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DETERMINATION OF FUEL FAILURE PROBABILITIES BY STATISTICAL EXPERIMENTS IN A TEST REACTOR WITH REGARD TO THE UNLOADING PROCESS EMPLOYED

by

G. BLAESSER, W. MATTHES, V. RAIEVSKI

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JOINT NUCLEAR RESEARCH CENTER
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Reactor Physics Department

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DETERMINATION OF FUEL FAILURE PROBABILITIES BY STATISTICAL EXPERIMENTS IN A TEST REACTOR WITH REGARD TO THE UNLOADING PROCESS EMPLOYED

by G. Blaesser (*), W. Matthes (*) and V. Raievski (**)

SUMMARY.

For many types of reactors the failure of fuel elements is still the quantity limiting the maximum obtainable burn-up. Since the number of failed fuel elements that are allowed to be present in the reactor and the capacity of the fuel handling machine are limited in all practical cases, one has to study the stochastic process of failure and renewal of the fuel elements. This is done in the present report. Also are discussed methods of statistical evaluation of experiments for the determination of the fuel failure probability.

I. — GENERAL CONSIDERATIONS.

Up to the present, the failure of fuel elements — and not the reactivity limit — is still the quantity limiting the maximum obtainable burn-up for many types of reactors. The number of failed fuel elements allowed to be present simultaneously in the reactor is usually rather limited; if this number gets too high, this may result in operational difficulties due to contamination by fission products. In such a case the reactor operation has to be interrupted to replace all the failed fuel elements by new ones.

This leads to the following problem where continuous loading and unloading of the reactor is envisaged : to avoid a shutdown of the reactor caused by the accumulation of too many failed fuel rods, it may be stipulated that the fuel handling machine must replace a fuel rod as soon as it fails. But we have to admit that a machine with a limited capacity which replaces only failed rods gives us no guarantee that no forced reactor shutdown will occur. This is because the fuel elements in the reactor get older and the failure probability increases with the age of a fuel element. As a consequence, the

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failure rate would increase in such a way that the machine could no longer cope with the changing of these failed fuel elements. The machine has therefore to replace not only failed fuel rods but also those fuel elements whose age exceeds a certain limit T.

In view of this, we can state that each fuel element spends on an average the time T in the reactor before it is taken out either as failed or an overaged fuel element. Now we have two conflicting facts :

On the one hand, we want the time T to be as high as possible, in order to get a good burn-out of the fuel elements. On the other hand, a very high T leads to a high failure rate which cannot be handled by the discharging machine.

The problem therefore consists in finding such a value of T as will fulfil both conditions satisfactorily, i.e., T must be large enough to result in a high burn-up but not so large as to give a failure rate which cannot be handled by the machine.

II. — THEORETICAL EXPRESSION OF THE FAILURE RATE AS A FUNCTION OF THE FAILURE PROBABILITY.

Let p(t) denote the failure probability to time t, that is, the probability that a fuel element fails before it reaches the age t. The probability that a fuel element reaches the age t without failure is then simply $1-p(t)$.

Let us now assume that a reactor containing fuel elements of the same kind, i.e. all characterized by the same failure probability distribution p(t), is operated in such a way that only failed fuel elements are replaced by new ones, but each element is replaced as soon as it fails. After a certain transition period, a stationary age distribution within the reactor will be attained. Let us call H(u) the age distribution function, i.e. the probability that a fuel element chosen at random within the reactor has an age $< u$. The probability that the randomly chosen fuel element has an age which falls within the interval $(u, u + du)$ is thus

$$dH(u) = \frac{dH(u)}{du} du$$

This latter probability must be proportional to the probability that a fuel element reaches the age u without failure. Thus :

[1] $$dH(u) = r_{\infty} \Big(1-p(u) \Big) du$$

where r_{∞} is a constant factor of proportionality. Integration of (1) leads to :

[2] $$H(u) = r_{\infty} \int_0^u \Big(1-p(t) \Big) dt$$

r_{∞} can be determined by the normalisation condition :

[3] $$\lim_{u \to \infty} H(u) = 1$$

which yields :

[4] $$r_{\infty}^{-1} = \int_0^{\infty} \Big(1-p(t) \Big) dt$$

So far we have dealt with the unloading of failed fuel elements only. If in addition fuel elements that exceed a certain age limit T are replaced by fresh fuel elements, this case can be described within the framework of the above formulae by introducing a fictitious "overaging failure", i.e. by also considering the passing of the age limit T as a certain type of "failure". Thus the probability $p_T(t)$ of both types of "failures" before time t is expressed as :

[5] $$p_T(t) = \begin{cases} p(t) & t < T \\ 1 & t \geqslant T \end{cases}$$

We then obtain, instead of eq. [2].

[6] $$H_T(u) = r_T \int_0^u \Big(1 - p_T(t)\Big) dt$$

with

[7] $$r_T^{-1} = \int_0^T \Big(1 - p(t)\Big) dt$$

since $H_T(T) = 1$

It is immediately seen that equations [2] and [4] are contained in equations [6] and [7] as the limiting case $T \to \infty$.

Equation [7] can be given another form by partial integration, from which is obtained :

[8] $$r_T^{-1} = T\Big(1 - p(T)\Big) + \int_0^T t\, dp(t) = T\Big(1 - p(T)\Big) + \bar{t}_T$$

since $\int_0^T t dp(t)$ is the mean value of t in the interval (O, T) which we denote by $\bar{t}_T$. Obviously $\lim_{T\to\infty} \bar{t}_T = \bar{t}$ = mean life time of the fuel. Since p(t) is a probability distribution function, $\lim_{t\to\infty} p(t) = 1$. But in order to have a finite r_∞^{-1} we also require the supplementary condition :

$$\lim_{T\to\infty} T\Big(1 - p(T)\Big) = a < \infty$$

In all practical cases, it is even found that a = 0. This we shall assume from now on, unless otherwise stated. Consequently, r_∞ simply becomes :

[9] $$r_\infty = \frac{1}{\bar{t}}$$

For very small values of T, in such a way that $p(T) \ll 1$, the integration in equation [7] simply yields :

[10] $$r_T = \frac{1}{T} \text{ for T so small that } p(T) \ll 1$$

Thus r_T is a monotonously decreasing function of T, falling off at first like $\frac{1}{T}$ and approaching for $T \geqslant \bar{t}$ the asymptotic value $\frac{1}{\bar{t}}$.

