Martina Kubitzke

Lefschetz Elements for Stanley-Reisner Rings

and

Annihilator Numbers

Philipps Universität Marburg

Lefschetz Elements for Stanley-Reisner Rings and Annihilator Numbers

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Martina Kubitzke
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Abstract

This thesis is composed of three big parts. In the first two chapters we provide the basic definitions and facts which are needed in the subsequent chapters. Chapter 1 is concerned with basic algebraic principles whereas in the second chapter we treat simplicial complexes. The actual results of this thesis are presented in Chapter 3 through 5. In particular, we dedicate Chapters 3 and 4 to the g-conjecture, the g-theorem, further related results as well as the Lefschetz property for barycentric subdivisions of shellable simplicial complexes. In Chapter 5, which constitutes the second main topic, we consider the symmetric and the exterior depth of finitely generated modules as well as the symmetric and the exterior annihilator numbers. In the following we dwell on the two main topics in more detail.

One of the most classical and most studied problems in combinatorial commutative algebra and discrete geometry is the characterization of f-vectors of special classes of simplicial complexes. Kruskal [Kru60] and Katona [Kat68] succeeded to describe all possible vectors which can occur as the f-vector of a simplicial complex. Based on this classification one might ask if in addition it is possible to extract those vectors which belong to specific classes of simplicial complexes, such as Gorenstein* complexes, boundary complexes of simplicial polytopes or simplicial spheres. In 1971 McMullen [McM71] formulated the so-called g-conjecture for precisely this latter class of simplicial complexes. This conjecture, which he originally proposed only for boundary complexes of simplicial polytopes, was proven by Stanley and Billera/Lee in 1979, respectively. The result is widely known as the g-theorem.

Theorem. *[BL81, Sta80](g-theorem)*
Let $h = (h_0, \dots, h_d) \in \mathbb{N}^{d+1}$ and let $g = (1, h_1 - h_0, \dots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$. Then h is the h-vector of the boundary complex of a simplicial d-polytope if and only if g is an M-sequence.

Accessorily to the classical g-theorem there is a multitude of results showing a Lefschetz property for special classes of simplicial complexes or investigating the behavior of this property when performing a certain operation on the simplicial complex. In order to mention just some of those results we want to cite the results of Swartz for independence complexes of matroids and for simplicial complexes featuring a convex ear decomposition [Swa03, Swa06]. There are further achievements by Nevo and Babson [Nev07, NB08] for the join, the union, the connected sum and stellar subdivisions of simplicial complexes as well as results by Murai for strongly edge decomposable simplicial complexes [Mur07]. Additionally, there are algebraic results characterizing the Lefschetz property or providing equivalent conditions, see e.g. [HW07], [CP07] and [Wie04].

The main result of this thesis which was compassed in joint work with Eran Nevo shows the (so-called) almost strong Lefschetz property not only for barycentric subdivisions of shellable simplicial complexes but also for barycentric subdivisions of shellable polytopal complexes. The motivation for studying barycentric subdivisions of shellable simplicial complexes originates from results by Brenti and Welker [BW06]. They showed amongst other things that the h-vector of the barycentric subdivision of a Cohen-Macaulay complex is unimodal. This can also be deduced if a simplicial complex exhibits the almost strong Lefschetz property. Brenti and Welker therefore conjectured that this is the case for the barycentric subdivision of a Cohen-Macaulay simplicial complex. In collaboration with Eran Nevo the following result could be proven.

Theorem 0.0.1. *Let Δ be a shellable $(d-1)$-dimensional simplicial complex and let k be an infinite field. Let further $\mathrm{sd}(\Delta)$ be the barycentric subdivision of Δ. Then $\mathrm{sd}(\Delta)$ is almost strong Lefschetz over k.*
If Δ is a shellable polytopal complex, then $\mathrm{sd}(\Delta)$ is almost strong Lefschetz over $\mathbb{R}$.

The above result in particular implies that the h-vectors of barycentric subdivisions of Cohen-Macaulay simplicial complexes are M-sequences. We want to emphasize at this point that it is quite remarkable that the numerical result is true in the greater generality of Cohen-Macaulay complexes, even though the algebraic result does only hold for shellable simplicial complexes. The crucial fact which is used for proving this result is that Cohen-Macaulay complexes and shellable simplicial complexes possess the same set of h-vectors [Sta96]. Note that Theorem 0.0.1 in particular shows the g-conjecture for barycentric subdivisions of simplicial spheres, Gorenstein* complexes and 2-Cohen-Macaulay complexes. Furthermore, Brenti and Welker show in [BW06] that the entries of the h-vector of the barycentric subdivision of a simplicial complex can be expressed as positive linear combinations of the entries of the h-vector of the original complex. The coefficients emerging in this transformation are a certain refinement of the usual Eulerian statistics on permutations, see e.g. [FS70]. More precisely, permutations are grouped according to their number of descents and the image of 1.
Using the results for barycentric subdivisions of shellable simplicial complexes – the algebraic as well as the numerical ones – we are able to analyze those numbers in great detail. We first study their behavior when increasing the number of descents while keeping the image of 1 fixed. By dint of those results we can deduce further inequalities for those numbers when changing the image of 1 while fixing the number of descents.

In a second big group of topics of this thesis we compare algebraic invariants over the polynomial ring with their counterparts over the exterior algebra. The main focus here lies on the symmetric and the exterior depth as well as on the symmetric and the exterior annihilator numbers. It is used that there exists an equivalence of categories between squarefree modules over S and squarefree modules over E, see [AAH00] and [Röm01]. Here $S := k[x_1,\dots,x_n]$ denotes the polynomial ring in n variables over a field k and $E := k\langle e_1,\dots,e_n\rangle$ denotes the exterior algebra. By means of the mentioned equivalence we can associate to every squarefree S-module a squarefree E-module, e.g., the exterior Stanley-Reisner ring of a simplicial complex is assigned to the (symmetric) Stanley-Reisner ring of the same complex. This correspondence allows the comparison of corresponding invariants.

Aramova, Avramov and Herzog introduced in [AAH00] the notion of the exterior depth of an E-module which is defined analogously as the symmetric depth of an S-module. In collaboration with Gesa Kämpf, we could show, amongst other things, that the symmetric depth of an S-module can never be smaller than the exterior depth of the associated E-module. Moreover, we are able to characterize simplicial complexes whose exterior Stanley-Reisner ring exhibits a specified exterior depth in terms of their exterior shifting.

The so-called symmetric annihilator numbers of an S-module with respect to a sequence of linear forms were originally defined by Trung in [Tru87]. Those numbers can be considered as an iteration of the concept of the symmetric depth. It can be shown that they are independent of the particular chosen sequence if the latter one originates from a certain non-empty Zariski-open set. This gives rise to the definition of the symmetric generic annihilator numbers. Those are strongly related to the graded Betti numbers over S. Indeed, as was proven by Conca, Herzog and Hibi in [CHH04], the symmetric graded Betti numbers of an S-module of the form S/I, where $I \subseteq S$ is a graded ideal, are bounded from above by positive linear combinations of the symmetric generic annihilator numbers. This bound is tight if and only if I is a componentwise linear ideal.
We carry over the concept of the symmetric annihilator numbers with respect to a sequence to the situation in the exterior algebra. Bearing in mind that each element of an E-module is a zero-divisor we introduce the exterior annihilator numbers with respect to a sequence of linear forms. Our aim is to translate some properties of the symmetric annihilator numbers into properties of the exterior annihilator numbers. In doing so it emerges that, as in the symmetric case, the exterior annihilator numbers with respect to different sequences coincide if the latter ones stem from a certain non-empty Zariski-open set. In the following we therefore only examine the so-called exterior generic annihilator numbers. Along the lines of the situation over the polynomial ring, it can be shown that positive linear combinations of those numbers serve as upper bounds for the graded Cartan-Betti numbers of an E-module of the form E/J, where $J \subseteq E$ is a graded ideal. This in particular provides us with an upper bound for the ordinary graded Betti numbers over E. As in the symmetric case, equality is attained only for componentwise linear ideals.
Besides the mere conferment of the results over the polynomial ring to the exterior algebra, additional results can be achieved. For E-modules of the form E/J it turns out that the exterior generic annihilator numbers count certain generators of the generic initial ideal of J with respect to the reverse lexicographic order. In the special case of simplicial complexes this result can be used in order to demonstrate that the exterior generic annihilator numbers equal the numbers of certain minimal generators of the symmetric and the exterior Stanley-Reisner ideal.

Looking at the generic annihilator numbers in more detail, at the symmetric as well as at the exterior ones, the question occurs if those numbers stand out due to something compared to the annihilator numbers with respect to a particular sequence. Herzog predicted that they are the minimal ones under all annihilator numbers with respect to a sequence. We construct for the symmetric as well as for the exterior annihilator numbers a counterexample to this conjecture.

Zusammenfassung

Diese Arbeit gliedert sich in drei große Teile. Bei den ersten beiden Kapiteln handelt es sich um Grundlagenkapitel. Wir behandeln im ersten Kapitel algebraische Grundlagen, während sich das zweite mit simplizialen Komplexen befasst. Kapitel 3 bis 5 stellen den eigentlichen Ergebnisteil dieser Arbeit da. Dabei sind Kapitel 3 und 4 der g-Vermutung, dem g-Theorem, damit verwandten Ergebnissen, sowie der Lefschetz-Eigenschaft für baryzentrische Unterteilungen schälbarer simplizialer Komplexe gewidmet. Kapitel 5, als zweiter Themenkomplex, behandelt die symmetrische und äußere Tiefe von endlich erzeugten Moduln, sowie symmetrische und äußere Annulatorzahlen. Wir gehen im Folgenden genauer auf die beiden Ergebnisteile ein und fassen die erhaltenen Resultate kurz zusammen.

Wohl eines der klassischsten und meist untersuchten Probleme im Bereich der kombinatorischen kommutativen Algebra und der diskreten Geometrie ist die Charakterisierung von f-Vektoren spezieller Klassen simplizialer Komplexe. Kruskal [Kru60] und Katona [Kat68] gelang es, alle Vektoren zu beschreiben, die als f-Vektoren simplizialer Komplexe auftreten können. Ausgehend von dieser Klassifizierung stellt sich die Frage, ob es möglich ist, noch einmal die Vektoren zu extrahieren, die zu bestimmten Klassen simplizialer Komplexe gehören, wie z. B. Gorenstein* Komplexen, Randkomplexen simplizialer Polytopen oder simplizialen Sphären. Für letztere Klasse simplizialer Komplexe formulierte McMullen [McM71] 1971 die sog. g-Vermutung. Diese ursprünglich nur für Randkomplexe simplizialer Polytope aufgestellte Vermutung wurde 1979 von Stanley [Sta80] bzw. Billera und Lee [BL81] bewiesen. Das Resultat ist als g-Theorem bekannt.

Theorem. *[BL81], [Sta80](g-Theorem)*
Sei $h = (h_0, \ldots, h_d) \in \mathbb{N}^{d+1}$ und sei $g = (1, h_1 - h_0, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$. Dann ist h genau dann der h-Vektor des Randkomplexes eines simplizialen d-Polytops wenn g eine M-Sequenz ist.

Zusätzlich zu dem klassischen g-Theorem gibt es eine Vielzahl von Ergebnissen, die eine Lefschetz-Eigenschaft für spezielle Klassen simplizialer Komplexe zeigen oder das Verhalten dieser Eigenschaft bei Durchführung bestimmter Operationen untersuchen. Hierbei sind z. B. die Ergebnisse von Swartz für Matroid-Komplexe und Komplexe mit einer konvexen Ohrenzerlegung [Swa03, Swa06] oder die Ergebnisse von Nevo und Babson [Nev07, NB08] für den Join, die Vereinigung, die zusammenhängende Summe und stellare Unterteilungen simplizialer Komplexe, sowie die Ergebnisse von Murai für stark Kanten-zerlegbare Komplexe [Mur07] zu nennen. Zusätzlich gibt es noch algebraische Ergebnisse, die die

Lefschetz-Eigenschaft charakterisieren oder zu ihr äquivalente Bedingungen liefern, siehe z. B. [HW07], [CP07] und [Wie04].

Das in Zusammenarbeit mit Eran Nevo erzielte Hauptergebnis dieser Arbeit zeigt die sog. fast starke Lefschetz-Eigenschaft sowohl für baryzentrische Unterteilungen schälbarer simplizialer Komplexe als auch für baryzentrische Unterteilungen schälbarer polytopaler Komplexe. Die Motivation, baryzentrische Unterteilungen von schälbaren simplizialen Komplexen zu betrachten, stammt von Ergebnissen von Brenti und Welker in [BW06]. Diese zeigen u. a., dass der h-Vektor der baryzentrischen Unterteilung eines Cohen-Macaulay Komplexes unimodal ist. Dies kann auch gefolgert werden, wenn ein simplizialer Komplex die fast starke Lefschetz-Eigenschaft besitzt. Brenti und Welker vermuteten daher, dass dies für die baryzentrische Unterteilung eines Cohen-Macaulay Komplexes der Fall ist. Es gelang folgendes Ergebnis zu zeigen.

Theorem 0.0.2. *Sei Δ ein schälbarer $(d-1)$-dimensionaler simplizialer Komplex und sei k ein unendlicher Körper. Sei $\mathrm{sd}(\Delta)$ die baryzentrische Unterteilung von Δ. Dann ist $\mathrm{sd}(\Delta)$ fast stark Lefschetz über k.*
Ist Δ ein schälbarer polytopaler Komplex, so ist $\mathrm{sd}(\Delta)$ fast stark Lefschetz über $\mathbb{R}$.

Daraus folgt insbesondere, dass es sich bei den h-Vektoren baryzentrischer Unterteilungen von Cohen-Macaulay Komplexen um M-Sequenzen handelt. Bemerkenswert ist, dass – auch wenn das algebraische Resultat nur für schälbare Komplexe gilt – das numerische Resultat für die größere Klasse von Cohen-Macaulay Komplexen gezeigt werden kann. Dabei wird verwendet, dass Cohen-Macaulay Komplexe und schälbare Komplexe die gleiche Menge an h-Vektoren besitzen, siehe [Sta96]. Theorem 0.0.2 zeigt insbesondere die g-Vermutung für baryzentrische Unterteilungen von simplizialen Sphären, Gorenstein* Komplexen und 2-Cohen-Macaulay Komplexen.
In [BW06] zeigen Brenti und Welker des Weiteren, dass sich die Einträge des h-Vektors der baryzentrischen Unterteilung eines simplizialen Komplexes als positive Linearkombinationen der ursprünglichen h-Vektor-Einträge schreiben lassen. Bei den in dieser Transformation auftretenden Koeffizienten handelt es sich um eine Verfeinerung der Eulerschen Statistik auf Permutationen, siehe z. B. [FS70]. Genauer zählen die Koeffizienten die Anzahl der Permutationen der S_n mit einer gewissen Anzahl an Abstiegen und vorgegebenem Bild von 1.
Unter Verwendung der Ergebnisse für baryzentrische Unterteilungen schälbarer Komplexe – sowohl der algebraischen als auch der numerischen – sind wir in der Lage diese Anzahlen genauer zu analysieren. Es wird zunächst ihr Verhalten bei Erhöhung der Anzahl von Abstiegen und festem Bild von 1 untersucht. Mit Hilfe dieser Ergebnisse können weitere Ungleichungen für diese Anzahlen gezeigt werden, wenn bei gleichbleibender Anzahl von Abstiegen das Bild von 1 verändert wird.

In einem zweiten großen Themenkomplex dieser Arbeit werden algebraische Invarianten über dem Polynomring mit ihren Entsprechungen über der äußeren Algebra verglichen. Der Schwerpunkt liegt hierbei auf der symmetrischen und der äußeren Tiefe, sowie auf den symmetrischen und äußeren Annulatorzahlen. Es wird dabei verwendet, dass es eine Äquivalenz zwischen den Kategorien quadratfreier S-Moduln und quadratfreier E-Moduln gibt, siehe [AAH00] und [Röm01]. Hierbei ist $S := k[x_1, \ldots, x_n]$ der Polynomring in n Variablen über einem Körper k und $E := k\langle e_1, \ldots, e_n \rangle$ bezeichnet die äußere Algebra. Mittels der genannten Äquivalenz lässt sich zu einem quadratfreien S-Modul ein quadratfreies E-Modul assoziieren. So wird beispielsweise dem (symmetrischen) Stanley-Reisner-Ring eines simplizialen Komplexes dessen äußerer Stanley-Reisner-Ring zugewiesen. Dies ermöglicht den Vergleich entsprechender Invarianten.

Aramova, Avramov und Herzog führten in [AAH00] in Analogie zur (symmetrischen) Tiefe den Begriff der äußeren Tiefe ein. In Zusammenarbeit mit Gesa Kämpf gelang es u. a. zu zeigen, dass die symmetrische Tiefe eines S-Moduls niemals kleiner als die äußere Tiefe des entsprechenden E-Moduls ist. Ferner sind wir in der Lage, simpliziale Komplexe, deren äußerer Stanley-Reisner-Ring eine vorgegebene äußere Tiefe hat, mittels ihres äußeren Shiftings zu charakterisieren.

In [Tru87] führte Trung die sog. symmetrischen Annulatorzahlen eines S-Moduls bzgl. einer Sequenz von Linearformen ein. Diese können als eine Art Iteration des Konzepts der symmetrischen Tiefe betrachtet werden. Es kann gezeigt werden, dass diese Anzahlen unabhängig von der betrachteten Sequenz sind, wenn letztere aus einer gewissen Zariski-offenen Menge gewählt wird. Dies führt zur Definition der symmetrischen generischen Annulatorzahlen. Diese stehen in engem Zusammenhang zu den graduierten Betti-Zahlen über S. Wie von Conca, Herzog und Hibi in [CHH04] gezeigt, sind die symmetrischen graduierten Betti-Zahlen eines S-Moduls der Form S/I, wobei $I \subseteq S$ ein graduiertes Ideal ist, durch positive Linearkombinationen der symmetrischen generischen Annulatorzahlen nach oben beschränkt. Dabei gilt Gleichheit genau dann, wenn I ein komponentenweise lineares Ideal ist.

Wir übertragen das Konzept der symmetrischen Annulatorzahlen bzgl. einer Sequenz auf die Situation in der äußeren Algebra. Unter Berücksichtigung der Tatsache, dass jedes Element eines E-Moduls ein Nullteiler ist, führen wir äußere Annulatorzahlen bzgl. einer Sequenz von Linearformen ein. Das Ziel ist es, Eigenschaften der symmetrischen Annulatorzahlen auch auf die äußeren Annulatorzahlen zu übertragen. Es stellt sich dabei heraus, dass – wie im symmetrischen Fall – die äußeren Annulatorzahlen bzgl. verschiedener Sequenzen übereinstimmen, wenn letztere aus einer gewissen nicht-leeren Zariski-offenen Menge stammen. Im Folgenden werden daher nur noch die sog. äußeren generischen Annulatorzahlen betrachtet. Analog zu der Situation über dem Polynomring kann gezeigt werden, dass positive Linearkombinationen dieser Zahlen als obere Schranken für die graduierten Cartan-Betti-Zahlen eines E-Moduls der Form E/J dienen. Hierbei ist $J \subseteq E$ ein graduiertes Ideal. Dies liefert insbesondere eine obere Schranke für die gewöhnlichen graduierten Betti-Zahlen über E. Gleichheit wird, wie im symmetrischen Fall, nur für komponentenweise lineare Ideale erreicht.

Neben der bloßen Übertragung der Ergebnisse über dem Polynomring auf die äußere Algebra können für die äußeren generischen Annulatorzahlen noch weitere Resultate erzielt werden. Betrachtet man E-Moduln der Form E/J, so ergeben sich die äußeren generischen Annulatorzahlen als gewisse Anzahlen von Erzeugern des generischen Initialideals von J bzgl. der umgekehrt lexikographischen Ordnung. Speziell für simpliziale Komplexe lässt sich damit zeigen, dass die äußeren generischen Annulatorzahlen bestimmte minimale Erzeuger des symmetrischen bzw. äußeren Stanley-Reisner-Ideals zählen.

Bei der Definition und näheren Betrachtung der generischen Annulatorzahlen – sowohl der symmetrischen als auch der äußeren – stellt sich die Frage, ob diese sich gegenüber den Annulatorzahlen bzgl. einer bestimmten Sequenz in irgendeiner Form auszeichnen. Herzog vermutete, dass sie die minimalen Annulatorzahlen unter allen Annulatorzahlen bzgl. einer beliebigen Sequenz sind. Wir konstruieren sowohl für die Annulatorzahlen über dem Polynomring als auch für die Annulatorzahlen über der äußeren Algebra ein Gegenbeispiel zu dieser Vermutung.

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Contents

Part I

Introduction

1 Basic algebraic definitions and constructions

In this chapter our aim is to present the algebraic background which is needed in the subsequent parts of this thesis. Even though some familiarity with notion from commutative and homological algebra is assumed we try to give a brief but preferably complete exposure of the theory required for the understanding of the following chapters. We also mention references which can be used for a deeper study of certain topics.
Algebraic notions which are not defined in this chapter but which are needed at some point in one of the following chapters are given in the context of use.
The first section of this chapter is devoted to some homological algebra. We first recall the notion of complexes, some theory about the homology and the cohomology of a complex and then go on to considering exact complexes, i.e., resolutions. After having treated both free and injective resolutions we look at special resolutions in more detail – including the Eliahou-Kervaire resolution for stable ideals and the Cartan resolution, which is the exterior analogue of the Koszul resolution.
The second section focuses on the construction of the generic initial ideal over the polynomial ring and its basic properties. We start with a short recapitulation of some background about initial ideals. Without giving the proofs we then state the behavior of certain algebraic invariants when passing from an ideal to its generic initial ideal. We conclude this section with the construction of the generic initial ideal over the exterior algebra. The presentation in this part is rather skarce since most facts can be carried over literally from the situation over the polynomial ring. Our main purpose of this part therefore lies in emphasizing the differences between the situations over the polynomial ring and the exterior algebra.

1.1 Some homological algebra

1.1.1 Free resolutions

Throughout this section let R be a graded k-algebra where k is an arbitrary field. For our purposes we usually have $R = k[x_1, \dots, x_n]$, the polynomial ring in n variables, or $R = k\langle e_1, \dots, e_n\rangle$, the exterior algebra. Since all definitions and constructions are valid over an arbitrary graded algebra and since we will use them for the polynomial ring as well as for the exterior algebra we do not want to restrict ourselves to one of the cases in general. Nevertheless, we want to fix some notation. Throughout this and the following sections let

$S := k[x_1,\dots,x_n]$ denote the polynomial ring in n variables over k and let $E := k\langle e_1,\dots,e_n\rangle$ denote the exterior algebra over V, where V is an n-dimensional k-vector space with basis $e_1,\dots,e_n$.

Definition 1.1.1. A *complex* is a collection of finitely generated R-modules $(F_i)_{i\in\mathbb{Z}}$ and homogeneous R-linear maps $\partial_i : F_i \to F_{i-1}$ of degree 0 such that $\partial_i \circ \partial_{i+1} = 0$ for $i \in \mathbb{Z}$. We denote it by $(\mathscr{F},\partial)$.

To each complex

$$\mathscr{F} : \dots \xrightarrow{\partial_{i+2}} F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} \dots$$

one can assign its *homology groups* $H_i(\mathscr{F}) := \operatorname{Ker}(\partial_i)/\operatorname{Im}(\partial_{i+1})$ for $i \in \mathbb{Z}$.
A complex $(\mathscr{F},\partial)$ is called *exact* if $H_i(\mathscr{F}) = 0$ for all $i \in \mathbb{Z}$, i.e., $\operatorname{Ker}(\partial_i) = \operatorname{Im}(\partial_{i+1})$ for all $i \in \mathbb{Z}$.
Note that the condition $\partial_i \circ \partial_{i+1} = 0$ is equivalent to $\operatorname{Im}(\partial_{i+1}) \subseteq \operatorname{Ker}(\partial_i)$. Thus, for a complex to be exact we only have to require $\operatorname{Im}(\partial_{i+1}) \supseteq \operatorname{Ker}(\partial_i)$ for $i \in \mathbb{Z}$.

Definition 1.1.2. Let M be a finitely generated graded R-module. A complex

$$\mathscr{F} : \dots \to F_i \to F_{i-1} \to \dots \to F_1 \to F_0 \to 0$$

is called a *graded free* (respectively *projective*) *resolution* of M if $H_i(\mathscr{F}) = 0$ for $i > 0$ and $H_0(\mathscr{F}) = M$ and the F_i are free (respectively projective) R-modules.

Note that for the polynomial ring the notion of free and projective resolutions coincide since in this situation every finitely generated projective S-module is a free module, as was shown by Quillen and Suslin [Lam06, Theorem V.2.9]. Since in general it also holds that every projective R-module is free, the finitely generated free S-modules are exactly the finitely generated projective S-modules. Often, we are interested in resolutions featuring special properties.

Definition 1.1.3. A graded free resolution $(\mathscr{F},\partial)$ of a finitely generated graded R-module M is called *minimal* if $\partial_{i+1}(F_{i+1}) \subseteq \mathfrak{m}F_i$ for all $i \geq 0$. Here $\mathfrak{m}$ denotes the graded maximal ideal of R. Equivalently, the matrices of the maps in the resolution do not contain any element of the field k.

It can be shown that up to an isomorphism of complexes the minimal free resolution of a finitely generated R-module M is unique [Eis95, Theorem 20.2]. Since the modules in a minimal free resolution $(\mathscr{F},\partial)$ of a finitely generated R-module are free modules they can be written as a direct sum $F_i = \bigoplus_{j\in\mathbb{Z}} R(-j)^{\beta^R_{i,j}(M)}$. From the uniqueness of the minimal free resolution we get that the numbers $\beta^R_{i,j}(M)$ are uniquely determined. They are called the *graded Betti numbers* of M over R.
The numbers $\beta^R_i(M) := \sum_{j\in\mathbb{Z}} \beta^R_{i,i+j}(M)$ are called the *total Betti numbers* of M over R. It follows directly from the definition that $\beta^R_i(M) = \operatorname{rank}(F_i)$ for $i \in \mathbb{Z}$.

Depending on the range of the shifts of the graded Betti numbers it is customary to distinguish different types of resolutions.

Definition 1.1.4. Let M be a finitely generated graded R-module and let $\beta^R_{i,i+j}(M)$ be the graded Betti numbers of M.

(i) The minimal free resolution of M is called *linear* if there exists $r \in \mathbb{Z}$ such that $\beta^R_{i,i+j}(M) = 0$ for $i \geq 0$ and $j \neq r$. In this case, in order to emphasize where the shift occurs, the resolution is also called *r-linear*.

(ii) M is called *componentwise linear* if for all $j \in \mathbb{Z}$ the module $M_{\langle j \rangle}$ has a linear resolution. Here $M_{\langle j \rangle}$ denotes the submodule of M which is generated by the elements of degree j.

Several invariants of an R-module M are encoded by its minimal graded free resolution and can be directly derived from it.

Definition 1.1.5. Let M be a finitely generated graded R-module.

(i) $\operatorname{proj}\dim_R(M) := \sup\{i \mid \beta^R_{i,j}(M) \neq 0 \text{ for some } j \in \mathbb{Z}\}$ is called the *projective dimension* of M over R.

(ii) $\operatorname{reg}_R(M) = \sup\{j \mid \beta^R_{i,i+j}(M) \neq 0 \text{ for some } i \in \mathbb{Z}\}$ is called the *(Castelnuovo-Mumford) regularity* of M over R.

It is well-known that the minimal free resolution of an E-module is infinite except for the case of a free E-module. Thus, even though it makes sense to define the projective dimension for E-modules this notion is almost meaningless in this case. Note that those modules are included in the definition.
On the contrary, for finitely generated graded S-modules over the polynomial ring, it is a classical result (Hilbert Syzygy Theorem) that the minimal free resolution has length at most n, see e.g., [Eis95, Theorem 1.13]. In this case, we additionally have the well-known Auslander-Buchsbaum formula (Theorem 1.1.6) which relates the projective dimension of an S-module with its depth and which provides an explicit formula for the projective dimension. Although this formula holds in greater generality than for S-modules we only state the special version for S-modules since this one suffices for our purposes.

Theorem 1.1.6. *[Eis95, Theorem 19.9] (Auslander-Buchsbaum)*
Let $S = k[x_1, \ldots, x_n]$ and let M be a finitely generated S-module. Then

$$\operatorname{proj}\dim_S(M) = n - \operatorname{depth}_S(M),$$

where $\operatorname{depth}_S(M)$ *denotes the depth of M over S.*

Although the result is rather standard we have included it since we give an exterior analogue of it in Chapter 5. For more details on free and projective resolutions see e.g., [Eis95] and [BH98].

1.1.2 Cochain complexes and injective resolutions

Dualizing the concept of chain complexes and projective and free resolutions, respectively, we obtain the concept of cochain complexes and injective resolutions. We now give the most important definitions and facts concerning this construction. We mostly restrict ourselves to those notion which differ from the projective situation. For more details on cochain complexes and injective resolutions see e.g., [Eis95] and [BH98].

Definition 1.1.7. Let $(F^i)_{i\in\mathbb{Z}}$ be a family of finitely generated graded R-modules and let $(\partial^i : F^i \to F^{i+1})_{i\in\mathbb{Z}}$ be a family of R-linear homogeneous maps of degree 0. If $\partial^{i+1}\circ\partial^i = 0$ for $i\in\mathbb{Z}$ then $(F^i,\partial^i)_{i\in\mathbb{Z}}$ is called a *cochain complex*.

To each cochain complex

$$\mathscr{F}:\ \dots \xrightarrow{\partial^{i-2}} F^{i-1} \xrightarrow{\partial^{i-1}} F^i \xrightarrow{\partial^i} F^{i+1} \xrightarrow{\partial^{i+1}} \dots$$

one can associate its *cohomology groups* $H^i(\mathscr{F}) := \operatorname{Ker}(\partial^i)/\operatorname{Im}(\partial^{i-1})$ for $i\in\mathbb{Z}$.

The notion of exactness and resolution are defined in exactly the same way as for chain complexes. We therefore skip those definitions but directly state what is understood by an injective resolution.

Definition 1.1.8. Let M be a finitely generated graded R-module. A cochain complex

$$\mathscr{I}:\ 0 \to I^0 \to I^1 \to \dots \to I^i \to I^{i+1} \to \dots$$

is called a graded *injective resolution* of M if $H^i(\mathscr{I}) = 0$ for $i>0$ and $H^0(\mathscr{I}) = M$ and the I^i are injective R-modules.

As for projective resolutions it can be shown that for each R-module M there exists an injective resolution. Crucial in proving this fact is that every R-module can be embedded in an injective R-module [BH98, Theorem 3.1.8]. Furthermore, under all injective resolutions of an R-module one can distinguish the minimal ones. Those are, as for free resolutions, unique up to an isomorphism of complexes, see e.g., [BH98, Proposition 3.2.4]. This gives rise to the definition of the graded Bass numbers of an R-module, which are the dual of the graded Betti numbers. We state the exact definition only for the special case of E-modules. If M is an E-module and $(\mathscr{I},\partial)$ is the minimal injective resolution of M then we can write the injective modules appearing in the resolution as $I^i = \bigoplus_{j\in\mathbb{Z}} E(n-j)^{\mu^E_{i,j}(M)}$ for $i\geq 0$. The numbers $\mu^E_{i,j}(M)$ are uniquely determined and depend only on the module M. They are called the *graded Bass numbers* of M over E, see e.g., [BH98, Chapter 3.2].
As for free resolutions we call the minimal injective resolution of M *linear* if there exists $r\in\mathbb{Z}$ such that $\mu^E_{i,i+j}(M) = 0$ for $i\geq 0$ and $j\neq r$. There exist several equivalent conditions for an injective resolution to be linear. One of those will be important for us in Chapter 5 and will be provided there.

1.1.3 Tor- and Ext-groups

In this section we recall some basic facts concerning the Tor- and the Ext-functor. A more detailed treatment of those functors can be found in [Eis95].

Definition 1.1.9. Let M be an R-module.

(i) The left-derived functor of the functor $M \otimes_R \cdot$ is called $\mathrm{Tor}_i^R(M, \cdot)$.

(ii) The right-derived functor of the functor $\mathrm{Hom}_R(M, \cdot)$ is called $\mathrm{Ext}_i^R(M, \cdot)$.

Let M, N be R-modules. In order to compute $\mathrm{Tor}_i^R(M,N)$ one starts with the minimal free resolution $\mathscr{F}$ of N, applies the functor $M \otimes_R \cdot$ to it and then computes the i-th homology group. It is well-known that $\mathrm{Tor}_i^R(M,N) = H_i(M \otimes_R \mathscr{F})$ for $i \in \mathbb{Z}$.
Similarly, for the computation of $\mathrm{Ext}_R^i(M,N)$ we take the minimal injective resolution $\mathscr{I}$ of N and apply the functor $\mathrm{Hom}_R(M, \cdot)$ to this exact sequence. It is a classical result from homological algebra that the i-th cohomology group of the newly obtained sequence equals $\mathrm{Ext}_R^i(M,N)$, i.e., $\mathrm{Ext}_R^i(M,N) = H^i(\mathrm{Hom}_R(M, \mathscr{I}))$ for $i \in \mathbb{Z}$, see e.g., [Wei94].
We now list some properties of the Tor-functor.

Theorem 1.1.10. *[Eis95] Let M, N be R-modules. Then*

(i) $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$

(ii) $\mathrm{Tor}_i^R(M,N) = 0$ *for* $i > 0$ *if* M *or* N *is free.*

(iii) $\mathrm{Tor}_i^R(M,N) = \mathrm{Tor}_i^R(N,M)$ *for* $i \geq 0$.

(iv) For any short exact sequence of R-modules M, M' *and* M''

$$0 \to M' \to M \to M'' \to 0$$

there is a long exact sequence of Tor*-groups*

$$\begin{aligned} \ldots \to \mathrm{Tor}_i^R(M',N) &\to \mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M'',N) \\ &\to \ldots \to \mathrm{Tor}_1^R(M',N) \to \mathrm{Tor}_1^R(M,N) \to \mathrm{Tor}_1^R(M'',N) \\ &\to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0. \end{aligned}$$

Since the Ext-functor is the dual functor to the Tor-functor, the analogous properties also hold for the Ext-functor. The statement of Theorem 1.1.10 (iii) essentially follows from the commutativity of the tensor product. In particular it tells us that we can use both, the minimal free resolution of M and the minimal free resolution of N, for the computation of $\mathrm{Tor}_i^R(M,N)$. Depending on the particular modules, taking advantage of this fact may facilitate the computation of $\mathrm{Tor}_i^R(M,N)$.

We conclude this part with the relation between the graded Betti and Bass numbers of an R-module M and the Tor- and Ext-groups of M, respectively.

Proposition 1.1.11. *[BH98, Proposition 1.3.1, Proposition 3.2.9] Let M be a finitely generated graded R-module. Let $\beta^R_{i,j}(M)$ denote the graded Betti numbers of M. Then*

(i) $\dim_k \operatorname{Tor}^R_i(M,k)_j = \beta^R_{i,j}(M)$ *for $i \geq 0$ and $j \in \mathbb{Z}$.*

(ii) Let $R := E$ be the exterior algebra and let $\mu^E_{i,j}(M)$ denote the graded Bass numbers of M over E. Then $\dim_k \operatorname{Ext}^i_E(M,k)_j = \mu^E_{i,j}(M)$ *for $i \geq 0$ and $j \in \mathbb{Z}$.*

1.1.4 The Eliahou-Kervaire resolution

In this section we explicitly construct the minimal free resolution for a special class of S-modules. Even though there are several constructions which yield a free resolution of an ideal, e.g., the Taylor resolution for monomial ideals, in general those constructions do not lead to minimal resolutions. However, for special classes of ideals in the polynomial ring as well as in the exterior algebra it is possible to write down an explicit minimal free resolution. From this resolution the graded Betti numbers of the ideal and the corresponding quotient ring can be directly read off. For the class of so-called stable ideals there exists the well-known Eliahou-Kervaire resolution. From this resolution formulas for the graded Betti numbers of the ideal can be obtained. We now give the definition of stable ideals in the polynomial ring.

Definition 1.1.12. Let $I \subseteq S$ be a monomial ideal.

(i) I is called *stable* if for all monomials $u \in I$ and all $i > m := \min\{l \mid x_l \text{ divides } u\}$, one has $x_i \frac{u}{x_m} \in I$.

(ii) I is called *squarefree stable* if for all squarefree monomials $u \in I$ and all $i > m := \min\{l \mid x_l \text{ divides } u\}$ such that x_i does not divide u, one has $x_i \frac{u}{x_m} \in I$.

