Stability and Hermitian-Einstein metrics for vector bundles on framed manifolds

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1 Introduction

This thesis is a contribution to algebraic geometry using transcendental methods. The so-called *Kobayashi-Hitchin correspondence*, which has been known since the 80s of the 20th century, establishes a connection between algebraic geometry and analysis by giving a relation between the algebraic-geometric notion of *stability* of a holomorphic vector bundle on an (in the classical case) compact Kähler manifold and the analytic notion of a *Hermitian-Einstein metric* in such a vector bundle.

The notion of stability considered here is the one introduced by Takemoto in [Ta72], which is also known as *slope-stability* or *Mumford-Takemoto stability*. Given a compact Kähler manifold (X, g) of complex dimension n, it can be formulated as follows. The *g-degree* of a torsion-free coherent analytic sheaf $\mathcal{F}$ on X is defined as

$$\deg_g(\mathcal{F}) = \int_X c_1(\mathcal{F}) \wedge \omega^{n-1},$$

where $c_1(\mathcal{F})$ denotes the first Chern class of $\mathcal{F}$ and ω is the fundamental form of the Kähler metric g. If $\mathcal{F}$ is non-trivial, the ratio

$$\mu_g(\mathcal{F}) = \frac{\deg_g(\mathcal{F})}{\operatorname{rank}(\mathcal{F})}$$

of the g-degree of $\mathcal{F}$ and its rank is called the *g-slope* of $\mathcal{F}$. A torsion-free coherent analytic sheaf $\mathcal{E}$ on X is then called *g-semistable* if

$$\mu_g(\mathcal{F}) \leqslant \mu_g(\mathcal{E})$$

holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F})$. If, moreover, the strict inequality

$$\mu_g(\mathcal{F}) < \mu_g(\mathcal{E})$$

holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$, then $\mathcal{E}$ is called *g-stable*. The notion of stability can be applied to a holomorphic vector bundle E on X by considering its sheaf $\mathcal{E} = \mathcal{O}_X(E)$ of holomorphic sections. Every stable holomorphic vector bundle on a compact Kähler manifold is *simple*, i. e. the only holomorphic sections of its endomorphism bundle are the homotheties. A Hermitian metric h in E is called a *g-Hermitian-Einstein metric* if

$$\sqrt{-1}\Lambda_g F_h = \lambda_h \operatorname{id}_E$$

with a real constant λ_h, where $\sqrt{-1}\Lambda_g$ is the contraction with ω, F_h is the curvature form of the Chern connection of the Hermitian holomorphic vector bundle (E, h) and id_E is the identity endomorphism of E. In this case, λ_h is called the *Einstein factor* of h and (E, h) is called a *g-Hermitian-Einstein vector bundle*. The Einstein factor only depends on the Kähler manifold (X, g) and the vector bundle E. In fact, we have

$$\lambda_h = \frac{2\pi\mu_g(E)}{(n-1)!\,\mathrm{vol}_g(X)},$$

where $\mathrm{vol}_g(X)$ is the volume of X with respect to g. The notion of a Hermitian-Einstein metric was introduced by S. Kobayashi in [Kb80] as a generalization of the notion of a Kähler-Einstein metric in the tangent bundle of a compact Kähler manifold.

The Kobayashi-Hitchin correspondence states that an irreducible holomorphic vector bundle admits a g-Hermitian-Einstein metric if and only if it is g-stable. The proof of the g-stability of an irreducible g-Hermitian-Einstein vector bundle is due to S. Kobayashi [Kb82] and Lübke [Lue83]. The other implication, namely the existence of a g-Hermitian-Einstein metric in a g-stable holomorphic vector bundle, was shown for compact Riemann surfaces by Donaldson in [Do83], who gave a new proof of a famous theorem of Narasimhan and Seshadri [NS65]. He later proved the statement for projective-algebraic surfaces in [Do85] and, more generally, for projective-algebraic manifolds of arbitrary dimension in [Do87]. Finally, Uhlenbeck and Yau treated the general case of a compact Kähler manifold in [UY86] (see also [UY89]). All proofs are based on the fact that, given a smooth Hermitian metric h_0 in E (the so-called *background metric*), any Hermitian metric h in E can be written as $h = h_0 f$, i. e.

$$h(s, t) = h_0(f(s), t)$$

for all sections s and t of E, where f is a smooth endomorphism of E which is positive definite and self-adjoint with respect to h_0. One observes that h is a g-Hermitian-Einstein metric if and only if f satisfies a certain non-linear partial differential equation. Donaldson, in his proof, considers an evolution equation of the heat conduction type involving a real parameter t. After he obtains a solution defined for all non-negative values of t, he shows the convergence of the solution as t goes to infinity by using the stability of the vector bundle and an induction argument on the dimension of the complex manifold. The limit is a solution of the partial differential equation and thus yields the desired Hermitian-Einstein metric. Uhlenbeck and Yau, in their proof, consider a perturbed version of the partial differential equation depending on a real parameter ε. They show that it has solutions for every small positive ε. If these solutions converge in a good sense as ε approaches zero, the limit yields a Hermitian-Einstein metric. If the solutions are, however, divergent, this produces a coherent subsheaf contradicting the stability of the vector bundle.

The Kobayashi-Hitchin correspondence has been subject to many generalizations and adaptations for additional structures on the holomorphic vector bundle and the underlying complex manifold. Li and Yau proved a generalization for non-Kähler manifolds in [LY87], which was independently proved for the surface case by Buchdahl in [Bu88]. Hitchin [Hi87] and Simpson [Si88] introduced the notion of a *Higgs bundle* on a complex manifold X, which is a pair (E, θ) consisting of a holomorphic vector bundle E and a bundle map $\theta : E \to E \otimes \Omega^1_X$. They generalized the notions of stability and Hermitian-Einstein metrics to Higgs bundles and proved a Kobayashi-Hitchin correspondence under the integrability condition $0 = \theta \wedge \theta : E \to E \otimes \Omega^2_X$. Bando and Siu extended the notion of a Hermitian-Einstein metric to the case of reflexive sheaves in [BaS94] and proved a Kobayashi-Hitchin correspondence for this situation. The two generalizations for Higgs bundles and reflexive sheaves were recently combined into a generalization for *Higgs sheaves* by Biswas and Schumacher in [BsS09]. Moreover, the Kobayashi-Hitchin correspondence has been considered for the situation of a *holomorphic pair*, which is a holomorphic vector bundle together with a global holomorphic section as introduced by Bradlow in [Br94], and a *holomorphic triple*, which is a pair of two holomorphic vector bundles together with a global holomorphic homomorphism between them as introduced by Bradlow and García-Prada in [BG96].

In this thesis, we consider the situation of a *framed manifold.*

Definition 1.1.

(i) A *framed manifold* is a pair (X, D) consisting of a compact complex manifold X and a smooth divisor D in X.

(ii) A framed manifold (X, D) is called *canonically polarized* if the line bundle $K_X \otimes [D]$ is ample, where K_X denotes the canonical line bundle of X and $[D]$ is the line bundle associated to the divisor D.

The notion of a framed manifold, which is also referred to as a *logarithmic pair*, is introduced e. g. in [Sch98a] and [Sch98b] (see also [ST04]) in analogy to the concept of a *framed vector bundle* (cf. [Le93], [Lue93] and [LOS93]). A simple example of a canonically polarized framed manifold is $(\mathbb{P}^n, V)$, where $\mathbb{P}^n$ is the n-dimensional complex-projective space and V is a smooth hypersurface in $\mathbb{P}^n$ of degree $\geqslant n+2$. Given a canonically polarized framed manifold (X, D), one obtains a special Kähler metric on the complement $X' := X \setminus D$ of D in X.

Theorem 1.2 (R. Kobayashi, [Ko84])**.** *If (X, D) is a canonically polarized framed manifold, there is a unique (up to a constant multiple) complete Kähler-Einstein metric on X' with negative Ricci curvature.*

This is an analogue to the classical theorem of Yau saying that every compact complex manifold with ample canonical bundle possesses a unique (up to a constant multiple) Kähler-Einstein metric with negative Ricci curvature, cf. [Yau78b]. The metric from theorem 1.2, which is of Poincaré-type growth near the divisor D and will therefore be referred to as the *Poincaré metric*, is a natural choice when looking for a suitable Kähler metric on X'.

In [Ko84], R. Kobayashi introduces special "coordinate systems" on X' called *quasi-coordinates.* These are in a certian sense very well adapted to the Poincaré metric. One says that X' together with the Poincaré metric is of *bounded geometry.* This concept has also been investigated by Cheng and Yau in [CY80] and by Tian and Yau in [TY87]. It will be of great importance for the results of this thesis that the asymptotic behaviour of the Poincaré metric is well-known. In fact, in [Sch98a], Schumacher gives an explicit description of its volume form in terms of the quasi-coordinates.

Theorem 1.3 (Schumacher, [Sch98a], theorem 2)**.** *There is a number $0 < \alpha \leqslant 1$ such that for all $k \in \{0, 1, \ldots\}$ and $\beta \in (0, 1)$, the volume form of the Poincaré metric is of the form*

$$\frac{2\Omega}{||\sigma||^2 \log^2(1/||\sigma||^2)} \left(1 + \frac{\nu}{\log^{\alpha}(1/||\sigma||^2)}\right) \quad \textit{with } \nu \in \mathcal{C}^{k,\beta}(X'),$$

where Ω is a smooth volume form on X, σ is a canonical section of $[D]$, $||\cdot||$ is the norm induced by a Hermitian metric in $[D]$ and $\mathcal{C}^{k,\beta}(X')$ is the Hölder space of $\mathcal{C}^{k,\beta}$ functions with respect to the quasi-coordinates.

Moreover, in [Sch98a], Schumacher shows that the fundamental form of the Poincaré metric converges to a Kähler-Einstein metric on D locally uniformly when restricted to coordinate directions parallel to D. From this, one obtains the following result on the asymptotics of the Poincaré metric. Let σ be a canonical section of $[D]$, which can be regarded as a local coordinate in a neighbourhood of a point $p \in D$. Then we can choose local coordinates $(\sigma, z^2, \ldots, z^n)$ near p such that if $g_{\sigma\bar{\sigma}}$, $g_{\sigma\bar{j}}$ etc. denote the coefficients of the fundamental form of the Poincaré metric and $g^{\bar{\sigma}\sigma}$ etc. denote the entries of the corresponding inverse matrix, we have the following statement from [Sch02].

Proposition 1.4. *With $0 < \alpha \leqslant 1$ from theorem 1.3, we have*

(i) $g^{\bar{\sigma}\sigma} \sim |\sigma|^2 \log^2(1/|\sigma|^2)$,

(ii) $g^{\bar{\sigma}i}, g^{\bar{j}\sigma} = O\left(|\sigma| \log^{1-\alpha}(1/|\sigma|^2)\right)$, $i, j = 2, \ldots, n$,

(iii) $g^{\bar{\imath}i} \sim 1$, $i = 2, \ldots, n$ *and*

(iv) $g^{\bar{\jmath}i} \to 0$ *as* $\sigma \to 0$, $i, j = 2, \ldots, n$, $i \neq j$.

We will use the above estimates in order to establish the relevant notions for a Kobayashi-Hitchin correspondence for vector bundles on framed manifolds. In order to do this, one can proceed in several directions. One way is to consider *parabolic bundles* as introduced by Mehta and Seshadri in [MS80] on Riemann surfaces and generalized to higher-dimensional varieties by Maruyama and Yokogawa in [MY92] (see also [Bs95], [Bs97a], [Bs97b]). Let (X, D) be a framed manifold and $\mathcal{E}$ a torsion-free coherent analytic sheaf on X. A *quasi-parabolic structure* on $\mathcal{E}$ with respect to D is a filtration

$$\mathcal{E} = \mathcal{F}_1(\mathcal{E}) \supset \mathcal{F}_2(\mathcal{E}) \supset \cdots \supset \mathcal{F}_l(\mathcal{E}) \supset \mathcal{F}_{l+1}(\mathcal{E}) = \mathcal{E}(-D)$$

by coherent subsheaves, where $\mathcal{E}(-D)$ is the image of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ in $\mathcal{E}$. The integer l is called the *length of the filtration*. A *parabolic structure* is a quasi-parabolic structure together with a system of *parabolic weights* $\{\alpha_1, \ldots, \alpha_l\}$ such that $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_l < 1$. The weight α_i corresponds to $\mathcal{F}_i(\mathcal{E})$. The sheaf $\mathcal{E}$ together with these data is then called a *parabolic sheaf* and denoted by $(\mathcal{E}, \mathcal{F}_*, \alpha_*)$ or simply by $\mathcal{E}_*$. If g is a Kähler metric on X, the notion of g-stability can be adapted for parabolic sheaves. Given a parabolic sheaf $(\mathcal{E}, \mathcal{F}_*, \alpha_*)$, let

$$\mathcal{E}_t = \mathcal{F}_i(\mathcal{E})(-\lfloor t \rfloor D) \quad \text{for any } t \in \mathbb{R},$$

where $\lfloor t \rfloor$ is the integral part of t and $i \in \{1, \ldots, l+1\}$ is such that

$$\alpha_{i-1} < t - \lfloor t \rfloor \leqslant \alpha_i,$$

where $\alpha_0 := \alpha_l - 1$ and $\alpha_{l+1} := 1$. The filtration $(\mathcal{E}_t)_{t \in \mathbb{R}}$ thus defined has the following properties.

- It is decreasing, i. e. $\mathcal{E}_t \subset \mathcal{E}_{t'}$ for all real numbers $t \geqslant t'$.
- It is continuous from the left, i. e. there is an $\varepsilon > 0$ such that $\mathcal{E}_{t-\varepsilon} = \mathcal{E}_t$ for all $t \in \mathbb{R}$.
- It has a jump at $t \in \mathbb{R}$, i. e. $\mathcal{E}_{t+\varepsilon} \neq \mathcal{E}_t$ for any $\varepsilon > 0$, if and only if $t - \lfloor t \rfloor = \alpha_i$ for some $i \in \{1, \ldots, l\}$.
- It completely determines the parabolic structure $(\mathcal{E}, \mathcal{F}_*, \alpha_*)$.

This filtration allows us to adapt the ordinary notions of subsheaves, g-degree, g-slope and, finally, g-stability for the parabolic situation as follows. A parabolic sheaf $\mathcal{S}_*$ is called a *parabolic subsheaf* of $\mathcal{E}_*$ if the following conditions are satisfied.

(i) $\mathcal{S}$ is a subsheaf of $\mathcal{E}$ with $\mathcal{E}/\mathcal{S}$ being torsion-free,

(ii) $\mathcal{S}_t \subset \mathcal{E}_t$ for all $t \in \mathbb{R}$ and,

(iii) if $\mathcal{S}_s \subset \mathcal{E}_t$ for any $s, t \in \mathbb{R}$ with $t > s$, then $\mathcal{S}_s = \mathcal{S}_t$.

The *parabolic g-degree* of a parabolic sheaf $\mathcal{E}_*$ is defined as

$$\operatorname{pardeg}_g(\mathcal{E}_*) = \int_0^1 \deg_g(\mathcal{E}_t)\, dt + \operatorname{rank}(\mathcal{E}) \deg D.$$

Then, proceeding as in the standard situation, the *parabolic g-slope* of a parabolic sheaf $\mathcal{E}_*$ with $\operatorname{rank}(\mathcal{E}) > 0$ is defined to be

$$\text{par-}\mu_g(\mathcal{E}_*) = \frac{\operatorname{pardeg}_g(\mathcal{E}_*)}{\operatorname{rank}(\mathcal{E})}.$$

A parabolic sheaf $\mathcal{E}_*$ is called *parabolic g-semistable* if

$$\text{par-}\mu_g(\mathcal{S}_*) \leqslant \text{par-}\mu_g(\mathcal{E}_*)$$

holds for every parabolic subsheaf $\mathcal{S}_*$ of $\mathcal{E}_*$ with $0 < \operatorname{rank}(\mathcal{S})$. If, moreover, the strict inequality

$$\text{par-}\mu_g(\mathcal{S}_*) < \text{par-}\mu_g(\mathcal{E}_*)$$

holds for every parabolic subsheaf $\mathcal{S}_*$ of $\mathcal{E}_*$ with $0 < \operatorname{rank}(\mathcal{S}) < \operatorname{rank}(\mathcal{E})$, then $\mathcal{E}_*$ is called *parabolic g-stable.* Note that in [Bs97b], Biswas describes a relation between parabolic bundles on X with respect to D and so-called *orbifold bundles* on a finite covering $p : Y \to X$ ramified along D.

In [LN99], Li and Narasimhan establish a Kobayashi-Hitchin correspondence for rank-2 parabolic vector bundles on framed manifolds of complex dimension 2 by showing the equivalence between parabolic stability and the existence of a Hermitian-Einstein metric in the restriction $E' := E|_{X'}$ of E to X'. Here, the Hermitian-Einstein condition is considered with respect to a Kähler metric on X' which is the restriction to X' of a smooth Kähler metric on X and Hermitian-Einstein metrics are required to satisfy an additional condition called *compatibility with the parabolic structure.* These considerations, therefore, do not involve the special Kähler metric on X' from theorem 1.2 which is available in the canonically polarized case. This approach is also covered by Simpson in [Si88] for the case of Higgs bundles. In contrast to that, in [Bi97], Biquard deals with the relation between parabolic stability and the existence of a Hermitian metric in E' which is Hermitian-Einstein with respect to the Poincaré metric. He introduces a combination of parabolic bundles and Higgs bundles called *logarithmic bundles* and establishes a Kobayashi-Hitchin correspondence in this situation.

In this thesis, we follow an alternative way to define the notions needed for a Kobayashi-Hitchin correspondence for vector bundles on framed manifolds. Our method does not involve parabolic structures. Instead, we directly adapt the ordinary notions of stability and Hermitian-Einstein metrics to the framed situation. Given a canonically polarized framed manifold (X, D), there are two approaches to "stability in the framed sense" of a torsion-free coherent analytic sheaf $\mathcal{E}$ on X with respect to the framed manifold (X, D) that seem reasonable to us. Firstly, there is the standard notion of stability of $\mathcal{E}$ with respect to the polarization $K_X \otimes [D]$ of X. This means that the degree of a coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ is computed with respect to a Kähler metric on X whose fundamental form is the curvature form of a positive smooth Hermitian metric in the line bundle $K_X \otimes [D]$. Regarding the second approach, we consider coherent subsheaves $\mathcal{F}$ of $\mathcal{E}$ again but this time compute their degree on X' with respect to the Poincaré metric. Note, however, that this does not yield the standard notion of stability on X' with respect to the Poincaré metric since we only consider subsheaves of $\mathcal{E}$ on X instead of X'. Fortunately, using Schumacher's theorem 1.3 on the asymptotic behaviour of the Poincaré metric, we can show that these two approaches are equivalent, which is a strong evidence that the notion of *stability in the framed sense* or *framed stability* obtained this way is reasonable in view of a Kobayashi-Hitchin correspondence. In fact, since framed stability is a special case of stability in the ordinary sense, the framed stability of a holomorphic vector bundle on X implies its simplicity. It does, however, not necessarily imply the simplicity of its restriction to X'.

We have to pay special attention on what a *Hermitian-Einstein metric in the framed sense* or a *framed Hermitian-Einstein metric* should be. We are interested in Hermitian metrics in E' satisfying the Hermitian-Einstein condition with respect to the Poincaré metric. However, a look at the proof of the uniqueness (up to a constant multiple) of such a Hermitian-Einstein metric shows that this condition is not sufficient in order to obtain a sensible notion of a framed Hermitian-Einstein metric. Indeed, the classical uniqueness proof makes use of the simplicity of a stable vector bundle. Thus, since the framed stability of E only implies the simplicity of E and not that of E', this does not give us the uniqueness of an arbitrary Hermitian-Einstein metric in E' with respect to the Poincaré metric. Instead, we additionally require a condition of compatibility with a smooth Hermitian metric in E over the compact manifold X, which is the one introduced by Simpson in [Si88]. This condition is, in fact, similar to the condition of compatibility with the parabolic structure mentioned above.

We prove that every holomorphic vector bundle on a canonically polarized framed manifold which is stable in the framed sense possesses a unique (up to a constant multiple) framed Hermitian-Einstein metric. Our methods are as follows. The concept of bounded geometry mentioned above allows us to apply Simpson's heat equation method from [Si88] (done there, among others, in the compact case) to our situation as long as all analytic considerations are expressed in terms of quasi-coordinates. Simpson, like Donaldson, solves an evolution equation of the heat conduction type for all non-negative values of a real parameter t. If the solution converges as t goes to infinity, the limit yields the desired Hermitian-Einstein metric. There is only one critical point about the application of Simpson's proof to our situation, namely the construction of a destabilizing subsheaf of $\mathcal{E} = \mathcal{O}_X(E)$ for the case that the solution does not converge. One first obtains a so-called *weakly holomorphic subbundle* of E (or E'), which means a measurable section π of $\mathrm{End}(E)$ lying in the Sobolev space of L^2 sections with L^2 first-order weak derivatives and additionally satisfying the conditions

$$\pi = \pi^* = \pi^2 \quad \text{and} \quad (\mathrm{id}_E - \pi) \circ \nabla'' \pi = 0,$$

where π^* denotes the adjoint of π with respect to a Hermitian metric in E and ∇'' is the $(0,1)$ part of the associated Chern connection. In their paper [UY86], Uhlenbeck and Yau show that this actually defines a coherent subsheaf of $\mathcal{E}$ and, implicitly, a holomorphic subbundle of E outside an analytic subset of X of codimension $\geqslant 2$. An alternative proof of this statement based on current theory is given by Popovici in [Po05]. In our situation, the section π from Simpson's proof satisfies the L^2 conditions with respect to the Poincaré metric. Using the results from proposition 1.4, we can show that these already imply the L^2 conditions in the ordinary sense. Consequently, the theorem of Uhlenbeck-Yau-Popovici can be applied to our situation without change.

We would like to remark that "asymptotic" versions of our result have been obtained by Ni and Ren in [NR01] and Xi in [Xi05]. Here, the authors consider certain classes of complete, non-compact Hermitian manifolds (X, g). In order to be able to show the existence of a Hermitian-Einstein metric in a holomorphic vector bundle E on X, they do not suppose that the vector bundle is stable. Instead, they require the existence of a Hermitian metric h_0 in E that is *asymptotically Hermitian-Einstein*, which is a condition on the growth of $|\sqrt{-1}\Lambda_g F_{h_0} - \lambda_{h_0} \mathrm{id}_E|_{h_0}$.

The content of this thesis is organized as follows. In chapter 2, we define the notion of a Kähler metric on X' with Poincaré-type growth near the divisor D and present a construction of such a metric due to Griffiths. After introducing the concept of local quasi-coordinates and bounded geometry following R. Kobayashi, we present a proof of the existence and uniqueness (up to a constant multiple) of a complete Kähler-Einstein metric on X' with negative Ricci curvature. This metric also has Poincaré-type growth and will be called the *Poincaré metric* later on. Finally, we show that the square-integrability conditions for functions and 1-forms with respect to the Poincaré metric imply the corresponding conditions in the ordinary sense.

Chapter 3 is the central part of this thesis. After giving a short review of the concepts of stability and Hermitian-Einstein metrics in the compact case, we develop the corresponding notions in the framed situation. In particular, we show that the two approaches to framed stability mentioned above are equivalent. Moreover, we show the uniqueness (up to a constant multiple) of a framed Hermitian-Einstein metric in a simple bundle.

Chapter 4 contains the existence result for framed Hermitian-Einstein metrics in a holomorphic vector bundle on a canonically polarized framed manifold which is stable in the framed sense. Here, we give a summary of Donaldson's existence proof for a solution of the evolution equation defined for all finite non-negative values of the time parameter and a review of Simpson's approach to the convergence of this solution in infinite time. Moreover, we summarize Popovici's proof of the regularity theorem for weakly holomorphic subbundles, which can be applied to our situation because of the result on the square-integrability conditions from chapter 2.

Finally, in chapter 5, we outline some further thoughts based on the results of this thesis. Starting from the work [TY87] of Tian and Yau, one is led to conjecture that the unique framed Hermitian-Einstein metric obtained in chapter 4 can also be seen as the limit of a sequence of Hermitian-Einstein metrics on X' with respect to certain non-complete Kähler-Einstein metrics constructed by Tian and Yau. This problem is, however, still open.

Let us now fix some notations used throughout the text. Unless otherwise stated, X is always a compact complex manifold of complex dimension n and D is a smooth divisor (or, more generally, a divisor with simple normal crossings) in X. We denote the canonical line bundle of X by K_X and the line bundle associated to the divisor D by $[D]$. We write $X' = X \setminus D$ for the complement of D in X. Kähler metrics are always denoted by the letter g and their fundamental forms by the letter ω. A subscript of X, X' or D indicates the manifold on which they are defined. The subscript will occasionally be dropped when no confusion is likely to arise. As usual, Λ_g denotes the formal adjoint of forming the $\wedge$-product with ω. When comparing integrability conditions with respect to the Poincaré metric to those in the ordinary sense, dV denotes the Euclidean volume element and dV_g denotes the volume element of the Poincaré metric. In the same spirit, regarding L^2 spaces, the letter g indicates the use of the Poincaré metric, whereas its absence hints at the use of a smooth Kähler metric on the compact manifold X. A holomorphic vector bundle on X is denoted by E and its restriction to X' by E'. We write $\mathcal{E} = \mathcal{O}_X(E)$ for its sheaf of holomorphic sections and use the letter $\mathcal{F}$ to indicate a coherent subsheaf of $\mathcal{E}$. The letters h and h' are used for Hermitian metrics in E and E', respectively. We denote the covariant derivative with respect to the Chern connection of a Hermitian holomorphic vector bundle (E, h) by $\nabla = \nabla' + \nabla''$, where ∇' and ∇'' are its $(1, 0)$ and $(0, 1)$ components. Finally, we use the summation convention wherever it is unlikely to cause any confusion.

2 Poincaré metrics and quasi-coordinates

In this chapter we introduce the notion of a Kähler metric on X' with Poincaré-type growth near the divisor D (Poincaré metric for short). An investigation of the properties of such a metric shows that metrics of this type are in a certain sense a natural choice when studying framed manifolds. We present a construction due to Griffiths asserting the existence of a Poincaré metric on X' in the canonically polarized case, i. e. when $K_X \otimes [D]$ is ample.

We then define the notion of a local quasi-coordinate and describe the construction of a quasi-coordinate system on X' due to R. Kobayashi. The relevant function spaces defined with respect to these quasi-coordinates will turn out to be very well adapted to Poincaré metrics on X'. In fact, the notion of bounded geometry, which goes together with the quasi-coordinates, will be a very powerful tool when doing analysis with respect to a Poincaré metric. In particular, although the complex manifold X' is only complete, it behaves as if it were compact, provided that analytic considerations are always expressed in terms of quasi-coordinates.

We then present a result by R. Kobayashi on the existence of a Poincaré-type Kähler-Einstein metric on X' in the canonically polarized case. This metric is actually unique up to a constant multiple and therefore represents a natural choice of Kähler metric on X' for our later studies. We further quote a result of Schumacher which expresses the volume form of this Kähler-Einstein metric in a rather explicit way.

Finally, for later application, we consider the condition of square-integrability for functions and differential forms of degree 1 on X' with respect to our Poincaré-type Kähler-Einstein metric. We shall discover that this condition actually implies the square-integrability in the ordinary sense, a statement that will enable us to apply the regularity theorem for weakly holomorphic subbundles in the ordinary sense to our Poincaré-type situation. This is proved by using results of Schumacher on the asymptotic behaviour of the Poincaré metric near the divisor D.

Poincaré metrics have been an object of study for many years. The consideration of such metrics is essentially due to Zucker ([Zu79], [Zu82]) and Saper ([Sa85], [Sa92]), who dealt with Poincaré metrics in their works about L^2 cohomology on singular Kähler varieties. The existence and uniqueness result for a Poincaré-type Kähler-Einstein metric has later been generalized by Tian and Yau [TY87]. Poincaré metrics have also been considered by Biquard in his work on logarithmic vector bundles [Bi97], Grant and Milman [GM95] in their work on L^2 cohomology and other authors.

2.1 Definition and existence of Poincaré metrics

In this and the following two sections, we can relax the assumption on the divisor D. Instead of assuming D to be a smooth (irreducible) divisor, we impose the condition of *simple normal crossings*, meaning that $D = D_1 + \cdots + D_m$ is an effective divisor such that its irreducible components $D_1, \ldots, D_m$ are smooth and every two of them intersect at most transversally.

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $\Delta^* = \Delta \setminus \{0\}$ be the unit disc, respectively the punctured unit disc, in $\mathbb{C}$ with coordinate z. The fundamental form of the Poincaré metric on Δ^* is given by

$$\omega_{\Delta^*} = -\sqrt{-1}\partial\bar{\partial} \log \log^2 |z|^2 = \frac{2\sqrt{-1}dz \wedge d\bar{z}}{|z|^2 \log^2(1/|z|^2)}. \tag{2.1}$$

Definition 2.1 (Quasi-isometricity)**.** Two Hermitian metrics g_1 and g_2 on a complex manifold are called *quasi-isometric*, written $g_1 \sim g_2$, if there is a constant $c > 0$ such that

$$\frac{1}{c} g_1 \leqslant g_2 \leqslant c g_1.$$

The corresponding notion for functions and differential forms is defined in the same way.

Definition 2.2 (Poincaré metric)**.** A Kähler metric g on X' is said to have *Poincaré-type growth* near the divisor D (or to be a *Poincaré metric* on X') if for every point $p \in D$ there is a coordinate neighbourhood $U(p) \subset X$ of p with $U(p) \cap X' \cong (\Delta^*)^k \times \Delta^{n-k}$, $1 \leqslant k \leqslant n$, such that in these coordinates, g is quasi-isometric to a product of k copies of the Poincaré metric on Δ^* and $n - k$ copies of the Euclidean metric on Δ.