In order to calculate the mean rate of (real) failures in the reactor we must know the probability $\mu(t)dt$ that a given fuel element of age t fails within the time interval dt at t. This probability is not simply p'(t)dt, since p'(t)dt is the probability for a fresh fuel element to die afterwards in (t,t+dt). But since we consider a fuel element already of age t, we have thus to divide p'(t)dt by the probability of reaching the point t, that is, by 1−p(t). Thus the failure rate for a given fuel element at an age t is

[11] $$\mu(t) = \frac{p'(t)}{1 - p(t)}$$

The fraction of fuel elements within the reactor that have an age between t and

$t + dt$ is $dH_T(t) = \frac{dH_T(t)}{dt} dt$. If we thus integrate the product of these two factors over all ages from 0 to T, we obtain the mean rate of failures per fuel element

[12] $$c_T = r_T \int_0^T p'(t)\,dt = r_T p(T)$$

In the case of $T \to \infty$, we obtain

$$\lim_{T \to \infty} c_T = r_\infty = \frac{1}{\bar{t}}$$

Thus, if unloading is confined mainly to the failed fuel elements, the mean failure rate per fuel element will be given by r_∞ and is therefore simply the inverse mean lifetime. On the other hand. if it is mostly unfailed fuel, i.e., for $p(T) \ll 1$, that is unloaded we obtain

$$c_T = \frac{p(T)}{T} \quad \text{for } p(T) \ll 1$$

The mean failure rate in the reactor, i.e. the mean number of fuel elements that fail per unit time, is

[13] $$\lambda = \lambda(T) = M c_T$$

where M is the total number of fuel elements in the reactor.

III. — RELATION BETWEEN MEAN FAILURE RATE, CAPACITY OF THE UNLOADING DEVICE AND MAXIMUM ADMISSIBLE NUMBER OF FAILED FUEL ELEMENTS IN THE REACTOR.

The capacity of the fuel handling mechanism, briefly denoted as unloading device, is determined by the maximum number h of "services" per unit time that the device can execute. By a "service" we mean the complete exchange of a failed fuel element for a fresh one. If s, the "service time", is the duration of a service, then $h = 1/s$. We shall take s as a constant for the reactor with given fuel handling machine, although one can also easily generalize the theory to include statistical fluctuations in the duration of a service time.

If there are fuel failures during the time that the machine is engaged in exchanging already failed fuel elements, the fuel elements waiting to be unloaded form a "waiting line" or "queue". Let $P_N(t)$ be the probability of having exactly N failed elements in the queue at time t. Let p_k be the probability that during the time interval s, i.e. during a service time, there will be k new failures. If $N \neq 0$, the machine replaces one failed fuel element by a new one during the time s. Thus the balance equation for the probability $P_N(t)$ becomes

[14] $$P_N(t + s) = \sum_{k=0}^{N} p_k P_{N+1-k}(t) + p_N P_o(t)$$

Under steady state conditions equation [14] is invariant against translations of the time and we may as well drop this variable. In order to be able to solve equation [14] we have to know the p_k. For its determination we assume that the failures of the fuel elements appear statistically independent of one another. The failure probability during the service time for any fuel element chosen at random within the reactor is

[15] $$\gamma = c_T s$$

where C_T as defined in equation [12] is the mean rate of failure per fuel element. Thus the probability that of the M elements in the reactor k fail during the time interval s is given by the binomial law

[16] $$p_k = \binom{M}{k} (1-\gamma)^{M-k} \gamma^k$$

which, for large M and small γ, but finite $\gamma M = \lambda s$, can be approximated by Poisson's law

[17] $$p_k = \frac{e^{-\lambda s} (\lambda s)^k}{k!}$$

The general solution of equation [14] for arbitrary values of (λs) is given in the appendix. Here we are especially interested in its solution for small values of (λs), so small that terms of the order of $(\lambda s)^{k+1}$ can be neglected in comparison with terms $(\lambda s)^k$.

Then we can write approximately

$$p_k \approx \frac{(\lambda s)^k}{k!}$$

Thus $p_k = 0\left((\lambda s)^k\right)$. As is shown by the general solution in the appendix

[18] $$P_o = (1-\lambda s) \approx 1$$

Therefore $P_o = 0(1)$. We prove by complete induction (i) that $P_N = 0\left((\lambda s)^N\right)$ and even (ii) that

[19] $$P_N = p_N P_o + 0\left((\lambda s)^{N+1}\right)$$

Equation [19] holds good for $N = 0$. Let us assume that it is valid up to P_N. We then read directly from [14] that

$$p_N P_o + 0\left((\lambda s)^{N+1}\right) = p_o P_{N+1} + \sum_{k=1}^{N} p_k P_{N+1-k} + p_N P_o$$

