, we set $p = p(j)$. We show that $\|\cdot\|_{p,j}$ is continuous in $y$. Note that, due to Definition 2.7, we have to show that for each open neighborhood $W$ of $\|y\|_{p,j}$ there exist an open neighborhood $V$ of $y$ such that $\|V\|_{p,j} \subset W$. Now, since the sets $V_{j,r}(y)$ are open due to *Step 1*, it suffices to show that for each $\varepsilon > 0$ there exists an $r > 0$ such that

$$\left| \|y\|_{p,j} - \|z\|_{p,j} \right| < \varepsilon \qquad \text{for all} \qquad z \in V_{j,r}(y). \tag{A.3.1}$$

Now, let $\varepsilon > 0$. In case $y = 0$, we can choose $r = \varepsilon$. Hence, let $y \neq 0$ and w.l.o.g. $0 < \varepsilon < \|y\|_{p,j}$. Choose $r > 0$, such that

$$r < \left(1 - \left(1 - \frac{2^{-1/p}\varepsilon}{\|y\|_{p,j}}\right)^p\right)^{1/p} \|y\|_{p,j}\,. \tag{A.3.2}$$

Let $z \in V_{j,r}(y)$. We get

$$\left| \|y\|_{p,j}^p - \|z\|_{p,j}^p \right| \leq \|y-z\|_{p,j}^p < r^p. \tag{A.3.3}$$

To show (A.3.1), we distinguish two cases. In case $\|z\|_{p,j} \leq \|y\|_{p,j}$, we estimate

$$\|y\|_{p,j}^p - \|z\|_{p,j}^p < r^p < \left(1 - \left(1 - \frac{2^{-1/p}\varepsilon}{\|y\|_{p,j}}\right)^p\right) \|y\|_{p,j}^p\,,$$

hence

$$\left(1 - \frac{2^{-1/p}\varepsilon}{\|y\|_{p,j}}\right)^p \|y\|_{p,j}^p < \|z\|_{p,j}^p\,,$$

and we conclude that

$$\|y\|_{p,j} - \|z\|_{p,j} < 2^{-1/p}\varepsilon < \varepsilon.$$

Now we consider $\|z\|_{p,j} > \|y\|_{p,j}$. From (A.3.2) and (A.3.3) we get

$$\|z\|_{p,j}^p - \|y\|_{p,j}^p < \left(1 - \left(1 - \frac{2^{-1/p}\varepsilon}{\|y\|_{p,j}}\right)^p\right) \|z\|_{p,j}^p\,,$$

where we used that $\|y\|_{p,j} < \|z\|_{p,j}$. Hence,

$$\left(1 - \frac{2^{-1/p}\varepsilon}{\|y\|_{p,j}}\right)^p \|z\|_{p,j}^p < \|y\|_{p,j}^p\,,$$

and we conclude that

$$\|z\|_{p,j} - \|y\|_{p,j} < 2^{-1/p}\varepsilon\,\frac{\|z\|_{p,j}}{\|y\|_{p,j}}. \tag{A.3.4}$$

Now, using again (A.3.3) as well as Lemma A.13(i), we estimate

$$\|z\|_{p,j}^p < \|y\|_{p,j}^p + r^p \le 2^{1-p}\left(\|y\|_{p,j} + r\right)^p,$$

i.e.,

$$\|z\|_{p,j} < 2^{(1-p)/p}\left(\|y\|_{p,j} + r\right). \tag{A.3.5}$$

Finally, combining (A.3.4) and (A.3.5) we arrive at

$$\|z\|_{p,j} - \|y\|_{p,j} < \frac{\varepsilon}{2}\left(1 + \frac{r}{\|y\|_{p,j}}\right) < \varepsilon,$$

where in the last step we used that $r < \|y\|_{p,j}$. □

# A.4 Alternative Proofs

## A.4.1 Proof of Corollary 5.4

The following proof of Corollary 5.4 closely follows the lines of [76, Theorem 3].

*Proof.* Let $u \in \mathcal{K}^\ell_{p,a}(\Omega) \cap B^s_p(L_p(\Omega))$ and let $\mathcal{E}_S$ denote the extension operator, which simultaneously extends the spaces $\mathcal{K}^\ell_{p,a}(\Omega)$ and $B^s_p(L_p(\Omega))$ to $\mathcal{K}^\ell_{p,a}(\mathbb{R}^2)$ and $B^s_p(L_p(\mathbb{R}^2))$, respectively, see Remark 1.7. We emphasize that $\mathcal{K}^\ell_{p,a}(\Omega)$ and $\mathcal{K}^\ell_{p,a}(\mathbb{R}^2)$ admit the same singular set $S$, consisting of the vertices of $\Omega$. Furthermore, let $\psi_I$ denote the system of Daubechies wavelets, see Section 1.6 for details, where we choose $m \in \mathbb{N}$ sufficiently large such that $m \ge \ell$ and $\phi_m, D_m \in C^r(\mathbb{R})$ for some $r \in \mathbb{N}$ with $r > s$ and $r \ge \ell$. Now, first note that we can extend $u$ to some $\mathcal{E}_S u \in \mathcal{K}^\ell_{p,a}(\mathbb{R}^2) \cap B^s_p(L_p(\mathbb{R}^2))$. Since in particular $\mathcal{E}_S u \in L_p(\mathbb{R}^2)$ and the system $\psi_I$ forms a basis for $L_p(\mathbb{R}^2)$, it can be written as

$$\begin{aligned}\mathcal{E}_S u &= P_0(\mathcal{E}_S u) + \sum_{(I,\psi)\in\mathcal{I}^+\times\Psi^M} \langle \mathcal{E}_S u, \psi_I\rangle \psi_I \\ &= P_0(\mathcal{E}_S u) + \sum_{(I,\psi)\in\mathcal{I}^+\times\Psi^M} \langle \mathcal{E}_S u, \psi_{I,p'}\rangle \psi_{I,p}.\end{aligned}$$

Next we restrict this expansion and consider only those wavelets for which $(I,\psi)$ belongs to

$$\Lambda^\rho := \bigcup_{j\ge 0} \Lambda^\rho_j,$$

where we set

$$\Lambda_j^\rho := \left\{(I,\psi) \in \mathcal{I}^+ \times \Psi^M \,\middle|\, |I| = 2^{-2j} \quad \text{and} \quad Q(I)\cap\Omega \neq \emptyset\right\}.$$

Next we split up the index sets $\Lambda_j^\rho$ once more and write

$$\Lambda_j^\rho = \bigcup_{n\geq 0} \Lambda_{j,n}^\rho, \quad \text{with} \quad \Lambda_{j,n}^\rho := \left\{(I,\psi) \in \Lambda_j^\rho \,\middle|\, n2^{-j} \leq \rho_I < (n+1)2^{-j}\right\},$$

where

$$\rho_I := \inf_{x\in Q(I)} \rho(x)$$

and $\rho$ denotes the smooth distance function from the definition of the Babuska-Kondratiev spaces $\mathcal{K}_{p,a}^\ell(\mathbb{R}^2)$, see Section 1.3. We define

$$P_{\text{reg}}u := \sum_{j\geq 0}\sum_{n>0}\sum_{(I,\psi)\in\Lambda_{j,n}^\rho} \langle \mathcal{E}_S u, \psi_{I,p'}\rangle \psi_{I,p}$$

and

$$P_{\text{sing}}u := \sum_{j\geq 0}\sum_{(I,\psi)\in\Lambda_{j,0}^\rho} \langle \mathcal{E}_S u, \psi_{I,p'}\rangle \psi_{I,p},$$

we have

$$u = \mathcal{E}_S u\,|\,\Omega = P_0(\mathcal{E}_S u)\,|\,\Omega + P_{\text{reg}}u\,|\,\Omega + P_{\text{sing}}u\,|\,\Omega. \tag{A.4.1}$$

*Step 1.* We show that $\|P_0(\mathcal{E}_S u)\,|\,B_\tau^\sigma(L_\tau(\mathbb{R}^2))\| \lesssim \left\|u\,\middle|\, B_p^s(L_p(\Omega))\right\|$. Note that since $P_0$ is the projection onto the coarse levels, it holds that $P_0(\mathcal{E}_S u) \perp \psi_{I,p'}$ for all $I \in \mathcal{I}^+$ and $\psi \in \Psi^M$, i.e., $P_0^2(\mathcal{E}_S u) = P_0(\mathcal{E}_S u)$. Now, Proposition 1.22, i.e., the wavelet characterization of $B_\tau^\sigma(L_\tau(\mathbb{R}^2))$, yields

$$\begin{aligned}\left\|P_0(\mathcal{E}_S u)\,\middle|\, B_\tau^\sigma(L_\tau(\mathbb{R}^2))\right\| &\sim \left\|P_0^2(\mathcal{E}_S u)\,\middle|\, L_\tau(\mathbb{R}^2)\right\| + \left(\sum_{I\in\mathcal{I}^+}\sum_{\psi\in\Psi^M} |\langle P_0(\mathcal{E}_S u), \psi_{I,p'}\rangle|^\tau\right)^{1/\tau} \\ &= \left\|P_0(\mathcal{E}_S u)\,\middle|\, L_\tau(\mathbb{R}^2)\right\|. \end{aligned} \tag{A.4.2}$$

Next, $P_0(\mathcal{E}_S u)$ has compact support in $\mathbb{R}^2$, since $\mathcal{E}_S$ is local, and therefore

$$\left\|P_0(\mathcal{E}_S u)\,\middle|\, L_\tau(\mathbb{R}^2)\right\| \lesssim \left\|P_0(\mathcal{E}_S u)\,\middle|\, L_p(\mathbb{R}^2)\right\| \tag{A.4.3}$$

due to $\tau < p$. Further, from the wavelet characterization of $B_p^s(L_p(\mathbb{R}^2))$, see Lemma 1.20, we conclude

$$\left\|P_0(\mathcal{E}_S u)\,\middle|\, L_p(\mathbb{R}^2)\right\| \lesssim \left\|\mathcal{E}_S u\,\middle|\, B_p^s(L_p(\mathbb{R}^2))\right\|. \tag{A.4.4}$$

