Algebraic Discrete Morse Theory and Applications to Commutative Algebra

(Algebraische Diskrete Morse-Theorie und Anwendungen in der Kommutativen Algebra)

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Contents

The following text is a PhD thesis in Algebraic Combinatorics. It consists of two parts and an appendix. In the first part, "Algebraic Discrete Morse Theory and Applications to Commutative Algebra", we generalize Forman's Discrete Morse theory and give several applications to problems in commutative algebra.

In the second part we present results on two related problems in Algebraic Combinatorics, namely "Homology of Nilpotent Lie Algebras of Finite Type" and the "Neggers-Stanley Conjecture".

The appendix consists of the German abstract, acknowledgments, curriculum vitae, and the declaration of authorship.

Part 1

Algebraic Discrete Morse Theory and Applications to Commutative Algebra

Introduction

In linear algebra there is the fundamental concept of linear independence. The situation turns out to be simple due to the fact that all vector spaces V over a field k are free k-modules. In particular, the two conditions “maximal independent” and “minimal generating” for a set of vectors are equivalent.

The concept of dependence of polynomials $p_1, \ldots, p_r \in S := k[x_1, \ldots, x_n]$ is more complex. For example the conditions “maximal independent” and “minimal generating” are not equivalent anymore.

In order to measure dependence of polynomials, one considers free resolutions of the ideal $\langle p_1, \ldots, p_r \rangle \trianglelefteq S$, especially minimal free resolutions. Even though minimal free resolutions always exist, even in the monomial case, it is still an open problem to explicitly construct a minimal free resolution in general.

For monomial ideals there exist many explicit free resolutions, but they are mostly not minimal. One central idea of our work is to extract from a given free resolution of a monomial ideal a minimal resolution, by dividing out acyclic subcomplexes.

In commutative algebra one considers, besides minimal resolutions of monomial ideals, many other invariants, for example regularity, Poincaré-Betti series, Ext, Tor, which are calculated as well by the homology of chain complexes of free R-modules. Often we are in the situation that the homology of a given chain complex calculates an invariant, but the complex is very large in the sense that there exist homotopy-equivalent chain complexes such that the modules have a smaller rank. For example, the homology of the Taylor resolution of a monomial ideal tensored with the field k calculates the Betti numbers, but if one had the minimal resolution at hand, this calculation would be more efficient. Theoretically, one can minimize each chain complex by dividing out acyclic subcomplexes, but in praxis there does not exist any efficient algorithm which minimizes a given chain complex. The concept of cellular resolutions is a good tool for minimizing free resolutions with topological arguments, namely Forman’s Discrete Morse theory (see [**21**], [**22**]). The main idea is that a given cellular resolution can be made smaller by finding a so-called acyclic matching on the supporting CW-complex. This method was studied by Batzies in his PhD thesis (see [**4**]). But this concept is limited. First, the theory only works if the given chain complex is supported by a regular CW-complex, which is not always the case in applications. Moreover, it is still an open problem if any monomial ideal admits a minimal cellular resolution. But even if the given chain complex is supported by a regular CW-complex, it can happen that, after applying Discrete Morse theory once, the resulting Morse complex is not minimal and the CW-complex is not regular anymore. This obstructs a further application of this theory.

In this thesis we define an algebraic version of Forman's Discrete Morse theory. We call this generalization "Algebraic Discrete Morse theory". It generalizes the idea of "matching down" given chain complexes to smaller ones from the realm of cellular chain complexes to all algebraic chain complexes. Another advantage of this theory over the Discrete Morse theory for cellular resolutions is the fact that it can be applied iteratively.

We apply our method to several problems in commutative algebra. Similar to Discrete Morse theory for cellular resolutions, one major field of application for Algebraic Discrete Morse theory is finding minimal resolutions of monomial ideals in the commutative polynomial ring. For example, in this thesis we are able to give new minimal resolutions for a subclass of p-Borel fixed ideals.