Remark 2.3. For a point $p \in D$, let $D_1, \ldots, D_k$ be the irreducible components of D going through p and consider a neighbourhood $U(p) \subset X$ of p such that no other components of D intersect the closure $\overline{U(p)}$. A coordinate system $(z^1, \ldots, z^n)$ on $U(p)$ is called *normal* with respect to D if D_i is locally given by the equation $z^i = 0$, $1 \leqslant i \leqslant k$. In such a coordinate system, the fundamental form ω of a Poincaré metric on X' satisfies

$$\omega \sim 2\sqrt{-1} \left(\sum_{i=1}^{k} \frac{dz^i \wedge d\bar{z}^i}{|z^i|^2 \log^2(1/|z^i|^2)} + \sum_{i=k+1}^{n} dz^i \wedge d\bar{z}^i \right).$$

We discuss some fundamental properties of Poincaré metrics.

Definition 2.4 (Completeness)**.** A Kähler manifold (X', g) is said to be *complete* if (X', δ) is complete as a metric space, where δ is the geodesic distance on X' induced by g.

Proposition 2.5. *If g is a Poincaré metric on X', then (X', g) is a complete Kähler manifold with finite volume.*

Proof. Since X is compact, we only need to consider small neighbourhoods of points of D.

By the definition of a Poincaré metric on X' and the completeness of the Poincaré metric on Δ^* near the origin, it follows that the length of any curve in X' approaching a point of D measured by g is infinity, which implies the completeness of (X', g).

Since

$$\int_{0<|z|<\varepsilon} \frac{2\sqrt{-1}dz \wedge d\bar{z}}{|z|^2 \log^2(1/|z|^2)} = -\frac{2\pi}{\log r}\bigg|_{r=0}^{\varepsilon} = -\frac{2\pi}{\log \varepsilon} < \infty \quad \text{for } 0 < \varepsilon < 1,$$

the volume of (X', g) is finite. □

We prove a lemma asserting the existence of a Poincaré-type Kähler metric (without the Kähler-Einstein condition) on X' whose fundamental form is defined to be $-\operatorname{Ric}\Psi$, where Ric denotes the Ricci curvature and Ψ is a volume form on X with singularities along the divisor D. This metric will later serve as a background metric for the construction of a Poincaré-type Kähler-Einstein metric on X'.

Lemma 2.6 (Griffiths, [Gr76])**.** *Let X be a compact complex manifold and let D be a divisor in X with simple normal crossings such that $K_X \otimes [D]$ is ample. Then there is a volume form Ψ on X' with the following properties.*

(i) $-\operatorname{Ric}\Psi$ *is a closed positive definite real* $(1,1)$*-form on X' and the associated Kähler metric on X' is a Poincaré metric.*

(ii) There is a constant $c > 0$ such that

$$\frac{1}{c} < \frac{\Psi}{(-\operatorname{Ric}\Psi)^n} < c \quad \text{on } X'.$$

Proof. Let $D = D_1 + \cdots + D_m$ be the decomposition of D into its irreducible components and for $1 \leqslant i \leqslant m$ let $\sigma_i \in \Gamma(X, [D_i])$ be a canonical holomorphic section of $[D_i]$, i. e. such that $D_i = V(\sigma_i)$ is the vanishing locus of σ_i. Given a Hermitian metric on each $[D_i]$, let $||\cdot||$ denote the induced norm on each $[D_i]$ as well as the product norm on $[D] = [D_1] \otimes \cdots \otimes [D_m]$. We can assume that $||\sigma_i||^2 < 1$ for $1 \leqslant i \leqslant m$. By the assumption that $K_X \otimes [D]$ is ample, there is a positive Hermitian metric on $K_X \otimes [D]$, which means that there is a smooth volume form Ω on X and a Hermitian metric on each $[D_i]$ such that

$$\eta := -\operatorname{Ric}\Omega - \sum_{i=1}^{m} \sqrt{-1}\partial\bar{\partial}\log||\sigma_i||^2$$

is positive definite on X. Now define a volume form Ψ on X' by

$$\Psi = \frac{2\Omega}{\prod_{i=1}^{m} ||\sigma_i||^2 \log^2(1/||\sigma_i||^2)}.$$

We show that $-\operatorname{Ric}\Psi$ can be made positive definite on X'. A direct computation yields

$$\begin{aligned} -\operatorname{Ric}\Psi &= -\operatorname{Ric}\Omega - \sum_{i=1}^{m} \sqrt{-1}\partial\bar{\partial}\log||\sigma_i||^2 - 2\sum_{i=1}^{m} \sqrt{-1}\partial\bar{\partial}\log\log(1/||\sigma_i||^2) \\ &= \eta - 2\sum_{i=1}^{m} \frac{\sqrt{-1}\partial\bar{\partial}\log||\sigma_i||^2}{\log||\sigma_i||^2} + 2\sum_{i=1}^{m} \frac{\sqrt{-1}\partial\log||\sigma_i||^2 \wedge \bar{\partial}\log||\sigma_i||^2}{\log^2(1/||\sigma_i||^2)}. \end{aligned} \tag{2.2}$$

The third term in this expression is positive semidefinite. Thus, by replacing $||\cdot||$ by $\varepsilon||\cdot||$ with a sufficiently small $\varepsilon > 0$ such that

$$\eta - \sum_{i=1}^{m} \frac{\sqrt{-1}\partial\bar{\partial}\log||\sigma_i||^2}{\log||\sigma_i||^2} \geqslant \frac{\eta}{2}$$

in the sense of positive semidefiniteness, we make $-\operatorname{Ric}\Psi$ a positive definite form.

We now show that the Kähler metric with fundamental form $-\operatorname{Ric}\Psi$ is a Poincaré metric. Let $p \in D$ and assume that $p \in (D_1 \cap \cdots \cap D_k) \setminus (D_{k+1} \cup \cdots \cup D_m)$ with $1 \leqslant k \leqslant m$. Since D is a divisor with simple normal crossings, there is a coordinate neighbourhood $U(p) \subset X$ of p such that $U(p) \cap D \cong \bigcup_{i=1}^k \{(z^1, \ldots, z^n) \in \Delta^n : z^i = 0\}$. Thus $U(p) \cap X' \cong (\Delta^*)^k \times \Delta^{n-k}$. In this coordinate system, $||\sigma_i||^2$ is given by $|z^i|^2/h_i$, where h_i is a smooth positive function on Δ^n. Since we are only interested in the asymptotic behaviour of $-\operatorname{Ric}\Psi$ (i. e. up to quasi-isometricity), we can neglect all expressions that are bounded near D. In particular, we only need to consider the last term in (2.2). Except for some positive semidefinite smooth terms coming from the coordinate directions $z^{k+1}, \ldots, z^n$, it locally looks like

$$2\sum_{i=1}^{k} \frac{\sqrt{-1}dz^i \wedge d\bar{z}^i + |z^i|^2\alpha_i}{|z^i|^2(\log|z^i|^2 - \log h_i)^2} \tag{2.3}$$

with

$$\alpha_i = -\frac{dz^i \wedge \bar{\partial}\log h_i}{z^i} - \frac{\partial \log h_i \wedge d\bar{z}^i}{\bar{z}^i} + \partial \log h_i \wedge \bar{\partial}\log h_i, \quad 1 \leqslant i \leqslant k.$$

Comparing (2.3) with the fundamental form (2.1) of the Poincaré metric in the punctured unit disc, we see that the Kähler metric given by $-\operatorname{Ric}\Psi$ is a Poincaré metric, which proves (i). Assertion (ii) follows in a similar way from (2.2), (2.3) and the definition of Ψ. □

2.2 Quasi-coordinates and Hölder spaces

We first introduce the notion of a local quasi-coordinate of X'.

Definition 2.7. A holomorphic map from an open set $V \subset \mathbb{C}^n$ into X' is called a *quasi-coordinate map* if it is of maximal rank everywhere in V. In this case, V together with the Euclidean coordinates of $\mathbb{C}^n$ is called a *local quasi-coordinate* of X'.

We now describe a family of local quasi-coordinates of X', which is shown to be very well adapted to the Poincaré metric described in the previous section. This construction is due to R. Kobayashi. Similar quasi-coordinate systems have been introduced in [CY80] and [TY87].

Fix a point $p \in D$ such that $p \in (D_1 \cap \cdots \cap D_k) \setminus (D_{k+1} \cup \cdots \cup D_m)$ with $1 \leqslant k \leqslant m$. As above, there is an open neighbourhood $U(p) \subset X$ of p such that $((\Delta^*)^k \times \Delta^{n-k}; z^1, \ldots, z^n)$ is a coordinate for X' on $U(p)$ and locally,

- the Poincaré metric given by $-\operatorname{Ric}\Psi$ is quasi-isometric to a product of k copies of the Poincaré metric on Δ^* and $n-k$ copies of the Euclidean metric on Δ and
- if $D_1, \ldots, D_k$ are the irreducible components of D going through p, D_i is given by $z^i = 0$, $1 \leqslant i \leqslant k$.

We need the following two auxiliary constructions.

(i) There is a universal covering map

$$\begin{array}{rcl} \Delta^n = \Delta^k \times \Delta^{n-k} & \longrightarrow & (\Delta^*)^k \times \Delta^{n-k} \\ (w^1, \ldots, w^k, w^{k+1}, \ldots, w^n) & \longmapsto & (z^1, \ldots, z^k, z^{k+1}, \ldots, z^n) \end{array}$$

$$\text{with}\quad z^i = \begin{cases} \exp\left(\dfrac{w^i+1}{w^i-1}\right) & \text{if } 1 \leqslant i \leqslant k, \\ w^i & \text{if } k+1 \leqslant i \leqslant n. \end{cases}$$

(ii) We introduce coordinate systems on open sets in Δ close to 1 as follows. Fix a real number R with $\frac{1}{2} < R < 1$ and a real number $a \in \Delta$ close to 1. We have to remark that while a will vary in a neighbourhood of 1, the number R will stay fixed throughout the whole construction. There is a biholomorphic map

$$\Phi_a : \begin{cases} \Delta & \longrightarrow & \Delta \\ w & \longmapsto & \dfrac{w-a}{1-aw} \end{cases}$$

with $\Phi_a(a) = 0$. If we let $B_R(0) := \{v \in \mathbb{C} : |v| < R\}$, the inverse image $\Phi_a^{-1}(B_R(0))$ is an open neighbourhood of a and we can define a coordinate function

$$\begin{array}{rcl} \Phi_a^{-1}(B_R(0)) & \longrightarrow & B_R(0) \\ w & \longmapsto & v = \Phi_a(w) \end{array} .$$

Figure 2.1 shows $B_R(0)$ and the inverse images $\Phi_a^{-1}(B_R(0))$ (shaded areas) for some values of a close to 1.

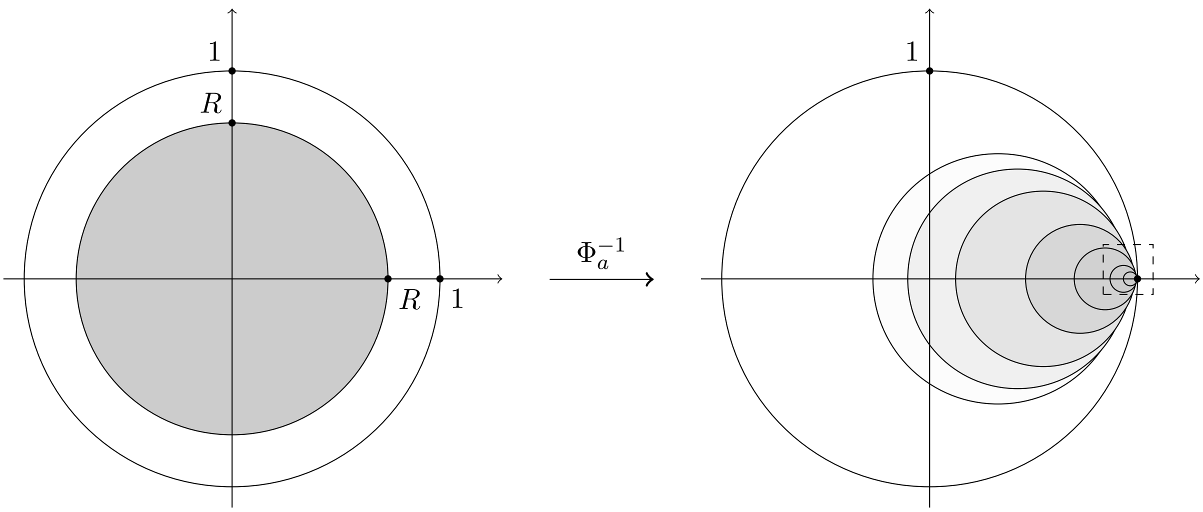


Figure 2.1: Mapping $B_R(0)$ with Φ_a^{-1} for some values of a.

Let $z \in (\Delta^*)^k \times \Delta^{n-k}$ such that z^i is close to 0 for $1 \leqslant i \leqslant k$. By the universal covering map (i), there is a point $(w^1, \dots, w^n) \in \Delta^n$ which is projected onto $(z^1, \dots, z^n)$. Then w^i is close to

1 for $1 \leqslant i \leqslant k$ and we can use the local coordinates described in (ii) by choosing suitable real numbers $a^i \in \Delta$ close to 1 for $1 \leqslant i \leqslant k$ and letting

$$v^i = \begin{cases} \Phi_{a^i}(w^i) = \dfrac{w^i - a^i}{1 - a^i w^i} & \text{if } 1 \leqslant i \leqslant k, \\ w^i & \text{if } k+1 \leqslant i \leqslant n. \end{cases}$$

The fact that with the a^i chosen sufficiently close to 1, we can actually cover an open neighbourhood of p in X', is contained in the following lemma.

Lemma 2.8. *The set* $\bigcup_a \Phi_a^{-1}(B_R(0))$*, where the union is taken over real numbers* $a \in \Delta$ *close to* 1*, covers the punctured neighbourhoods of* 1 *in fundamental domains of the universal covering map* $\Delta \to \Delta^*$*.*

The situation of the lemma is shown in figure 2.2. The left-hand side is a larger version of the dashed rectangle in figure 2.1. It shows the sets $\Phi_a^{-1}(B_R(0))$ (shaded areas) for some values of a close to 1 as well as a fundamental domain of the universal covering map $\Delta \to \Delta^*$, which is the domain bounded by two geodesics tending to 1. The arrows illustrate how the fundamental domain is mapped onto the punctured disc, which is depicted on the right-hand side.

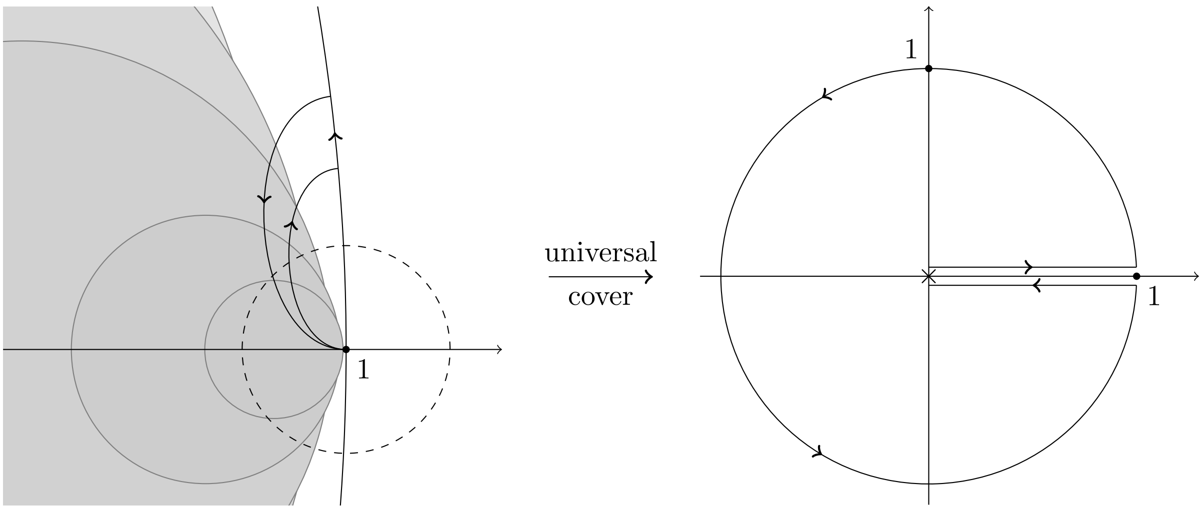


Figure 2.2: The situation of lemma 2.8.

Proof of lemma 2.8. Take v on the boundary of $B_R(0)$, namely $v = Re^{i\theta}$ with $\theta \in [0, 2\pi)$. Then

$$w = \Phi_a^{-1}(v) = \frac{v + a}{1 + av}$$

is on the boundary of $\Phi_a^{-1}(B_R(0))$ and we have

$$\operatorname{Re}(w) = \frac{a(1+R^2)+(1+a^2)R\cos\theta}{1+a^2R^2+2aR\cos\theta},$$
$$\operatorname{Im}(w) = \frac{(1-a^2)R\sin\theta}{1+a^2R^2+2aR\cos\theta}.$$

In particular, for $\theta = \frac{\pi}{2}$, we get some w on the boundary of $\Phi_a^{-1}(B_R(0))$ satisfying

$$\operatorname{Re}(w) = \frac{a(1+R^2)}{1+a^2R^2} \geqslant a,$$
$$\operatorname{Im}(w) = \frac{R(1-a^2)}{1+a^2R^2} \geqslant \frac{R}{2}(1-a^2)$$

and thus lying above the parabola $a \mapsto \frac{R}{2}(1-a^2)$. The claim follows. □

Therefore we can define a "coordinate" $(B_R(0)^k \times \Delta^{n-k}; v^1, \ldots, v^n)$ of $U(p)$ by letting

$$v^i = \begin{cases} \Phi_{a^i}(w^i) = \dfrac{w^i - a^i}{1 - a^i w^i} & \text{if } 1 \leqslant i \leqslant k, \\ w^i & \text{if } k+1 \leqslant i \leqslant n, \end{cases}$$

$$\text{where} \quad z^i = \begin{cases} \exp\left(\dfrac{w^i+1}{w^i-1}\right) & \text{if } 1 \leqslant i \leqslant k, \\ w^i & \text{if } k+1 \leqslant i \leqslant n \end{cases}$$

and the a^i are real numbers in Δ close to 1. Although this is not a coordinate system in the ordinary sense because of the universal covering map, it makes sense to talk about the components of a tensor field on $U(p)$ (or $(\Delta^*)^k \times \Delta^{n-k}$) with respect to the "coordinates" v^i by first lifting it to a tensor field on Δ^n. The behaviour of a function on $U(p)$ can thus be examined by looking at the (locally) lifted function in a neighbourhood of $(1, \ldots, 1, *, \ldots, *)$ in Δ^n. So $(B_R(0)^k \times \Delta^{n-k}; v^1, \ldots, v^n)$ is a local quasi-coordinate in the sense of definition 2.7.

Keeping this in mind, we can cover the whole of X' with a family of local quasi-coordinates with respect to which the Poincaré metric defined by $-\operatorname{Ric}\Psi$ behaves nicely. Indeed, let

$$-\operatorname{Ric}\Psi = \sqrt{-1} g_{i\bar{j}} dv^i \wedge d\bar{v}^j$$

be a local representation of its fundamental form with respect to the quasi-coordinates v^i. Then we have the following proposition.

Proposition 2.9. *There is a family $\mathcal{V} = \{(V; v^1, \ldots, v^n)\}$ of local quasi-coordinates of X' with the following properties.*

(i) X' is covered by the images of the quasi-coordinates in $\mathcal{V}$.

(ii) The complement of some open neighbourhood of D in X is covered by the images of finitely many of the quasi-coordinates in $\mathcal{V}$ which are local coordinates in the usual sense.

(iii) For each $(V; v^1, \dots, v^n) \in \mathcal{V}$, $V \subset \mathbb{C}^n$ *contains an open ball of radius* $\frac{1}{2}$.

(iv) There are constants $c > 0$ *and* $A_k > 0$, $k = 0, 1, \dots$, *such that for every* $(V; v^1, \dots, v^n) \in \mathcal{V}$, *the following inequalities hold.*

- *We have*

$$\frac{1}{c}(\delta_{i\bar{j}}) < (g_{i\bar{j}}) < c(\delta_{i\bar{j}})$$

  as matrices in the sense of positive definiteness, where $\delta_{i\bar{j}}$ *is the Kronecker delta.*
- *For any multiindices* $I = (i_1, \dots, i_p)$ *and* $J = (j_1, \dots, j_q)$ *of order* $|I| = i_1 + \cdots + i_p$ *respectively* $|J| = j_1 + \cdots + j_q$, *we have*

$$\left| \frac{\partial^{|I|+|J|} g_{i\bar{j}}}{\partial v^I \partial \bar{v}^J} \right| < A_{|I|+|J|},$$

  where $\partial v^I = (\partial v^1)^{i_1} \cdots (\partial v^p)^{i_p}$ *and* $\partial \bar{v}^J = (\partial \bar{v}^1)^{j_1} \cdots (\partial \bar{v}^q)^{j_q}$.

Remark 2.10. According to [CY80] and [TY87], a complete Kähler manifold (X', g) which admits a family $\mathcal{V}$ of local quasi-coordinates satisfying the conditions of proposition 2.9 is called of *bounded geometry* (of order ∞).

Proof of proposition 2.9. There is an open neighbourhood U of D in X such that $U \setminus D$ can be covered with the local quasi-coordinates $(V; v^1, \dots, v^n)$ constructed above. Since X is compact, $X \setminus U$ can be covered by finitely many local coordinates $(B; z^1, \dots, z^n)$, where $B \subset \mathbb{C}^n$ is a unit ball. This yields a family $\mathcal{V}$ covering X' as mentioned in the proposition satisfying (i), (ii) and (iii). Regarding (iv), we have to consider the essential part (2.3) of the local expression of $-\operatorname{Ric}\Psi$ again and translate it into our new "coordinates" v^i. From

$$z^i = \exp\left(\frac{w^i+1}{w^i-1}\right) = \exp\left(\frac{(1+a^i)(v^i+1)}{(1-a^i)(v^i-1)}\right), \quad 1 \leqslant i \leqslant k,$$

it follows that

$$dz^i = t_i dv^i, \qquad d\bar{z}^i = \bar{t}_i d\bar{v}^i, \tag{2.4}$$

$$\frac{\partial}{\partial v^i} = t_i \frac{\partial}{\partial z^i}, \qquad \frac{\partial}{\partial \bar{v}^i} = \bar{t}_i \frac{\partial}{\partial \bar{z}^i}, \tag{2.5}$$

where

$$t_i = \frac{-2(1+a^i)z^i}{(1-a^i)(v^i-1)^2}$$

and

$$\log|z^i|^2 = \frac{2(1+a^i)(|v^i|^2-1)}{(1-a^i)|v^i-1|^2}. \tag{2.6}$$

From (2.4) and (2.6), we obtain

$$\begin{aligned}\frac{dz^i\wedge d\bar z^i}{|z^i|^2(\log|z^i|^2-\log h_i)^2}&=\frac{4dv^i\wedge d\bar v^i}{\left(2(|v^i|^2-1)-\frac{(1-a^i)|v^i-1|^2\log h_i}{1+a^i}\right)^2},\\ \frac{dz^i\wedge\bar\partial\log h_i}{z^i(\log|z^i|^2-\log h_i)^2}&=\frac{-2(1-a^i)(1+a^i)dv^i\wedge\bar\partial\log h_i}{\left(\frac{2(1+a^i)(|v^i|^2-1)}{|v^i-1|^2}-(1-a^i)\log h_i\right)^2(v^i-1)^2}.\end{aligned}\tag{2.7}$$

Since we always have $|v^i|\leqslant R$ and $a^i\to 1$, the first inequality of (iv) follows by substituting (2.7) into (2.3). The second inequality follows from this and (2.5) because of the fact that $\lim_{x\to\infty}e^{-x}x^t=0$ for any $t\in\mathbb{R}$. □

Remark 2.11. For later use, we would like to remark that the above proof yields a representation of $\frac{\partial}{\partial v^i}$ and dv^i in terms of $\frac{\partial}{\partial z^i}$ and dz^i, respectively, which does not directly involve the numbers a^i. In fact, from (2.4), (2.5) and (2.6), we obtain

$$\begin{aligned}\frac{\partial}{\partial v^i}&=\frac{\bar v^i-1}{(|v^i|^2-1)(v^i-1)}\,z^i\log(1/|z^i|^2)\,\frac{\partial}{\partial z^i},\\ dv^i&=\frac{(|v^i|^2-1)(v^i-1)}{\bar v^i-1}\,\frac{dz^i}{z^i\log(1/|z^i|^2)}\end{aligned}$$

for $1\leqslant i\leqslant k$.

Hölder spaces

We are now ready to define the Hölder spaces of $\mathcal{C}^{k,\beta}$ functions on X' in terms of the quasi-coordinates just described. These spaces will be useful in the construction of a Kähler-Einstein Poincaré metric in the following section.

Definition 2.12. Let $k\in\{0,1,\dots\}$ and $\beta\in(0,1)$ and denote by $\mathcal{C}^k(X')$ the space of k-times differentiable functions $u:X'\to\mathbb{C}$.

(i) For $u\in\mathcal{C}^k(X')$, let

$$||u||_{k,\beta}=\sup_{(V;v^1,\dots,v^n)\in\mathcal{V}}\left(\sup_{z\in V}\sum_{|I|+|J|\leqslant k}\left|\partial_v^I\bar\partial_v^J u(z)\right|+\sup_{z,z'\in V}\sum_{|I|+|J|=k}\frac{\left|\partial_v^I\bar\partial_v^J u(z)-\partial_v^I\bar\partial_v^J u(z')\right|}{|z-z'|^\beta}\right)$$

be the $\mathcal{C}^{k,\beta}$ *norm* of u, where $\partial_v^I\bar\partial_v^J=\frac{\partial^{|I|+|J|}}{\partial v^I\partial\bar v^J}$.

(ii) Let

$$\mathcal{C}^{k,\beta}(X')=\left\{u\in\mathcal{C}^k(X'):||u||_{k,\beta}<\infty\right\}$$

be the function space of $\mathcal{C}^{k,\beta}$ *functions* on X' with respect to $\mathcal{V}$.

Remark 2.13.

(i) $||\cdot||_{k,\beta}$ is a norm on $\mathcal{C}^{k,\beta}(X')$ and $(\mathcal{C}^{k,\beta}(X'), ||\cdot||_{k,\beta})$ is a Banach space.

(ii) The fact that the $\mathcal{C}^{k,\beta}$ condition is considered with respect to the quasi-coordinates is useful in the Schauder estimate on X'. In fact, in the interior Schauder estimate

$$||u||_{\mathcal{C}^{k,\beta}(V')} \leqslant c\left(||u||_{\mathcal{C}^{0}(V)} + ||Lu||_{\mathcal{C}^{k-2,\beta}(V)}\right) \quad \text{for } V' \Subset V \subset \mathbb{R}^{2n} \tag{2.8}$$

(see chapter 6 of [GT01]) for a linear elliptic operator L, the constant c is determined by n, k, the ellipticity of L, the $\mathcal{C}^{k-2,\beta}$ norms of the coefficients of L and the distance between V' and the boundary of V. Thus, because of (iii) and (iv) in proposition 2.9, the interior Schauder estimate on X' can be reduced to that on a bounded domain in Euclidean space.

2.3 A Kähler-Einstein Poincaré metric

Definition 2.14 (Kähler-Einstein metric)**.** A Kähler metric g on X' is called *Kähler-Einstein* if its fundamental form ω satisfies

$$\operatorname{Ric}\omega^n = \lambda\omega$$

with a constant $\lambda \in \mathbb{C}$.

We have the following classical existence theorem for Kähler-Einstein metrics on compact complex manifolds by Yau.

Theorem 2.15 (Yau, [Yau78b])**.** *Let X be a compact complex manifold such that K_X is ample. Then there is a unique (up to a constant multiple) Kähler-Einstein metric on X with negative Ricci curvature.*

The main objective of this section will be the proof of the corresponding result for the case of a framed manifold due to R. Kobayashi, which actually asserts the existence of a Poincaré-type Kähler-Einstein metric.

Theorem 2.16 (R. Kobayashi, [Ko84])**.** *Let X be a compact complex manifold and let D be a divisor in X with simple normal crossings such that $K_X \otimes [D]$ is ample. Then there is a unique (up to a constant multiple) complete Kähler-Einstein metric on X' with negative Ricci curvature. This metric has Poincaré-type growth near the divisor D.*

As in the compact case, such a Kähler-Einstein metric can be found as the limit of a deformation along the complex Monge-Ampère equation. In the framed situation, however, we use the Poincaré-type Kähler metric constructed in section 2.1 as a starting metric for such a deformation.

Set $\omega_0 := -\operatorname{Ric}\Psi$ and consider the deformation

$$\omega_0 \rightsquigarrow \omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}u$$

with a smooth function u on X'. Suppose that u satisfies the conditions

$$\left.\begin{array}{l}\omega_0+\sqrt{-1}\partial\bar{\partial}u \text{ is positive definite,}\\ (\omega_0+\sqrt{-1}\partial\bar{\partial}u)^n = e^u\Psi\end{array}\right\} \text{ on } X'. \tag{2.9}$$

Then we get

$$\begin{aligned}-\operatorname{Ric}\omega^n &= -\operatorname{Ric}(\omega_0+\sqrt{-1}\partial\bar{\partial}u)^n\\ &= -\operatorname{Ric}(e^u\Psi)\\ &= -\operatorname{Ric}\Psi+\sqrt{-1}\partial\bar{\partial}u\\ &= \omega_0+\sqrt{-1}\partial\bar{\partial}u\\ &= \omega,\end{aligned}$$

so that by definition 2.14, ω is the fundamental form of a Kähler-Einstein metric g on X' with negative Ricci curvature. To ensure that g is still a Poincaré metric, we define an open subset $U\subset\mathcal{C}^{k,\beta}(X')$ by

$$U=\{u\in\mathcal{C}^{k,\beta}(X'):\omega_0+\sqrt{-1}\partial\bar{\partial}u\sim\omega_0\}.$$

If $u\in U$ satisfies conditions (2.9), then $\omega=\omega_0+\sqrt{-1}\partial\bar{\partial}u$ is the fundamental form of a Kähler-Einstein Poincaré metric g on X'. For the proof of theorem 2.16, we need the following two results by Yau.