Finally, the continuity of $\mathcal{E}_S$, together with (A.4.2), (A.4.3) and (A.4.4) yields

$$\left\|P_0(\mathcal{E}_S u)\,\middle|\, B_\tau^\sigma(L_\tau(\mathbb{R}^2))\right\| \lesssim \left\|u\,\middle|\, B_p^s(L_p(\Omega))\right\|.$$

*Step 2.* We show that $\|P_{\text{reg}}u\,|\,B^\sigma_\tau(L_\tau(\mathbb{R}^2))\| \lesssim \left\|u\,\middle|\,\mathcal{K}^\ell_{p,a}(\Omega)\right\|$. Therefore, w.l.o.g. we may assume that $a \le \ell$, since $\mathcal{K}^\ell_{p,\tilde{a}}(\Omega) \hookrightarrow \mathcal{K}^\ell_{p,a}(\Omega)$ for all $\tilde{a} \ge a$. Now, let $n > 0$ and $(I,\psi) \in \Lambda^\rho_{j,n}$. First note that from Proposition A.1 we know, that for each $I$ there exists a polynomial $P_I$ of degree less than $\ell$, such that

$$\begin{aligned}\|\mathcal{E}_S u - P_I\,|\,L_p(Q(I))\| &\le c_0\,|Q(I)|^{\ell/2}\,|\mathcal{E}_S u|_{W^\ell(L_p(Q(I)))}\\ &\le c_1\,|I|^{\ell/2}\,|\mathcal{E}_S u|_{W^\ell(L_p(Q(I)))}\,.\end{aligned} \tag{A.4.5}$$

Now, the vanishing moment property of $\psi_{I,p'}$, together with an application of Hölder's inequality yields

$$\begin{aligned}|\langle\mathcal{E}_S u,\psi_{I,p'}\rangle| &= |\langle\mathcal{E}_S u - P_I,\psi_{I,p'}\rangle|\\ &\le \|\mathcal{E}_S u - P_I\,|\,L_p(Q(I))\|\cdot\|\psi_{I,p'}\,|\,L_{p'}(Q(I))\|\,.\end{aligned}$$

Using (A.4.5) and $\ell - a \ge 0$, we further estimate

$$\begin{aligned}|\langle\mathcal{E}_S u,\psi_{I,p'}\rangle| &\le c_1\,|I|^{\ell/2}\,|\mathcal{E}_S u|_{W^\ell(L_p(Q(I)))}\\ &= c_1\,|I|^{\ell/2}\left(\sum_{|\nu|=\ell}\int_{Q(I)}|D^\nu\mathcal{E}_S u|^p\,\mathrm{d}x\right)^{1/p}\\ &\le c_1\,|I|^{\ell/2}\,\rho_I^{-\ell+a}\left(\sum_{|\nu|=\ell}\int_{Q(I)}\left|\rho(x)^{\ell-a}D^\nu\mathcal{E}_S u\right|^p\,\mathrm{d}x\right)^{1/p}.\end{aligned}$$

With

$$\mu_I := \left(\sum_{|\nu|=\ell}\int_{Q(I)}\left|\rho(x)^{\ell-a}D^\nu\mathcal{E}_S u\right|^p\,\mathrm{d}x\right)^{1/p}$$

we hence get

$$|\langle\mathcal{E}_S u,\psi_{I,p'}\rangle| \le c_1\,|I|^{\ell/2}\,\rho_I^{-\ell+a}\mu_I.$$

Now, for fixed $j \ge 0$, we estimate

$$\begin{aligned}\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}|\langle\mathcal{E}_S u,\psi_{I,p'}\rangle|^\tau &\le c_1^\tau\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\left(|I|^{\ell/2}\,\rho_I^{-\ell+a}\mu_I\right)^\tau\\ &\le c_1^\tau\left(\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\left(|I|^{\ell\tau/2}\,\rho_I^{(a-\ell)\tau}\right)^{p/(p-\tau)}\right)^{(p-\tau)/p}\left(\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\mu_I^p\right)^{\tau/p},\end{aligned}$$

where in the last step we applied Hölder's inequality once again. Since for fixed $j$ every $x \in \Omega$ is contained in a finite number of cubes $Q(I)$, and this number is bounded by some constant independent of $x$ and $j$, we get

$$\begin{aligned}\left(\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\mu_I^p\right)^{1/p} &= \left(\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\sum_{|\nu|=\ell}\int_{Q(I)}\left|\rho(x)^{\ell-a}D^\nu(\mathcal{E}_S u)(x)\right|^p\,\mathrm{d}x\right)^{1/p}\\ &\le c_2\left(\sum_{|\nu|=\ell}\int_{\mathbb{R}^2}\left|\rho(x)^{\ell-a}D^\nu(\mathcal{E}_S u)(x)\right|^p\,\mathrm{d}x\right)^{1/p}\\ &\le c_2\left\|\mathcal{E}_S u\,\middle|\,\mathcal{K}^\ell_{p,a}(\mathbb{R}^2)\right\|\\ &\le c_3\left\|u\,\middle|\,\mathcal{K}^\ell_{p,a}(\Omega)\right\|.\end{aligned}$$

Since $\rho_I \le 1$, it holds $\Lambda^\rho_{j,n} = \emptyset$ for $n > 2^j$. Furthermore, $\#\Lambda^\rho_{j,n} \lesssim n$. With this we estimate

$$\begin{aligned}&\left(\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\left(|I|^{\ell\tau/2}\rho_I^{(a-\ell)\tau}\right)^{p/(p-\tau)}\right)^{(p-\tau)/p}\\ &\qquad\le\left(\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\left(|I|^{\ell\tau/2}n^{(a-\ell)\tau}2^{-j(a-\ell)\tau}\right)^{p/(p-\tau)}\right)^{(p-\tau)/p}\\ &\qquad\le\left(\sum_{n=1}^{2^j}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\left(2^{-ja\tau}n^{(a-\ell)\tau}\right)^{p/(p-\tau)}\right)^{(p-\tau)/p}\\ &\qquad\lesssim\left(2^{-jap\tau/(p-\tau)}\sum_{n=1}^{2^j}n^{(a-\ell)p\tau/(p-\tau)+1}\right)^{(p-\tau)/p}\\ &\qquad= 2^{-ja\tau}\left(\sum_{n=1}^{2^j}n^{(a-\ell)p\tau/(p-\tau)+1}\right)^{(p-\tau)/p}.\end{aligned}$$

Note that

$$(a-\ell)\frac{p\tau}{p-\tau}+1 = \frac{p\tau}{p-\tau}\left(a-\ell+\frac{1}{\tau}-\frac{1}{p}\right) = \frac{p\tau}{p-\tau}\left(a-\ell+\frac{\sigma}{2}\right) > -1$$

if and only if

$$a-\ell+\frac{\sigma}{2} > \frac{1}{p}-\frac{1}{\tau} = -\frac{\sigma}{2},$$

which is equivalent to $a-\ell+\sigma>0$. We distinguish between three cases.

$$\left(\sum_{n>0}\sum_{(I,\psi)\in\Lambda^\rho_{j,n}}\left(|I|^{\ell\tau/2}\rho_I^{(a-\ell)\tau}\right)^{p/(p-\tau)}\right)^{(p-\tau)/p} \lesssim 2^{-ja\tau}\begin{cases}2^{j(a-\ell)\tau+2j(p-\tau)/p}, & a-\ell+\sigma>0,\\ (j+1)^{(p-\tau)/p}, & a-\ell+\sigma=0,\\ 1, & a-\ell+\sigma<0.\end{cases}$$

In the first case we get

$$\sum_{j\geq 0}\sum_{n>0}\sum_{(I,\psi)\in\Lambda^{\rho}_{j,n}} |\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{\tau} \lesssim \sum_{j\geq 0} 2^{-j\ell\tau+2j(p-\tau)/p} \left\| u \,\middle|\, \mathcal{K}^{\ell}_{p,a}(\Omega)\right\|^{\tau}$$
$$\lesssim \left\| u \,\middle|\, \mathcal{K}^{\ell}_{p,a}(\Omega)\right\|^{\tau},$$

where the last estimate holds if and only if $-\ell\tau + 2(p-\tau)/p < 0$, which is equivalent to

$$\ell > 2\left(\frac{1}{\tau}-\frac{1}{p}\right) = \sigma.$$

In case $a-\ell+\sigma = 0$ we get

$$\sum_{j\geq 0}\sum_{n>0}\sum_{(I,\psi)\in\Lambda^{\rho}_{j,n}} |\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{\tau} \lesssim \sum_{j\geq 0} 2^{-ja\tau}\,(j+1)^{(p-\tau)/p} \left\| u \,\middle|\, \mathcal{K}^{\ell}_{p,a}(\Omega)\right\|^{\tau}$$
$$\lesssim \left\| u \,\middle|\, \mathcal{K}^{\ell}_{p,a}(\Omega)\right\|^{\tau},$$

where the last estimate holds if and only if $a = \ell - \sigma > 0$, thus if $\ell > \sigma$. For the last case we get

$$\sum_{j\geq 0}\sum_{n>0}\sum_{(I,\psi)\in\Lambda^{\rho}_{j,n}} |\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{\tau} \lesssim \sum_{j\geq 0} 2^{-ja\tau} \left\| u \,\middle|\, \mathcal{K}^{\ell}_{p,a}(\Omega)\right\|^{\tau}$$
$$\lesssim \left\| u \,\middle|\, \mathcal{K}^{\ell}_{p,a}(\Omega)\right\|^{\tau},$$

since $a\tau > 0$ due to our assumptions.