The first and the last terms cancel. The sum is $0\left((\lambda s)^{N+1}\right)$ even for arbitrarily large N since

$$p_k P_{N+1-k} = \frac{(\lambda s)^k P_{N+1-k}}{k!} = \frac{0\left((\lambda s)^{N+1}\right)}{k!}$$

and the sum $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges. Thus

$$p_o P_{N+1} = 0\left((\lambda s)^{N+1}\right), \text{ whence } P_{N+1} = 0\left((\lambda s)^{N+1}\right)$$

and the first assertion (i) is proved. The second assertion (ii), then, is trivial, since in

$$P_{N+1} = \sum_{k=0}^{N+1} p_k P_{N+2-k} + p_{N+1} P_o$$

the sum is $0\left((\lambda s)^{N+2}\right)$ by the already proved assertion (i), while $p_{N+1} P_o = 0\left((\lambda s)^{N+1}\right)$ so that the sum can be neglected in comparison with this term and we have

$$P_{N+1} = p_{N+1} P_o + 0\left((\lambda s)^{N+2}\right)$$

and (ii) is proved. This completes the proof of [19]. Since [19] can also be written

$$P_N = p_N + 0\left((\lambda s)^{N+1}\right)$$

we see that the distribution P_N is also a Poisson distribution for very small λs. For larger values of λs the departures of P_N from the Poisson distribution become appreciable. Also, in the same approximation, the probability S_G that the number of failed elements in the queue is larger than G turns out to be :

$$S_G = \sum_{N=G+1}^{\infty} P_N = \sum_{N=G+1}^{\infty} p_N + 0\left((\lambda s)^{G+2} \right) = p_{G+1} + 0\left((\lambda s)^{G+2} \right) \qquad [20]$$

Thus

$$S_G = \frac{(\lambda s)^{G+1}}{(G+1)!} + 0\left((\lambda s)^{G+2} \right)$$

We now assume that at most G failed fuel elements can be tolerated in the reactor at the same time, but if there are more than G failed fuel elements, the reactor has to be shut down in order to limit contamination by fission products. Since such a shut-down is a costly process in the case of a power reactor operating on a continuous load-unload cycle, it is desirable to limit the probability of such an event to a low value δ. Since our description of the queuing process in equation [14] is essentially a sampling of the reactor state at discrete points of time corresponding to multiples of the service time s, a probability δ means that the event occurs with a mean rate of $\delta/s = h\delta$ times during the unit time interval. Thus if we want, for example, to have a shut-down only once in τ days and we take the day to be our unit of time, we have to choose $h\delta = 1/\tau$ or $\delta = 1/h\tau$. Thus for a given τ — which will be determined by economic considerations — a given h — which will be fixed by the engineering lay-out of the unloading device — and a given G — which is to be found by safety and operational arguments — we can determine λ by inverting the relation

$$\frac{1}{h\tau} = \delta \approx \frac{(\lambda s)^{G+1}}{(G+1)!} = \frac{(\lambda/h)^{G+1}}{(G+1)!}$$

obtaining

$$\lambda = h\left[\frac{(G+1)!}{h\tau}\right]^{1/(G+1)} = h^{G/(G+1)}\left[\frac{(G+1)!}{\tau}\right]^{1/(G+1)}$$

Using equation [13] we finally have

$$h^{G/(G+1)} = A\, c_T \qquad [21]$$

with

$$A = M\left[\frac{\tau}{(G+1)!}\right]^{1/(G+1)}$$

One can interpret equation [21] either as an equation for T if h is given or as an equation for h if T is given. The first procedure would correspond to an operational limitation for a given unloading device, whereas the second would correspond to a condition on the unloading device for a chosen operational procedure.

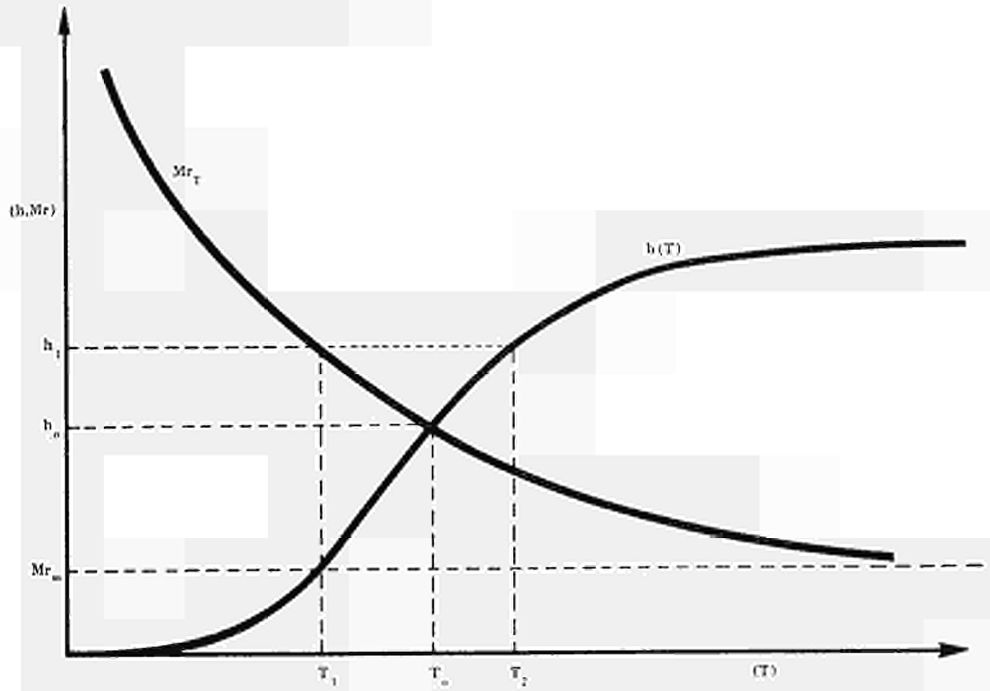
But in addition to the capacity of the device necessary to deal with the fluctuations, the capacity of the device also has to be sufficient to unload the fuel (failed on overaged) at a constant mean rate $r_T M$. Thus for a given irradiation time T the capacity h_o necessary to have a continuous operation at least for a time τ is given by

$$h_o = \left(h(T), M r_T \right) \qquad [22]$$

Since, from equations [21], [12] and [7], it is clear that h(T) is a monotonously increasing function of T, while, as we have seen above, r_T decreases monotonously with T, and since, for small T, $h(T) \to 0$ as $T \to 0$ while $r_T \approx \frac{1}{T} \to \infty$. there exists one unique point T_o where