Another field of applications are minimal resolutions of the residue class field $k \cong A/\mathfrak{m}$ over standard graded k-algebras A, where $\mathfrak{m}$ is the unique graded maximal ideal in A. Once we have found a minimal resolution of k, it allows us to calculate the multigraded Hilbert and Poincaré-Betti series. For example we were able to specify and prove a conjecture about the Poincaré-Betti series made by Charalambous and Reeves (see [**13**]). We get new connections between Hilbert and Poincaré-Betti series and find interesting corollaries for the Golod property.

The generalization of Forman's theory and the application to resolutions of the field k is a joint work with Volkmar Welker and appears in [**36**]. The results about Hilbert and Poincaré-Betti series, as well as the corollaries for the Golod property, can be found in my paper [**35**].

Finally, we want to mention that the generalization of Forman's Discrete Morse theory to an algebraic version was independently developed by Sköldberg in a preprint from 2004 (see [**41**]).

The structure of Part I of this thesis is as follows:

In Chapter 2 we give a short introduction to the tools of commutative algebra which are used in this thesis. We define the concept of free multigraded resolutions and cellular resolutions. We further define the multigraded Hilbert and Poincaré-Betti series and show how they are connected to each other. In the second paragraph we list some particular chain complexes, namely

- The Taylor and Scarf resolution for monomial ideals.
- The poset resolution for monomial ideals.
- The Koszul complex $K_\bullet$ with respect to a regular sequence $f_1, \ldots, f_r$.
- The Bar and the normalized Bar resolution.
- The Hochschild and normalized Hochschild resolution.

In the last paragraph we introduce the Eagon complex. We recall how it is constructed and define the Massey operations on the Koszul homology. We show that if all Massey operations vanish, the differential of the Eagon complex can be built from using this fact. In this case, the Eagon complex defines a minimal multigraded free resolution of the residue class field k. We introduce the Golod property and recall that Golodness is equivalent to the fact that the Poincaré-Betti series of k takes a specific form.

In Chapter 3 we develop the generalization of Forman's Discrete Morse theory. In fact, it is a generalization of the approach of Chari [**15**] using acyclic matchings on the directed cell graph of the CW-complex. We view any algebraic chain complex as a directed, weighted graph and show that a method similar to Discrete Morse theory can be applied in order to minimize the graph. We obtain a smaller complex, which is homotopic to the original complex.
In the third paragraph of Chapter 3 we show that the normalized Bar resolution, as well as the normalized Hochschild resolution, can be obtained by an acyclic matching from the Bar, resp. the Hochschild, resolution.

Chapter 4 up to Chapter 6 contain the applications.

In Chapter 4 we apply our method to resolutions of monomial ideals. In its first paragraph we consider acyclic matchings on the Taylor resolution of monomial ideals. This paragraph is essentially a preparation for Chapter 6, where we use these results in order to prove our conjecture about the minimal free resolution of the residue class field k and about the form of the multigraded Poincaré-Betti series.

- We develop a special sequence of acyclic matchings on the Taylor resolution for any monomial ideal such that the resulting Morse complex - though not explicitly constructed - is minimal. Moreover, some additional properties of this resolution will be used in Chapter 6. We call such a sequence a standard matching and prove that it always exists.
- For monomial ideals generated in degree two, we show via Algebraic Discrete Morse theory that there exists a subcomplex of the Taylor resolution consisting of all those subsets of the generating system of the ideal which contain no broken circuit, which defines a resolution of the ideal. Here a subset of the generating system contains no broken circuit if and only if the associated graph fulfills this property. The associated graph has vertices $1, \dots, n$ and two vertices i, j are joined by an edge if and only if $x_i x_j$ lies in the subset.
We denote this subcomplex with $T_{\mathbf{nbc}}$.
- For Stanley Reisner ideals of the order complex of a partially ordered set, we construct on the subcomplex $T_{\mathbf{nbc}}$ a matching which in general is not acyclic. Thus, the Morse complex is not necessarily defined. But if we use the same notation as for acyclic matchings, the critical cells with respect to this matching still help us to calculate the Poincaré-Betti series of k over the Stanley Reisner ring $k[\Delta]$.
- We introduce two new properties for monomial ideals depending on the minimal generating system, namely the gcd-condition and the strong gcd-condition.