The same definition carries over literally to monomial ideals in the exterior algebra. But note that over E the notion of stable and squarefree stable ideals coincide since all monomials in E are squarefree.
One class of stable ideals in the polynomial ring are the generic initial ideals, defined in Section 1.2, at least if the field k is of characteristic 0. The same is true for generic initial ideals in the exterior algebra without the additional assumption on the characteristic of the field. We now give the explicit construction of the Eliahou-Kervaire resolution.

Construction 1.1.13. [EK90, PS08]
Let $I \subseteq S$ be a monomial ideal. We denote the minimal system of generators of I by $G(I)$. Note that $G(I)$ is unique and finite. For $j \geq 0$ let $G(I)_j$ denote the set of monomials in $G(I)$ which are of degree j. For a monomial $u \in S$ we write $\max(u)$ and $\min(u)$ for the maximal and the minimal $1 \leq i \leq n$ such that x_i divides u, respectively, i.e.,

$$\max(u) := \max\{i \mid x_i \text{ divides } u\} \text{ and } \min(u) := \min\{i \mid x_i \text{ divides } u\}.$$

Let $u \in I$ be a monomial. It can be shown that u can be uniquely decomposed in the following form:
There exists $v \in G(I)$ and $w \in S$ such that $u = wv$ and $\min(v) \geq \max(w)$.
We denote v by $e(u)$ and call it the *ending segment* of u. For fixed $i \geq 0$ set

$$\mathscr{B}_i := \{(l_1, \ldots, l_i; u) \mid \min(u) < l_1 < \ldots < l_i \leq n,\ u \in G(I)\}$$

and let F_i be the free S-module with basis $\mathscr{B}_i$. The modules F_i are $\mathbb{N}^n$-graded by setting $\deg((l_1, \ldots, l_i; u)) = \sum_{r=1}^{i} \mathbf{e}_{l_r} + \deg(u)$, where $\mathbf{e}_j$ denotes the j-th unit vector in $\mathbb{R}^n$ for $1 \leq j \leq n$ and $\deg(u)$ is the usual $\mathbb{N}^n$-degree of a monomial $u \in S$. Note that the set of generators of F_0 are in bijection with the elements of $G(I)$. This gives an S-module homomorphism $\partial_0 : F_0 \to I$ by mapping $(\emptyset; u)$ to u.
For $i \geq 1$ we define homomorphisms $F_i \to F_{i-1}$ of graded S-modules by

$$\gamma_i((l_1, \ldots, l_i; u)) := \sum_{r=1}^{i} (-1)^r x_{l_r} (l_1, \ldots, \hat{l}_r, \ldots, l_i; u)$$
$$\vartheta_i((l_1, \ldots, l_i; u)) := \sum_{r=1}^{i} (-1)^r \frac{u x_{l_r}}{e(u x_{l_r})} (l_1, \ldots, \hat{l}_r, \ldots, l_i; e(u x_{l_r})).$$

Here $(l_1, \ldots, \hat{l}_r, \ldots, l_i)$ denotes the sequence in which l_r has been omitted. We set $\partial_i := \gamma_i + \vartheta_i : F_i \to F_{i-1}$.

In [EK90] it is shown that $\mathscr{F} = (F_i, \partial_i)$ is a complex of $\mathbb{N}^n$-graded S-modules. Furthermore, Eliahou and Kervaire proved that under certain conditions on I this complex is indeed a minimal resolution of I.

Theorem 1.1.14. *[EK90] Let $I \subseteq S$ be a stable monomial ideal. Then the complex $\mathscr{F} := (F_i, \partial_i)$ is a minimal free $\mathbb{N}^n$-graded resolution of I over S. This resolution is called the* Eliahou-Kervaire resolution *of I.*

From the Eliahou-Kervaire resolution one can easily derive formulas for the graded Betti numbers of a stable ideal. For a monomial $u \in G(I)$, where I is a stable ideal, we have $\binom{n-\min(u)}{i}$ possibilities to choose a sequence $(l_1, \ldots, l_i)$ such that $(l_1, \ldots, l_i; u) \in \mathscr{B}_i$. Taking into account that $(l_1, \ldots, l_i; u)$ has $\mathbb{N}$-degree $\deg_{\mathbb{N}}(u) + i$ where $\deg_{\mathbb{N}}(u)$ means the total degree of u we get the following formula

Corollary 1.1.15. *Let $I \subseteq S$ be a stable ideal. Let $\beta^S_{i,i+j}(I)$ be the $\mathbb{N}$-graded Betti numbers of I over S. Then*

$$\beta^S_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{n - \min(u)}{i}.$$

Aramova, Herzog and Hibi considered in [AHH98] the class of squarefree stable ideals. They showed that a construction similar to the Eliahou-Kervaire resolution works for this class of ideals. Using their construction one obtains formulas for the graded Betti numbers of squarefree stable ideals.

Corollary 1.1.16. *[AHH98, Corollary 2.3] Let $I \subseteq S$ be a squarefree stable ideal and let $\beta^S_{i,i+j}(I)$ be the $\mathbb{N}$-graded Betti numbers of I over S. Then*

$$\beta^S_{i,i+j}(I) = \sum_{u \in G(I)_j} \binom{n - \min(u) - j + 1}{i}.$$

The construction of Aramova, Herzog and Hibi can also be applied to (squarefree) stable ideals in the exterior algebra. Thus, for those ideals the graded Betti numbers over E are given by the formulas in Corollary 1.1.16.

1.1.5 The Cartan complex

In this section we describe the construction of the Cartan complex. It is a very helpful complex over the exterior algebra $E = k\langle e_1, \ldots, e_n \rangle$ which plays a similar role as the Koszul complex for the polynomial ring. A detailed description of the Cartan complex and its properties can be found in [AH00] and [HH08]. For a sequence $\mathbf{v} := v_1, \ldots, v_m \in E$ of elements of degree 1 let $C_{\bullet}(\mathbf{v}; E) := C_{\bullet}(v_1, \ldots, v_m; E)$ be the free divided power algebra $E\langle x_1, \ldots, x_m \rangle$. It is generated by the divided powers $x_i^{(j)}$ for $1 \leq i \leq m$ and $j \geq 0$ which satisfy the relations $x_i^{(j)} x_i^{(k)} = ((j+k)!/(j!k!)) x_i^{(j+k)}$. Thus $C_i(\mathbf{v}; E)$ is a free E-module with basis $x^{(a)} = x_1^{(a_1)} \cdot \ldots \cdot x_m^{(a_m)}$, $a \in \mathbb{N}^m$, $|a| = i$. The E-linear differential on $C_{\bullet}(v_1, \ldots, v_m; E)$ is

$$\begin{aligned} \partial_i : C_i(v_1, \ldots, v_m; E) &\longrightarrow C_{i-1}(v_1, \ldots, v_m; E) \\ x^{(a)} &\mapsto \sum_{a_j > 0} v_j x_1^{(a_1)} \cdot \ldots \cdot x_j^{(a_j - 1)} \cdot \ldots \cdot x_m^{(a_m)}. \end{aligned}$$

Direct computation shows that $\partial_i \circ \partial_{i+1} = 0$. Therefore the above construction indeed yields a complex.
Let $\mathscr{M}$ be the category of finitely generated graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for homogeneous elements $a \in E$, $m \in M$. For example, if $J \subseteq E$ is a graded ideal, then E/J belongs to $\mathscr{M}$.

Definition 1.1.17. Let $M \in \mathscr{M}$. The complex

$$C_{\bullet}(\mathbf{v}; M) := C_{\bullet}(\mathbf{v}; E) \otimes_E M$$

is called the *Cartan complex* of $\mathbf{v}$ with values in M. The corresponding homology modules

$$H_i(\mathbf{v}; M) := H_i(C_{\bullet}(\mathbf{v}; M))$$

are called the *Cartan homology* of $\mathbf{v}$ with values in M.

Cartan homology can be computed inductively as there is a long exact sequence connecting the homologies of $v_1, \ldots, v_j$ and $v_1, \ldots, v_j, v_{j+1}$.

Proposition 1.1.18. *[AH00, Propositions 4.1, 4.3] Let $M \in \mathscr{M}$ and let $\mathbf{v} = v_1, \dots, v_m \in E_1$. For all $1 \leq j \leq m$ there exists a long exact sequence of graded E-modules*

$$\dots \longrightarrow H_i(v_1,\dots,v_j;M) \longrightarrow H_i(v_1,\dots,v_{j+1};M) \longrightarrow H_{i-1}(v_1,\dots,v_{j+1};M)(-1)$$

$$\longrightarrow H_{i-1}(v_1,\dots,v_j;M) \longrightarrow H_{i-1}(v_1,\dots,v_{j+1};M) \longrightarrow \dots.$$

Setting $\deg(x_i) = 1$ induces a grading on the complex and its homologies.
It is well-known that the Cartan complex $C_{\bullet}(v_1,\dots,v_m;E)$ with values in E is exact if the linear forms $v_1,\dots,v_m$ are k-linearly independent. Hence, analogously to the results for the Koszul complex over S, see e.g., [AH00] and [BH98], one gets the following result.

Theorem 1.1.19. *Let $\mathbf{v} = v_1,\dots,v_m$ be a sequence of linear forms in E whose elements are k-linearly independent. Then the Cartan complex $C_{\bullet}(v_1,\dots,v_m;E)$ is the minimal graded free resolution of $H_0(v_1,\dots,v_m;E) = E/(v_1,\dots,v_m)$ over E.*

From the above theorem we immediately deduce that the Cartan complex can be used to compute $\operatorname{Tor}_i^E(E/(v_1,\dots,v_m),\cdot)$.

Proposition 1.1.20. *[AHH97, Theorem 2.2] Let $M \in \mathscr{M}$ and let $\mathbf{v} = v_1,\dots,v_m \in E_1$ be linearly independent over k. There are isomorphisms of graded E-modules*

$$\operatorname{Tor}_i^E(E/(v_1,\dots,v_m),M) \cong H_i(\mathbf{v};M) \qquad \text{for all } i \geq 0.$$

Nagel, Römer and Vinai defined in [NRV08] the so-called *Cartan-Betti numbers* which measure the homology of the Cartan complex with respect to a sequence of linear forms $v_1,\dots,v_n$.

Definition 1.1.21. Let $J \subseteq E$ be a graded ideal and let $v_1,\dots,v_n$ be a basis of E_1. We set

$$h_{i,j}(r)(v_1,\dots,v_n;E/J) := \dim_k H_i(v_1,\dots,v_r;E/J)_j,$$

where $H_i(v_1,\dots,v_r;E/J)$ denotes the i-th Cartan homology.

Nagel, Römer and Vinai remarked that there exists a non-empty Zariki-open set W such that when choosing a basis of E_1 from this set the $h_{i,j}$ are constant on it. Therefore they make the following definition.

Definition 1.1.22. Let $J \subseteq E$ be a graded ideal and let $v_1,\dots,v_n$ be a basis of E_1. We set

$$h_{i,j}(r)(E/J) := h_{i,j}(r)(v_1,\dots,v_n;E/J)$$

for $(v_1,\dots,v_n) \in W$ as above and call these numbers the *Cartan-Betti numbers* of E/J.

For $r = n$, we obtain from Proposition 1.1.20 that the Cartan-Betti numbers of E/J are the usual exterior graded Betti numbers of E/J, i.e., $h_{i,j}(n)(E/J) = \beta_{i,j}^E(E/J)$.

1.2 The generic initial ideal

In this section we study the generic initial ideal of an ideal in S and E, respectively, and its basic properties. In order to give the definition of the generic initial ideal we first need to recall some background about initial ideals. We work over S and it is only in the last part of this section where we transfer the main definitions to the situation in the exterior algebra. For more details on generic initial ideals see e.g., [HH08].
For $\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we denote by x^α the monomial $x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ in S. Let $\mathscr{M}(S)$ denote the set of monomials of S.

1.2.1 The basic construction

Definition 1.2.1. A *monomial order* or *term order* on S is a total order $\prec$ on $\mathscr{M}(S)$ such that the following two conditions are satisfied:

(i) $1 \prec x^\alpha$ for all $1 \neq x^\alpha \in \mathscr{M}(S)$.

(ii) If $x^\alpha, x^\beta \in \mathscr{M}(S)$ and $x^\alpha \prec x^\beta$ then $x^\alpha x^\gamma \prec x^\beta x^\gamma$ for all $x^\gamma \in \mathscr{M}(S)$.

Term orders which are considered in this thesis are the lexicographic and the reverse lexicographic order.

Example 1.2.2. (i) The *degree lexicographic order* with respect to $x_1 < \ldots < x_n$ is defined in the following way. We set $x^\alpha <_{\text{lex}} x^\beta$ if either $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, or $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and the left-most non-zero component of $\alpha - \beta$ is positive.

(ii) The *degree reverse lexicographic order* with respect to $x_1 < \ldots < x_n$ is defined in the following way. We set $x^\alpha <_{\text{rlex}} x^\beta$ if either $\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$, or $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ and the right-most non-zero component of $\alpha - \beta$ is negative.

Having defined a term order on the set of monomials of S we can define the so-called initial ideal of an ideal with respect to this particular term order.

Definition 1.2.3. Let $\prec$ be a term order on S.

(i) For a polynomial $f := \sum_{\alpha \in \mathbb{N}^n} a_\alpha x^\alpha \in S$ with $a_\alpha \in k$ we set $\text{in}_\prec(f) := \max\{x^\alpha \mid a_\alpha \neq 0\}$ and call $\text{in}_\prec(f)$ the *initial monomial* of f, i.e., $\text{in}_\prec(f)$ is the biggest monomial appearing in f with non-zero coefficient with respect to $\prec$.

(ii) Let $I \subseteq S$ be an ideal. Then $\text{in}_\prec(I) := (\text{in}_\prec(f) \mid f \in I)$ is called the *initial ideal* of I with respect to $\prec$.

It is an interesting issue to ask how several algebraic invariants behave under the passage to the initial ideal. Since in Chapter 5 our proofs use some of these behaviors, we summarize the relations between the most common invariants in the following theorem.

Theorem 1.2.4. *[HH08, Theorem 3.3.4] Let $I \subseteq S$ be a graded ideal and let $\prec$ be a term order on S. Let further $\mathrm{Hilb}(S/I,t)$ and $\mathrm{Hilb}(S/\operatorname{in}_\prec(I),t)$ denote the Hilbert series of S/I and $S/\operatorname{in}_\prec(I)$, respectively. Then*

(i) $\mathrm{Hilb}(S/I,t) = \mathrm{Hilb}(S/\operatorname{in}_\prec(I),t)$,

(ii) $\dim(S/I) = \dim(S/\operatorname{in}_\prec(I))$,

(iii) $\operatorname{proj}\dim(S/I) \leq \operatorname{proj}\dim(S/\operatorname{in}_\prec(I))$,

(iv) $\operatorname{reg}_S(S/i) \leq \operatorname{reg}_S(S/\operatorname{in}_\prec(I))$,

(v) $\operatorname{depth}_S(S/I) \geq \operatorname{depth}_S(S/\operatorname{in}_\prec(I))$.

We do not give the proof of the above theorem but we want to emphasize that in order to show (iii), (iv) and (v) it is crucial to know that the Betti numbers of I over S can only increase when passing to the initial ideal of I, i.e., $\beta^S_{i,i+j}(I) \leq \beta^S_{i,i+j}(\operatorname{in}_\prec(I))$. For more details see for example [HH08].

Let now $GL_n(k)$ be the general linear group of k^n, i.e., $GL_n(k) := \{A \in k^{n\times n} \mid \det(A) \neq 0\}$. For $g = (\gamma_{i,j})_{1\leq i,j\leq n} \in GL_n(k)$ we get an automorphism $g\colon S \to S$ which is induced by $g(x_j) = \sum_{i=1}^n \gamma_{i,j}x_i$ for $1 \leq j \leq n$. The result which justifies the definition of the so-called generic initial ideal of an ideal is the following.

Theorem 1.2.5. *[HH08, Theorem 4.1.1] Let $I \subseteq S$ be a graded ideal and let $\prec$ be a term order on S. Then there exists a non-empty Zariski-open subset $U \subseteq GL_n(k)$ such that $\operatorname{in}_\prec(g(I)) = \operatorname{in}_\prec(g'(I))$ for all $g, g' \in U$.*

Now the following definition makes sense.

Definition 1.2.6. The ideal $\operatorname{in}_\prec(g(I))$ with $g \in U$ (as in Theorem 1.2.5) is called the *generic initial ideal* of I with respect to $\prec$. It is denoted by $\operatorname{gin}_\prec(I)$.

1.2.2 Main properties

Even though their computation is rather elaborate generic initial ideals turn out to have nice properties. One of the most important properties certainly is that they are stable under the action of the Borel subgroup. The *Borel subgroup $\mathscr{B}$* of $GL_n(k)$ is the subgroup of all invertible upper triangular matrices. An ideal $I \subseteq S$ is called *Borel-fixed* if it is fixed under the action of $\mathscr{B}$, i.e., $a(I) = I$ for all $a \in \mathscr{B}$. The following result is due to Galligo and Bayer/Stillman, respectively.

Theorem 1.2.7. *[HH08, Theorem 4.2.1] (Galligo, Bayer-Stillman)*
Let $I \subseteq S$ be a graded ideal and let $\prec$ be a term order on S. Then $\operatorname{gin}_\prec(I)$ is Borel-fixed.

If we work over a field k of characteristic 0, the property of being Borel-fixed can be characterized in a way which makes it easier to check if an ideal is Borel-fixed.

Definition 1.2.8. Let $I \subseteq S$ be a monomial ideal, i.e., I is generated by monomials. Then I is called *strongly stable* if for all monomials $u \in I$ and all $i > j$ such that x_j divides u, one has $x_i \frac{u}{x_j} \in I$.

Note that if an ideal $I \subseteq S$ is strongly stable then it is in particular stable. In general, being Borel-fixed is a weaker condition than being strongly stable. The following proposition gives the exact relation between those two properties and tells us some consequences and further results for the generic initial ideal.

Proposition 1.2.9. *[HH08, Proposition 4.2.3, Proposition 4.2.5] Let $I \subseteq S$ be a graded ideal and let $\prec$ be a term order on S. Then the following holds.*

(i) Let I be strongly stable. Then I is Borel-fixed.

(ii) Let $\mathrm{char}(k) = 0$ *or* $|k| = infty$*. If I is Borel-fixed then it is strongly stable. In particular,* $\mathrm{gin}_\prec(I)$ *is strongly stable if* $\mathrm{char}(k) = 0$ *or* $|k| = \infty$*.*

(iii) (Conca) $\mathrm{gin}_\prec(I) = I$ *if and only if I is Borel-fixed. In particular,* $\mathrm{gin}_\prec(\mathrm{gin}_\prec(I)) = \mathrm{gin}_\prec(I)$.

1.2.3 Algebraic invariants of the generic initial ideal with respect to the reverse lexicographic order

Note that for an ideal $I \subseteq S$ and $g = (\gamma_{i,j}) \in GL_n(k)$ it holds that I and $g(I)$ are isomorphic as S-modules. Therefore I and $g(I)$ share several common invariants, e.g., the Hilbert series, the Krull dimension, the projective dimension, the regularity and the depth. From this observation we get that Theorem 1.2.4 carries over verbatim to the generic initial ideal of I. However, if we consider the generic initial ideal with respect to the reverse lexicographic order it turns out that those generic initial ideals behave particularly nicely. The following classical result which is due to Bayer and Stillman describes this behavior exactly.

Theorem 1.2.10. *[BS87] (Bayer, Stillman)*
Let $I \subseteq S$ be a graded ideal. Let $<_{\mathrm{rlex}}$ denote the reverse lexicographic order. Then

(i) $\mathrm{proj\,dim}(S/I) = \mathrm{proj\,dim}(S/\mathrm{gin}_{<_{\mathrm{rlex}}}(I))$,

(ii) $\mathrm{depth}_S(S/I) = \mathrm{depth}_S(S/\mathrm{gin}_{<_{\mathrm{rlex}}}(I))$,

(iii) $\mathrm{reg}_S(S/I) = \mathrm{reg}_S(S/\mathrm{gin}_{<_{\mathrm{rlex}}}(I))$,

(iv) S/I is Cohen-Macaulay if and only if $S/\mathrm{gin}_{<_{\mathrm{rlex}}}(I)$ is Cohen-Macaulay.

For arbitrary (generic) initial ideals the full statement of Theorem 1.2.10 (iv) does not hold. But in either case, independent of the chosen term order, it is true that if $S/\mathrm{in}_\prec(I)$ is Cohen-Macaulay then S/I is Cohen-Macaulay also, e.g., see [HH08, Corollary 3.3.5]. Recall that an S-module M is Cohen-Macaulay if and only if $\dim_S(M) = \mathrm{depth}_S(M)$.

1.2.4 The generic initial ideal over the exterior algebra

We conclude this chapter with some facts showing that the generic initial ideal can also be defined for ideals in the exterior algebra and that those ideals behave even more nicely than their counterparts in the polynomial ring.
As usual let $E := k\langle e_1,\dots,e_n\rangle$ denote the exterior algebra over V. For $F = \{i_1,\dots,i_r\} \subseteq [n]$ we set $e_F := e_{i_1} \wedge \dots \wedge e_{i_r}$. We usually assume that $1 \leq i_1 < \dots < i_r \leq n$. The elements e_F are called *monomials* in E. Now the notion of term order, initial monomial and initial ideal over the exterior algebra can be introduced verbatim as over the polynomial ring. Furthermore, the same proof as over the polynomial ring shows the following.

Theorem 1.2.11. *[HH08, Theorem 5.2.8] Let $J \subseteq E$ be a graded ideal and let $\prec$ be a term order on E. Then there exists a non-empty Zariski-open subset $U \subseteq GL_n(k)$ such that* $\mathrm{in}_{\prec}(g(J)) = \mathrm{in}_{\prec}(g'(J))$ *for all $g, g' \in U$. Furthermore, for $g \in U$ the ideal* $\mathrm{in}_{\prec}(g(J))$ *is Borel-fixed.*

The ideal $\mathrm{in}_{\prec}(g(J))$ for $g \in U$ (as above) is called the *generic initial ideal* of J. It is denoted by $\mathrm{gin}_{\prec}(I)$. Similar to the situation over the polynomial ring it is possible to characterize the property of being Borel-fixed for generic initial ideals over the exterior algebra. However, although over the polynomial ring in order to get a characterization which is easy to handle we need to require that the characteristic of the field is 0, over the exterior algebra this is not necessary.

Proposition 1.2.12. *[HH08, Proposition 5.2.10] Let $J \subseteq E$ be a graded ideal and let $\prec$ be a term order on E. Then* $\mathrm{gin}_{\prec}(J)$ *is strongly stable.*

2 Simplicial complexes

Throughout this chapter we study basic properties of simplicial complexes. In the first section we give the basic definitions and recall some background. Besides some combinatorial invariants as the f- and the h-vector of a simplicial complex, we introduce the Stanley-Reisner ring (the symmetric as well as the exterior one) of a simplicial complex. Many combinatorial invariants of a simplicial complex are encoded in the ring-theoretic invariants of its Stanley-Reisner ring.
The second part of this chapter treats different classes of simplicial complexes such as Cohen-Macaulay simplicial complexes and shellable simplicial complexes. We mainly stick to the definitions or equivalent characterizations and do not get into too much detail.
The last part of this chapter finally focuses on operations and constructions which can be performed on simplicial complexes. We consider such simple constructions as the join of several complexes or the link of a face, but also more elaborate ones like the barycentric subdivision of a simplicial complex and the exterior shifting. The latter ones are studied in more detail accounting for their more complicated structure and their importance for the remaining parts of this thesis.
A presentation of the most important notion and facts concerning simplicial complex can be found in [Sta96].

2.1 Simplicial complexes – the basic definition

Definition 2.1.1. A *simplicial complex* Δ on vertex set Ω is a collection of subsets of Ω such that the following properties hold:

(i) $\emptyset \in \Delta$

(ii) If $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

Throughout this thesis we always assume that if Δ is a simplicial complex on vertex set Ω then it holds that $\{v\} \in \Delta$ for all $v \in \Omega$. As vertex set we usually take the set $[n] := \{1, \ldots, n\}$ for some positive integer n. The elements of a simplicial complex Δ are called *faces* and the *dimension* of a face $F \in \Delta$ is defined to be the cardinality of the face minus one, i.e., $\dim(F) = |F| - 1$. Having defined the dimension of a face of Δ, the *dimension* of the simplicial complex itself is defined as

$$\dim(\Delta) := \max\{\dim(F) \mid F \in \Delta\},$$

i.e., the dimension of Δ is the maximal dimension of all of its faces.
The 0-dimensional faces of a simplicial complex Δ are called *vertices* and 1-dimensional faces are called *edges*. Moreover, faces F of Δ which are inclusionwise maximal are called *facets* of Δ. Simplicial complexes whose facets are all of the same dimension are called *pure*.

Example 2.1.2. (i) Let $\Delta^{(n)} := 2^{[n]}$ be the set of subsets of $[n]$. Then $\Delta^{(n)}$ is called the $(n-1)$-*simplex*. $\Delta^{(n)}$ is pure and of dimension $n-1$.

(ii) Let Δ be the simplicial complex having facets $\{1,2,3\}$ and $\{3,4,5\}$ and an additional vertex $\{6\}$. Δ is a non-pure 2-dimensional simplicial complex.

Let $\mathbf{e}_1,\dots,\mathbf{e}_n$ denote the standard basis vectors in $\mathbb{R}^n$. To a set $F \subseteq [n]$ we can associate the following so-called *geometric simplex* $|F| := \mathrm{conv}\{\mathbf{e}_i \mid i \in F\}$, where $\mathrm{conv}(S)$ denotes the convex hull of a set $S \subseteq \mathbb{R}^n$. Applying this construction to an arbitrary simplicial complex Δ on vertex set $[n]$ we can define $|\Delta| := \bigcup_{F\in\Delta} |F|$. We call $|\Delta|$ a *geometric realization* of Δ. There are several ways to construct a geometric realization of a simplicial complex. However, an important fact to know is that different geometric realizations of a simplicial complex are homeomorphic and they are combinatorially equivalent to the original simplicial complex. For more details on the geometric realization of simplicial complexes, see e.g., [Sta96].
One classical problem in combinatorics is the classification of simplicial complexes in terms of their face numbers, i.e., one wants to know in order to be a simplicial complex what restrictions and conditions are there on the number of faces. For this aim one considers the so-called *f-vector* $f^\Delta = (f^\Delta_{-1}, f^\Delta_0, \dots, f^\Delta_{\dim(\Delta)})$ of a simplicial complex Δ, where f^Δ_i counts the number of i-dimensional faces of Δ, i.e.,

$$f^\Delta_i := |\{F \in \Delta \mid \dim(F) = i\}|$$

for $-1 \le i \le \dim(\Delta)$. In particular, $f^\Delta_{-1} = 1$ counts the empty set and $f^\Delta_0 = |\Omega|$ by the assumption that each $v \in \Omega$ is a vertex of Δ. The polynomial $f^\Delta(t) := \sum_{i=0}^{\dim(\Delta)+1} f^\Delta_{i-1} t^{\dim(\Delta)+1-i}$ is referred to as the *f-polynomial* of Δ. Kruskal and Katona [Kru60, Kat68] determined all possible vectors that can occur as f-vectors of simplicial complexes. However it is still of particular interest to determine the possible f-vectors for special classes of simplicial complexes. We will treat this issue in more detail in Chapters 3 and 4.
Instead of looking at the f-vector of a simplicial complex Δ it is often more convenient to consider the so-called h-vector of Δ. The vector $h^\Delta = (h^\Delta_0, h^\Delta_1, \dots, h^\Delta_{\dim(\Delta)+1})$ defined by

$$\sum_{i=0}^{\dim(\Delta)+1} h^\Delta_i t^i = \sum_{i=0}^{\dim(\Delta)+1} f^\Delta_{i-1} t^i (1-t)^{\dim(\Delta)+1-i} \tag{2.1}$$

is referred to as the *h-vector* of the simplicial complex Δ and the polynomial $h^\Delta(t) := \sum_{i=0}^{\dim(\Delta)+1} h^\Delta_i t^{\dim(\Delta)+1-i}$ is called the *h-polynomial* of Δ. By straightforward computation it follows from Equation 2.1 that the f- and the h-polynomial of a simplicial complex Δ are related by $f^{\Delta}(t-1) = h^{\Delta}(t)$. In particular, the f- and the h-vectors of a simplicial complex Δ carry the same combinatorial information about Δ, and the f-vector of Δ can be obtained from the knowledge of the h-vector of Δ and vice versa.

Lemma 2.1.3. *[BH98, Lemma 5.1.8, Corollary 5.1.9] Let Δ be a $(d-1)$-dimensional simplicial complex with f-vector $f^{\Delta} = (f^{\Delta}_{-1}, f^{\Delta}_{0}, \dots, f^{\Delta}_{d-1})$ and h-vector $h^{\Delta} = (h^{\Delta}_0, h^{\Delta}_1, \dots, h^{\Delta}_d)$. Then,*

$$h^{\Delta}_j = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f^{\Delta}_{i-1}$$

and

$$f^{\Delta}_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h^{\Delta}_i.$$

In particular, $h^{\Delta}_0 = 1$, $h^{\Delta}_1 = f^{\Delta}_0 - d$ and $\sum_{i=0}^{d} h^{\Delta}_i = f^{\Delta}_{d-1}$.

Example 2.1.4. (i) The f-vector of the $(n-1)$-simplex $\Delta^{(n)}$ is the vector $f^{\Delta^{(n)}} = (f^{\Delta^{(n)}}_0, f^{\Delta^{(n)}}_1, \dots, f^{\Delta^{(n)}}_{n-1})$, where $f^{\Delta^{(n)}}_i = \binom{n}{i+1}$ counts all $(i+1)$-element subsets of $[n]$. The h-vector of $\Delta^{(n)}$ is $h^{\Delta^{(n)}} = (1, 0 \dots, 0)$.

(ii) Consider the simplicial complex Δ specified in Example 2.1.2 (ii). Its f-vector is $f^{\Delta} = (1, 6, 6, 2)$. As h-vector we obtain $h^{\Delta} = (1, 3, -3, 1)$. Observe that this example shows in particular that the entries of the h-vector of a simplicial complex need not be positive.

Many combinatorial invariants of a simplicial complex Δ are encoded in the *Stanley-Reisner ring* or *face ring* of Δ which can be defined over the polynomial ring as well as over the exterior algebra.

Definition 2.1.5. Let Δ be a simplicial complex on vertex set $[n]$. The *Stanley-Reisner ideal* of Δ is the ideal $I_{\Delta} \subsetneq k[x_1, \dots, x_n]$ generated by the monomials $x_F := \prod_{i \in F} x_i$, where $F \subseteq [n]$ and $F \notin \Delta$.
Similarly, the ideal $J_{\Delta} = (e_F \mid F \notin \Delta) \subsetneq k\langle x_1, \dots, x_n \rangle$ is called the *exterior Stanley-Reisner ideal* of Δ.
The standard graded k-algebras $k[\Delta] = k[x_1, \dots, x_n]/I_{\Delta}$ and $k\{\Delta\} = k\langle x_1, \dots, x_n \rangle / J_{\Delta}$ are referred to as the *(symmetric)* and *exterior*, respectively, *Stanley-Reisner ring* (or *face ring*).

Note that neither the symmetric nor the exterior Stanley-Reisner ideal contain any variable since we assume that $\{j\} \in \Delta$ for all $j \in [n]$.

Example 2.1.6. (i) The $(n-1)$-simplex does not feature any non-face. Thus $k[\Delta^{(n)}] = k[x_1, \dots, x_n]$, the usual polynomial ring in n variables.

(ii) The minimal non-faces of the simplicial complex defined in Example 2.1.2 (ii) are $\{1,6\}$, $\{2,6\}$, $\{3,6\}$, $\{4,6\}$, $\{5,6\}$, $\{1,4\}$, $\{1,5\}$, $\{2,4\}$ and $\{2,5\}$. Thus, the Stanley-Reisner ideal of Δ is $I_\Delta = (x_1x_6, x_2x_6, x_3x_6, x_4x_6, x_5x_6, x_1x_4, x_1x_5, x_2x_4, x_2x_5)$. As Stanley-Reisner ring of Δ we therefore obtain $k[\Delta] = k[x_1,x_2,x_3,x_4,x_5,x_6]/I_\Delta$.

In the following we recall a well-known result which clarifies the relation between the f- and the h-vectors of a simplicial complex and its (symmetric) Stanley-Reisner ring. In fact, the following holds

Theorem 2.1.7. *[BH98, Theorem 5.1.7] Let Δ be a $(d-1)$-dimensional simplicial complex and let $f^\Delta = (f^\Delta_{-1}, f^\Delta_0, \ldots, f^\Delta_{d-1})$ denote its f-vector. Let further* $\mathrm{Hilb}(k[\Delta],t)$ *denote the Hilbert series of $k[\Delta]$. Then*

$$\mathrm{Hilb}(k[\Delta],t) = \sum_{i=-1}^{d-1} \frac{f_i^\Delta t^{i+1}}{(1-t)^{i+1}}. \tag{2.2}$$

Using Equation 2.1 and bringing down Equation 2.2 to the denominator $(1-t)^d$ yields the desired relation between the h-vector of Δ and its Stanley-Reisner ring. More precisely,

$$\mathrm{Hilb}(k[\Delta],t) = \frac{h_0^\Delta + h_1^\Delta t + \ldots + h_d^\Delta t^d}{(1-t)^d}.$$

Thus, on one hand it is possible to construct the Stanley-Reisner ring of a simplicial complex just from the knowledge of the h-vector. On the other hand the h-vector of a simplicial complex is uniquely determined by its Stanley-Reisner ring.

2.2 Classes of simplicial complexes

There exist a lot of different classes of simplicial complexes. In this section we restrict ourselves to the treatment of Cohen-Macaulay complexes and shellable simplicial complexes. We do not only give the definitions but also state equivalent conditions and how these properties relate to each other. Our main results of Chapter 4 do hold for exactly those two classes of simplicial complexes and therefore those two classes must be treated in more detail than any other class appearing somewhere in this thesis.

2.2.1 Cohen-Macaulay complexes

We first give the definition of Cohen-Macaulay complexes and then give two topological characterizations which enable us to check more easily whether or not a simplicial complex is Cohen-Macaulay.

Definition 2.2.1. Let Δ be a simplicial complex on vertex set $[n]$ and let k be an arbitrary field. Then Δ is called a *Cohen-Macaulay complex* over k if its Stanley-Reisner ring $k[\Delta]$ is

Cohen-Macaulay over k, i.e., $\dim(k[\Delta]) = \operatorname{depth}(k[\Delta])$. Here, $\dim(k[\Delta])$ denotes the Krull dimension of $k[\Delta]$.
The simplicial complex Δ is called a *Cohen-Macaulay complex* if Δ is Cohen-Macaulay over some field k.

From the pure definition it is often rather difficult to tell whether a simplicial complex is Cohen-Macaulay or not. However, it turns out that being Cohen-Macaulay is a purely topological property which does only depend on the geometric realization of the simplicial complex. More precisely, we have the following characterizations due to Reisner and Munkres/Stanley, respectively.

Theorem 2.2.2. *[BH98, Corollary 5.3.9, Corollary 5.4.6] Let Δ be a simplicial complex on vertex set $[n]$ and let X be the geometric realization of Δ. For $F \in \Delta$ let further $\mathrm{lk}_\Delta(F) := \{G \in \Delta \mid G \cup F \in \Delta,\ G \cap F = \emptyset\}$ denote the* link *of F in Δ. Then the following conditions are equivalent:*

(i) Δ is Cohen-Macaulay over k.

(ii) (Reisner) $\widetilde{H}_i(\mathrm{lk}_\Delta(F);k) = 0$ for all $F \in \Delta$ and all $i < \dim(\mathrm{lk}_\Delta(F))$.

(iii) (Munkres, Stanley) $\widetilde{H}_i(X;k) = H_i(X, X \setminus \{p\};k) = 0$ for all $p \in X$ and all $i < \dim X$.

Note that the above criteria imply that the Cohen-Macaulay property might indeed depend on the particular field, i.e., its characteristic. Using Reisner's characterization one can show that every Cohen-Macaulay simplicial complex is pure [BH98, Corollary 5.1.5].

Example 2.2.3. (i) The $(n-1)$-simplex is a Cohen-Macaulay complex. All links of faces of $\Delta^{(n)}$ are lower dimensional simplices. Therefore they are simplicial balls, thus having homology 0 in each dimension.

(ii) Consider the complex Δ having facets $\{1,2,3\}$ and $\{3,4,5\}$.
Then $\mathrm{lk}_\Delta(\{3\}) = \{\{1,2\},\{4,5\},\{1\},\{2\},\{4\},\{5\},\emptyset\}$ is the simplicial complex having the two isolated edges $\{1,2\}$ and $\{4,5\}$ as facets. This complex is disconnected and therefore it holds that $\widetilde{H}_0(\mathrm{lk}_\Delta(\{3\});k) = k$, which contradicts Reisner's criterion. This shows that Δ is not Cohen-Macaulay.