[23] $$h_o = h(T_o) = Mr_{T_o} (= h_o \text{ say})$$

At this point, evidently, the necessary capacity h_o has the minimum. With an unloading device of smaller capacity than h_o a continuous reactor operation for τ days is impossible under the stated conditions. If the unloading device has a capacity $h_1 > h_o$ one has multiple choices. One might fix the fuel lifetime at a minimum value T_1 given by $h = Mr_{T_1}$; this choice is not likely for power reactors since the minimum value of T is economically unattractive. It contains, however, an additional safety margin reducing still further the number of elements that fail before being unloaded. The other extreme choice would be T_2 such that $h = h(T_2)$. Then one goes as far as one can in the economically promising direction of higher burn-up. But one is at the limit of safety tolerance. All choices between these extreme values, that is, all values $T_1 \leqslant T \leqslant T_2$ are admissible. The situation is sketched in fig. 1.



Numerical example :

We consider a reactor consisting of 4000 elements and operating at a specific power of 25 MW/$T_{nat.U}$. We assume that the lifetime distribution of the fuel is a truncated Gaussian:

$$p(T) = N\left[\Phi\left(\frac{T-\xi}{\sigma}\right) - \Phi\left(-\frac{\xi}{\sigma}\right)\right] \quad \text{with } \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy$$

and $\quad N = [1-\Phi(-\xi/\sigma)]^{-1}$

The mean ξ shall correspond to 5000 MWd/$T_{nat.U}$, the standard deviation σ to 1000 MWd/$T_{nat.U}$. Expressed in days this corresponds to $\xi = 200$ d, $\sigma = 40$ d. Since $\xi/\sigma = 5$, $\Phi(-\xi/\sigma) \approx 3.10^{-7}$ can be neglected. Thus we can use simply $p(T) = \Phi\left(\frac{T-\xi}{\sigma}\right)$.

In the case of a Gaussian distribution

$$\overline{t_T} = \int_0^T t\,\frac{dp(t)}{dt}\,dt = \frac{1}{\sqrt{2\pi}}\int_0^T t\,e^{-\frac{(t-\xi)^2}{2\sigma^2}}\,\frac{dt}{\sigma}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^T (t-\xi) e^{-\frac{(t-\xi)^2}{2\sigma^2}} \frac{dt}{\sigma} + \frac{\xi}{\sqrt{2\pi}} \int_0^T e^{-\frac{(t-\xi)^2}{2\sigma^2}} \frac{dt}{\sigma}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left\{ e^{-\frac{\xi^2}{2\sigma^2}} - e^{-\frac{(t-\xi)^2}{2\sigma^2}} \right\} + \xi \left[\Phi\left(\frac{T-\xi}{\sigma}\right) - \Phi\left(-\frac{\xi}{\sigma}\right) \right]$$

Since the terms $e^{-\xi^2/2\sigma^2}$ and $\Phi(-\xi/\sigma)$ can be neglected, we obtain

$$\bar{t}_T = -\frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(T-\xi)^2}{2\sigma^2}} + \xi \Phi\left(\frac{T-\xi}{\sigma}\right)$$

One sees immediately that for $T \to \infty \ \bar{t}_T \to \bar{t} = \xi$ as it should.

Thus :

$$r_T^{-1} = T\left(1 - \Phi\left(\frac{T-\xi}{\sigma}\right)\right) + \xi \Phi\left(\frac{T-\xi}{\sigma}\right) - \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{(T-\xi)^2}{2\sigma^2}}$$

The values of r_T, c_T, $h(T)$, Mr_T and $h_e(T)$ for different values of T are given in Table 1 :

TABLE 1

T(d)	40	120	160	200	400
r_T (d^{-1})	2.5×10^{-2}	8.36×10^{-3}	6.38×10^{-3}	5.44×10^{-3}	5.00×10^{-3}
C_T (d^{-1})	0.8×10^{-6}	2.4×10^{-4}	1.01×10^{-3}	2.7×10^{-3}	5.00×10^{-3}
$h(T)$ (d^{-1})	7.4×10^{-4}	3.8	33	145	365
Mr_T (d^{-1})	100	33.4	25.6	21.8	20.0
$h_e(T)$ (d^{-1})	100	33.4	33	145	365

IV. — EXPERIMENTAL DETERMINATION OF THE PROBABILITY p(T).

Up to now, the failure probability p(T) has been assumed to be accurately known. For practical purposes, it will have to be determined for a particular type of fuel under operating conditions, possibly simulated in a test reactor or in a prototype. There an attempt is made to find an estimate of the function p(T) by making an experiment with a sample of some fuel elements all under the same conditions as they are in the reactor. To establish these same conditions, we have to locate these elements in a realistic lattice arrangement in a test reactor within a flux of the same magnitude as in the real power reactor, and to replace an element which has failed during the time of this experiment by a new element. This way we observe the lifetimes of fuel elements (original or replaced) $L_1, L_2, \dots L_n$, where we assume n to be the number of failed elements at the end of the experiment. Also, at the end of the experiment there are m unfailed elements with irradiations l_{n+1}, l_{n+m} as a result of the replacement procedure.

There are two different ways of extracting an estimation of p(T) from such an "ensemble" of observed lifetimes :

a) For different discrete values of times T_i we count the number r_i of elements, for which the lifetime is larger than T_i : let r_{1i} be the number of elements k for which $L_k > T_i$; $1 \leqslant k \leqslant n$; let r_{2i} be the number of elements k for which $l_k > T_i$; $n + 1 \leqslant k \leqslant n + m$; then $r_i = r_{1i} + r_{2i}$.

Let n_i be determined by $n_i = n - r_{1i}$. Define h_i by

$$h_i = \frac{n_i}{r_i + n_i} = \frac{n_i}{n + r_{2i}}.$$ Then h_i can be taken as an estimation of $p(T_i)$.

b) Often from theoretical considerations the functional form of p(T) is known except for some undetermined parameters. In such a case it is better to use the experimental results for an estimation of those parameters.