Theorem 2.17 (Yau's maximum principle, [Yau78a], theorem 1)**.** *Let (M,ds_M^2) be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a $\mathcal{C}^2$ function on M which is bounded from below. Then for any $\varepsilon>0$, there is a point $p\in M$ such that*

$$|\operatorname{grad} f(p)|<\varepsilon,\quad \Delta f(p)>-\varepsilon,\quad f(p)<\inf_{x\in M} f(x)+\varepsilon.$$

Theorem 2.18 (Yau's Schwarz lemma, [Yau78a], theorem 2)**.** *Let (M,ds_M^2) be a complete Kähler manifold with Ricci curvature bounded from below by a constant K_1. Let (N,ds_N^2) be another Hermitian manifold with holomorphic bisectional curvature bounded from above by a negative constant K_2. Then, if there is a non-constant holomorphic map $f:M\to N$, we have $K_1\leqslant 0$ and*

$$f^*ds_N^2\leqslant\frac{K_1}{K_2}\,ds_M^2.$$

In particular, if $K_1\geqslant 0$, there is no non-constant holomorphic map $f:M\to N$.

Proof of theorem 2.16. The uniqueness of a complete Kähler-Einstein metric with negative Ricci curvature up to a constant multiple follows from theorem 2.18 by letting (M,ds_M^2) and (N,ds_N^2) be the underlying Riemannian manifolds of (X',ω_1) and (X',ω_2), respectively, where ω_1 and ω_2 are two complete Kähler-Einstein metrics on X' with negative Ricci curvature and $f=\mathrm{id}_{X'}$.

The existence proof amounts to showing that there is some $u\in U$ satisfying conditions (2.9). This is accomplished by solving a complex Monge-Ampère equation by the continuity method as in [Yau78b] and [CY80]. We give a brief review of this method as described in [Ko84].

For $k \geqslant 2$ and $\beta \in (0,1)$, consider the map Φ defined by

$$\Phi : \begin{cases} \mathcal{C}^{k,\beta}(X') & \longrightarrow \quad \mathcal{C}^{k-2,\beta}(X') \\ \qquad u & \longmapsto \quad e^{-u}(\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n/\omega_0^n \end{cases}.$$

The fact that Φ is a well-defined map from $\mathcal{C}^{k,\beta}(X')$ into $\mathcal{C}^{k-2,\beta}(X')$ can be verified in local coordinates.

We claim that for $k \geqslant 6$ and any $F \in \mathcal{C}^{k-2,\beta}(X')$, there is a solution $u \in U$ of

$$\Phi(u) = e^F, \quad \text{i. e.} \quad (\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n = e^{u+F}\omega_0^n. \tag{2.10}$$

This is called the *complex Monge-Ampère equation*. Using the continuity method to solve this equation means to show that the set $T \subset [0,1]$ defined by

$$T = \left\{t \in [0,1] : \text{There is a solution } u \in U \text{ of } \Phi(u) = e^{tF}\right\}$$

is both open and closed in $[0,1]$. Indeed, since $u = 0 \in \mathcal{C}^{k,\beta}(X')$ solves $\Phi(u) = e^{tF}$ with $t = 0$, we have $0 \in T$ and thus $T \neq \emptyset$. So, by the connectedness of $[0,1]$ we would obtain $T = [0,1]$ and, in particular, $1 \in T$, which proves the existence of a solution to (2.10).

The openness of T is shown using the usual inverse mapping theorem for Banach spaces. Let $t_0 \in T$ and $u_0 \in U$ with $\Phi(u_0) = e^{t_0F}$. The Fréchet derivative of Φ at u_0 is then a bounded linear operator given by

$$\Phi'(u_0) : \begin{cases} \mathcal{C}^{k,\beta}(X') & \longrightarrow \quad \mathcal{C}^{k-2,\beta}(X') \\ \qquad h & \longmapsto \quad e^{t_0F}(\tilde{\Delta}h - h) \end{cases},$$

where $\tilde{\Delta}$ is the Laplacian with respect to the Kähler metric on X' given by $\omega_0 + \sqrt{-1}\partial\bar{\partial}u_0$. It suffices to show that $\Phi'(u_0)$ has a bounded linear inverse. In fact, the inverse mapping theorem then implies that Φ maps an open neighbourhood of u_0 in U diffeomorphically onto an open neighbourhood of t_0F in $\mathcal{C}^{k-2,\beta}(X')$, so that there is an open neighbourhood of t_0 in $[0,1]$ in which the equation $\Phi(u) = e^{tF}$ is solvable. We have to show that for any $v \in \mathcal{C}^{k-2,\beta}(X')$, there is a unique solution $h \in \mathcal{C}^{k,\beta}(X')$ of

$$\tilde{\Delta}h - h = v$$

such that there is an estimate

$$||h||_{k,\beta} \leqslant c||v||_{k-2,\beta} \quad \text{with a constant } c > 0 \text{ independent of } v. \tag{2.11}$$

To achieve this, we consider the Dirichlet problem

$$\begin{cases} \tilde{\Delta}h - h = v & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega \end{cases}$$

for a relatively compact domain $\Omega \Subset X'$. This problem has a unique solution (see e. g. [GT01], Theorem 6.13) and in [CY80], p. 521, Cheng and Yau apply this to $\Omega = \Omega_i$, where $(\Omega_i)_i$ is an exhaustion of X' by relatively compact domains, to obtain a sequence $(h_i)_i \subset \mathcal{C}^{k,\beta}(X')$. It remains to show the convergence of this sequence as well as the above estimate (2.11). As in [GT01], Corollary 6.3, this follows by using the interior Schauder estimate (2.8) with respect to our quasi-coordinates.

The proof of the closedness of T essentially involves an a-priori estimate of the Monge-Ampère equation (2.10) and the interior Schauder estimate of the linearized version of (2.10). The former estimate can be shown as in [CY80] using our quasi-coordinates. In the latter estimate, proposition 2.9 plays an essential role. We give an alternative proof to [CY80] of the $\mathcal{C}^0$ estimate of (2.10), which can be found in [Ko84].

Let $u \in U$ be a solution of (2.10), i. e. with $\Phi(u) = e^F$, satisfying conditions (2.9). With respect to local coordinates $z^1, \dots, z^n$, we write

$$\omega_0 = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$
$$u_{i\bar{j}} = \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}.$$

Then

$$\begin{aligned} u + F &= \log\det(g_{i\bar{j}} + u_{i\bar{j}})_{i,j} - \log\det(g_{i\bar{j}})_{i,j} \\ &= \int_0^1 \frac{d}{dt} \log\det(g_{i\bar{j}} + tu_{i\bar{j}})_{i,j}\, dt \\ &= \int_0^1 (g+tu)^{\bar{j}i} u_{i\bar{j}}\, dt, \end{aligned}$$

where $((g+tu)^{\bar{j}i})_{j,i}$ denotes the inverse matrix of $(g_{i\bar{j}} + tu_{i\bar{j}})_{i,j}$. At a point $p \in X'$, we can assume that $g_{i\bar{j}} = \delta_{i\bar{j}}$ is the identity matrix and $u_{i\bar{j}} = \delta_{i\bar{j}} u_{i\bar{i}}$ is a diagonal matrix. Thus, if Δ denotes the Laplacian with respect to ω_0 and $\tilde{\Delta}$ denotes the Laplacian with respect to $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}u$ as above, we have

$$(g+tu)^{\bar{j}i} u_{i\bar{j}} = \sum_{i=1}^n \frac{u_{i\bar{i}}}{1+tu_{i\bar{i}}} \begin{cases} = \sum_{i=1}^n \left(u_{i\bar{i}} - \frac{tu_{i\bar{i}}^2}{1+tu_{i\bar{i}}} \right) \leqslant \sum_{i=1}^n u_{i\bar{i}} = \Delta u, \\ = \sum_{i=1}^n \left(\frac{u_{i\bar{i}}}{1+u_{i\bar{i}}} + \frac{(1-t)u_{i\bar{i}}^2}{(1+u_{i\bar{i}})(1+tu_{i\bar{i}})} \right) \geqslant \sum_{i=1}^n \frac{u_{i\bar{i}}}{1+u_{i\bar{i}}} = \tilde{\Delta} u \end{cases}$$

for $t \in [0,1]$, making use of the fact that $(\delta_{i\bar{j}}(1+u_{i\bar{i}}))_{i,j} = \omega_0 + \sqrt{-1}\partial\bar{\partial}u$ is positive definite by (2.9). It follows that $u + F \leqslant \Delta u$ and $u + F \geqslant \tilde{\Delta} u$. Since $u \in U$, both ω_0 and $\omega = \omega_0 + \sqrt{-1}\partial\bar{\partial}u$ define a complete Riemannian metric on X' with bounded curvature and thus, in particular, with Ricci curvature bounded from below. By Yau's maximum principle, theorem 2.17, we obtain

$$\sup_{X'} u \leqslant \sup_{X'} |F| \quad \text{and} \quad \inf_{X'} u \geqslant -\sup_{X'} |F|.$$

Altogether, we know that for $k \geqslant 6$ and any $F \in \mathcal{C}^{k-2,\beta}(X')$, there is a solution $u \in U$ of (2.10). We can apply this to $F_0 := \log(\Psi/\omega_0^n)$. In fact, we have $F \in \mathcal{C}^{k-2,\beta}(X')$ for any $k \geqslant 2$ and $\beta \in (0,1)$ because

- by lemma 2.6 (ii), F_0 is a bounded smooth function on X' and
- by remark 2.11, the derivatives of F_0 with respect to the quasi-coordinates $v^1, \dots, v^n$ are bounded as well.

Equation (2.10) then reads

$$e^{-u}(\omega_0 + \sqrt{-1}\partial\bar{\partial}u)^n/\omega_0^n = \Psi/\omega_0^n,$$

which implies (2.9). □

In what follows, "Poincaré metric" always means the Poincaré-type Kähler-Einstein metric on X' constructed in theorem 2.16. It will be denoted by $g_{X'}$ and its fundamental form by $\omega_{X'}$. Furthermore, we return to the assumption of a smooth divisor D in X.

We give an explicit description of the volume form of $g_{X'}$. Let D be a smooth divisor in X such that $K_X \otimes [D]$ is ample and, as above, let $\sigma \in \Gamma(X, [D])$ be a canonical section of $[D]$ such that $||\sigma||^2 < 1$ for the norm $||\cdot||$ induced by a Hermitian metric in $[D]$. By abuse of notation, we regard σ as a local coordinate function near a point $p \in D$. Choose a smooth volume form Ω on X and a Hermitian metric in $[D]$ such that

$$\eta = -\operatorname{Ric}\Omega - \sqrt{-1}\partial\bar{\partial}\log||\sigma||^2$$

is positive definite on X.

Theorem 2.19 (Schumacher, [Sch98a], theorem 2)**.** *There is a number $0 < \alpha \leqslant 1$ such that for all $k \in \{0, 1, \dots\}$ and $\beta \in (0,1)$, the volume form of $g_{X'}$ is of the form*

$$\frac{2\Omega}{||\sigma||^2 \log^2(1/||\sigma||^2)}\left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right) \quad \textit{with } \nu \in \mathcal{C}^{k,\beta}(X').$$

By the adjunction formula, we see that

$$K_D = (K_X \otimes [D])|_D \quad \text{is ample},$$

so that by theorem 2.15 there is a unique (up to a constant multiple) Kähler-Einstein metric g_D on D with negative Ricci curvature. Let ω_D be its fundamental form. When we restrict $\omega_{X'}$ to the locally defined sets $D_{\sigma_0} := \{\sigma = \sigma_0\}$ for small $\sigma_0 > 0$, there is a notion of locally uniform convergence of $\omega_{X'}|D_{\sigma_0}$ for $\sigma_0 \to 0$ and we have the following convergence theorem by Schumacher.

Theorem 2.20 (Schumacher, [Sch98a], theorem 1)**.** *$\omega_{X'}|D_{\sigma_0}$ converges to ω_D locally uniformly as $\sigma_0 \to 0$.*

Remark 2.21. In his dissertation [Koe01], B. Koehler generalizes this result to the setting of a two-component divisor with simple normal crossings. Using his theorem, the results of this thesis could, in fact, be formulated for this more general situation. However, for the sake of simplicity, we stick to the case of a smooth divisor.

2.4 Square-integrability for the Poincaré metric

In this section we define the space of L^2 sections and the Sobolev space of L^2 sections with L^2 first-order weak derivatives with values in a holomorphic vector bundle. Since L^2 sections are only defined almost everywhere and the divisor D has measure zero, there is no difference between considering L^2 sections on X and on X', so we introduce all notions on the compact manifold X. We show that the square-integrability conditions with respect to the Poincaré metric are stronger than those in the ordinary sense. This will later be helpful in the regularity statement for L^2 weakly holomorphic subbundles.

In what follows, all estimates can be done in a small neighbourhood $U \subset X$ of an arbitrary point $p \in D$. This neighbourhood will be shrinked several times as needed throughout the computation. We can choose coordinates $z^2, \dots, z^n$ for D on $U \cap D$ such that

$$\omega_D = \sqrt{-1} \sum_{i=2}^{n} dz^i \wedge d\bar{z}^i$$

is diagonal at p. Let the section $\sigma \in \Gamma(X, [D])$ be as above and regard σ as a local coordinate. Then we have local coordinates $(\sigma, z^2, \dots, z^n)$ on U. We write dV for the Euclidean volume element and dV_g for the volume element of the Poincaré metric $g = g_{X'}$. Then locally we have

$$dV = \left(\frac{\sqrt{-1}}{2}\right)^n d\sigma \wedge d\bar{\sigma} \wedge dz^2 \wedge d\bar{z}^2 \wedge \cdots \wedge dz^n \wedge d\bar{z}^n \quad \text{and} \quad dV_g \sim \frac{dV}{|\sigma|^2 \log^2(1/|\sigma|^2)}.$$

Let E be a holomorphic vector bundle on X with a smooth Hermitian metric h. We write $\langle \cdot, \cdot \rangle$ for the scalar product in the fibres of E induced by h and $||\cdot||$ for the corresponding norm in the fibres of E.

Definition 2.22 (L^2 spaces)**.**

(i) Let

$$L^2(X, E, g) = \left\{ s \text{ measurable section of } E : \int_X ||s||^2 dV_g < \infty \right\}$$

be the space of L^2 sections of E with respect to the Poincaré metric $g_{X'}$ with the L^2 norm

$$||s||_{L^2(X,E,g)} = \left(\int_X ||s||^2 dV_g \right)^{1/2}.$$

(ii) Let

$$L^2_1(X, E, g) = \left\{ s \in L^2(X, E, g) : \nabla s \in L^2(X, T^*_X \otimes E, g) \right\}$$

be the Sobolev space of L^2 sections of E with L^2 first-order weak derivatives with respect to the Poincaré metric $g_{X'}$ with the Sobolev norm

$$||s||_{L^2_1(X,E,g)} = \left(||s||^2_{L^2(X,E,g)} + ||\nabla s||^2_{L^2(X,T^*_X \otimes E,g)} \right)^{1/2}.$$

Here, ∇ denotes the covariant derivative with respect to the Chern connection of the Hermitian holomorphic vector bundle (E, h), where ∇s is computed in the sense of currents, T^*_X denotes the cotangent bundle of X and the bundle $T^*_X \otimes E$ is endowed with the product of the dual of the Poincaré metric in T^*_X and the Hermitian metric h in E.

The spaces $L^2(X, E)$ and $L^2_1(X, E)$ are defined in the ordinary sense, i. e. with respect to a smooth Kähler metric on X.

Remark 2.23. Let $\nabla = \nabla' + \nabla''$ be the decomposition of ∇ into its $(1, 0)$ and $(0, 1)$ parts. Then for a section $s \in L^2(X, E, g)$, we have $s \in L^2_1(X, E, g)$ if and only if $\nabla' s \in L^2(X, \Lambda^{1,0}T^*_X \otimes E, g)$ and $\nabla'' s \in L^2(X, \Lambda^{0,1}T^*_X \otimes E, g)$. In what follows, we only consider $\nabla' s$ since then everything follows for $\nabla'' s$ in an analogue way.

We locally write the fundamental form $\omega_{X'}$ of the Poincaré metric $g_{X'}$ as

$$\omega_{X'} = \sqrt{-1}\left(g_{\sigma\bar{\sigma}} d\sigma \wedge d\bar{\sigma} + \sum_{j=2}^{n} g_{\sigma\bar{j}} d\sigma \wedge d\bar{z}^j + \sum_{i=2}^{n} g_{i\bar{\sigma}} dz^i \wedge d\bar{\sigma} + \sum_{i,j=2}^{n} g_{i\bar{j}} dz^i \wedge d\bar{z}^j \right)$$

and let

$$\begin{pmatrix} g^{\bar{\sigma}\sigma} & g^{\bar{\sigma}2} & \cdots & g^{\bar{\sigma}n} \\ g^{\bar{2}\sigma} & & & \\ \vdots & & (g^{\bar{j}i})_{j,i=2,\dots,n} & \\ g^{\bar{n}\sigma} & & & \end{pmatrix} \quad \text{be the inverse matrix of} \quad \begin{pmatrix} g_{\sigma\bar{\sigma}} & g_{\sigma\bar{2}} & \cdots & g_{\sigma\bar{n}} \\ g_{2\bar{\sigma}} & & & \\ \vdots & & (g_{i\bar{j}})_{i,j=2,\dots,n} & \\ g_{n\bar{\sigma}} & & & \end{pmatrix}.$$

Then, writing

$$\nabla' s = s_\sigma d\sigma + \sum_{i=2}^{n} s_i dz^i$$

with local sections s_σ, s_i of E, $i = 2, \dots, n$, the condition $\nabla' s \in L^2(X, \Lambda^{1,0}T^*_X \otimes E, g)$ reads

$$\int \left(\langle s_\sigma, s_\sigma \rangle g^{\bar{\sigma}\sigma} + \sum_{j=2}^{n} \langle s_\sigma, s_j \rangle g^{\bar{j}\sigma} + \sum_{i=2}^{n} \langle s_i, s_\sigma \rangle g^{\bar{\sigma}i} + \sum_{i,j=2}^{n} \langle s_i, s_j \rangle g^{\bar{j}i} \right) \frac{dV}{|\sigma|^2 \log^2(1/|\sigma|^2)} < \infty.$$

Proposition 2.24. *The square-integrability conditions defined above with respect to the Poincaré metric imply the corresponding conditions in the ordinary sense, i. e. we have*

(i) $L^2(X, E, g) \subset L^2(X, E)$ *and*

(ii) $L^2_1(X, E, g) \subset L^2_1(X, E)$.

First we need to make a remark about the asymptotic behaviour of the Poincaré metric. Using Schumacher's convergence theorem 2.20 and the fact that ω_D is diagonal at p, we see that $g^{\bar{j}i}$ approaches 0 for $i, j = 2, \dots, n$ and $i \neq j$ as $\sigma \to 0$. Together with proposition 1 from [Sch02], which is stated there in the surface case but holds analogously in higher dimensions, we obtain the following proposition.

Proposition 2.25. *With $0 < \alpha \leqslant 1$ from theorem 2.19, we have*

(i) $g^{\bar\sigma\sigma} \sim |\sigma|^2 \log^2(1/|\sigma|^2)$,

(ii) $g^{\bar\sigma i}, g^{\bar j\sigma} = O\left(|\sigma| \log^{1-\alpha}(1/|\sigma|^2)\right)$, $i,j = 2,\dots,n$,

(iii) $g^{\bar i i} \sim 1$, $i = 2,\dots,n$ *and*

(iv) $g^{\bar j i} \to 0$ *as* $\sigma \to 0$, $i,j = 2,\dots,n$, $i \neq j$.

Proof of proposition 2.24. Since the terms coming from the smooth Hermitian metric h in E do not influence the following computations, we can assume that E is the trivial line bundle on X and ∇ is the ordinary exterior derivative $d = \partial + \bar\partial$.

We first observe that since $|\sigma|^2 \log^2(1/|\sigma|^2) \to 0$ as $\sigma \to 0$, we can assume (after possibly shrinking U) that

$$|\sigma|^2 \log^2(1/|\sigma|^2) \leqslant 1. \tag{2.12}$$

Therefore, for every measurable function s, we have

$$\int |s|^2 \frac{dV}{|\sigma|^2 \log^2(1/|\sigma|^2)} \geqslant \int |s|^2 dV,$$

which implies (i). In order to show (ii), we only consider ∂s. Since $2\operatorname{Re}(z\overline{w}) \leqslant |z|^2 + |w|^2$ for any complex numbers z and w, we have

$$\begin{aligned}
\int |\partial s|^2 dV &= \int \Big(|s_\sigma|^2 + \sum_{j=2}^n s_\sigma \overline{s_j} + \sum_{i=2}^n s_i \overline{s_\sigma} + \sum_{i,j=2}^n s_i \overline{s_j}\Big) dV \\
&= \int \Big(|s_\sigma|^2 + \sum_{i=2}^n |s_i|^2 + \sum_{j=2}^n 2\operatorname{Re}(s_\sigma \overline{s_j}) + \sum_{\substack{i,j=2 \\ i<j}}^n 2\operatorname{Re}(s_i \overline{s_j})\Big) dV \\
&\leqslant n \int \Big(|s_\sigma|^2 + \sum_{i=2}^n |s_i|^2\Big) dV,
\end{aligned} \tag{2.13}$$

so it suffices to show that this integral is dominated by the Poincaré-L^2-norm of ∂s. Now we have

$$\begin{aligned}
&\int \Big(|s_\sigma|^2 g^{\bar\sigma\sigma} + \sum_{j=2}^n s_\sigma \overline{s_j} g^{\bar j\sigma} + \sum_{i=2}^n s_i \overline{s_\sigma} g^{\bar\sigma i} + \sum_{i,j=2}^n s_i \overline{s_j} g^{\bar j i}\Big) \frac{dV}{|\sigma|^2 \log^2(1/|\sigma|^2)} \\
&= \int \Bigg(\sum_{j=2}^n \Big(\frac{|s_\sigma|^2 g^{\bar\sigma\sigma}}{n-1} + \frac{|s_j|^2 g^{\bar j j}}{n-1} + 2\operatorname{Re}(s_\sigma \overline{s_j} g^{\bar j\sigma})\Big) \\
&\quad + \sum_{\substack{i,j=2 \\ i<j}}^n \Big(\frac{|s_i|^2 g^{\bar i i}}{n-1} + \frac{|s_j|^2 g^{\bar j j}}{n-1} + 2\operatorname{Re}(s_i \overline{s_j} g^{\bar j i})\Big)\Bigg) \frac{dV}{|\sigma|^2 \log^2(1/|\sigma|^2)}.
\end{aligned} \tag{2.14}$$

We estimate the two sums in this expression separately. By proposition 2.25 (i)–(iii), there are constants $c, c' > 0$ such that

$$\begin{aligned} g^{\bar\sigma\sigma} &\geqslant c|\sigma|^2 \log^2(1/|\sigma|^2), \\ g^{\bar j j} &\geqslant c, \\ |g^{\bar j \sigma}| &\leqslant c'|\sigma| \log^{1-\alpha}(1/|\sigma|^2) \end{aligned}$$

for $2 \leqslant j \leqslant n$. It follows that

$$\begin{aligned} &\frac{1}{|\sigma|^2 \log^2(1/|\sigma|^2)} \left(\frac{|s_\sigma|^2 g^{\bar\sigma\sigma}}{n-1} + \frac{|s_j|^2 g^{\bar j j}}{n-1} + 2\operatorname{Re}(s_\sigma \overline{s_j} g^{\bar j \sigma}) \right) \\ &\geqslant \frac{1}{(n-1)|\sigma|^2 \log^2(1/|\sigma|^2)} \left(c|s_\sigma|^2 |\sigma|^2 \log^2(1/|\sigma|^2) + c|s_j|^2 - 2c'(n-1)|s_\sigma||s_j||\sigma| \log^{1-\alpha}(1/|\sigma|^2) \right) \\ &= \frac{c}{n-1} \left(|s_\sigma|^2 + \left(\frac{|s_j|}{|\sigma| \log(1/|\sigma|^2)} \right)^2 - \frac{2c'(n-1)|s_\sigma||s_j|}{c|\sigma| \log^{1+\alpha}(1/|\sigma|^2)} \right). \end{aligned}$$

Since $\alpha > 0$, $\log^\alpha(1/|\sigma|^2)$ tends to infinity as σ approaches 0. Thus we can assume (after possibly shrinking U) that $\log^\alpha(1/|\sigma|^2) \geqslant 2c'(n-1)/c$. Together with the estimate

$$a^2 + b^2 - ab = \frac{a^2+b^2}{2} + \frac{(a-b)^2}{2} \geqslant \frac{a^2+b^2}{2} \quad \text{for real numbers } a \text{ and } b$$

and (2.12), we obtain

$$\begin{aligned} &\frac{1}{|\sigma|^2 \log^2(1/|\sigma|^2)} \left(\frac{|s_\sigma|^2 g^{\bar\sigma\sigma}}{n-1} + \frac{|s_j|^2 g^{\bar j j}}{n-1} + 2\operatorname{Re}(s_\sigma \overline{s_j} g^{\bar j \sigma}) \right) \\ &\geqslant \frac{c}{2(n-1)} \left(|s_\sigma|^2 + \frac{|s_j|^2}{|\sigma|^2 \log^2(1/|\sigma|^2)} \right) \\ &\geqslant \frac{c}{2(n-1)} \left(|s_\sigma|^2 + |s_j|^2 \right). \end{aligned} \tag{2.15}$$

The second sum in (2.14) can be estimated similarly to the first. Here we note that by proposition 2.25 (iv) we can assume (again after possibly shrinking U) that

$$|g^{\bar j i}| \leqslant \frac{c}{2(n-1)}$$

for $2 \leqslant i < j \leqslant n$. As above, it follows that

$$\begin{aligned}
&\frac{1}{|\sigma|^2 \log^2(1/|\sigma|^2)}\left(\frac{|s_i|^2 g^{\bar{\imath}i}}{n-1} + \frac{|s_j|^2 g^{\bar{\jmath}j}}{n-1} + 2\operatorname{Re}(s_i\overline{s_j}g^{\bar{\jmath}i})\right)\\
&\geqslant \frac{c}{(n-1)|\sigma|^2 \log^2(1/|\sigma|^2)}\left(|s_i|^2 + |s_j|^2 - \frac{2(n-1)|s_i||s_j||g^{\bar{\jmath}i}|}{c}\right)\\
&\geqslant \frac{c\left(|s_i|^2 + |s_j|^2\right)}{2(n-1)|\sigma|^2 \log^2(1/|\sigma|^2)}\\
&\geqslant \frac{c}{2(n-1)}\left(|s_i|^2 + |s_j|^2\right).
\end{aligned} \tag{2.16}$$

Substituting (2.15) and (2.16) into (2.14), we finally obtain

$$\begin{aligned}
&\int\left(|s_\sigma|^2 g^{\bar{\sigma}\sigma} + \sum_{j=2}^n s_\sigma\overline{s_j}g^{\bar{\jmath}\sigma} + \sum_{i=2}^n s_i\overline{s_\sigma}g^{\bar{\sigma}i} + \sum_{i,j=2}^n s_i\overline{s_j}g^{\bar{\jmath}i}\right)\frac{dV}{|\sigma|^2 \log^2(1/|\sigma|^2)}\\
&\geqslant \frac{c}{2(n-1)}\int\left(\sum_{j=2}^n \left(|s_\sigma|^2 + |s_j|^2\right) + \sum_{\substack{i,j=2\\ i<j}}^n \left(|s_i|^2 + |s_j|^2\right)\right)dV\\
&= \frac{c}{2}\int\left(|s_\sigma|^2 + \sum_{i=2}^n |s_i|^2\right)dV,
\end{aligned}$$

which equals (2.13) up to a constant. This proves the claim. □

3 Stability and Hermitian-Einstein metrics

In this chapter we discuss the concepts of stability of a holomorphic vector bundle and Hermitian-Einstein metrics in such a bundle. We give a short review of the notion of stability for the case of a compact Kähler manifold and, in particular, of a compact projective-algebraic manifold. We observe that every stable holomorphic vector bundle on a compact Kähler manifold is simple, i. e. it admits only homotheties as its holomorphic endomorphisms. Then we introduce the notion of a Hermitian-Einstein metric in a holomorphic vector bundle, which is a generalization of a Kähler-Einstein metric in the tangent bundle of a compact Kähler manifold. A classical result is the so-called Kobayashi-Hitchin correspondence, which says that an irreducible holomorphic vector bundle on a compact Kähler manifold is stable if and only if it admits a Hermitian-Einstein metric.

The next step is the adaptation of these concepts for the case of a framed manifold. In the canonically polarized case, i. e. when $K_X \otimes [D]$ is ample, there are two natural notions of "stability in the framed sense" for a holomorphic vector bundle E on X. On the one hand, from the algebraic point of view, one can define the degree of coherent subsheaves of $\mathcal{E} = \mathcal{O}_X(E)$ in terms of an intersection number with the ample line bundle $K_X \otimes [D]$. On the other hand, one can use the Poincaré metric on X' constructed in the previous chapter to define such a degree. Fortunately, the two notions of stability implied by these definitions turn out to be equivalent, a statement that is proved in this chapter. Moreover, the notion of framed stability of a holomorphic vector bundle E on X obtained in this way again implies that E is simple.