(iii) Let Δ be a triangulation of the projective plane $\mathbb{P}^2$. Then we have

$$\widetilde{H}_1(\Delta;k) = \begin{cases} k, & \text{if } \mathrm{char}(k) = 2 \\ 0, & \text{otherwise} \end{cases}$$

Note that $\Delta = \mathrm{lk}_\Delta(\emptyset)$. This shows that Δ is not Cohen-Macaulay over k if k is a field of characteristic 2. Using further arguments it can be shown that Δ is indeed Cohen-Macaulay over k if $\mathrm{char}(k) \neq 2$. Thus the triangulation of the projective plane provides an example of the dependency of the Cohen-Macaulay property on the characteristic of the field.

2.2.2 Shellable complexes

In this section we consider shellable simplicial complexes and their basic properties. Shellability is a property which is stronger than being Cohen-Macaulay and which does not depend on the characteristic of the field k.

Definition 2.2.4. Let Δ be a pure simplicial complex. Then Δ is called *shellable* if one of the following equivalent conditions is satisfied. There exists a linear order $F_1,\dots,F_m$ of the facets of Δ such that:

(i) $\langle F_i\rangle \cap \langle F_1,\dots,F_{i-1}\rangle$ is generated by a non-empty set of maximal proper faces of $\langle F_i\rangle$ for all $2 \le i \le m$. Here $\langle F_i\rangle$ and $\langle F_1,\dots,F_{i-1}\rangle$ denote the simplicial complexes whose faces are subsets of F_i and $F_1,\dots,F_{i-1}$, respectively.

(ii) The set $\{F \mid F \in \langle F_1,\dots,F_i\rangle \setminus \langle F_1,\dots,F_{i-1}\rangle\}$ has a unique minimal element for all $2 \le i \le m$. This element is called the *restriction face* of F_i and is denoted by $\mathrm{res}(F)$.

(iii) For all $1 \le j < i \le m$, there exists some $v \in F_i \setminus F_j$ and some $1 \le k \le i-1$ with $F_i \setminus F_k = \{v\}$.

A linear order of the facets satisfying the equivalent conditions (i), (ii) and (iii) is called a *shelling* of Δ.

As already mentioned shellability is a property which is independent of the field k. The following theorem states how it relates to the Cohen-Macaulay property.

Theorem 2.2.5. *[BH98, Theorem 5.1.13] Let Δ be a shellable simplicial complex. Then Δ is Cohen-Macaulay over every field.*

The converse of Theorem 2.2.5 is not true.

Example 2.2.6. (i) Let Δ be the simplicial complex which is determined by the facets $F_1 := \{1,2,5\}$, $F_2 := \{2,3,5\}$, $F_3 := \{3,4,5\}$ and $F_4 := \{1,4,5\}$. Then Δ is a shellable complex. A possible shelling of Δ is the ordering F_1, F_2, F_3, F_4. On the contrary, no shelling order starts with the facets F_1, F_3 since their intersection is only the vertex 3, being of dimension 0. In a shelling order the intersection should be of dimension 1.

(ii) Let Δ be a triangulation of the projective plane. Then Δ is not Cohen-Macaulay over k if $\mathrm{char}(k) = 2$. Therefore, Δ is not shellable by Theorem 2.2.5.

Even though there are Cohen-Macaulay simplicial complexes which are not shellable, Stanley showed that the classes of h-vectors of Cohen-Macaulay simplicial complexes and those of shellable simplicial complexes coincide. We will strongly take advantage of this fact in Chapter 4 when we show that the h-vector inequalities we are able to conclude from our algebraic result – holding only for shellable simplicial complexes – are also true for Cohen-Macaulay complexes. We explicitly state Stanley's result in this context.

For shellable simplicial complexes McMullen and Walkup gave a combinatorial interpretation of the h-vector in terms of a shelling of the complex which will be used later on.

Proposition 2.2.7. *[BH98, Corollary 5.1.14], [MW71] Let Δ be a $(d-1)$-dimensional shellable simplicial complex on vertex set $[n]$ with shelling $F_1,\dots,F_m$. For $2 \le j \le m$ let r_j be the number of facets of $\langle F_j\rangle \cap \langle F_1,\dots,F_{j-1}\rangle$ and set $r_1 := 0$. Let further $h^\Delta = (h_0^\Delta, h_1^\Delta, \dots, h_d^\Delta)$ be the h-vector of Δ. Then*

$$h_i = |\{j \mid r_j = i\}| \text{ for } 0 \le i \le d.$$

In particular, up to their order, the numbers r_j do not depend on the particular shelling.

Proof. For $1 \le j \le m$ let $\Delta_j := \langle F_1,\dots,F_j\rangle$ be the simplicial complex whose faces are the subsets of $F_1,\dots,F_j$. Then there is the following exact sequence of $k[x_1,\dots,x_n]$-modules:

$$0 \to k[\Delta_j] \to k[\Delta_{j-1}] \oplus k[\langle F_j\rangle] \to k[\langle F_j\rangle \cap \Delta_{j-1}] \to 0. \tag{2.3}$$

Here the injection on the left-hand side is given by $\alpha \mapsto (\tilde{\alpha}, -\tilde{\alpha})$ and the surjection on the right-hand side by $(\beta,\gamma) \mapsto \tilde{\beta} + \tilde{\gamma}$, where $\tilde{a}$ denotes the obvious projection of a on the appropriate quotient module. From the exact sequence (2.3) we obtain for the Hilbert series of the occuring modules

$$\mathrm{Hilb}(k[\Delta_j],t) = \mathrm{Hilb}(k[\Delta_{j-1}],t) + \mathrm{Hilb}(k[\langle F_j\rangle],t) - \mathrm{Hilb}(k[\langle F_j\rangle \cap \Delta_{j-1}],t). \tag{2.4}$$

We set $\mathrm{Hilb}(k[\Delta_j],t) = \frac{Q_j(t)}{(1-t)^d}$ for $1 \le j \le m$ and $\mathrm{Hilb}(k[\langle F_j\rangle \cap \Delta_{j-1}],t) = \frac{P_j(t)}{(1-t)^{d-1}}$ for $1 \le j \le m$. Here, $Q_j(t)$ and $P_j(t)$, respectively, are polynomials of degree not bigger than d and $d-1$, respectively. Multiplying Equation (2.4) with $(1-t)^d$ yields

$$Q_j(t) = Q_{j-1}(t) + 1 - P_j(t)(1-t). \tag{2.5}$$

Straightforward computation shows that $P_j(t) = 1 + t + \ldots + t^{r_j - 1}$. By plugging this into Equation (2.5) we finally obtain $Q_j(t) = Q_{j-1}(t) + t^{r_j}$. From $Q_1(t) = 1$ we deduce that $Q_m(t) = \sum_{j=1}^m t^{r_j}$. The assertion now follows since the coefficient vector of $Q_m(t)$ is just the h-vector of Δ. □

Remark 2.2.8. It follows from the definition of a restriction face that $r_j = |\mathrm{res}(F_j)|$. In fact, let Δ be a shellable simplicial complex and let $F_1,\dots,F_m$ be a shelling of Δ. Let j be fixed. For a facet F of $\langle F_j\rangle \cap \langle F_1,\dots,F_{j-1}\rangle$ we have by definition of the restriction face of F_j that $\mathrm{res}(F_j) \nsubseteq F$. Since F is a facet of $\langle F_j\rangle$ it follows that $F = F_j \setminus \{v\}$ for some $v \in \mathrm{res}(F_j)$. From the minimality of the restriction face $\mathrm{res}(F_j)$ we conclude that for each $v \in \mathrm{res}(F_j)$ the face $F_j \setminus \{v\}$ is a facet of $\langle F_j\rangle \cap \langle F_1,\dots,F_{j-1}\rangle$. Thus, $r_j = |\mathrm{res}(F_j)|$ for $2 \le j \le m$. By setting $\mathrm{res}(F_1) = \emptyset$ we also get $r_1 = |\mathrm{res}(F_1)|$.

2.3 Operations and constructions on simplicial complexes

2.3.1 Several standard operations

Starting with one or several simplicial complexes one can perform several operations on those complexes in order to get new simplicial complexes. We first consider operations which are applied to a single simplicial complex.

Definition 2.3.1. Let Δ be a simplicial complex on vertex set Ω and let w be an additional vertex not in Ω.

(i) For $F \in \Delta$, the simplicial complex $\mathrm{lk}_\Delta(F) := \{G \in \Delta \mid G \cup F \in \Delta,\ G \cap F = \emptyset\}$ is called the *link* of F in Δ.

(ii) For a subset $W \subseteq \Omega$, the simplicial complex $\Delta_W := \{F \in \Delta \mid F \subseteq W\}$ is called the *restriction* of Δ to W.

(iii) For a vertex $v \in \Omega$, we call the simplicial complex $\Delta - \{v\} := \{F \in \Delta \mid v \notin F\}$ the *deletion* of v from Δ.

(iv) The simplicial complex $\mathrm{cone}_w(\Delta) := \{F \cup \{w\} \mid F \in \Delta\} \cup \Delta$ is called the *cone* of Δ with *apex* w.

It can be checked that the collections of subsets defined above are indeed simplicial complexes.

If we have several (more than one) simplicial complexes at our disposal there are further constructions which can be performed.

Definition 2.3.2. (i) Let Δ_1 and Δ_2 be simplicial complexes on vertex sets Ω_1 and Ω_2, respectively, such that $\Omega_1 \cap \Omega_2 = \emptyset$. Then the simplicial complex

$$\Delta_1 * \Delta_2 := \{F \cup G \mid F \in \Delta_1,\ G \in \Delta_2\}$$

on vertex set $\Omega_1 \cup \Omega_2$ is called the *join* of Δ_1 and Δ_2.
Note that if Δ_2 consists of a single vertex v then the join of Δ_1 and Δ_2 is the cone of Δ_1 with apex v.

(ii) Let Δ_1 and Δ_2 be simplicial complexes intersecting in a common facet F. The simplicial complex $\Delta_1 \#_F \Delta_2 := (\Delta_1 \cup \Delta_2) \setminus \{F\}$ is called the *connected sum* of Δ_1 and Δ_2 over F.

2.3.2 The barycentric subdivision

Besides the operations on simplicial complexes defined in the previous section there exist several subdivision operations for simplicial complexes, e.g., edgewise subdivisions and barycentric subdivision. In this section our interests focus on the barycentric subdivision of a simplicial complex Δ. We study this operation in more detail than the previously defined operations and show how certain properties of simplicial complexes are preserved after performing this operation. The treatment of this topic is rather extensive since our main results of Chapter 4 hold for exactly this operation.

Definition 2.3.3. Let Δ be a simplicial complex on vertex set $[n]$. The *barycentric subdivision* of Δ is the simplicial complex on vertex set $\Delta \setminus \{\emptyset\}$ whose faces are chains

$$\emptyset \neq A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_l,$$

where $A_i \in \Delta$ for $0 \leq i \leq l$. It is denoted by $\mathrm{sd}(\Delta)$.

Note that since the dimension of a face of the barycentric subdivision of Δ equals the length of the corresponding chain minus one, it directly follows from the definition that Δ and $\mathrm{sd}(\Delta)$ have the same dimension. Start with a dimension-maximal face of Δ and subsequently remove one vertex after the other in order to obtain a maximal chain of $\mathrm{sd}(\Delta)$.

Example 2.3.4. Let $\mathrm{sd}(\Delta^{(3)})$ be the barycentric subdivision of the 2-simplex. Then $\mathrm{sd}(\Delta^{(3)})$ is the cone over the boundary of a 6-gon, the apex being the vertex $\{1,2,3\}$. More precisely,

$$\begin{aligned}
\mathrm{sd}(\Delta^{(3)}) = \{&\{1\} \subsetneq \{1,2\} \subsetneq \{1,2,3\}, \{2\} \subsetneq \{1,2\} \subsetneq \{1,2,3\},\\
&\{2\} \subsetneq \{2,3\} \subsetneq \{1,2,3\}, \{3\} \subsetneq \{2,3\} \subsetneq \{1,2,3\},\\
&\{1\} \subsetneq \{1,3\} \subsetneq \{1,2,3\}, \{3\} \subsetneq \{1,3\} \subsetneq \{1,2,3\},\\
&\{1\} \subsetneq \{1,2\}, \{1\} \subsetneq \{1,3\}, \{1\} \subsetneq \{1,2,3\},\\
&\{2\} \subsetneq \{1,2\}, \{2\} \subsetneq \{2,3\}, \{2\} \subsetneq \{1,2,3\},\\
&\{3\} \subsetneq \{1,3\}, \{3\} \subsetneq \{2,3\}, \{3\} \subsetneq \{1,2,3\},\\
&\{1,2\} \subsetneq \{1,2,3\}, \{1,3\} \subsetneq \{1,2,3\}, \{2,3\} \subsetneq \{1,2,3\},\\
&\{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}, \emptyset\}.
\end{aligned}$$

One first thing to notice is that a simplicial complex Δ and its barycentric subdivision $\mathrm{sd}(\Delta)$ have homeomorphic geometric realizations, see e.g., [Bjö95]. For this reason topological properties are preserved when passing from a simplicial complex Δ to its barycentric subdivision $\mathrm{sd}(\Delta)$. One such example is the Cohen-Macaulay property. By Stanley and Munkres's criterion this property depends only on the geometric realization of the simplicial complex. Thus, the following holds.

Proposition 2.3.5. *[BG81, Corollary 6.3] Let Δ be a simplicial complex and $\mathrm{sd}(\Delta)$ be its barycentric subdivision. Then Δ is a Cohen-Macaulay complex if and only if $\mathrm{sd}(\Delta)$ is a Cohen-Macaulay complex.*

But topological properties are not the only properties which are unaltered under barycentric subdivision. Also some algebraic invariants do not change, e.g., the Krull dimension, or at least they show a rather nice behavior, e.g., the regularity. For a detailed listing of how the most common algebraic invariants behave under barycentric subdivision see for example [KW08]. In Chapter 4 our proofs will strongly use the fact that the barycentric subdivision of a shellable simplicial complex is shellable as well.
Indeed, let Δ be a $(d-1)$-dimensional shellable simplicial complex and let $F_1,\dots,F_m$ be a shelling order of Δ. Starting with the shelling of Δ it is possible to construct a shelling of the barycentric subdivision $\mathrm{sd}(\Delta)$. One first has to check that the barycentric subdivision of a standard simplex is shellable. This implies that the barycentric subdivision of each facet F_i of Δ, $1 \leq i \leq m$, is shellable. In order to construct a shelling of $\mathrm{sd}(\Delta)$ one takes a shelling of $\mathrm{sd}(\langle F_1\rangle)$. Since $F_1,\dots,F_m$ is a shelling of Δ we know that $F_1 \cap F_2$ is a pure $(d-2)$-dimensional subcomplex of Δ. This subcomplex is shellable, as well. Let now G be a facet of the barycentric subdivision of F_2 such that $\mathrm{sd}(F_1 \cap F_2) \cap G$ is a $(d-2)$-dimensional subcomplex of $\mathrm{sd}(\Delta)$. It is possible to show that G can be extended to a shelling of the barycentric subdivision of F_2, in such a way that the whole sequence of facets of $\mathrm{sd}(\Delta)$ constructed so far is a shelling of the barycentric subdivision of $F_1 \cup F_2$. In order to prove this fact it is crucial to know that $\mathrm{sd}(F_1 \cap F_2) = \mathrm{sd}(F_1) \cap \mathrm{sd}(F_2)$ is itself shellable, as can be assumed if one proceeds by induction over the dimension of the original simplicial complex. We do not want to give any further details of the proof. We only wanted to make clear its main idea. To summarize we have the following.

Proposition 2.3.6. *Let Δ be a shellable simplicial complex. Then the barycentric subdivision* $\mathrm{sd}(\Delta)$ *is shellable as well.*

2.3.3 Algebraic shifting: The exterior shifting of a simplicial complex

In this section we discuss exterior algebraic shifting which was first introduced by Kalai, see e.g., [Kal01]. There exist two kinds of algebraic shifting for simplicial complexes: a symmetric and an exterior version, but we stick to the latter one. We first introduce the exterior algebraic shifting of a simplicial complex and then state some of the main properties of the newly constructed simplicial complex.
Recall that a simplicial complex Δ on vertex set $[n]$ is *shifted* if for $F \in \Delta$, $j \in F$ and $i < j$ it holds that $(F \setminus \{j\}) \cup \{i\} \in \Delta$.

Example 2.3.7. (i) The $(n-1)$-simplex is a shifted complex.

(ii) Let Δ be the simplicial complex on [5] with facets $\{1,2,3\}$, $\{1,2,4\}$ and $\{1,2,5\}$. Then one can easily check that Δ is shifted.

(iii) Let Δ be the simplicial complex on [5] with facets $\{1,2,3\}$, $\{1,3,4\}$ and $\{1,2,5\}$. Then Δ is not shifted since both the facet $\{1,2,5\}$ and the facet $\{1,3,4\}$ would force $\{1,2,4\}$ to lie in the complex.

We now describe the construction of the exterior algebraic shifting of a simplicial complex Δ which will be denoted by Δ^e. Let k be an infinite field and let Δ be a simplicial complex on vertex set $[n]$. Let V be an n-dimensional k-vector space and let $E := \bigoplus_{l=0}^{n} \bigwedge^l V$ be the exterior algebra of V. Let further $f_1, \ldots, f_n$ be a basis of E_1 which is generic over k with respect to $e_1, \ldots e_n$. This means if $f_i = \sum_{j=1}^{n} a_{i,j} e_j$ for $1 \leq i \leq n$ then the coefficients $a_{i,j}$ are algebraically independent over k. For $A = \{i_1 < \ldots < i_r\} \subseteq [n]$ we set $f_A := f_{i_1} \wedge \ldots \wedge f_{i_r} \in E$ and by $\bar{f}_A$ we denote the image of f_A in $k\{\Delta\}$. Let further $<_{\text{lex}}$ denote the lexicographic order on subsets of $\mathbb{N}$ of the same size, i.e., $A <_{\text{lex}} B$ if and only if $\min((A \setminus B) \cup (B \setminus A)) \in A$. In order to define the exterior shifting of Δ we define the shifting of a family of sets of equal cardinality. We set

$$\Delta_i^e := \left\{ A \in \binom{[n]}{i} \;\middle|\; \bar{f}_A \notin \operatorname{span}\{\bar{f}_{A'} \mid A' <_{\text{lex}} A\} \right\}$$

for $0 \leq i \leq \dim \Delta + 1$. Finally, the exterior shifting of Δ is the simplicial complex $\Delta^e := \bigcup_{i=0}^{\dim \Delta + 1} \Delta_i^e$. It can be shown that the above construction is independent of the chosen matrix $(a_{ij})_{1 \leq i,j \leq n}$ and that Δ^e is indeed a shifted simplicial complex. Using the definition of the generic initial ideal of an ideal $J \subseteq E$ (see Chapter 1) one can easily show that the exterior Stanley-Reisner ideal is given by $J_{\Delta^e} = \operatorname{gin}_{<_{\text{rlex}}}(J_\Delta)$. Here $<_{\text{rlex}}$ denotes the reverse lexicographic order with respect to $e_1 < \ldots < e_n$. We therefore get another possible method to determine the exterior shifting of a simplicial complex just by computing the generic initial ideal of the exterior face ideal J_Δ. Note in particular that the exterior face ideal of Δ^e is a strongly stable ideal. From this observation it follows that Δ^e is shifted.
The exterior shifting of a simplicial complex Δ tends to have a simpler structure than the original complex. In fact, the exterior shifting Δ^e is always homeomorphic to a wedge of spheres. Furthermore, some combinatorial and topological properties are not altered under exterior algebraic shifting. Without giving the proofs we now summarize the main properties of exterior algebraic shifting which we will need in the remaining chapters of this thesis. Further terminology and results concerning algebraic shifting can be found in [HH08] and [Kal01].

Theorem 2.3.8. *[Kal01, Theorem 2.1, Lemma 3.1], [HH08, Proposition 10.2.1] Let Δ be a simplicial complex on vertex set $[n]$ and let Δ^e be its exterior algebraic shifting. Then,*

(i) *Δ^e is a shifted simplicial complex.*

(ii) *Δ and Δ^e have the same f-vector.*

(iii) *If Δ is shifted, then it holds that $\Delta^e = \Delta$. In particular, $(\Delta^e)^e = \Delta^e$.*

(iv) *Δ and Δ^e have isomorphic reduced simplicial homology groups.*

(v) *If $\Gamma \subseteq \Delta$ is a subcomplex of Δ, then $\Gamma^e \subseteq \Delta^e$.*

Part II

Lefschetz Properties for Classes of Simplicial Complexes

3 The Lefschetz property: classical and more recent results

This chapter focuses on the classical g-theorem for boundary complexes of simplicial polytopes, the g-conjecture for simplicial spheres and developments in the process of attacking this conjecture. We have divided the chapter into three main sections. In the first one we provide necessary definitions and present the classical g-theorem which was proved by Stanley and Billera/Lee in 1979. We further describe and give the necessary background of the g-conjecture which was first raised by McMullen in 1971 for simplicial spheres [McM71] and later generalized by Björner and Swartz to homology spheres and 2-Cohen-Macaulay complexes [Swa06].
The second section cites the most important results which had been obtained in the last decades and which had been steps towards solving the g-conjecture, see e.g., [Swa06], [Swa03], [Mur07], [Nev07], [Nev05], [Kar04]. We do not want to give a complete list of results but to illustrate how many attention this field of research has attracted and how many progress has been made. Most results are obtained by showing a certain type of Lefschetz property for special classes of simplicial complexes. From this property certain conditions and restrictions for the h-vectors of those classes of simplicial complexes can be derived.
In the third section we want to give an abridgement of the algebraic progress which had been attained in showing Lefschetz properties for certain classes of algebras. There are some results by Wiebe [Wie04], Harima [HW07] and Cho [JMAP06], [CP07] which provide equivalent or at least sufficient or necessary conditions for being Lefschetz. On one side those results could hint at further methods which could help to solve the g-conjecture but on the other side they could even provide us with tools for proving the g-conjecture or related problems.

3.1 The classical g-theorem and the g-conjecture

One classical problem in combinatorics is the characterization and classification of special classes of simplicial complexes in terms of their numbers of faces or, equivalently, in terms of the entries of the h-vector. Instead of considering the h-vector itself one often looks at the differences between its entries.

Definition 3.1.1. Let Δ be a $(d-1)$-dimensional simplicial complex with h-vector $h^\Delta = (h_0^\Delta, h_1^\Delta, \ldots, h_d^\Delta)$. Then the vector $g^\Delta := (g_0^\Delta, g_1^\Delta, \ldots, g_{\lfloor \frac{d}{2} \rfloor}^\Delta)$, where $g_0^\Delta := 1$ and $g_i^\Delta := h_i^\Delta - h_{i-1}^\Delta$

for $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$, is called the *g-vector* of Δ.

If Δ has a symmetric h-vector, which for instance is the case if Δ is a Gorenstein* complex or the boundary complex of a simplicial polytope [BH98, Sta96], then the h- and the f-vectors of Δ can be recovered from the g-vector.

Maybe, the longest outstanding conjecture in the field of face enumeration of simplicial complexes is the so-called g-conjecture which was first raised by McMullen in 1971 [McM71]. This conjecture aims at giving a complete characterization of the g-vectors of simplicial spheres. Before stating the conjecture we need to give some further definitions.

Definition 3.1.2. Let Δ be a $(d-1)$-dimensional simplicial complex. Then Δ is called a *simplicial sphere* if Δ is homeomorphic to a $(d-1)$-sphere.

In order to understand the g-conjecture we additionally need to define what is understood by an M-sequence and how this property can be characterized numerically.

Definition 3.1.3. Let $a = (a_0, \ldots, a_s) \in \mathbb{N}^{s+1}$ be a sequence of non-negative integers. Then the sequence a is called an *M-sequence* if a is the Hilbert function of a standard graded Artinian k-algebra.

Macaulay gave a characterization of an M-sequence by means of numerical conditions on its elements. He showed that given any positive integer i every non-negative integer a possesses a unique so-called *i-binomial representation*.

Lemma 3.1.4. *[BH98, Lemma 4.2.6] Let i be a positive integer. Any $a \in \mathbb{N}$ can be written uniquely in the form*

$$a = \binom{k_i}{i} + \binom{k_{i-1}}{i-1} + \ldots + \binom{k_s}{s}, \tag{3.1}$$

where $k_i > k_{i-1} > \ldots > k_s \geq s$.

The representation from Equation 3.1 is called the i-th *Macaulay representation* of a. We further need the following obstruction of the i-th Macaulay representation of an integer a. We set

$$a^{\langle i \rangle} := \binom{k_i+1}{i+1} + \binom{k_{i-1}+1}{i} + \ldots + \binom{k_s+1}{s+1}$$

and $0^{\langle i \rangle} = 0$. Macaulay's characterization of an M-sequence is the following.

Theorem 3.1.5. *[BH98, Theorem 4.2.10] (Macaulay)*
Let $a = (a_0, \ldots, a_s)$ be a sequence of non-negative integers. Then the following conditions are equivalent:

(i) a is an M-sequence.

(ii) $a_0 = 1$ *and* $a_{i+1} \leq a_i^{\langle i \rangle}$ *for* $0 \leq i \leq s-1$.

We are now able to state the classical g-conjecture due to McMullen.

Conjecture 3.1.6. [McM71] (g-conjecture)
Let Δ be a simplicial sphere. Then its g-vector is an M-sequence.

Originally, McMullen formulated the g-conjecture only for boundary complexes of simplicial polytopes but later extended it to simplicial spheres. He further predicted that to every M-sequence a there exists a simplicial complex Δ which is the boundary of a simplicial polytope such that $g^\Delta = a$.
In 1979, Billera and Lee [BL81] succeeded in constructing a simplicial polytope whose boundary has as g-vector a given M-sequence. In the same year, it was Stanley [Sta80] who – using the Hard Lefschetz Theorem for toric varieties – further showed that the g-vector of the boundary of a simplicial polytope is an M-sequence.

Theorem 3.1.7. *[BL81, Sta80] (g-theorem)*
Let $h = (h_0, \ldots, h_d) \in \mathbb{N}^{d+1}$ *and let* $g = (1, h_1 - h_0, \ldots, h_{\lfloor \frac{d}{2} \rfloor} - h_{\lfloor \frac{d}{2} \rfloor - 1})$. *Then h is the h-vector of the boundary complex of a simplicial d-polytope if and only if g is an M-sequence.*

From Stanley's proof of the necessity part of the g-theorem follows an algebraic version of the g-theorem which holds over any field of characteristic 0. We state this version in the next section after having provided the necessary definitions.
In 2005, Björner and Swartz generalized McMullen's g-conjecture to Gorenstein* and even to 2-Cohen-Macaulay complexes [Swa06].

Definition 3.1.8. Let Δ be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$ and let k be some field.

(i) Δ is called a 2-*Cohen-Macaulay* complex over k if Δ is Cohen-Macaulay over k and for every vertex $v \in [n]$ the deletion $\Delta - \{v\}$ of v from Δ is Cohen-Macaulay over k, as well.

(ii) Δ is called a *Gorenstein** complex or a *homology sphere* over k if for all $F \in \Delta$

$$\widetilde{H}_i(\mathrm{lk}_\Delta(F); k) = \begin{cases} k, & \text{if } i = \dim(\mathrm{lk}_\Delta(F)) \\ 0, & \text{otherwise} \end{cases}$$

Note that a simplicial complex Δ is Gorenstein* if Δ has the homology of a sphere and if the link of each face of Δ has the homology of a sphere of the appropriate dimension. In particular, all simplicial spheres are homology spheres. Furthermore, it can be shown – using the topological characterization of the Cohen-Macaulay property – that all Gorenstein* complexes are2-Cohen-Macaulay complexes. We close this section with the generalized g-conjecture of Björner and Swartz.

Conjecture 3.1.9. (generalized g-conjecture)
Let Δ be a 2-Cohen-Macaulay simplicial complex. Then its g-vector is an M-sequence.

This conjecture as well as the classical g-conjecture have attracted a lot of attention from researchers working in different fields of mathematics. As we want to show in the following section, lots of partial results and results in the scope of the g-conjecture have been attained in the last decades. However, people are still far from proving those conjectures and as a consequence there are still a lot of people working hard towards solving these conjectures. The progress which is made is most often achieved for special classes of simplicial complexes which feature useful properties. Another class of results which should be distinguished from the just mentioned ones are results which show that several constructions preserve the so-called Lefschetz property (for more details see Section 3.2). A third class of results are algebraic results which give equivalent formulations of the conditions of being an M-sequence or which give methods by hand which can be used and might be helpful to attack the g-conjecture. We will state some of these results in the next section.

3.2 More recent results

As we will explain in more detail in Chapter 4 one method for proving results which are in the context of the g-conjecture is to show some type of Lefschetz property for the Stanley-Reisner ring of a simplicial complex. Before we state the main results which had been attained towards solving the g-conjecture we give the necessary definitions which are needed in order to understand the results.

We first need the notion of a linear system of parameters for the face ring of a simplicial complex.

Definition 3.2.1. Let Δ be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$. Let further $\theta_1,\dots,\theta_d$ be linear forms of degree 1 in $k[\Delta]$. Then $\{\theta_1,\dots,\theta_d\}$ is called a *maximal linear system of parameters* of $k[\Delta]$, l.s.o.p. for short, if θ_i is a non-zero divisor on $k[\Delta]/(\theta_1,\dots,\theta_{i-1})$ for $1 \le i \le d$ and $k[\Delta]/(\theta_1,\dots,\theta_d) \neq 0$.

It is a well-known fact that an l.s.o.p. is always a regular sequence (see the definition of a regular sequence in Chapter 5). Vice versa, if a regular sequence consists of degree 1 elements such a sequence is an l.s.o.p. An l.s.o.p. is maximal in the sense that it cannot be extended to a sequence $\theta_1,\dots,\theta_d,\theta_{d+1}$ satisfying the same conditions. A priori, it is not clear that the Stanley-Reisner ring of any simplicial complex admits a maximal linear system of parameters and in fact, this is not true. But if Δ is Cohen-Macaulay it directly follows from the definition that $k[\Delta]$ possesses a maximal $k[\Delta]$-regular sequence. It then can be shown that such a sequence can be chosen from elements of degree 1, see e.g., [BH98] and [Sta75, Proposition 4.1]. By the previous remarks such a sequence is an l.s.o.p. for $k[\Delta]$. The property of being an l.s.o.p. can be reformulated in various ways, see e.g., [Swa06].

Proposition 3.2.2. *Let Δ be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$. Let $\theta_i := \theta_{i,1}x_1 + \ldots + \theta_{i,n}x_n \in k[\Delta]$ linear forms, $1 \le i \le d$. Consider the matrix $\Theta := (\theta_{i,j})_{\substack{1\le i\le d\\ 1\le j\le n}}$. For each facet F in Δ let Θ_F be the set of columns of Θ corresponding to the vertices in F. Then $\{\theta_1,\ldots,\theta_d\}$ is an l.s.o.p. for $k[\Delta]$ if and only if Θ_F is linearly independent for all $F \in \Delta$.*

Using this characterization of an l.s.o.p. it can be shown that the set of maximal linear systems of parameters of the Stanley-Reisner ring of a simplicial complex is a non-empty Zariski-open set in $(k[x_1,\ldots,x_n]_1)^d$. More precisely, the following holds.

Theorem 3.2.3. *[Swa06, Proposition 3.6] Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex on vertex set $[n]$. Then there exists a non-empty Zariski-open subset $U \subseteq GL_n(k)$ such that $\{\theta_{1,1}x_1 + \ldots + \theta_{1,n}x_n, \ldots, \theta_{d,1}x_1 + \ldots + \theta_{d,n}x_n\}$ is an l.s.o.p. for $k[\Delta]$ for all $(\theta_{i,j})_{1\le i,j\le n} \in U$.*

Proof. We only want to give a short sketch of the proof.
The proof relies on the fact that a set of columns is linearly independent if and only if there exists a submatrix having maximal rank. The latter condition can be expressed as the non-vanishing of a certain determinant. This is a Zariski-open condition. Finally, we have to intersect finitely many Zariski-open sets – possessing a simplicial complex only finitely many facets – which again yields a Zariski-open set. This set is non-empty since for every Cohen-Macaulay complex it is always possible to find an l.s.o.p. $\square$

From the definition of an l.s.o.p. $\{\theta_1,\ldots,\theta_d\}$ for $k[\Delta]$ we deduce that the quotient ring $k[\Delta]/(\theta_1,\ldots,\theta_d)$ has Krull dimension 0. In particular, it is finite-dimensional as a k-vector space and can be written in the form $k[\Delta]/(\theta_1,\ldots,\theta_d) = \bigoplus_{i=0}^{s} A_i$ for some s. Here A_i denotes the k-vector space which is generated by the elements of degree i in $k[\Delta]/(\theta_1,\ldots,\theta_d)$. The following well-known fact gives the connection between the Hilbert series of the Stanley-Reisner ring of a simplicial complex Δ and the one of the quotient of the Stanley-Reisner ring with respect to an l.s.o.p. for $k[\Delta]$.

Theorem 3.2.4. *[BH98, Remark 4.1.11] Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex with h-vector $h^\Delta = (h_0^\Delta,\ldots,h_d^\Delta)$ and let $\{\theta_1,\ldots,\theta_d\}$ be an l.s.o.p. for $k[\Delta]$. Then*

$$\mathrm{Hilb}(k[\Delta]/(\theta_1,\ldots,\theta_d),t) = (1-t)^d\mathrm{Hilb}(k[\Delta],t) = \sum_{i=0}^{d} h_i^\Delta t^i.$$

In particular, $\dim_k(A_i) = h_i^\Delta$*, where* $k[\Delta]/(\theta_1,\ldots,\theta_d) = \bigoplus_{i=0}^{d} A_i$.

We have now laid the required background for the definition of the weak and the strong Lefschetz property, respectively.

Definition 3.2.5. Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex on vertex set $[n]$. Let further $\{\theta_1,\ldots,\theta_d\}$ be an l.s.o.p. for $k[\Delta]$.

(i) A degree 1 element $\omega \in k[x_1,\dots,x_n]$ is called a *weak Lefschetz element* for $k[\Delta]/(\theta_1,\dots,\theta_d)$ if the multiplication

$$\omega : (k[\Delta]/(\theta_1,\dots,\theta_d))_{i-1} \to (k[\Delta]/(\theta_1,\dots,\theta_d))_i : \; m \mapsto \omega m$$

is an injection for $1 \le i \le \lfloor \frac{d}{2} \rfloor$.

(ii) A degree 1 element $\omega \in k[x_1,\dots,x_n]$ is called a *strong Lefschetz element* for $k[\Delta]/(\theta_1,\dots,\theta_d)$ if the multiplication

$$\omega^{d-2i} : (k[\Delta]/(\theta_1,\dots,\theta_d))_i \to (k[\Delta]/(\theta_1,\dots,\theta_d))_{d-i} : \; m \mapsto \omega^{d-2i} m$$

is an injection for $0 \le i \le \lfloor \frac{d}{2} \rfloor$.

If a simplicial complex Δ admits an l.s.o.p. $\{\theta_1,\dots,\theta_d\}$ and a weak and strong Lefschetz element for $k[\Delta]/(\theta_1,\dots,\theta_d)$, then we say that Δ (or also $k[\Delta]$) has the *weak* and the *strong Lefschetz property*, respectively.

The following result is well-known and a proof can be found in [Swa06].

Theorem 3.2.6. *[Swa06, Proposition 3.6] Let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex on vertex set $[n]$. Let*

$$\begin{aligned}\Omega := \{((\theta_{i,j})_{\substack{1\le i\le d\\ 1\le j\le n}}, (\omega_1,\dots,\omega_n)) \in k^{d\times n} \times k^n \mid \\ \{\theta_1 = \theta_{1,1}x_1 + \ldots + \theta_{1,n}x_n, \dots, \theta_d = \theta_{d,1}x_1 + \ldots + \theta_{d,n}x_n\} \text{ is an l.s.o.p. for } k[\Delta] \\ \text{and } \omega_1 x_1 + \ldots + \omega_n x_n \text{ is a Lefschetz element for } k[\Delta]/(\theta_1,\dots,\theta_d)\}.\end{aligned}$$

Then Ω is Zariski-open in k^{dn+n}.

Thus, in order to show that a simplicial complex is weak and strong Lefschetz, respectively, it is necessary and sufficient to show that the set Ω, defined in Theorem 3.2.6, is non-empty.
We have now provided the main definitions we need in order to give the algebraic version of the g-theorem and a summary of some results which are in the scope of the g-conjecture. It follows from Stanley's proof of the necessity part of the g-theorem that the following algebraic version holds.