*Step 3.* We show that $\|P_{\mathrm{sing}}u \,|\, B^{\sigma}_{\tau}(L_{\tau}(\mathbb{R}^2))\| \lesssim \left\| u \,\middle|\, B^{s}_{p}(L_p(\Omega))\right\|$. To do so, first note that $\#\Lambda^{\rho}_{j,0} \leq C$. An application of Hölder's inequality yields

$$\sum_{(I,\psi)\in\Lambda^{\rho}_{j,0}} |\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{\tau} \lesssim \left(\sum_{(I,\psi)\in\Lambda^{\rho}_{j,0}} |\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{p}\right)^{\tau/p}$$
$$= 2^{-js\tau}\left(\sum_{(I,\psi)\in\Lambda^{\rho}_{j,0}} |I|^{-sp/2}\,|\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{p}\right)^{\tau/p}.$$

Now, summing over $j$ and applying Hölder's inequality once more yields

$$\sum_{j\geq 0}\sum_{(I,\psi)\in\Lambda^{\rho}_{j,0}} |\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{\tau} \lesssim \sum_{j\geq 0} 2^{-js\tau}\left(\sum_{(I,\psi)\in\Lambda^{\rho}_{j,0}} |I|^{-sp/2}\,|\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{p}\right)^{\tau/p}$$
$$\leq \left(\sum_{j\geq 0} 2^{-js\tau p/(p-\tau)}\right)^{(p-\tau)/p}\left(\sum_{j\geq 0}\sum_{(I,\psi)\in\Lambda^{\rho}_{j,0}} |I|^{-sp/2}\,|\langle \mathcal{E}_S u, \psi_{I,p'}\rangle|^{p}\right)^{\tau/p}$$
$$\lesssim \left(\sum_{j\geq 0} 2^{-js\tau p/(p-\tau)}\right)^{(p-\tau)/p} \left\| \mathcal{E}_S u \,\middle|\, B^{s}_{p}(L_p(\mathbb{R}^2))\right\|^{\tau}$$
$$\lesssim \left\| u \,\middle|\, B^{s}_{p}(L_p(\Omega))\right\|^{\tau},$$

where the last inequality holds true because $s > 0$ and $0 < \tau < p$, thus $s\tau p/(p-\tau) > 0$.

*Step 4.* Finally, with (A.4.1) and *Step 1 - 3*, we estimate

$$\begin{aligned}\|u \,|\, B^\sigma_\tau(L_\tau(\Omega))\| &= \|P_0(\mathcal{E}_S u) + P_{\mathrm{reg}}u + P_{\mathrm{sing}}u \,|\, B^\sigma_\tau(L_\tau(\Omega))\| \\ &\lesssim \|P_0(\mathcal{E}_S u) \,|\, B^\sigma_\tau(L_\tau(\Omega))\| + \|P_{\mathrm{reg}}u \,|\, B^\sigma_\tau(L_\tau(\Omega))\| + \|P_{\mathrm{sing}}u \,|\, B^\sigma_\tau(L_\tau(\Omega))\| \\ &\lesssim \left\|P_0(\mathcal{E}_S u) \,\middle|\, B^\sigma_\tau(L_\tau(\mathbb{R}^2))\right\| + \left\|P_{\mathrm{reg}}u \,\middle|\, B^\sigma_\tau(L_\tau(\mathbb{R}^2))\right\| + \left\|P_{\mathrm{sing}}u \,\middle|\, B^\sigma_\tau(L_\tau(\mathbb{R}^2))\right\| \\ &\lesssim \left\|u \,\middle|\, \mathcal{K}^\ell_{p,a}(\Omega)\right\| + \left\|u \,\middle|\, B^s_p(L_p(\Omega))\right\|.\end{aligned}$$

□

## A.4.2 Proof of Lemma 7.4

To prove Lemma 7.4, we need a representation of the solution $t$ of problem (7.1.3) - as well as some other assertions - as derived by Dobrowolski in his proof of Lemma 7.2, see [57, Proof of Theorem 1]. Hence, at first we give a rigorous proof of Lemma 7.2, which essentially is a detailed version of the outline given in [57].

**Proof of Lemma 7.2.**

*Proof.* For simplicity, we will use the notation $t'$, $t''$ etc. for the derivatives of $t$ and when appropriate, we may as well omit the argument $\phi$. First note that problem (7.1.3) is invariant under translations, i.e., we may consider (7.1.3) on the interval $(-\omega/2, \omega/2)$ with boundary conditions $t(-\omega/2) = t(\omega/2) = 0$ and seek a positive solution $t \in C^2([-\omega/2, \omega/2])$. Furthermore, we impose the additional conditions $t'(0) = 0$, as well as $t'(-\omega/2) \neq 0$ and $t'(\omega/2) \neq 0$. In the following, we will show that for $\alpha$ as given by (7.1.4), this problem admits a unique (up to multiplication by scalars) solution $t$ .

*Step 1.* We derive a problem of a simpler shape. Therefore, at first we write (7.1.3) as

$$\begin{aligned}(p-2)\left[\alpha^2 t^2 + (t')^2\right]^{(p-4)/2}\left[\alpha^2 t t' + t' t''\right] t' + \left[\alpha^2 t^2 + (t')^2\right]^{(p-2)/2} t'' \\ + \alpha\left(\alpha\,(p-1) + 2 - p\right)\left[\alpha^2 t^2 + (t')^2\right]^{(p-2)/2} t \;=\; 0 \qquad \text{in} \qquad \left(-\frac{\omega}{2}, \frac{\omega}{2}\right).\end{aligned} \tag{A.4.6}$$

Before we proceed, note that $-1 \leq \Gamma < 0$, and therefore

$$\Theta = \frac{p(\Gamma-1) - 2\Gamma}{2\Gamma(p-1)} \geq \frac{2\Gamma p - 2\Gamma}{2\Gamma(p-1)} = 1. \tag{A.4.7}$$

Furthermore, it holds

$$\begin{aligned}\Theta^2 + \frac{1}{\Gamma} &= \frac{(\Gamma-1)^2 p^2 - 4\Gamma p(\Gamma-1) + 4\Gamma^2 + 4\Gamma(p-1)^2}{4\Gamma^2(p-1)^2} \\ &= \frac{(\Gamma+1)^2 p^2 - 4\Gamma(\Gamma+1)(p-1)}{4\Gamma^2(p-1)^2} \\ &\geq 0,\end{aligned} \tag{A.4.8}$$

and from (A.4.8) with $-1 \le \Gamma < 0$ we estimate

$$\begin{aligned}\Theta^2 + \frac{1}{\Gamma} &< \frac{(\Gamma+1)^2 p^2 - 4\Gamma(\Gamma+1)p + 4\Gamma^2}{4\Gamma^2(p-1)^2} \\ &= \left(\frac{2\Gamma - (\Gamma+1)p}{2\Gamma(p-1)}\right)^2 \\ &= \left(\frac{(\Gamma-1)p - 2\Gamma - 2\Gamma(p-2)}{2\Gamma(p-1)}\right)^2 \\ &= \left(\Theta - \frac{p-2}{p-1}\right)^2,\end{aligned}$$

i.e.,

$$0 \le \Theta^2 + \frac{1}{\Gamma} < \left(\Theta - \frac{p-2}{p-1}\right)^2.$$

Thus, we conclude

$$\Theta - \sqrt{\Theta^2 + \Gamma^{-1}} > \frac{p-2}{p-1},$$

and since $0 \le \sqrt{\Theta^2 + \Gamma^{-1}} < \Theta$, it holds

$$\alpha > \max\left\{0, \frac{p-2}{p-1}\right\}$$

for all $0 < \omega \le 2\pi$ and $1 < p < \infty$. Now, dividing equation (A.4.6) through $[\alpha^2 t^2 + (t')^2]^{(p-4)/2} > 0$ yields

$$\begin{aligned}0 &= (p-2)\left[\alpha^2 tt' + t't''\right]t' + \left[\alpha^2 t^2 + (t')^2\right]t'' + \alpha\left(\alpha\left(p-1\right) + 2 - p\right)\left[\alpha^2 t^2 + (t')^2\right]t \\ &= \left[\alpha^2(p-2) + \alpha\left(\alpha\left(p-1\right) + 2 - p\right)\right](t')^2 t + [p-2+1]\,(t')^2 t'' + \alpha^2 t^2 t'' \\ &\quad + \left[\alpha\left(\alpha\left(p-1\right) + 2 - p\right)\alpha^2\right]t^3 \\ &= \left[\alpha^2(2p-3) + \alpha(2-p)\right](t')^2 t + (p-1)(t')^2 t'' + \alpha^2 t^2 t'' \\ &\quad + \left[\alpha^4(p-1) + \alpha^3(2-p)\right]t^3\end{aligned}$$

in $(-\omega/2, \omega/2)$. Dividing through $t^3$ gives

$$0 = \left[\alpha^2(2p-3) + \alpha(2-p)\right]\left(\frac{t'}{t}\right)^2 + (p-1)\left(\frac{t'}{t}\right)^2 \frac{t''}{t} + \alpha^2 \frac{t''}{t} + \left[\alpha^4(p-1) + \alpha^3(2-p)\right]$$

in $(-\omega/2, \omega/2)$.

Next, we set $v(\phi) := t'(\phi)/t(\phi)$ for $\phi \in (-\omega/2, \omega/2)$. Since $t(0) > 0$ and $t'(0) = 0$, we have $v(0) = 0$. Since $t(-\omega/2) = t(\omega/2) = 0$, as well as $t'(-\omega/2) > 0$ and $t'(\omega/2) < 0$, it holds $\lim_{\phi \to -\omega/2} v(\phi) = \infty$ and $\lim_{\phi \to \omega/2} v(\phi) = -\infty$. Using that $t'' = t(v' + v^2)$, this substitution yields

$$\begin{aligned}0 &= \left[\alpha^2(2p-3) + \alpha(2-p)\right]v^2 + (p-1)v^2(v'+v^2) + \alpha^2(v'+v^2) + \left[\alpha^4(p-1) + \alpha^3(2-p)\right] \\ &= (p-1)v^4 + \left[\alpha^2(2p-2) + \alpha(2-p)\right]v^2 + \left[(p-1)v^2 + \alpha^2\right]v' + \left[\alpha^4(p-1) + \alpha^3(2-p)\right]\end{aligned}$$

in $(-\omega/2, \omega/2)$. Next, dividing through $(p-1)v^2 + \alpha^2$ gives

$$\begin{aligned} -v' &= \frac{(p-1)v^4 + \left[\alpha^2(2p-2) + \alpha(2-p)\right] v^2 + \left[\alpha^4(p-1) + \alpha^3(2-p)\right]}{(p-1)v^2 + \alpha^2} \\ &= \frac{v^4 + \left[2\alpha^2 + \alpha(2-p)/(p-1)\right] v^2 + \left[\alpha^4 + \alpha^3(2-p)/(p-1)\right]}{v^2 + \alpha^2/(p-1)} \quad \text{in} \quad \left(-\frac{\omega}{2}, \frac{\omega}{2}\right). \end{aligned}$$