Given a holomorphic vector bundle E on X which is stable in the framed sense, the classical Kobayashi-Hitchin correspondence yields a Hermitian-Einstein metric in E with respect to a Kähler metric on X whose fundamental form is the curvature form of a positive Hermitian metric in $K_X \otimes [D]$. We are, however, interested in smooth Hermitian metrics in the restriction E' of E to X' satisfying the Hermitian-Einstein condition with respect to the Poincaré metric on X'. Here, the classical methods cannot be applied directly since X' is not compact. We need to impose additional conditions on smooth Hermitian metrics in E' to the effect that they behave nicely near the divisor D. In this case we call them Hermitian-Einstein metrics on E in the framed sense. This turns out to be the correct notion in order to obtain the existence of such a metric in the case of framed stability. The first evidence of this is the proof of the uniqueness of a framed Hermitian-Einstein metric up to a constant multiple if E is simple.

3.1 Review of the compact case

We give a brief review of the concept of stability on a compact Kähler manifold (X, g). A more thorough treatment of the subject can be found in S. Kobayashi's monograph [Kb87], chapter V.

Recall that for a coherent analytic sheaf $\mathcal{F}$ on X, there is a well-defined *determinant line bundle* $\det \mathcal{F}$ of $\mathcal{F}$ defined by

$$(\det \mathcal{F})|_U = \bigotimes_{i=0}^{n} (\det F_i)^{\otimes (-1)^i}$$

on open neighbourhoods $U \subset X$, where

$$0 \longrightarrow \mathcal{F}_n \longrightarrow \cdots \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

is a resolution of $\mathcal{F}|_U$ by locally free coherent sheaves $\mathcal{F}_i$, F_i is the vector bundle corresponding to $\mathcal{F}_i$ and $\det F_i$ is the determinant line bundle of F_i, $i = 0, \ldots, n$. Then the *first Chern class* of $\mathcal{F}$ is defined as

$$c_1(\mathcal{F}) = c_1(\det \mathcal{F}),$$

where $c_1(\det \mathcal{F})$ denotes the first Chern class of the line bundle $\det \mathcal{F}$. Denote by ω the fundamental form of the Kähler metric g.

Definition 3.1 (Degree, slope)**.** Let $\mathcal{F}$ be a torsion-free coherent analytic sheaf on X.

(i) The *g-degree* of $\mathcal{F}$ is defined to be

$$\deg_g(\mathcal{F}) = \int_X c_1(\mathcal{F}) \wedge \omega^{n-1},$$

where, by abuse of notation, $c_1(\mathcal{F})$ also denotes a closed smooth real $(1,1)$-form representing the first Chern class $c_1(\mathcal{F})$.

(ii) If $\operatorname{rank}(\mathcal{F}) > 0$, the *g-slope* of $\mathcal{F}$ is defined to be

$$\mu_g(\mathcal{F}) = \frac{\deg_g(\mathcal{F})}{\operatorname{rank}(\mathcal{F})},$$

where $\operatorname{rank}(\mathcal{F})$ is defined to be the rank of $\mathcal{F}$ outside the singularity set $S_{n-1}(\mathcal{F})$, where $\mathcal{F}$ is locally free.

Remark 3.2. The definition of $\deg_g(\mathcal{F})$ is independent of the choice of a closed smooth real $(1,1)$-form representing $c_1(\mathcal{F})$ by Stokes' theorem. In particular, if $\mathcal{F} = \mathcal{O}_X(E)$ is the sheaf of holomorphic sections of a holomorphic vector bundle E on X, we have

$$\deg_g(E) = \int_X \frac{\sqrt{-1}}{2\pi} \operatorname{tr}(F_h) \wedge \omega^{n-1},$$

where F_h is the curvature form of the Chern connection of the Hermitian holomorphic vector bundle (E, h), where h is a smooth Hermitian metric in E.

With these definitions at hand, we can now define the notion of (semi-)stability following Takemoto [Ta72].

Definition 3.3 ((Semi-)stability)**.** A torsion-free coherent analytic sheaf $\mathcal{E}$ on X is said to be *g-semistable* if for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F})$, the inequality

$$\mu_g(\mathcal{F}) \leqslant \mu_g(\mathcal{E})$$

holds. If, moreover, the strict inequality

$$\mu_g(\mathcal{F}) < \mu_g(\mathcal{E})$$

holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$, we say that $\mathcal{E}$ is *g-stable*.

Remark 3.4.

(i) In definition 3.3, it suffices to consider coherent subsheaves $\mathcal{F}$ of $\mathcal{E}$ such that the quotient $\mathcal{E}/\mathcal{F}$ is torsion-free.

(ii) The notion of (semi-)stability is, of course, also defined for a holomorphic vector bundle E on X by considering its sheaf of holomorphic sections $\mathcal{E} = \mathcal{O}_X(E)$, which is a locally free and, therefore, torsion-free coherent analytic sheaf on X. Note, however, that even if we are only interested in the stability of a holomorphic vector bundle, we need to consider not only subbundles but arbitrary coherent subsheaves.

(iii) In the projective-algebraic case, when we are given an ample line bundle H on X and let ω be the curvature form of a positive Hermitian metric in H, the g-degree of a torsion-free coherent analytic sheaf $\mathcal{F}$ on X is also called the *H-degree* and it can be written as

$$\deg_H(\mathcal{F}) = (c_1(\mathcal{F}) \cup c_1(H)^{n-1}) \cap [X].$$

Note that this is independent of the choice of a positive Hermitian metric in H. Since $c_1(\mathcal{F})$ and $c_1(H)$ are integral classes, in this case the H-degree $\deg_H(\mathcal{F})$ is an integer. It can also be regarded as an intersection number of line bundles $\det(\mathcal{F}) \cdot H^{n-1}$. We also talk about the *H-slope* $\mu_H(\mathcal{F})$ and the *H-(semi-)stability* of a torsion-free coherent analytic sheaf $\mathcal{E}$ or a vector bundle E on X.

(iv) In order to distinguish it from other notions of stability, the notion of stability defined here is also referred to as *slope-stability* or *Mumford-Takemoto stability.*

An important consequence of the stability of a holomorphic vector bundle E on X is the simplicity of E.

Definition 3.5 (Simplicity)**.** A holomorphic vector bundle E on X is called *simple* if every holomorphic section of $\operatorname{End}(E) = E^* \otimes E$ is a scalar multiple of the identity endomorphism.

We have the following general statement.

Proposition 3.6 (S. Kobayashi, [Kb87], (V.7.12))**.** *Let E_1 and E_2 be g-semistable holomorphic vector bundles on X of the same rank and degree. If E_1 or E_2 is g-stable, then every non-zero sheaf homomorphism $f : E_1 \to E_2$ is an isomorphism.*

Corollary 3.7. *If E is a g-stable holomorphic vector bundle on X, then E is simple.*

Proof. Given a holomorphic section f of $\mathrm{End}(E)$, fix a point $p \in X$ and let a be an eigenvalue of the endomorphism $f_p : E_p \to E_p$ of the fibre E_p of E at p. Then the sheaf homomorphism $f - a\,\mathrm{id}_E : E \to E$ is not an isomorphism. Applying proposition 3.6 with $E_1 = E_2 = E$ yields $f - a\,\mathrm{id}_E = 0$ and thus $f = a\,\mathrm{id}_E$. □

We now review the concept of a Hermitian-Einstein metric in a holomorphic vector bundle E on a compact Kähler manifold (X, g). Recall that there is an operator Λ_g mapping (p, q)-forms (possibly with values in a vector bundle) onto $(p-1, q-1)$-forms for $p, q \geqslant 1$ which is formally adjoint to the operation of forming the $\wedge$-product with ω. In particular, if we are given a $(1,1)$-form η on X and, in local coordinates $z^1, \dots, z^n$, we write

$$\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$
$$\eta = \eta_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$

we have

$$\sqrt{-1}\Lambda_g \eta = g^{\bar{j}i} \eta_{i\bar{j}}, \tag{3.1}$$

where $(g^{\bar{j}i})_{j,i=1,\dots,n}$ denotes the inverse matrix of $(g_{i\bar{j}})_{i,j=1,\dots,n}$. Given a smooth Hermitian metric h in E, denote by F_h the curvature form of the Chern connection of the Hermitian vector bundle (E, h). Then F_h is a smooth $(1,1)$-form on X with values in $\mathrm{End}(E)$. The following notion was introduced by Kobayashi [Kb80] as a generalization of a Kähler-Einstein metric in the tangent bundle of a compact Kähler manifold.

Definition 3.8 (Hermitian-Einstein metric)**.** A smooth Hermitian metric h in a holomorphic vector bundle E on a compact Kähler manifold (X, g) is called a *g-Hermitian-Einstein metric* if

$$\sqrt{-1}\Lambda_g F_h = \lambda_h \,\mathrm{id}_E \tag{3.2}$$

with a constant $\lambda_h \in \mathbb{R}$, which is then called the *Einstein factor* of h. A Hermitian holomorphic vector bundle (E, h), where h is a g-Hermitian-Einstein metric in E, is also called a *g-Hermitian-Einstein vector bundle.*

In fact, the Einstein factor λ_h depends only on the Kähler manifold (X, g) and the vector bundle E, as is shown in the following lemma. In particular, it is independent of the Hermitian-Einstein metric h.

Lemma 3.9. *If h is a g-Hermitian-Einstein metric in E with Einstein factor λ_h, we have*

$$\lambda_h = \frac{2\pi \mu_g(E)}{(n-1)!\,\mathrm{vol}_g(X)},$$

where $\mathrm{vol}_g(X) = \int_X \frac{\omega^n}{n!}$ *is the* volume *of X with respect to g.*

Proof. Taking the trace of the Hermitian-Einstein equation (3.2) and integrating against the volume form $\frac{\omega^n}{n!}$ yields

$$\sqrt{-1}\int_X \Lambda_g \operatorname{tr}(F_h)\frac{\omega^n}{n!} = \lambda_h \operatorname{rank}(E)\operatorname{vol}_g(X).$$

Since, in general, for $(1,1)$-forms η we have $(\Lambda_g\eta)\frac{\omega^n}{n!} = \eta\wedge\frac{\omega^{n-1}}{(n-1)!}$, the left-hand side equals

$$\sqrt{-1}\int_X \operatorname{tr}(F_h)\wedge\frac{\omega^{n-1}}{(n-1)!} = \frac{2\pi}{(n-1)!}\int_X \frac{\sqrt{-1}}{2\pi}\operatorname{tr}(F_h)\wedge\omega^{n-1} = \frac{2\pi}{(n-1)!}\deg_g(E)$$

by remark 3.2. By the definition of $\mu_g(E)$, the claim follows. □

The now classical relation between the g-stability of a holomorphic vector bundle E and the existence of a g-Hermitian-Einstein metric in E can be stated as follows.

Theorem 3.10 (Kobayashi-Hitchin correspondence)**.** *Let E be a holomorphic vector bundle on a compact Kähler manifold (X, g).*

(i) If E is g-stable, then there is a g-Hermitian-Einstein metric in E.

(ii) If there is a g-Hermitian-Einstein metric in E, then E is g-polystable *in the sense that E is g-semistable and is a direct sum*

$$E = E_1 \oplus \cdots \oplus E_m$$

of g-stable subbundles E_k of E with $\mu_g(E_k) = \mu_g(E)$, $k = 1, \dots, m$. In particular, if E is irreducible, then it is g-stable.

The proof of this theorem has developed over many years. In 1982, S. Kobayashi [Kb82] proved that an irreducible Hermitian-Einstein vector bundle on a compact Kähler manifold is in fact stable. An alternative proof was given by Lübke [Lue83] in 1983. Shortly after that, Donaldson [Do83] showed that the two notions of stable and Hermitian-Einstein vector bundles are actually equivalent for the case of X being a compact Riemann surface, thus giving a new proof of a theorem of Narasimhan and Seshadri [NS65] from 1965. Around that time, S. Kobayashi and N. Hitchin independently conjectured that this equivalence holds in the more general case of X being a compact Kähler manifold of arbitrary dimension. In 1985, Donaldson [Do85] proved the existence of a Hermitian-Einstein metric in a stable holomorphic vector bundle on a projective-algebraic surface. In the method he used, the Hermitian-Einstein metric was found as the limit of a deformation of a background metric along a heat-type equation. This method, which he later generalized for the case of a projective-algebraic manifold of arbitrary dimension [Do87], also is the one employed in this thesis. Finally, in 1986, Uhlenbeck and Yau [UY86], [UY89] were able to prove the theorem for an arbitrary compact Kähler manifold. We also refer the reader to S. Kobayashi [Kb87], Lübke and Teleman [LT95] and Simpson [Si88].

3.2 Adaptation for the framed case

We discuss an adaptation of the notion of stability for the case of a canonically polarized framed manifold (X, D). As before, this means that X is a compact complex manifold and D is a smooth divisor in X such that $K_X \otimes [D]$ is ample. Let E be a holomorphic vector bundle on X and denote by $E' := E|_{X'}$ its restriction to $X' := X \setminus D$. If one wants to find a good notion of "framed stability" of E with respect to the framed manifold (X, D), the critical aspect is the choice of a Kähler metric to be used to define the degree of a coherent subsheaf $\mathcal{F}$ of $\mathcal{E} = \mathcal{O}_X(E)$. The following two notions turn up when thinking about stability in the framed sense.

- Since $H := K_X \otimes [D]$ is an ample line bundle on X, by remark 3.4 (iii) there is the notion of $(K_X \otimes [D])$-stability of E. In this case, we define the degree of a torsion-free coherent analytic sheaf $\mathcal{F}$ on X as

  $$\deg_H(\mathcal{F}) = (c_1(\mathcal{F}) \cup c_1(H)^{n-1}) \cap [X] = \det(\mathcal{F}) \cdot H^{n-1}.$$

  This means that the degree is computed with respect to a Kähler metric on X whose fundamental form is the curvature form of a positive Hermitian metric in $K_X \otimes [D]$.

- As was shown in chapter 2, there is a unique (up to a constant multiple) complete Kähler-Einstein metric $g_{X'}$ on X' with negative Ricci curvature and Poincaré-type growth near the divisor D. We can thus define the degree of a torsion-free coherent analytic sheaf $\mathcal{F}$ on X as

  $$\deg_{X'}(\mathcal{F}) = \int_{X'} c_1(\mathcal{F}) \wedge \omega_{X'}^{n-1},$$

  where $\omega_{X'}$ is the fundamental form of $g_{X'}$ and $c_1(\mathcal{F})$ is a closed smooth real $(1,1)$-form representing the first Chern class of $\mathcal{F}$. When following this approach, we have to make sure that the integral is well-defined and, in particular, independent of the choice of such a $(1,1)$-form.

In fact, as we will see below, these two ways of computing the degree of a torsion-free coherent analytic sheaf on X are equivalent and so there is only one notion of "framed stability" of E. Note that while the first approach is a special case of stability in the ordinary (un-framed) sense on X (namely, with respect to a special polarization), the second approach is not a special case of stability in the ordinary sense on X' because here one only considers subsheaves of $\mathcal{E}$ on X instead of X'.

In order to show the well-definedness of $\deg_{X'}(\mathcal{F})$, we need the following lemma.

Lemma 3.11. *If η is a smooth real $(1,1)$-form on X, we have*

$$\int_{X'} |\Lambda_g \eta| dV_g < \infty,$$

where $g = g_{X'}$ is the Poincaré metric on X' with volume form dV_g and Λ_g is the formal adjoint of forming the $\wedge$-product with the fundamental form $\omega_{X'}$ of $g_{X'}$ as in the previous section.

Proof. Using local coordinates $z^1, \ldots, z^n$ on an open neighbourhood $U \subset X$ of a point $p \in D$ and writing

$$\omega_{X'} = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j,$$
$$\eta = \eta_{i\bar{j}} dz^i \wedge d\bar{z}^j$$

with smooth local functions $\eta_{i\bar{j}}$, $i, j = 1, \ldots, n$, we have, as in (3.1),

$$\sqrt{-1}\Lambda_g \eta = g^{\bar{j}i} \eta_{i\bar{j}}$$

and thus

$$|\Lambda_g \eta|^2 = g^{\bar{j}i} \eta_{i\bar{j}} g^{\bar{\ell}k} \eta_{k\bar{\ell}}.$$

If, in particular, $(z^1, z^2, \ldots, z^n) = (\sigma, z^2, \ldots, z^n)$ is the coordinate system of section 2.4, proposition 2.25 implies that $g^{\bar{j}i}$ is bounded for $i, j = 1, \ldots, n$. Since the $\eta_{i\bar{j}}$ are smooth functions, we obtain that $|\Lambda_g \eta|$ is bounded. As $(X', g_{X'})$ has finite volume by proposition 2.5, the claim follows. □

Furthermore, we need the following generalization of Stokes' theorem for complete Riemannian manifolds due to Gaffney.

Theorem 3.12 (Gaffney, [Ga54]). *Let (M, ds^2_M) be an orientable complete Riemannian manifold of real dimension $2n$ whose Riemann tensor is of class $\mathcal{C}^2$. Let γ be a $(2n-1)$-form on M of class $\mathcal{C}^1$ such that both γ and $d\gamma$ are in L^1. Then*

$$\int_M d\gamma = 0.$$

Lemma 3.13. *If $\mathcal{F}$ is a torsion-free coherent analytic sheaf on X, the integral*

$$\deg_{X'}(\mathcal{F}) = \int_{X'} c_1(\mathcal{F}) \wedge \omega_{X'}^{n-1} \tag{3.3}$$

is well-defined and, in particular, independent of the choice of a closed smooth real $(1,1)$-form $c_1(\mathcal{F})$ representing the first Chern class of $\mathcal{F}$.

Proof. Let η be a closed smooth real $(1,1)$-form on X representing $c_1(\mathcal{F})$. Then we have

$$\eta \wedge \omega_{X'}^{n-1} = (n-1)!(\Lambda_g \eta)\frac{\omega_{X'}^n}{n!}$$

and Lemma 3.11 implies the existence of the integral (3.3).

Now if $\tilde{\eta}$ is another such $(1,1)$-form representing $c_1(\mathcal{F})$, we have $\eta - \tilde{\eta} = d\zeta$ for a smooth 1-form ζ on X. It follows that

$$\int_{X'} \eta \wedge \omega_{X'}^{n-1} - \int_{X'} \tilde{\eta} \wedge \omega_{X'}^{n-1} = \int_{X'} d\zeta \wedge \omega_{X'}^{n-1} = \int_{X'} d\gamma,$$

where $\gamma := \zeta \wedge \omega_{X'}^{n-1}$ is a smooth $(2n-1)$-form on X' such that $d\gamma$ and (as can be shown analogously) γ itself are in L^1. Now apply Gaffney's theorem 3.12 with (M, ds_M^2) being the underlying Riemannian manifold of $(X', g_{X'})$, which is complete by proposition 2.5. This implies $\int_{X'} d\gamma = 0$, thus proving the claim. □

We can now prove the equivalence of the two notions of degree discussed above.

Proposition 3.14. *Let $\mathcal{F}$ be a torsion-free coherent analytic sheaf on X. Then*

$$\deg_H(\mathcal{F}) = \deg_{X'}(\mathcal{F}),$$

where $H := K_X \otimes [D]$.

Proof. Let η be a closed smooth real $(1,1)$-form on X representing $c_1(\mathcal{F})$. Then we have

$$\deg_H(\mathcal{F}) = \int_X \eta \wedge \omega_X^{n-1},$$

where ω_X is the curvature form of a positive Hermitian metric in $H = K_X \otimes [D]$, i. e.

$$\omega_X = -\operatorname{Ric}\left(\frac{\Omega}{||\sigma||^2}\right) = \sqrt{-1}\partial\bar{\partial}\log\left(\frac{\Omega}{||\sigma||^2}\right)$$

with a smooth volume form Ω on X and a smooth Hermitian metric h in $[D]$ with induced norm $||\cdot||$ such that ω_X is positive definite. Here, as above, σ denotes a canonical holomorphic section of $[D]$. On the other hand, we have

$$\deg_{X'}(\mathcal{F}) = \int_{X'} \eta \wedge \omega_{X'}^{n-1},$$

where $\omega_{X'}$ is the fundamental form of the Poincaré metric $g_{X'}$ on X'. By theorem 2.19 and the fact that $g_{X'}$ is Kähler-Einstein, there is a number $0 < \alpha \leqslant 1$ such that (in particular) for all $k \geqslant 2$ and $\beta \in (0,1)$, we have

$$\omega_{X'} = -\operatorname{Ric}(\omega_{X'}^n) = \sqrt{-1}\partial\bar{\partial}\log\left(\frac{2\Omega}{||\sigma||^2\log^2(1/||\sigma||^2)}\left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right)\right)$$

with a function $\nu \in \mathcal{C}^{k,\beta}(X')$. A comparison of ω_X and $\omega_{X'}$ yields

$$\begin{aligned}\omega_{X'} &= \sqrt{-1}\partial\bar{\partial}\log\left(\frac{2\Omega}{||\sigma||^2\log^2(1/||\sigma||^2)}\left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right)\right)\\ &= \sqrt{-1}\partial\bar{\partial}\log\left(\frac{\Omega}{||\sigma||^2}\right) - 2\sqrt{-1}\partial\bar{\partial}\log\log(1/||\sigma||^2) + \sqrt{-1}\partial\bar{\partial}\log\left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right)\end{aligned}$$

and thus

$$\omega_{X'} = \omega_X|_{X'} - 2\sqrt{-1}\partial\bar{\partial}\log\log(1/||\sigma||^2) + \sqrt{-1}\partial\bar{\partial}\log\left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right). \tag{3.4}$$

For notational convenience, we first do the proof for the case of $n = 2$ and then explain the necessary changes for the proof to work in higher dimensions as well.

Since $X' = \bigcup_{\varepsilon>0} X_\varepsilon$ with $X_\varepsilon = \{x \in X : ||\sigma(x)|| > \varepsilon\}$, we have

$$\deg_H(\mathcal{F}) = \lim_{\varepsilon\to 0}\int_{X_\varepsilon} \eta\wedge\omega_X \quad \text{and} \quad \deg_{X'}(\mathcal{F}) = \lim_{\varepsilon\to 0}\int_{X_\varepsilon} \eta\wedge\omega_{X'}$$

and, therefore,

$$\begin{aligned}
\deg_{X'}(\mathcal{F}) &= \deg_H(\mathcal{F}) - 2\sqrt{-1}\lim_{\varepsilon\to 0}\int_{X_\varepsilon} \eta\wedge\partial\bar\partial\log\log(1/||\sigma||^2) \\
&\quad + \sqrt{-1}\lim_{\varepsilon\to 0}\int_{X_\varepsilon} \eta\wedge\partial\bar\partial\log\left(1+\frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right) \\
&= \deg_H(\mathcal{F}) + 2\sqrt{-1}\lim_{\varepsilon\to 0}\int_{X_\varepsilon} d\left(\eta\wedge\partial\log\log(1/||\sigma||^2)\right) \\
&\quad - \sqrt{-1}\lim_{\varepsilon\to 0}\int_{X_\varepsilon} d\left(\eta\wedge\partial\log\left(1+\frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right)\right) \\
&= \deg_H(\mathcal{F}) + 2\sqrt{-1}\lim_{\varepsilon\to 0}\int_{\partial X_\varepsilon} \eta\wedge\partial\log\log(1/||\sigma||^2) \\
&\quad - \sqrt{-1}\lim_{\varepsilon\to 0}\int_{\partial X_\varepsilon} \eta\wedge\partial\log\left(1+\frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right)
\end{aligned}$$

by Stokes' theorem. It remains to show that

$$\lim_{\varepsilon\to 0}\int_{\partial X_\varepsilon} \eta\wedge\partial\log\log(1/||\sigma||^2) = 0, \tag{3.5}$$

$$\lim_{\varepsilon\to 0}\int_{\partial X_\varepsilon} \eta\wedge\partial\log\left(1+\frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right) = 0. \tag{3.6}$$

We have $\partial X_\varepsilon = \{x \in X : ||\sigma(x)|| = \varepsilon\}$. By abuse of notation, we regard σ as a local coordinate on an open neighbourhood $U \subset X$ of a point $p \in D$ and regard h as a smooth positive function on U. Then we have local coordinates (σ, z) on U such that $||\sigma||^2 = |\sigma|^2/h$. In (3.5), we have

$$\partial\log\log(1/||\sigma||^2) = \frac{\partial\log(1/||\sigma||^2)}{\log(1/||\sigma||^2)} = \frac{\partial\log h - \partial\log|\sigma|^2}{\log(1/||\sigma||^2)} = \frac{\partial\log h - \frac{d\sigma}{\sigma}}{\log(1/\varepsilon^2)} \quad \text{on } \partial X_\varepsilon,$$

and thus

$$\int_{\partial X_\varepsilon} \eta\wedge\partial\log\log(1/||\sigma||^2) = \frac{1}{\log(1/\varepsilon^2)}\left(\int_{\partial X_\varepsilon}\eta\wedge\partial\log h - \int_{\partial X_\varepsilon}\eta\wedge\frac{d\sigma}{\sigma}\right).$$

The first integral is clearly bounded uniformly in ε. The second integral can be estimated as follows. By Fubini's theorem, it suffices to estimate a one-dimensional line integral of the form

$$\int_{||\sigma||=\varepsilon} \frac{f(\sigma)d\sigma}{\sigma},$$

where f is a smooth locally defined function involving the coefficients of η. Since by the $\mathcal{C}^1$ version of Cauchy's integral formula (see, e. g., Hörmander [Hoe90], theorem 1.2.1), we have

$$f(0) = \frac{1}{2\pi\sqrt{-1}} \int_{||\sigma||=\varepsilon} \frac{f(\sigma)d\sigma}{\sigma} + \frac{1}{2\pi\sqrt{-1}} \iint_{||\sigma||<\varepsilon} \frac{\partial f}{\partial\bar{\sigma}} \frac{d\sigma \wedge d\bar{\sigma}}{\sigma}$$

and $f(0)$ is a finite number, it suffices to estimate the area integral. The latter is, however, bounded uniformly in ε since f is smooth and, writing $\sigma = re^{i\varphi}$ in polar coordinates, we have

$$\left|\frac{d\sigma \wedge d\bar{\sigma}}{\sigma}\right| = \left|\frac{-2\sqrt{-1}r dr \wedge d\varphi}{re^{i\varphi}}\right| = 2|dr \wedge d\varphi|.$$

As $\log(1/\varepsilon^2) \to \infty$ as $\varepsilon \to 0$, we obtain (3.5). In (3.6), we have

$$\partial \log\left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right) = \frac{1}{1 + \frac{\nu}{\log^\alpha(1/\varepsilon^2)}} \left(\frac{\partial\nu}{\log^\alpha(1/\varepsilon^2)} - \frac{\alpha\nu\left(\partial \log h - \frac{d\sigma}{\sigma}\right)}{\log^{\alpha+1}(1/\varepsilon^2)}\right) \quad \text{on } \partial X_\varepsilon,$$

and thus

$$\begin{aligned} \int_{\partial X_\varepsilon} \eta \wedge \partial \log\left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right) = {} & \frac{1}{\log^\alpha(1/\varepsilon^2)} \int_{\partial X_\varepsilon} \frac{\eta \wedge \partial\nu}{1 + \frac{\nu}{\log^\alpha(1/\varepsilon^2)}} \\ & - \frac{\alpha}{\log^{\alpha+1}(1/\varepsilon^2)} \int_{\partial X_\varepsilon} \frac{\eta \wedge \nu\partial \log h}{1 + \frac{\nu}{\log^\alpha(1/\varepsilon^2)}} \\ & + \frac{\alpha}{\log^{\alpha+1}(1/\varepsilon^2)} \int_{\partial X_\varepsilon} \frac{\eta \wedge \nu\frac{d\sigma}{\sigma}}{1 + \frac{\nu}{\log^\alpha(1/\varepsilon^2)}}. \end{aligned}$$

Again, by Fubini's theorem, it suffices to consider the one-dimensional situation. Since ν is in $\mathcal{C}^{k,\beta}(X')$ with $k \geqslant 2$, ν is (in particular) bounded on X' and so

$$\sup_{\partial X_\varepsilon} \left|\frac{1}{1 + \frac{\nu}{\log^\alpha(1/\varepsilon^2)}}\right|$$

is bounded uniformly in ε and so is the second integral above. Moreover, if v is the quasi-coordinate corresponding to σ, we have

$$\partial\nu = \frac{\partial\nu}{\partial v} dv = \frac{\partial\nu}{\partial v} \frac{(|v|^2 - 1)(v - 1)}{(\bar{v} - 1)\log(1/|\sigma|^2)} \frac{d\sigma}{\sigma}$$

by remark 2.11, where $\frac{\partial \nu}{\partial v}$ is bounded on X'. Consequently, the other two integrals can be bounded by using Cauchy's integral formula as above. Since

$$\log^{\alpha}(1/\varepsilon^2) \to \infty \quad \text{and} \quad \log^{\alpha+1}(1/\varepsilon^2) \to \infty \quad \text{as } \varepsilon \to 0,$$

we obtain (3.6). This concludes the proof for the case of $n = 2$.

In dimension $n > 2$, one expands the expression $\omega_{X'}^{n-1}$, where $\omega_{X'} = \omega_X|_{X'} + \xi$ is written as in (3.4) with a closed smooth real $(1,1)$-form ξ on X'. Then one has to show the vanishing for $\varepsilon \to 0$ of several integrals of the forms (3.5) and (3.6) with additional terms which are either equal to ω_X or to ξ. Since ω_X is smooth on X, it does not destroy the convergence. Concerning ξ, an argument similar to the one in the proof of lemma 3.13 shows that this does not influence the convergence either. Thus the proof works in any dimension. □

We can now proceed in parallel to the compact case.