Theorem 3.2.7. *[Sta80] (algebraic g-theorem)*
Let Δ be the boundary complex of a simplicial d-polytope and let k *be a field of characteristic* 0. *Then Δ is strong Lefschetz over* k.

3.2.1 The strong Lefschetz property for matroid complexes

There exists a result due to Ed Swartz showing that the independence complex of a matroid satisfies the strong Lefschetz property. Before stating his result we give the definition of a matroid and its independence complex. For more details on matroids see e.g., [Oxl92].

Definition 3.2.8. [Oxl92] A *matroid* M is an ordered pair $(E, \mathscr{I})$ consisting of a finite set E and a collection $\mathscr{I}$ of subsets of E satisfying the following three conditions:

(i) $\emptyset \in \mathscr{I}$.

(ii) If $I \in \mathscr{I}$ and $I' \subseteq I$, then $I' \in \mathscr{I}$.

(iii) If I_1 and I_2 are in $\mathscr{I}$ and $|I_1| < |I_2|$, then there is an element $e \in I_2 \setminus I_1$ such that $I_1 \cup \{e\} \in \mathscr{I}$.

The third condition is called the *independence augmentation axiom*. Note that the first two conditions imply that $\mathscr{I}$ is a simplicial complex on vertex set E, even though we do not necessarily have that $\{e\} \in \mathscr{I}$ for all $e \in E$. The simplicial complex $\mathscr{I}$ is called the *independence complex* of the matroid M. We denote it by $\Delta(M)$. It follows from the third condition that $\Delta(M)$ is a pure simplicial complex. Even more is true. It is well-known that the independence complex of a matroid is a Cohen-Macaulay simplicial complex, see e.g., [Swa03]. Thus, it is possible to choose an l.s.o.p. for the Stanley-Reisner ring of an independence complex of a matroid. We are now able to formulate Swartz's result.

Theorem 3.2.9. *[Swa03, Theorem 4.2] Let $\Delta(M)$ be the independence complex of a matroid M. Then $\Delta(M)$ is strong Lefschetz.*

Note that Theorem 3.2.9 does not depend on the characteristic of the field.

3.2.2 The strong Lefschetz property for simplicial complexes admitting a convex ear decomposition

In addition to his result for independence complexes of matroids Swartz obtained a slightly weaker result for simplicial complexes admitting a so-called convex ear decomposition. We first give the definition of a convex ear decomposition which was originally introduced by Chari [Cha97].

Definition 3.2.10. [Swa06] Let Δ be a $(d-1)$-dimensional simplicial complex. A *convex ear decomposition* of Δ is an ordered sequence $\Delta_1, \ldots, \Delta_m$ of pure $(d-1)$-dimensional subcomplexes of Δ such that:

(i) Δ_1 is the boundary of a simplicial d-polytope. For each $2 \leq j \leq m$, Δ_j is a $(d-1)$-ball which is a proper subcomplex of the boundary of a simplicial d-polytope.

(ii) For $j \geq 2$, $\Delta_j \cap (\bigcup_{i=1}^{j-1} \Delta_i) = \partial \Delta_j$.

(iii) $\bigcup_{i=1}^{m} \Delta_i = \Delta$.

Δ_1 is called the *initial subcomplex*. Each Δ_i for $i \geq 2$ is called an *ear* of the decomposition.

There are several types of simplicial complexes known which admit a convex ear decomposition, e.g., order complexes of geometric lattices [NS04], finite buildings and independence complexes of matroids [Swa06]. On the other hand, simplicial complexes admitting a convex ear decomposition are in particular Cohen-Macaulay. They satisfy an even stronger property, the 2-Cohen-Macaulay property [Swa06, Theorem 4.1]. Thus, for simplicial complexes which have a convex ear decomposition it is always possible to find an l.s.o.p. for their face ring. The result of Swartz is the following.

Theorem 3.2.11. *[Swa06, Theorem 3.9] Let Δ be a $(d-1)$-dimensional simplicial complex and let k be a field of characteristic* 0 *which has a convex ear decomposition. Then $k[\Delta]$ is strong Lefschetz.*

We would like to remark that in contrary to Theorem 3.2.9 the result concerning simplicial complexes having a convex ear decomposition depends on the characteristic of the field. With this additional assumption it is paid credit for that the proof of Theorem 3.2.11 uses the classical g-theorem which does only hold in characteristic 0.
As already mentioned, simplicial complexes having a convex ear decomposition are 2-Cohen-Macaulay. So, a natural question to ask is if Theorem 3.2.11 does hold in greater generality also for 2-Cohen-Macaulay complexes. An affirmative answer would show the g-conjecture for Gorenstein* complexes since all Gorenstein* complexes are in particular 2-Cohen-Macaulay. There is a partial answer by Nevo [Nev08] pointing in this direction. He shows that for a generic l.s.o.p. $\{\theta_1, \ldots, \theta_d\}$ and a generic degree 1 element ω multiplication from the degree 1 to the degree 2 part of $k[\Delta]/(\theta_1, \ldots, \theta_d)$ with ω is an injection if Δ is a 2-Cohen-Macaulay complex of dimension at least 2. From this result it follows that the beginning sequence $(g_0^\Delta, g_1^\Delta, g_2^\Delta)$ of the g-vector of Δ is an M-sequence.

3.2.3 The behavior of Lefschetz properties under join, union and connected sum

Nevo together with Babson and Iron, respectively, showed that under certain conditions Lefschetz properties are maintained when performing several constructions on simplicial complexes, such as join, connected sum and gluing, see [Nev07] and [NB08].

Nevo in joint work with Babson investigated how the Lefschetz property behaves when taking the join of two Gorenstein* simplicial complexes [NB08]. First note that the h-vector of a Gorenstein* complex is symmetric. Thus, if a Gorenstein* complex has the strong Lefschetz property then the usual injections we get by multiplying with a strong Lefschetz element are indeed isomorphisms. Babson and Nevo obtained the following result.

Theorem 3.2.12. *[NB08, Theorem 2.2] Let Δ_1 and Δ_2 be two simplicial complexes on disjoint vertex sets which are Gorenstein* over $\mathbb{R}$ and let* $\dim\Delta_1 = d_1$ *and* $\dim\Delta_2 = d_2$. *Assume that both Δ_1 and Δ_2 have the strong Lefschetz property over $\mathbb{R}$. Let further Θ_1 and Θ_2 be an l.s.o.p. for Δ_1 and Δ_2, respectively, and let ω_1 and ω_2 be strong Lefschetz elements for $\mathbb{R}[\Delta_1]/\Theta_1$ and $\mathbb{R}[\Delta_2]/\Theta_2$, respectively. Then*

(i) $\Delta_1 * \Delta_2$ *has a symmetric h-vector and* $\dim(\Delta_1 * \Delta_2) = d_1 + d_2 - 1$.

(ii) $\Theta_1 \cup \Theta_2$ *is an l.s.o.p. for* $\mathbb{R}[\Delta_1 * \Delta_2]$.

(iii) $\omega_1 + \omega_2$ *is a strong Lefschetz element for* $\mathbb{R}[\Delta_1 * \Delta_2]/(\Theta_1 \cup \Theta_2)$.

The above theorem tells us that the strong Lefschetz property is preserved when taking the join of two Gorenstein* simplicial complexes. As for Theorem 3.2.11 the proof of Theorem 3.2.12 uses the classical g-theorem. As a consequence the result does not hold over any field.

A natural question which arises when having obtained a result in the spirit of Theorem 3.2.12 is if there exist further constructions on simplicial complexes which do not destroy the strong or at least the weak Lefschetz property. Nevo, in joint work with Iron, considered the union of simplicial complexes of the same dimension. Under certain circumstances a result similar to Theorem 3.2.12 holds.

Theorem 3.2.13. *[Nev07, Proposition 4.3.1] Let Δ_1 and Δ_2 be $(d-1)$-dimensional simplicial complexes which are weak Lefschetz. If $\Delta_1 \cap \Delta_2$ is a $(d-1)$-dimensional Cohen-Macaulay simplicial complex, then $\Delta_1 \cup \Delta_2$ is weak Lefschetz.*

Note that the above result does not depend on the characteristic of the field.
Even more in the spirit of Theorem 3.2.12 is the following result by Nevo which tells us how the Lefschetz property behaves when taking the connected sum of two simplicial complexes.

Theorem 3.2.14. *[NB08, Theorem 6.1] Let Δ_1 and Δ_2 be $(d-1)$-dimensional Gorenstein* complexes over k which intersect in a common facet F. Then*

(i) $\Delta_1 \#_F \Delta_2$ *is a $(d-1)$-dimensional Gorenstein* complex. In particular, its h-vector is symmetric.*

(ii) $\Delta_1 \#_F \Delta_2$ *is strong Lefschetz.*

Note that the result of Theorem 3.2.14 is independent of the characteristic of the field.

3.2.4 The behavior of Lefschetz properties under stellar subdivisions of simplicial complexes

Besides the already defined barycentric subdivision of a simplicial complex Δ there exists an operation called stellar subdivision which is a subdivision operation performed only on specific faces of the simplicial complex.

Definition 3.2.15. Let Δ be a simplicial complex on vertex set $[n]$ and let $F \in \Delta$. The *stellar subdivision* at F is the operation $\Delta \mapsto \mathrm{Stellar}(\Delta, F)$, where

$$\mathrm{Stellar}(\Delta, F) := (\Delta \setminus (F * \mathrm{lk}_\Delta(F))) \cup (\{v_F\} * \partial(F) * \mathrm{lk}_\Delta(F)).$$

Here, v_F is a vertex which is not contained in $[n]$ and $\partial(F)$ denotes the boundary of the simplex which is generated by F.

Stellar subdivisions and the barycentric subdivision of a $(d-1)$-dimensional simplicial complex Δ are strongly related to each other. We can obtain the barycentric subdivision of Δ by subsequently performing stellar subdivisions on all the faces of Δ, starting with the facets, proceeding with the $(d-2)$-dimensional faces and eventually subdividing the edges. Nevo, together with Babson, showed that under certain conditions the strong Lefschetz property is preserved when taking stellar subdivisions of a Gorenstein* complex. This result is independent of the characteristic of the field.

Theorem 3.2.16. *[NB08, Corollary 5.4] Let Δ be a Gorenstein* complex and let $F \in \Delta$. If Δ and $\mathrm{lk}_\Delta(F)$ are strong Lefschetz then $\mathrm{Stellar}(\Delta, F)$ is strong Lefschetz.*

3.2.5 Lefschetz properties for strongly edge decomposable complexes

Nevo introduced in [Nev07] so-called strongly edge decomposable spheres. Murai extended this notion to arbitrary simplicial complexes. As shown by Murai in [Mur07] this class of simplicial complexes shows nice behaviors regarding the strong Lefschetz property. Before we state Murai's exact result we have to introduce some notion including the one of strongly edge decomposable simplicial complexes.

We first need to define a further operation on a simplicial complex. Let Δ be a simplicial complex on vertex set $[n]$ and let $1 \leq i < j \leq n$ be integers. The *contraction $\mathscr{C}_\Delta(ij)$ of Δ with respect to $\{i, j\}$* is the simplicial complex on $[n] \setminus \{i\}$ which is obtained from Δ by identifying the vertices i and j, i.e.,

$$\mathscr{C}_\Delta(ij) := \{F \in \Delta \mid i \notin F\} \cup \{(F \setminus \{i\}) \cup \{j\} \mid i \in F \in \Delta\}.$$

Let $\{i, j\} \in \Delta$ be an edge. We say that Δ satisfies the *link condition with respect to $\{i, j\}$* if

$$\mathrm{lk}_\Delta(\{i\}) \cap \mathrm{lk}_\Delta(\{j\}) = \mathrm{lk}_\Delta(\{i, j\}).$$

Note that it always holds that $\mathrm{lk}_\Delta(\{i\}) \cap \mathrm{lk}_\Delta(\{j\}) \supseteq \mathrm{lk}_\Delta(\{i, j\})$. We have now all notion by hand we need in order to give the definition of a strongly edge decomposable simplicial complex. These complexes are defined inductively.

Definition 3.2.17. [Mur07, Definition 1.1] A pure simplicial complex is said to be *strongly edge decomposable* if either

(i) $\Delta = \{\emptyset\}$, or

(ii) Δ is the boundary of a simplex, or recursively,

(iii) there exists an edge $\{i, j\} \in \Delta$ such that Δ satisfies the link condition with respect to $\{i, j\}$ and both $\mathrm{lk}_\Delta(\{i, j\})$ and $\mathscr{C}_\Delta(ij)$ are strongly edge decomposable.

It is known and was proven by Murai in [Mur07, Corollary 3.5] that strongly edge decomposable complexes are Cohen-Macaulay and that their h-vectors are symmetric. The first statement implies that we can always find an l.s.o.p. for the Stanley-Reisner rings of those simplicial complexes.

The following result can essentially be deduced from the proof of Theorem 3.2.16 but was also shown by Murai using more algebraic methods.

Proposition 3.2.18. *[Nev07, Theorem 4.6.5],[Mur07, Proposition 3.2] Let Δ be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$ satisfying the link condition with respect to $\{i, j\}$, where $1 \le i < j \le n$. Suppose $\dim \mathscr{C}_\Delta(ij) = d-1$ and $\dim \mathrm{lk}_\Delta(\{i, j\}) = d-3$. If $\mathscr{C}_\Delta(ij)$ and $\mathrm{lk}_\Delta(\{i, j\})$ have the strong Lefschetz property then Δ is strong Lefschetz.*

Combined with an additional inductive argument Murai's proof in particular shows the following property of strongly edge decomposable complexes.

Corollary 3.2.19. *[Mur07, Corollary 3.5] Let Δ be a strongly edge decomposable simplicial complex. Then Δ is strong Lefschetz.*

3.2.6 The non-negativity of the cd-index

In this part we state a result by Karu [Kar04] which differs from the results mentioned so far in the sense that it is not an algebraic but a numeric result. In order to state the result we need to introduce some further definitions.
We start with some poset terminology. Let P be a finite graded poset of rank $n+1$ with minimal element $\hat{0}$ and maximal element $\hat{1}$. Let further ρ be its rank funktion. For a subset $S \subseteq [n]$ we denote by P_S the S-rank selected subposet of P, i.e.,

$$P_S := \{x \in P \mid \rho(x) \in S\}.$$

We denote by $\alpha_P(S)$ the number of maximal chains in P_S. The function $\alpha_P : 2^{[n]} \to \mathbb{N}$ is called the *flag f-vector* of P. The function $\beta_P : 2^{[n]} \to \mathbb{N}$, which is given by

$$\beta_P(S) := \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha(T),$$

is called the *flag h-vector* of P.
The next step is to define polynomials in non-commutative variables which encode the flag f- and the flag h-vectors. Let $k\langle a, b\rangle$ be the polynomial ring with non-commuting variables

a and b. To a subset $S \subseteq [n]$ we assign the monomial $u_S := u_1 \cdot \ldots \cdot u_n$, where $u_i = a$ if $a \in S$ and $u_i = b$ if $i \notin S$. We now consider the polynomials

$$\Upsilon_P(a,b) := \sum_{S \subseteq [n]} \alpha_P(S) u_S$$
$$\Psi_P(a,b) := \sum_{S \subseteq [n]} \beta_P(S) u_S.$$

We introduce new variables c and d by setting $c := a+b$ and $d := ab+ba$. Fine [BK91] showed that if the poset is Eulerian then $\Psi_P(a,b)$ can be written as a polynomial in c and d. This polynomial is denoted by $\Phi_P(c,d)$ and is called the *cd-index* of P. Recall that a poset P is called *Eulerian* if $\mu(x,y) = (-1)^{\rho(y)-\rho(x)}$ for all $x \leq y \in P$, where μ denotes the Möbius function of P.
To a poset P one can associate its order complex $\Delta(P)$ which is the simplicial complex consisting of the chains in P. Note that the barycentric subdivision of a simplicial complex Δ is an order complex. Just consider the poset which is induced by Δ by ordering its faces with respect to inclusion. A poset P with $\hat{0}$ and $\hat{1}$ is called a *Gorenstein** poset over k – being k an arbitrary field – if the order complex $\Delta(P \setminus \{\hat{0}, \hat{1}\})$ is a Gorenstein* simplicial complex. Such a poset is in particular Eulerian which means that one can associate a cd-index to it. Stanley [Sta94] conjectured that the coefficients of the cd-index of a Gorenstein* poset are non-negative. This conjecture was finally proven by Karu in [Kar04].

Theorem 3.2.20. *[Kar04, Theorem 1.3] Let P be a Gorenstein* poset. Then the cd-index $\Psi_P(c,d)$ of P has non-negative integer coefficients.*

From the non-negativity of the cd-index of a Gorenstein* poset it can be deduced that the g-vector of the order complex of a Gorenstein* poset has non-negative entries. In particular it follows that the g-vector of the barycentric subdivision of a Gorenstein* complex is non-negative. If the barycentric subdivision of a Gorenstein* complex was strong Lefschetz then its g-vector would be an M-sequence and in particular its entries would be non-negative. To summarize, both – the property of having a non-negative cd-index and the property of being strong Lefschetz – imply that the g-vector is non-negative. Thus, Karu's result together with the results of Brenti and Welker [BW06], which we consider in the next chapter in great detail, give support to the conjecture that barycentric subdivisions of Gorenstein* complexes are strong Lefschetz.

3.3 Algebraic methods

The aim of this section is to present some algebraic results which give methods by hand for showing Lefschetz properties for certain k-algebras. The main idea is that certain constructions on k-algebras (comparable to those performed on simplicial complexes described in the previous subsections) maintain the Lefschetz property. In addition, we mention some results yielding equivalent conditions for being Lefschetz which in some situations may be easier to prove.

We first want to generalize our definitions of the weak and the strong Lefschetz property, respectively, to Artinian standard graded k-algebras, i.e., to standard graded k-algebras having Krull dimension 0.

Definition 3.3.1. Let $A := \bigoplus_{i=0}^{s} A_i$ be a standard graded 0-dimensional k-algebra with $A_s \neq 0$. By standard graded we mean that $A_0 = k$ and A is generated in degree 1, $A_iA_j \subseteq A_{i+j}$ and $\dim_k A_i < \infty$.

(i) An element $\omega \in A_1$ is called a *weak Lefschetz element* for A if the multiplication

$$\omega : A_{i-1} \to A_i : v \mapsto \omega v$$

is an injection for $1 \leq i \leq \lfloor \frac{s}{2} \rfloor$.

(ii) An element $\omega \in A_1$ is called a *strong Lefschetz element* for A if the multiplication

$$\omega^{s-2i} : A_i \to A_{s-i} : v \mapsto \omega^{s-2i} v$$

is an injection for $0 \leq i \leq \lfloor \frac{s}{2} \rfloor$.

There are further notion of Lefschetz type properties which can be found in the literature. Sometimes it is required in order to be a weak and a strong Lefschetz element, respectively, that the multiplication maps are of full rank.
Wiebe showed in [Wie04] that an $\mathfrak{m}$-primary homogeneous ideal in $k[x_1,\ldots,x_n]$ inherits the strong Lefschetz property from its initial ideal (with respect to any term order). Here, $\mathfrak{m} = (x_1,\ldots,x_n)$ is the unique homogeneous maximal ideal of $k[x_1,\ldots,x_n]$. Murai remarked in [Mur07] that Wiebe's result could be strengthened to arbitrary homogeneous ideals combining Wiebe's proof with an additional argument by Conca [Con03, Theorem 1.1].

Lemma 3.3.2. *[Wie04, Proposition 2.9], [Mur07, Lemma 3.3] Let $I \subseteq k[x_1,\ldots,x_n]$ be a homogeneous ideal and let $\prec$ be a term order. If $k[x_1,\ldots,x_]/\operatorname{in}_\prec(I)$ has the strong Lefschetz property then $k[x_1,\ldots,x_n]/I$ has the strong Lefschetz property.*

In general, the converse of Lemma 3.3.2 is not true. However, if we consider the generic initial ideal with respect to the reverse lexicographic order we obtain the following result.

Theorem 3.3.3. *[Wie04, Proposition 2.8] Let $I \subseteq k[x_1,\ldots,x_n]$ be a homogeneous ideal, such that $k[x_1,\ldots,x_n]/I$ is an Artinian k-algebra. Let $\operatorname{gin}_{<_{\text{rlex}}}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order. Then $k[x_1,\ldots,x_n]/I$ has the weak and the strong Lefschetz property if and only if $k[x_1,\ldots,x_n]/\operatorname{gin}_{<_{\text{rlex}}}(I)$ has the weak and the strong Lefschetz property, respectively.*

Besides Wiebe's result which gives a method for proving the Lefschetz property by showing this property for a hopefully easier to understand k-algebra there are also results which give characterizations of the different types of Lefschetz properties. Since it would go beyond the scope of this thesis we do not state all of those results but stick to one result and its consequences. For this aim, we need the definition of an almost reverse lexicographic ideal.

Definition 3.3.4. [CP07] Let $I \subseteq k[x_1,\ldots,x_n]$ be a monomial ideal and let $G(I)$ denote the unique minimal set of generators of I. We call I an *almost reverse lexicographic* ideal if for each monomial $u \in I$ and $v \in k[x_1,\ldots,x_n]$ with $\deg(u) = \deg(v)$ and $v >_{\mathrm{rlex}} u$ it holds that $v \in I$.

Harima and Wachi [HW07] showed the following.

Proposition 3.3.5. *[HW07, Corollary 14] Let $I \subseteq k[x_1,\ldots,x_n]$ be an almost reverse lexicographic ideal such that $k[x_1,\ldots,x_n]/I$ is an Artinian k-algebra. Then $k[x_1,\ldots,x_n]/I$ is strong Lefschetz and x_n is a strong Lefschetz element for $k[x_1,\ldots,x_n]/I$.*

It is also known [HW07, Lemma 12] that if I is an almost reverse lexicographic ideal then I is Borel-fixed. Combining Theorem 3.3.3 and Proposition 3.3.5 it follows that in order to show that $k[x_1,\ldots,x_n]/I$ is strong Lefschetz one can also show that $\mathrm{gin}_{<_{\mathrm{rlex}}}(I)$ is an almost reverse lexicographic ideal.

For ideals which are generated by generic forms, Cho and Ahn further proved a sufficient condition such that the ideal has an almost reverse lexicographic generic initial ideal.

Theorem 3.3.6. *[CP07, Theorem 2.10] Let $I \subseteq k[x_1,\ldots,x_n]$ be a homogeneous ideal which has an almost reverse lexicographic generic initial ideal (with respect to the reverse lexicographic order). Let $J \subseteq k[x_1,\ldots,x_n]$ be a homogeneous ideal generated by generic forms. If $\mathrm{Hilb}(k[x_1,\ldots,x_n]/I,t) = \mathrm{Hilb}(k[x_1,\ldots,x_n]/J,t)$ then the generic initial ideal of J with respect to the reverse lexicographic order is an almost reverse lexicographic ideal.*

From the proof of the above theorem it follows that the statement can be sharpened to hold for ideals containing any (at least one) generic form in the minimal set of generators. Thus the following holds.

Corollary 3.3.7. *Let $I \subseteq k[x_1,\ldots,x_n]$ be a homogeneous ideal which has an almost reverse lexicographic generic initial ideal (with respect to the reverse lexicographic order). Let $J \subseteq k[x_1,\ldots,x_n]$ be a homogeneous ideal such that the minimal set of generators of J contains at least one generic form. If $\mathrm{Hilb}(k[x_1,\ldots,x_n]/I,t) = \mathrm{Hilb}(k[x_1,\ldots,x_n]/J,t)$ then the generic initial ideal of J with respect to the reverse lexicographic order is an almost reverse lexicographic ideal.*

The above corollary could yield a method to show that the Stanley-Reisner ring of a simplicial complex – not being the empty complex – has the strong Lefschetz property. In this situation the ideal which is considered contains at least one generic form (under its minimal generators), since an l.s.o.p. consists of at least one element. Thus, by Lemma 3.3.2, Proposition 3.3.5 and Corollary 3.3.7, showing that there exists an almost reverse lexicographic ideal such that the corresponding quotient has the same Hilbert function as $k[\Delta]$ would be sufficient for being $k[\Delta]$ strong Lefschetz. However, it is not sure if this result is applicable in many cases since being an almost reverse lexicographic ideal has turned out to be a much stronger property than being strong Lefschetz, see [HW07]. In fact, Harima and Wachi showed in [HW07] that being an almost reverse lexicographic ideal in $k[x_1,\dots,x_n]$ is equivalent to being the quotient n-times strong Lefschetz.

4 The Lefschetz property for barycentric subdivisions of simplicial complexes

In this chapter we present our main result concerning classes of simplicial complexes which have a certain type of Lefschetz property. To be more precise we show that the barycentric subdivision of a shellable simplicial complex has a – what we call – almost strong Lefschetz property.
The motivation for studying barycentric subdivisions of simplicial complexes originally came from results obtained by Brenti and Welker [BW06] who investigated the transformation of the h-vector of a simplicial complex into the h-vector of the barycentric subdivision of this simplicial complex. From their results inequalities for the h-vector entries of the barycentric subdivision can be deduced – suggerating that the h-vector of the barycentric subdivision of a shellable simplicial complex might be an M-sequence. We present Brenti and Welker's results in the first section of this chapter.

The following section is devoted to the proof of our main result, i.e., the almost strong Lefschetz property for barycentric subdivisions of shellable simplicial complexes. We further show that almost the same proof works for barycentric subdivisions of shellable polytopal complexes. The necessary definitions are provided within this section.

If a simplicial complex satisfies an almost strong Lefschetz property one can easily deduce consequences for its h-vector. We state those consequences, such as unimodality and being an M-sequence, in the third section, with an emphasis on those results which could not already be concluded from the results by Brenti and Welker [BW06]. Furthermore, we show that the inequalities between the h-vector entries which are obtained do not only hold for barycentric subdivisions of shellable simplicial complexes but in greater generality for barycentric subdivisions of Cohen-Macaulay complexes.

In the fourth section we scrutinize the coefficients which appear in the transformation of the h-vector of a simplicial complex into the h-vector of its barycentric subdivision. Those coefficients are a certain refinement of the Eulerian statistics on permutations. Using not only the inequalities between the h-vector entries but also the algebraic result of Section 4.2 we are able to prove a lot of inequalities between those numbers.

The last section of this chapter delves into open problems and conjectures concerning not only the barycentric subdivision of a simplicial complex but also other kinds of subdivision operations, such as edgewise subdivisions. Although we have obtained the numerical result for the h-vector entries of the barycentric subdivision of a simplicial complex for the whole class of Cohen-Macaulay simplicial complexes it is still an open question if the algebraic

result can be expanded to this class as well.
Furthermore, there are several other subdivision operations, e.g., edgewise subdivisions of a simplicial complex, for which the h-vector transformation is known [BW08] and which exhibit similar or even the same properties as the h-vector transformation of the barycentric subdivision, thus suggesting similar algebraic and numerical results as those for the barycentric subdivision to hold.

The results which are obtained in this chapter are results of joint work with Eran Nevo and can be found in [KN08].

4.1 The motivation for studying barycentric subdivisions of simplicial complexes

In [BW06] Brenti and Welker studied the behavior of the f- and the h-vectors of a simplicial complex when passing to its barycentric subdivision. Their results do not only hold for simplicial complexes but in the greater generality of Boolean cell complexes. They showed that the h-vector entries of the barycentric subdivision of a Boolean cell complex and in particular those of the barycentric subdivision of a simplicial complex can be expressed as positive linear combinations of the h-vector entries of the original complex. The coefficients that occur in this representation are a refinement of the Eulerian statistics on permutations. We now expound the required definitions and then state Brenti and Welker's exact result – providing the mentioned transformation.

For $d \in \mathbb{N}$ let $S_d := \{\sigma : [d] \to [d] \mid \sigma \text{ is bijective}\}$ denote the symmetric group on $[d]$. We say that a permutation $\sigma \in S_d$ has a *descent* at position i for $1 \le i \le d-1$ if $\sigma(i) > \sigma(i+1)$. We write $\mathrm{des}(\sigma) := \{i \mid i \text{ is a descent of } \sigma\}$ for the descent set of σ. For $0 \le i \le d-1$ and $1 \le j \le d$ we set $A(d,i,j) := |\{\ \sigma \in S_d \mid |\mathrm{des}(\sigma)| = i,\ \sigma(1) = j\}|$ and $A(d,i,j) := 0$ if $i \le -1$ or $i \ge d$. Recall that the usual Eulerian statistics on permutations count the number of permutations in S_d having a certain number of descents. Thus the just defined numbers are a refinement of the usual Eulerian statistics on permutations.
Before giving the explicit h-vector transformation when passing to the barycentric subdivision of a Boolean cell complex we want to recall the definition of a Boolean cell complex.

Definition 4.1.1. A regular CW-complex Δ is called a *Boolean cell complex* if for each $F \in \Delta$ the lower interval $[\emptyset, F] := \{G \in \Delta \mid \emptyset \le_\Delta G \le_\Delta F\}$ is a Boolean lattice, where $F \le_\Delta F'$ if F is contained in the closure of F' for $F, F' \in \Delta$.

The barycentric subdivision of a Boolean cell complex is defined verbatim as for simplicial complexes. The following theorem finally establishes the relation between the h-vector of a Boolean cell complex Δ and the h-vector of its barycentric subdivision $\mathrm{sd}(\Delta)$.

Theorem 4.1.2. *[BW06, Theorem 2.2] Let Δ be a $(d-1)$-dimensional Boolean cell complex* *and let* $\mathrm{sd}(\Delta)$ *be its barycentric subdivision. Then*

$$h_j^{\mathrm{sd}(\Delta)} = \sum_{r=0}^{d} A(d+1, j, r+1) h_r^{\Delta}$$

for $0 \leq j \leq d$.

As mentioned in the introduction of this chapter accessorily to the h-vector transformation Brenti and Welker showed some numerical constraints for the h-vector of the barycentric subdivision of a simplicial complex. Imposing several restrictions on the original simplicial complex they were able to prove that the h-vector of the barycentric subdivision of this simplicial complex is unimodal and log-concave.
In order to show that a certain sequence is log-concave and unimodal a common method is to look at the polynomial whose coefficients are given by the particular sequence. If this polynomial is real-rooted then it is a well-known result, see e.g., [Bre89], that its coefficient sequence is log-concave. Before pursuing with Brenti and Welker's result from [BW06] we recall the notion of unimodality and log-concavity.

Definition 4.1.3. Let $(a_0, \ldots, a_d)$ be a sequence of real numbers.

(i) $(a_0, \ldots, a_d)$ is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq d-1$.

(ii) We say that $(a_0, \ldots, a_d)$ has *internal zeros* if there are $0 \leq i+1 < j \leq d$ such that $a_i, a_j \neq 0$ and $a_{i+1} = a_{i+2} = \ldots = a_{j-1} = 0$.

(iii) $(a_0, \ldots, a_d)$ is called *unimodal* if there exists $0 \leq j \leq d$ such that $a_0 \leq \ldots \leq a_j \geq \ldots \geq a_d$. We call a_j a *peak* of this sequence and say that it is at *position* j.

Note that a unimodal sequence $(a_0, \ldots, a_d)$ could feature several peaks if it stays constant for a while after having attained its maximum value. Moreover, it is a classical and straightforward to prove result that a log-concave sequence $(a_0, \ldots, a_d)$ without internal zeros is unimodal. Besides the transformation of the h-vector of a simplicial complex when passing to its barycentric subdivision, Brenti and Welker showed that under certain conditions on the original simplicial complex the h-polynomial of the barycentric subdivision has only real roots.

Theorem 4.1.4. *[BW06, Theorem 3.1, Corollary 3.5] Let* Δ *be a* $(d-1)$*-dimensional Boolean cell complex with* $h_i^{\Delta} \geq 0$ *for* $0 \leq i \leq d-1$ *and let* $\mathrm{sd}(\Delta)$ *be the barycentric subdivision of* Δ*. Let further* $h^{\mathrm{sd}(\Delta)} = (h_0^{\mathrm{sd}(\Delta)}, \ldots, h_d^{\mathrm{sd}(\Delta)})$ *be the h-vector of* $\mathrm{sd}(\Delta)$*. Then the h-polynomial of* $\mathrm{sd}(\Delta)$

$$h^{\mathrm{sd}(\Delta)}(t) = \sum_{i=0}^{d} h_i^{\mathrm{sd}(\Delta)} t^{d-i}$$

has only simple and real zeros. In particular, $h^{\mathrm{sd}(\Delta)} = (h_0^{\mathrm{sd}(\Delta)}, \ldots, h_d^{\mathrm{sd}(\Delta)})$ *is a log-concave and unimodal sequence.*

In particular, all conclusions hold for Cohen-Macaulay, Gorenstein, Gorenstein* simplicial complexes (over some field k), for simplicial spheres and for boundary complexes of simplicial polytopes.

The part of Theorem 4.1.4 concerning Cohen-Macaulay complexes follows from the fact that the h-vector entries of a Cohen-Macaulay complex Δ are non-negative. Indeed, if Θ is an l.s.o.p. for $k[\Delta]$ then it follows from Theorem 3.2.4 that $\dim_k(k[\Delta]/\Theta)_i = h_i^\Delta$. Thus, the h-vector entries of Δ equal the dimensions of finite dimensional k-vector spaces which immediately implies $h_i^\Delta \geq 0$ for $0 \leq i \leq d-1$.

On the one hand, it is a direct consequence of Theorem 4.1.4 that the h-vector of the barycentric subdivision of a Cohen-Macaulay complex is unimodal. On the other hand, this fact also follows if a simplicial complex Δ has the strong Lefschetz property. Thus, a natural question which arises when having obtained a result such as Theorem 4.1.4 is if the barycentric subdivision of a Cohen-Macaulay complex is strong Lefschetz or exhibits a similar property. One can further suspect that the g-vector of such a complex is an M-sequence. This conjecture gets additional support from Karu's result (see Chapter 3.2.6) – showing the non-negativity of the cd-index for barycentric subdivisions of Gorenstein* complexes – which implies that the g-vector of such a complex is non-negative. We attack those issues in the next section and answer them (mostly) in the affirmative.

4.2 The main theorem: The almost strong Lefschetz property for barycentric subdivisions of shellable complexes

In this section we show that the barycentric subdivision of a shellable simplicial complex is almost strong Lefschetz. After providing some definitions we state the main result and set out its proof. From the proof it will be clear why we were not able to show a strong Lefschetz property, i.e., why actually this property does not hold. Ancillary, we stress this point.

In the following we introduce polytopal complexes and explain what shellability means for those complexes. Using the same methods as in the proof of the almost strong Lefschetz property for barycentric subdivisions of shellable simplicial complexes we are able to obtain an analogous result for shellable polytopal complexes. We do not give the proof in great detail but emphasize in which points it differs from the one for barycentric subdivisions of shellable simplicial complexes.

Definition 4.2.1. Let Δ be a $(d-1)$-dimensional simplicial complex which is Cohen-Macaulay over some field k and let $\Theta = \{\theta_1 \ldots, \theta_d\}$ be an l.s.o.p. for its face ring $k[\Delta]$.

(i) A degree one element in the polynomial ring $\omega \in A = k[x_1, \ldots, x_n]$ is called an s-

Lefschetz element for $k[\Delta]/\Theta$ if the multiplication

$$\omega^{s-2i}: \ (k[\Delta]/\Theta)_i \longrightarrow (k[\Delta]/\Theta)_{s-i}$$

is an injection for $0 \leq i \leq \lfloor \frac{s-1}{2} \rfloor$.

(ii) A $(\dim\Delta)$-Lefschetz element is called an *almost strong Lefschetz element for* $k[\Delta]/\Theta$.

We call Δ and $k[\Delta]$, as well, *s-Lefschetz* and *almost strong Lefschetz* if there exists an l.s.o.p. Θ of $k[\Delta]$ such that $k[\Delta]/\Theta$ has an s-Lefschetz and an almost strong Lefschetz element, respectively.
Recall that from the above definition it directly follows that Δ is strong Lefschetz if and only if it is $(\dim\Delta+1)$-Lefschetz.

Remark 4.2.2. Imitating the proof of Theorem 3.2.6 it can be shown that for a Cohen-Macaulay complex Δ the set

$$\begin{aligned}\Omega^{(s)} := \{((\theta_{i,j})_{\substack{1\leq i\leq d\\ 1\leq j\leq n}}, (\omega_1,\ldots,\omega_n)) \in k^{d\times n}\times k^n \mid \\ \{\theta_1 = \theta_{1,1}x_1+\ldots+\theta_{1,n}x_n,\ldots,\theta_d = \theta_{d,1}x_1+\ldots+\theta_{d,n}x_n\} \text{ is an l.s.o.p. for } k[\Delta] \\ \text{and } \omega_1x_1+\ldots+\omega_nx_n \text{ is an } s\text{-Lefschetz element for } k[\Delta]/(\theta_1,\ldots,\theta_d)\}\end{aligned}$$

is Zariski-open in k^{dn+n}. Thus, a Cohen-Macaulay complex Δ is s-Lefschetz if $\Omega^{(s)}$ is a non-empty set.