> An acyclic matching on the Taylor resolution for ideals satisfying the strong gcd-condition is constructed. From the structure of the resulting Morse complex - though not explicitly constructed - we draw corollaries on the Golod property of the quotient ring (see Chapter 6).

In the second paragraph we apply our theory to the poset resolution of monomial ideals. First we show that this resolution can be obtained from the Taylor resolution by an acyclic matching. We then construct several acyclic matchings in order to minimize this resolution and turn these procedures into effective algorithms, which produce rather small cellular resolutions from the poset resolution. We finally discuss which properties of the partially ordered set imply minimality of the resolution.

In Paragraph 3 of Chapter 4 we consider Borel and p-Borel fixed ideals. Minimal resolutions for Borel ideals are well known (see [**19**], [**2**]), even cellular resolutions for this type of ideals were constructed (see [**4**]). In the first part we construct for principal Borel fixed ideals a new minimal cellular resolution which is a generalization of the hypersimplical resolution, introduced by Batzies [**4**] in order to get cellular resolutions of powers of the maximal ideal.
In the second part we develop via Algebraic Discrete Morse theory new minimal resolutions for classes of p-Borel fixed ideals. Minimal resolutions for p-Borel fixed ideals are only known in the case where the ideal is also Cohen-Macaulay (see [**2**]). In [**4**] it is proved that this resolution is even cellular. We prove the existence of minimal cellular resolutions for a larger class of p-Borel fixed ideals. In addition, we give a recursive formula for the calculation of the multigraded Poincaré-Betti series. Finally, we calculate the regularity of the above mentioned subclass of p-Borel fixed ideals, which reproves and generalizes known results on the regularity.

Chapter 5 contains the following applications, which are in joint work with Volkmar Welker (see [**36**]).
We construct minimal resolutions of the residue class field k, viewed as an A-module, where $A = k\langle x_1, \dots, x_n\rangle/\mathfrak{a}$ is the quotient ring of the (not necessarily commutative) polynomial ring $R := k\langle x_1, \dots, x_n\rangle$ divided by an ideal $\mathfrak{a}$. We choose a Gröbner basis of the ideal $\mathfrak{a}$ and define an acyclic matching on the normalized Bar resolution. In the first paragraph we consider the case where R is commutative. We give another description of the Morse complex, which can be viewed as a generalization of the Anick resolution [**1**] to the commutative case. We prove minimality if the initial ideal is either generated in degree two or a complete intersection. Finally, we draw some corollaries on the multigraded Poincaré-Betti series.
In the second paragraph we generalize our results of the first paragraph to the case where R is not commutative. In this case, the Morse complex is isomorphic to the Anick resolution. This result was also obtained by Sköldberg (see [**41**]), but in addition we prove minimality in special cases, which proves the rationality of the Poincaré-Betti series for these cases. In the section "Examples" we discuss three interesting examples including a proof of a conjecture by Sturmfels [**42**].
In the last paragraph we apply the acyclic matching, developed in the first two paragraphs, to the Hochschild complex. Again, we obtain minimality in some special cases, which allows to calculate the Hochschild homology with

coefficients in k. In addition, we give for some cases an explicit description of the minimal resolution, which reproves and generalizes a result obtained in [**9**].

Chapter 6 discusses the multigraded Poincaré-Betti series of monomial rings. The contents of this chapter coincides with my article [**35**].
We view the field k as an A-module, where $A = S/\mathfrak{a}$ is the quotient of the commutative polynomial ring $S = k[x_1, \ldots, x_n]$ divided by a monomial ideal $\mathfrak{a}$, and ask for an explicit form of the multigraded Poincaré-Betti series $P_k^A(\underline{x}, t) := \sum_{i,\alpha} \dim_k \left(\mathrm{Tor}_i^A(k,k)_\alpha \right) \underline{x}^\alpha \, t^i$. Backelin proved in 1982 [**3**] that in this case the multigraded Poincaré-Betti series is always a rational function, but an explicit description is still not known. In the case where the Taylor resolution of $\mathfrak{a}$ is minimal, Charalambous and Reeves gave in 1995 an explicit form of $P_k^A(\underline{x}, t)$ (see [**13**]). They conjectured that in general the series has a similar form. With our standard matching on the Taylor resolution, developed in Chapter 4, we formulate a conjecture about the minimal free resolution of k as an A-module, which we prove for several types of algebras A. This conjecture gives an explicit form of the Poincaré-Betti series, which specifies and implies the conjecture by Charalambous and Reeves. With the Euler characteristic we get in addition an explicit form of the multigraded Hilbert series of A and a general connection between these two series.