Definition 3.15 (Framed degree, framed slope)**.** Let $\mathcal{F}$ be a torsion-free coherent analytic sheaf on X.

(i) We call the integer

$$\deg_{(X,D)}(\mathcal{F}) := \deg_H(\mathcal{F}) = \deg_{X'}(\mathcal{F})$$

from proposition 3.14 the *framed degree* or the *degree in the framed sense* of $\mathcal{F}$ with respect to the framed manifold (X, D).

(ii) If $\operatorname{rank}(\mathcal{F}) > 0$, we call

$$\mu_{(X,D)}(\mathcal{F}) := \frac{\deg_{(X,D)}(\mathcal{F})}{\operatorname{rank}(\mathcal{F})}$$

the *framed slope* or the *slope in the framed sense* of $\mathcal{F}$ with respect to the framed manifold (X, D).

Definition 3.16 (Framed (semi-)stability)**.** A torsion-free coherent analytic sheaf $\mathcal{E}$ on X is said to be *semistable in the framed sense* with respect to the framed manifold (X, D) if for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F})$, the inequality

$$\mu_{(X,D)}(\mathcal{F}) \leqslant \mu_{(X,D)}(\mathcal{E})$$

holds. If, moreover, the strict inequality

$$\mu_{(X,D)}(\mathcal{F}) < \mu_{(X,D)}(\mathcal{E})$$

holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$, we say that $\mathcal{E}$ is *stable in the framed sense* with respect to the framed manifold (X, D).

Of course, statements (i) and (ii) of remark 3.4 also hold in the framed case. Since the framed stability of E with respect to (X, D) is a special case of the stability of E in the ordinary sense on X (namely, it is the stability with respect to the polarization $K_X \otimes [D]$), corollary 3.7 can be applied to the framed situation.

Corollary 3.17. *If E is a stable holomorphic vector bundle on X in the framed sense with respect to (X, D), then E is simple.*

Remark 3.18. Note, however, that the framed stability of E with respect to (X, D) does not necessarily imply the simplicity of $E' = E|_{X'}$. Thus, given a holomorphic section of $\mathrm{End}(E)$ over X', one has to make sure that it can be holomorphically extended to the whole of X in order to be able to conclude that it is a scalar multiple of the identity endomorphism.

We now introduce a suitable notion of a Hermitian-Einstein metric in the framed sense. Our interest lies on smooth Hermitian metrics in the holomorphic vector bundle E' on X' which satisfy the Hermitian-Einstein condition with respect to the Poincaré metric on X'. In order to ensure that everything is well-defined in the following considerations, we first have to make a restriction on the class of smooth Hermitian metrics in E', which is the one employed by Simpson in [Si88]. Denote by $\mathcal{P}$ the space of smooth Hermitian metrics h' in E' such that

$$\int_{X'} |\Lambda_g F_{h'}|_{h'} dV_g < \infty,$$

where $F_{h'}$ is the curvature form of the Chern connection of the Hermitian holomorphic vector bundle (E', h') on X'. First of all, if h' is the restriction to E' of a smooth Hermitian metric h in E, we have $h' \in \mathcal{P}$ by lemma 3.11. Now the definition of $\mathcal{P}$ is such that for any $h' \in \mathcal{P}$, the integral

$$\deg_{X'}(E', h') := \int_{X'} \frac{\sqrt{-1}}{2\pi} \operatorname{tr}(F_{h'}) \wedge \omega_{X'}^{n-1}$$

is well-defined. However, in order to ensure that it equals the framed degree $\deg_{(X,D)}(E)$ of E with respect to (X, D), we have to impose an additional condition on h'. Following Simpson [Si88], we denote by $S_{h'}$ the bundle of endomorphisms of E' which are self-adjoint with respect to h'. Furthermore, we let $P(S_{h'})$ be the space of smooth sections s of $S_{h'}$ such that

$$||s||_P := \sup_{X'} |s|_{h'} + ||\nabla'' s||_{L^2(X, \Lambda^{0,1}T_X^* \otimes \mathrm{End}(E), g)} + ||\Delta' s||_{L^1(X, \mathrm{End}(E), g)} < \infty,$$

where $\nabla = \nabla' + \nabla''$ is the covariant derivative on smooth sections of $\mathrm{End}(E')$ with respect to the Chern connection of the Hermitian holomorphic vector bundle (E', h') and $\Delta' = \sqrt{-1}\Lambda_g \nabla'' \nabla'$ is the ∇'-Laplacian on smooth sections of $\mathrm{End}(E')$ with respect to h' and the Poincaré metric. Here, the L^2 norm is as in chapter 2 and the L^1 norm is defined analogously, where $\mathrm{End}(E)$ is endowed with the metric h' over X'. Now, according to [Si88], $\mathcal{P}$ can be turned into an analytic manifold with local charts

$$\begin{array}{rcl} P(S_{h'}) & \longrightarrow & \mathcal{P} \\ s & \longmapsto & h' e^s \end{array}.$$

Divide $\mathcal{P}$ into maximal components such that each of these charts covers a component. Choose a smooth Hermitian metric h_0 in E and use the same notation h_0 for its restriction to E'. The component $\mathcal{P}_0$ of $\mathcal{P}$ containing h_0 is easily seen to be independent of the choice of h_0 because the restrictions to E' of any two smooth Hermitian metrics in E lie in the same component of $\mathcal{P}$. This space $\mathcal{P}_0$ turns out to be a suitable space in which to look for Hermitian-Einstein metrics with respect to the Poincaré metric.

Definition 3.19 (Framed Hermitian-Einstein metric)**.** A smooth Hermitian metric h' in E' is called a *framed Hermitian-Einstein metric* or *Hermitian-Einstein metric in the framed sense* in E with respect to the framed manifold (X, D) if $h' \in \mathcal{P}_0$ and

$$\sqrt{-1}\Lambda_g F_{h'} = \lambda_{h'} \operatorname{id}_{E'}$$

with a constant $\lambda_{h'} \in \mathbb{R}$, which is then called the *Einstein factor* of h'.

This definition leads to an analogue of lemma 3.9 for the framed situation.

Lemma 3.20. *If $h' \in \mathcal{P}_0$, we have*

$$\deg_{X'}(E', h') = \deg_{(X,D)}(E).$$

In particular, if h' is a framed Hermitian-Einstein metric in E with respect to (X, D) and Einstein factor $\lambda_{h'}$, we have

$$\lambda_{h'} = \frac{2\pi\mu_{(X,D)}(E)}{(n-1)! \operatorname{vol}_g(X')},$$

where $\operatorname{vol}_g(X') = \int_{X'} \frac{\omega_{X'}^n}{n!}$ *is the* volume *of X' with respect to $g_{X'}$.*

Proof. First of all, because of $h' \in \mathcal{P}_0 \subset \mathcal{P}$, the integral

$$\deg_{X'}(E', h') = \int_{X'} \frac{\sqrt{-1}}{2\pi} \operatorname{tr}(F_{h'}) \wedge \omega_{X'}^{n-1} = \int_{X'} \frac{\sqrt{-1}(n-1)!}{2\pi} \operatorname{tr}(\Lambda_g F_{h'}) \frac{\omega_{X'}^n}{n!}$$

is well-defined. Furthermore, $\frac{\sqrt{-1}}{2\pi} \operatorname{tr}(F_{h_0})$ is a closed smooth real $(1,1)$-form on X representing the first Chern class $c_1(E)$ and thus

$$\deg_{(X,D)}(E) = \deg_{X'}(E) = \int_{X'} \frac{\sqrt{-1}}{2\pi} \operatorname{tr}(F_{h_0}) \wedge \omega_{X'}^{n-1} = \int_{X'} \frac{\sqrt{-1}(n-1)!}{2\pi} \operatorname{tr}(\Lambda_g F_{h_0}) \frac{\omega_{X'}^n}{n!}.$$

We therefore have to show that

$$\int_{X'} (\operatorname{tr}(\Lambda_g F_{h'}) - \operatorname{tr}(\Lambda_g F_{h_0})) \frac{\omega_{X'}^n}{n!} = 0. \tag{3.7}$$

Because of $h' \in \mathcal{P}_0$, we have $h' = h_0 e^s$ with $s \in P(S_{h_0})$. By the standard theory on Hermitian holomorphic vector bundles, we know that

$$\operatorname{tr}(\Lambda_g F_{h'}) - \operatorname{tr}(\Lambda_g F_{h_0}) = \Lambda_g \bar{\partial}\partial \operatorname{tr}(s).$$

From $h', h_0 \in \mathcal{P}$, it follows that $\Lambda_g \bar{\partial}\partial \operatorname{tr}(s)$ is integrable on X'. Also, because of $s \in P(S_{h_0})$, we know that $\bar{\partial} \operatorname{tr}(s) = \operatorname{tr}(\nabla'' s)$ is integrable on X'. By Gaffney's theorem 3.12, (3.7) follows.

The expression of the Einstein factor of a framed Hermitian-Einstein metric in terms of the framed slope then follows exactly as in lemma 3.9. □

As mentioned above, a framed Hermitian-Einstein metric in a simple bundle is unique up to a constant multiple.

Proposition 3.21 (Uniqueness of framed Hermitian-Einstein metrics)**.** *Let E be a simple holomorphic vector bundle on a canonically polarized framed manifold (X, D). Then if h_0' and h_1' are Hermitian-Einstein metrics in E in the framed sense with respect to (X, D), there is a constant $c > 0$ such that $h_1' = ch_0'$.*

Proof. First of all, we have

$$\sqrt{-1}\Lambda_g F_{h_0'} = \lambda\,\mathrm{id}_{E'} = \sqrt{-1}\Lambda_g F_{h_1'} \quad \text{with } \lambda = \frac{2\pi\mu_{(X,D)}(E)}{(n-1)!\,\mathrm{vol}_g(X')} \tag{3.8}$$

by lemma 3.20. Since h_0' and h_1' lie in the same component $\mathcal{P}_0$ of $\mathcal{P}$, we know that $h_1' = h_0' e^s$ for some $s \in P(S_{h_0'})$. Join h_0' and h_1' by the path $h_t' = h_0' e^{ts}$ for $t \in [0,1]$ and define the function $L : [0,1] \to \mathbb{C}$ by

$$L(t) = \int_{X'} \int_0^t \mathrm{tr}\left(s(\sqrt{-1}\Lambda_g F_{h_u'} - \lambda\,\mathrm{id}_{E'})\right) du \frac{\omega_{X'}^n}{n!}.$$

This is a special version of Donaldson's functional as it will be used in the existence proof in the following chapter. It is well-defined since for every $t \in [0,1]$, we have

$$\begin{aligned}\left|\int_0^t \mathrm{tr}\left(s(\sqrt{-1}\Lambda_g F_{h_u'} - \lambda\,\mathrm{id}_{E'})\right) du\right| &\leqslant t \sup_{u\in[0,t]} \left|\left\langle \sqrt{-1}\Lambda_g F_{h_u'} - \lambda\,\mathrm{id}_{E'}, s\right\rangle_{h_u'}\right| \\ &= t\left|\left\langle \sqrt{-1}\Lambda_g F_{h_{u_0}'} - \lambda\,\mathrm{id}_{E'}, s\right\rangle_{h_{u_0}'}\right| \\ &\leqslant t\left|\sqrt{-1}\Lambda_g F_{h_{u_0}'} - \lambda\,\mathrm{id}_{E'}\right|_{h_{u_0}'} |s|_{h_{u_0}'} \\ &\leqslant t\left(|\Lambda_g F_{h_{u_0}'}|_{h_{u_0}'} + |\lambda|\sqrt{\mathrm{rank}(E)}\right) ||s||_P\end{aligned}$$

for some $u_0 \in [0,t]$, where the last expression is integrable over X' with respect to the Poincaré metric because of $h_{u_0}' \in \mathcal{P}_0 \subset \mathcal{P}$, $s \in P(S_{h_{u_0}'})$ and the finite volume of $(X', g_{X'})$. The first derivative of L is

$$L'(t) = \int_{X'} \mathrm{tr}\left(s(\sqrt{-1}\Lambda_g F_{h_t'} - \lambda\,\mathrm{id}_{E'})\right) \frac{\omega_{X'}^n}{n!}$$

and the Hermitian-Einstein condition (3.8) yields $L'(0) = 0 = L'(1)$. By the standard theory, we know that

$$\frac{d}{dt}(\Lambda_g F_{h_t'}) = \Lambda_g \nabla'' \nabla_{h_t'}' s,$$

where

$$\nabla_{h_t'} = \nabla_{h_t'}' + \nabla''$$

is the covariant derivative on smooth sections of $\mathrm{End}(E')$ with respect to the Chern connection of the Hermitian holomorphic vector bundle (E', h_t'). Consequently, the second derivative of L is

$$L''(t) = \int_{X'} \mathrm{tr}\left(s(\sqrt{-1}\Lambda_g \nabla'' \nabla_{h_t'}' s)\right) \frac{\omega_{X'}^n}{n!}$$

$$
\begin{aligned}
&= \sqrt{-1}\int_{X'} \operatorname{tr}(s\nabla''\nabla'_{h'_t}s)\wedge\frac{\omega_{X'}^{n-1}}{(n-1)!}\\
&= -\sqrt{-1}\int_{X'} \operatorname{tr}(\nabla''s\wedge\nabla'_{h'_t}s)\wedge\frac{\omega_{X'}^{n-1}}{(n-1)!}+\sqrt{-1}\int_{X'}\bar\partial\operatorname{tr}(s\nabla'_{h'_t}s)\wedge\frac{\omega_{X'}^{n-1}}{(n-1)!}\\
&= ||\nabla''s||_{L^2}^2+\sqrt{-1}\int_{X'}d\gamma
\end{aligned}
$$

since s is self-adjoint with respect to h'_t, where the L^2 norm is as above and

$$\gamma = \operatorname{tr}(s\nabla'_{h'_t}s)\wedge\frac{\omega_{X'}^{n-1}}{(n-1)!}$$

is a smooth $(2n-1)$-form on X'. We are going to verify the hypotheses of Gaffney's theorem 3.12. We have

$$|\operatorname{tr}(s\nabla'_{h'_t}s)|\leqslant|\nabla'_{h'_t}s|_{h'_t}|s|_{h'_t}=|\nabla''s|_{h'_t}|s|_{h'_t}$$

and from $s\in P(S_{h'_t})$, we know that $|\nabla''s|_{h'_t}$ is L^2 and $|s|_{h'_t}$ is bounded on X'. It follows that γ is L^2 on X' and, in particular, L^1 due to the finite volume of $(X', g_{X'})$. Moreover, we know that

$$|\Delta'_{h'_t}s|_{h'_t}=|\Lambda_g\nabla''\nabla'_{h'_t}s|_{h'_t}$$

is L^1 on X'. Thus, $d\gamma$ is seen to be L^1 on X' as well. By Gaffney's theorem, it follows that $\int_{X'}d\gamma=0$ and we obtain

$$L''(t)=||\nabla''s||_{L^2}^2$$

for all $t\in[0,1]$. In particular, $L''(t)$ is independent of t. From $L'(0)=0=L'(1)$, it follows that $L'\equiv 0$ and thus $L''\equiv 0$ on $[0,1]$. This implies that $\nabla''s=0$, i. e. s is a holomorphic section of $\operatorname{End}(E')$. As above, let h_0 be a smooth Hermitian metric in E. Then h_0 and h'_0 lie in the same component $\mathcal{P}_0$ of $\mathcal{P}$ and the boundedness of $|s|_{h'_0}$ implies the boundedness of $|s|_{h_0}$. By Riemann's extension theorem, s can be extended to a holomorphic section of $\operatorname{End}(E)$ over X. Since the bundle E is simple by hypothesis, we have $s = a\,\mathrm{id}_E$ for some number a, which must be real as s is self-adjoint. Finally, we obtain

$$h'_1=h'_0e^s=ch'_0\quad\text{with } c=e^a>0$$

as claimed. □

To conclude this chapter, we state the existence and uniqueness result for a framed Hermitian-Einstein metric in a holomorphic vector bundle which is stable in the framed sense.

Theorem 3.22. *Let E be a holomorphic vector bundle on a canonically polarized framed manifold (X,D) such that E is stable in the framed sense with respect to (X,D). Then there is a unique (up to a constant multiple) Hermitian-Einstein metric in E in the framed sense with respect to (X,D).*

The uniqueness follows from corollary 3.17 and proposition 3.21. The existence will be proved in chapter 4.

4 Solution of the heat equation

In this chapter, we introduce the evolution equation considered by Donaldson. We first present an overview of his existence proof for a solution defined for all finite non-negative values of the time parameter, cf. [Do85]. Then we review Simpson's proof of the convergence of this solution to a Hermitian-Einstein metric in infinite time if the bundle is stable, cf. [Si88]. This involves an estimate regarding Donaldson's functional which is shown by constructing a special weakly holomorphic subbundle for the case that the estimate does not hold. We then summarize Popovici's proof of a theorem by Uhlenbeck and Yau which states that one actually obtains a coherent subsheaf contradicting the stability of the bundle, cf. [UY86], [UY89], [Po05].

The methods introduced in chapter 2, especially the notions of quasi-coordinates and bounded geometry, together with Gaffney's theorem 3.12, enable us to apply the known arguments from the compact case to our framed situation. Therefore, in this chapter, it suffices to consider the compact situation.

Let (X, g) be a compact Kähler manifold and let E be a holomorphic vector bundle on X. Choose a smooth Hermitian metric h_0 in E as a background metric. Then the space of smooth Hermitian metrics in E can be identified with the space of smooth sections of $\mathrm{End}(E)$ which are positive definite and self-adjoint with respect to h_0. Such a section f corresponds to the Hermitian metric $h = h_0 f$ in E defined by

$$h(s,t) = h_0(f(s), t)$$

for all sections s and t of E. One also writes $f = h_0^{-1}h$. The evolution equation for a family $(h_t)_t$ of smooth Hermitian metrics depending smoothly on a real time parameter t can be written as

$$h_t^{-1}\dot{h}_t = -(\sqrt{-1}\Lambda_g F_{h_t} - \lambda\,\mathrm{id}_E), \tag{4.1}$$

where $\dot{h}_t = \frac{dh_t}{dt}$ denotes the time derivative of h_t, F_{h_t} is the curvature form of the Chern connection of the Hermitian holomorphic vector bundle (E, h_t) and

$$\lambda = \frac{2\pi\mu_g(E)}{(n-1)!\,\mathrm{vol}_g(X)}$$

is as in the previous chapter.

4.1 Existence for finite times

In order to show that (4.1) has a solution defined for all $0 \leqslant t < \infty$, we use the continuity method. Writing $h_t = h_0 f_t$ with a family $(f_t)_t$ of smooth endomorphisms of E as explained above, (4.1) is equivalent to the equation

$$\left.\begin{aligned}\left(\frac{d}{dt} + \Delta'_{h_0}\right) f_t &= -f_t(\sqrt{-1}\Lambda_g F_{h_0} - \lambda \operatorname{id}_E) + \sqrt{-1}\Lambda_g(\nabla'' f_t \wedge f_t^{-1}\nabla'_{h_0} f_t),\\ f_0 &= \operatorname{id}_E,\end{aligned}\right\}$$

which is a non-linear parabolic partial differential equation. The general theory on such equations is explained, e. g., in [Ha75], part III and section 11 of part IV. In particular, it guarantees the existence of a short-time solution.

Proposition 4.1 ([Do85], proposition 11)**.** *For a sufficiently small $\varepsilon > 0$, equation* (4.1) *has a smooth solution defined for* $0 \leqslant t < \varepsilon$.

We have to show that the solution can be continued for all positive times. In [Do85], Donaldson introduces the following measure of the "distance" between two Hermitian metrics.

Definition 4.2. If h and k are smooth Hermitian metrics in E, set

$$\left.\begin{aligned}\tau(h,k) &= \operatorname{tr}(h^{-1}k),\\ \sigma(h,k) &= \tau(h,k) + \tau(k,h) - 2\operatorname{rank}(E)\end{aligned}\right\} \in \mathcal{C}^\infty(X).$$

Then σ is symmetric and from the inequality

$$\alpha + \alpha^{-1} \geqslant 2 \quad \text{for all } \alpha > 0,$$

it follows that $\sigma(h,k) \geqslant 0$ for any Hermitian metrics h and k with equality if and only if $h = k$. Although σ is not a metric, one shows that a sequence $(h_i)_i$ of Hermitian metrics converges to a Hermitian metric h in the usual $\mathcal{C}^0$ topology if and only if $\sup_X \sigma(h_i, h)$ converges to zero. Moreover, the existence of such a $\mathcal{C}^0$ limit is equivalent to the condition that the sequence $(h_i)_i$ is a "Cauchy sequence" with respect to σ, i. e. that for any $\varepsilon > 0$, there is some i_0 such that $\sup_X \sigma(h_i, h_j) < \varepsilon$ if $i, j \geqslant i_0$.

Proposition 4.3. *If $(h_t)_t$ and $(k_t)_t$ are solutions of the evolution equation* (4.1) *defined on some open interval, then*

$$\left(\frac{d}{dt} + \Delta'\right)\sigma(h_t, k_t) \leqslant 0$$

on X for all t, where $\Delta' = \sqrt{-1}\Lambda_g\bar\partial\partial$ is the usual ∂-Laplacian.

Proof. For notational convenience, we drop the index t and write $\tau = \tau(h,k)$ for short. It suffices to show that $\left(\frac{d}{dt} + \Delta'\right)\tau \leqslant 0$. Writing $f = h^{-1}k$, we have

$$\begin{aligned} \frac{d\tau}{dt} &= \operatorname{tr}(-h^{-1}\dot{h}h^{-1}k + h^{-1}\dot{k}) \\ &= \operatorname{tr}\big((\sqrt{-1}\Lambda_g F_h - \lambda\,\mathrm{id}_E)f - f(\sqrt{-1}\Lambda_g F_k - \lambda\,\mathrm{id}_E)\big) \\ &= \operatorname{tr}\big(f(\sqrt{-1}\Lambda_g F_h - \sqrt{-1}\Lambda_g F_k)\big) \end{aligned} \tag{4.2}$$

by equation (4.1). Moreover, we have

$$\begin{aligned} \sqrt{-1}\Lambda_g F_h - \sqrt{-1}\Lambda_g F_k &= -\sqrt{-1}\Lambda_g \nabla''(f^{-1}\nabla'_h f) \\ &= f^{-1}\big(-\Delta'_h f + \sqrt{-1}\Lambda_g(\nabla'' f \wedge f^{-1}\nabla'_h f)\big). \end{aligned} \tag{4.3}$$

From (4.2), (4.3) and $\operatorname{tr}(\Delta'_h f) = \Delta'\tau$, it follows that

$$\left(\frac{d}{dt} + \Delta'\right)\tau = \sqrt{-1}\Lambda_g \operatorname{tr}(\nabla'' f \wedge f^{-1}\nabla'_h f),$$

which is non-positive. □

Corollary 4.4. *Let $(h_t)_t$ and $(k_t)_t$ be two solutions of (4.1) which are defined for $0 \leqslant t < \varepsilon$, are continuous at $t = 0$ and satisfy the same initial condition $h_0 = k_0$. Then they agree for all $0 \leqslant t < \varepsilon$.*

Proof. This follows by applying the maximum principle for the heat operator $\frac{d}{dt} + \Delta'$ (see, e. g., [Ha75], p. 101) to $\sigma(h_t, k_t)$ and using proposition 4.3. □

Corollary 4.5. *If $(h_t)_t$ is a smooth solution of (4.1) defined for $0 \leqslant t < T$, then h_t converges in $\mathcal{C}^0$ to a continuous Hermitian metric h_T as $t \to T$.*

Proof. It suffices to show that for any $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\sup_X \sigma(h_t, h_{t'}) < \varepsilon \quad \text{for } T - \delta < t, t' < T. \tag{4.4}$$

However, by the continuity of $(h_t)_t$ at $t = 0$, there is some $\delta > 0$ such that

$$\sup_X \sigma(h_t, h_{t'}) < \varepsilon \quad \text{for } 0 < t, t' < \delta. \tag{4.5}$$

Now since for any small $\alpha > 0$, $(h_{t+\alpha})_t$ is another solution of (4.1), proposition 4.3 yields

$$\left(\frac{d}{dt} + \Delta'\right)\sigma(h_t, h_{t+\alpha}) \leqslant 0$$

and by the maximum principle, it follows that $t \mapsto \sigma(h_t, h_{t+\alpha})$ is monotonically decreasing. Thus (4.5) can be carried over from the interval $(0, \delta)$ to $(T - \delta, T)$, which proves (4.4). □

In order to obtain a solution defined for all positive times, we need the $\mathcal{C}^\infty$ convergence of a solution $(h_t)_t$ defined for $0 \leqslant t < T$ as $t \to T$. To achieve this, we have to investigate the behaviour of the curvature form F_{h_t}. This involves some technical arguments, which will not be repeated here in full detail. Instead, we quote the main results from [Do85]. Given a family $(h_t)_t$ of smooth Hermitian metrics, define functions on X depending on the time parameter by

$$\begin{aligned} e &= |F_{h_t}|^2_{h_0}, \\ \hat{e} &= |\Lambda_g F_{h_t}|^2_{h_0}, \\ e_k &= |\nabla^k_{h_t} F_{h_t}|^2_{h_0} \quad \text{for } k \geqslant 0, \end{aligned}$$

where $\nabla^k_{h_t}$ is the k-th iterated covariant derivative.

Proposition 4.6 ([Do85], proposition 16)**.** *If $(h_t)_t$ is a solution of* (4.1)*, then*

(i) $\left(\frac{d}{dt} + \Delta'\right) \operatorname{tr}(F_{h_t}) = 0$,

(ii) $\left(\frac{d}{dt} + \Delta'\right) e \leqslant c(e^{3/2} + 3)$,

(iii) $\left(\frac{d}{dt} + \Delta'\right) \hat{e} \leqslant 0$,

(iv) $\left(\frac{d}{dt} + \Delta'\right) e_k \leqslant c_k e_k^{1/2} \sum_{i+j=k} e_i^{1/2}(e_j^{1/2} + 1)$

with constants $c, c_k > 0$ depending only on the Riemannian metric on X.

Corollary 4.7 ([Do85], corollary 17)**.** *Let $(h_t)_t$ be a solution of* (4.1) *defined for $0 \leqslant t < T$. Then the following statements hold.*

(i) $\sup_X |\operatorname{tr}(F_{h_t})|$ *and* $\sup_X \hat{e}$ *are both uniformly bounded for $0 \leqslant t < T$.*

(ii) If also $\sup_X e$ *is uniformly bounded for $0 \leqslant t < T$, then, for all $k \geqslant 0$,* $\sup_X e_k$ *is uniformly bounded for $0 \leqslant t < T$.*

Lemma 4.8 ([Do85], lemma 18)**.** *Suppose that $(h_t)_t$ is a solution of* (4.1) *defined for $0 < t < T$ and that F_{h_t} is bounded in L^q for some $q > 6$ uniformly in $0 < t < T$. Then, in fact, F_{h_t} is bounded in $\mathcal{C}^0$ uniformly in $0 < t < T$.*

Lemma 4.9 ([Do85], lemma 19)**.** *Let $(h_t)_t$ be a family of smooth Hermitian metrics defined for $0 \leqslant t < T$ such that*

(i) h_t converges in $\mathcal{C}^0$ to some continuous Hermitian metric h_T as $t \to T$,

(ii) $\sup_X \hat{e}$ *is uniformly bounded for $0 \leqslant t < T$.*

Then h_t is bounded in $\mathcal{C}^1$ and F_{h_t} is bounded in L^p for each $p < \infty$, both uniformly in $0 \leqslant t < T$.

Using these results, we can now prove the existence of a solution defined for all positive times.

Proposition 4.10. *The evolution equation* (4.1),

$$h_t^{-1}\dot{h}_t = -(\sqrt{-1}\Lambda_g F_{h_t} - \lambda\,\mathrm{id}_E),$$

has a unique smooth solution defined for $0 \leqslant t < \infty$.

Proof. A solution exists for short time by proposition 4.1 and is unique by corollary 4.4. Suppose that it can only be continued to a solution $(h_t)_t$ defined on some maximal interval $0 \leqslant t < T$. By corollaries 4.5 and 4.7 (i), the hypotheses of lemma 4.9 apply. Thus, F_{h_t} is bounded in L^p for any $p < \infty$ uniformly in $0 \leqslant t < T$. By lemma 4.8, it is in fact bounded in $\mathcal{C}^0$ uniformly in $0 \leqslant t < T$, so, by corollary 4.7 (ii), the iterated covariant derivatives of F_{h_t} are also bounded in $\mathcal{C}^0$ uniformly in $0 \leqslant t < T$. From the local expression

$$F_{h_t} = \bar{\partial}(h_t^{-1}\partial h_t),$$

we see that

$$\sqrt{-1}\Lambda_g F_{h_t} = h_t^{-1}\big(\Delta' h_t - \sqrt{-1}\Lambda_g(\bar{\partial}h_t \wedge h_t^{-1}\partial h_t)\big).$$

By the elliptic estimates for the Laplacian Δ' and an induction argument starting from the uniform $\mathcal{C}^1$ bound of h_t from lemma 4.9, it follows that h_t is bounded in $\mathcal{C}^k$ uniformly in $0 \leqslant t < T$ for each k. Thus the h_t, which we know converge in $\mathcal{C}^0$ as $t \to T$, in fact converge in $\mathcal{C}^\infty$. Using the short time existence from proposition 4.1 starting with h_T, the solution can be extended for $0 \leqslant t < T + \varepsilon$ with some $\varepsilon > 0$, contradicting the maximality of T. This proves the claim. □

Remark 4.11. By applying a suitable conformal change to the background metric h_0, one can achieve that the solution $(h_t)_t$ of (4.1) from proposition 4.10 satisfies $\det(h_t) = \det(h_0)$, i. e. $\det(f_t) = 1$, for all $0 \leqslant t < \infty$.