We have now defined everything we need in order to give our main result.

Theorem 4.2.3. *Let Δ be a shellable $(d-1)$-dimensional simplicial complex and let k be an infinite field. Let* $\mathrm{sd}(\Delta)$ *be the barycentric subdivision of* Δ*. Then* $\mathrm{sd}(\Delta)$ *is almost strong Lefschetz over k.*

The proof of Theorem 4.2.3 is composed of several parts and proceeds by double induction on the dimension and the number of facets of Δ. We first show how the s-Lefschetz property behaves when taking the cone of a simplicial complex Δ over a new vertex v. The next step – which provides part of the base of the induction – is to prove that Theorem 4.2.3 holds if Δ is the $(d-1)$-simplex. Finally, there is the actual proof of Theorem 4.2.3 which employs the first two steps.
We now describe the effect of coning on the s-Lefschetz property.

Lemma 4.2.4. *Let Δ be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$ and let v be an additional vertex not in $[n]$. If Δ is s-Lefschetz over* k *then the same is true for* $\mathrm{cone}_v(\Delta)$.

Proof. Let $\Theta := \{\theta_1, \dots, \theta_d\}$ be an l.s.o.p. for $k[\Delta]$. Then $\widetilde{\Theta} := \Theta \cup \{x_v\}$ is an l.s.o.p. for $k[\mathrm{cone}_v(\Delta)]$. Indeed, considered as modules over $k[x_1, \dots, x_n, x_v] \cong k[x_1, \dots, x_n] \otimes_k k[x_v]$ we have the isomorphism $k[\mathrm{cone}_v(\Delta)] \cong k[\Delta] \otimes_k k[x_v]$. Thus, x_v is a non-zero divisor on $k[\mathrm{cone}_v(\Delta)]/\Theta \cong (k[\Delta]/\Theta) \otimes_k k[x_v]$ and the claim follows.
Furthermore, it holds that $k[\Delta]/\Theta \cong k[\mathrm{cone}_v(\Delta)]/\widetilde{\Theta}$ as S-modules, where $S = k[x_1, \dots, x_n, x_v]$ and $x_v \cdot k[\Delta] = 0$. Hence, for any pair (Θ, ω) such that Θ is an l.s.o.p. for $k[\Delta]$ and ω is an s-Lefschetz element for $k[\Delta]/\Theta$ we have that $\widetilde{\Theta}$ is an l.s.o.p. for $k[\mathrm{cone}_v(\Delta)]$ and ω is an s-Lefschetz element for $k[\mathrm{cone}_v(\Delta)]/\Theta$. This shows the assertion of the lemma. $\square$

Note that if Δ is strong Lefschetz, i.e., $(\dim\Delta + 1)$-Lefschetz, then $\mathrm{cone}_v(\Delta)$ is $(\dim\Delta + 1)$-Lefschetz, i.e., $\mathrm{cone}_v(\Delta)$ is $\dim(\mathrm{cone}_v(\Delta))$-Lefschetz, i.e., $\mathrm{cone}_v(\Delta)$ is almost strong Lefschetz. Thus taking the cone provokes that a formerly strong Lefschetz property is weakened to an almost strong Lefschetz property.

Using Lemma 4.2.4 we are able to show the next step in the proof of Theorem 4.2.3.

Theorem 4.2.5. *Let Δ be the $(d-1)$-simplex on vertex set $[d]$ and let $\mathrm{sd}(\Delta)$ be its barycentric subdivision. Let further* k *be an infinite field. Then $\mathrm{sd}(\Delta)$ is almost strong Lefschetz over* k.

Proof. Note that the boundary complex $\partial(\mathrm{sd}(\Delta))$ is obtained from $\partial(\Delta)$ by a sequence of stellar subdivisions – order the faces of $\partial(\Delta)$ by decreasing dimension and perform a stellar subdivision at each of them according to this order to obtain $\partial(\mathrm{sd}(\Delta))$. In particular, $\partial(\mathrm{sd}(\Delta))$ is strongly edge decomposable, as the inverse stellar moves when going backwards in this sequence of complexes demonstrate. It follows from Murai's results for strongly edge decomposable complexes (see Corollary 3.2.19) that $\partial(\mathrm{sd}(\Delta))$ has the strong Lefschetz property. It further holds that $\mathrm{sd}(\Delta) = \mathrm{cone}_{[d]}(\partial(\mathrm{sd}(\Delta)))$, where we take the cone over the vertex of $\mathrm{sd}(\Delta)$ corresponding to the set $[d]$. By Lemma 4.2.4 we conclude that $\mathrm{sd}(\Delta)$ is $(d-1)$-Lefschetz, i.e., almost strong Lefschetz over k. $\square$

A natural question which occurs is if the result of Theorem 4.2.5 can be improved to the strong Lefschetz property. Indeed, this is not the case since $h_d^{\mathrm{sd}(\Delta)} = 0$ if Δ is the $(d-1)$-simplex. This can be seen by straightforward computation or by looking at a shelling of $\mathrm{sd}(\Delta)$ since no facet in a shelling has a restriction face of cardinality d. Being the barycentric subdivision of the $(d-1)$-simplex not strong Lefschetz already tells us that in general we cannot expect the barycentric subdivision of a shellable simplicial complex to be strong Lefschetz.
We are now in the position to give the proof of Theorem 4.2.3.

Proof of Theorem 4.2.3. The proof is by double induction, on the number of facets $f_{\dim\Delta}^{\Delta}$ of Δ and on the dimension of Δ. Let $\dim\Delta \geq 0$ be arbitrary and let $f_{\dim\Delta}^{\Delta} = 1$, i.e., Δ is a $(d-1)$-simplex, and by Theorem 4.2.5 we are done. Let $\dim\Delta = 0$, i.e., Δ as well as $\mathrm{sd}(\Delta)$ consist of vertices only. Since $h_0^{\mathrm{sd}(\Delta)} = h_{1-1-0}^{\mathrm{sd}(\Delta)}$ there is nothing to show. This provides the base of the induction.

For the induction step let $\dim\Delta \geq 1$. Let $n := f_0^{\mathrm{sd}(\Delta)}$ and let $S := k[x_1,\ldots,x_n]$ be the polynomial ring in n variables. Let $F_1,\ldots,F_m$ be a shelling of Δ with $m \geq 2$ and let $\widetilde{\Delta} := \langle F_1,\ldots,F_{m-1}\rangle$. Then $\sigma := \widetilde{\Delta} \cap \langle F_m\rangle$ is a pure $(d-2)$-dimensional subcomplex of $\partial(F_m)$. The barycentric subdivision $\mathrm{sd}(\Delta)$ of Δ is given by $\mathrm{sd}(\Delta) = \mathrm{sd}(\widetilde{\Delta}) \cup \mathrm{sd}(\langle F_m\rangle)$ and $\mathrm{sd}(\sigma) = \mathrm{sd}(\widetilde{\Delta}) \cap \mathrm{sd}(\langle F_m\rangle)$.

We get the following Mayer-Vietoris exact sequence of S-modules:

$$0 \to k[\mathrm{sd}(\Delta)] \to k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(\langle F_m\rangle)] \to k[\mathrm{sd}(\sigma)] \to 0. \tag{4.1}$$

Here the injection on the left-hand side is given by $\alpha \mapsto (\tilde{\alpha}, -\tilde{\alpha})$ and the surjection on the right-hand side by $(\beta,\gamma) \mapsto \tilde{\beta} + \tilde{\gamma}$, where $\tilde{a}$ denotes the obvious projection of a on the appropriate quotient module. (For a subcomplex Γ of Δ and a vertex $\{v\} \in \Delta \setminus \Gamma$ it holds that $x_v \cdot k[\Gamma] = 0$.)

Let $\Theta = \{\theta_1,\ldots,\theta_d\}$ be an l.s.o.p. for $k[\mathrm{sd}(\Delta)]$, $k[\mathrm{sd}(\widetilde{\Delta})]$ and $k[\mathrm{sd}(F_m)]$, and such that $\{\theta_1,\ldots,\theta_{d-1}\}$ is an l.s.o.p. for $k[\sigma]$. This is possible, as the intersection of finitely many non-empty Zariski-open sets is non-empty (for $k[\sigma]$, its set of l.s.o.p. times k^n (for θ_d) is a non-empty Zariski-open set in k^{dn}). Dividing out by Θ in the short exact sequence (4.1), which is equivalent to tensoring with $-\otimes_S S/\Theta$, yields the following Tor-long exact sequence:

$$\begin{aligned}
\ldots \quad & \to \mathrm{Tor}_1(k[\mathrm{sd}(\Delta)], S/\Theta) \to \mathrm{Tor}_1(k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(\langle F_m\rangle)], S/\Theta)\\
\to \quad & \mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta) \xrightarrow{\delta} \mathrm{Tor}_0(k[\mathrm{sd}(\Delta)], S/\Theta)\\
\to \quad & \mathrm{Tor}_0(k[\mathrm{sd}(\widetilde{\Delta})] \oplus k[\mathrm{sd}(\langle F_m\rangle)], S/\Theta) \to \mathrm{Tor}_0(k[\mathrm{sd}(\sigma)], S/\Theta) \to 0,
\end{aligned}$$

where $\delta:\ \mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta) \to \mathrm{Tor}_0(k[\mathrm{sd}(\Delta)], S/\Theta)$ is the connecting homomorphism. In order to simplify notation we set

$$\begin{aligned}
k(\mathrm{sd}(\Delta)) &:= k[\mathrm{sd}(\Delta)]/\Theta, \qquad k(\mathrm{sd}(\widetilde{\Delta})) := k[\mathrm{sd}(\widetilde{\Delta})]/\Theta,\\
k(\mathrm{sd}(\sigma)) &:= k[\mathrm{sd}(\sigma)]/\Theta \text{ and } k(\mathrm{sd}(\langle F_m\rangle)) := k[\mathrm{sd}(\langle F_m\rangle)]/\Theta.
\end{aligned}$$

Using that for R-modules M, N and Q it holds that $\mathrm{Tor}_0(M,N) \cong M \otimes_R N$ (see Theorem 1.1.10 (ii) in Chapter 1), that $(M \oplus N) \otimes_R Q \cong (M \otimes_R Q) \oplus (N \otimes_R Q)$ and that $M/IM \cong M \otimes_R R/I$ for an ideal $I \lhd R$, we get the following exact sequence of S-modules:

$$\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta) \xrightarrow{\delta} k(\mathrm{sd}(\Delta)) \to k(\mathrm{sd}(\widetilde{\Delta})) \oplus k(\mathrm{sd}(\langle F_m\rangle)) \to k(\mathrm{sd}(\sigma)) \to 0.$$

Note that all the maps in this sequence are grading preserving, where $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)$ inherits the grading from (a projective grading preserving resolution of) the sequence (4.1). From this we deduce the following commutative diagram:

$$\begin{array}{ccccc}
\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)_i & \xrightarrow{\delta} & k(\mathrm{sd}(\Delta))_i & \to & k(\mathrm{sd}(\widetilde{\Delta}))_i \oplus k(\mathrm{sd}(\langle F_m\rangle))_i\\
 & & \downarrow \omega^{d-2i-1} & & \downarrow (\omega^{d-2i-1}, \omega^{d-2i-1})\\
 & & k(\mathrm{sd}(\Delta))_{d-1-i} & \to & k(\mathrm{sd}(\widetilde{\Delta}))_{d-1-i} \oplus k(\mathrm{sd}(\langle F_m\rangle))_{d-1-i}
\end{array}$$

where ω is a degree one element in S. Since F_m is a $(d-1)$-simplex we know from the base of the induction that the multiplication

$$\omega^{d-2i-1}: \ k(\mathrm{sd}(\langle F_m\rangle))_i \to k(\mathrm{sd}(\langle F_m\rangle))_{d-1-i}$$

is an injection for a generic degree 1 element ω in S. Note that if G is a Zariski-open set in $k[x_v \mid v \text{ vertex of } F_m]_1$ then $G \times k[x_v \mid v \in \Delta \setminus F_m \text{ vertex}]_1$ is Zariski-open in S_1. By construction, $\widetilde{\Delta}$ is shellable and therefore by the induction hypothesis the multiplication

$$\omega^{d-2i-1}: \ k(\mathrm{sd}(\widetilde{\Delta}))_i \to k(\mathrm{sd}(\widetilde{\Delta}))_{d-1-i}$$

is an injection for generic ω. Since the intersection of two non-empty Zariski-open sets is non-empty, the multiplication

$$(\omega^{d-2i-1}, \omega^{d-2i-1}) : k(\mathrm{sd}(\widetilde{\Delta}))_i \oplus k(\mathrm{sd}(\langle F_m\rangle))_i \to k(\mathrm{sd}(\widetilde{\Delta}))_{d-1-i} \oplus k(\mathrm{sd}(\langle F_m\rangle))_{d-1-i}$$

is an injection for a generic $\omega \in S_1$.

Our aim is to show that $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)_i = 0$ for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$. As soon as this is shown, the above commutative diagram implies that the multiplication

$$\omega^{d-2i-1}: \ k(\mathrm{sd}(\Delta))_i \to k(\mathrm{sd}(\Delta))_{d-1-i}$$

is injective for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$ and ω as above. For the computation of $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)$ we consider the following exact sequence of S-modules:

$$0 \to \Theta S \to S \to S/\Theta \to 0.$$

Since $\mathrm{Tor}_0(M,N) \cong M \otimes_R N$ and $\mathrm{Tor}_1(R,M) = 0$ for R-modules M and N, we get the following Tor-long exact sequence (see Theorem 1.1.10 in Chapter 1)

$$0 \to \mathrm{Tor}_1(S/\Theta, k[\mathrm{sd}(\sigma)]) \to \Theta S \otimes_S k[\mathrm{sd}(\sigma)] \to k[\mathrm{sd}(\sigma)] \to S/\Theta \otimes_S k[\mathrm{sd}(\sigma)] \to 0.$$

From the exactness of this sequence we deduce

$$\mathrm{Tor}_1(S/\Theta, k[\mathrm{sd}(\sigma)]) = \mathrm{Ker}(\Theta S \otimes_S k[\mathrm{sd}(\sigma)] \to k[\mathrm{sd}(\sigma)]).$$

Since we have $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta) \cong \mathrm{Tor}_1(S/\Theta, k[\mathrm{sd}(\sigma)])$, and by the fact that the isomorphism is grading preserving, we finally get that

$$\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta) \cong \mathrm{Ker}(\Theta S \otimes_S k[\mathrm{sd}(\sigma)] \to k[\mathrm{sd}(\sigma)])$$

as graded S-modules. The grading of $\Theta S \otimes_S k[\mathrm{sd}(\sigma)]$ is given by $\deg(f \otimes_S g) = \deg_S(f) + \deg_S(g)$, where $\deg_S$ refers to the grading induced by S.

As aforementioned, for generic Θ, $\widetilde{\Theta} := \{\theta_1, \ldots, \theta_{d-1}\}$ is an l.s.o.p. for $k[\mathrm{sd}(\sigma)]$. Thus the kernel of the map

$$(\Theta S \otimes_S k[\mathrm{sd}(\sigma)])_i \to (k[\mathrm{sd}(\sigma)])_i : \ b \otimes f \mapsto bf$$

is zero if and only if the kernel of the map

$$((\theta_d) \otimes_S (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta}))_i \to (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_i : \ \theta_d \otimes f \mapsto \theta_d f$$

is zero, which is the case if and only if the kernel of the multiplication map

$$\theta_d : \ (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_{i-1} \to (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_i : \ f \mapsto \theta_d f$$

is zero. We have a shift (-1) in the grading since the last map θ_d increases the degree by one. By construction, σ is a pure subcomplex of the boundary of a $(d-1)$-simplex and thus it is shellable. Since $\dim(\sigma) = d-2$ the induction hypothesis applies to $\mathrm{sd}(\sigma)$. Thus, the multiplication

$$\theta_d^{d-2i-2} : \ (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_i \to (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_{d-i-2}$$

is an injection for $0 \le i \le \lfloor \frac{d-3}{2} \rfloor$ for a generic degree one element $\theta_d \in S$. In particular, the multiplication

$$\theta_d : \ (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_i \to (k[\mathrm{sd}(\sigma)]/\widetilde{\Theta})_{i+1}$$

is injective as well. Thus, $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)_i = 0$ for $1 \le i \le \lfloor \frac{d-3}{2} \rfloor + 1 = \lfloor \frac{d-1}{2} \rfloor$. In particular, $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)_i = 0$ for $1 \le i \le \lfloor \frac{d-2}{2} \rfloor$. Note that $(\Theta S \otimes_S k[\mathrm{sd}(\sigma)])_0 = 0$, hence $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)_0 = 0$. To summarize, $\mathrm{Tor}_1(k[\mathrm{sd}(\sigma)], S/\Theta)_i = 0$ for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$, which completes the proof. □

As we expatiate in the next section Theorem 4.2.3 has many consequences for the h-vectors of barycentric subdivisions of shellable simplicial complexes.

Our next aim is to introduce so-called polytopal complexes and to describe how the notion of shellability carries over to those complexes. A similar proof as the one of Theorem 4.2.3 – differing mostly in the base of the induction – can be used to show an almost strong Lefschetz property for the barycentric subdivision of a polytopal complex as well. We now state the necessary definitions.

Definition 4.2.6. A *polytopal complex* is a finite, non-empty collection $\mathscr{C}$ of polytopes in some $\mathbb{R}^t$ that contains all the faces of its polytopes, and such that the intersection of two of its polytopes is a face of each of them. The elements of $\mathscr{C}$ are called *faces* of $\mathscr{C}$.

Along the lines of simplicial complexes we can define notion such as facets, dimension, pureness and barycentric subdivision for polytopal complexes. Moreover, the property of being shellable – originally defined for simplicial complexes – can be extended to polytopal complexes in a canonical way. For more details on polytopal complexes see e.g., [Zie95].

Definition 4.2.7. Let $\mathscr{C}$ be a pure $(d-1)$-dimensional polytopal complex. A *shelling* of $\mathscr{C}$ is a linear ordering $F_1, F_2, \ldots, F_m$ of the facets of $\mathscr{C}$ such that either $\mathscr{C}$ is 0-dimensional, or it satisfies the following conditions:

(i) The boundary complex $\mathscr{C}(\partial F_1)$ of the first facet F_1 has a shelling.

(ii) For $1 < j \leq m$ the intersection of the facet F_j with the previous facets is non-empty and is the beginning segment of a shelling of the $(d-2)$-dimensional boundary complex of F_j, that is,

$$F_j \cap \left(\bigcup_{i=1}^{j-1} F_i \right) = G_1 \cup G_2 \cup \ldots \cup G_r$$

for some shelling $G_1, G_2, \ldots, G_r, \ldots, G_t$ of $\mathscr{C}(\partial F_j)$, and $1 \leq r \leq t$.

A polytopal complex is called *shellable* if it is pure and has a shelling.

We can now give the analogous result to Theorem 4.2.3.

Theorem 4.2.8. *Let Δ be a shellable $(d-1)$-dimensional polytopal complex. Then* $\mathrm{sd}(\Delta)$ *is almost strong Lefschetz over $\mathbb{R}$.*

Proof. We give a sketch of the proof, indicating the needed modifications with respect to the proof of Theorem 4.2.3.

We use double induction on the dimension of Δ and the number of facets f_{d-1}^{Δ} of Δ. For $f_{d-1}^{\Delta} = 1$, note that the barycentric subdivision of a polytope is combinatorially isomorphic to a simplicial polytope (see [ES74]). Indeed, let P be a polytope on vertex set $[n]$. First we subdivide the facets of $\partial(P)$ by coning from their barycenters. After slightly lifting the barycenters we assure that convexity is maintained. We then do the same with the codimension 1 faces of $\partial(P)$, then with the codimension 2 faces of $\partial(P)$, and so on. In the end, the boundary complex of the resulting polytope is combinatorially isomorphic to $\mathrm{sd}(\partial(P))$. Clearly it is simplicial, since it is an order complex.
Theorem 3.2.7 implies that $\mathrm{sd}(\partial(P))$ is $(d-1)$-Lefschetz over $\mathbb{R}$. By Lemma 4.2.4 the same holds for $\mathrm{cone}_{[n]}(\mathrm{sd}(\partial(P))) = \mathrm{sd}(P)$, where we take the cone over the vertex corresponding to the barycenter of P, i.e., corresponding to $[n]$. Together with the $\dim \Delta = 0$ case, this provides the base of the induction.
The induction step works as in the proof of Theorem 4.2.3. $\square$

Note that in the above proof we really need the algebraic version of the classical g-theorem (Theorem 3.2.7), whereas in the proof of Theorem 4.2.3 it was not required. We pay credit for this by obtaining the almost strong Lefschetz property for barycentric subdivisions of polytopal complexes only over $\mathbb{R}$.

4.3 Numerical consequences for the h-vector

In this section our aim is to derive inequalities for the h-vector entries of the barycentric subdivision of a shellable simplicial complex and the ones of the barycentric subdivision of a shellable polytopal complex. Using a result of Stanley we are able to extend those combinatorial consequences to Cohen-Macaulay simplicial complexes.

We have seen in Theorem 2.2.5 of Chapter 2 that shellable simplicial complexes are Cohen-Macaulay. While the converse is not true, Stanley showed that these two families of complexes have the same set of h-vectors.

Theorem 4.3.1. *[Sta96, Theorem II.3.3] Let $s = (s_0, \dots, s_d)$ be a sequence of integers. The following conditions are equivalent.*

(i) s is the h-vector of a shellable simplicial complex.

(ii) s is the h-vector of a Cohen-Macaulay simplicial complex.

(iii) s is an M-sequence.

From Theorems 4.2.3 and 4.2.8 we are able to derive the following consequences.

Corollary 4.3.2. *(i) Let Δ be a Cohen-Macaulay complex (over some field). Then the g-vector of the barycentric subdivision of Δ is an M-sequence. In particular, the g-conjecture holds for barycentric subdivisions of simplicial spheres, of Gorenstein* complexes and of 2-Cohen-Macaulay complexes.*

(ii) Let P be a shellable polytopal complex. Then the g-vector of the barycentric subdivision of P is an M-sequence.

Before we give the proof of this result note that Corollary 4.3.2 verifies the generalized g-conjecture, suggested by Björner and Swartz (see Conjecture 3.1.9), in the special case of barycentric subdivisions of 2-Cohen-Macaulay complexes, Gorenstein* complexes and even Cohen-Macaulay complexes – the latter class of simplicial complexes being not included in the proper conjecture. As already mentioned the non-negativity of the g-vector of barycentric subdivisions of Gorenstein* complexes can also be deduced from Karu's result – showing the non-negativity of the cd-index for order complexes of Gorenstein* posets (see Theorem 3.2.20). We now give the proof of Corollary 4.3.2.

Proof of Corollary 4.3.2. We only prove (i) since the proof of (ii) is verbatim the same.
If Δ is a $(d-1)$-dimensional Cohen-Macaulay complex with h-vector $h^\Delta = (h_0^\Delta, \dots, h_d^\Delta)$ then by Theorem 4.3.1 there exists a shellable simplicial complex Γ having the same h-vector as Δ. Using the transformation of the h-vector due to Brenti and Welker (see Theorem 4.1.2) it directly follows that the barycentric subdivisions $\mathrm{sd}(\Delta)$ and $\mathrm{sd}(\Gamma)$ exhibit the same h-vector. Hence, we can assume that Δ is shellable.

By Theorem 4.2.3, for a generic l.s.o.p. Θ and a generic degree one element ω, the multiplication

$$\omega^{d-1-2i}: \ (k[\mathrm{sd}(\Delta)]/\Theta)_i \to (k[\mathrm{sd}(\Delta)]/\Theta)_{d-1-i}$$

is an injection for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$, hence the multiplication

$$\omega: \ (k[\mathrm{sd}(\Delta)]/\Theta)_i \to (k[\mathrm{sd}(\Delta)]/\Theta)_{i+1}$$

is an injection as well. (This conclusion is vacuous for $d \le 1$.) Since $\mathrm{sd}(\Delta)$ is Cohen-Macaulay it holds that $h_i^{\mathrm{sd}(\Delta)} = \dim_k(k[\mathrm{sd}(\Delta)]/\Theta)_i$. Using the just shown injections, we deduce that $g_i^{\mathrm{sd}(\Delta)} = \dim_k(k[\mathrm{sd}(\Delta)]/(\Theta,\omega))_i$ for $0 \le i \le \lfloor \frac{d}{2} \rfloor$. Hence $g^{\mathrm{sd}(\Delta)}$ is an M-sequence. $\square$

Note that it directly follows from the non-negativity of the g-vector of $\mathrm{sd}(\Delta)$ that for a $(d-1)$-dimensional Cohen-Macaulay complex Δ it holds that $h_{i-1}^{\mathrm{sd}(\Delta)} \le h_i^{\mathrm{sd}(\Delta)}$ for $0 \le i \le \lfloor \frac{d}{2} \rfloor$ which yields the first half of the unimodality property of the h-vector.
Using Theorem 4.2.3 we are further able to deduce the following h-vector inequalities for barycentric subdivisions of Cohen-Macaulay complexes.

Corollary 4.3.3. *Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex. Then $h_{d-i-1}^{\mathrm{sd}(\Delta)} \ge h_i^{\mathrm{sd}(\Delta)}$ for any $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$.*

Proof. By the same reasoning as in the beginning of the proof of Corollary 4.3.2 we can assume that Δ is a shellable complex. Let Θ be an l.s.o.p. for $k[\mathrm{sd}(\Delta)]$ and let ω be a degree one element in $k[x_1,\ldots,x_n]$, where $n := f_0^{\mathrm{sd}(\Delta)}$. From Theorem 4.2.3 and Remark 4.2.2 we conclude that if Θ and ω have been chosen as generic elements, the multiplication

$$\omega^{d-1-2i}: \ (k[\mathrm{sd}(\Delta)]/\Theta)_i \to (k[\mathrm{sd}(\Delta)]/\Theta)_{d-1-i}$$

is an injection for $1 \le i \le \lfloor \frac{d-2}{2} \rfloor$. Since $h_i^{\mathrm{sd}(\Delta)} = \dim_k(k[\mathrm{sd}(\Delta)]/\Theta)_i$ this implies $h_i^{\mathrm{sd}(\Delta)} \le h_{d-1-i}^{\mathrm{sd}(\Delta)}$. $\square$

4.4 Inequalities for a special refinement of the Eulerian numbers

In this section we discuss consequences which can be deduced from Theorem 4.2.3 for the coefficients appearing in the transformation of the h-vector of a simplicial complex into the h-vector of its barycentric subdivision (see Theorem 4.1.2). Recall from Chapter 4.1 that the numbers $A(d,i,j)$ are a refinement of the Eulerian statistics on permutations, counting permutations in S_d having i descents and being the image of 1 equal to j. In [BW06] Brenti and Welker showed that those numbers feature the following symmetry.

Lemma 4.4.1. *[BW06, Lemma 2.5]*

$$A(d,i,j) = A(d,d-1-i,d+1-j)$$

for $d \geq 1$, $1 \leq j \leq d$ *and* $0 \leq i \leq d-1$.

The following corollary tells us how the $A(d,i,j)$ behave when increasing the number of descents while keeping the image of 1 fixed. Crucial in the proof of this corollary are Theorem 4.2.3 and Lemma 4.4.1.

Corollary 4.4.2. *(i)*

$$A(d+1,j,r) \leq A(d+1,d-1-j,r)$$

for $d \geq 0$, $1 \leq r \leq d+1$ *and* $0 \leq j \leq \left\lfloor \frac{d-2}{2} \right\rfloor$.

(ii)

$$A(d+1,0,r+1) \leq A(d+1,1,r+1) \leq \ldots \leq A(d+1,\left\lfloor \frac{d}{2} \right\rfloor,r+1)$$

and

$$A(d+1,d,r+1) \leq A(d+1,d-1,r+1) \leq \ldots \leq A(d+1,\left\lceil \frac{d}{2} \right\rceil,r+1)$$

for $d \geq 1$ *and* $1 \leq r \leq d$. *For d odd,* $A(d+1,\left\lfloor \frac{d}{2} \right\rfloor,r+1)$ *may be larger or smaller then* $A(d+1,\left\lceil \frac{d}{2} \right\rceil,r+1)$.

Proof. Let Δ be a shellable $(d-1)$-dimensional simplicial complex. Let $F_1,\ldots,F_m$ be a shelling of Δ with $m \geq 2$ and set $\widetilde{\Delta} := \langle F_1,\ldots,F_{m-1}\rangle$, i.e., $\widetilde{\Delta}$ is the simplicial complex obtained by restricting Δ to the first $m-1$ facets in the shelling. Let further $n := f_0^{\mathrm{sd}(\Delta)}$ and $S := k[x_1,\ldots,x_n]$. Since $\mathrm{sd}(\widetilde{\Delta})$ is a subcomplex of $\mathrm{sd}(\Delta)$ we get the following short exact sequence of S-modules:

$$0 \to I \to k[\mathrm{sd}(\Delta)] \to k[\mathrm{sd}(\widetilde{\Delta})] \to 0,$$

where I denotes the kernel of the projection on the right-hand side. Let $\Theta := \{\theta_1,\ldots,\theta_d\}$ be an l.s.o.p. for both $k[\mathrm{sd}(\Delta)]$ and $k[\mathrm{sd}(\widetilde{\Delta})]$. It is possible to choose such a system since the set of l.s.o.p. for $k[\mathrm{sd}(\Delta)]$ and for $k[\mathrm{sd}(\widetilde{\Delta})]$, respectively, is a non-empty Zariski-open set. Hence, their intersection is a non-empty Zariski-open set as well. As $\widetilde{\Delta}$ is shellable, it is Cohen-Macaulay and therefore $\mathrm{sd}(\widetilde{\Delta})$ is Cohen-Macaulay as well. Hence dividing out by Θ yields the following exact sequence of S-modules:

$$0 \to I/(I \cap \Theta) \to k[\mathrm{sd}(\Delta)]/\Theta \to k[\mathrm{sd}(\widetilde{\Delta})]/\Theta \to 0. \tag{4.2}$$

Consider the following commutative diagram

$$\begin{array}{ccccc}
0 & \to & (I/(I\cap\Theta))_i & \to & (k[\mathrm{sd}(\Delta)]/\Theta)_i \\
 & & \downarrow \omega^{d-1-2i} & & \downarrow \omega^{d-1-2i} \\
0 & \to & (I/(I\cap\Theta))_{d-1-i} & \to & (k[\mathrm{sd}(\Delta)]/\Theta)_{d-1-i}
\end{array}$$

where ω is a degree one element in S. By Theorem 4.2.3 the multiplication

$$\omega^{d-2i-1}: \ (k[\mathrm{sd}(\Delta)]/\Theta)_i \to (k[\mathrm{sd}(\Delta)]/\Theta)_{d-1-i}$$

is an injection for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$ and generic ω. It hence follows that also the multiplication

$$\omega^{d-1-2i}: \ (I/(I\cap\Theta))_i \to (I/(I\cap\Theta))_{d-1-i} \tag{4.3}$$

is an injection for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$. Furthermore, we deduce from the sequence (4.2) that $\dim_k(I/(I\cap\Theta))_t = h_t^{\mathrm{sd}(\Delta)} - h_t^{\mathrm{sd}(\widetilde{\Delta})}$ for $0 \le t \le d$. In order to compute this difference we determine the change in the h-vector of $\widetilde{\Delta}$ when adding the last facet F_m of the shelling. Let $r_m := |\mathrm{res}(F_m)|$. Proposition 2.2.7 implies that $h_{r_m}^{\Delta} = h_{r_m}^{\widetilde{\Delta}} + 1$ and $h_i^{\Delta} = h_i^{\widetilde{\Delta}}$ for $i \neq r_m$. Using Theorem 4.1.2 we deduce

$$\begin{aligned}
h_i^{\mathrm{sd}(\Delta)} &= \sum_{r=0}^{d} A(d+1,i,r+1)h_r^{\Delta} \\
&= \sum_{r=0}^{d} A(d+1,i,r+1)h_r^{\widetilde{\Delta}} + A(d+1,i,r_m+1) \\
&= h_i^{\mathrm{sd}(\widetilde{\Delta})} + A(d+1,i,r_m+1).
\end{aligned}$$

Thus $\dim_k(I/(I\cap\Theta))_i = A(d+1,i,r_m+1)$ for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$. From (4.3) it follows that $A(d+1,i,r_m+1) \le A(d+1,d-1-i,r_m+1)$. Take Δ to be the boundary of the d-simplex. Since in this case $h_i^{\Delta} \ge 1$ for $0 \le i \le d$, i.e., restriction faces of all possible sizes occur in a shelling of Δ, it follows that $A(d+1,i,r) \le A(d+1,d-1-i,r)$ for every $1 \le r \le d+1$ and $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$. This shows (i).
To show (ii) we use that the injections in (4.3) induce injections

$$\omega: \ (I/(I\cap\Theta))_i \to (I/(I\cap\Theta))_{i+1}$$

for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$. Thus, $A(d+1,i,r_m+1) \le A(d+1,i+1,r_m+1)$. The same reasoning as in (i) shows that $A(d+1,i,r) \le A(d+1,i+1,r+1)$ for $0 \le i \le \lfloor \frac{d-2}{2} \rfloor$ and $1 \le r \le d$. The second part of (ii) follows from the first one using Lemma 4.4.1. $\square$

We want to emphasize that we were not able to find a purely combinatorial proof of Corollary 4.4.2 which does not use Theorem 4.2.3. It would be interesting to know if there exists such a proof.

The next example shows that we cannot say in general which of the numbers $A(d+1,\lfloor\frac{d}{2}\rfloor,r+1)$ and $A(d+1,\lceil\frac{d}{2}\rceil,r+1)$ is the larger one.

Example 4.4.3. One computes that $A(6,2,3)=60>48=A(6,3,3)$ while $A(6,2,4)=48<60=A(6,3,4)$. This shows that for d odd $A(d+1,\lfloor\frac{d}{2}\rfloor,r+1)$ may be larger or smaller then $A(d+1,\lceil\frac{d}{2}\rceil,r+1)$.

Remark 4.4.4. Let $A_{d,r} := (A(d+1,0,r+1),\ldots,A(d+1,d,r+1))$. Corollary 4.4.2 in particular tells us that the sequence of numbers $A_{d,r}$ is unimodal. This fact can already be deduced from [BW06]. Applying the linear transformation of Theorem 4.1.2 to the $(r+1)$-st unit vector of $\mathbb{R}^d$ yields the sequence $A_{d,r}$. It then follows from [BW06, Theorem 3.1, Remark 3.3] (and Theorem 4.1.4) that the generating polynomial of this sequence is real-rooted. Since $A(d+1,i,r+1)\geq 1$ for $i\geq 1$ the sequence $A_{d,r}$ has no internal zeros. Together with the real-rootedness this implies that $A_{d,r}$ is unimodal. However, this argument tells us nothing about the position of a peak. From Corollary 4.4.2 it follows that the sequence $A_{d,r}$ has a peak at position $\lfloor\frac{d}{2}\rfloor$ or $\lceil\frac{d}{2}\rceil$. But we cannot exclude that the sequence is endowed with several peaks, i.e., it could happen that the sequence stays constant on a stretch after having attained its maximal value.

In the following our discussion focuses on determining a peak of the h-vector of the barycentric subdivision of a Cohen-Macaulay complex. In Theorem 4.1.4 it is stated that the h-vector of the barycentric subdivision of a Boolean cell complex with non-negative h-vector entries is unimodal. What remains open is the location of its peaks. Knowing the location of a peak of the sequence $(A(d+1,0,r+1),\ldots,A(d+1,d,r+1))$ (Corollary 4.4.2) helps us to determine the location of a peak of the h-vector of the barycentric subdivision of a Boolean cell complex, although not uniquely. Using further arguments we are finally able to show that under certain conditions the h-vector of the barycentric subdivision of a $(d-1)$-dimensional Boolean cell complex has at most two peaks.

Corollary 4.4.5. *Let Δ be a $(d-1)$-dimensional Boolean cell complex with $h_i^\Delta\geq 0$ for $0\leq i\leq d$. Then $h^{\mathrm{sd}(\Delta)}$ has a peak at position $\frac{d}{2}$ if d is even and at position $\frac{d-1}{2}$ or $\frac{d+1}{2}$ if d is odd. In both situations $h^{\mathrm{sd}(\Delta)}$ has at most two peaks.*
In particular, all assertions hold for Cohen-Macaulay complexes.