We shall discuss both methods below. Method *a)* has the advantage that it can be used without any a priori knowledge of the form of the distribution p(T), and also that it includes even the unfailed rods in the estimation procedure. The drawback consists in the high possibility of error due to statistical fluctuations which has to be offset by a large safety margin. Therefore, if possible, method *b)* is preferable. But here, usually, only the failed rods can be used for estimation purposes.

Case a : The problem of estimating probabilities on the basis of frequencies is a classical problem of statistics. It is a well-known theorem of probability theory that for a large number of observations the observed frequency tends towards the probability ("law of large numbers"). Also. there exist already tabulations of fiducial limits (Ref. 1, 2) which give values p_N as a function of the frequency h_N observed in N independent trials, such that the true probability p is not likely to exceed p_N (i.e. the probability that $p > p_N$ is smaller than a given small value ϵ; in the following table $\epsilon = 0.025$). A few values for such fiducial upper limits are given in the following table. In the application of our method *a*, h_N is h_i, N is $n + r_{2i}$ and p_N is the fiducial upper limit of $p(T_i)$.

TABLE 2

Fiducial upper limits of p.

N	h_N										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
8	0.369	0.499	0.605	0.695	0.775	0.843	0.900	0.950	0.980	0.998	1.00
12	0.265	0.405	0.520	0.622	0.710	0.785	0.860	0.923	0.965	0.995	1.00
20	0.168	0.320	0.443	0.545	0.640	0.730	0.835	0.880	0.945	0.985	1.00
50	0.079	0.225	0.340	0.450	0.550	0.645	0.745	0.825	0.900	0.970	1.00
100	0.036	0.179	0.285	0.400	0.505	0.600	0.697	0.785	0.875	0.950	1.00
500	0.007	0.130	0.230	0.340	0.445	0.540	0.648	0.740	0.838	0.925	1.00
∞	0.000	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	1.00

We shall not, however, apply this table in its most general form for the interpretation of low frequencies, but we shall make the hypothesis that $\mu(t)$, the failure rate, is a monotonously increasing function of the age t. Then $\mu(t) \geqslant \mu(0)$ for all t and thus, for small values of p, from eq. (11), $\mu(t) \approx p'(t) \geqslant \mu(0)$, so that $p(t) \geqslant \mu(0)t$ for small values of t (such that p(t) is small). If, in this region, a value $p(t_o)$ is known, then for all $t \leqslant t_o$, $p(t) \leqslant (t/t_o)$, $p(t_o)$. We apply this by choosing T_o to be the highest of the T_i for which $h_i = 0$. If then p_o is the fiducial upper limit of $p(T_o)$ we take the fiducial upper limits for all p(T) with $T \leqslant t_o$ to be $(T/T_o)p_o$.

We illustrate method *a)* by two examples :

Example 1 :

The true distribution p(T) which is to be determined by the experiment is assumed to be

$$p(T) = \begin{cases} T/300 & \text{for } 0 \leqslant T \leqslant 300 \text{ days} \\ 1 & \text{for } T > 300 \text{ days} \end{cases}$$

In order to obtain an idea of the divergences in the experimental results, four independent experiments, using 8 elements each, have been simulated by drawing lifetimes from the distribution p(T) using a table of uniformly distributed random numbers. The "experimental" results are given in table 3.

TABLE 3
Observed lifetimes (days).

Experiment			
1	2	3	4
45	255	141	39
231	120	207	78
3	153	105	261
192	120	270	120
207	30	285	60
207	45	48	120
174	99	51	243
120	282	135	138

This yields the following table for the fiducial probabilities as compared with the true probability p(T) :

TABLE 4
Experimental fiducial probabilities.

T (days)	Experiment				p (T)
	1	2	3	4	
10	0,527	0,185	0,123	0,123	0,033
20	0,527	0,369	0,246	0,246	0,066
30	0,527	0,527	0,369	0,369	0,100
50	0,651	0,651	0,527	0,527	0,167
80	0,651	0,651	0,651	0,651	0,267
100	0,651	0,755	0,651	0,755	0,333
120	0.755	0,915	0,755	0,915	0,400
150	0.755	0,915	0,915	0,968	0,500
200	0.915	0,968	0,915	0,968	0,666
250	1,000	0,968	0,968	0,997	0,833
300	1,000	1,000	1,000	1,000	1,000

Example 2 :

The true distribution p(T) is assumed to be Gaussian with a mean of 200 days and a standard deviation σ of 40 days. Again. we have simulated four experiments, using 8 elements each, by sampling from a table of Gaussian distributed random numbers. The resulting lifetimes are given in table 5.

TABLE 5
Observed lifetimes (days).

Experiment			
1	2	3	4
179.5	197.3	137.7	231.4
179.0	211.8	207.5	240.9
223.8	188.4	259.4	181.1
235.2	202.4	185.8	251.2
218.6	99.0	174.6	340.8
205.5	178.8	227.9	222.8
298.2	122.4	237.0	126.0
187.1	221.7	255.0	207.6

We obtain the following table for the fiducial probabilities as compared with the true probability p(T).

TABLE 6
Experimental fiducial probabilities.