4.2 Convergence in infinite time

In order to show that the unique solution $(h_t)_t$ for $0 \leqslant t < \infty$ of the evolution equation (4.1) constructed in the previous section yields a Hermitian-Einstein metric as its limit in infinite time, we use Donaldson's functional as it is defined by Simpson in [Si88]. For this, we need a few preparations. Fix h_0 as the Hermitian metric in E and let $S = S_{h_0}$ be the bundle of self-adjoint endomorphisms of E. Given a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, define a bundle map

$$\varphi : S \longrightarrow S$$

as follows: Let $s \in S$ and near each point of X, choose a local orthonormal basis $\{e_i\}$ of E such that $s(e_i) = \lambda_i e_i$ with real numbers λ_i. Then set

$$\varphi(s)(e_i) = \varphi(\lambda_i)e_i.$$

This is well-defined and smooth in s. Now let $S(\mathrm{End}(E))$ be the bundle of endomorphisms of $\mathrm{End}(E)$ which are self-adjoint with respect to the Hermitian metric in $\mathrm{End}(E)$ induced by h_0. Given a smooth function $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, define a bundle map

$$\Psi : S \longrightarrow S(\mathrm{End}(E))$$

as follows: Choose a local orthonormal basis $\{e_i\}$ of E as above, let $\{e^i\}$ be the dual basis of E^* and set

$$\Psi(s)(e^i \otimes e_j) = \Psi(\lambda_i, \lambda_j) e^i \otimes e_j.$$

Again this is well-defined and smooth in s. The construction Ψ can be used to express the derivatives of the construction φ. More precisely, given a smooth function $\varphi : \mathbb{R} \to \mathbb{R}$, define a smooth function $d\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$d\varphi(\lambda_1, \lambda_2) = \begin{cases} \frac{\varphi(\lambda_1) - \varphi(\lambda_2)}{\lambda_1 - \lambda_2} & \text{if } \lambda_1 \neq \lambda_2, \\ \varphi'(\lambda_1) & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Then one shows that

$$\nabla''(\varphi(s)) = d\varphi(s)(\nabla'' s), \tag{4.6}$$

where $d\varphi$ is extended to form coefficients in the second variable in the obvious way.

Suppose again that $\varphi : \mathbb{R} \to \mathbb{R}$ and $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are smooth functions. Then, according to [Si88], there are extensions

$$\varphi : L^2_1(S) \longrightarrow L^2_1(S) \quad \text{and} \quad \Psi : L^2(S) \longrightarrow L^2(S(\mathrm{End}(E))),$$

which still satisfy relation (4.6).

With these constructions at hand, we can define Donaldson's functional as follows. Given two smooth Hermitian metrics h and k in E, write $h = ke^s$ with a smooth endomorphism s of E which is self-adjoint with respect to k and define

$$M(k, h) = \frac{\sqrt{-1}}{2\pi n} \int_X \mathrm{tr}(s \Lambda_g F_{h_0}) \omega^n + \frac{1}{2\pi n} \int_X \left\langle \Psi(s)(\nabla'' s), \nabla'' s \right\rangle_{h_0} \omega^n,$$

where Ψ is constructed as above from the smooth function

$$\Psi(\lambda_1, \lambda_2) = \begin{cases} \frac{e^{\lambda_2 - \lambda_1} - (\lambda_2 - \lambda_1) - 1}{(\lambda_2 - \lambda_1)^2} & \text{if } \lambda_1 \neq \lambda_2, \\ \frac{1}{2} & \text{if } \lambda_1 = \lambda_2. \end{cases}$$

Donaldson's functional satisfies the following simple formula.

Proposition 4.12 ([Si88], proposition 5.1)**.** *If h_1, h_2 and h_3 are smooth Hermitian metrics in E, then*

$$M(h_1, h_2) + M(h_2, h_3) = M(h_1, h_3).$$

If E is g-stable, we have the following main estimate involving M.

Proposition 4.13 ([Si88], proposition 5.3)**.** *Suppose that E is g-stable. Then for every real number B with $\sup_X |\Lambda_g F_{h_0}|_{h_0} \leqslant B$, there are positive constants C_1 and C_2 such that*

$$\sup_X |s|_{h_0} \leqslant C_1 + C_2 M(h_0, h_0 e^s)$$

for any smooth endomorphism s of E which is trace-free, self-adjoint with respect to h_0 and satisfies $\sup_X |\Lambda_g F_{h_0 e^s}|_{h_0} \leqslant B$.

Before we give a proof of this proposition, we first explain how it implies the convergence of $(h_t)_t$ to a g-Hermitian-Einstein metric in E as $t \to \infty$. Let $(h_t)_t$ be the solution of the evolution equation (4.1) defined for $0 \leqslant t < \infty$ from proposition 4.10. By proposition 4.6 (iii), we have $\left(\frac{d}{dt} + \Delta'\right) \hat{e} \leqslant 0$, so by the maximum principle for the heat operator $\frac{d}{dt} + \Delta'$, we see that $\sup_X \hat{e}$ is monotonically decreasing in t. In particular, there is a constant $B > 0$ such that

$$\sup_X |\Lambda_g F_{h_t}|_{h_0} \leqslant B \quad \text{uniformly in } 0 \leqslant t < \infty. \tag{4.7}$$

Lemma 4.14 ([Si88], lemma 7.1)**.** *The function $t \mapsto M(h_0, h_t)$ is continuously differentiable with*

$$\frac{d}{dt} M(h_0, h_t) = -\frac{1}{2\pi n} \int_X |\sqrt{-1}\Lambda_g F_{h_t} - \lambda \operatorname{id}_E|_{h_t}^2 \omega^n.$$

By proposition 4.12, this can be reduced to the case of $t = 0$, which is then proved using the evolution equation (4.1) and the uniform bound (4.7) of $|\Lambda_g F_{h_t}|_{h_0}$.

Now let E be g-stable. In this case, one can show the convergence of the solution $(h_t)_t$ to a g-Hermitian-Einstein metric as follows. Write $h_t = h_0 e^{s_t}$ as above with a family $(s_t)_t$ of smooth endomorphisms of E which are self-adjoint with respect to h_0. By remark 4.11, we have $\det(e^{s_t}) = 1$ for all t, which is equivalent to $\operatorname{tr}(s_t) = 0$. Moreover, by (4.7), we have $\sup_X |\Lambda_g F_{h_0 e^{s_t}}|_{h_0} \leqslant B$ uniformly in t. By proposition 4.13, there are positive constants C_1 and C_2 independent of t such that

$$\sup_X |s_t|_{h_0} \leqslant C_1 + C_2 M(h_0, h_t)$$

for all $0 \leqslant t < \infty$. Since $M(h_0, h_t)$ is decreasing in t by lemma 4.14, we have

$$\sup_X |s_t|_{h_0} \leqslant C \quad \text{uniformly in } 0 \leqslant t < \infty \tag{4.8}$$

with a positive constant C. Moreover, we see that $M(h_0, h_t)$ is bounded from below, so there is a sequence of times $(t_i)_i$ with $t_i \to \infty$ and, writing $h_i = h_{t_i}$,

$$\int_X |\sqrt{-1}\Lambda_g F_{h_i} - \lambda \operatorname{id}_E|_{h_i}^2 \omega^n \to 0 \quad \text{as } i \to \infty.$$

Since the norms $|\cdot|_{h_i}$ are bounded with respect to h_0 uniformly in i by (4.8), this means that $\sqrt{-1}\Lambda_g F_{h_i}$ converges to $\lambda \operatorname{id}_E$ in L^2. Then one shows that, after restricting to a subsequence of $(t_i)_i$, the sequence $(h_i)_i$ converges in $\mathcal{C}^0$ to a continuous Hermitian metric h_∞. Moreover, one can see that h_i converges to h_∞ weakly in the Sobolev space of functions whose weak derivatives up to the second order are locally L^p. It follows that $\sqrt{-1}\Lambda_g F_{h_\infty}$ is defined in the weak sense and satisfies $\sqrt{-1}\Lambda_g F_{h_\infty} = \lambda \operatorname{id}_E$. By an elliptic regularity argument similar to the one given above, this implies that h_∞ is a smooth Hermitian metric in E satisfying the g-Hermitian-Einstein condition.

We now turn to the proof of proposition 4.13. Our method is the one employed by Uhlenbeck and Yau in [UY86]: Under the assumption that the required estimate does not hold, we produce a coherent subsheaf of $\mathcal{E} = \mathcal{O}_X(E)$ contradicting the stability. This subsheaf will first be obtained as a so-called weakly holomorphic subbundle, which is then shown to define a coherent subsheaf in section 4.3.

Definition 4.15 (Weakly holomorphic subbundle)**.** Let (E, h_0) be a Hermitian holomorphic vector bundle on a compact Kähler manifold (X, g). A *weakly holomorphic subbundle* of E is a section $\pi \in L^2_1(\operatorname{End}(E))$ lying in the Sobolev space of L^2 sections of $\operatorname{End}(E)$ with L^2 first-order weak derivatives and satisfying

$$\pi = \pi^* = \pi^2 \quad \text{and} \quad (\operatorname{id}_E - \pi) \circ \nabla'' \pi = 0, \tag{4.9}$$

where π^* denotes the adjoint of π with respect to h_0 and $\nabla''\pi$ is computed in the sense of currents using the $(0,1)$ part of the Chern connection of (E, h_0).

This notion is motivated as follows. If $\mathcal{F}$ is a coherent subsheaf of $\mathcal{E}$, it is torsion-free (as a coherent subsheaf of a torsion-free sheaf) and thus locally free outside an analytic subset of X of codimension $\geqslant 2$ (see, e. g., [Kb87], V.5). More precisely, there is an analytic subset $S \subset X$ of codimension $\geqslant 2$ and a holomorphic vector bundle F on $X \setminus S$ such that

$$\mathcal{F}|_{X\setminus S} = \mathcal{O}(F).$$

Then F is a subbundle of $E|_{X\setminus S}$ and there is an orthogonal projection $\pi : E|_{X\setminus S} \to F$ with respect to h_0. This can be seen as a smooth section of $\operatorname{End}(E)$ over $X \setminus S$ satisfying the conditions (4.9). The second condition means that the holomorphic structure of F is the restriction of the holomorphic structure of $E|_{X\setminus S}$ to F. One can show that, in particular, π belongs to the space $L^2_1(\operatorname{End}(E))$ and thus it is a weakly holomorphic subbundle of E. Moreover, one can express the g-degree of $\mathcal{F}$ in terms of π.

Lemma 4.16 (Chern-Weil formula)**.** *In the above situation, we have*

$$\deg_g(\mathcal{F}) = \frac{\sqrt{-1}}{2\pi n} \int_X \operatorname{tr}(\pi \Lambda_g F_{h_0}) \omega^n - \frac{1}{2\pi n} \int_X |\nabla''\pi|^2_{h_0} \omega^n.$$

Since the right-hand side is well-defined even if we only require π to be an L^2_1 section instead of a $\mathcal{C}^\infty$ section, the following definition makes sense.

Definition 4.17. Let π be a weakly holomorphic subbundle of (E, h_0). Then the *g-degree* of π is defined as

$$\deg_g(\pi) = \frac{\sqrt{-1}}{2\pi n} \int_X \operatorname{tr}(\pi \Lambda_g F_{h_0}) \omega^n - \frac{1}{2\pi n} \int_X |\nabla''\pi|^2_{h_0} \omega^n.$$

The Chern-Weil formula makes sure that if π is the projection onto a coherent subsheaf of $\mathcal{E}$, this coincides with our previous definition of the g-degree of such a subsheaf.

For the proof of the estimate in proposition 4.13, first one shows that from $\sup_X |\Lambda_g F_{h_0}|_{h_0} \leqslant B$ and $\sup_X |\Lambda_g F_{h_0 e^s}|_{h_0} \leqslant B$, it follows that there are positive constants C_1 and C_2 such that

$$\sup_X |s|_{h_0} \leqslant C_1 + C_2 ||s||_{L^1}$$

for the sections s considered in the proposition. Now suppose that the required estimate does not hold. By choosing a sequence of constants $(C_i)_i$ with $C_i \to \infty$, one sees that there is a sequence $(s_i)_i$ of sections with the properties mentioned in the proposition which satisfies

$$||s_i||_{L^1} \to \infty \quad \text{and} \quad ||s_i||_{L^1} \geqslant C_i M(h_0, h_0 e^{s_i}).$$

Set $l_i = ||s_i||_{L^1}$ and $u_i = l_i^{-1} s_i$. Then we have $||u_i||_{L^1} = 1$ and $\sup_X |u_i|_{h_0} \leqslant C$ for all i with a positive constant C. We quote some technical lemmas from [Si88].

Lemma 4.18 ([Si88], lemma 5.4)**.** *After going to a subsequence, u_i converges to some u_∞ weakly in $L^2_1(S)$. The limit u_∞ is non-trivial. If $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a positive smooth function such that $\Phi(\lambda_1, \lambda_2) \leqslant (\lambda_1 - \lambda_2)^{-1}$ whenever $\lambda_1 > \lambda_2$, then*

$$\frac{\sqrt{-1}}{2\pi n} \int_X \operatorname{tr}(u_\infty \Lambda_g F_{h_0}) \omega^n + \frac{1}{2\pi n} \int_X \left\langle \Phi(u_\infty)(\nabla'' u_\infty), \nabla'' u_\infty \right\rangle_{h_0} \omega^n \leqslant 0.$$

Lemma 4.19 ([Si88], lemma 5.5)**.** *The eigenvalues of u_∞ are constant, i. e. there are $\lambda_1, \ldots, \lambda_r$, $r = \operatorname{rank}(E)$, which are the eigenvalues of $u_\infty(p)$ for almost all $p \in X$. The λ_i are not all equal.*

A consequence of this is that if $\varphi : \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are smooth functions, $\varphi(u_\infty)$ and $\Phi(u_\infty)$ depend only on $\varphi(\lambda_i)$ and $\Phi(\lambda_i, \lambda_j)$ for $1 \leqslant i, j \leqslant r$, respectively. Moreover, we have the following lemma.

Lemma 4.20 ([Si88], lemma 5.6)**.** *If $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a smooth function such that $\Phi(\lambda_i, \lambda_j) = 0$ whenever $\lambda_i > \lambda_j$, $1 \leqslant i, j \leqslant r$, then*

$$\Phi(u_\infty)(\nabla'' u_\infty) = 0.$$

We can now use u_∞ to construct a flag of weakly holomorphic subbundles of E as follows. Let $\{\gamma\}$ be the set of intervals between the eigenvalues $\lambda_1, \ldots, \lambda_r$ of u_∞. Since by lemma 4.19, the eigenvalues are not all equal, there are at least 1 and at most $r-1$ of these intervals. For each γ, choose a smooth function $p_\gamma : \mathbb{R} \to \mathbb{R}$ with $p_\gamma(\lambda_i) = 1$ for all λ_i less than γ and $p_\gamma(\lambda_i) = 0$ for all λ_i greater than γ. Setting $\pi_\gamma = p_\gamma(u_\infty)$, we obtain well-defined sections lying in $L^2_1(S)$. In fact, these are weakly holomorphic subbundles of E. Indeed, we know that $\pi_\gamma = \pi_\gamma^*$ and, since $p_\gamma^2 - p_\gamma$ vanishes at $\lambda_1, \ldots, \lambda_r$, we also have $\pi_\gamma = \pi_\gamma^2$. Now set

$$\Phi_\gamma(y_1, y_2) = (1 - p_\gamma)(y_2) \cdot dp_\gamma(y_1, y_2).$$

Then one sees that, on the one hand,

$$(\mathrm{id}_E - \pi_\gamma) \circ \nabla'' \pi_\gamma = \Phi_\gamma(u_\infty)(\nabla'' u_\infty).$$

On the other hand, $\Phi_\gamma(\lambda_i, \lambda_j) = 0$ if $\lambda_i > \lambda_j$. Thus, by lemma 4.20, we have $(\mathrm{id}_E - \pi_\gamma) \circ \nabla'' \pi_\gamma = 0$ and so π_γ is a weakly holomorphic subbundle of E.

Lemma 4.21 ([Si88], lemma 5.7)**.** *For at least one γ, the weakly holomorphic subbundle π_γ formally contradicts the g-stability of E in the sense that*

$$\frac{\deg_g(\pi_\gamma)}{\mathrm{tr}(\pi_\gamma)} \geqslant \frac{\deg_g(E)}{\mathrm{rank}(E)}.$$

Proof. First we remark that $\mathrm{tr}(\pi_\gamma)$ is between 0 and $\mathrm{rank}(E)$ because γ is an interval between two eigenvalues of u_∞. Now let a be the maximal eigenvalue of u_∞ and denote by a_γ the width of the interval γ. Then we have

$$\lambda_i = a - \sum_\gamma a_\gamma p_\gamma(\lambda_i)$$

for all $1 \leqslant i \leqslant r$ and thus

$$u_\infty = a\,\mathrm{id}_E - \sum_\gamma a_\gamma \pi_\gamma. \tag{4.10}$$

Consider the combination of degrees

$$\begin{aligned} W &= a \deg_g(E) - \sum_\gamma a_\gamma \deg_g(\pi_\gamma) \\ &= \frac{\sqrt{-1}}{2\pi n} \int_X \mathrm{tr}(u_\infty \Lambda_g F_{h_0}) \omega^n + \frac{1}{2\pi n} \int_X \sum_\gamma |\nabla'' \pi_\gamma|^2_{h_0} \omega^n. \end{aligned}$$

From (4.6), we know that $\nabla'' \pi_\gamma = dp_\gamma(u_\infty)(\nabla'' u_\infty)$. Moreover, the endomorphism $dp_\gamma(u_\infty)$ of $\mathrm{End}(E)$ is self-adjoint with respect to the Hermitian metric in $\mathrm{End}(E)$ induced by h_0. Hence we have

$$W = \frac{\sqrt{-1}}{2\pi n} \int_X \mathrm{tr}(u_\infty \Lambda_g F_{h_0}) \omega^n + \frac{1}{2\pi n} \int_X \Big\langle \sum_\gamma a_\gamma (dp_\gamma)^2(u_\infty)(\nabla'' u_\infty), \nabla'' u_\infty \Big\rangle_{h_0} \omega^n.$$

For $1 \leqslant i, j \leqslant r$ with $\lambda_i > \lambda_j$, we have

$$(dp_\gamma)^2(\lambda_i, \lambda_j) = \begin{cases} (\lambda_i - \lambda_j)^{-2} & \text{if } \gamma \text{ is between } \lambda_j \text{ and } \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

Since the sum of the a_γ for γ between λ_j and λ_i equals $\lambda_i - \lambda_j$, it follows that

$$\sum_\gamma a_\gamma (dp_\gamma)^2(\lambda_i, \lambda_j) = (\lambda_i - \lambda_j)^{-1}.$$

Lemma 4.18 then yields $W \leqslant 0$. On the other hand, u_∞ is trace-free as the limit of the s_i and by taking the trace of (4.10), we see that

$$a \operatorname{rank}(E) - \sum_\gamma a_\gamma \operatorname{tr}(\pi_\gamma) = \operatorname{tr}(u_\infty) = 0.$$

This implies that we must have

$$\frac{\deg_g(\pi_\gamma)}{\operatorname{tr}(\pi_\gamma)} \geqslant \frac{\deg_g(E)}{\operatorname{rank}(E)}$$

for at least one γ. □

The proof of proposition 4.13 is done if we can show that the weakly holomorphic subbundle $\pi = \pi_\gamma$ of (E, h_0) obtained in lemma 4.21 defines a coherent subsheaf $\mathcal{F}$ of $\mathcal{E} = \mathcal{O}_X(E)$. Indeed, we then have $\deg_g(\mathcal{F}) = \deg_g(\pi)$ by the Chern-Weil formula (lemma 4.16) and by lemma 4.21, $\mathcal{F}$ contradicts the g-stability of E. The existence of such a coherent subsheaf will be shown in the following section.

4.3 Regularity of weakly holomorphic subbundles

In this section, we show that a weakly holomorphic subbundle π of a Hermitian holomorphic vector bundle (E, h) on a compact Kähler manifold defines a coherent subsheaf $\mathcal{F}$ of $\mathcal{E} = \mathcal{O}_X(E)$. Note that when applying Simpson's construction described in the previous section to our framed situation, we obtain a weakly holomorphic subbundle satisfying the L_1^2 condition with respect to the Poincaré metric. This, however, by proposition 2.24 implies the L_1^2 condition in the ordinary sense, i. e. with respect to a smooth Kähler metric on the compact manifold X. Consequently, in order to get a coherent subsheaf contradicting the stability in the framed sense, it suffices to prove the regularity of weakly holomorphic subbundles in the compact situation.

This was first done by Uhlenbeck and Yau, who gave a very technical argument in their original work on the Kobayashi-Hitchin correspondence on compact Kähler manifolds, cf. [UY86] and [UY89]. Here, we would like to review an alternative proof based on the theory of currents, which was later given by Popovici in [Po05]. The theorem can be stated as follows.

Theorem 4.22. *Let (E, h) be a Hermitian holomorphic vector bundle on a compact Kähler manifold (X, g) and let π be a weakly holomorphic subbundle of (E, h), i. e. $\pi \in L_1^2(X, \operatorname{End}(E))$ such that*

$$\pi = \pi^* = \pi^2 \quad \text{and} \quad (\operatorname{id}_E - \pi) \circ \nabla'' \pi = 0. \tag{4.11}$$

Then there is a coherent subsheaf $\mathcal{F}$ of $\mathcal{E} = \mathcal{O}_X(E)$ and an analytic subset $S \subset X$ of codimension $\geqslant 2$ such that

(i) $\pi|_{X \setminus S} \in \mathcal{C}^\infty(X \setminus S, \operatorname{End}(E))$,

(ii) $\pi = \pi^* = \pi^2$ *and* $(\operatorname{id}_E - \pi) \circ \nabla''\pi = 0$ *on* $X \setminus S$,

(iii) $\mathcal{F}|_{X \setminus S} = \pi|_{X \setminus S}(E|_{X \setminus S}) \hookrightarrow E|_{X \setminus S}$ *is a holomorphic subbundle of* $E|_{X \setminus S}$.

Note that the first condition in (4.11) implies that we even have

$$\pi \in L^2_1(X, \operatorname{End}(E)) \cap L^\infty(X, \operatorname{End}(E)).$$

Before beginning the proof, we have to make some preliminary remarks. There is a subbundle $F = \operatorname{Im} \pi$ of E which is defined almost everywhere as an L^2 bundle, i. e. its fibre F_x is defined as $\operatorname{Im} \pi_x$ for almost all points $x \in X$ and the transition functions are measurable. In the same way, the quotient bundle $Q = E/F$ is defined almost everywhere as an L^2 bundle. There are L^2 currents β and β^* of bidegree $(1, 0)$ with values in $\operatorname{Hom}(F, Q)$ and of bidegree $(0, 1)$ with values in $\operatorname{Hom}(Q, F)$, respectively, which are uniquely determined by the equations

$$\nabla'\pi = \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix}, \quad \nabla''\pi = \begin{pmatrix} 0 & \beta^* \\ 0 & 0 \end{pmatrix}$$

at almost all points $x \in X$ with respect to the decomposition $E_x \simeq F_x \oplus Q_x$, where ∇' and ∇'' are the components of the covariant derivative with respect to h and $\nabla'\pi$ and $\nabla''\pi$ are computed in the sense of currents. If π happens to be $\mathcal{C}^\infty$, the current β is the second fundamental form of the exact sequence

$$0 \longrightarrow F \xrightarrow{j} E \xrightarrow{g} Q \longrightarrow 0,$$

where j is the inclusion and g is the projection. Details on exact sequences of Hermitian holomorphic vector bundles can be found in [Gr69] or chapter V, §14 of Demailly's book [De09].

The idea of the proof of theorem 4.22 is as follows. We have to show that the L^2 bundle F is holomorphic outside an analytic subset of codimension $\geqslant 2$. Using the fact that meromorphic maps are holomorphic outside an analytic subset of codimension $\geqslant 2$, we see that it suffices to construct local meromorphic sections of F which span F locally. This is accomplished by constructing local holomorphic sections of $F \otimes \det Q$ which span $F \otimes \det Q$ locally, as well as a local holomorphic section of $\det Q$ which spans $\det Q$ locally. Dividing these holomorphic sections then yields the desired meromorphic sections of F.

Writing $\sqrt{-1}\Theta(E) = \sqrt{-1}\Theta_h(E)$ for the curvature form of the Chern connection of a Hermitian holomorphic vector bundle (E, h), in the $\mathcal{C}^\infty$ case we know that

$$\begin{aligned} \sqrt{-1}\Theta(\det Q) &= \operatorname{tr}(\sqrt{-1}\nabla'\pi \wedge \nabla''\pi + \sqrt{-1}\Theta(E)|_Q) \\ &= \operatorname{tr}(\sqrt{-1}\beta \wedge \beta^* + (\operatorname{id}_E - \pi) \circ \sqrt{-1}\Theta(E) \circ (\operatorname{id}_E - \pi)), \end{aligned}$$

where $\det Q$ is endowed with the Hermitian metric induced by h (see, e. g., [Gr69]). Although in the situation of theorem 4.22, the right-hand side cannot be seen as the curvature form of $\det Q$ in advance, it does exist as an L^1 current of bidegree $(1, 1)$. In particular, its restriction to almost every complex line L contained in a coordinate neighbourhood of X defines a d-closed current since it exists as an L^1 current by Fubini's theorem and it is d-closed as a current of maximal bidegree on L. This current will play a role in the construction of a local holomorphic section of $\det Q$.

Proving theorem 4.22 is a local problem. According to [Po05], one can assume that locally, the curvature $\sqrt{-1}\Theta(E)$ is positive for otherwise one can apply a suitable conformal change to the Hermitian metric in E. Then one has the following lemma.

Lemma 4.23 ([Po05], corollary 0.2.3)**.** *The current*

$$\operatorname{tr}(\sqrt{-1}\beta \wedge \beta^* + (\operatorname{id}_E - \pi) \circ \sqrt{-1}\Theta(E) \circ (\operatorname{id}_E - \pi))$$

of bidegree $(1,1)$ *admits a local subharmonic potential on almost every complex line contained in a coordinate neighbourhood of* X*. This means that for every point* $x \in X$ *and almost every complex line* L *with respect to a local coordinate system in a neighbourhood of* x*, there is a subharmonic function* φ_L *such that*

$$\sqrt{-1}\partial\bar{\partial}\varphi_L = \operatorname{tr}(\sqrt{-1}\beta \wedge \beta^* + (\operatorname{id}_E - \pi) \circ \sqrt{-1}\Theta(E) \circ (\operatorname{id}_E - \pi))$$

locally on L*.*

The main difficulty in the proof of theorem 4.22 arises from the insufficient regularity of π. One has to be careful when forming wedge products of currents since their coefficients are distributions and thus cannot be multiplied in general. We will not give the details on this here. Instead, we review the main steps of Popovici's proof and refer the reader to [Po05] for a more thorough treatment of the regularity question.

We begin with the reduction of the problem to the case that the Hermitian holomorphic vector bundle (E,h) is flat, i. e. that its curvature vanishes identically. We have the following elementary result from linear algebra.

Lemma 4.24 ([Po05], lemma 0.3.1)**.** *Let* E *be a finite-dimensional complex vector space and* F *a vector subspace of* E*. Consider two Hermitian metrics* h *and* h_0 *in* E *and let* π *and* π_0 *be the respective orthogonal projections of* E *onto* F*. If*

$$E = F \oplus F_h^\perp \quad \textit{and} \quad E = F \oplus F_{h_0}^\perp$$

are the respective orthogonal decompositions of E*, there is an automorphism* $v : E \to E$ *such that*

$$v(F) = F, \quad v(F_h^\perp) = F_{h_0}^\perp \quad \textit{and} \quad h(s,t) = h_0(v(s), v(t)) \textit{ for all } s,t \in E.$$

Moreover, for every such v *the projections* π *and* π_0 *are related by* $\pi_0 = v \circ \pi \circ v^{-1}$*.*

Corollary 4.25 ([Po05], corollary 0.3.2)**.** *Let* (E,h) *be a Hermitian holomorphic vector bundle of rank* r *on a compact Kähler manifold* (X,g) *and let* π *be a weakly holomorphic subbundle of* (E,h)*. Set* $F = \operatorname{Im}\pi$*. Let* U *be a trivializing open set for* E *and let* h_0 *be the trivial flat metric on* $E|_U \simeq U \times \mathbb{C}^r$*. Let* $\pi_0 \in L_1^2(U, \operatorname{End}(E))$ *be the orthogonal projection of* $E|_U$ *onto* $F|_U$ *with respect to* h_0*. Then there is some* $v \in \mathcal{C}^\infty(U, \operatorname{End}(E))$ *such that*

$$(\operatorname{id}_E - \pi) \circ v \circ \pi = 0 \quad \textit{and} \quad \pi_0 \circ v \circ (\operatorname{id}_E - \pi) = 0$$

almost everywhere on U *and*

$$h(s,t) = h_0(v(s), v(t))$$

for all sections s *and* t *of* E *over* U*. Furthermore,* $\pi_0 = v \circ \pi \circ v^{-1}$ *almost everywhere on* U*.*

Lemma 4.26 ([Po05], lemma 0.3.3)**.** *Under the hypotheses of corollary 4.25, we have*

$$(\mathrm{id}_E - \pi_0) \circ \nabla'' \pi_0 = 0$$

almost everywhere on U.

These facts enable us to locally reduce the problem to the case of a flat vector bundle: Since the problem is local, by replacing locally the metric h with the trivial flat metric h_0, we can assume that $\sqrt{-1}\Theta_h(E) = 0$ on the trivializing open set U.

We would like to show that the L^2 bundle $F \otimes \det Q$ is locally generated by its local holomorphic sections. Since the projection of E onto F is not holomorphic in general, we show that $F \otimes \det Q$ can also be realized as the image of a holomorphic projection from $\Lambda^{q+1}E$, the $(q+1)$-th exterior power of E. In the $\mathcal{C}^\infty$ situation, we have the following lemma.