Setting $a := \alpha^2/(p-1)$, $f := \alpha^2$ and $g := \alpha^2 + \alpha(2-p)/(p-1)$, we arrive at

$$-v' = \frac{v^4 + (f+g)v^2 + fg}{v^2 + a} = \frac{(v^2+f)(v^2+g)}{v^2+a} \qquad \text{in} \qquad \left(-\frac{\omega}{2}, \frac{\omega}{2}\right).$$

Hence, we have to find a solution $v \in C^1((-\omega/2, \omega/2))$ of the problem

$$v'(\phi) = -\frac{(v(\phi)^2 + f)(v(\phi)^2 + g)}{v(\phi)^2 + a} \qquad \text{for all} \qquad \phi \in \left(-\frac{\omega}{2}, \frac{\omega}{2}\right), \tag{A.4.9}$$

which satisfies

$$v(0) = 0, \qquad \lim_{\phi \to -\omega/2} v(\phi) = \infty, \qquad \lim_{\phi \to \omega/2} v(\phi) = -\infty. \tag{A.4.10}$$

*Step 2.* Before we solve problem (A.4.9), (A.4.10), note that $a, f$ and $g$ are positive, since $\alpha > \max\{0, (p-2)/(p-1)\}$. Hence, from (A.4.9) it follows that the derivative $v'$ is strictly negative, i.e., $v$ is strictly decreasing in $(-\omega/2, \omega/2)$. Thus, $v$ has a unique inverse, and for $y \in \mathbb{R}$ it holds

$$(v^{-1})'(y) = \frac{1}{v'(v^{-1}(y))} = -\frac{y^2 + a}{(y^2+f)(y^2+g)}.$$

From the condition $v(0) = 0$, i.e. $v^{-1}(0) = 0$, it follows

$$v^{-1}(y) = \int_0^y (v^{-1})'(\eta)\,\mathrm{d}\eta.$$

Note that $v^{-1}$ is odd and therefore $v$ is odd as well. Hence, it remains to check one of the boundary conditions in (A.4.10).

We consider $\phi \to \omega/2$. From the condition $\lim_{\phi \to \omega/2} v(\phi) = -\infty$, we have to solve $\lim_{y \to -\infty} v^{-1}(y) = \omega/2$. For $v^{-1}$ we compute

$$\begin{aligned} v^{-1}(y) &= -\int_0^y \frac{\eta^2 + a}{(\eta^2+f)(\eta^2+g)}\,\mathrm{d}\eta \\ &= -\int_0^y \frac{\eta^2+f}{(\eta^2+f)(\eta^2+g)} + \frac{a-f}{(\eta^2+f)(\eta^2+g)}\,\mathrm{d}\eta \\ &= -\int_0^y \frac{1}{\eta^2+g}\,\mathrm{d}\eta - \int_0^y \frac{a-f}{(\eta^2+f)(\eta^2+g)}\,\mathrm{d}\eta, \qquad y \in \mathbb{R}. \end{aligned}$$

In case $p \neq 2$, a partial fraction decomposition yields

$$\frac{a-f}{(\eta^2+f)(\eta^2+g)} = \frac{f-a}{f-g} \cdot \frac{1}{\eta^2+f} + \frac{a-f}{f-g} \cdot \frac{1}{\eta^2+g},$$

and with $\arctan(\eta)' = 1/(1+\eta^2)$ we get

$$\begin{aligned}
v^{-1}(y) &= -\int_0^y \frac{1}{\eta^2+g}\,\mathrm{d}\eta - \frac{a-f}{g-f}\int_0^y \frac{1}{\eta^2+f}\,\mathrm{d}\eta - \frac{a-f}{f-g}\int_0^y \frac{1}{\eta^2+g}\,\mathrm{d}\eta \\
&= -\int_0^y \left(\frac{1}{\sqrt{g}}\arctan\left(\frac{\eta}{\sqrt{g}}\right)\right)'\mathrm{d}\eta - \frac{a-f}{g-f}\int_0^y \left(\frac{1}{\sqrt{f}}\arctan\left(\frac{\eta}{\sqrt{f}}\right)\right)'\mathrm{d}\eta \\
&\quad - \frac{a-f}{f-g}\int_0^y \left(\frac{1}{\sqrt{g}}\arctan\left(\frac{\eta}{\sqrt{g}}\right)\right)'\mathrm{d}\eta \\
&= -\frac{1}{\sqrt{g}}\frac{a-g}{f-g}\int_0^y \left(\arctan\left(\frac{\eta}{\sqrt{g}}\right)\right)'\mathrm{d}\eta - \frac{1}{\sqrt{f}}\frac{a-f}{g-f}\int_0^y \left(\arctan\left(\frac{\eta}{\sqrt{f}}\right)\right)'\mathrm{d}\eta.
\end{aligned}$$

Now, it holds $(a-g)/(f-g) = 1-\alpha$ and $(a-f)/(g-f) = \alpha$, and we arrive at

$$v^{-1}(y) = -\frac{1-\alpha}{\sqrt{g}}\arctan\left(\frac{y}{\sqrt{g}}\right) - \arctan\left(\frac{y}{\alpha}\right), \qquad y \in \mathbb{R}. \tag{A.4.11}$$

A short computation shows that (A.4.11) also holds for $p = 2$. Now, from the condition $\lim_{y\to-\infty} v^{-1}(y) = \omega/2$ we conclude

$$\frac{\omega}{2} = \left[\frac{1-\alpha}{\sqrt{g}} + 1\right]\frac{\pi}{2}, \tag{A.4.12}$$

and thus the boundary condition is satisfied if

$$\frac{\omega}{\pi} - 1 = \frac{1-\alpha}{\sqrt{\alpha^2 + \alpha(2-p)/(p-1)}}. \tag{A.4.13}$$

For $\omega = 2\pi$, a short computation yields $\alpha = (p-1)/p$, as stated in (7.1.4). Note that from (A.4.13) it follows

$$\alpha \geq 1, \qquad \text{if } 0 < \omega \leq \pi, \tag{A.4.14}$$

$$\alpha < 1, \qquad \text{if } \pi < \omega \leq 2\pi. \tag{A.4.15}$$

In case $0 < \omega < 2\pi$, using $\Gamma = \Gamma(\omega) = (\omega/\pi - 1)^2 - 1$, we further get

$$\Gamma + 1 = \frac{(1-\alpha)^2}{\alpha^2 + \alpha(2-p)/(p-1)},$$

hence,

$$\begin{aligned}
0 &= (\Gamma+1)\left(\alpha^2 + \alpha\frac{2-p}{p-1}\right) - 1 + 2\alpha - \alpha^2 \\
&= \alpha^2\Gamma + \alpha\left((\Gamma+1)\frac{2-p}{p-1} + 2\right) - 1 \\
&= \alpha^2\Gamma + \alpha\,\frac{(\Gamma+1)(2-p) + 2p - 2}{p-1} - 1 \\
&= \alpha^2\Gamma + \alpha\,\frac{p(1-\Gamma) + 2\Gamma}{p-1} - 1.
\end{aligned}$$

Dividing through $\Gamma$ and using $\Theta = \Theta(\Gamma, p) = [p(\Gamma - 1) - 2\Gamma] / [2\Gamma(p-1)]$ yields

$$0 = \alpha^2 - 2\Theta\alpha - \frac{1}{\Gamma},$$

i.e.,

$$\alpha = \Theta \pm \sqrt{\Theta^2 + \Gamma^{-1}}. \tag{A.4.16}$$

Note that for $\omega = \pi$, we get $\Gamma = -1$ and $\Theta = 1$, hence $\alpha = 1$ as stated in (7.1.4). In the following, let $\omega \in (0, 2\pi) \setminus \{\pi\}$. Since then $-1 < \Gamma < 0$, from (A.4.7) we know that $\Theta > 1$ for all $p > 1$. Furthermore, it holds

$$\frac{1}{\Gamma} > \frac{1}{\Gamma} \cdot \frac{p + \Gamma}{p - 1} = \frac{(1 - \Gamma)p + 2\Gamma}{\Gamma(p-1)} + 1 = -2\Theta + 1,$$

and therefore

$$(\Theta - 1)^2 < \Theta^2 + \Gamma^{-1}.$$

Now, since $\Theta > 1$, this implies

$$0 < \Theta - 1 < \sqrt{\Theta^2 + \Gamma^{-1}},$$

thus

$$\Theta - \sqrt{\Theta^2 + \Gamma^{-1}} < 1 \tag{A.4.17}$$

and

$$\Theta + \sqrt{\Theta^2 + \Gamma^{-1}} > 1 \tag{A.4.18}$$

for all $p > 1$ and $\omega \in (0, 2\pi) \setminus \{\pi\}$. Finally, the conditions (A.4.14), (A.4.15) on $\alpha$, together with (A.4.16), (A.4.17) and (A.4.18) show that $\alpha$ as stated in (7.1.4) uniquely solves equation (A.4.13). Thus, we have shown that $\lim_{y \to -\infty} v^{-1}(y) = \omega/2$, and $v$ is a solution of problem (A.4.9), (A.4.10). Note that since $v^{-1}$ is uniquely determined, $v$ is unique as well.

*Step 3.* Next, we build the solution $t$ of the original problem with the help of $v$. Since

$$(\ln \circ t)'(\phi) = \frac{t'(\phi)}{t(\phi)} = v(\phi)$$

for all $\phi \in (-\omega/2, \omega/2)$, we have

$$\ln(t(\phi)) - \ln(t(0)) = \int_0^\phi (\ln \circ t)'(\xi)\, \mathrm{d}\xi = \int_0^\phi v(\xi)\, \mathrm{d}\xi,$$

hence,

$$t(\phi) = t(0) \exp\left( \int_0^\phi v(\xi)\, \mathrm{d}\xi \right), \qquad \phi \in \left( -\frac{\omega}{2}, \frac{\omega}{2} \right).$$

Now we have found a solution $t$ of (7.1.3) on the interval $(-\omega/2, \omega/2)$. For the sake of simplicity, we set $t(0) := 1$. Note that $t > 0$ on $(-\omega/2, \omega/2)$, and that $t'(0) = t(0)v(0) = 0$, since $v(0) = 0$.

*Step 4.* Next, we show that the boundary conditions for $t$ and $t'$ are satisfied. To do so, note that $v$ is odd, and therefore $t$ is an even function, and $t'$ is odd. Hence, it suffices to consider $\phi = \omega/2$.