In the first paragraph we formulate our conjecture and draw the above corollaries. In the next paragraph we construct a new graded commutative polynomial ring depending on the standard matching and prove that this ring is as an algebra isomorphic to the Koszul homology. This result will later be used in the proof of our conjecture.
Using the matching, constructed in Chapter 4, on the Taylor resolution for Stanley Reisner ideals of a partially ordered set, we can prove in the third paragraph our conjecture about the Poincaré-Betti series of k over the Stanley Reisner ring $k[\Delta]$.
In Paragraph 4 we use the results obtained in the second paragraph and prove the conjecture about the minimal resolution of k in the case where the ideal $\mathfrak{a}$ is generated in degree two. We prove our conjecture for some further classes of algebras A. In the last part of this paragraph we justify our conjecture: We generalize the Massey operations on the Koszul homology in order to get an explicit description of the Eagon complex. We then define an acyclic matching on the Eagon complex. If the resulting Morse complex is minimal, one only has to find an isomorphism to the conjectured complex, and the conjecture is proved. In general, we do not have a good description of the minimized Eagon complex. Therefore, we cannot construct this isomorphism. But we present an approach to construct this isomorphism. This approach justifies our conjecture.

In the last paragraph we get, under the assumption of the conjecture, some corollaries on the Golod property of monomial rings. For example, if our conjecture is true, a ring is Golod if and only if the product on the Koszul homology vanishes, which is a strong simplification of the definition of Golodness. We further prove, under the assumption of the conjecture that ideals satisfying the strong gcd-condition are Golod (we call an ideal $\mathfrak{a}$ Golod if $S/\mathfrak{a}$ is Golod). In fact, we conjecture that this is an equivalence. This means that in the monomial case Golodness is a purely combinatoric condition on the generating system of the ideal. In particular, if true, Golodness does not depend on the characteristic of k.

Finally, we give a new criterion for ideals generated in degree $l \geq 2$ to be Golod. This result does not depend on our conjecture and generalizes a theorem proved by Herzog, Reiner, and Welker [**29**].

Using Algebraic Discrete Morse theory, in [**41**] Sköldberg calculates the homology of the nilpotent Lie algebra generated by $\{x_1, \ldots, x_n, y_1, \ldots, y_n, z\}$ with the only nonvanishing Lie bracket being $[z, x_i] = y_i$ over a field of characteristic 2. Note that Lie algebra studied by Sköldberg is quasi-isomorphic to the Heisenberg Lie algebra. This shows how large is the field of applications of Algebraic Discrete Morse theory. We believe that there are still many open problems which can be solved with our theory. Another interesting application is currently studied by Jan Brähler: In his Diplomarbeit, he applies Algebraic Discrete Morse theory to the chain complex calculating the Grassmann homology of a field k. Results in this direction relate to the algebraic k-theory of k. Clearly, our theory has its limits and finding a suitable matching can be an unsurmountable task. For example, we tried to calculate the homology of the nilpotent part of the Lie algebra associated to the root system A_n (see Part 2, Chapter 2), but in this case we were not able to find a "good" acyclic matching. The main difficulties of the theory are first to find a "good" acyclic matching, then to prove acyclicity, and finally to have a "good" control over the differential of the Morse complex.