Lemma 4.27 ([Po05], lemma 0.3.4)**.** *Let (E, h) be a flat Hermitian holomorphic vector bundle of rank r on a compact Kähler manifold (X, g) and let $\pi \in \mathcal{C}^\infty(X, \mathrm{End}(E))$ be such that $\pi = \pi^* = \pi^2$ and $(\mathrm{id}_E - \pi) \circ \nabla'' \pi = 0$. Denote by p the rank of π and let $q = r - p$. Consider the holomorphic subbundle $F = \mathrm{Im}\,\pi$ of E and the exact sequence of holomorphic vector bundles*

$$0 \longrightarrow F \xrightarrow{j} E \xrightarrow{g} Q \longrightarrow 0,$$

where j is the inclusion and g is the projection onto the quotient bundle $Q = E/F$. Then there is a holomorphic bundle morphism

$$\sigma : \Lambda^{q+1} E \otimes \Lambda^q Q^* \longrightarrow E$$

whose image is F. More precisely, if $(e_1, \dots, e_r)$ is a local orthonormal holomorphic frame of E and $K = (k_1 < \cdots < k_q)$ is a multiindex, consider the local holomorphic section of $\det Q = \Lambda^q Q$ defined as

$$v_K = (\mathrm{id}_E - \pi)(e_{k_1}) \wedge \cdots \wedge (\mathrm{id}_E - \pi)(e_{k_q}) = \sum_J D_{JK} e_J,$$

where the sum is taken over all multiindices $J = (j_1 < \cdots < j_q)$, D_{JK} is the minor corresponding to the rows $J = (j_1 < \cdots < j_q)$ and the columns $K = (k_1 < \cdots < k_q)$ of the matrix representing $\mathrm{id}_E - \pi$ in the frame $(e_1, \dots, e_r)$ and $e_J := e_{j_1} \wedge \cdots \wedge e_{j_q}$. Associate with v_K the local holomorphic section of $\Lambda^q Q^*$ defined as

$$v_K^{-1} = \frac{\sum_J \bar{D}_{JK} e_J^*}{\sum_J |D_{JK}|^2}.$$

Then for all multiindices $I = (i_1 < \cdots < i_{q+1})$ and $K = (k_1 < \cdots < k_q)$, the morphism σ is locally defined by

$$\sigma(e_I \otimes v_K^{-1}) = \sum_{\ell=1}^{q+1} (-1)^\ell \frac{\sum_J \bar{D}_{JK} e_J^*(e_{I \setminus \{i_\ell\}})}{\sum_J |D_{JK}|^2} e_{i_\ell}. \tag{4.12}$$

In particular, by tensoring σ on the right by $\det Q = \Lambda^q Q$, one obtains a holomorphic bundle morphism

$$u : \Lambda^{q+1} E \longrightarrow E \otimes \det Q$$

whose image is $F \otimes \det Q$. The morphisms σ and u are locally related by

$$\sigma(e_I \otimes v_K^{-1}) = \frac{u(e_I)}{v_K},$$

where the division is performed in the line bundle $\det Q$.

Returning to our situation where π is only L_1^2, one first shows that the rank of π equals a constant p almost everywhere on X. Denote by r the rank of E and let $q = r - p$ as above. Fix a local holomorphic frame $(e_1, \dots, e_r)$ of E on an open set U. For a fixed point $x_0 \in U$ we can assume that $e_1(x_0), \dots, e_q(x_0)$ is a basis of Q_{x_0} and $e_{q+1}(x_0), \dots e_r(x_0)$ is a basis of F_{x_0}. Then we have

$$(\mathrm{id}_E - \pi)(e_j(x_0)) = \begin{cases} e_j(x_0) & \text{if } 1 \leqslant j \leqslant q, \\ 0 & \text{if } q+1 \leqslant j \leqslant r. \end{cases}$$

For every matrix $A = (a_{kj})_{1 \leqslant k \leqslant q, 1 \leqslant j \leqslant r} \in \mathbb{C}^{q \times r}$, where $(a_{kj})_{1 \leqslant k \leqslant q, 1 \leqslant j \leqslant q}$ is the $(q \times q)$ identity matrix, define local holomorphic sections of E over U by

$$s_k = \sum_{j=1}^{r} a_{kj} e_j \quad \text{for } k = 1, \dots, q$$

and a local section of $\Lambda^q E$ over U by

$$\tau_A = (\mathrm{id}_E - \pi)(s_1) \wedge \cdots \wedge (\mathrm{id}_E - \pi)(s_q) \in L_1^2(U, \Lambda^q E) \cap L^\infty(U, \Lambda^q E).$$

This is a linear combination of the sections v_K of $\det Q$ considered in lemma 4.27. Once $\det Q$ is realized as a holomorphic vector bundle, τ_A will be a local holomorphic section of $\det Q$. Moreover, we have $\tau_A(x_0) = e_1(x_0) \wedge \cdots \wedge e_q(x_0)$ and therefore $|\tau_A(x_0)| \neq 0$. Imitating formula (4.12) of lemma 4.27, we obtain the following statement.

Corollary 4.28 ([Po05], corollary 0.3.5)**.** *Let (E, h) be a Hermitian holomorphic vector bundle of rank r on a compact Kähler manifold (X, g) and let π be a weakly holomorphic subbundle of (E, h). Using the same notation as in lemma 4.27, consider the local bundle morphism $v : \Lambda^{q+1} E|_U \to E|_U$ defined by*

$$v : e_I = e_{i_1} \wedge \cdots \wedge e_{i_{q+1}} \longmapsto \sigma(e_I \otimes v_K^{-1}) = \frac{u(e_I)}{\tau_A} \tag{4.13}$$

for all multiindices $I = (i_1 < \cdots < i_{q+1})$. Then its image is $\mathrm{Im}\, v = \mathrm{Im}\, \pi|_U$.

Having realized $F = \mathrm{Im}\, \pi$ locally as the image of a projection v from $\Lambda^{q+1} E$, in order to see that F is a holomorphic subbundle of E outside an analytic subset of X of codimension $\geqslant 2$, it would suffice to show that $\nabla''(v(e_I)) = 0$ holds in the sense of currents for every multiindex $I = (i_1 < \cdots < i_{q+1})$. However, although the equation $\nabla''(v(e_I)) = 0$ is formally true, it is not well-defined since $1/\tau_A$ does not necessarily define a distribution because the coefficients of τ_A are L_1^2 functions and hence their inverses are only measurable. Popovici overcomes this difficulty by proving the following lemma, which is the main technical argument in [Po05].

Lemma 4.29 ([Po05], lemma 0.3.6). *For all $\delta > 0$, we have the following inequality of $(1,1)$ forms.*

$$\sqrt{-1}\partial\bar{\partial}\log(|\tau_A|^2 + \delta^2) \geqslant -\frac{|\tau_A|^2}{|\tau_A|^2 + \delta^2}\operatorname{tr}(\sqrt{-1}\beta \wedge \beta^*).$$

We can now prove that $F = \operatorname{Im}\pi$ defines a holomorphic subbundle of E almost everywhere on almost every complex line in a local coordinate neighbourhood of X. Fix a point $x_0 \in X$ and a coordinate neighbourhood U of x_0 such that E is trivial on U. Let L be a complex line with respect to the coordinate system of U such that the restriction of $\operatorname{tr}(\sqrt{-1}\beta \wedge \beta^*)$ to L is a well-defined $(1,1)$ current. This is true for almost every choice of L. From corollary 4.23 and the assumption that the curvature of E vanishes identically, we know that there is a subharmonic potential $\varphi = \varphi_L$ on $U \cap L$ such that

$$\sqrt{-1}\partial\bar{\partial}\varphi = \operatorname{tr}(\sqrt{-1}\beta \wedge \beta^*)|_{U\cap L}.$$

By lemma 4.29 and the positivity of $\sqrt{-1}\partial\bar{\partial}\varphi$, it follows that

$$\sqrt{-1}\partial\bar{\partial}\log(|\tau_A|^2 + \delta^2) \geqslant -\frac{|\tau_A|^2}{|\tau_A|^2 + \delta^2}\sqrt{-1}\partial\bar{\partial}\varphi \geqslant -\sqrt{-1}\partial\bar{\partial}\varphi \quad \text{for all } \delta > 0$$

on $U\cap L$. This implies that the function $\log(|\tau_A|^2 e^{\varphi} + \delta^2 e^{\varphi})$ is subharmonic on $U\cap L$ for all $\delta > 0$. Thus, the function $\log(|\tau_A|^2 e^{\varphi})$ is subharmonic on $U \cap L$ as a decreasing limit of subharmonic functions. In particular, the function

$$\psi = \log(|\tau_A| e^{\frac{\varphi}{2}})$$

is subharmonic and not identically $-\infty$ on $U \cap L$. We can then choose a holomorphic function $f : U \cap L \to \mathbb{C}$ which is not identically zero and satisfies

$$\int_{U\cap L} |f|^2 e^{-2\psi} d\lambda < \infty,$$

where $d\lambda$ denotes the Lebesgue measure on $U \cap L$. Consequently, the function

$$|f|e^{-\psi} = \frac{|f|}{|\tau_A|e^{\frac{\varphi}{2}}}$$

is L^2 on $U \cap L$. In particular, $f/(\tau_A e^{\frac{\varphi}{2}})$ is an L^2 section of $(\det Q)^{-1}$ over $U \cap L$. Moreover, we know that $e^{\frac{\varphi}{2}}$ is subharmonic and L^∞ on $U \cap L$, so we finally obtain that

$$\frac{f}{\tau_A} = e^{\frac{\varphi}{2}} \frac{f}{\tau_A e^{\frac{\varphi}{2}}}$$

is an L^2 section of $(\det Q)^{-1}$ over $U \cap L$. In particular, $\nabla''(f/\tau_A)$ is well-defined in the sense of currents and we have

$$\nabla''\left(\frac{f}{\tau_A}\right) = 0.$$

The bundle morphism v defined by (4.13) can be redefined on $U \cap L$ as

$$v : \Lambda^{q+1} E \longrightarrow E, \quad e_I \longmapsto \frac{f u(e_I)}{\tau_A}$$

for all multiindices $I = (i_1 < \cdots < i_{q+1})$. We know that $u(e_I)$ is a ∇''-closed L^2 section of $E \otimes \det Q$ over $U \cap L$. This implies that

$$\frac{f u(e_I)}{\tau_A} \in L^1(U \cap L, E) \quad \text{and} \quad \nabla'' \left(\frac{f u(e_I)}{\tau_A} \right) = \nabla'' \left(\frac{f}{\tau_A} \right) u(e_I) + \frac{f}{\tau_A} \nabla'' u(e_I) = 0$$

for all I. Hence the L^2 bundle $F = \operatorname{Im} \pi = \operatorname{Im} v$ is locally generated by its local meromorphic sections on almost every complex line with respect to a local coordinate system.

Finally, we have to get rid of the restriction to complex lines. If U is a trivializing open set for E, r is the rank of E, p is the rank of π almost everywhere and $\operatorname{Gr}(p, r)$ denotes the Grassmannian of p-dimensional vector subspaces of $\mathbb{C}^r$, there is a map

$$\Phi : U \longrightarrow \operatorname{Gr}(p, r),$$

where for almost every $x \in U$, $\Phi(x)$ is the p-dimensional subspace of $\mathbb{C}^r$ corresponding to the p-dimensional subspace $\operatorname{Im} \pi_x$ of E_x via the given trivialization. What we have shown so far means that the components of Φ have almost everywhere meromorphic restrictions to almost all complex lines L. We can thus apply the following Hartogs-type theorem due to Shiffman.

Theorem 4.30 ([Sh86], corollary 2)**.** *Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in $\mathbb{C}$ and let $f : \Delta^n \to \mathbb{C}$ be a measurable function such that for all $1 \leqslant j \leqslant n$ and almost all $(z_1, \ldots, \widehat{z_j}, \ldots, z_n)$, the map $\Delta \ni z_j \mapsto f(z_1, \ldots, z_n)$ is equal almost everywhere to a meromorphic function on Δ. Then f is equal almost everywhere to a meromorphic function.*

Our map Φ satisfies the hypotheses of theorem 4.30 and even stronger ones: Its components are L^2_1 and meromorphic almost everywhere along almost all complex lines. Theorem 4.30 then implies that the components of Φ and hence Φ itself are meromorphic almost everywhere. Since every meromorphic map is holomorphic outside an analytic subset of codimension $\geqslant 2$, it follows that $F = \operatorname{Im} \pi$ is a holomorphic subbundle of E outside such an exceptional set. This completes the proof of theorem 4.22.

5 Further aspects

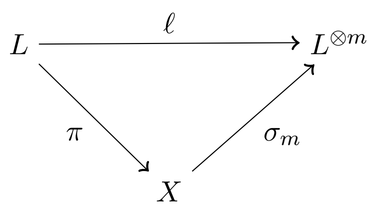
In this final chapter, we would like to discuss some additional ideas based on the work [TY87] of Tian and Yau regarding the theory developed in the previous chapters. These ideas have not yet been fully elaborated and may thus serve as a basis for further research in this area.

As before, let (X, D) be a canonically polarized framed manifold. For this chapter, besides assuming the ampleness of $K_X \otimes [D]$, we require the divisor D to be ample as well. First we explain the construction of *cyclic coverings* of X as it is done, for instance, by Schumacher and Tsuji in [ST04], section 4.

Since D is ample, there is a number m_0 such that for every $m \geqslant m_0$, the effective divisor mD is very ample. In particular, the associated linear system $|mD|$ is base point free. By Bertini's theorem, we can then choose a smooth divisor $D_m \in |mD|$, i. e. such that D_m is linearly equivalent to mD. Moreover, the ampleness of $K_X \otimes [D]$ implies that

$$K_X \otimes [D]^{\otimes(m-1)}$$

is ample for every $m \geqslant 2$. In the terminology of [ST04], this means that for every $m \geqslant m_0$ (which we choose to be $\geqslant 2$), the framed manifold (X, D_m) is *m-framed*. In what follows, we always assume that $m \geqslant m_0$. Now let $L = [D]$ be the line bundle associated to the divisor D. Then we have $[D_m] = [mD] = L^{\otimes m}$. Consider the following diagram.



Here, $\ell : L \to L^{\otimes m}$ is the bundle morphism which, in a local trivialization $L|_U \simeq U \times \mathbb{C} \simeq L^{\otimes m}|_U$, sends an element $(p, \alpha) \in U \times \mathbb{C}$ to (p, α^m). Furthermore, $\pi : L \to X$ is the bundle projection and $\sigma_m : X \to L^{\otimes m}$ is a canonical section of $L^{\otimes m} = [D_m]$, i. e. with vanishing locus $D_m = V(\sigma_m)$. Let $X_m = V(\ell - \sigma_m \circ \pi)$ be the analytic subvariety of the bundle space of L defined as the zero locus of $\ell - \sigma_m \circ \pi : L \to L^{\otimes m}$. Since D_m is smooth, this is a compact complex manifold. By setting $\pi_m = \pi|_{X_m} : X_m \to X$, one obtains a Galois covering of X with branch locus $D_m \subset X$. The group of covering transformations is isomorphic to $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$ and X is isomorphic to the quotient $X_m/\mathbb{Z}_m$.

This construction can be described locally as follows. For a point $p \in D_m$, choose an open neighbourhood $U \subset X$ of p such that there is a trivialization $L|_U \simeq U \times \mathbb{C}$ of L over U and a coordinate system $(w^1, \ldots, w^n)$ on U which is normal with respect to the smooth divisor D_m, i. e. such that in U, D_m is given by the equation $w^1 = 0$. Then we have local coordinates

$(w^1, \dots, w^n, \alpha)$ for the bundle space of L on the open subset $\pi^{-1}(U) \subset L$, where α is a bundle coordinate. In these coordinates, X_m is given by the equation $w^1 = \alpha^m$. Letting

$$z^1 = \alpha \quad \text{and} \quad z^i = w^i \quad \text{for } 2 \leqslant i \leqslant n,$$

we obtain local coordinates $(z^1, \dots, z^n)$ for X_m on $X_m \cap \pi^{-1}(U)$, where the point $(z^1, \dots, z^n)$ of X_m corresponds to the point $((z^1)^m, z^2, \dots, z^n, z^1)$ of L. An element $k+m\mathbb{Z} \in \mathbb{Z}_m$, $0 \leqslant k \leqslant m-1$, acts on X_m by sending $(z^1, \dots, z^n)$ to $(\zeta^k z^1, z^2, \dots, z^n)$, where $\zeta \in \mathbb{C}$ is a primitive m-th root of unity. The projection π_m sends $(z^1, \dots, z^n)$ to $(w^1, \dots, w^n) = ((z^1)^m, z^2, \dots, z^n)$.

We write $X' = X \setminus D$ and $X'_m = X_m \setminus \pi_m^{-1}(D)$. By [TY87], §3, for every m there is a Kähler-Einstein metric g_m on X' and a Kähler-Einstein metric g_{X_m} on X_m such that $\pi_m^* g_m = g_{X_m}|_{X'_m}$. The metric g_m is constructed as in section 2.3 by solving a complex Monge-Ampère equation starting with the background metric which is given by its fundamental form

$$\sqrt{-1}\partial\bar{\partial} \log \left(\frac{2\Omega}{m^{n+1} ||\sigma||^{2(1-1/m)} (1 - ||\sigma||^{2/m})^{n+1}} \right),$$

where Ω is a smooth volume form on X, σ is a canonical section of the line bundle $[D]$ and $||\cdot||$ is the norm induced by a smooth Hermitian metric in $[D]$. As a matter of fact, by [ST04], we know that the canonical line bundle of X_m is given by

$$K_{X_m} = \pi_m^* \left(K_X \otimes [D]^{\otimes(m-1)} \right)$$

and that it is ample. The Kähler-Einstein metric g_{X_m} on X_m is the one obtained from the ampleness of K_{X_m} by Yau's theorem 2.15. Denote the fundamental form of g_m by ω_m. Recall that $\omega_{X'}$ is the fundamental form of the Poincaré metric on X'. We have the following convergence result by Tian and Yau.

Theorem 5.1 ([TY87], proposition 3.1). *The sequence $(\omega_m)_m$ converges to $\omega_{X'}$ on X' in $\mathcal{C}^{2,\beta}$ for some $\beta \in (0,1)$ with respect to the quasi-coordinates. Moreover, we have the inequality $\omega_m^n \leqslant \omega_{X'}^n$ on X' for all m.*

The theorem is shown by using the monotonicity property

$$\omega_m^n \leqslant \omega_{m'}^n \quad \text{for } m \leqslant m',$$

which follows from Yau's Schwarz lemma 2.18, and the estimates from the complex Monge-Ampère equation.

Now let E be a holomorphic vector bundle on X which is stable in the framed sense with respect to (X, D). As before, denote by E' its restriction to X'. We are looking for Hermitian metrics in E' satisfying the Hermitian-Einstein condition with respect to the Kähler-Einstein metrics g_m. Since the metrics g_m are incomplete (in particular, they are not of Poincaré-type growth near the divisor D), the methods of the previous chapters cannot be applied to this situation. Here, however, we can use the cyclic coverings in order to obtain Hermitian-Einstein metrics as follows. Choose a smooth Hermitian metric h_0 in E as a background metric. Pulling back to X_m, we obtain a holomorphic vector bundle $\pi_m^* E$ on X_m with a smooth Hermitian metric

$\pi_m^* h_0$. In order to find the correct notion of stability in this context, we introduce the concept of *orbifold sheaves* and their stability, which is described, for instance, by Biswas in [Bs97b]. Orbifold sheaves were first introduced on Riemann surfaces under the name of *π-bundles* by Seshadri in [Se70]. Note that the action of $\mathbb{Z}_m$ on X_m is faithful and is thus given by an injective group homomorphism

$$\rho : \mathbb{Z}_m \longrightarrow \mathrm{Aut}(X_m)$$

from $\mathbb{Z}_m$ into the group of automorphisms of X_m.

Definition 5.2 (Orbifold structures)**.**

(i) An *orbifold sheaf* on X_m is a torsion-free coherent analytic sheaf $\mathcal{E}$ on X_m together with a lift of the action of $\mathbb{Z}_m$ to $\mathcal{E}$. This means that $\mathbb{Z}_m$ acts on the total space of stalks of $\mathcal{E}$, and the automorphism of the total space of stalks given by an element $g \in \mathbb{Z}_m$ is a sheaf isomorphism between $\mathcal{E}$ and $\rho(g)^*\mathcal{E}$.

(ii) A coherent subsheaf $\mathcal{F}$ of an orbifold sheaf $\mathcal{E}$ on X_m is called *$\mathbb{Z}_m$-saturated* if $\mathcal{F}$ is invariant under the action of $\mathbb{Z}_m$.

(iii) A locally free orbifold sheaf $\mathcal{E}$ is called an *orbifold bundle*.

Note that if $\mathcal{F}$ is a $\mathbb{Z}_m$-saturated subsheaf of an orbifold sheaf $\mathcal{E}$ on X_m, then $\mathcal{F}$ carries an induced orbifold sheaf structure.

Definition 5.3 (Orbifold (semi-)stability)**.** Let g be a Kähler metric on X_m. An orbifold sheaf $\mathcal{E}$ on X_m is said to be *g-orbifold semistable* if for every $\mathbb{Z}_m$-saturated subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rank}(\mathcal{F})$, the inequality

$$\mu_g(\mathcal{F}) \leqslant \mu_g(\mathcal{E})$$

holds. If, moreover, the strict inequality

$$\mu_g(\mathcal{F}) < \mu_g(\mathcal{E})$$

holds for every $\mathbb{Z}_m$-saturated subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \mathrm{rank}(\mathcal{F}) < \mathrm{rank}(\mathcal{E})$, we say that $\mathcal{E}$ is *g-orbifold stable.*

Returning to our situation, we have a Hermitian holomorphic vector bundle $(\pi_m^* E, \pi_m^* h_0)$ on X_m. Since it is the pull-back of a bundle on X, it can be regarded as an orbifold bundle on X_m in a canonical way. Then the pull-back by π_m induces a one-to-one correspondence between the coherent subsheaves $\mathcal{F}$ of $\mathcal{E} = \mathcal{O}_X(E)$ and the $\mathbb{Z}_m$-saturated subsheaves of the orbifold sheaf $\pi_m^*\mathcal{E}$. Since we have $\pi_m^* g_m = g_{X_m}|_{X'_m}$ and $\pi_m : X_m \to X$ is an m-sheeted covering, the g_m-degree of a coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ can be computed by

$$\deg_{g_m}(\mathcal{F}) = \int_{X'} c_1(\mathcal{F}) \wedge \omega_m^{n-1} = \frac{1}{m}\int_{X_m} c_1(\pi_m^*\mathcal{F}) \wedge \omega_{X_m}^{n-1} = \frac{1}{m}\deg_{g_{X_m}}(\pi_m^*\mathcal{F}), \tag{5.1}$$

where ω_{X_m} is the fundamental form of g_{X_m}. In particular, this degree is well-defined. By the convergence theorem 5.1, we see that

$$\deg_{g_m}(\mathcal{F}) \to \deg_{X'}(\mathcal{F}) = \deg_{(X,D)}(\mathcal{F}) \quad \text{for } m \to \infty. \tag{5.2}$$

Now suppose that E is stable in the framed sense. Any $\mathbb{Z}_m$-saturated subsheaf of the orbifold sheaf $\pi_m^*\mathcal{E}$ on X_m is of the form $\pi_m^*\mathcal{F}$ for a coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$. If

$$0 < \operatorname{rank}(\pi_m^*\mathcal{F}) < \operatorname{rank}(\pi_m^*\mathcal{E}),$$

we have $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$ and the framed stability of E implies

$$\frac{\deg_{(X,D)}(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} < \frac{\deg_{(X,D)}(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}.$$

By (5.2), it follows that

$$\frac{\deg_{g_m}(\mathcal{F})}{\operatorname{rank}(\mathcal{F})} < \frac{\deg_{g_m}(\mathcal{E})}{\operatorname{rank}(\mathcal{E})}$$

if m is sufficiently large. This, in turn, by (5.1) means that

$$\frac{\deg_{g_{X_m}}(\pi_m^*\mathcal{F})}{\operatorname{rank}(\pi_m^*\mathcal{F})} < \frac{\deg_{g_{X_m}}(\pi_m^*\mathcal{E})}{\operatorname{rank}(\pi_m^*\mathcal{E})}.$$

Consequently, the bundle π_m^*E on X_m is g_{X_m}-orbifold stable for large m. Now we can apply the methods described in chapter 4 to construct a g_{X_m}-Hermitian-Einstein metric in π_m^*E. In fact, since the Kähler metric g_{X_m}, the bundle π_m^*E and the background metric $\pi_m^*h_0$ are invariant under the action of $\mathbb{Z}_m$, the solution of the heat equation is also invariant for all times by the uniqueness statement of corollary 4.4. The destabilizing subsheaf of $\pi_m^*\mathcal{E}$ from section 4.3 is then $\mathbb{Z}_m$-saturated and the orbifold stability of π_m^*E implies the existence of a g_{X_m}-Hermitian-Einstein metric in π_m^*E. This metric is $\mathbb{Z}_m$-invariant as well and so we obtain a g_m-Hermitian-Einstein metric h_m in E' over X'.

To sum up, given a holomorphic vector bundle E on X which is stable in the framed sense with respect to (X, D), we have found a sequence $(h_m)_m$ of Hermitian metrics in E' such that h_m satisfies the Hermitian-Einstein condition with respect to the incomplete Kähler-Einstein metric g_m on X'. In view of theorem 5.1, one could now conjecture that this sequence converges in $\mathcal{C}^{2,\beta}$ to the previously constructed framed Hermitian-Einstein metric on X'. This is still an open problem.

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A Deutsche Zusammenfassung

Diese Arbeit liefert einen Beitrag zur algebraischen Geometrie unter Benutzung transzendenter Methoden. Die sogenannte *Kobayashi-Hitchin-Korrespondenz*, die seit den 80er Jahren des 20. Jahrhunderts bekannt ist, stellt einen Zusammenhang zwischen algebraischer Geometrie und Analysis her, indem der algebraisch-geometrische Begriff der *Stabilität* eines holomorphen Vektorbündels auf einer (im klassischen Fall) kompakten Kähler-Mannigfaltigkeit mit dem analytischen Begriff der Hermite-Einstein-Metrik in einem solchen Vektorbündel in Beziehung gesetzt wird.

Der hier betrachtete Stabilitätsbegriff wurde von Takemoto in [Ta72] eingeführt und ist auch als *slope-Stabilität* oder *Mumford-Takemoto-Stabilität* bekannt. Ist eine kompakte Kähler-Mannigfaltigkeit (X, g) der komplexen Dimension n gegeben, so kann man die Stabilität wie folgt formulieren. Der *g-Grad* einer torsionsfreien kohärenten analytischen Garbe $\mathcal{F}$ auf X wird definiert als

$$\deg_g(\mathcal{F}) = \int_X c_1(\mathcal{F}) \wedge \omega^{n-1},$$

wobei $c_1(\mathcal{F})$ die erste Chernklasse von $\mathcal{F}$ und ω die Fundamentalform der Kähler-Metrik g bezeichnet. Ist $\mathcal{F}$ nicht-trivial, so wird das Verhältnis

$$\mu_g(\mathcal{F}) = \frac{\deg_g(\mathcal{F})}{\operatorname{rank}(\mathcal{F})}$$

des g-Grades der Garbe $\mathcal{F}$ zu ihrem Rang als *normierter g-Grad* (engl. *g-slope*) von $\mathcal{F}$ bezeichnet. Eine torsionsfreie kohärente analytische Garbe $\mathcal{E}$ auf X heißt dann *g-semistabil*, falls

$$\mu_g(\mathcal{F}) \leqslant \mu_g(\mathcal{E})$$

für jede kohärente Untergarbe $\mathcal{F}$ von $\mathcal{E}$ mit $0 < \operatorname{rank}(\mathcal{F})$ gilt. Gilt sogar die strikte Ungleichung

$$\mu_g(\mathcal{F}) < \mu_g(\mathcal{E})$$

für jede kohärente Untergarbe $\mathcal{F}$ von $\mathcal{E}$ mit $0 < \operatorname{rank}(\mathcal{F}) < \operatorname{rank}(\mathcal{E})$, so heißt $\mathcal{E}$ *g-stabil.* Der Begriff der Stabilität lässt sich auf ein holomorphes Vektorbündel E auf X anwenden, indem man die Garbe $\mathcal{E} = \mathcal{O}_X(E)$ seiner holomorphen Schnitte betrachtet. Jedes stabile holomorphe Vektorbündel auf einer kompakten Kähler-Mannigfaltigkeit ist *einfach*, d. h. die einzigen holomorphen Schnitte seines Endomorphismenbündels sind die Homothetien. Eine hermitesche Metrik h in E heißt *g-Hermite-Einstein-Metrik*, falls

$$\sqrt{-1}\Lambda_g F_h = \lambda_h \operatorname{id}_E$$

mit einer reellen Konstanten λ_h gilt, wobei $\sqrt{-1}\Lambda_g$ die Kontraktion mit ω, F_h die Krümmungsform des Chern-Zusammenhangs des hermiteschen holomorphen Vektorbündels (E, h) und id_E den identischen Endomorphismus von E bezeichnet. In diesem Fall nennt man λ_h den *Einstein-Faktor* von h und (E, h) ein *g-Hermite-Einstein-Vektorbündel*. Der Einstein-Faktor hängt nur von der Kähler-Mannigfaltigkeit (X, g) und dem Vektorbündel E ab. Tatsächlich gilt

$$\lambda_h = \frac{2\pi\mu_g(E)}{(n-1)!\,\mathrm{vol}_g(X)},$$

wobei $\mathrm{vol}_g(X)$ das Volumen von X bzgl. g bezeichnet. Der Begriff der Hermite-Einstein-Metrik wurde von S. Kobayashi in [Kb80] als Verallgemeinerung des Begriffs der Kähler-Einstein-Metrik im Tangentialbündel einer kompakten Kähler-Mannigfaltigkeit eingeführt.