T (days)	Experiment 1	2	3	4	p (T)
100	0.246	0.527	0.369	0.369	0.00621
150	0.369	0.651	0.527	0.527	0.1075
180	0.651	0.755	0.651	0.527	0.3085
200	0.755	0.915	0.755	0.651	0.5000
220	0.915	0.997	0.843	0.755	0.6915
240	0.997	1.000	0.968	0.915	0.8413
300	1.000	1.000	1.000	0.997	0.9938

Method b :

As a concrete example of method b, let us assume it to be known that p(T) is given by a Gaussian distribution $\Phi\left(\frac{T-\xi}{\sigma}\right)$ but that the parameters ξ, σ are unknown. On the basis of the n observed lifetimes $L_1 ..., L_n$ one can obtain the usual estimations for ξ and σ (we denote the estimators by an asterisk) :

[24] $$\xi^* = \frac{1}{n}\sum_{i=1}^{n} L_i \qquad \sigma^{*2} = \frac{1}{n-1}\sum_{i=1}^{n}(L_i-\xi^*)^2$$

Classical statistical theory (Ref. 3) gives us the following distributions which characterize these sampling values :

(i) the Student distribution $H(a, \nu)$ which is the distribution of $t = \sqrt{n}\,\frac{\xi^*-\xi}{\sigma^*}$ is defined by :

[25] $$H(a, \nu) \equiv \text{Prob}\,[t \leqslant a] = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)}\int_{-\infty}^{a}\left(1+\frac{t^2}{\nu}\right)^{-(\nu+1)/2} dt$$

where $\nu = n-1$ is called "number of degrees of freedom".

(ii) the χ^2– distribution $G(u, \nu)$ is the distribution of $\chi^2 = (n-1)(\sigma^*/\sigma)^2$ and is given by :

[26] $$G(u, \nu) \equiv \text{Prob}\,[\chi^2 \leqslant u] = \frac{1}{2^{\nu/2}\,\Gamma(\nu/2)}\int_{0}^{u} y^{\frac{\nu}{2}-1}\, e^{-y/2}\, dy$$

But in our case we are not so much interested in the separate knowledge of the distributions of the sample means and variances. Our problem is the following : the probability p(T) for a given T is calculated as $p(T) = \Phi(a)$ where a $(T-\xi)/\sigma$. If instead of a we use $a^+ = (T-\xi^*)/\sigma^*$ as argument of the error integral Φ, the value of the probability p(T) thus calculated will fluctuate from sample to sample, depending on the fluctuations of a^+.

We study therefore the deviation $a-a^+$ between the exact and the sample value of the argument of Φ. $a-a^+$ is, like a^+, a random variable containing T as additional parameter. From its explicit form :

[27] $$a-a^+=a\left(1-\frac{\sigma}{\sigma^*}\right)+\frac{\xi^*-\xi}{\sigma^*}=a\left(1-\frac{\sigma}{\sigma^*}\right)+\frac{t}{\sqrt{n}}$$

with t defined as above, we can see that the expectation $E(a-a^+)$ of $a-a^+$ equals

[28] $$E(a-a^+)=a\Big(1-E(\sigma/\sigma^*)\Big)$$

since

[29] $$E(t)=E\left(\frac{\xi^*-\xi}{\sigma}\right)=0$$

Using the definition of χ^2 as given in (ii), we have :

$$\frac{\sigma}{\sigma^*}=\sqrt{\frac{2\theta}{\chi^2}}$$

where we have put $\theta=\nu/2=(n-1)/2$. We assume from now on always $n \geqslant 4$. From eq. [26] $\left(\text{with the abbreviation } \alpha=1/2^{\theta}\Gamma(\theta)\right)$ we find :

$$E\left(\frac{\sigma}{\sigma^*}\right)=\alpha\sqrt{2\theta}\int_0^\infty u^{\theta-\frac{3}{2}}e^{-u/2}\,du$$

Introducing $y=\frac{u}{2}$ as a new variable we have

$$E\left(\frac{\sigma}{\sigma^*}\right)=\alpha\sqrt{\theta}\,2^{\theta}\int_0^\infty y^{\theta-\frac{3}{2}}e^{-y}\,dy$$

Comparing this with the definition of the gamma function :

$$\Gamma(z)=\int_0^\infty e^{-y}y^{z-1}\,dy$$

we find immediately :

[30] $$E\left(\frac{\sigma}{\sigma^*}\right)=\alpha\sqrt{\theta}\,2^{\theta}\,\Gamma\left(\theta-\frac{1}{2}\right)=\sqrt{\theta}\,\frac{\Gamma\left(\theta-\frac{1}{2}\right)}{\Gamma(\theta)}=g(\theta)$$

Thus the estimation of a by a^+ has the bias

[31] $$E(a-a^+)=-af(\theta)$$

where

[32] $$f(\theta)=g(\theta)-1=\sqrt{\theta}\,\frac{\Gamma\left(\theta-\frac{1}{2}\right)}{\Gamma(\theta)}-1$$

However, the bias $E(a-a^+)$ vanishes asymptotically for large θ, i.e. for large n : a general theorem (Ref. 4) on the asymptotic behaviour of the gamma function reads (for general a) :

$$\lim_{|z|\to\infty} \frac{\Gamma(z+a)}{\Gamma(z)} e^{-a \ln z} = 1$$

Let $a = -1/2$, $z = \theta$. we obtain :

[33] $$\lim_{\theta\to\infty} \sqrt{\theta} \frac{\Gamma\left(\theta-\frac{1}{2}\right)}{\Gamma(\theta)} = 1$$

and therefore $\lim_{n\to\infty} E(a-a^+) = 0$. In the following table $g(\theta)$ and $f(\theta)$ are given for a few values of θ :

TABLE 7

n	θ	$g(\theta)$	$f(\theta)$
4	3/2	1.383	0.383
8	7/2	1.1265	0.1265
12	11/2	1.0747	0.0747
∞	∞	1.0000	0.0000

An estimation for a, having no bias, is

[34] $$a_0 = \frac{a^+}{g} = \frac{T-\xi^*}{g\,\sigma^*}$$

Then

[35] $$a - a_0 = a\left(1-\frac{\sigma}{g\,\sigma^*}\right) + \frac{t}{g\sqrt{n}}$$

From now on, we use only a_0 as estimator for a. We calculate the variance of $a-a_0$. Since $E(a-a_0) = 0$, the variance is simply

[36] $$E\{(a-a_0)^2\} = E\{(a-a^*/g)^2\} = a^2\, E\left\{\left(1-\frac{\sigma}{g\,\sigma^*}\right)^2\right\} + \frac{E(t^2)}{ng^2}$$

because of the statistical independence of σ/σ^* and t and relation [29].