*Substep 4.1.* First, we prove that $\lim_{\phi\to\omega/2} t(\phi) = 0$, i.e.,

$$\lim_{\phi\to\omega/2} \int_0^\phi v(\xi)\,\mathrm{d}\xi = -\infty.$$

For $\phi \in (-\omega/2, \omega/2)$, a change of variables and integration by parts yields

$$\begin{aligned}\int_0^\phi v(\xi)\,\mathrm{d}\xi &= \int_0^{v(\phi)} y \cdot (v^{-1})'(y)\,\mathrm{d}y \\ &= v(\phi)\cdot\phi - \int_0^{v(\phi)} v^{-1}(y)\,\mathrm{d}y. \end{aligned} \tag{A.4.19}$$

Now, with

$$\int \arctan\left(\frac{y}{b}\right)\mathrm{d}y = y\arctan\left(\frac{y}{b}\right) - \frac{b}{2}\ln(b^2+y^2), \qquad b \in \mathbb{R},$$

from (A.4.11) we get

$$\begin{aligned}\int_0^{v(\phi)} v^{-1}(y)\,\mathrm{d}y &= -\frac{1-\alpha}{\sqrt{g}}\int_0^{v(\phi)} \arctan\left(\frac{y}{\sqrt{g}}\right)\mathrm{d}y - \int_0^{v(\phi)}\arctan\left(\frac{y}{\alpha}\right)\mathrm{d}y \\ &= -\frac{1-\alpha}{\sqrt{g}}\int_0^{v(\phi)}\left(y\arctan\left(\frac{y}{\sqrt{g}}\right) - \frac{\sqrt{g}}{2}\ln\left(g+y^2\right)\right)'\mathrm{d}y \\ &\quad - \int_0^{v(\phi)}\left(y\arctan\left(\frac{y}{\alpha}\right) - \frac{\alpha}{2}\ln\left(\alpha^2+y^2\right)\right)'\mathrm{d}y \\ &= -\frac{1-\alpha}{\sqrt{g}}\left(v(\phi)\arctan\left(\frac{v(\phi)}{\sqrt{g}}\right) - \frac{\sqrt{g}}{2}\ln\left(g+v(\phi)^2\right)\right) \\ &\quad -\left(v(\phi)\arctan\left(\frac{v(\phi)}{\alpha}\right) - \frac{\alpha}{2}\ln\left(\alpha^2+v(\phi)^2\right)\right) - \frac{1-\alpha}{2}\ln(g) - \alpha\ln(\alpha).\end{aligned}$$

Next, using this result and setting $C := (1-\alpha)/2\ln(g) + \alpha\ln(\alpha)$, from (A.4.19) we derive

$$\begin{aligned}\int_0^\phi v(\xi)\,\mathrm{d}\xi &= v(\phi)\cdot\phi + \frac{1-\alpha}{\sqrt{g}}\left(v(\phi)\arctan\left(\frac{v(\phi)}{\sqrt{g}}\right) - \frac{\sqrt{g}}{2}\ln\left(g+v(\phi)^2\right)\right) \\ &\quad + \left(v(\phi)\arctan\left(\frac{v(\phi)}{\alpha}\right) - \frac{\alpha}{2}\ln\left(\alpha^2+v(\phi)^2\right)\right) + C \\ &= v(\phi)A(\phi) + B(\phi) + C, \qquad \phi \in \left(-\frac{\omega}{2}, \frac{\omega}{2}\right),\end{aligned} \tag{A.4.20}$$

where we have set

$$A(\phi) = \phi + \frac{1-\alpha}{\sqrt{g}} \arctan\left(\frac{v(\phi)}{\sqrt{g}}\right) + \arctan\left(\frac{v(\phi)}{\alpha}\right),$$
$$B(\phi) = -\frac{1-\alpha}{2} \ln\left(g + v(\phi)^2\right) - \frac{\alpha}{2} \ln\left(\alpha^2 + v(\phi)^2\right).$$

Next we show that

$$\lim_{\phi\to\omega/2} v(\phi)A(\phi) = \lim_{\phi\to\omega/2} \frac{A(\phi)}{1/v(\phi)} = 0. \tag{A.4.21}$$

To do so, first note that $\lim_{\phi\to\omega/2} 1/v(\phi) = 0$, and from (A.4.12) we know that

$$\lim_{\phi\to\omega/2} A(\phi) = \omega/2 - \left(\frac{1-\alpha}{\sqrt{g}} + 1\right)\frac{\pi}{2} = 0.$$

Now, with (A.4.9) we derive

$$\left(\frac{1}{v}\right)'(\phi) = -\frac{v'(\phi)}{v(\phi)^2} = \frac{(v(\phi)^2+f)(v(\phi)^2+g)}{v(\phi)^2(v(\phi)^2+a)},$$

and thus

$$\lim_{\phi\to\omega/2} \left(\frac{1}{v}\right)'(\phi) = 1. \tag{A.4.22}$$

For the derivative of $A$ we again use (A.4.9) and $\alpha^2 = f$ to compute

$$\begin{aligned}
A'(\phi) &= 1 + \frac{1-\alpha}{v(\phi)^2+g} v'(\phi) + \frac{\alpha}{v(\phi)^2+\alpha^2} v'(\phi) \\
&= 1 - (1-\alpha)\frac{v(\phi)^2+f}{v(\phi)^2+a} - \alpha\frac{v(\phi)^2+g}{v(\phi)^2+a} \\
&= \frac{a-(1-\alpha)f-\alpha g}{v(\phi)^2+a} \\
&= 0 \qquad \text{for all} \qquad \phi \in \left(-\frac{\omega}{2}, \frac{\omega}{2}\right),
\end{aligned}$$

since $a - (1-\alpha)f - \alpha g = 0$. In particular it holds

$$\lim_{\phi\to\omega/2} A'(\phi) = 0, \tag{A.4.23}$$

and together with (A.4.22) an application of L'Hôpital's rule proves equation (A.4.21). Next, writing $B$ as

$$B(\phi) = -\frac{1}{2}\ln\left(v(\phi)^2+g\right) + \frac{\alpha}{2}\ln\left(\frac{v(\phi)^2+g}{v(\phi)^2+\alpha^2}\right), \tag{A.4.24}$$

we see that

$$\lim_{\phi\to\omega/2} B(\phi) = -\infty. \tag{A.4.25}$$

Hence, we have shown that

$$\lim_{\phi\to\omega/2}\int_0^\phi v(\xi)\,\mathrm{d}\xi = -\infty,$$

i.e., $\lim_{\phi\to\omega/2} t(\phi) = 0$.

*Substep 4.2.* Next, we compute $\lim_{\phi\to\omega/2} t'(\phi)$. First, from (A.4.20) and (A.4.24), we get

$$\begin{aligned} t(\phi) &= \exp\left(v(\phi)A(\phi) + B(\phi) + C\right) \\ &= \exp\left(v(\phi)A(\phi) + C\right)\cdot\left(\frac{g+v(\phi)^2}{\alpha^2+v(\phi)^2}\right)^{\alpha/2}\cdot\left(g+v(\phi)^2\right)^{-1/2}. \end{aligned}$$

Then, since $v(\phi) < 0$ and therefore $v(\phi) = -\sqrt{v(\phi)^2}$ for $\phi\in(0,\omega/2)$, we derive

$$\begin{aligned} t'(\phi) &= t(\phi)v(\phi) \\ &= -\exp\left(v(\phi)A(\phi) + C\right)\cdot\left(\frac{g+v(\phi)^2}{\alpha^2+v(\phi)^2}\right)^{\alpha/2}\cdot\left(\frac{v(\phi)^2}{g+v(\phi)^2}\right)^{1/2}. \end{aligned}$$

Now, since $v\cdot A\to 0$ for $\phi\to\omega/2$ due to (A.4.21), we see that

$$\lim_{\phi\to\omega/2} t'(\phi) = -\exp(C) = -g^{(1-\alpha)/2} - \alpha^\alpha < 0$$

is finite.

*Step 5.* Finally, for the second derivative of $t$ with the help of (A.4.9) we compute

$$\begin{aligned} t''(\phi) &= t(\phi)\left(v(\phi)^2 + v'(\phi)\right) \\ &= t(\phi)\left(v(\phi)^2 - \frac{(f+v(\phi)^2)(g+v(\phi)^2)}{a+v(\phi)^2}\right) \\ &= t(\phi)\frac{v(\phi)^2(a-f-g) - fg}{a+v(\phi)^2}, \end{aligned} \tag{A.4.26}$$

and therefore

$$\lim_{\phi\to\omega/2} t''(\phi) = (a-f-g)\lim_{\phi\to\omega/2} t(\phi) = 0.$$

Since $t''$ is even, it holds $\lim_{\phi\to-\omega/2} t''(\phi) = 0$, and we have shown that $t\in C^2([-\omega/2,\omega/2])$. □

Now we have everything at hand to prove Lemma 7.4.

**Proof of Lemma 7.4.**

*Proof.* We keep the notation from the proof of Lemma 7.2. In particular, we consider $t$ on the interval $(-\omega/2,\omega/2)$. First, we show $t\in C^\infty((-\omega/2,\omega/2))$ with

$$t^{(i)}(\phi) = t(\phi)\frac{P_i(v(\phi))}{Q_i(v(\phi))}, \tag{A.4.27}$$

where $P_i$ and $Q_i$ are polynomials. Here, $Q_i$ takes the form

$$Q_i(x) = \left(x^2 + a\right)^{k_i}, \tag{A.4.28}$$

where $k_i = 2^{i-1} - 1$ and $a = \alpha^2/(p-1) > 0$. For $i$ even, $P_i$ is an even polynomial with $deg(P_i) \leq deg(Q_i)$. For $i$ odd, $P_i$ is an odd polynomial with $deg(P_i) \leq deg(Q_i) + 1$.

*Step 1.* For the basis, we consider $i = 1$ and $i = 2$. Note that from Lemma 7.2 we know that $t \in C^2([-\omega/2, \omega/2])$. Since $t'(\phi) = t(\phi)v(\phi)$, we have $P_1(x) = x$ and $Q_1(x) = 1$. For $i = 2$, from (A.4.26) we know that

$$t''(\phi) = t(\phi)\frac{(a - f - g)v(\phi)^2 - fg}{v(\phi)^2 + a}.$$

Hence, $P_2(x) = (a - f - g)x^2 - fg$ and $Q_2(x) = x^2 + a$.