Die Kobayashi-Hitchin-Korrespondenz besagt nun, dass ein irreduzibles holomorphes Vektorbündel genau dann eine g-Hermite-Einstein-Metrik besitzt, wenn es g-stabil ist. Der Beweis der g-Stabilität eines irreduziblen g-Hermite-Einstein-Vektorbündels stammt von S. Kobayashi [Kb82] und Lübke [Lue83]. Die umgekehrte Implikation, d. h. die Existenz einer g-Hermite-Einstein-Metrik in einem g-stabilen holomorphen Vektorbündel, wurde für kompakte Riemannsche Flächen von Donaldson in [Do83] gezeigt, der einen neuen Beweis eines berühmten Satzes von Narasimhan und Seshadri [NS65] gab. Er bewies die Aussage später für projektiv-algebraische Flächen in [Do85] und allgemeiner für projektiv-algebraische Mannigfaltigkeiten beliebiger Dimension in [Do87]. Schließlich behandelten Uhlenbeck und Yau den allgemeinen Fall einer kompakten Kähler-Mannigfaltigkeit in [UY86] (siehe auch [UY89]). Alle Beweise basieren auf der Tatsache, dass sich bei Vorgabe einer glatten hermiteschen Metrik h_0 in E (der sogenannten *Hintergrundmetrik*) jede hermitesche Metrik h in E schreiben lässt als $h = h_0 f$, d. h.

$$h(s, t) = h_0(f(s), t)$$

für alle Schnitte s und t von E, wobei f ein glatter Endomorphismus von E ist, der bzgl. h_0 positiv definit und selbstadjungiert ist. Man bemerkt, dass h genau dann eine g-Hermite-Einstein-Metrik ist, wenn f eine gewisse nicht-lineare partielle Differentialgleichung löst. Donaldson betrachtet in seinem Beweis eine Evolutionsgleichung vom Wärmeleitungstyp mit einem reellen Parameter t. Nachdem er eine Lösung erhält, die für alle nicht-negativen Werte von t definiert ist, zeigt er die Konvergenz der Lösung für t gegen unendlich unter Benutzung der Stabilität des Vektorbündels und eines Induktionsarguments über die Dimension der komplexen Mannigfaltigkeit. Der Grenzwert ist eine Lösung der partiellen Differentialgleichung und liefert daher die gewünschte Hermite-Einstein-Metrik. Uhlenbeck und Yau betrachten in ihrem Beweis eine gestörte Version der partiellen Differentialgleichung, die von einem reellen Parameter ε abhängt. Sie zeigen, dass diese für jedes kleine positive ε lösbar ist. Konvergieren diese Lösungen in einem guten Sinne für ε gegen Null, so liefert der Grenzwert eine Hermite-Einstein-Metrik. Sind die Lösungen aber divergent, so produziert dies eine kohärente Untergarbe, die der Stabilität des Vektorbündels widerspricht.

Die Kobayashi-Hitchin-Korrespondenz war Gegenstand vieler Verallgemeinerungen und Anpassungen an zusätzliche Strukturen auf dem holomorphen Vektorbündel und der zugrunde liegenden komplexen Mannigfaltigkeit. Li und Yau bewiesen eine Verallgemeinerung für nicht-

Kähler-Mannigfaltigkeiten in [LY87], die unabhängig davon im Flächenfall von Buchdahl in [Bu88] bewiesen wurde. Hitchin [Hi87] und Simpson [Si88] führten den Begriff des *Higgs-Bündels* auf einer komplexen Mannigfaltigkeit X ein. Darunter versteht man ein Paar (E, θ), bestehend aus einem holomorphen Vektorbündel E und einer Bündelabbildung $\theta : E \to E \otimes \Omega^1_X$. Sie verallgemeinerten die Begriffe der Stabilität und der Hermite-Einstein-Metriken auf Higgs-Bündel und bewiesen eine Kobayashi-Hitchin-Korrespondenz unter der Integrabilitätsbedingung $0 = \theta \wedge \theta : E \to E \otimes \Omega^2_X$. Bando und Siu erweiterten den Begriff der Hermite-Einstein-Metrik in [BaS94] auf den Fall reflexiver Garben und bewiesen eine Kobayashi-Hitchin-Korrespondenz in dieser Situation. Die zwei Verallgemeinerungen für Higgs-Bündel und reflexive Garben wurden kürzlich durch Biswas und Schumacher kombiniert zu einer Verallgemeinerung für *Higgs-Garben*, siehe [BsS09]. Weitere Verallgemeinerungen umfassen die Situation eines *holomorphen Paars*, d. h. eines holomorphen Vektorbündels zusammen mit einem globalen holomorphen Schnitt entsprechend der Definition von Bradlow in [Br94], sowie eines *holomorphen Tripels*, d. h. eines Paars zweier holomorpher Vektorbündel zusammen mit einem globalen holomorphen Homomorphismus dazwischen entsprechend der Definition von Bradlow und García-Prada in [BG96].

In dieser Arbeit betrachten wir die Situation einer *gerahmten Mannigfaltigkeit.*

Definition A.1.

(i) Eine *gerahmte Mannigfaltigkeit* ist ein Paar (X, D), bestehend aus einer kompakten komplexen Mannigfaltigkeit X und einem glatten Divisor D in X.

(ii) Eine gerahmte Mannigfaltigkeit (X, D) heißt *kanonisch polarisiert*, falls das Geradenbündel $K_X \otimes [D]$ ampel ist, wobei K_X das kanonische Geradenbündel von X und $[D]$ das zum Divisor D gehörende Geradenbündel bezeichnet.

Der Begriff der gerahmten Mannigfaltigkeit, auch bekannt unter der Bezeichnung *logarithmisches Paar*, wird z. B. in [Sch98a] und [Sch98b] (siehe auch [ST04]) in Analogie zum Konzept des *gerahmten Vektorbündels* eingeführt (siehe [Le93], [Lue93] und [LOS93]). Ein einfaches Beispiel einer kanonisch polarisierten gerahmten Mannigfaltigkeit ist $(\mathbb{P}^n, V)$, wobei $\mathbb{P}^n$ der n-dimensionale komplex-projektive Raum ist und V eine Hyperfläche in $\mathbb{P}^n$ vom Grad $\geqslant n + 2$. Ist eine kanonisch polarisierte gerahmte Mannigfaltigkeit (X, D) gegeben, so erhält man eine spezielle Kähler-Metrik auf dem Komplement $X' := X \setminus D$ von D in X.

Theorem A.2 (R. Kobayashi, [Ko84])**.** *Ist (X, D) eine kanonisch polarisierte gerahmte Mannigfaltigkeit, so existiert eine (bis auf ein konstantes Vielfaches) eindeutig bestimmte vollständige Kähler-Einstein-Metrik auf X' mit negativer Ricci-Krümmung.*

Dies ist ein Analogon zu dem klassischen Satz von Yau, der besagt, dass jede kompakte komplexe Mannigfaltigkeit mit amplem kanonischen Bündel eine (bis auf ein konstantes Vielfaches) eindeutige Kähler-Einstein-Metrik mit negativer Ricci-Krümmung besitzt, vgl. [Yau78b]. Die Metrik aus Theorem A.2, die in der Nähe des Divisors D ein Wachstum vom Poincaré-Typ besitzt und daher als die *Poincaré-Metrik* bezeichnet wird, ist eine natürliche Wahl auf der Suche nach einer passenden Kähler-Metrik auf X'.

In [Ko84] führt R. Kobayashi spezielle „Koordinatensysteme“ auf X' ein, die *Quasi-Koordinaten* heißen. Diese sind in einem gewissen Sinn sehr gut an die Poincaré-Metrik angepasst. Man sagt, dass X' zusammen mit der Poincaré-Metrik von *beschränkter Geometrie* sei. Dieses Konzept wurde auch von Cheng und Yau in [CY80] und von Tian und Yau in [TY87] untersucht. Es wird für die Resultate dieser Arbeit sehr wichtig sein, dass das asymptotische Verhalten der Poincaré-Metrik gut bekannt ist. Tatsächlich gibt Schumacher in [Sch98a] eine explizite Beschreibung ihrer Volumenform auf Grundlage der Quasi-Koordinaten.

Theorem A.3 (Schumacher, [Sch98a], theorem 2)**.** *Es gibt eine Zahl $0 < \alpha \leqslant 1$, so dass für alle $k \in \{0, 1, \ldots\}$ und $\beta \in (0, 1)$ die Volumenform der Poincaré-Metrik von der Form*

$$\frac{2\Omega}{||\sigma||^2 \log^2(1/||\sigma||^2)} \left(1 + \frac{\nu}{\log^\alpha(1/||\sigma||^2)}\right) \quad \text{mit } \nu \in \mathcal{C}^{k,\beta}(X')$$

ist, wobei Ω eine glatte Volumenform auf X, σ ein kanonischer Schnitt von $[D]$, $||\cdot||$ die von einer hermiteschen Metrik in $[D]$ induzierte Norm und $\mathcal{C}^{k,\beta}(X')$ der Hölder-Raum der $\mathcal{C}^{k,\beta}$-Funktionen bzgl. der Quasi-Koordinaten ist.

Außerdem zeigt Schumacher in [Sch98a], dass die Fundamentalform der Poincaré-Metrik lokal gleichmäßig gegen eine Kähler-Einstein-Metrik auf D konvergiert, wenn man sie auf Koordinatenrichtungen parallel zu D einschränkt. Daraus erhält man das folgende Resultat über die Asymptotik der Poincaré-Metrik. Ist σ ein kanonischer Schnitt von $[D]$, so betrachte man σ als lokale Koordinate in einer Umgebung eines Punktes $p \in D$. Man kann dann lokale Koordinaten $(\sigma, z^2, \ldots, z^n)$ nahe p wählen, so dass man die folgende Aussage aus [Sch02] bekommt, wobei $g_{\sigma\bar{\sigma}}$, $g_{\sigma\bar{j}}$ etc. die Koeffizienten der Fundamentalform der Poincaré-Metrik und $g^{\bar{\sigma}\sigma}$ etc. die Einträge der entsprechenden inversen Matrix bezeichnen.

Proposition A.4. *Mit $0 < \alpha \leqslant 1$ aus Theorem A.3 gilt*

(i) $g^{\bar{\sigma}\sigma} \sim |\sigma|^2 \log^2(1/|\sigma|^2)$,

(ii) $g^{\bar{\sigma}i}, g^{\bar{j}\sigma} = O\left(|\sigma| \log^{1-\alpha}(1/|\sigma|^2)\right)$, $i, j = 2, \ldots, n$,

(iii) $g^{\bar{i}i} \sim 1$, $i = 2, \ldots, n$ *und*

(iv) $g^{\bar{j}i} \to 0$ *für* $\sigma \to 0$, $i, j = 2, \ldots, n$, $i \neq j$.

Wir werden die obigen Abschätzungen benutzen, um die Begriffe bereitzustellen, die für eine Kobayashi-Hitchin-Korrespondenz für Vektorbündel auf gerahmten Mannigfaltigkeiten relevant sind. Diesem Ziel kann man sich auf verschiedene Arten nähern. Eine Möglichkeit ist es, *parabolische Bündel* zu betrachten, wie sie durch Mehta und Seshadri in [MS80] auf Riemannschen Flächen eingeführt und durch Maruyama und Yokogawa in [MY92] auf höher-dimensionale Varietäten verallgemeinert wurden (siehe auch [Bs95], [Bs97a], [Bs97b]). Sei (X, D) eine gerahmte Mannigfaltigkeit und $\mathcal{E}$ eine torsionsfreie kohärente analytische Garbe auf X. Eine *quasi-parabolische Struktur* auf $\mathcal{E}$ bzgl. D ist eine Filtration

$$\mathcal{E} = \mathcal{F}_1(\mathcal{E}) \supset \mathcal{F}_2(\mathcal{E}) \supset \cdots \supset \mathcal{F}_l(\mathcal{E}) \supset \mathcal{F}_{l+1}(\mathcal{E}) = \mathcal{E}(-D)$$

durch kohärente Untergarben, wobei $\mathcal{E}(-D)$ das Bild von $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-D)$ in $\mathcal{E}$ bezeichnet. Die ganze Zahl l heißt die *Länge der Filtration.* Eine *parabolische Struktur* ist eine quasi-parabolische

Struktur zusammen mit einem System von *parabolischen Gewichten* $\{\alpha_1, \dots, \alpha_l\}$ mit der Eigenschaft, dass $0 \leqslant \alpha_1 < \alpha_2 < \dots < \alpha_l < 1$. Das Gewicht α_i korrespondiert zu $\mathcal{F}_i(\mathcal{E})$. Die Garbe $\mathcal{E}$ zusammen mit diesen Daten heißt dann *parabolische Garbe* und wird mit $(\mathcal{E}, \mathcal{F}_*, \alpha_*)$ oder einfach mit $\mathcal{E}_*$ bezeichnet. Ist g eine Kähler-Metrik auf X, so kann der Begriff der g-Stabilität für parabolische Garben angepasst werden. Für eine parabolische Garbe $(\mathcal{E}, \mathcal{F}_*, \alpha_*)$ setze man

$$\mathcal{E}_t = \mathcal{F}_i(\mathcal{E})(-\lfloor t \rfloor D) \quad \text{für jedes } t \in \mathbb{R},$$

wobei $\lfloor t \rfloor$ den ganzzahligen Anteil von t bezeichnet und $i \in \{1, \dots, l+1\}$ so gewählt sei, dass

$$\alpha_{i-1} < t - \lfloor t \rfloor \leqslant \alpha_i,$$

wobei $\alpha_0 := \alpha_l - 1$ und $\alpha_{l+1} := 1$. Die so definierte Filtration $(\mathcal{E}_t)_{t \in \mathbb{R}}$ hat die folgenden Eigenschaften.

- Sie ist absteigend, d. h. $\mathcal{E}_t \subset \mathcal{E}_{t'}$ für alle reellen Zahlen $t \geqslant t'$.
- Sie ist linksseitig stetig, d. h. es gibt ein $\varepsilon > 0$, so dass $\mathcal{E}_{t-\varepsilon} = \mathcal{E}_t$ für alle $t \in \mathbb{R}$.
- Sie hat eine Sprungstelle bei $t \in \mathbb{R}$, d. h. $\mathcal{E}_{t+\varepsilon} \neq \mathcal{E}_t$ für jedes $\varepsilon > 0$, genau dann, wenn $t - \lfloor t \rfloor = \alpha_i$ für ein $i \in \{1, \dots, l\}$.
- Sie bestimmt vollständig die parabolische Struktur $(\mathcal{E}, \mathcal{F}_*, \alpha_*)$.

Diese Filtration ermöglicht es uns, die gewöhnlichen Begriffe von Untergarben, g-Grad, normiertem g-Grad und schließlich der g-Stabilität für die parabolische Situation wie folgt anzupassen. Eine parabolische Garbe $\mathcal{S}_*$ heißt *parabolische Untergarbe* von $\mathcal{E}_*$, falls die folgenden Bedingungen erfüllt sind.

(i) $\mathcal{S}$ ist eine Untergarbe von $\mathcal{E}$ mit torsionsfreier Quotientengarbe $\mathcal{E}/\mathcal{S}$,

(ii) $\mathcal{S}_t \subset \mathcal{E}_t$ für alle $t \in \mathbb{R}$ und

(iii) ist $\mathcal{S}_s \subset \mathcal{E}_t$ für gewisse $s, t \in \mathbb{R}$ mit $t > s$, so gilt $\mathcal{S}_s = \mathcal{S}_t$.

Der *parabolische g-Grad* einer parabolischen Garbe $\mathcal{E}_*$ wird definiert als

$$\operatorname{pardeg}_g(\mathcal{E}_*) = \int_0^1 \deg_g(\mathcal{E}_t)\, dt + \operatorname{rank}(\mathcal{E}) \deg D.$$

Parallel zur Standardsituation wird dann der *parabolische normierte g-Grad* (engl. *parabolic g-slope*) einer parabolischen Garbe $\mathcal{E}_*$ mit $\operatorname{rank}(\mathcal{E}) > 0$ eingeführt als

$$\text{par-}\mu_g(\mathcal{E}_*) = \frac{\operatorname{pardeg}_g(\mathcal{E}_*)}{\operatorname{rank}(\mathcal{E})}.$$

Eine parabolische Garbe $\mathcal{E}_*$ heißt *parabolisch g-semistabil*, falls

$$\text{par-}\mu_g(\mathcal{S}_*) \leqslant \text{par-}\mu_g(\mathcal{E}_*)$$

für jede parabolische Untergarbe $\mathcal{S}_*$ von $\mathcal{E}_*$ mit $0 < \mathrm{rank}(\mathcal{S})$ gilt. Gilt sogar die strikte Ungleichung

$$\text{par-}\mu_g(\mathcal{S}_*) < \text{par-}\mu_g(\mathcal{E}_*)$$

für jede parabolische Untergarbe $\mathcal{S}_*$ von $\mathcal{E}_*$ mit $0 < \mathrm{rank}(\mathcal{S}) < \mathrm{rank}(\mathcal{E})$, so heißt $\mathcal{E}_*$ *parabolisch g-stabil*. Es sei darauf hingewiesen, dass Biswas in [Bs97b] eine Beziehung zwischen parabolischen Bündeln auf X bzgl. D und sogenannten *Orbifold-Bündeln* auf einer entlang D verzweigten endlich-blättrigen Überlagerung $p : Y \to X$ herstellt.

In [LN99] entwickeln Li und Narasimhan eine Kobayashi-Hitchin-Korrespondenz für parabolische Vektorbündel vom Rang 2 auf gerahmten Mannigfaltigkeiten der komplexen Dimension 2, indem sie die Äquivalenz zwischen parabolischer Stabilität und der Existenz einer Hermite-Einstein-Metrik in der Einschränkung $E' := E|_{X'}$ von E auf X' zeigen. Hier wird die Hermite-Einstein-Bedingung bzgl. einer Kähler-Metrik auf X' betrachtet, die sich durch Einschränkung auf X' aus einer glatten Kähler-Metrik auf X ergibt. Hermite-Einstein-Metriken sollen eine zusätzliche Bedingung erfüllen, die als *Kompatibilität mit der parabolischen Struktur* bezeichnet wird. Diese Betrachtungen beziehen daher die spezielle Kähler-Metrik auf X' aus Theorem A.2, die im kanonisch polarisierten Fall vorhanden ist, nicht mit ein. Dieser Ansatz wird auch von Simpson in [Si88] für den Fall von Higgs-Bündeln mit abgedeckt. Im Gegensatz dazu behandelt Biquard in [Bi97] die Beziehung zwischen parabolischer Stabilität und der Existenz einer hermiteschen Metrik in E', die die Hermite-Einstein-Bedingung bzgl. der Poincaré-Metrik erfüllt. Er führt unter der Bezeichnung *logarithmische Bündel* eine Kombination von parabolischen und Higgs-Bündeln ein und stellt in dieser Situation eine Kobayashi-Hitchin-Korrespondenz her.

In dieser Arbeit verfolgen wir einen alternativen Weg, um die Begriffe zu definieren, die für eine Kobayashi-Hitchin-Korrespondenz für Vektorbündel auf gerahmten Mannigfaltigkeiten gebraucht werden. Unsere Methode verwendet keine parabolischen Strukturen. Stattdessen passen wir direkt die gewöhnlichen Begriffe der Stabilität und der Hermite-Einstein-Metrik an die gerahmte Situation an. Ist eine kanonisch polarisierte gerahmte Mannigfaltigkeit (X, D) gegeben, so gibt es zwei Ansätze zur „Stabilität im gerahmten Sinn“ einer torsionsfreien kohärenten analytischen Garbe $\mathcal{E}$ auf X bzgl. der gerahmten Mannigfaltigkeit (X, D), die uns vernünftig erscheinen. Zum einen gibt es den Standardbegriff der Stabilität von $\mathcal{E}$ bzgl. der Polarisierung $K_X \otimes [D]$ von X. Das bedeutet, dass der Grad einer kohärenten Untergarbe $\mathcal{F}$ von $\mathcal{E}$ bzgl. einer Kähler-Metrik auf X berechnet wird, deren Fundamentalform die Krümmungsform einer positiven glatten hermiteschen Metrik im Geradenbündel $K_X \otimes [D]$ ist. Den zweiten Ansatz betreffend betrachten wir wieder kohärente Untergarben $\mathcal{F}$ von $\mathcal{E}$, aber berechnen ihren Grad nun auf X' bzgl. der Poincaré-Metrik. Man beachte allerdings, dass dies nicht den Standardbegriff der Stabilität auf X' bzgl. der Poincaré-Metrik liefert, da wir nur Untergarben von $\mathcal{E}$ auf X und nicht auf X' betrachten. Glücklicherweise können wir unter Benutzung von Schumachers Theorem A.3 über das asymptotische Verhalten der Poincaré-Metrik zeigen, dass diese zwei Ansätze äquivalent sind, was ein starkes Anzeichen dafür ist, dass der so erhaltene Begriff der *Stabilität im gerahmten Sinn* oder der *gerahmten Stabilität* vernünftig ist im Hinblick auf eine Kobayashi-Hitchin-Korrespondenz. Tatsächlich impliziert die gerahmte Stabilität eines Vektorbündels auf X seine Einfachheit, da die gerahmte Stabilität ein Spezialfall der Stabilität im gewöhnlichen Sinn ist. Sie impliziert dagegen nicht notwendigerweise die Einfachheit der Einschränkung des Bündels auf X'.

Besondere Aufmerksamkeit müssen wir der Frage widmen, was wir unter einer *Hermite-Einstein-Metrik im gerahmten Sinn* oder *gerahmten Hermite-Einstein-Metrik* verstehen wollen. Wir interessieren uns für hermitesche Metriken in E', die die Hermite-Einstein-Bedingung bzgl. der Poincaré-Metrik erfüllen. Ein Blick auf den Beweis der Eindeutigkeit (bis auf ein konstantes Vielfaches) einer solchen Hermite-Einstein-Metrik zeigt aber, dass diese Bedingung nicht ausreicht, um einen sinnvollen Begriff einer gerahmten Hermite-Einstein-Metrik zu erhalten. In der Tat benutzt der klassische Eindeutigkeitsbeweis die Einfachheit eines stabilen Vektorbündels. Da die gerahmte Stabilität von E aber nur die Einfachheit von E und nicht die von E' impliziert, gibt uns dies noch nicht die Eindeutigkeit einer beliebigen Hermite-Einstein-Metrik in E' bzgl. der Poincaré-Metrik. Stattdessen verlangen wir zusätzlich eine Bedingung der Kompatibilität mit einer glatten hermiteschen Metrik in E über der kompakten Mannigfaltigkeit X. Dabei handelt es sich um die von Simpson in [Si88] eingeführte Bedingung. Tatsächlich ähnelt sie der oben erwähnten Bedingung der Kompatibilität mit der parabolischen Struktur.

Wir zeigen, dass jedes holomorphe Vektorbündel auf einer kanonisch polarisierten gerahmten Mannigfaltigkeit, das im gerahmten Sinn stabil ist, eine (bis auf ein konstantes Vielfaches) eindeutig bestimmte gerahmte Hermite-Einstein-Metrik besitzt. Unsere Methoden sind die Folgenden. Das oben erwähnte Konzept der beschränkten Geometrie erlaubt es uns, Simpsons Methode der Wärmeleitungsgleichung aus [Si88] (die dort unter anderem im kompakten Fall behandelt wird) auf unsere Situation anzuwenden, solange alle analytischen Betrachtungen in Quasi-Koordinaten ausgedrückt werden. Simpson löst wie Donaldson eine Evolutionsgleichung vom Wärmeleitungstyp für alle nicht-negativen Werte eines reellen Parameters t. Konvergiert die Lösung für t gegen unendlich, so liefert der Grenzwert die gewünschte Hermite-Einstein-Metrik. Es besteht nur ein kritischer Punkt bei der Anwendung von Simpsons Methode auf unsere Situation, nämlich die Konstruktion einer destabilisierenden Untergarbe von $\mathcal{E} = \mathcal{O}_X(E)$ für den Fall, dass die Lösung nicht konvergiert. Man erhält zunächst ein sogenanntes *schwach holomorphes Unterbündel* von E (oder E'), d. h. einen messbaren Schnitt π von $\mathrm{End}(E)$, der im Sobolev-Raum der L^2-Schnitte liegt, die schwache Ableitungen erster Ordnung in L^2 besitzen, und zusätzlich die Bedingungen

$$\pi = \pi^* = \pi^2 \quad \text{und} \quad (\mathrm{id}_E - \pi) \circ \nabla'' \pi = 0$$

erfüllt, wobei π^* den zu π adjungierten Endomorphismus bzgl. einer hermiteschen Metrik in E und ∇'' den $(0, 1)$-Anteil des zugehörigen Chern-Zusammenhangs bezeichnen. In ihrer Arbeit [UY86] zeigen Uhlenbeck und Yau, dass ein solcher Schnitt in Wirklichkeit eine kohärente Untergarbe von $\mathcal{E}$ und implizit ein holomorphes Unterbündel von E außerhalb einer analytischen Teilmenge von X der Kodimension $\geqslant 2$ definiert. Popovici gibt in [Po05] einen alternativen Beweis dieser Aussage, der auf der Theorie der Ströme basiert. In unserer Situation erfüllt der Schnitt π aus Simpsons Beweis die L^2-Bedingungen bzgl. der Poincaré-Metrik. Mit Hilfe der Resultate aus Proposition A.4 können wir zeigen, dass diese bereits die L^2-Bedingungen im gewöhnlichen Sinn implizieren. Folglich kann der Satz von Uhlenbeck-Yau-Popovici ohne Veränderung auf unsere Situation angewandt werden.

Wir möchten anmerken, dass „asymptotische“ Versionen unseres Resultats von Ni und Ren in [NR01] sowie von Xi in [Xi05] erzielt wurden. Hier betrachten die Autoren bestimmte Klassen vollständiger, nicht-kompakter hermitescher Mannigfaltigkeiten (X, g). Um in der Lage zu sein, die Existenz einer Hermite-Einstein-Metrik in einem holomorphen Vektorbündel auf X zu beweisen, nehmen sie nicht die Stabilität des Bündels an. Stattdessen verlangen sie die Existenz einer hermiteschen Metrik h_0 in E, die *asymptotisch Hermite-Einstein* ist, wobei es sich um eine Bedingung an das Wachstum von $|\sqrt{-1}\Lambda_g F_{h_0} - \lambda_{h_0} \operatorname{id}_E|_{h_0}$ handelt.

Der Inhalt dieser Arbeit ist wie folgt organisiert. Kapitel 1 beinhaltet eine Einführung in das Thema der Arbeit. Hierbei handelt es sich im Wesentlichen um eine englische Version der vorliegenden Zusammenfassung.

In Kapitel 2 definieren wir den Begriff der Kähler-Metrik auf X' mit Wachstum vom Poincaré-Typ in der Nähe des Divisors D und präsentieren die Konstruktion einer solchen Metrik nach Griffiths. Nach einer Einführung in das Konzept der Quasi-Koordinaten und der beschränkten Geometrie nach R. Kobayashi stellen wir einen Beweis der Existenz und Eindeutigkeit (bis auf ein konstantes Vielfaches) einer vollständigen Kähler-Einstein-Metrik auf X' mit negativer Ricci-Krümmung vor. Diese Metrik hat ebenfalls Poincaré-Wachstum und wird später als die *Poincaré-Metrik* bezeichnet werden. Schließlich zeigen wir, dass die Quadratintegrierbarkeitsbedingungen für Funktionen und 1-Formen bzgl. der Poincaré-Metrik diejenigen im gewöhnlichen Sinn implizieren.

Kapitel 3 ist der zentrale Teil dieser Arbeit. Nachdem wir einen kurzen Überblick über die Konzepte der Stabilität und der Hermite-Einstein-Metriken im kompakten Fall gegeben haben, entwickeln wir die entsprechenden Begriffe in der gerahmten Situation. Insbesondere zeigen wir, dass die beiden oben erwähnten Ansätze zur gerahmten Stabilität äquivalent sind. Außerdem zeigen wir die Eindeutigkeit (bis auf ein konstantes Vielfaches) einer gerahmten Hermite-Einstein-Metrik in einem einfachen Bündel.

Kapitel 4 enthält das Existenzresultat für gerahmte Hermite-Einstein-Metriken in einem holomorphen Vektorbündel auf einer kanonisch polarisierten gerahmten Mannigfaltigkeit, das im gerahmten Sinn stabil ist. Hier geben wir eine Zusammenfassung von Donaldsons Existenzbeweis für eine Lösung der Evolutionsgleichung, die für alle endlichen nicht-negativen Werte des Zeitparameters definiert ist, sowie einen Überblick über Simpsons Ansatz zur Konvergenz dieser Lösung in unendlicher Zeit. Außerdem fassen wir Popovicis Beweis des Regularitätssatzes für schwach holomorphe Unterbündel zusammen, der wegen des Resultats über die Quadratintegrierbarkeitsbedingungen aus Kapitel 2 auf unsere Situation angewandt werden kann.

Schließlich skizzieren wir in Kapitel 5 einige weitere Gedanken, die auf den Resultaten dieser Arbeit basieren. Ausgehend von der Arbeit [TY87] von Tian und Yau könnte man vermuten, dass die in Kapitel 4 erhaltene gerahmte Hermite-Einstein-Metrik auch als Grenzwert einer Folge von Hermite-Einstein-Metriken auf X' bzgl. gewisser von Tian und Yau konstruierter unvollständiger Kähler-Einstein-Metriken angesehen werden kann. Dieses Problem ist aber noch offen.

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C Lebenslauf

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