Basics from Commutative Algebra

In this chapter we introduce some basic tools from Commutative Algebra which are used in this thesis. In the first paragraph we introduce the general theory of free and cellular resolutions of R-modules. In the second paragraph we introduce some examples of chain complexes and explain their applications. The last paragraph considers the Eagon complex and its applications. We explain the Eagon resolution, the Massey operations on the Koszul homology of R, and the Golod property for R. Finally, we outline the connections between these objects and properties.

Throughout this chapter let

$$R = \bigoplus_{\alpha \in \mathbb{N}^n} R_\alpha = (R, \mathfrak{m}, k)$$

be a standard $\mathbb{N}^n$-graded (not necessarily commutative) Noetherian k-algebra with unique graded maximal ideal

$$\mathfrak{m} = \bigoplus_{\alpha \in \mathbb{N}^n \setminus \{0\}} R_\alpha$$

and $k = R_0 = R/\mathfrak{m}$ the residue class field. It is clear that the set of units of R is given by $k = R_0$. Let

$$M = \bigoplus_{\alpha \in \mathbb{Z}^n} M_\alpha$$

be a $\mathbb{Z}^n$-graded left R-module.
The grading induces a degree function $\deg : R \to \mathbb{N}^n$ (resp. $\deg : M \to \mathbb{Z}^n$) for R (resp. M).

In the whole thesis all modules we consider are left R-modules and we denote for a natural number $n \in \mathbb{N}$ the set $\{1, 2, \ldots, n\}$ by $[n]$. An abstract simplex $\Delta([n])$ is the set of subsets of $[n]$:

$$\Delta([n]) := \big\{ J \subset [n] \big\}.$$

1. Free Resolutions of R-Modules

In this paragraph we introduce $\mathbb{Z}^n$-graded free R resolutions of left R-modules M and multigraded Hilbert and Poincaré-Betti series. We explain the definitions and give some basic properties. For more details see for example [**18**].

Definition 1.1. A $\mathbb{Z}^n$-graded chain complex $C_\bullet = (C_\bullet, \partial)$ of free $\mathbb{Z}^n$-graded R-modules is a family $C_i = \bigoplus_{\alpha\in\mathbb{Z}^n}(C_i)_\alpha$, $i \geq 0$, of free $\mathbb{Z}^n$-graded R-modules together with R-linear maps $\partial_i : C_i \to C_{i-1}$, $i \geq 1$ such that

(1) $\partial_i \circ \partial_{i+1} = 0$ for all $i \geq 1$ and
(2) the maps ∂_i are homogeneous, i.e. $\deg(\partial_i(m)) = \deg(m) = \alpha$ for all $m \in (C_i)_\alpha$ and all $\alpha \in \mathbb{Z}^n$.

The maps ∂_i are called differentials. We write $Z_i = \bigoplus_{\alpha\in\mathbb{Z}^n}(Z_i)_\alpha := \mathrm{Ker}(\partial_i)$ for the module of cycles and $B_i = \bigoplus_{\alpha\in\mathbb{Z}^n}(B_i)_\alpha := \mathrm{Im}(\partial_{i+1})$ for the module of boundaries of the complex $C_\bullet$.
The homology $H_i(C_\bullet) = \bigoplus_{\alpha\in\mathbb{Z}^n}(H_i)_\alpha$ is defined to be the quotient of the cycles Z_i and the boundaries B_i.

In addition to the grading $\deg \in \mathbb{Z}^n$, we sometimes also consider the total degree $\deg_t(m) := |\deg(m)| \in \mathbb{Z}$, where $|\cdot|$ is the sum over the coordinates of $\alpha \in \mathbb{Z}^n$, i.e. $|\alpha| = \sum_{i=1}^n \alpha_i$. Then an $\mathbb{N}^n$-graded ring R admits a decomposition of the following form and is called a bigraded ring:

$$R = \bigoplus_{i\in\mathbb{N}} \bigoplus_{\substack{\alpha\in\mathbb{Z}^n \\ |\alpha|=i}} R_\alpha.$$

Clearly, the bigrading of R induces a bigrading on all R-modules and all chain complexes of free R-modules. For simplification from now on we do not anymore specify the grading (or bigrading) and speak just of multigraded rings (modules, chain complexes, etc.)