*Step 2.* Inductive step. Since $a > 0$ and $t$ and $v$ are differentiable, from (A.4.27) we know that $t^{(i+1)}$ exists. We compute

$$\begin{aligned} t^{(i+1)} &= t' \,\frac{P_i(v)}{Q_i(v)} + t\,\frac{\frac{\partial}{\partial \phi}\left[P_i(v)\right] Q_i(v) - P_i(v) \frac{\partial}{\partial \phi}\left[Q_i(v)\right]}{Q_i(v)^2} \\ &= t \cdot \left( v\,\frac{P_i(v)}{Q_i(v)} + v'\,\frac{P_i'(v)Q_i(v) - P_i(v)Q_i'(v)}{Q_i(v)^2} \right), \end{aligned}$$

where we used the fact that $t' = tv$. Using equation (A.4.9) we further get

$$t^{(i+1)} = t \cdot \left( v\,\frac{P_i(v)}{Q_i(v)} - \frac{(v^2 + f)(v^2 + g)}{v^2 + a} \cdot \frac{P_i'(v)Q_i(v) - P_i(v)Q_i'(v)}{Q_i(v)^2} \right).$$

Hence,

$$P_{i+1}(v) = v(v^2 + a)Q_i(v)P_i(v) - (v^2 + f)(v^2 + g)\left(P_i'(v)Q_i(v) - P_i(v)Q_i'(v)\right), \tag{A.4.29}$$

and with (A.4.28) it holds

$$Q_{i+1}(v) = (v^2 + a)Q_i(v)^2 = (v^2 + a)^{2k_i + 1},$$

i.e., $k_{i+1} = 2k_i + 1 = 2^i - 1$.

Let $n = 2k_i$ denote the degree of $Q_i$. We first assume that $i$ is even. Due to our assumptions, $P_i$ is an even polynomial with $deg(P_i) \leq n$. From (A.4.29) we see that $P_{i+1}$ is an odd polynomial with $deg(P_{i+1}) \leq 2n + 3 = deg(Q_{i+1}) + 1$. To complete the inductive step, we next assume that $i$ is odd, i.e., $P_i$ is an odd polynomial with $deg(P_i) \leq n + 1$. First note from (A.4.29), that $P_{i+1}$ is an even polynomial in this case. We may write $P_i$ as

$$P_i(x) = a_i x^{n+1} + \tilde{P}_i(x), \tag{A.4.30}$$

where $a_i \in \mathbb{R}$ and $\tilde{P}_i$ is an odd polynomial with $deg(\tilde{P}_i) \leq n - 1$. Analogously, we write $Q_i$ as

$$Q_i(x) = x^n + \tilde{Q}_i(x), \tag{A.4.31}$$

where $\tilde{Q}_i$ is an even polynomial of degree $n-2$. Now with (A.4.29) we get

$$\begin{aligned} P_{i+1}(v) &= a_i v^{2n+4} - v^4\left((n+1)a_i v^{2n} - n a_i v^{2n}\right) + R(v) \\ &= R(v), \end{aligned}$$

where $R$ is an even polynomial with $deg(R) \le 2n+2 = deg(Q_{i+1})$. This proves the inductive step.

*Step 3.* We show $t \in C^i([-\omega/2, \omega/2])$. Since $t^{(i)}$ is continuous in $(-\omega/2, \omega/2)$, it suffices to consider the boundary points. Furthermore, from the proof of Lemma 7.2 we know that $t$ is an even function. Hence, for $i$ even, we know that $t^{(i)}$ is an even function, and for $i$ odd it holds that $t^{(i)}$ is odd. Thus, we can confine ourselves to check if $\lim_{\phi\to\omega/2} t^{(i)}(\phi)$ is bounded. To do so, recall that $v(\phi) < 0$ for $\phi > 0$, so that in this case we can write $t^{(i)}(\phi)$ as

$$t^{(i)}(\phi) = t'(\phi)\frac{P_i(v(\phi))}{v(\phi)Q_i(v(\phi))},$$

where we used that $t' = tv$. Note that $\lim_{\phi\to\omega/2} t'(\phi)$ exists, since $t \in C^2([-\omega/2, \omega/2])$ due to Lemma 7.2.

Let us first assume that $i$ is odd. Then, using (A.4.30) and (A.4.31) yields

$$t^{(i)}(\phi) = t'(\phi)\frac{a_i v(\phi)^{n+1} + \tilde{P}_i(v(\phi))}{v(\phi)^{n+1} + v(\phi)\tilde{Q}_i(v(\phi))},$$

where $deg(\tilde{P}_i) \le n-1$ and $deg(\tilde{Q}_i) = n-2$. Since $\lim_{\phi\to\omega/2} v(\phi) = -\infty$, we conclude

$$\lim_{\phi\to\omega/2} t^{(i)}(\phi) = a_i \lim_{\phi\to\omega/2} t'(\phi) = a_i\, t'(\omega/2).$$

Finally, for $i$ even, a proof completely analog to the one above shows that

$$\lim_{\phi\to\omega/2} t^{(i)}(\phi) = 0.$$

$\square$

# Zusammenfassung

Die vorliegende Arbeit befasst sich mit einer speziellen Klasse von quasilinearen elliptischen Differentialgleichungen: den *p-Poisson-Gleichungen*

$$-\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = f \qquad \text{in} \quad \Omega, \tag{PP}$$

wobei $1 < p < \infty$, $f \in W^{-1}(L_{p'}(\Omega))$ und $\Omega \subset \mathbb{R}^d$ für $d \geq 2$ ein beschränktes Lipschitz-Gebiet bezeichnet. Zentraler Aspekt der Arbeit ist die Analyse der Regularität von Lösungen $u$ der $p$-Poisson-Gleichungen in der sogenannten *Adaptivitätsskala*

$$B^\sigma_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p}, \quad \sigma > 0, \tag{$*$}$$

von Besov-Räumen. Einen weiteren Gegenstand bildet die Implementierung sowie das numerische Testen eines relaxierten Iterationsverfahrens vom Kačanov-Typ zur approximativen Lösung der $p$-Poisson-Gleichung (PP) mit homogenen Dirichlet-Randbedingungen.

### Hintergrund und Motivation

Bei der mathematischen Modellierung naturwissenschaftlicher Phänomene spielen *partielle Differentialgleichungen* (PDEs) häufig eine zentrale Rolle. Eigenschaften sowie Verfahren zur numerischen Lösung von linearen Gleichungen sind seit geraumer Zeit Gegenstand von Forschung, mit dem Resultat vielfältiger Ergebnisse; für einen Überblick siehe beispielsweise [23, 74]. Viele reale Zusammenhänge weisen allerdings nichtlineare Charakteristika auf, und die Beschreibung der zugrundeliegenden Situation führt oft auf natürliche Weise auf *nichtlineare* PDEs. Die Klasse der *quasilinearen* Gleichungen vom Typ $-\mathrm{div}(\alpha(\cdot, |\nabla u|)\nabla u) = F(u)$ kommt beispielsweise bei verschiedenen Problemen der Kontinuumsmechanik vor, insbesondere bei der mathematischen Modellierung von nichtnewtonschen Fluiden [95], aber auch beispielsweise in der Theorie nichtnewtonscher Filtration [88, 109], turbulenter Strömung eines Gases in einem porösen Medium [60] oder der Plastizitätstheorie [8]. Hierbei haben die $p$-Poisson-Gleichungen (PP) einen ähnlichen Modellcharakter für allgemeinere quasilineare elliptische Probleme wie die gewöhnliche Poisson-Gleichung für lineare elliptische Probleme. Für eine allgemeine Einführung sei auf [99] verwiesen.

Bedingungen für die Existenz und Eindeutigkeit von Lösungen von (PP) sind zwar wohlbekannt [100], allerdings ist eine explizite Darstellung der Lösung, d.h. ihre konkrete Gestalt, im Allgemeinen nicht gegeben. Aus diesem Grund werden numerische Verfahren zur konstruktiven Approximation der Lösung benötigt. Solche

Verfahren beruhen gewöhnlich auf einer Diskretisierung des Problems, beispielsweise basierend auf einem endlichen Gitter oder einer Triangulierung des Gebietes. Die numerische Berechnung hinreichend genauer Approximationen an die exakte Lösung einer (nichtlinearen) PDE führt in der Praxis typischerweise zu linearen Gleichungssystemen mit sehr vielen Unbekannten. Aus diesem Grund ist die Effizienz solcher Verfahren von entscheidender Bedeutung. Grundsätzlich wird zwischen uniformen und adaptiven numerischen Verfahren unterschieden.

Ein adaptives Verfahren ist im Wesentlichen eine Update-Strategie, welche iterativ eine Folge von Approximationen generiert, wobei zusätzliche Freiheitsgrade nur dort hinzugefügt werden, wo die Näherungslösung noch 'weit entfernt' von der exakten Lösung ist. Eine typische Eigenschaft von Lösungen von PDEs ist das Auftreten von Singularitäten, welche durch den Gebietsrand, den Quellterm oder den Operator induziert sein können ([46, 70, 71, 89]). Um in solchen Situationen eine hinreichend genaue Approximation bei gleichzeitiger Beschränkung der Anzahl an Freiheitsgraden zu erzielen, ist die Verwendung von hochgradig nicht-uniformen räumlichen Diskretisierungen oft unumgänglich. In diesem Zusammenhang zielen adaptive numerische Verfahren auf eine effiziente Auflösung der Singularitäten der (unbekannten) Lösung.

Obwohl das Konzept adaptiver Verfahren durchaus vielversprechend ist, sind diese Methoden meist schwieriger zu analysieren und implementieren, verglichen mit konventionelleren uniformen Methoden. Aus diesem Grund sind theoretische Grundlagen, welche die Entwicklung, Analyse und Implementierung von adaptiven Verfahren rechtfertigen, höchst wünschenswert. Hierzu ist zu untersuchen, ob adaptiven Verfahren prinzipiell eine höhere Konvergenzrate als uniforme Methoden erzielen können. Die Regularitätsanalyse in dieser Arbeit ist durch eben jenes Problem motiviert, insbesondere im Zusammenhang mit Waveletverfahren. Bei dieser Art Verfahren ist ein klares Verständis der maximal erzielbaren Konvergenzraten von adaptiven und uniformen Verfahren vorhanden.