Proof. Since $h_i^\Delta\geq 0$ for $0\leq i\leq d$, by Theorem 4.1.4 and Corollary 4.4.2 (ii) we deduce

$$\begin{aligned} h_j^{\mathrm{sd}(\Delta)} &= \sum_{r=0}^{d} A(d+1,j,r+1)h_r^\Delta \\ &\overset{Corollary\, 4.4.2(ii)}{\leq} \sum_{r=0}^{d} A(d+1,j+1,r+1)h_r^\Delta = h_{j+1}^{\mathrm{sd}(\Delta)} \end{aligned} \tag{4.4}$$

for $0 \leq j \leq \lfloor \frac{d-2}{2} \rfloor$. Thus $h_0^{\mathrm{sd}(\Delta)} \leq h_1^{\mathrm{sd}(\Delta)} \leq \ldots \leq h_{\lfloor \frac{d}{2} \rfloor}^{\mathrm{sd}(\Delta)}$.
Similarly one shows $h_{\lceil \frac{d}{2} \rceil}^{\mathrm{sd}(\Delta)} \geq h_{\lceil \frac{d}{2} \rceil+1}^{\mathrm{sd}(\Delta)} \geq \ldots \geq h_d^{\mathrm{sd}(\Delta)}$, when applying Corollary 4.4.2 (ii) for $j \geq \lceil \frac{d}{2} \rceil$.
If d is even it holds that $\lfloor \frac{d}{2} \rfloor = \lceil \frac{d}{2} \rceil = \frac{d}{2}$ and $h^{\mathrm{sd}(\Delta)}$ has a peak at position $\frac{d}{2}$.

It remains to prove that Δ features at most two peaks. For this aim, we show that we have the following strict inequalities if Δ is non-empty:

$$h_0^{\mathrm{sd}(\Delta)} < h_1^{\mathrm{sd}(\Delta)} < \ldots < h_{\lfloor \frac{d-1}{2} \rfloor}^{\mathrm{sd}(\Delta)} \text{ and } h_{\lceil \frac{d}{2} \rceil}^{\mathrm{sd}(\Delta)} > \ldots > h_d^{\mathrm{sd}(\Delta)}.$$

If $\sigma \in S_d$ is a permutation with $\sigma(1) = 1$ then σ has no descent at position 1. Therefore, $A(d+1, j, 1) = E(d, j)$, where $E(d, j)$ is the Eulerian number equal to the number of permutations on $[d]$ with j descents. Let $E_d(t) := \sum_{0 \leq j \leq d-1} E(d, j)t^j$ be the Euler polynomial. It is well-known that $E_d(t)$ has a non-negative integer expansion with respect to the basis $\{t^i(t+1)^{d-1-2i}\}_{0 \leq i \leq \lfloor \frac{d-1}{2} \rfloor}$, see e.g., [FS70] or [SWG83] for a combinatorial proof. Note that the coefficient of $(t+1)^{d-1}$ in this expansion equals 1 for $d \geq 1$. Comparing the coefficients in the two expansions of the Euler polynomial and using that for any non-negative integer k the binomial coefficients $\binom{k}{l}$ strictly increase for $0 \leq l \leq \lfloor \frac{k}{2} \rfloor$ and strictly decrease for $\lceil \frac{k}{2} \rceil \leq l \leq k$ we conclude that the numbers $A(d+1, j, 1)$ strictly increase for $0 \leq j \leq \lfloor \frac{d-1}{2} \rfloor$ and strictly decrease for $\lceil \frac{d-1}{2} \rceil \leq j \leq d$ (note that $A(d+1, d-1, 1) = 1 > 0 = A(d+1, d, 1)$). As Δ is non-empty, it holds that $h_0^{\Delta} = 1$. Comparing the $r = 0$ summands on both sides of Inequality 4.4 implies the desired strict inequalities. □

Example 4.4.6. If d is odd, depending on whether $h_{\lfloor \frac{d}{2} \rfloor}^{\mathrm{sd}(\Delta)} \leq h_{\lceil \frac{d}{2} \rceil}^{\mathrm{sd}(\Delta)}$ or vice versa the peak of $h^{\mathrm{sd}(\Delta)}$ is at position $\frac{d-1}{2}$ or $\frac{d+1}{2}$. For example, for $d = 3$ let Δ be the 2-skeleton of the 4-simplex. Then $h^{\Delta} = (1, 2, 3, 4)$ and $h^{\mathrm{sd}(\Delta)} = (1, 22, 33, 4)$, i.e., the peak is at position $\frac{3+1}{2} = 2$. If Δ consists of 2 triangles intersecting along one edge, i.e., $\Delta := \langle \{1, 2, 3\}, \{2, 3, 4\} \rangle$, then $h^{\Delta} = (1, 1, 0, 0)$ and $h^{\mathrm{sd}(\Delta)} = (1, 8, 3, 0)$. In this case the h-vector peaks at position $\frac{3-1}{2} = 1$.

If d is even, taking Δ to be a $(d-1)$-dimensional simplex shows that $h_{\frac{d}{2}-1}^{\mathrm{sd}(\Delta)} = h_{\frac{d}{2}}^{\mathrm{sd}(\Delta)}$ may occur.

Whereas Corollary 4.4.2 deals with the behavior of the numbers $A(d, i, j)$ when changing the number of descents i while keeping the image of 1 fixed, in the following we want to analyze the behavior of those numbers when changing the image of 1 while retaining the number of descents unchanged.

Corollary 4.4.7. *(i)*

$$A(d+1, j, 1) \leq A(d+1, j, 2) \leq \ldots \leq A(d+1, j, d+1)$$

for $\lceil \frac{d+1}{2} \rceil = \lfloor \frac{d+2}{2} \rfloor \leq j \leq d$.

(ii)
$$A(d+1,j,1) \geq A(d+1,j,2) \geq \ldots \geq A(d+1,j,d+1)$$
for $0 \leq j \leq \lfloor \frac{d-1}{2} \rfloor$.

(iii) $A(d+1,\frac{d}{2},1) \leq A(d+1,\frac{d}{2},2) \leq \ldots \leq A(d+1,\frac{d}{2},\frac{d}{2}+1) \geq A(d+1,\frac{d}{2},\frac{d}{2}+2) \geq \ldots \geq A(d+1,\frac{d}{2},d+1)$ *if* d *is even.*

(iv) $A(d+1,j,1) = A(d+1,j+1,d+1)$ *for* $0 \leq j \leq d-1$.

Proof. To prove (i) we need to show that $A(d+1,j,r) \leq A(d+1,j,r+1)$ for $1 \leq r \leq d$ and $\lfloor \frac{d+2}{2} \rfloor \leq j \leq d$. For $j = d$ this follows from $\{\ \sigma \in S_{d+1} \mid |\mathrm{des}(\sigma)| = d\} = \{\ (d+1)d\ldots 21\ \}$. Let $C^d_{j,r} := \{\ \sigma \in S_{d+1} \mid |\mathrm{des}(\sigma)| = j, \sigma(1) = r\}$. Consider the following map

$$\begin{array}{rcl} \phi^d_{j,r} : \{\sigma \in C^d_{j,r} \mid \sigma(2) \neq r+1\} & \to & \{\ \sigma \in C^d_{j,r+1} \mid \sigma(2) \neq r\} \\ \sigma & \mapsto & (r,r+1)\sigma. \end{array}$$

For $\sigma \in C^d_{j,r}$, if $|\mathrm{des}(\sigma)| = j$ and $\sigma(2) \neq r+1$ then σ and $(r,r+1)\sigma$ have the same descent set, hence $|\mathrm{des}((r,r+1)\sigma)| = j$ as well. As $((r,r+1)\sigma)(1) = r+1$, the function $\phi^d_{j,r}$ is well-defined. Since $(r,r+1)^2 = \mathrm{id}$ it follows that $\phi^d_{j,r}$ is invertible and therefore

$$|\{\ \sigma \in C^d_{j,r} \mid \sigma(2) \neq r+1\}| = |\{\ \sigma \in C^d_{j,r+1} \mid \sigma(2) \neq r\}|.$$

If $\sigma \in C^d_{j,r}$ and $\sigma(2) = r+1$, then all of the j descents must occur at position at least 2. The sequence $\tilde{\sigma} = (r+1)\sigma(3)\ldots\sigma(d+1)$ can be identified with a permutation τ in S_d with $\tau(1) = r$ and vice versa via the order preserving map $[d+1] \setminus \{r\} \to [d]$, hence the descent set is preserved under this identification. Therefore $|\{\ \sigma \in C^d_{j,r} \mid \sigma(2) = r+1\}| = |\{\ \sigma \in C^{d-1}_{j,r}\}| = A(d,j,r)$. On the other hand, if $\sigma \in C^d_{j,r+1}$ and $\sigma(2) = r$ then σ has exactly $j-1$ descents at positions $\{2,\ldots,d\}$. A similar argument then implies

$$|\{\ \sigma \in C^d_{j,r+1} \mid \sigma(2) = r\}| = |\{\ \sigma \in C^{d-1}_{j-1,r}\}| = A(d,j-1,r).$$

By Corollary 4.4.2 (ii) it holds that $A(d,j,r) \leq A(d,j-1,r)$ for $d-2 \geq j-1 \geq \lceil \frac{d-1}{2} \rceil$, i.e., $d-1 \geq j \geq \lceil \frac{d+1}{2} \rceil = \lfloor \frac{d+2}{2} \rfloor$. Combining the above, we obtain

$$A(d+1,j,r) \leq A(d+1,j,r+1) \text{ for } 1 \leq r \leq d \text{ and } \lfloor \frac{d+2}{2} \rfloor \leq j \leq d-1,$$

and (i) follows.

(ii) follows directly from (i) and Lemma 4.4.1.

For the proof of (iii) we only show

$$A(d+1,\frac{d}{2},1) \leq A(d+1,\frac{d}{2},2) \leq \ldots \leq A(d+1,\frac{d}{2},\frac{d}{2}+1).$$

The other inequalities in (iii) follow directly from this part by Lemma 4.4.1. The proof of (i) shows that $\left|\left\{ \sigma \in C^d_{\frac{d}{2},r} \mid \sigma(2) \neq r+1 \right\}\right| = \left|\left\{ \sigma \in C^d_{\frac{d}{2},r+1} \mid \sigma(2) \neq r \right\}\right|$. As in the proof of (i), it remains to prove that $A(d,\frac{d}{2},r) \le A(d,\frac{d}{2}-1,r)$ for $1 \le r \le \frac{d}{2}$. By Lemma 4.4.1 it holds that $A(d,\frac{d}{2},r) = A(d,\frac{d}{2}-1,d+1-r)$. For $1 \le r \le \frac{d}{2}$ we have $r \le d+1-r$ and (ii) then implies $A(d,\frac{d}{2}-1,r) \ge A(d,\frac{d}{2}-1,d+1-r)$ which finishes the proof of (iii).

In order to show (iv) note that by Lemma 4.4.1 we have that

$$A(d+1,j,1) = A(d+1,d-j,d+1).$$

If $\sigma = (d+1)\sigma(2)\dots\sigma(d+1) \in C^d_{d-j,d+1}$, then the reverse permutation $\widetilde{\sigma} := (d+1)\sigma(d+1)\dots\sigma(2)$ has a descent at position 1 and whenever there is an ascent in $\sigma(2)\dots\sigma(d+1)$. Since σ has $d-j-1$ descents at positions $\{2,\dots,d\}$ this implies $|\operatorname{des}(\widetilde{\sigma})| = 1+(d-1)-(d-j-1) = j+1$, i.e., $\widetilde{\sigma} \in C^d_{j+1,d+1}$. We recover σ by repeating this construction and hence $A(d+1,j,1) = A(d+1,j+1,d+1)$. $\square$

We conclude this section with an alternative and more compact representation of the results of Corollary 4.4.7.

For fixed $d \in \mathbb{N}$ let $\mathscr{A}(d) := (A(d,i,j))_{\substack{0\le i\le d-1\\1\le j\le d}}$ be the matrix with entries $A(d,i,j)$ for fixed d. For pairs $(i,j),(i',j')$ we set $(i,j) < (i',j')$ if either $i < i'$, or $i = i'$ and $j > j'$. This defines a total order on the set of pairs (i,j). Using this ordering for the indices of the entries of the matrix we can write the matrix $\mathscr{A}(d)$ as a vector $A(d)$.

From Corollary 4.4.7 and Lemma 4.4.1 we immediately get the following.

Corollary 4.4.8. *The sequence $A(d)$ is unimodal and symmetric for $d \ge 1$. In particular, a peak of $A(d)$ lies in the middle and there are at most two peaks.*

4.5 Open problems and conjectures

This section deals with some open problems and further conjectures which come to mind when having obtained the results of the previous sections.

The numerical results in Corollaries 4.3.2 and 4.4.5, in particular that the h-vector of the barycentric subdivision of a simplicial complex Δ decreases from the middle onwards, suggest that the barycentric subdivision of a Cohen-Macaulay complex might feature the following property which is stronger than being weak Lefschetz.

Definition 4.5.1. Let Δ be a $(d-1)$-dimensional simplicial complex on vertex set $[n]$. Then Δ is called *Lefschetz* if there exists an l.s.o.p. Θ for $k[\Delta]$ and a degree one element $\omega \in k[x_1,\dots,x_n]$ such that the multiplication maps

$$\omega:\ (k[\Delta]/\Theta)_i \longrightarrow (k[\Delta]/\Theta)_{i+1}$$

have full rank for every i.

In particular, for barycentric subdivisions of Cohen-Macaulay complexes this means that we have injections for $0 \leq i < \frac{d}{2}$, i.e., $\mathrm{sd}(\Delta)$ is weak Lefschetz, and surjections for $\lceil \frac{d}{2} \rceil \leq i \leq \dim \Delta$.

Conjecture 4.5.2. Let k be an infinite field and let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex over k. Then the barycentric subdivision of Δ is Lefschetz over k.

It already follows from Theorem 4.2.3 that we can attain injections up to the middle degree of $k[\mathrm{sd}(\Delta)]/\Theta$ if we require Δ not only to be Cohen-Macaulay but to be shellable. Note that injections from the degree i-part of $k[\mathrm{sd}(\Delta)]/\Theta$ to the degree $(d-1-i)$-part of $k[\mathrm{sd}(\Delta)]/\Theta$ induce injections from the degree i-part of $k[\mathrm{sd}(\Delta)]/\Theta$ to the degree $(i+1)$-part of $k[\mathrm{sd}(\Delta)]/\Theta$. But it is still open – even for shellable complexes – if we can choose an l.s.o.p. Θ and a degree one element ω such that in addition to the injections up to the middle degree of $k[\mathrm{sd}(\Delta)]/\Theta$ we have surjections from the middle degree of $k[\mathrm{sd}(\Delta)]/\Theta$ onwards. Furthermore, being the numerical results of Corollary 4.3.2 valid for Cohen-Macaulay complexes adumbrates that the algebraic result of Theorem 4.2.3 might be extendable to Cohen-Macaulay complexes.

Conjecture 4.5.3. Let k be an infinite field and let Δ be a $(d-1)$-dimensional Cohen-Macaulay complex over k. Then the barycentric subdivision of Δ is almost strong Lefschetz over k.

However, we do not know how the proof of this conjecture could work. An induction on the number of facets of Δ – similar to the one in the proof of Theorem 4.2.3 – does not seem to be possible since in general after removing a facet from an arbitrary Cohen-Macaulay complex the new complex does not remain Cohen-Macaulay.

Problem 4.5.4. The proofs of Corollaries 4.3.2 and 4.3.3 use the algebraic result of Theorem 4.2.3. It would be interesting to know if there exist purely combinatorial proofs for these results.

Besides the barycentric subdivision there exist several other subdivision operations on simplicial complexes, one of which is the so-called r-th edgewise subdivision of a simplicial complex, see e.g., [BR05]. This subdivision operation is strongly related to the Veronese construction on standard graded k-algebras. More precisely, the Stanley-Reisner ideal of the r-th edgewise subdivision of a simplicial complex Δ is a certain initial ideal of the defining ideal of the r-th Veronese algebra of $k[\Delta]$. For more details see [BR05]. Recall, that if $A = \bigoplus_{i \geq 0} A_i$ is a standard graded k-algebra – denoting A_i the i-th graded component of A – then the r-th *Veronese algebra* of A is defined to be $kA^{(r)} = \bigoplus_{i \geq 0} A_{ir}$. Brenti and Welker showed in [BW08] how the Hilbert series of a standard graded k-algebra is transformed into the one of its r-th Veronese algebra. This also yields a transformation of the h-vector of a simplicial complex into the h-vector of the r-th edgewise subdivision of this very complex. From this transformation they were able to deduce combinatorial consequences for the Hilbert series of Veronese algebras and the h-vectors of r-th edgewise subdivisions of simplicial complexes. Those results are in the same spirit as Brenti and Welker's results for h-vectors of barycentric subdivisions. In particular, they showed that the h-polynomial of the r-th edgewise subdivision of a Cohen-Macaulay simplicial complex is real-rooted for r sufficiently large. This implies (using further arguments) log-concavity and unimodality for the h-vector of the r-th edgewise subdivision of a Cohen-Macaulay complex if r is sufficiently large. This yields the following conjecture.

Conjecture 4.5.5. Let Δ be a $(d-1)$-dimensional Cohen-Macaulay simplicial complex. Then its r-th edgewise subdivision is almost strong Lefschetz for r sufficiently large.

We are already working on this conjecture in the case that Δ is shellable and that $r \geq \dim\Delta + 1$. There is a paper in preparation where Conjecture 4.5.5 is proved for exactly this situation [KWon].

Part III

Notion of Depth and Annihilator Numbers

5 Exterior depth and generic annihilator numbers

This chapter is topically independent from the previous chapter. Maybe one of the most classical notion in commutative algebra is the (symmetric) depth of an S-module, where $S := k[x_1,\dots,x_n]$ denotes the polynomial ring in n variables over an arbitrary field k. Aramova, Avramov and Herzog introduced in [AAH00] an exterior analogue of this classical notion, the so-called exterior depth of an E-module, where $E := k\langle e_1,\dots,e_n\rangle$ denotes the exterior algebra. We study this notion and compare it to the symmetric depth of an S-module in the case of squarefree modules. This comparison is sensible since there exists an equivalence of categories between the category of squarefree S-modules and the one of squarefree E-modules, as was shown by Römer [Röm01]. We use this relation between squarefree S- and squarefree E-modules in order to contrast certain symmetric invariants with the corresponding exterior ones.
To give a first example, under the equivalence of categories, just mentioned, the symmetric Stanley-Reisner ring $k[\Delta]$ of a simplicial complex Δ corresponds to the exterior Stanley-Reisner ring $k\{\Delta\}$ of Δ. Several algebraic and homological invariants of $k[\Delta]$ over S are analogous to invariants of $k\{\Delta\}$ over E.
In the first section of this chapter we cover the just contemplated topics. Besides the pure statement of the relations between symmetric and exterior invariants, with an emphasis on the symmetric and the exterior depth, we also try to explain why it is not always possible or sensible to transfer a certain notion over S verbatim to the exterior algebra E. As an example of such an invariant we mention the projective dimension of an S-module.

In the second section of this chapter we address the so-called annihilator numbers of a module which can be considered as an iteration of the concept of depth. Conca, Herzog and Hibi [CHH04] and Trung [Tru87] were the first ones to introduce annihilator numbers with respect to a sequence over the polynomial ring. We extend and slightly change – in order to get adapted to the situation over E – their definition to annihilator numbers with respect to a sequence over the exterior algebra. Despite the differences in the definitions those numbers share several properties, e.g., they do not depend on the particular sequence when choosing this sequence from a certain non-empty Zariski-open set. In both situations, this gives rise to the definition of the so-called symmetric and exterior generic annihilator numbers, respectively.
In the sequel, we therefore restrict our studies to the generic annihilator numbers. If $J \subseteq E$ is a graded ideal we can give a combinatorial characterization of the exterior generic

annihilator numbers. Furthermore, in both situations – in the symmetric as well as in the exterior one – positive linear combinations of those numbers even though not the same can be used to bound the graded Betti numbers over S and over E, respectively.

The third section is concerned with a conjecture by Herzog concerning the symmetric and the exterior generic annihilator numbers, respectively. He conjectured that those numbers are the minimal ones under all symmetric and exterior generic annihilator numbers with respect to a particular sequence. We disprove this conjecture by giving a counterexample to it. The same example including some minor differences works over the symmetric as well as over the exterior algebra.

In the last section we consider the symmetric and the exterior Stanley-Reisner ring of a simplicial complex. We try to convey the results concerning the symmetric and the exterior depth and the generic annihilator numbers to this particular situation and to interpret them for simplicial complexes. We give a characterization of simplicial complexes whose exterior Stanley-Reisner ring has a certain exterior depth in terms of the exterior algebraic shifting of the simplicial complex. From the combinatorial description of the exterior generic annihilator numbers obtained in Section 5.2 we derive a combinatorial description of those numbers for the special case of the exterior Stanley-Reisner ring of a simplicial complex. Using this description we are able to express the graded symmetric Betti numbers of the symmetric Stanley-Reisner ring of the exterior shifting of a simplicial complex as positive linear combinations of the exterior generic annihilator numbers of the exterior Stanley-Reisner ring of Δ.

Throughout this chapter, we use $S := k[x_1, \dots, x_n]$ to denote the polynomial ring in n variables and we use $E := k\langle e_1, \dots, e_n \rangle$ to denote the exterior algebra over a field k. If not otherwise mentioned we assume throughout the whole chapter that k is infinite.
We further use $\mathscr{M}$ to denote the category of finitely generated graded left and right E-modules M satisfying $am = (-1)^{\deg a \deg m} ma$ for homogeneous elements $a \in E$, $m \in M$. For example, if $J \subseteq E$ is a graded ideal then E/J belongs to $\mathscr{M}$. Note that every left ideal is a right ideal and vice versa. The relation $am = (-1)^{\deg a \deg m} ma$ follows just by changing the order of the elements of am.

The work which yielded the results of this chapter was carried out with Gesa Kämpf. All results can be found in [KK09a] and [KK09b].

5.1 The exterior depth

The aim of this section is to compare several algebraic definitions and invariants, defined over the symmetric algebra, with the corresponding definitions and invariants over the exterior algebra. The main focus in doing so lies on the symmetric depth of a squarefree S-module and the exterior depth of an associated squarefree E-module. In order to make such a comparison reasonable it is essential that to each squarefree S-module we can assign a squarefree E-module in a canonical way and that this assignment behaves in a certain sense well. Indeed, Römer [Röm01] showed that there exists an equivalence of categories between the categories of squarefree S-modules and the category of squarefree E-modules. Before stating this equivalence explicitly we need to introduce the notion of squarefree modules over S and squarefree modules over E. Squarefree S-modules were first introduced by Yanagawa as a generalization of squarefree monomial ideals in [Yan00, Definition 2.1]. Römer defined in [Röm01, Definition 1.4] the corresponding notion of a squarefree E-module.

Definition 5.1.1. (i) A finitely generated $\mathbb{N}^n$-graded S-module $N = \oplus_{a\in\mathbb{N}^n} N_a$ is called *squarefree* if the multiplication map $N_a \to N_{a+\varepsilon_i}: \ y \mapsto x_i y$ is bijective for any $a \in \mathbb{N}^n$ and for all $i \in \operatorname{supp}(a)$, where $\operatorname{supp}(a) = \{j \mid a_j \neq 0\}$.

(ii) A finitely generated $\mathbb{N}^n$-graded E-module $M = \oplus_{a\in\mathbb{N}^n} M_a$ is called *squarefree* if it has only squarefree non-zero components.

Example 5.1.2. Let Δ be a simplicial complex on vertex set $[n]$. Then its symmetric Stanley-Reisner ring $k[\Delta]$ is a squarefree S-module whereas its exterior face ring $k\{\Delta\}$ is a squarefree E-module.

Aramova, Avramov and Herzog and Römer construct in [AAH00] and [Röm01], respectively, a minimal free resolution of a squarefree E-module N_E starting with the minimal free resolution of a squarefree S-module N which is relate to N_E. The assignment $N \mapsto N_E$ induces an equivalence between the categories of squarefree S-modules and squarefree E-modules (where the morphisms are the $\mathbb{N}^n$-graded homomorphisms). The construction is as follows.

Construction 5.1.3. Let $(\mathfrak{F}_\bullet, \theta)$ be an acyclic complex of free $\mathbb{N}^n$-graded S-modules. By *acyclic* we mean, that for

$$\mathfrak{F}_\bullet: \ \ldots \to F_m \xrightarrow{\theta} F_{m-1} \to \ldots \to F_1 \xrightarrow{\theta} F_0 \to 0$$

it holds that $H_i(\mathfrak{F}_\bullet) = 0$ for all $i > 0$.
For each F_i we choose a homogeneous basis B_i such that $\deg(f)$ is squarefree for all $f \in B_i$. To $a \in \mathbb{N}^n$ and to $f \in B_i$ we assign the symbol $y^{(a)} f$ and we set $\deg(y^{(a)} f) := a + \deg(f)$. Let now G_l be the free $\mathbb{N}^n$-graded right E-module with basis

$$\left\{ y^{(a)} f \;\middle|\; a \in \mathbb{N}^n,\ f \in B_i,\ \operatorname{supp}(a) \subseteq \operatorname{supp}(f),\ l = |a| + i \right\}.$$

If the differential θ at F_i in the complex $(\mathfrak{F}_\bullet, \theta)$ is given by

$$\theta(f) := \sum_{j:\, f_j \in B_{i-1}} \lambda_j x^{b-b_j} f_j \quad \text{with} \quad \lambda_j \in k,\ b = \deg(f),\ b_j = \deg(f_j),$$

we define homomorphisms $G_l \to G_{l-1}$ of $\mathbb{N}^n$-graded E-modules by

$$\gamma(y^{(a)}f) := (-1)^{|b|} \sum_{j \in \mathrm{supp}(a)} y^{(a-\varepsilon_j)} f e_j,$$
$$\vartheta(y^{(a)}f) := (-1)^{|a|} \sum_{j:\, f_j \in B_{i-1}} y^{(a)} f_j \lambda_j e_{b_j}^{-1} e_b.$$

We set $\delta := \gamma + \vartheta :\ G_l \to G_{l-1}$.

In [AAH00, Theorem 1.3] it is proved that $(G_\bullet, \delta)$ is a complex of free $\mathbb{N}^n$-graded E-modules in $\mathscr{M}$. Furthermore, if $(G^{'}_\bullet, \delta^{'})$ is the complex obtained from different homogeneous bases $B^{'}_i$ of F_i then $G_\bullet$ and $G^{'}_\bullet$ are isomorphic as complexes of $\mathbb{N}^n$-graded modules. In special situations, one can say even more about the complex $(G_\bullet, \delta)$. The following theorem was first proven by Aramova, Avramov and Herzog in [AAH00] for squarefree S-modules of the form $N = S/I$. Römer noted in [Röm01] that the same proof works for the more general case of squarefree S-modules.

Theorem 5.1.4. *[Röm01, Theorem 1.2] If $(\mathfrak{F}_\bullet, \theta)$ is the minimal free $\mathbb{N}^n$-graded S-resolution of a squarefree S-module N, then $(G_\bullet, \delta)$ is the minimal free $\mathbb{N}^n$-graded E-resolution of $N_E := \mathrm{Coker}(G_1 \to G_0)$.*

Thus, starting with the minimal free graded resolution of an S-module N we can construct the minimal graded free resolution of an associated E-module N_E. The following remark gives an easy example of how the equivalence of categories works for the case of squarefree monomial ideals.

Remark 5.1.5. Let $I \subsetneq S$ be a squarefree monomial ideal, i.e., $I = (x_F \mid F \in A)$ for $A \subseteq 2^{[n]}$. In this case $(S/I)_E = E/J$, where $J = (e_F \mid F \in A)$. Here, for $F = \{i_1 < \ldots < i_r\} \subseteq [n]$ we set $x_F := x_{i_1} \cdot \ldots \cdot x_{i_r}$ and $e_F := e_{i_1} \wedge \ldots \wedge e_{i_r}$.
In particular, if Δ is a simplicial complex on vertex set $[n]$, then – as mentioned in the introduction of this chapter – under the equivalence of categories we have $(k[\Delta])_E = k\{\Delta\}$.

We now want to define the exterior depth of an E-module. For this aim, we need to introduce the notion of regular elements of an E-module N. Recall, that for a graded S-module N a linear form $y \in S_1$ is called *N-regular* if y is not a zero-divisor on N. Note that on the contrary, for every element $v \in E$ it always holds that $v^2 = 0$. Thus, every linear form in E_1 is a zero-divisor. It is therefore not reasonable to define regular elements over the exterior algebra in exactly the same way as over the polynomial ring. For an element $v \in E$ to be regular we therefore demand that any annihilation $vm = 0$ is a consequence of $v^2 = 0$.

Definition 5.1.6. [AAH00] Let $N \in \mathscr{M}$ be an E-module. A linear form $v \in E_1$ is called *N-regular* if $0 :_N v = vN$. A sequence $v_1, \ldots, v_r$ of linear forms in E_1 is called an *N-regular sequence* if v_i is an $N/(v_1, \ldots, v_{i-1})N$-regular element for $1 \le i \le r$ and $N/(v_1, \ldots, v_r)N \neq 0$.

Note that a linear form $v \in E_1$ is N-regular if and only if the annihilator of v in N is the smallest possible. The following examples illustrates the notion of regular elements over S and over E and also alludes to possible differences.

Example 5.1.7. Let Δ be the simplicial complex on vertex set $[3]$ whose maximal faces are the edges $\{1,2\}$ and $\{2,3\}$. Then $x_1 + x_3$ and x_2 are $k[\Delta]$-regular. They even constitute a $k[\Delta]$-regular sequence. On the contrary, $e_1 + e_3$ is not $k\{\Delta\}$-regular since for example $(e_1 + e_3) \wedge e_3 = e_1 \wedge e_3 \in J_\Delta = (e_1 \wedge e_3)$. But e_2 is $k\{\Delta\}$-regular.

It is a classical result that for an S-module N every N-regular sequence (defined verbatim as over E) can be extended to a maximal one and that all maximal N-regular sequences have the same length, which is called the (symmetric) depth of N over S, see e.g., [BH98]. Aramova, Avramov and Herzog proved in [AAH00] that the analogous result is true over the exterior algebra.

Theorem 5.1.8. *[AAH00] Let $N \in \mathscr{M}$ be an E-module. Every N-regular sequence can be extended to a maximal N-regular sequence. Furthermore, all maximal N-regular sequences have the same length.*

The above theorem justifies the following definition.

Definition 5.1.9. Let $N \in \mathscr{M}$. The common length of all maximal N-regular sequences is called the *depth* of N over E, or the *exterior depth* of N, and it is denoted by $\operatorname{depth}_E N$.

The symmetric and the exterior depth, respectively, behave quite similarly in many cases, e.g., neither of them does change when passing to the generic initial ideal with respect to the reverse lexicographic order. We make this more explicit now.
If we are working over the polynomial ring the following result can be deduced from a result of Bayer and Stillman [BS87].

Theorem 5.1.10. *[Eis95, Corollary 19.11] Let $I \subseteq S$ be a graded ideal and let $\operatorname{gin}_{<_{\text{rlex}}}(I)$ be the generic initial ideal of I with respect to the reverse lexicographic order, where $x_1 < \ldots < x_n$. Then*

$$\operatorname{depth}_S(S/I) = \operatorname{depth}_S(S/\operatorname{gin}_{<_{\text{rlex}}}(I)).$$

In the exterior situation Herzog and Terai showed in [HT99] that the exterior depth remains the same under the passage to the generic initial ideal.

Theorem 5.1.11. *[HT99, Proposition 2.3] Let $J \subseteq E$ be a graded ideal and let $\operatorname{gin}_{<_{\text{rlex}}}(J)$ be the generic initial ideal of J with respect to the reverse lexicographic order, where $e_1 < \ldots < e_n$. Then*

$$\operatorname{depth}_E(E/J) = \operatorname{depth}_E(E/\operatorname{gin}_{<_{\text{rlex}}}(J)).$$

In the following we compare the symmetric depth of a squarefree S-module N with the exterior depth of the corresponding E-module N_E. Our aim is to show that the symmetric depth of N is always greater or equal than the exterior depth of N_E. One of the crucial facts we need in order to prove this result is the following relation between the graded Betti numbers of N over S and the graded Betti numbers of N_E over E, which is an immediate consequence of Construction 5.1.3, see [Röm01].

Corollary 5.1.12. *[Röm01, Corollary 1.3] Let N be a squarefree S-module and let N_E be the associated squarefree E-module. Let β^E and β^S denote the graded Betti numbers over E and S, respectively. Then*

$$\beta^E_{i,i+j}(N_E) = \sum_{k=0}^{i} \binom{i+j-1}{j+k-1} \beta^S_{k,k+j}(N).$$

Remark 5.1.13. From the above Corollary it follows conductively that the graded Betti numbers of N_E over E – having a certain shift j – are determined only by those graded Betti numbers of N over S having the same shift j. Being the coefficients in Equation 5.1.12 positive this implies

$$\operatorname{reg}_S(N) = \operatorname{reg}_E(N_E).$$

In order to state and prove our result we still need to introduce the notion of the complexity of an E-module, see e.g., [AAH00] and [KR08].

Definition 5.1.14. Let $N \in \mathscr{M}$ be an E-module. Then

$$\operatorname{cx}_E(N) := \inf\left\{c \in \mathbb{Z} \mid \beta^E_i(N) \leq \alpha i^{c-1} \text{ for some } \alpha \in \mathbb{R} \text{ and for all } i \geq 1\right\}$$

is called the *complexity* of N.

The complexity gauges the polynomial growth of the exterior Betti numbers of N and is therefore a measure for the size of a minimal free resolution of N by free E-modules. It can thus be regarded as the exterior analogue of the projective dimension. As mentioned in Chapter 1 even though it makes sense to define the projective dimension for E-modules – and indeed those modules are included in the definition – this notion is almost meaningless in this case since the minimal free resolution of an E-module is infinite (unless in the case of free modules). Over the polynomial ring we have the classical Auslander-Buchsbaum formula which relates the projective dimension of an S-module to its symmetric depth, see Theorem 1.1.6. Over the exterior algebra, Aramova, Avramov and Herzog [AAH00] proved a similiar result, correlating the complexity of an E-module and the exterior depth.

Theorem 5.1.15. *[AAH00, Theorem 3.2] Let $N \in \mathscr{M}$. Then*

$$\operatorname{depth}_E(N) + \operatorname{cx}_E(N) = n.$$

We have now provided all notion and facts we need to prove the desired inequality between the symmetric depth of an S-module and the exterior depth of the associated E-module.

Theorem 5.1.16. *Let N be a finitely generated $\mathbb{N}^n$-graded squarefree S-module. Then*

$$\operatorname{depth}_E(N_E) \le \operatorname{depth}_S(N). \tag{5.1}$$

Proof. From Theorem 5.1.15 it follows that

$$\operatorname{depth}_E(N_E) + \operatorname{cx}_E(N_E) = n.$$

By Auslander-Buchsbaum (Theorem 1.1.6) it holds that

$$\operatorname{proj}\dim_S(N) + \operatorname{depth}_S(N) = n.$$

Therefore, it suffices to show $\operatorname{proj}\dim_S(N) \le \operatorname{cx}_E(N_E)$. From Corollary 5.1.12 it follows that

$$\begin{aligned}\beta_i^E(N_E) &= \sum_{j\ge 0} \beta^E_{i,i+j}(N_E)\\ &= \sum_{j\ge 0}\sum_{k=0}^{i} \binom{i+j-1}{j+k-1} \beta^S_{k,k+j}(N).\end{aligned}$$

Let $m_j^{(i)} := \max\left\{k+j \;\middle|\; \beta^S_{k,k+j}(N) \neq 0,\ 0 \le k \le i\right\}$. From the above formula for the Betti numbers we conclude that $\beta^E_{i,i+j}(N_E)$ is a polynomial in i of degree $m_j^{(i)} - 1$. It hence follows that $\beta_i^E(N_E)$ is a polynomial in i of degree $m^{(i)} - 1$, where $m^{(i)} := \max\left\{m_j^{(i)} \;\middle|\; j \ge 0\right\}$. This yields for the complexity

$$\begin{aligned}\operatorname{cx}_E(N_E) &= \sup\{m^{(i)} \mid i \ge 0\}\\ &= \sup\{k+j \mid \beta^S_{k,k+j}(N) \neq 0,\ k\ge 0,\ j \ge 0\}\\ &= \max\{l \mid \beta^S_{i,l}(N) \neq 0,\ i \ge 0,\ l \ge 0\},\end{aligned} \tag{5.2}$$

where the last equality holds because N has a finite S-resolution. Let $p := \operatorname{proj}\dim_S(N)$. Then by definition of p there exists $k \ge 0$ such that $\beta^S_{p,p+k}(N) \neq 0$. This implies $\operatorname{cx}_E(N_E) \ge p+k \ge p = \operatorname{proj}\dim_S(N)$. This finally shows the claim. $\square$

Remark 5.1.17. The above proof shows that in general the following inequalities hold

$$\operatorname{cx}_E(N_E) \ge \operatorname{proj}\dim_S(N) \tag{5.3}$$

and

$$\operatorname{depth}_S(N) - \operatorname{depth}_E(N_E) = \operatorname{cx}_E(N_E) - \operatorname{proj}\dim_S(N) \le \operatorname{reg}_S(N). \tag{5.4}$$

It is a natural matter to ask if there are classes of squarefree modules for which equality holds in Inequality (5.1). In general, we cannot give an answer to this issue. Nevertheless, in Section 5.4 we identify one such class in the special case of Stanley-Reisner rings of simplicial complexes.