We write $R(-\alpha)$ for the ring R as an R-module, shifted with $\alpha \in \mathbb{Z}^n$, i.e.

$$R(-\alpha) = \bigoplus_{\beta\in\mathbb{N}^n} R_{\alpha+\beta}.$$

For a multigraded chain complex $C_\bullet$ we then write

$$C_\bullet : \cdots \to \bigoplus_{\alpha\in\mathbb{Z}^n} R(-\alpha)^{\beta_{i,\alpha}} \to \bigoplus_{\alpha\in\mathbb{Z}^n} R(-\alpha)^{\beta_{i-1,\alpha}} \to \cdots.$$

Definition 1.2. A multigraded chain complex $C_\bullet$ is called a multigraded R-free resolution of M if

(1) $H_i(C_\bullet) = 0$ for all $i \geq 1$ and
(2) $H_0(C_\bullet) := \mathrm{Coker}(\partial_1) = C_0/\mathrm{Im}(\partial_1) \cong M$.

We say that a multigraded free resolution is minimal if $\partial_i(C_i) \in \mathfrak{m}C_{i-1}$ for all $i \geq 1$.
If for a multigraded chain complex we fix for each module C_i a basis $\mathcal{B}_i$, then we can write the differential for $e_\alpha \in \mathcal{B}_i$ in terms of the basis:

$$\partial(e_\alpha) = \sum_{e_\beta\in\mathcal{B}_{i-1}} [e_\alpha : e_\beta]\, e_\beta,$$

where $[e_\alpha : e_\beta] \in R$.
Then it is easy to prove that the following criterion for minimality holds.

Proposition 1.3. *The resolution $C_\bullet$ is minimal if and only if the differential has no unit as coefficient, i.e. $[e_\alpha, e_\beta] \notin R^* = R_0 = k$ for all $e_\alpha, e_\beta \in \mathcal{B}$.* □

The following corollary provides the main source of interest in minimal multigraded free resolutions:

Corollary 1.4. *If $C_\bullet$ is a multigraded minimal R-free resolution of an R-module M such that $C_i = \bigoplus_{\alpha\in\mathbb{Z}^n} R(-\alpha)^{\beta_{i,\alpha}}$, then*

$$\mathrm{Tor}_i^R(M,k)_\alpha \cong H_i(C_\bullet \otimes_R k)_\alpha \cong k^{\beta_{i,\alpha}}.$$

In particular, $\dim_k(\mathrm{Tor}_i^R(M,k)_\alpha) = \beta_{i,\alpha}$. □

1.1. Cellular Resolutions. Here we give a very short introduction to the concept of cellular resolutions. For more detail see [**6**], where this concept was introduced for the first time, and [**4**].

Let X be a CW-complex and $X_* = \cup_i X_*^{(i)}$ the set of cells ($X_*^{(i)}$ is the set of cells of dimension i). On this set we define a partial order by $\sigma \prec \tau$, for $\sigma, \tau \in X_*$, off for the topological cells σ, τ we have that σ lies in the closure of τ ($\sigma \subset \overline{\tau}$). If $(P, \prec)$ is a partially ordered set and $\mathrm{gr} : X_* \to P$ an order-preserving map, we call the tuple (X, gr) a P-graded CW-complex.

The homology $H_\bullet(X,R)$ of the CW-complex X with coefficients in R is defined as the homology of the cellular chain complex $C_\bullet(X)$, where the modules C_i are the free R-modules generated by the cells of dimension i, and the differential is given by

$$\partial(e_\sigma) = \sum_{\tau\subset\overline{\sigma}} [\sigma : \tau] e_\tau,$$

where $[\sigma : \tau]$, for an i-cell σ, is the topological degree of the map

$$S^{i-1} \overset{f_{\partial\sigma}}{\to} X^{(i-1)} \overset{\pi_\tau}{\to} S^{i-1}.$$

Here S^{i-1} denotes the $(i-1)$-sphere, $X^{(i-1)}$ the $(i-1)$-skeleton of X, $f_{\partial\sigma}$ the characteristic map, and π_τ the canonical projection. For more details of the homology of CW-complexes see [**48**].