Wavelets werden typischerweise derart konstruiert, dass sie eine spezielle Multiskalenbasis des zugrundeliegenden Funktionenraums bilden. Hierbei wird jedes Basiselement durch dyadische Dilatation und ganzzahlige Translation eines oder mehrerer Mother-Wavelets erzeugt ([45, 108, 135]). Ein großer Vorteil von Wavelets liegt in ihren starken analytischen Eigenschaften. So können viele Funktionenräume wie beispielsweise Sobolev- und Besov-Räume mit Hilfe von Wavlelets charakterisiert werden, im Sinne einer Äquivalenzen der jeweiligen Norm zu gewichteten Folgennormen der Wavelet-Entwicklungskoeffizienten. Wavelets haben sich - neben ihrer Verwendung in der Signal/Bild-Analyse und Verarbeitung [103] - ebenfalls als geeignetes Mittel zur adaptiven Lösung verschiedener Operatorgleichungen erwiesen ([6, 18, 19, 20, 27, 28, 34, 38, 66, 67, 84, 96, 111, 119, 134]).

Im Wavelet-Setting existiert ein natürliches Benchmark-Verfahren für Adaptivität, genannt 'best $n$-term'-Approximation. Bei diesem Verfahren stammen die Approximierenden nicht aus linearen Räumen, sondern aus nichtlinearen Mannigfaltigkeiten $S_n$, bestehend aus Funktionen der Form

$$S = \sum_{\lambda \in \overline{\Lambda}} c_\lambda \psi_\lambda,$$

wobei $\{\psi_\lambda \,|\, \lambda \in \Lambda\}$ eine gegebene Waveletbasis bezeichnet und $\overline{\Lambda} \subset \Lambda$ mit $\#\overline{\Lambda} = n$.

Eine 'best $n$-term'-Waveletapproximation kann gebildet werden, indem die $n$ größten Koeffizienten der Waveletentwicklung von der (unbekannten) Funktion ausgewählt werden, welche approximiert werden soll. Auf der einen Seite kann solch ein Verfahren natürlich niemals numerisch realisiert werden, da dies die Berechnung *aller* Waveletkoeffizienten und die Auswahl der $n$ größten erfordern würde. Auf der anderen Seite ist das Optimum was wir von einem adaptiven Waveletalgorithmus erwarten können, dass er (asymptotisch) die Konvergenzrate der 'best $n$-term'-Approximation realisiert. In diesem Sinne ist die Verwendung von adaptiven Waveletmethoden theoretisch gerechtfertigt, falls die 'best $n$-term'-Waveletapproximation eine signifikant höhere Konvergenzrate realisiert als konventionellere, uniforme Approximationsverfahren. Diese maximalen Konvergenzraten werden nun wiederum von der Regularität der exakten Lösung in bestimmten Skalen von Funktionenräumen bestimmt.

Im Wavelet-Setting ist bekannt, dass die Konvergenzrate von uniformen Verfahren bezüglich $L_p$ von der Glattheit der Zielfunktion, welche approximiert werden soll, in der Skala $W^s(L_p(\Omega))$ von $L_p$-Sobolev-Räumen abhängt. Im Unterschied dazu hängt die Konvergenzrate der 'best $n$-term'-Waveletapproximation in $L_p$ von der Regularität in der *Adaptivitätsskala* $(*)$ von Besov-Räumen ab ([17, 26, 47, 76]). Ein ähnlicher Zusammenhang wurde kürzlich im Zusammenhang von Finite-Elemente-Approximationen nachgewiesen [68], siehe auch [10].

Demzufolge ist die Verwendung von adaptiven (Wavelet-)Algorithmen für (PP) gerechtfertigt, falls die Besov-Regularität $\sigma$ der exakten Lösung in der Adaptivitätsskala von Besov-Räumen höher ist als ihre Sobolev-Regularität $s$.

### Ziele der Arbeit

Das erste grundlegende Ziel dieser Arbeit ist durch das oben beschriebene Problem motiviert, das heißt die Frage, ob adaptive Verfahren für die $p$-Poisson-Gleichung das Potential besitzen, uniforme Verfahren hinsichtlich ihrer Effizienz zu übertreffen. Um eine fundierte Antwort geben zu können, muss untersucht werden, ob die Besov-Glattheit $\sigma$ der Lösungen in der Adaptivitätsskala von Besov-Räumen höher ist als ihre Sobolev-Regularität $s$.

In der vorliegenden Arbeit soll diesbezüglich ein erstes positives Resultat für quasilineare elliptische Gleichungen, d.h. für die $p$-Poisson-Gleichung (PP), gezeigt werden. Ergebnisse von Savaré [116] zeigen, dass auf allgemeinen Lipschitz-Gebieten die Sobolev-Regularität der Lösungen von (PP) durch $s^* = 1 + 1/p$ für $2 \leq p < \infty$, sowie durch $s^* = 3/2$ für $1 < p < 2$ gegeben ist. Das erste zentrale Ziel dieser Arbeit kann somit wie folgt formuliert werden.

**(O1)** Es sollen Regularitäts-Abschätzungen für Lösungen der $p$-Poisson-Gleichung (PP) in der Adaptivitätsskala $(*)$ von Besov-Räumen hergeleitet werden. Hierbei soll geklärt werden, ob $u \in B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/d + 1/p$, für ein $\sigma > s^*$. Um eine substanzielle Klasse von Problemen abzudecken, soll sowohl $1 < p < \infty$, also auch die allgemeine Klasse der beschränkten Lipschitz-Gebiete betrachtet werden. Desweiteren sollen zusätzlich explizite Aussagen für die praktisch relevante Klasse der polygonalen Gebiete bewiesen werden.

Das zweite zentrale Thema dieser Arbeit ist die numerische Lösung der $p$-Poisson-Gleichung (PP). In [53] haben Diening et al. ein iteratives Linearisierungsverfahren für die $p$-Poisson-Gleichung mit homogenen Dirichlet-Randbedingungen vorgeschlagen. Insbesondere wird der Fall $1 < p \leq 2$ behandelt. Das wesentliche Merkmal dieses Algorithmus, welcher als relaxierte Kačanov-Iteration interpretiert werden kann, ist, dass in jeder Iteration nur noch ein *lineares* elliptisches Problem gelöst werden muss. Diese linearen (Unter-)Probleme können numerisch auf stabile und bewährte Art gelöst werden, bspw. mittels eines Finite-Elemente- oder Wavelet-Verfahrens. In der vorliegenden Arbeit soll dieses Verfahren implementiert und getestet werden, in Verbindung mit einem geeigneten adaptiven Wavelet-Frame-Verfahren zur Lösung der linearen Teilprobleme.

Das zweite wesentliche Ziel dieser Arbeit kann also wie folgt fomuliert werden.

**(O2)** Es soll ein neuer adaptiver Löser für die $p$-Poisson-Gleichung (PP), für alle $1 < p < 2$, entwickelt und implementiert werden. Dieses Verfahren soll auf der relaxierten Kačanov-Iteration basieren, wobei für die numerische Lösung der linearen elliptischen Teilprobleme ein adaptives Wavelet Verfahren verwendet werden soll. Die praktischen Eigenschaften des neuen Algorithmus sollen schließlich in einer Reihe numerischer Tests analysiert werden.

## (O1) Regularität in der Skala $(*)$ von Besov-Räumen

Um Besov-Regularitäts-Abschätzungen für Lösungen der $p$-Poisson-Gleichung herzuleiten, werden in dieser Arbeit zwei Ansätze verfolgt.

Der erste Ansatz macht von der Tatsache Gebrauch, dass die Lösungen von (PP) unter gewissen Voraussetzungen eine höhere Regularität im Innern des Gebiets besitzen, in dem Sinne dass sie lokal Hölder-stetig sind; siehe beispielsweise [52, 61, 127, 132, 133]. Dabei können im Allgemeinen bei Annäherung an den Gebietsrand die lokalen Hölder-Seminormen explodieren, jedoch kann dieses singuläre Verhalten durch eine gewisse Potenz des Abstandes zum Gebietsrand kontrolliert werden ([54, 93, 97, 98]). Derartige Eigenschaften gelten sehr oft im Zusammenhang mit elliptischen Randwertproblemen auf nichtglatten Gebieten, siehe beispielsweise [105].

Es stellt sich heraus, dass die Kombination von globaler Sobolev-Glattheit und lokaler Hölder-Regularität dazu verwendet werden kann, um Besov-Glattheit in der Skala $(*)$ für die Lösungen von (PP) nachzuweisen. Wie gezeigt wird, ist in vielen Fällen die Besov-Glattheit $\sigma$ deutlich höher als die Sobolev-Glattheit $s^* = 1 + 1/p$ beziehungsweise $s^* = 3/2$, so dass die Entwicklung von adaptiven Verfahren für das $p$-Poisson-Problem gerechtfertigt ist.

Auf der einen Seite ist dieser universelle Ansatz für die allgemeine Klasse von Lipschitz-Gebieten anwendbar. Auf der anderen Seite ist für Lösungen von partiellen PDEs auf polygonalen Gebieten bekannt, dass die kritischen Singularitäten gewöhnlich nur in den Ecken des Gebietes vorkommen. In der Tat existieren für (nicht-negative) Lösungen der $p$-Poisson-Gleichung auf endlichen Kegeln Ergebnisse zur singulären Entwicklung bezüglich der Ecke [57, 126]. Im Wesentlichen kann die Lösung (bzw. ihre Ableitungen) durch eine Potenz des Abstandes zur Ecke abgeschätzt werden. Infolgedessen könnte man - unter Verwendung dieser stärkeren (lokalen) Ergebnisse - bessere Besov-Regularitäts-Abschätzungen auf polygonalen Gebieten erwarten.