We want to conclude this section with an example illustrating the results obtained in this section.

Example 5.1.18. Let Δ be a simplicial complex which consists of two edges intersecting in a common vertex. Since Δ is Cohen-Macaulay it holds that $\operatorname{depth}_S(k[\Delta]) = 2$. The only non-zero graded Betti numbers of $k[\Delta]$ are $\beta^S_{0,0}(k[\Delta]) = 1$ and $\beta^S_{1,2}(k[\Delta]) = 1$. Furthermore, the minimal free resolution of $k[\Delta]$ has length 1, i.e., $\operatorname{proj}\dim_S(k[\Delta]) = 1$. Thus,

$$\operatorname{proj}\dim_S(k[\Delta]) + \operatorname{depth}_S(k[\Delta]) = 1 + 2 = 3,$$

which is exactly the equality we can deduce from Auslander-Buchsbaum.
On the other hand we have $\operatorname{depth}_E(k\{\Delta\}) = 1$. Note that $ae_1 + be_2 + ce_3$ is a $k\{\Delta\}$-regular element if and only if $b \neq 0$. Thus, $\operatorname{depth}_S(k[\Delta]) = 2 \geq 1 = \operatorname{depth}_E(k\{\Delta\})$. From Equation (5.2) in the proof of Theorem 5.1.16 it follows that

$$\operatorname{cx}_E(k\{\Delta\}) = \max\left\{ l \;\middle|\; \beta_{i,l} \neq 0,\ i \geq 0,\ l \geq 0 \right\} = 2.$$

Hence,

$$\operatorname{cx}_E(k\{\Delta\}) + \operatorname{depth}_E(k\{\Delta\}) = 2 + 1,$$

which also follows from Theorem 5.1.15.
Comparing the projective dimension of $k[\Delta]$ with the complexity of $k\{\Delta\}$ we obtain

$$\operatorname{cx}_E(k\{\Delta\}) = 2 \geq 1 = \operatorname{proj}\dim_S(k[\Delta]).$$

This inequality already follows from Inequality (5.3) in Remark 5.1.17.

5.2 Annihilator numbers

In this section our interest lies on the symmetric and the exterior annihilator numbers of an S- and E-module, respectively. Those numbers can be considered as an iteration of the concept of depth.

In the first part, we consider the symmetric annihilator numbers of an S-module, originally introduced by Trung in [Tru87] and subsequently studied by Conca, Herzog and Hibi in [CHH04]. We state the definition as well as some fundamental results for those numbers – including a certain genericity property giving rise to the definition of the so-called symmetric generic annihilator numbers. We omit the proofs of these results and just want to remark at this point that – whenever there are similar results over the exterior algebra – those are proven in a similar way.

In the second part we focus on the exterior annihilator numbers of an E-module with respect to a sequence. Analogously to the symmetric annihilator numbers we are able to show that those numbers do not depend on the particular chosen sequence when choosing this very one from a certain non-empty Zariski-open set. This fact leads us to the definition of the exterior generic annihilator numbers. In the following we derive a combinatorial description of these numbers for E-modules of the form E/J, where $J \subseteq E$ is a graded ideal. We are further able to relate those invariants to the symmetric Betti numbers of the corresponding S-module (via the equivalence of categories given in Section 5.1).

In the last part of this section we use almost regular sequences and annihilator numbers in order to give a proof of the well-known fact that being a regular sequence is a non-empty Zariski-open condition.

5.2.1 Symmetric annihilator numbers

It was Trung [Tru87] who first introduced the so-called symmetric annihilator numbers of an S-module. They can be considered as a measure for how strongly a sequence of linear forms fails to be regular. In addition, it turns out that those numbers are closely related to the graded Betti numbers over S.
If $y \in S_1$ is a regular element on a finitely generated graded S-module N, then the multiplication by y is injective, hence $0 :_N y = 0$. This property is weakened if one only requires the annihilator $0 :_N y$ to have finite length, i.e., to be of finite k-vector space dimension.

Definition 5.2.1. Let N be a finitely generated graded S-module. A linear form $y \in S_1$ is called *almost regular* on N if $0 :_N y$, i.e., the kernel of the multiplication with y, has finite length. A sequence $y_1, \dots, y_r$ of linear forms is called an *almost regular sequence* on N if y_i is an almost regular element on $N/(y_1, \dots, y_{i-1})N$ for $1 \le i \le r$.

As was shown by Herzog and Hibi in [HH08, Corollary 4.3.2] it is always possible to find a k-basis of S_1 which is an almost regular sequence. In the special case of S-modules which are quotients of Borel-fixed ideals an explicit (canonical) almost regular sequence is known.

Lemma 5.2.2. *[HH08, Proposition 4.3.3] Let $I \subseteq S$ be a Borel-fixed ideal. Then $x_1, \dots, x_n$ is an almost regular sequence on S/I.*

The symmetric annihilator numbers of an S-module with respect to a sequence are defined in the following way.

Definition 5.2.3. [CHH04, Tru87] Let $y_1, \dots, y_n$ be a sequence of linear forms in S_1 and let N be a finitely generated graded S-module. We denote by $A_i(y_1, \dots, y_n; N)$ the graded module

$$0 :_{N/(y_1,\dots,y_{i-1})N} y_i.$$

We write A_i instead of $A_i(y_1,\dots,y_n;N)$ if it is clear from the context which sequence is used. The k-vector space dimension of $(A_i)_j$ is denoted by

$$\alpha_{i,j}(y_1,\dots,y_n;N) := \dim_K(A_i)_j$$

for $j \in \mathbb{Z}$ and $1 \le i \le n$. The numbers $\alpha_{i,j}(y_1,\dots,y_n;N)$ are called the *symmetric annihilator numbers* of N with respect to the sequence $y_1,\dots,y_n$.

Note that it directly follows from the definition of an almost regular sequence that for such a sequence there exist only finitely many symmetric annihilator numbers which are non-zero. So far, the symmetric annihilator numbers of an S-module may actually depend on the chosen sequence. Herzog and Hibi showed in [HH08] that this is not the case if one picks those numbers from a certain non-empty Zariski-open set.

Theorem 5.2.4. *[HH08, Theorem 4.3.6] Let $I \subseteq S$ be a graded ideal. Then there exists a non-empty Zariski-open set $U \subseteq GL_n(k)$ such that $\gamma(x) = (\sum_{i=1}^n \gamma_{i,1}x_i,\dots,\sum_{i=1}^n \gamma_{i,n}x_i)$ is almost regular for all $\gamma = (\gamma_{i,j})_{1\le i,j\le n} \in U$. Moreover,*

$$\alpha_{i,j}(\gamma(x);S/I) = \alpha_{i,j}(x_1,\dots,x_n;S/\operatorname{gin}_{<_{\text{rlex}}}(I))$$

for all $\gamma \in U$.

Theorem 5.2.4 justifies the following definition.

Definition 5.2.5. Let $I \subseteq S$ be a graded ideal in S. We set

$$\alpha_{i,j}(S/I) := \alpha_{i,j}(x_1,\dots,x_n;S/\operatorname{gin}_{<_{\text{rlex}}}(I))$$

for $j \in \mathbb{Z}$ and $1 \le i \le n$ and call these numbers the *generic annihilator numbers* of S/I over S.

Herzog and Hibi [HH08] posed the question when the symmetric generic annihilator numbers vanish.

Proposition 5.2.6. *[HH08, Proposition 4.3.4] Let N be a finitely generated graded S-module and let $y_1,\dots,y_n \in S_1$ be a k-basis of S_1 which is almost regular on N. Let further $\alpha_i(\mathbf{y};N) := \sum_{j\in\mathbb{Z}} \alpha_{i,j}(\mathbf{y};N)$ for all i. Then $\alpha_i(\mathbf{y};N) = 0$ if and only if $i < \operatorname{depth}_S(N)$.*

Remark 5.2.7. Let N be a finitely generated graded S-module and let $y_1,\dots,y_n$ be a k-basis of S_1 which is almost regular on N. From Proposition 5.2.6 it in particular follows that $y_1,\dots,y_r$ is a regular sequence on N, where $r = \operatorname{depth}_S(N)$.

In [CHH04] Conca, Herzog and Hibi further established a relation between the symmetric Betti numbers of an S-module and its generic annihilator numbers over S. They showed that the symmetric Betti numbers can be bounded from above by positive linear combinations of certain of the annihilator numbers.

Theorem 5.2.8. *[CHH04, Corollary 1.2], [HH08, Proposition 4.3.11] Let $I \subseteq S$ be a graded ideal and let $y_1, \ldots, y_n$ be a k-basis of S_1 which is almost regular on S/I. Then*

$$\beta^S_{i,i+j}(S/I) \leq \sum_{l=1}^{n-i+1} \binom{n-l}{i-1} \alpha_{l,j}(y_1, \ldots, y_n; S/I).$$

For the special class of componentwise linear ideals, see Chapter 1 for the definition, and for generic sequences of linear forms, i.e., sequences which can be used to determine the symmetric generic annihilator numbers, the above bound is tight.

Corollary 5.2.9. *[CHH04, Theorem 1.5], [HH08] Let $I \subseteq S$ be a componentwise linear ideal in S. Then*

$$\beta^S_{i,i+j}(S/I) = \sum_{l=1}^{n-i+1} \binom{n-l}{i-1} \alpha_{l,j}(S/I).$$

5.2.2 Exterior annihilator numbers

In this section we introduce the so-called exterior annihilator numbers of an E-module with respect to a sequence. They are defined verbatim the same as over the polynomial ring. Whenever it is possible we try to transfer the results for the symmetric annihilator numbers, stated in Section 5.2.1, to the exterior annihilator numbers. As in the symmetric case it turns out that the exterior annihilator numbers with respect to different sequences coincide when taking the latter ones from a certain non-empty Zariski-open set. This helps us to get rid of the sequence and justifies the definition of the exterior generic annihilator numbers. In addition to the analogous results of the results in Section 5.2.1 we derive a combinatorial description of the exterior generic annihilator numbers.

Over the polynomial ring generic annihilator numbers were defined using almost regular sequences. Over the exterior algebra the notion of almost regular elements is superfluous. Since the exterior algebra E has only finitely many graded components every finitely generated graded E-module is of finite k-vector space dimension. Thus, being an almost regular sequence is no sensible condition over E. We can therefore directly give the definition of the exterior annihilator numbers of an E-module with respect to a sequence. They are defined in almost the same way as their symmetric counterpart. We only have to keep in mind that every element in E_1 is a zero-divisor. This issue is accounted for by dividing out the image of the multiplication with the particular linear form.

Definition 5.2.10. Let $v_1, \ldots, v_n$ be a basis of E_1 and let $N \in \mathscr{M}$. The numbers

$$\alpha_{i,j}(v_1, \ldots, v_n; N) := \dim_k \left(\left(0 :_{N/(v_1,\ldots,v_{i-1})N} v_i\right) / \left(v_i N/(v_1, \ldots, v_{i-1})N\right) \right)_j$$

for $j \in \mathbb{Z}$ and $1 \leq i \leq n$ are called the *exterior annihilator numbers* of N with respect to $v_1, \ldots, v_n$.

For a linear form $v \in E_1$ and $N \in \mathscr{M}$ we have the complex

$$(N,v): \ \ldots \to N_{j-1} \xrightarrow{\cdot v} N_j \xrightarrow{\cdot v} N_{j+1} \to \ldots,$$

which we get by considering the multiplication with v on N. Since $v^2 = 0$ for every element in E this is indeed a complex. If we denote by $H^j(N,v)$ the homology modules of this complex it follows directly from the definition of the exterior annihilator numbers that

$$\alpha_{i,j}(v_1,\ldots,v_n;N) = \dim_k H^j(N/(v_1,\ldots,v_{i-1})N, v_i).$$

From the pure definition the exterior annihilator numbers do depend on the chosen sequence. However, as in the symmetric situation we are able to show that the sequence does not matter if we take it from a certain non-empty Zariski-open set. The proof of this fact follows exactly the same steps as the one of the corresponding result for the symmetric annihilator numbers (see Theorem 5.2.4).

Theorem 5.2.11. *Let $J \subseteq E$ be a graded ideal. Then there exists a non-empty Zariski-open set $U \subseteq \mathrm{GL}_n(k)$ such that*

$$\alpha_{i,j}(\gamma(e_1,\ldots,e_n);E/J) = \alpha_{i,j}(e_1,\ldots,e_n;E/\operatorname{gin}_{<_{\mathrm{rlex}}}(J))$$

for all $\gamma = (\gamma_{i,j})_{1\le i,j\le n} \in U$, where

$$\gamma(e_1,\ldots,e_n) = (\gamma_{1,1}e_1 + \ldots + \gamma_{n,1}e_n, \ldots, \gamma_{1,n}e_1 + \ldots + \gamma_{n,n}e_n)$$

and $<_{\mathrm{rlex}}$ denotes the reverse lexicographic order with respect to $e_1 < \ldots < e_n$.

Proof. Let

$$U' := \{\varphi \in \mathrm{GL}_n(K) \mid \operatorname{in}_{<_{\mathrm{rlex}}}(\varphi(J)) = \operatorname{gin}_{<_{\mathrm{rlex}}}(J)\}$$

be the non-empty Zariski-open set of linear transformations that can be used to compute the generic initial ideal of J. Set $U := \{\varphi^{-1} \mid \varphi \in U'\}$. Let $\gamma := \varphi^{-1} \in U$ and set $v_i := \gamma(e_i)$ for $1 \le i \le n$, i.e., $\varphi(v_i) = e_i$. As φ is an automorphism, $E/(J + (v_1,\ldots,v_i))$ and $E/(\varphi(J) + (e_1,\ldots,e_i))$ have the same Hilbert function. [AH00, Proposition 5.1] implies that

$$\operatorname{in}_{<_{\mathrm{rlex}}}(\varphi(J) + (e_1,\ldots,e_i)) = \operatorname{gin}_{<_{\mathrm{rlex}}}(J) + (e_1,\ldots,e_i)$$

(observe that we use the reversed order on $[n]$). Therefore also $E/(J + (v_1,\ldots,v_i))$ and $E/\big(\operatorname{gin}_{<_{\mathrm{rlex}}}(J) + (e_1,\ldots,e_i)\big)$ have the same Hilbert function. The sequences

$$\begin{aligned} 0 \ \to \ & H^j(E/(J+(v_1,\ldots,v_{i-1})),v_i) \to (E/(J+(v_1,\ldots,v_i)))_j \\ \xrightarrow{\cdot v_i} \ & (E/(J+(v_1,\ldots,v_{i-1})))_{j+1} \to (E/(J+(v_1,\ldots,v_i)))_{j+1} \to 0 \end{aligned}$$

and

$$\begin{aligned} 0 \quad &\to \quad H^j\left(E/\left(\operatorname{gin}_{<_{\mathrm{rlex}}}(J)+(e_1,\ldots,e_{i-1})\right),e_i\right) \to \left(E/\left(\operatorname{gin}_{<_{\mathrm{rlex}}}(J)+(e_1,\ldots,e_i)\right)\right)_j \\ &\xrightarrow{\cdot e_i} \quad \left(E/\left(\operatorname{gin}_{<_{\mathrm{rlex}}}(J)+(e_1,\ldots,e_{i-1})\right)\right)_{j+1} \to \left(E/\left(\operatorname{gin}_{<_{\mathrm{rlex}}}(J)+(e_1,\ldots,e_i)\right)\right)_{j+1} \to 0 \end{aligned}$$

are exact sequences of k-vector spaces. The vector space dimensions of the three latter vector spaces in the two sequences coincide. Hence it holds that

$$\begin{aligned} \alpha_{i,j}(v_1,\ldots,v_n;E/J) &= \dim_k H^j(E/(J+(v_1,\ldots,v_{i-1})),v_i) \\ &= \dim_k H^j(E/\left(\operatorname{gin}_{<_{\mathrm{rlex}}}(J)+(e_1,\ldots,e_{i-1})\right),e_i) \\ &= \alpha_{i,j}(e_1,\ldots,e_n;E/\operatorname{gin}_{<_{\mathrm{rlex}}}(J)). \end{aligned}$$

□

As mentioned beforehand, now the following definition makes sense.

Definition 5.2.12. Let $J \subseteq E$ be a graded ideal. We set

$$\alpha_{i,j}(E/J) := \alpha_{i,j}(e_1,\ldots,e_n;E/\operatorname{gin}_{<_{\mathrm{rlex}}}(J))$$

for $j \in \mathbb{Z}$ and $1 \leq i \leq n$ and call these numbers the *exterior generic annihilator numbers* of E/J.

As Herzog and Hibi did for the symmetric annihilator numbers (Proposition 5.2.6) we can give a criterion for the vanishing and the non-vanishing of the exterior generic annihilator numbers.

Proposition 5.2.13. *Let $J \subseteq E$ be a graded ideal. Set $\alpha_i(E/J) := \sum_{j\in\mathbb{Z}} \alpha_{i,j}(E/J)$ and let $1 \leq r \leq n$. Then $\alpha_i(E/J) = 0$ for all $i \leq r$ if and only if $r \leq \operatorname{depth}_E(E/J)$.*

Proof. By definition, it holds that

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(E/\operatorname{gin}_{<_{\mathrm{rlex}}}(J)).$$

Thus, it suffices to show the result for the generic initial ideal with respect to the reverse lexicographic order. In Section 5.4 (see Lemma 5.4.3) we show that $e_1,\ldots,e_i$ is a regular sequence on $E/\operatorname{gin}_{<_{\mathrm{rlex}}}(J)$ if and only if $i \leq \operatorname{depth}_E(E/\operatorname{gin}_{<_{\mathrm{rlex}}}(J)) = \operatorname{depth}_E(E/J)$. This directly implies the claim. □

Our next aim is to find a combinatorial description of the exterior generic annihilator numbers.

Theorem 5.2.14. *Let $J \subseteq E$ be a graded ideal. Then*

$$\alpha_{i,j}(E/J) = \left|\{\overline{e_F} \in E/\operatorname{gin}_{<_{\mathrm{rlex}}}(J) \mid \deg \overline{e_F} = j,\ \min F \geq i+1,\ \overline{e_F} \neq 0,\ \overline{e_i e_F} = 0\}\right|.$$

Here $\overline{e_F}$ denotes the projection of $e_F \in E$ on $E/\operatorname{gin}_{<_{\mathrm{rlex}}}(J)$.

Proof. Since $\alpha_{i,j}(E/J) = \alpha_{i,j}(E/\operatorname{gin}_{<_{\text{rlex}}}(J))$ we may assume that $J = \operatorname{gin}_{<_{\text{rlex}}}(J)$. Then $\alpha_{i,j}(E/J)$ can be computed using the sequence $e_1, \dots, e_n$, i.e.,

$$\alpha_{i,j}(E/J) = \dim_k H^j(E/J + (e_1, \dots, e_{i-1}), e_i) = \dim_k \left(\frac{(J + (e_1, \dots, e_{i-1})) : e_i}{J + (e_1, \dots, e_i)} \right)_j .$$

Thus

$$\alpha_{i,j}(E/J) = \left| \left\{ e_F \;\middle|\; \deg e_F = j, e_F \in (J + (e_1, \dots, e_{i-1})) : e_i,\ e_F \notin J + (e_1, \dots, e_i) \right\} \right| .$$

Let $e_F \in (J + (e_1, \dots, e_{i-1})) : e_i$ of degree j. Then $e_i e_F \in J + (e_1, \dots, e_{i-1})$. Since J is a monomial ideal, either $e_i e_F \in (e_1, \dots, e_{i-1})$ or $e_i e_F \in J$. In the first case it follows that $e_F \in (e_1, \dots, e_{i-1})$ such that we do not need to count it. In the second case, $e_F \notin J + (e_1, \dots, e_i)$ is equivalent to $e_F \notin J$ and $e_F \notin (e_1, \dots, e_i)$ or equivalently $\overline{e_F} \neq 0$ and $\min F \geq i + 1$. □

In order to get a result similar to Theorem 5.2.8 which relates the Betti numbers over E with the exterior generic annihilator numbers we use the Cartan-Betti numbers (see Chapter 1 for the definition).

Theorem 5.2.15. *Let $J \subseteq E$ be a graded ideal. Then*

$$h_{i,i+j}(r) \leq \sum_{k=1}^{r} \binom{r + i - k - 1}{i - 1} \alpha_{k,j}(E/J) \quad i \geq 1, j \geq 0$$

and equality holds for all $i \geq 1$ and $1 \leq r \leq n$ if and only if J is componentwise linear.

Proof. Let $v_1, \dots, v_n$ be a sequence of linear forms that can be used to compute the exterior generic annihilator numbers and the Cartan-Betti numbers of E/J, as well. Such a sequence exists as both conditions specify non-empty Zariski-open sets and the intersection of two non-empty Zariski-open sets remains non-empty. Set $\alpha_{i,j} := \alpha_{i,j}(E/J)$ and

$$A_i := \operatorname{Ker}\left(E/\left(J + (v_1, \dots, v_i)\right) \xrightarrow{\cdot v_i} E/\left(J + (v_1, \dots, v_{i-1})\right) \right)$$

for $1 \leq i \leq n$. Then A_i is a graded E-module and the k-vector space dimension of the j-th graded piece equals $\alpha_{i,j}(E/J)$. The above map occurs in the long exact sequence of Cartan homologies (see Proposition 1.1.18) since the 0-th Cartan homology is

$$H_0(v_1, \dots, v_r; E/J) = E/\left(J + (v_1, \dots, v_r)\right).$$

In order to simplify notation, in the following we write $H_i(r)$ for $H_i(v_1, \dots, v_r; E/J)$. By the above for $i = 1$ and $r = 1$ we obtain from the long exact Cartan homology sequence the following exact sequence

$$H_1(1)(-1)_{j+1} \to H_1(0)_{j+1} \to H_1(1)_{j+1} \to A_1(-1)_{j+1} \to 0.$$

Since $H_i(0) = 0$ for $i \geq 1$ this yields

$$h_{1,j+1}(1) = \alpha_{1,j}.$$

For $r \geq 1$ we have the exact sequence

$$H_1(r+1)(-1)_{j+1} \to H_1(r)_{j+1} \to H_1(r+1)_{j+1} \to A_{r+1}(-1)_{j+1} \to 0.$$

From this sequence we conclude by induction hypothesis on r

$$\begin{aligned} h_{1,j+1}(r+1) &\leq \alpha_{r+1,j} + h_{1,j+1}(r) \\ &\leq \alpha_{r+1,j} + \sum_{k=1}^{r} \binom{r-k}{0} \alpha_{k,j} \\ &= \sum_{k=1}^{r+1} \alpha_{k,j}. \end{aligned}$$

Now let $i > 1$. For $r = 1$ there is the exact sequence

$$H_i(0)_{i+j} \to H_i(1)_{i+j} \to H_{i-1}(1)(-1)_{i+j} \to H_{i-1}(0)_{i+j}.$$

The outer spaces in the sequence are zero, hence

$$h_{i,i+j}(1) = h_{i-1,i+j-1}(1) \leq \alpha_{1,j}$$

by induction hypothesis on i.

Let now $r \geq 1$. There is the exact sequence

$$H_i(r)_{i+j} \to H_i(r+1)_{i+j} \to H_{i-1}(r+1)(-1)_{i+j} \to H_{i-1}(r)_{i+j}.$$

We conclude by induction hypothesis on r and i

$$\begin{aligned} &h_{i,i+j}(r+1) \leq h_{i,i+j}(r) + h_{i-1,i-1+j}(r+1) \\ &\quad\leq \sum_{k=1}^{r} \binom{r+i-k-1}{i-1} \alpha_{k,j} + \sum_{k=1}^{r+1} \binom{r+1+i-1-k-1}{i-2} \alpha_{k,j} \\ &\quad= \sum_{k=1}^{r} \left(\binom{r+i-k-1}{i-1} + \binom{r+i-k-1}{i-2} \right) \alpha_{k,j} + \binom{i-2}{i-2} \alpha_{r+1,j} \\ &\quad= \sum_{k=1}^{r+1} \binom{r+i-k}{i-1} \alpha_{k,j}. \end{aligned}$$

The inequalities in the proof are all equalities if and only if the long exact sequence is split exact, i.e., the maps

$$H_{i-1}(r)(-1)_{i+j} \to H_{i-1}(r-1)_{i+j}$$

are the zero maps. In this case the sequence $v_1, \ldots, v_n$ is called a *proper sequence* for E/J. In [NRV08, Theorem 2.10] it is shown that this is the case if and only if J is a componentwise linear ideal. $\square$

We would like to emphasize at this point that the above result is the same as [NRV08, Theorem 2.4 (i)] which is a direct consequence of the construction of the Cartan homology for stable ideals in [AHH97, Proposition 3.1]. To see this, one just has to substitute the generic annihilator numbers by their description in terms of the minimal generators of $\mathrm{gin}_{<_{\mathrm{rlex}}}(J)$ and to take into account that we use the reversed order on $[n]$.

Remark 5.2.16. The above results in particular imply that the graded Betti numbers of a componentwise linear ideal J are bounded from above by linear combinations of the annihilator numbers $\alpha_{i,j}(y_1,\dots,y_n;E/J)$ for any k-basis $y_1,\dots,y_n$. If in addition the sequence is generic (in the sense that it computes the generic annihilator numbers), the graded Betti numbers of E/J are equal to this linear combination. Here we use that for $r = n$ the Cartan-Betti numbers of E/J specialize to the usual graded Betti numbers of E/J, i.e., $h_{i,j}(n)(E/J) = \beta^E_{i,j}(E/J)$. This shows that this linear combination is minimized by the generic annihilator numbers. The same is true for a componentwise linear ideal in the polynomial ring.

5.2.3 An application of almost regular sequences and generic annihilator numbers

It is widely used that being a regular sequence is a Zariski-open condition over the polynomial ring as well as over the exterior algebra. In [Swa06] Swartz gives a proof for the special case of (symmetric) Stanley-Reisner rings of simplicial complexes. Nevertheless, in the general situation we were not able to give a reference for this fact. Therefore we include a short proof for the symmetric situation following ideas from Herzog using almost regular sequences. For regular sequences over the exterior algebra a similar proof using exterior generic annihilator numbers works.
We first consider regular sequences over the polynomial ring.

Proposition 5.2.17. *Let N be a finitely generated graded S-module and let* $\mathrm{depth}_S(N) = t$. *There exists a Zariski-open set $U \subseteq GL_n(k)$ such that*

$$\gamma(x_1,\dots,x_n) := (\gamma_{1,1}x_1 + \ldots + \gamma_{n,1}x_n, \dots, \gamma_{1,t}x_1 + \ldots + \gamma_{n,t}x_n)$$

is an N-regular sequence for all $\gamma = (\gamma_{i,j})_{1\le i,j\le n} \in U$.

Proof. Herzog and Hibi proved in [HH08] that the set of almost regular sequences on N is a non-empty Zariski-open set (Proposition 5.2.6). They further showed that $v_1,\dots,v_t$ is an N-regular sequence if $v_1,\dots,v_n$ is an almost regular sequence on N and $\mathrm{depth}_S(N) = t$ (see Remark 5.2.7). Hence the assertion follows. □

We now recall the result for regular sequences over the exterior algebra.

Proposition 5.2.18. *Let $J \subseteq E$ be a graded ideal and let* $\operatorname{depth}_E E/J = t$. *Then there exists a non-empty Zariski-open set $U \subseteq GL_n(k)$ such that*

$$\gamma(e_1,\dots,e_n) := (\gamma_{1,1}e_1 + \ldots + \gamma_{n,1}e_n, \dots, \gamma_{1,t}e_1 + \ldots + \gamma_{n,t}e_n)$$

is an (E/J)-regular sequence for all $\gamma = (\gamma_{i,j})_{1\le i,j\le n} \in U$.

Proof. Let U be the non-empty Zariski-open set as in Proposition 5.2.11, i.e., such that the annihilator numbers with respect to sequences $v_1,\dots,v_n$ induced by U equal the generic annihilator numbers. Following Lemma 5.4.3 $e_1,\dots,e_t$ is a regular sequence on $E/\operatorname{gin}_{<_{\text{rlex}}}(J)$ and therefore

$$\alpha_{i,j}(v_1,\dots,v_n;E/J) = \alpha_{i,j}(e_1,\dots,e_n;E/\operatorname{gin}_{<_{\text{rlex}}}(J)) = 0$$

for $i \le t$. Thus $v_1,\dots,v_t$ is regular on E/J. $\square$

The above result can be proved directly using that a sequence of linear forms is regular if and only if the first Cartan homology vanishes and showing that the last condition is a non-empty Zariski-open condition. But using the generic annihilator numbers provides a shorter proof.

5.3 A counterexample to a minimality conjecture of Herzog

A natural question to put is whether the exterior generic annihilator numbers play a special role among the exterior annihilator numbers of E/J with respect to a certain sequence. From Theorem 5.2.15 it follows that the exterior generic annihilator numbers minimize certain positive linear combinations of exterior annihilator numbers with respect to a sequence (see Remark 5.2.16). More generally, one could wonder if they are even the minimal ones among all the annihilator numbers. Herzog posed this question and predicted it to be true. However, in the attempt of proving this conjecture it turned out to be wrong. In order to clarify this unexpected result we do not only give a counterexample of the conjecture but we also give a sketch of the original idea of the proof and explain how we came up with the example. This also gives a hint at how to construct further counterexamples. After some slight changes our example also serves as a counterexample of the corresponding conjecture for the symmetric generic annihilator numbers. For the sake of completeness we first state the conjecture.

Conjecture 5.3.1. Let $J \subseteq E$ be a graded ideal. For any basis $v_1,\dots,v_n$ of E_1 it holds that

$$\alpha_{i,j}(E/J) \le \alpha_{i,j}(v_1,\dots,v_n;E/J)$$

for $1 \le i \le n$ and $j \ge 0$.

Thus, our aim was to prove that the annihilator numbers are minimal on a non-empty Zariski-open set. For $i=1$ this is known to be true. Just take the non-empty Zariski-open set such that the ranks of the matrices of the maps of the complex

$$(N,v):\ \ldots \to N_{j-1} \xrightarrow{\cdot v} N_j \xrightarrow{\cdot v} N_{j+1} \to \ldots$$

are maximal (note that $N_j = 0$ for almost all j). To prove this for longer sequences we tried to show that the sets

$$U_{i,j} := \left\{ (v_1,\ldots,v_n) \subseteq E_1 \text{ basis} \;\middle|\; \begin{array}{c} \alpha_{i,j}(v_1,\ldots,v_n;E/J) \leq \alpha_{i,j}(w_1,\ldots,w_n;E/J) \\ \text{for any basis } (w_1,\ldots,w_n) \subseteq E_1 \end{array} \right\}$$

were non-empty Zariski-open sets for $1 \leq i \leq n$, $0 \leq j \leq n$. The intersection of those sets would have been a non-empty Zariski-open set having the required property, since only finitely many sets are intersected. In order to compute a certain annihilator number $\alpha_{i,j}(v_1,\ldots,v_n;E/J)$ we used the exact sequence

$$\begin{aligned} 0 &\to H^j(E/(J+(v_1,\ldots,v_{i-1})),v_i) \to (E/(J+(v_1,\ldots,v_i)))_j \\ &\xrightarrow{\cdot v_i} (E/(J+(v_1,\ldots,v_{i-1})))_{j+1} \to (E/(J+(v_1,\ldots,v_i)))_{j+1} \to 0, \end{aligned}$$

which yields

$$\begin{aligned} \alpha_{i,j}(v_1,\ldots,v_n;E/J) = {} & \dim_k (E/(J+(v_1,\ldots,v_i)))_j \\ & -(\dim_k (E/(J+(v_1,\ldots,v_{i-1})))_{j+1} - \dim_k (E/(J+(v_1,\ldots,v_i)))_{j+1}). \end{aligned}$$

The idea was to minimize $\dim_k (E/(J+(v_1,\ldots,v_i)))_j$ and to maximize the difference $(\dim_k (E/(J+(v_1,\ldots,v_{i-1})))_{j+1} - \dim_k (E/(J+(v_1,\ldots,v_i)))_{j+1})$ in order to get a minimal annihilator number $\alpha_{i,j}(v_1,\ldots,v_n;E/J)$.
To simplify notation in the sequel we write $D_{i-1,j+1}$ for $(\dim_k (E/(J+(v_1,\ldots,v_{i-1})))_{j+1} - \dim_k (E/(J+(v_1,\ldots,v_i)))_{j+1})$.
One can show that there exists a non-empty Zariski-open subset of

$$\{(v_1,\ldots,v_n) \subseteq E_1 \mid (v_1,\ldots,v_n) \text{ basis of } E_1\}$$

such that $\dim_k (E/(J+(v_1,\ldots,v_i)))_j$ is minimal. If the property of maximizing the difference $D_{i-1,j+1}$ were a non-empty Zariski-open condition on the sequence the claim would follow. But this would mean that this very difference would be maximized whenever $\dim_k (E/(J+(v_1,\ldots,v_{i-1})))_{j+1}$ and $\dim_k (E/(J+(v_1,\ldots,v_i)))_{j+1}$ are minimized.
But what happens for instance if $\dim_k (E/(J+(v_1,\ldots,v_{i-1})))_{j+1} = 0$? The maximal difference should be 0 in this case. The idea we came up with was to construct a sequence such that $\dim_k (E/(J+(v_1,\ldots,v_i)))_j$ and $\dim_k (E/(J+(v_1,\ldots,v_i)))_{j+1} = 0$ are minimal but $\dim_k (E/(J+(v_1,\ldots,v_{i-1})))_{j+1}$ is not. This would yield a smaller annihilator number.

Example 5.3.2. Let $1 \le i, j \le n$ and let

$$J := (e_{l_1} \cdot \ldots \cdot e_{l_{j+1}} \mid i \le l_1 < l_2 < \ldots < l_{j+1} \le n) \subseteq E$$

be a graded ideal. By construction, J is strongly stable. It therefore holds that $\mathrm{gin}_{<_{\mathrm{rlex}}}(J) = J$. Then

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(e_1, \ldots, e_n; E/\mathrm{gin}_{<_{\mathrm{rlex}}}(J)) = \alpha_{i,j}(e_1, \ldots, e_n; E/J),$$

i.e., we can use the sequence $e_1, \ldots, e_n$ to compute the generic annihilator numbers of E/J. From the exact sequence

$$\begin{aligned} 0 &\to H^j(E/(J + (e_1, \ldots, e_{i-1})), e_i) \to (E/(J + (e_1, \ldots, e_i)))_j \\ &\stackrel{\cdot e_i}{\to} (E/(J + (e_1, \ldots, e_{i-1})))_{j+1} \to (E/(J + (e_1, \ldots, e_i)))_{j+1} \to 0 \end{aligned}$$

we deduce

$$\begin{aligned} \alpha_{i,j}(E/J) = {} & \dim_k (E/(J + (e_1, \ldots, e_i)))_j \\ & -(\dim_k (E/(J + (e_1, \ldots, e_{i-1})))_{j+1} - \dim_k (E/(J + (e_1, \ldots, e_i)))_{j+1}). \end{aligned}$$

In the following we consider the sequence $\mathbf{e} := e_1, \ldots, e_{i-2}, e_i, e_{i-1}, e_{i+1}, \ldots, e_n$ and compute the exterior annihilator numbers of E/J with respect to this sequence. As before, we have the exact sequence

$$\begin{aligned} 0 &\to H^j(E/(J + (e_1, \ldots, e_{i-2}, e_i)), e_{i-1}) \to (E/(J + (e_1, \ldots, e_i)))_j \\ &\stackrel{\cdot e_{i-1}}{\to} (E/(J + (e_1, \ldots, e_{i-2}, e_i)))_{j+1} \to (E/(J + (e_1, \ldots, e_i)))_{j+1} \to 0, \end{aligned}$$

which leads to

$$\begin{aligned} \alpha_{i,j}(\mathbf{e}; E/J) = {} & \dim_k (E/(J + (e_1, \ldots, e_i)))_j \\ & -(\dim_k (E/(J + (e_1, \ldots, e_{i-2}, e_i)))_{j+1} - \dim_k (E/(J + (e_1, \ldots, e_i)))_{j+1}). \end{aligned}$$

Our aim is to show that $\alpha_{i,j}(E/J) > \alpha_{i,j}(\mathbf{e}; E/J)$. We therefore need to show that

$$\dim_k (E/(J + (e_1, \ldots, e_{i-2}, e_i)))_{j+1} > \dim_k (E/(J + (e_1, \ldots, e_{i-2}, e_{i-1})))_{j+1}.$$

Let $m := e_{l_1} \cdot \ldots \cdot e_{l_{j+1}} \in E_{j+1}$ with $l_1 < \ldots < l_{j+1}$. If $l_1 \le i - 1$, it already holds that $m \in (e_1, \ldots, e_{i-1})_{j+1} \subseteq (J + (e_1, \ldots, e_{i-1}))_{j+1}$. If $l_1 \ge i$, we have $i \le l_1 < \ldots < l_{j+1}$ and therefore $m \in J_{j+1} \subseteq (J + (e_1, \ldots, e_{i-1}))_{j+1}$. Thus, $m \in (J + (e_1, \ldots, e_{i-1}))_{j+1}$ in either case and therefore $(J + (e_1, \ldots, e_{i-1}))_{j+1} = E_{j+1}$ and $\dim_k (E/(J + (e_1, \ldots, e_{i-1})))_{j+1} = 0$.