Now

$$E\left\{\left(1-\frac{\sigma}{g\sigma^*}\right)^2\right\} = 1-2\frac{E(\sigma/\sigma^*)}{g} + \frac{E\{(\sigma/\sigma^*)^2\}}{g^2} = \frac{E\{(\sigma/\sigma^*)^2\}}{g^2} - 1$$

Since $(\sigma/\sigma^*)^2 = 2\theta/\chi^2$ we obtain from eq. [26]

$$E\{(\sigma/\sigma^*)^2\} = 2\theta \int_0^\infty u^{\theta-2} e^{-u/2} du$$

Again replacing u by $y = u/2$ and using the definition of the gamma function we obtain

$$[37] \qquad E\{(\sigma/\sigma^*)^2\} = \theta\, \alpha\, 2^{\theta}\, \Gamma(\theta-1) = \theta \frac{\Gamma(\theta-1)}{\Gamma(\theta)} = \frac{\theta}{\theta-1}$$

therefore

$$[38] \qquad h^2(\theta) \equiv E\left\{\left(1 - \frac{\sigma}{g\sigma^*}\right)^2\right\} = \frac{\Gamma^2(\theta)}{(\theta-1)\Gamma^2\left(\theta-\frac{1}{2}\right)} - 1 = \frac{\Gamma(\theta)\Gamma(\theta-1)}{\Gamma^2\left(\theta-\frac{1}{2}\right)} - 1$$

Concerning the second term in [36] we use the known relation for the Student distribution

$$[39] \qquad E(t^2) = \frac{\nu}{\nu-2} = \frac{\theta}{\theta-1}$$

Therefore

$$[40] \quad \Delta^2(\lambda) \equiv \frac{E(t^2)}{ng^2} = \frac{\Gamma^2(\theta)}{(2\theta+1)(\theta-1)\Gamma^2\left(\theta-\frac{1}{2}\right)} = \frac{\Gamma(\theta)\Gamma(\theta-1)}{(2\theta+1)\Gamma^2\left(\theta-\frac{1}{2}\right)} = \frac{h^2(\theta)+1}{2\theta+1}$$

Finally

$$[41] \qquad E\{(a-a_0)^2\} = a^2 h^2(\lambda) + \Delta^2(\lambda)$$

Thus the variance consists of two terms both of which vanish asymptotically; the first one is proportional to a^2 and thus important for values of T far away from ξ, while the second one is independent of a and thus of T. $h^2(\theta)$ and $\Delta^2(\theta)$ are shown in table 8 for a few values of θ.

TABLE 8

n	θ	$h^2(\theta)$	$\Delta^2(\theta)$
4	3/2	0.5707	0.3927
8	7/2	0.1044	0.1381
12	11/2	0.0569	0.0881
∞	∞	0.0000	0.0000

It can be demonstrated that the distribution of $a-a_0$ is asymptotically Gaussian so that the value of $E\{(a-a_0)^2\}$ already characterizes the distribution of $a-a_0$. Knowing this distribution, we can then define a δ_ϵ such that Prob. $[a_0 + \delta_0 < a] < \epsilon$. Since Φ is a monotonously increasing function, a δ_ϵ so defined has the following property : if instead of the true value of a we use the estimator a_0 plus the δ_ϵ as argument of the Φ — function, we obtain a value of p(T) which — with a probability $1-\epsilon$ — will be higher than the true value, thus being a fiducial upper limit of p(T) corresponding to an error probability of less than ϵ.

Example :

We consider again the observed lifetimes of table 5, corresponding to a Gaussian distribution. We obtain in this case the following values for ξ^* and σ^* in the four experiments.

	Experiment			
	1	2	3	4
ξ^*	215.9	177.7	210.6	225.3
σ^{*2}	954.6	1920.4	1460.2	3775.1
σ^*	30.9	43.8	38.4	61.4

Since the "experiments" have been carried out with 8 elements, we have $g(\theta) = 1.1265$ and $h^2(\theta) = 0.1044$, $\Delta^2(\theta) = 0.1381$.

In table 9 we give the values for a_e and $\sigma(a-a_e) = \sqrt{E\{(a-a_e)^2\}}$ for different values of T. As a is in practice unknown, it has to be replaced in the calculation of σ by its estimate a_e. This is sometimes rather crude but since $\sigma(a-a_e)$ will give us only an idea of the spread of the values of a_e around a this procedure is sufficient.

TABLE 9

T (days)	Exp. 1		Exp. 2		Exp. 3		Exp. 4		a
	a_e	$\sigma(a-a_e)$	a_e	$\sigma(a-a_e)$	a_e	$\sigma(a-a_e)$	a_e	$\sigma(a-a_e)$	a
100	— 3.33	1.14	— 1.58	0.63	—25.5	0.90	— 1.81	0.69	— 2.50
150	— 1.89	0.72	— 0.56	0.41	— 1.40	0.59	— 1.09	0.51	— 1.25
180	— 1.03	0.50	+ 0.05	0.37	— 0.71	0.44	— 0.66	0.43	— 0.50
200	— 0.46	0.40	+ 0.45	0.40	— 0.25	0.38	— 0.37	0.39	± 0.00
220	+ 0.12	0.37	+ 0.86	0.46	+ 0.22	0.38	+ 0.09	0.37	+ 0.50
240	+ 0.69	0.43	+ 1.26	0.55	+ 0.68	0.43	+ 0.21	0.38	+ 1.00
300	+ 2.43	0.87	+ 2.48	0.88	+ 2.05	0.76	+ 1.08	0.51	+ 2.50

If we simply take $a_e + \sigma(a-a_e)$ as argument of the Φ—function, we risk with a probability of a little more than 16 % that $a_e + \sigma(a-a_e) < a$. But for purposes of illustration it suffices to use $a_e + \sigma(a-a_e)$ as argument of the function in order to derive an "upper boundary" of the probability p(T) which will not be frequently exceeded. These values $\Phi[a_e + \sigma(a-a_e)]$ are compared with $\Phi(a)$ in table 10.