Das Ziel des zweiten Ansatzes ist es, einen ersten Schritt zur Verbesserung einiger der Besov-Regularitäts-Ergebnisse zu machen, welche mittels des ersten Ansatzes für polygonale Gebiete hergeleitet wurden. Dabei ist der natürliche erste Schritt, wie oben dargestellt, die Untersuchung der Regularität von Lösungen in einer Umgebung der Gebietsecken $x_0 \in \partial\Omega$. Dazu wird die Glattheit von Lösungen $u$ von (PP) in einem kleinen Kegel $\mathfrak{C} \subset \Omega$ mit Spitze $x_0$, in der Adaptivitätsskala $B^\sigma_\tau(L_\tau(\mathfrak{C}))$, $1/\tau = \sigma/2 + 1/p$, von Besov-Räumen, untersucht. Wie gezeigt wird, führt dieser Ansatz tatsächlich zu Regularitätsaussagen, welche - in einem lokalen Sinn, d.h. bei Betrachtung kleiner Umgebungen der Ecken - in einigen Fällen stärker sind als jene mittels des ersten Ansatzes hergeleiteten.

Die Resultate des ersten Ansatzes werden in zwei Schritten dargelegt. Zunächst wird ein allgemeines Einbettungstheorem bewiesen, welches besagt dass der Schnitt eines klassischen Sobolev-Raums mit einem Hölder-Raum mit den oben beschriebenen Eigenschaften in gewisse Besov-Räume in der Adaptivitätsskala $1/\tau = \sigma/d + 1/p$ eingebettet werden kann, d.h., dass unter geeigneten Bedingungen an die Parameter

$$C^{\ell,\alpha}_{\gamma,\mathrm{loc}}(\Omega) \cap W^s(L_p(\Omega)) \hookrightarrow B^\sigma_\tau(L_\tau(\Omega)), \quad \frac{1}{\tau} = \frac{\sigma}{d} + \frac{1}{p} \tag{E1}$$

gilt. Dabei stellt sich heraus, dass für einen großen Bereich von Parametern die Besov-Regularität $\sigma$ signifikant höher ist als die Sobolev-Glattheit $s$. Der Beweis dieses Einbettungs-Theorems beruht auf Fortsetzungsargumenten in Verbindung mit der Charakterisierung von Besov-Räumen mittels Wavelet-Entwicklungskoeffizienten. Im Anschluss wird verifiziert, dass in vielen Fällen die Lösungen von (PP) in der Tat die Voraussetzungen des Einbettungs-Theorems erfüllen, so dass seine Anwendung das gewünschte Regularitätsresultat liefert.

Die Beweise der mittels des zweiten Ansatzes hergeleiteten Ergebnisse basieren auf bekannten Resultaten über die singuläre Entwicklung der Lösung $u$ in einer Umgebung eines konischen Randpunktes, sowie auf Einbettungen des Typs $\mathcal{K}^\ell_{p,a}(\Omega) \cap B^s_p(L_p(\Omega)) \hookrightarrow B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$, wobei $\mathcal{K}^\ell_{p,a}(\Omega)$ spezielle gewichtete Sobolev-Räume bezeichnen, Babuska-Kondratiev-Räume genannt (siehe Section 1.3). Wie in Chapter 7 gezeigt wird, besitzen in einigen Fällen die Lösungen von (PP) beliebig hohe gewichtete Sobolev-Regularität $\ell$ in einer Umgebung der Ecken.

Aufgrund dieser Tatsache wird zusätzlich der Grenzfall $\ell \to \infty$ der obigen Einbettung analysiert. In diesem Zusammenhang werden die topologischen Vektorräume $H^{\infty,s}_a(L_p(\Omega)) := \cap_{\ell=1}^\infty \mathcal{K}^\ell_{p,a}(\Omega) \cap B^s_p(L_p(\Omega))$ und $B^\infty_{\mathrm{NL}}(L_p(\Omega)) := \cap_{\sigma>0} B^\sigma_\tau(L_\tau(\Omega))$, $1/\tau = \sigma/2 + 1/p$ betrachtet, ihre topologischen Eigenschaften untersucht (bzgl. lokaler Konvexität, Metrisierbarkeit und Vollständigkeit), und schließlich gezeigt, dass die Einbettungen vom Typ

$$H^{\infty,s}_a(L_p(\Omega)) \hookrightarrow B^\infty_{\mathrm{NL}}(L_p(\Omega)), \tag{E2}$$

stetig sind (im Sinne von stetigen Abbildungen zwischen topologischen Vektorräumen).

Es ist erwähnenswert, dass mit den obigen Einbettungen insbesondere universelle funktionalanalytische Tools bereitgestellt werden, welche es erlauben, das Problem der Herleitung von Besov-Regularitäts-Aussagen in der Skala $(*)$ zurückzuführen auf die Analyse geeignet gewichteter Hölder- beziehungsweise Sobolev-Glattheit (vergleiche auch Remark 5.10). Folglich könnten sich diese Einbettungen - neben der Verwendung in dieser Arbeit im Zusammenhang mit der $p$-Poisson-Gleichung - auch für die Regularitätsanalyse von verschiedenen weiteren Problemen als nützlich erweisen.

### (O2) Numerische Lösung der $p$-Poisson Gleichung

Das klassische Kačanov-Verfahren ist eine iterative Methode zur näherungsweisen Lösung bestimmter nichtlinearer Probleme mittels Linearisierung [86]. Für quasilineare elliptische Gleichungen vom Typ

$$-\mathrm{div}(\alpha\,(|\nabla u|)\,\nabla u) = f \qquad \text{in } \Omega,$$

hat die Kačanov-Iteration die folgende Gestalt. Für eine gegebene Funktion $u_0$, ist die neue Iterierte $u_{n+1}$ rekursiv definiert als die Lösung von

$$-\mathrm{div}(\alpha\,(|\nabla u_n|)\,\nabla u_{n+1}) = f \qquad \text{in } \Omega, \quad n \geq 0.$$

Entscheidend ist hierbei, dass in jeder Iteration nur noch ein *lineares* Problem gelöst werden muss. In [136] wurde unter gewissen Annahmen an $\alpha$ bewiesen, dass die Kačanov-Iteration gegen einen Fixpunkt $u$ konvergiert, welcher das ursprüngliche quasilineare Problem löst. Ein a-posteriori-Fehlerschätzer wurde in [75] hergeleitet. Zum einen erfüllt die $p$-Poisson-Gleichung, d.h. $\alpha(\xi) = \xi^{p-2}$, diese Bedingungen *nicht*, zum anderen sind die linearen Gleichungen, welche im Verlauf der Kačanov-Iteration gelöst werden müssen, numerisch nicht stabil lösbar, falls $|\nabla u_n|$ an gewissen Punkten verschwindet, da in diesem Fall das Gewicht $|\nabla u_n|^{p-2}$ degeneriert. Ein Ansatz zur Überwindung dieses Problems ist die Gewichtsfunktion $\alpha$ abzuschneiden. Unter Verwendung der Schreibweise $\varepsilon_- \vee x \wedge \varepsilon_+ := \max\{\varepsilon_-, \min\{x, \varepsilon_+\}\}$ für $0 < \varepsilon_- \leq \varepsilon_+ < \infty$ und $x \in \mathbb{R}$, hat die *relaxierte Kačanov-Iteration* die folgende Gestalt

$$-\mathrm{div}\Big((\varepsilon_- \vee |\nabla u_n| \wedge \varepsilon_+)^{p-2}\,\nabla u_{n+1}\Big) = f \qquad \text{in } \Omega, \quad n \geq 0. \tag{RKI}$$

Unter geeigneter Vergrößerung des Abschneide-Intervalls $[\varepsilon_-, \varepsilon_+]$ im Verlauf der Iteration konnte in [53] die Konvergenz für dieses Verfahren gezeigt werden, unter der Annahme, dass die linearen elliptischen PDEs in jeder Iteration *exakt* gelöst werden. Um einen vollständig implementierbaren Algorithmus zu erhalten, müssen diese Teilprobleme allerdings approximativ, beispielsweise mit einem Finite-Elemente- oder Waveletverfahren, gelöst werden. In dieser Arbeit wird ein adaptiver Ansatz basierend auf einer Wavelet-Diskretisierungen verfolgt.

Für die numerische Lösung der linearen elliptischen Teilprobleme wird insbesondere sowohl das adaptive Wavelet-Galerkin-Verfahren aus [18] verwendet, als auch das adaptive multiplikative Schwarz-Frame-Verfahren aus [124] (siehe Section 8.4, und Section 1.5 für eine Einführung in Wavelet Frames). Für beide Verfahren wurde in den angegebenen Quellen gezeigt, dass sie von asymptotisch optimaler Komplexität sind, in dem Sinne dass sie tatsächlich die gleiche Konvergenzrate erzielen wie die ‘best $n$-term’-Approximation, während die Anzahl der Floating-Point-Operationen und Speicherplätze, welche zur Berechnung einer Approximation benötigt werden, proportional zur jeweiligen Anzahl an Freiheitsgraden bleibt.

Der resultierende Algorithmus wird in einer Reihe von numerischen Tests untersucht. Hierbei werden verschiedene nicht-triviale zweidimensionale $p$-Poisson-Probleme mit homogenen Dirichlet-Randbedingungen für $1 < p < 2$ betrachtet, auf einem konvexen als auch auf einem nicht-konvexen polygonalen Gebiet mit einer einspringenden Ecke. Hierbei zeigt sich, dass der implementierte Algorithmus vom Kačanov-Typ in der Praxis ein stabiles Konvergenzverhalten aufweist. Die in [53] theoretisch bewiesene Konvergenz des *exakten* Verfahrens (RKI) wird somit auch in der Praxis bei approximativer Lösung der Teilprobleme nachgewiesen.

Diese Arbeit enthält eine einheitliche Darstellung der Regularitäts-Ergebnisse aus [30, 78], ergänzt durch einige weitere Resultate, alternative Beweise, sowie die Ergebnisse in Bezug auf (O2), d.h. die Implementierung und numerischen Tests des relaxierten Kačanov-Iterationsverfahrens.

# List of Figures

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# Erklärung

Ich versichere, dass ich die vorliegende Dissertation zum Thema

*The p-Poisson Equation:*
*Regularity Analysis and Adaptive Wavelet Frame Approximation*

selbstständig, ohne unerlaubte Hilfe angefertigt und mich dabei keiner anderen als der von mir ausdrücklich bezeichneten Quellen und Hilfen bedient habe.

Die Dissertation wurde in der jetzigen oder einer ähnlichen Form noch bei keiner anderen Hochschule eingereicht und hat noch keinen sonstigen Prüfungszwecken gedient.

Marburg, 19. Dezember 2017

(Christoph Hartmann)