Consider now $\widetilde{m} := e_{i-1} e_{i+1} \cdot \ldots \cdot e_{i+j} \in E_{j+1}$. By definition, it holds that $\widetilde{m} \notin J$ and $\widetilde{m} \notin (e_1, \ldots, e_{i-2}, e_i)$. Since J is a monomial ideal this implies $\widetilde{m} \notin (J + (e_1, \ldots, e_{i-2}, e_i))_{j+1}$. We therefore get $\dim_k (E/(J + (e_1, \ldots, e_{i-2}, e_i)))_{j+1} > 0$. This finally shows

$$\alpha_{i,j}(E/J) > \alpha_{i,j}(\mathbf{e}; E/J).$$

We also compute $\alpha_{i-1,j}(E/J)$ and $\alpha_{i-1,j}(e_1,\ldots,e_{i-2},e_i,e_{i-1},e_{i+1},\ldots,e_n;E/J)$ in this special case and show that those numbers are related to each other the other way round, i.e., we have

$$\alpha_{i-1,j}(E/J) < \alpha_{i-1,j}(\mathbf{e};E/J). \tag{5.5}$$

This suggests that, in order to have a chance to become smaller than the generic annihilator numbers, the annihilator numbers with respect to **e** first have to become greater. Similar to the i-th annihilator numbers of E/J we can compute the $(i-1)$-st annihilator numbers using the corresponding exact sequence. We therefore get

$$\begin{aligned}\alpha_{i-1,j}(E/J) = {} & \dim_k\left(E/(J+(e_1,\ldots,e_{i-1}))\right)_j \\ & -(\dim_k\left(E/(J+(e_1,\ldots,e_{i-2}))\right)_{j+1} - \dim_k\left(E/(J+(e_1,\ldots,e_{i-1}))\right)_{j+1})\end{aligned}$$

for the $(i-1)$-st generic annihilator number of E/J in degree j. In the same way, we obtain

$$\begin{aligned}\alpha_{i-1,j}(\mathbf{e};E/J) = {} & \dim_k\left(E/(J+(e_1,\ldots,e_{i-2},e_i))\right)_j \\ & -(\dim_k\left(E/(J+(e_1,\ldots,e_{i-2}))\right)_{j+1} - \dim_k\left(E/(J+(e_1,\ldots,e_{i-2},e_i))\right)_{j+1})\end{aligned}$$

for the $(i-1)$-st exterior annihilator number of E/J in degree j with respect to the sequence **e**. Since J is generated by monomials of degree strictly larger than j it holds that

$$\dim_k\left(E/(J+(e_1,\ldots,e_{i-1}))\right)_j = \dim_k\left(E/(e_1,\ldots,e_{i-1})\right)_j$$

and

$$\dim_k\left(E/(J+(e_1,\ldots,e_{i-2},e_i))\right)_j = \dim_k\left(E/(e_1,\ldots,e_{i-2},e_i)\right)_j.$$

Since

$$\dim_k\left(E/(J+(e_1,\ldots,e_{i-1}))\right)_j = \dim_k\left(E/(J+(e_1,\ldots,e_{i-2},e_i))\right)_j,$$

in order to show (5.5) we only need to prove that

$$\dim_k\left(E/(J+(e_1,\ldots,e_{i-1}))\right)_{j+1} < \dim_k\left(E/(J+(e_1,\ldots,e_{i-2},e_i))\right)_{j+1}.$$

This follows from

$$(J+(e_1,\ldots,e_{i-1}))_{j+1} \supsetneq (J+(e_1,\ldots,e_{i-2},e_i))_{j+1}. \tag{5.6}$$

To show (5.6) let $m = e_{l_1}\cdot\ldots\cdot e_{l_{j+1}} \in (J+(e_1,\ldots,e_{i-2},e_i))_{j+1}$ with $l_1 < \ldots < l_{j+1}$. If $l_1 \geq i$ it follows that $m \in J_{j+1} \subseteq (J+(e_1,\ldots,e_{i-1}))_{j+1}$. If $l_1 \leq i-1$ it already holds that $e_{l_1} \in (e_1,\ldots,e_{i-1})$ and thus $m \in (J+(e_1,\ldots,e_{i-1}))_{j+1}$. Since $e_{i-1}e_{i+1}\cdot\ldots\cdot e_{i+j} \in (J+(e_1,\ldots,e_{i-1}))_{j+1}$ but $e_{i-1}e_{i+1}\cdot\ldots\cdot e_{i+j} \notin (J+(e_1,\ldots,e_{i-2},e_i))_{j+1}$ we obtain (5.6).

Thus, although we have seen that the exterior generic annihilator numbers are minimal in a certain global sense (see Remark 5.2.16) the above example shows that the single ones are not.
We now compute Example 5.3.2 for the special case $n = 5$.

Example 5.3.3. Let $n = 5$ and $i = 3$. We consider the ideal $J := (e_3 \wedge e_4, e_3 \wedge e_5, e_4 \wedge e_5)$ in the exterior algebra $E := k\langle e_1, \dots, e_5\rangle$. We use the sequence e_1, e_2, e_3, e_4, e_5 to compute the generic annihilator numbers $\alpha_{i,j} := \alpha_{i,j}(E/J)$. Let $\alpha_i := \sum_{j\in\mathbb{Z}} \alpha_{i,j}$ denote the sum over all annihilator numbers computed in step i. Since e_1 and e_2 do not appear among the generators of J, these two form a regular sequence on E/J. This implies that the corresponding annihilator numbers are zero, i.e., $\alpha_1 = \alpha_2 = 0$. To compute $\alpha_{3,j}$ we have to compute the vector space dimension of

$$((\bar{J} : e_3)/(\bar{J} + (e_3)))_j = ((e_1, e_2, e_3, e_4, e_5)/(e_1, e_2, e_3, e_4 \wedge e_5))_j,$$

where $\bar{J} := J + (e_1, e_2)$. Therefore $\alpha_{3,1} = 2$ and in all other degrees the annihilator number is zero. In the next step we look at

$$((\bar{J} : e_4)/(\bar{J} + (e_4)))_j = ((e_1, e_2, e_3, e_4, e_5)/(e_1, e_2, e_3, e_4))_j,$$

where now $\bar{J} := J + (e_1, e_2, e_3)$. Thus $\alpha_{4,1} = 1$ and $\alpha_{4,j} = 0$ for $j \neq 1$. In the last step we obtain $\alpha_5 = 0$. Note that the last generic annihilator number is always zero, as it is the dimension of a quotient over $J + (e_1, \dots, e_n) = (e_1, \dots, e_n)$. In particular, we see that $\operatorname{depth}_E E/J = 2$.

Now we compute the annihilator numbers $\alpha'_{i,j}$ of E/J with respect to the sequence e_1, e_3, e_2, e_4, e_5. Again e_1 is an (E/J)-regular element and thus $\alpha'_1 = 0$. But $\alpha'_2 \neq 0$ in contrary to α_2 since

$$((J + (e_1)) : e_3) / (J + (e_1, e_3)) = (e_1, e_3, e_4, e_5)/(e_1, e_3, e_4 \wedge e_5).$$

Therefore $\alpha'_{2,1} = 2$, $\alpha'_{2,2} = 2$ and zero otherwise. Now e_2 is still regular on $E/(J + (e_1, e_3))$, whence $\alpha'_3 = 0 < \alpha_3$. Thus in this case the generic annihilator number is the greater one. The last two steps are the same as above, i.e., $\alpha'_{4,j} = \alpha_{4,j}$ which is 1 for $j = 1$ and zero otherwise and $\alpha'_5 = \alpha_5 = 0$. If one compares the respective sets of non-zero annihilator numbers, one has in the first case, i.e., for the generic numbers, $\{2, 1\}$ and in the second case $\{2, 2, 1\}$. Therefore the generic annihilator numbers seem to be minimal in a certain global sense, meaning that when comparing the sets of non-zero annihilator numbers the one for the generic annihilator numbers is lexicographically smaller.

Besides the minimality conjecture for the exterior generic annihilator numbers Herzog conjectured the commensurate result to be true over the polynomial ring, i.e., that the symmetric generic annihilator numbers are the minimal ones among all symmetric annihilator numbers with respect to a sequence. As already mentioned, this is not the case. In fact, after slight modifications Example 5.3.2 serves as a counterexample.

Example 5.3.4. Let $1 \leq i \leq j \leq n$ and let $I := (x_{l_1} \cdot \ldots \cdot x_{l_{j+1}} \mid i \leq l_1 \leq \ldots \leq l_{j+1}) \subseteq S$ be a graded ideal. Analogously to Example 5.3.2 we can use the sequence $x_1, \ldots, x_n$ to compute the symmetric generic annihilator numbers of S/I. From the exact sequence

$$\begin{aligned} 0 &\to A_i(x_1,\ldots,x_n;S/I)_j \to \left(S/\left(I+(x_1,\ldots,x_{i-1})\right)\right)_j \\ &\xrightarrow{\cdot x_i} \left(S/(I+(x_1,\ldots,x_{i-1}))\right)_{j+1} \to \left(S/(I+(x_1,\ldots,x_i))\right)_{j+1} \to 0 \end{aligned}$$

we deduce

$$\begin{aligned} \alpha_{i,j}(S/I) = {} & \dim_k \left(S/(I+(x_1,\ldots,x_{i-1}))\right)_j \\ & -(\dim_k \left(S/(I+(x_1,\ldots,x_{i-1}))\right)_{j+1} - \dim_k \left(S/(I+(x_1,\ldots,x_i))\right)_{j+1}). \end{aligned}$$

Using the same exact sequence for the sequence $\mathbf{x} = x_1,\ldots,x_{i-2},x_i,x_{i-1},x_{i+1},\ldots,x_n$ we get

$$\begin{aligned} \alpha_{i,j}(\mathbf{x};S/I) = {} & \dim_k \left(S/(I+(x_1,\ldots,x_{i-2},x_i))\right)_j \\ & -(\dim_k \left(S/(I+(x_1,\ldots,x_{i-2},x_i))\right)_{j+1} - \dim_k \left(S/(I+(x_1,\ldots,x_i))\right)_{j+1}. \end{aligned}$$

Since I is generated in degree $j+1$ it holds that

$$\begin{aligned} \dim_k \left(S/(I+(x_1,\ldots,x_{i-1}))\right)_j &= \dim_k \left(S/(x_1,\ldots,x_{i-1})\right)_j \\ &= \dim_k \left(S/(x_1,\ldots,x_{i-2},x_i)\right)_j \\ &= \dim_k \left(S/(I+(x_1,\ldots,x_{i-2},x_i))\right)_j. \end{aligned}$$

One easily shows that $(I+(x_1,\ldots,x_{i-2},x_i))_{j+1} \subsetneq (I+(x_1,\ldots,x_{i-1}))_{j+1}$. This implies

$$\dim_k \left(S/(I+(x_1,\ldots,x_{i-1}))\right)_{j+1} < \dim_k \left(S/(I+(x_1,\ldots,x_{i-2},x_i))\right)_{j+1}. \tag{5.7}$$

As in the exterior case we thus obtain

$$\alpha_{i,j}(S/I) > \alpha_{i,j}(\mathbf{x};S/I).$$

We now compute the $(i-1)$-st annihilator number in degree j and see what happens in this case. The same arguments as before show that

$$\begin{aligned} \alpha_{i-1,j}(S/I) = {} & \dim_k \left(S/(I+(x_1,\ldots,x_{i-2}))\right)_j \\ & -(\dim_k \left(S/(I+(x_1,\ldots,x_{i-2}))\right)_{j+1} - \dim_k \left(S/(I+(x_1,\ldots,x_{i-1}))\right)_{j+1}) \end{aligned}$$

and

$$\begin{aligned} \alpha_{i-1,j}(\mathbf{x};S/I) = {} & \dim_k \left(S/(I+(x_1,\ldots,x_{i-2}))\right)_j \\ & -(\dim_k \left(S/(I+(x_1,\ldots,x_{i-2}))\right)_{j+1} - \dim_k \left(S/(I+(x_1,\ldots,x_{i-2},x_i))\right)_{j+1}). \end{aligned}$$

Using Equation (5.7) we thus obtain

$$\alpha_{i-1,j}(S/I) < \alpha_{i-1,j}(\mathbf{x};S/I).$$

From Examples 5.3.2 and 5.3.4 it follows that the annihilator numbers with respect to a sequence – the symmetric as well as the exterior ones – may depend on the order of the sequence. However, we conjecture that when taking the sequence from a certain non-empty Zariski-open set the order of the elements of the sequence does not matter. Therefore, it is a bit surprising, that there are examples where the order plays a role when we consider the perhaps most natural sequences $e_1,\dots,e_n$ and $x_1,\dots,x_n$ for the exterior and the symmetric annihilator numbers, respectively.

Conjecture 5.3.5. Let $J \subseteq E$ be a graded ideal. Then there exists a non-empty Zariski-open set $V \subseteq GL_n(K)$ such that

$$\alpha_{i,j}(E/J) = \alpha_{i,j}(\gamma(e_{\sigma(1)},\dots,e_{\sigma(n)});E/J)$$

for all $\gamma \in V$ and all $\sigma \in S_n$, where S_n denotes the symmetric group on $[n]$.

5.4 The exterior depth and exterior annihilator numbers for Stanley-Reisner rings of simplicial complexes

The aim of this section is to transfer the results of the previous sections of this chapter to the special class of Stanley-Reisner rings of simplicial complexes. Besides the mere conferment of the results we also derive additional ones – including a characterization of simplicial complexes whose exterior Stanley-Reisner ring has a certain exterior depth. Those simplicial complexes can be characterized in terms of their exterior algebraic shifting. From this description we obtain alternative proofs for two of the previously obtained results.

After having dealt with the exterior depth of Stanley-Reisner rings we consider the exterior generic annihilator numbers for this class of rings. The combinatorial description of the exterior generic annihilator numbers turns out to be particularly nice for exterior face rings of simplicial complexes. Using this characterization we manage to express the graded Betti numbers of the symmetric Stanley-Reisner ring of a simplicial complex as positive linear combinations of certain exterior generic annihilator numbers of the Stanley-Reisner ring of the exterior shifting of the simplicial complex.

5.4.1 The exterior depth for Stanley-Reisner rings of simplicial complexes

In Section 5.1 we have shown that the exterior depth of an E-module is always bounded from above by the symmetric depth of the associated S-module. An interesting issue to examine is if there are classes of squarefree modules for which equality holds in Inequality (5.1). In the special case of Stanley-Reisner rings of simplicial complexes we can identify at least one such class.

Lemma 5.4.1. *Let Δ be a Cohen-Macaulay simplicial complex over k. Then*

$$\operatorname{depth}_S(k[\Delta]) - \operatorname{depth}_E(k\{\Delta\}) = \operatorname{cx}_E(k\{\Delta\}) - \operatorname{proj\,dim}_S(k[\Delta]) = \operatorname{reg}_S(k[\Delta]).$$

Proof. The first equality is always true as follows from the proof of Theorem 5.1.16. Let $\dim\Delta = d-1$. As Δ is Cohen-Macaulay over k, its Stanley-Reisner ring $k[\Delta]$ has depth d over S and the face ring $k\{\Delta\}$ has a d-linear injective resolution over E. In fact, Aramova and Herzog showed in [AH00, Corollary 7.6] that a simplicial complex Δ is Cohen-Macaulay if and only if the exterior Stanley-Reisner ideal of its Alexander dual $\Delta^* := \{F \subseteq [n] \mid ([n] \setminus F) \notin \Delta\}$ has a linear projective resolution over E. In [KR08, Example 5.1] Kämpf and Römer prove that $k\{\Delta\}$ is the dual of J_{Δ^*}. Thus, by dualizing the projective resolution of J_{Δ^*} we get an injective resolution of $k\{\Delta\}$. Being the projective resolution of J_{Δ^*} d-linear then implies that $k\{\Delta\}$ has a d-linear injective resolution. Following [KR08, Theorem 5.3] it holds that

$$d = \operatorname{depth}_E(k\{\Delta\}) + \operatorname{reg}_E(k\{\Delta\}).$$

The regularity over E is the same as the regularity over S (see Remark 5.1.13), hence

$$\operatorname{reg}_S(k[\Delta]) = d - \operatorname{depth}_E(k\{\Delta\}) = \operatorname{depth}_S(k[\Delta]) - \operatorname{depth}_E(k\{\Delta\}).$$

□

The following example shows that in general the converse of Lemma 5.4.1 is not true.

Example 5.4.2. Let Δ be the simplicial complex consisting of two isolated edges. Then Δ is not Cohen-Macaulay since it is disconnected. But, as one easily sees, $\operatorname{depth}_E(k\{\Delta\}) = 0$, $\operatorname{depth}_S(k[\Delta]) = 1$ and $\operatorname{reg}_S(k[\Delta]) = 1$. Therefore,

$$\operatorname{depth}_S(k[\Delta]) - \operatorname{depth}_E(k\{\Delta\}) = 1 = \operatorname{reg}_S(k[\Delta]).$$

Our next aim is to characterize simplicial complexes whose exterior Stanley-Reisner ring has a certain exterior depth. We are able to describe the exterior shifting of those complexes. From Theorem 5.1.11 it follows that the exterior Stanley-Reisner ring of a simplicial complex and the one of its exterior shifting have the same exterior depth. In order to give the desired characterization we need to recall some further results. It is well-known that for stable ideals an initial segment of $e_1, \dots, e_n$ is a regular sequence. However, for the sake of completeness we include a proof of it.

Lemma 5.4.3. *Let $J \subseteq E$ be a stable ideal of exterior depth t. Then $e_1, \dots, e_t$ is a regular sequence on E/J.*

Proof. If $t = 0$ there is nothing to prove, thus we may assume $t > 0$. Let e_A be a monomial in $J : e_1$. We show that $e_A \in (J + (e_1))$. Then the claim follows by induction on t. Note that since J is stable, $(J + (e_1))/(e_1)$ is stable in $E/(e_1) \cong k\langle e_2, \dots, e_n\rangle$, as well.

Since $e_1 e_A \in J$ there exists a minimal monomial generator $u \in G(J)$ such that u divides $e_1 e_A$. If u is not divisible by e_1 then u divides e_A and therefore it follows that $e_A \in J$. If not, u would be a minimal generator of J which is divisible by e_1. Aramova, Avramov and Herzog [AAH00, Theorem 3.2] as well as Kämpf and Römer [KR08, Proposition 3.4] described the depth of a stable ideal in the following way

$$\operatorname{depth}_E E/J = n - (n - \min\{\min(v) \mid v \in G(J)\} - 1) = \min\{\min(v) \mid v \in G(J)\} - 1$$

(according to the reversed order on $[n]$). Thus it follows that $\operatorname{depth}_E E/J = 0$, a contradiction to the assumption $t > 0$. $\square$

Before we can state the desired characterization we need to introduce the notion of a non-acyclic simplicial complex.

Definition 5.4.4. Let Δ be a simplicial complex. Δ is called *non-acyclic* if there exists $0 \leq i \leq \dim \Delta$ such that $\widetilde{H}_i(\Delta; k) \neq 0$.

It was shown by Aramova and Herzog [AH00, Lemma 3.3] that the simplicial homology of a simplicial complex Δ can be computed using the complex

$$(k\{\Delta\}, v) : \ \ldots \to k\{\Delta\}_{j-1} \xrightarrow{\cdot v} k\{\Delta\}_j \xrightarrow{\cdot v} k\{\Delta\}_{j+1} \to \ldots,$$

where $v \in E$ is a generic element of degree one. More precisely, we have $H^i(k\{\Delta\}, v) = \widetilde{H}_i(\Delta; k)$ for $0 \leq i \leq \dim \Delta$. Since $H^i(k\{\Delta\}, v) = 0$ for $0 \leq i \leq \dim \Delta$ if v is a $k\{\Delta\}$-regular element, it follows that $\operatorname{depth}_E(k\{\Delta\}) = 0$ if and only if Δ is non-acyclic. Using this characterization of non-acyclic simplicial complexes we can prove the following.

Theorem 5.4.5. *Let Δ be a simplicial complex on vertex set $[n]$. Then $\operatorname{depth}_E(k\{\Delta\}) = r$ if and only if $\Delta^e = 2^{[r]} * \Gamma$, where Γ is a non-acyclic simplicial complex, $2^{[r]}$ the $(r-1)$-simplex and $\dim \Gamma = \dim \Delta - r$.*

Proof. We first assume that $\operatorname{depth}_E(k\{\Delta\}) = r$. In order to prove the statement we need to show that for $F \in \Delta^e$ it holds that $F \cup [t]$ is a face of Δ^e for any $t \leq r$. We give two different proofs, an algebraic one and a combinatorial one.
Algebraic proof:
Let $F \in \Delta^e$. Without loss of generality we may assume that $F \cap [t] = \emptyset$. We can take $e_1, \ldots, e_t$ as a regular sequence for the exterior face ring $k\{\Delta^e\} = E/\operatorname{gin}_{<_{\text{rlex}}}(J_\Delta)$. Suppose $F \in \Delta^e$ such that $F \cap [t] = \emptyset$ and $F \cup [t] \notin \Delta^e$. Thus, $e_t \wedge \ldots \wedge e_1 \wedge e_F = 0$ in $k\{\Delta^e\}$. By the definition of a regular sequence it follows that

$$e_{t-1} \wedge \ldots \wedge e_1 \wedge e_F \in (e_t) \subseteq k\{\Delta^e\}.$$

Therefore, $e_{t-1} \wedge \ldots \wedge e_1 \wedge e_F = 0 \in E/(\operatorname{gin}_{<_{\text{rlex}}}(J_\Delta) + (e_t))$. Inductively, we get $e_F = 0 \in E/(\operatorname{gin}_{<_{\text{rlex}}}(J_\Delta) + (e_t, \ldots, e_1))$. Since $F \cap [t] = \emptyset$ by assumption, we have $e_F \in \operatorname{gin}_{<_{\text{rlex}}}(J_\Delta)$, i.e., $F \notin \Delta^e$. This is a contradiction.

Combinatorial proof:
Let $\{f_1, \ldots, f_n\} \subseteq E_1$ be a generic basis which is used in the computation of the exterior shifting Δ^e. We know that $f_t, \ldots, f_1$ is a regular sequence for $k\{\Delta^e\}$. Suppose $F \in \Delta^e$ such that $F \cap [t] = \emptyset$ and $F \cup [t] \notin \Delta^e$. Thus

$$\begin{aligned} f_1 \wedge \ldots \wedge f_t \wedge f_F &= \sum_{T <_{\text{lex}} [t] \cup F} \alpha_T f_T \\ &= \sum_{T' <_{\text{lex}} F} \alpha_{T'} f_1 \wedge \ldots \wedge f_t \wedge f_{T'} \\ &= f_1 \wedge \ldots \wedge f_t \wedge \big(\sum_{T' <_{\text{lex}} F} \alpha_{T'} f_{T'} \big), \end{aligned}$$

i.e., $f_1 \wedge \ldots \wedge f_t \wedge (f_F - \sum_{T' <_{\text{lex}} F} \alpha_{T'} f_{T'}) = 0$. If $f_t \wedge (f_F - \sum_{T' <_{\text{lex}} F} \alpha_{T'} f_{T'}) = 0$ it follows that $f_F - \sum_{T' <_{\text{lex}} F} \alpha_{T'} f_{T'} \in \operatorname{Im}(f_t)$, i.e., $f_F - \sum_{T' <_{\text{lex}} F} \alpha_{T'} f_{T'} = f_t \wedge (\sum_{G \subseteq [n]} \alpha_G f_G)$. This implies $f_F = \sum_{T' <_{\text{lex}} F} \alpha_{T'} f_{T'} + \sum_{G \subseteq [n]} \alpha_G (f_t \wedge f_G)$. Since $F \cap [t] = \emptyset$ it follows that $G \cup \{t\} <_{\text{lex}} F$ and therefore $F \notin \Delta^e$. Hence, a contradiction.
If $f_F - \sum_{T' <_{\text{lex}} F} \alpha_{T'} f_{T'} \notin \operatorname{Ker}(f_t)$, then $f_t \wedge (f_F - \sum_{T' <_{\text{lex}} F} \alpha_{T'} f_{T'}) \in \operatorname{Ker}(f_1 \wedge \ldots \wedge f_{t-1})$ and by induction we conclude $F \notin \Delta^e$.

Both proofs show $\Delta^e = 2^{[r]} * \Gamma$ for some $(\dim \Delta - r)$-dimensional simplicial complex Γ with $J_\Gamma = \operatorname{gin}_{<_{\text{rlex}}}(J_\Delta) + (e_1, \ldots, e_t)$. By definition of the exterior depth it holds that $\operatorname{depth}_E(k\{\Gamma\}) = \operatorname{depth}_E(k\{\Delta^e\}) - r = 0$. We therefore conclude that $\widetilde{H}_i(\Gamma; k) \neq 0$ for some $0 \leq i \leq \dim(\Gamma)$, i.e., Γ is non-acyclic.

To prove the sufficiency part recall that it holds that $\operatorname{depth}_E(k\{\Delta\}) = \operatorname{depth}_E(k\{\Delta^e\})$. Thus, we only need to show that $\operatorname{depth}_E(k\{\Delta^e\}) = r$. By assumption, we have that $\Delta^e = 2^{[r]} * \Gamma$, where Γ is a non-acyclic simplicial complex. Then the sequence $e_1, \ldots, e_r$ is regular on $k\{\Delta^e\}$ by Lemma 5.4.3. This implies $\operatorname{depth}_E(k\{\Delta^e\}) \geq r$. It further holds that $k\{\Delta^e\}/(e_1, \ldots, e_r) \cong k\{\Gamma\}$. Since Γ is non-acyclic we know from the remarks preceding this theorem that $\operatorname{depth}_E(k\{\Gamma\}) = 0$, i.e., there does not exist any regular element on $k\{\Gamma\}$. Using that each regular sequence on $k\{\Delta^e\}$ can be extended to a maximal one, we therefore deduce that $e_1, \ldots, e_r$ is already maximal and thus it follows that $\operatorname{depth}_E(k\{\Delta^e\}) = r$. $\square$

Remark 5.4.6. The above theorem can be used to deduce two of the previous results in the special case of simplicial complexes.

(i) Note that from the above proof the inequality

$$\operatorname{depth}_E(k\{\Delta\}) \leq \operatorname{depth}_S(k[\Delta])$$

between the exterior and the symmetric depth can be deduced. If $\operatorname{depth}_E(k\{\Delta\}) = r$, then Theorem 5.4.5 implies $\Delta^e = 2^{[r]} * \Gamma$ and therefore $x_1, \ldots, x_r$ is a regular sequence for $k[\Delta^e]$, i.e., $\operatorname{depth}_S(k[\Delta^e]) \geq r$. Since by Theorem 1.2.10 it holds that $\operatorname{depth}_S(k[\Delta]) = \operatorname{depth}_S(k[\Delta^e])$ we obtain the required inequality.

(ii) Using the characterization of Theorem 5.4.5 we can give a second proof of Lemma 5.4.1. If Δ is non-acyclic, i.e., $\widetilde{H}_i(\Delta;k) \neq 0$ for some $0 \leq i \leq \dim\Delta$, it follows that $\operatorname{depth}_E(k\{\Delta\}) = 0$. Since Δ is Cohen-Macaulay it follows from Reisner's criterion that $\widetilde{H}_i(\Delta;k) = 0$ for $i < \dim\Delta$. Thus, $\widetilde{H}_{\dim\Delta}(\Delta;k) \neq 0$. In [KW08, Proposition 2.6] Welker and the author of this thesis showed that in this case it holds that $\operatorname{reg}_S(k[\Delta]) = \dim\Delta + 1$. Combining these two facts and using that Δ is Cohen-Macaulay, i.e., $\operatorname{depth}_S(k[\Delta]) = \dim_S(k[\Delta])$, we obtain

$$\begin{aligned}\operatorname{depth}_S(k[\Delta]) - \operatorname{depth}_E(k\{\Delta\}) &= \operatorname{depth}_S(k[\Delta]) \\ &= \dim_S k[\Delta] = \dim\Delta + 1 = \operatorname{reg}_S(k[\Delta]).\end{aligned}$$

Let us now assume that Δ is an acyclic simplicial complex and let $\operatorname{depth}_E(k\{\Delta\}) = r$. We may assume that $\Delta = \Delta^e$ since the Stanley-Reisner rings of Δ and Δ^e have the same symmetric and exterior depth, respectively, and the same regularity (see [Röm01, Corollary 1.3]). From Theorem 5.4.5 we know that $\Delta^e = 2^{[r]} * \Gamma$, where Γ is a non-acyclic simplicial complex. In particular, since Δ^e is Cohen-Macaulay, so is Γ. This implies

$$\operatorname{depth}_S(k[\Gamma]) = \dim_S(k[\Gamma]) = \dim_S(k[\Delta]) - r = \operatorname{depth}_S(k[\Delta]) - r.$$

Using that $\operatorname{depth}_E(k\{\Gamma\}) = 0 = \operatorname{depth}_E(k\{\Delta\}) - r$ we deduce

$$\operatorname{depth}_S(k[\Delta]) - \operatorname{depth}_E(k\{\Delta\}) = \operatorname{depth}_S(k[\Gamma]) - \operatorname{depth}_E(k\{\Gamma\}).$$

Since Γ is a non-acyclic simplicial complex we know from the first part of our considerations that $\operatorname{depth}_S(k[\Gamma]) - \operatorname{depth}_E(k\{\Gamma\}) = \operatorname{reg}_S(k[\Gamma])$. We further have that $k[\Gamma] \cong k[\Delta]/(x_1,\dots,x_r)$ and $x_1,\dots,x_r$ is a regular sequence on $k[\Delta]$. Since reducing modulo a regular sequence leaves the regularity unchanged it holds that $\operatorname{reg}_S(k[\Delta]) = \operatorname{reg}_S(k[\Gamma])$. This finally shows the claim.

A natural issue which occurs is for which pairs of numbers $(s \leq t)$ does there exist a simplicial complex Δ with the property that $\operatorname{depth}_E(k\{\Delta\}) = s$ and $\operatorname{depth}_S(k[\Delta]) = t$. We can solve this matter by showing that all pairs of numbers are possible.

Example 5.4.7. Indeed, let $(s,t) \in \mathbb{N}^2$ with $s \leq t$. Consider $\Delta = 2^{[s]} * \partial(2^{[t-s+2]})$, where $\partial(2^{[t-s+2]})$ denotes the boundary of the $(t-s+2)$-simplex. Then $\operatorname{depth}_E(k\{\Delta\}) = s$ and since Δ is Cohen-Macaulay $\operatorname{depth}_S(k[\Delta]) = t$, as required.

The above example in particular shows that the inequality between the symmetric depth of an S-module and the exterior depth of the corresponding E-module cannot be improved and that the symmetric depth cannot be bounded from above in terms of the exterior depth.

5.4.2 Annihilator numbers for Stanley-Reisner rings of simplicial complexes

In Section 5.2.2 we have shown that the exterior generic annihilator numbers of an E-module E/J can be expressed in terms of certain generators of the generic initial ideal of J. Considering the special case of the exterior Stanley-Reisner ring of a simplicial complex this description has a nice combinatorial interpretation in terms of the simplicial complex.

Corollary 5.4.8. *Let Δ be a simplicial complex and let Δ^e be its exterior shifting. Then*

$$\alpha_{i,j}(k\{\Delta\}) = |\{F \in \Delta^e \mid |F| = j,\ [i] \cap F = \emptyset,\ F \cup \{i\} \notin \Delta^e\}|.$$

From the above formula for the exterior generic annihilator numbers we can derive a second combinatorial characterization of those numbers in terms of the minimal generators of the symmetric and the exterior Stanley-Reisner ideal of the exterior shifting of a simplicial complex, respectively. Using this second description we are able to couch the symmetric Betti numbers of the Stanley-Reisner ring of the exterior shifting of a simplicial complex as linear combinations of certain generic annihilator numbers. The proof of this formula uses the Eliahou-Kervaire formula for the symmetric Betti numbers for stable ideals (see Theorem 1.1.14 in Chapter 1).

Proposition 5.4.9. *Let Δ be a simplicial complex and Δ^e be its exterior shifting. Then*

$$\alpha_{l,j}(E/J_\Delta) = \left|\left\{u \in G(I_{\Delta^e})_{j+1} \;\middle|\; \min(u) = l\right\}\right| = \left|\left\{u \in G(J_{\Delta^e})_{j+1} \;\middle|\; \min(u) = l\right\}\right|.$$

In particular,

$$\beta^S_{i,i+j}(k[\Delta^e]) = \sum_{l=1}^{n} \binom{n-l-j}{i-1} \alpha_{l,j}(k\{\Delta\}).$$

Proof. As shown in Corollary 5.4.8 the number $\alpha_{l,j}(E/J_\Delta)$ counts the cardinality of the set

$$\mathscr{A} = \{A \in \Delta^e \mid |A| = j,\ [l] \cap A = \emptyset,\ A \cup \{l\} \notin \Delta^e\}.$$

On the other hand the minimal generators of I_{Δ^e} or J_{Δ^e} are the monomials corresponding to minimal non-faces of Δ^e, i.e., the elements of $\{u \in G(I_{\Delta^e})_{j+1} \mid \min(u) = l\}$ are the monomials x_B such that B lies in the set

$$\mathscr{B} = \{B \notin \Delta^e \mid |B| = j+1,\ \min(B) = l,\ \partial(B) \subseteq \Delta^e\},$$

where $\partial(B) = \{F \subset B \mid F \neq B\}$ denotes the boundary of B. We show that there is a one-to-one correspondence between $\mathscr{A}$ and $\mathscr{B}$. Let $B \in \mathscr{B}$. Then $l \in B$ and $A = B \setminus \{l\}$ is an element in $\mathscr{A}$. Conversely, if $A \in \mathscr{A}$ then $B = A \cup \{l\} \in \mathscr{B}$. The only non-trivial point here is to see that the boundary of B is contained in Δ^e. This holds since Δ^e is shifted and $\min(B) = l$.

The statement about the Betti numbers then follows from the Eliahou-Kervaire formula for squarefree stable ideals (Theorem 1.1.14). □

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Wissenschaftlicher Werdegang

Persönliche Angaben

Name:	Martina Kubitzke
Addresse:	Großseelheimer Str. 61
	35039 Marburg
	+49 (6421) 94 88 334
	kubitzke@mathematik.uni-marburg.de
Geburtsdatum:	28.12.1981

Schulbildung und akademische Laufbahn

1992 – 2001:	Besuch des Augustiner-Gymnasiums, Friedberg
Juni 2001:	Abitur am Augustiner-Gymnasium, Friedberg
Oktober 2001 – August 2006:	Studium der Mathematik mit Abschluss Diplom an der Philipps-Universität Marburg
September 2004 – Februar 2005:	Aufenthalt an der Universidad Complutense de Madrid im Rahmen des Erasmus-Programms
August 2006:	Diplom der Mathematik an der Philipps-Universität Marburg
seit Oktober 2006:	wissenschaftliche Mitarbeiter am Fachbereich Mathematik und Informatik der Philipps-Universität
August 2007 – Februar 2008:	Forschungsaufenthalt an der Cornell University im Rahmen eines DAAD-Doktorandenstipendiums
Mai 2009:	Abgabe der Dissertation *Lefschetz Elements for Stanley-Reisner Rings and Annihilator Numbers*