Definition 1.5. A $\mathbb{Z}^n$-graded R-free resolution $C_\bullet$ of M is called a cellular resolution if there exists a $\mathbb{Z}^n$-graded CW-complex (X, gr) such that

(1) for each $\alpha \in \mathbb{Z}^n, i \geq 0$ there exists a basis $\{c_\sigma \,|\, \sigma \in X_*^{(i)} \text{ and } \mathrm{gr}(\sigma) = \alpha\}$ of $(C_i)_\alpha$,

(2) for each $\sigma \in X_*^{(i)}$ we have:

$$\partial_i(c_\sigma) = \sum_{\sigma\succ\tau\in X_*^{(i-1)}} [\sigma : \tau]\, x^{\mathrm{gr}(\sigma)-\mathrm{gr}(\tau)} c_\tau,$$

where $[\sigma : \tau]$ is the coefficient in the cellular chain complex.

The concept of cellular resolutions is very useful since one can try to minimize cellular resolutions with topological arguments and for ideals which admit minimal cellular resolutions the concept gives information about possible Betti numbers. These questions have been studied by Batzies in [**4**]. He used Forman's Discrete Morse theory to deduce from a given cellular resolution a minimal resolution and showed for many classes of ideals that they admit a minimal cellular resolution. In general, it is still an open problem if every monomial ideal admits a cellular minimal resolution. The resolutions which we construct in Chapter 4 are all cellular.

The problem of this approach is that one only can minimize cellular resolutions supported by a regular CW-complex. Our idea was to generalize Forman's theory to arbitrary chain complexes (see Chapter 3) and apply it to several problems in Commutative Algebra.

1.2. Hilbert and Poincaré-Betti Series. In this section we introduce the Hilbert and Poincaré-Betti series for multigraded k-algebras R. Here we assume that all modules are bigraded and have a decomposition into their graded parts of the following form:

$$M = \bigoplus_{i\in\mathbb{N}} \bigoplus_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=i}} M_{i,\alpha}.$$

Definition 1.6. (1) The multigraded Hilbert series $\mathrm{Hilb}_M^R(\underline{x},t)$ of M is given by

$$\mathrm{Hilb}_M^R(\underline{x},t) := \sum_{\substack{i\in\mathbb{N} \\ \alpha\in\mathbb{N}^n}} \dim_k \Big(M_{i,\alpha}\Big)\, \underline{x}^\alpha\, t^i.$$

(2) The multigraded Poincaré-Betti series $P_M^R(\underline{x},t)$ is given by

$$P_M^R(\underline{x},t) := \sum_{\substack{i\in\mathbb{N} \\ \alpha\in\mathbb{N}^n}} \dim_k \Big(\mathrm{Tor}_i^R(M,k)_\alpha\Big)\, \underline{x}^\alpha\, t^i.$$

If R is clear from the context, we sometimes write $P_M(\underline{x},t)$ (resp. $\mathrm{Hilb}_M(\underline{x},t)$) instead of $P_M^R(\underline{x},t)$ (resp. $\mathrm{Hilb}_M^R(\underline{x},t)$).

For a long time it was an open problem if these series are rational functions. For the Poincaré-Betti series $P_k^R(\underline{x},t)$ it was first proved by Golod, if the ring R is Golod (see [**24**]), in 1982 Backelin [**3**] proved the rationality for $P_k^R(\underline{x},t)$ if R is the quotient of a commutative polynomial ring by a monomial ideal. An explicit form of $P_k^R(\underline{x},t)$ for this case is studied in Chapter 6.
Sturmfels showed in 1998 [**40**] that if R is the quotient of a commutative polynomial ring by a special binomial ideal, then $P_k^R(\underline{x},t)$ is irrational.

If R is the quotient of a (non-)commutative polynomial ring by an ideal $\mathfrak{a}$, it is still open in which cases the Poincaré-Betti series is rational. In Chapter 5.2 we come back to this question.