TABLE 10

T (days)	Experiment				Φ (a)
	1	2	3	4	
100	0.01426	0.1711	0.04947	0.1314	0.00621
150	0.1210	0.4404	0.2090	0.2810	0.1075
180	0.2981	0.6628	0.3936	0.4090	0.3085
200	0.4761	0.8023	0.5517	0.5080	0.5000
220	0.6879	0.9066	0.7257	0.6772	0.6915
240	0.8686	0.9649	0.8665	0.7224	0.8413
300	0.9995	0.9996	0.9975	0.9441	0.9938

APPENDIX

Derivation of the solution of eq. [14] :

It is more convenient to calculate the probability generating function (p.g.f) $F(z)$ of P_N instead of the probabilities P_N themselves. As is well known, a p.g.f. of a distribution P_N is defined as the expectation of z^N corresponding to this distribution, that is

$$F(z) = E(z^N) = \sum_{N=0}^{\infty} P_N z^N$$

Also, let $f(z)$ be the corresponding p.g.f. of p_k :

$$f(z) = \sum_{k=0}^{\infty} p_k z^k$$

If we multiply eq. [14] by z^N and sum over N we obtain

$$F(z) = z^{-1} \sum_{N=0}^{\infty} \sum_{k=0}^{N} P_{N+1-k} z^{N+1-k} p_k z^k + P_0 f(z) = z^{-1} \sum_{\mu=1}^{\infty} P_\mu z^\mu f(z) + P_0 f(z)$$

$$= z^{-1} F(z) f(z) - z^{-1} P_0 f(z) + P_0 f(z)$$

Thus

$$F(z) = P_0 \frac{1-z}{1-z/f(z)}$$

As it is obvious from its definition, a p.g.f. $F(z)$ satisfies the condition $F(1)=1$. Using this condition we easily determine P_0 :

$$1 = P_0 \lim_{z \to 1} \frac{1-z}{1-z/f(z)} = P_0 \left[\frac{1}{\dfrac{1}{f(z)} - \dfrac{z}{f^2(z)} f'(z)} \right]_{z=1} = \frac{P_0}{1-\bar{k}}$$

taking into account the fact that $f'(1) = \sum_{k=0}^{\infty} k p_k - \bar{k}$. Thus $P_0 = 1-\bar{k}$. From eq. [17] $\bar{k} = \lambda s$ so that $P_0 = 1-\lambda s$ which is the relation [18].

The function $f(z)$ corresponding to eq. [17] is

$$f(z) = e^{-\lambda s(1-z)}$$

Thus the function $F(z)$ becomes

$$F(z) = \frac{(1-\lambda s)(1-z)}{1-z\, e^{\lambda s(1-z)}}$$

Expanding this expression in a power series in z about the point $z = 0$ yields the P_N as coefficients of z^N. But we are more interested in the $S_G = \sum_{N=G+1}^{\infty} P_N$. The p.g.f. $Q(z)$ of S_G. defined as $\sum_{G=0}^{\infty} S_G z^G$, can be expressed in terms of $F(z)$ as follows :

It is $P_G = S_{G-1} - S_G$ and therefore

$$F(z) = \sum_{G=0}^{\infty} S_{G-1} z^G - Q(z) = 1 + z \sum_{G=1}^{\infty} S_{G-1} z^{G-1} - Q(z)$$

Thus $F(z) = 1 + z\,Q(z) - Q(z)$ or

$$Q(z) = \frac{1-F(z)}{1-z} = \frac{1}{1-z} - \frac{1-\lambda s}{1-z\,e^{\lambda s\,(1-z)}}$$

We obtain the S_G from the expansion of $Q(z)$. Let a_G be the coefficient of z^G in the expansion of $\left(1-ze^{\lambda s\,(1-z)}\right)^{-1}$; then we have

$$S_G = 1-(1-\lambda s)\, a_G$$

We thus only have to calculate the a_G. We find by direct expansion

$$\left(1-ze^{\lambda s}e^{-\lambda sz}\right)^{-1} = \sum_{n=0}^{\infty} e^{n\lambda s} z^n e^{-n\lambda sz} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-n\lambda s} z^n \frac{(-n\lambda sz)^m}{m\,!}$$

$$= \sum_{G=0}^{\infty} z^G \sum_{m=0}^{G} \frac{e^{(G-m)\lambda s} (G-m)^m (-\lambda s)^m}{m\,!}$$

Thus

$$a_G = \sum_{m=0}^{G} \frac{e^{(G-m)\lambda s} (G-m)^m (-\lambda s)^m}{m\,!}$$

The first few coefficients a_G, $G = 0, \ldots, 5$ are given explicitly as follows :

$a_0 = 1$

$a_1 = e^{\lambda s}$

$a_2 = e^{2\lambda s} - \lambda s e^{\lambda s}$

$a^3 = e^{3\lambda s} - 2\lambda s e^{2\lambda s} + 0{,}5\,(\lambda s)^2 e^{\lambda s}$

$a_4 = e^{4\lambda s} - 3\lambda s e^{3\lambda s} + 2\,(\lambda s)^2 e^{2\lambda s} - 0{,}17\,(\lambda s)^3 e^{\lambda s}$

$a_5 = e^{5\lambda s} - 4\lambda s e^{4\lambda s} + 4{,}5\,(\lambda s)^2 e^{3\lambda s} - 1{,}33\,(\lambda s)^3 e^{2\lambda s} + 0{,}04\,(\lambda s)^4 e^{\lambda s}$

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