The Hilbert series $\mathrm{Hilb}_R(\underline{x},t)$ of a commutative polynomial ring divided by an ideal $\mathfrak{a}$ is always rational. In the non-commutative case, it is also still open in which cases it is rational. But there exist examples for which the Hilbert series is irrational.
In Chapter 5.2 we prove the rationality of the Hilbert series $\mathrm{Hilb}_R(\underline{x},t)$ and Poincaré-Betti series $P_k^R(\underline{x},t)$ if $R = k\langle x_1,\dots,x_n\rangle/\mathfrak{a}$ and $\mathfrak{a}$ admits a quadratic Gröbner basis.

We finally want to give a relation between the Hilbert and the Poincaré-Betti series for Koszul rings.

Definition 1.7. A ring R is called Koszul if the (multigraded) minimal resolution of k – viewed as quotient of R by the maximal ideal $\mathfrak{m}$ – is linear, i.e.

$$\dim_k \left(\operatorname{Tor}_i^R(k,k)_\alpha \right) = \begin{cases} \neq 0 & , \quad |\alpha| = i+1 \\ 0 & , \quad |\alpha| \neq i+1. \end{cases}$$

Theorem 1.8. *If R is Koszul, then*

$$\operatorname{Hilb}_R(\underline{x},t) P_k^R(\underline{x},-t) = 1.$$

Proof. Let $C_\bullet$ be a minimal free resolution of k. Since R is Koszul, we have

$$C_i = \bigoplus_{j\in\mathbb{N}} \bigoplus_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=j}}^{\beta_{i,\alpha}} R(-\alpha) = \bigoplus_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=i}}^{\beta_{i,\alpha}} R(-\alpha)$$

and therefore

$$\operatorname{Hilb}_{C_i}(\underline{x},t) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=i}} \beta_{i,\alpha} \operatorname{Hilb}_{R(-\alpha)}(\underline{x},t) = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=i}} \beta_{i,\alpha} \underline{x}^\alpha t^i \operatorname{Hilb}_R(\underline{x},t).$$

Calculating the Euler characteristic of $C_\bullet$ we get:

$$\begin{aligned} 1 &= \sum_{i\geq 0} (-1)^i \operatorname{Hilb}_{C_i}(\underline{x},t) \\ &= \sum_{i\geq 0} (-1)^i \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=i}} \beta_{i,\alpha} \underline{x}^\alpha t^i \operatorname{Hilb}_R(\underline{x},t) \\ &= \sum_{i\geq 0} \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=i}} \beta_{i,\alpha} (-t)^i \underline{x}^\alpha \operatorname{Hilb}_R(\underline{x},t) \\ &= P_k^R(\underline{x},-t) \operatorname{Hilb}_R(\underline{x},t). \end{aligned}$$

□

A general relation between the Hilbert and the Poincaré-Betti series is studied in Chapter 6.

2. Examples for Chain Complexes in Commutative Algebra

In this paragraph we introduce some chain complexes which we use in this thesis. For each example, we only give the definition and some basic applications. For more detail see the given references.

2.1. Taylor and Scarf Complex. (see [**18**]) Let $S := k[x_1,\ldots,x_n]$ be the commutative polynomial ring and $\mathfrak{a} \trianglelefteq S$ a monomial ideal in S with minimal monomial generating system $\operatorname{MinGen}(\mathfrak{a}) = \{\mathrm{m}_1,\ldots,\mathrm{m}_l\}$. For a subset $I \subset \operatorname{MinGen}(\mathfrak{a})$ we write $m_I := \operatorname{lcm}\left\{m \in I\right\}$ for the least common multiple of the monomials in I. For simplification we sometimes may regard $J \subset \operatorname{MinGen}(\mathfrak{a})$ as subset of the index set $[l]$.

The *Taylor complex* $T_\bullet$ is given by

(1) T_i is the free S-module with basis e_J indexed by $J \subset [l]$ with $|J| = i$ for $i \geq 1$,
(2) $T_0 = S$,
(3) the differential $\partial_i : T_i \to T_{i-1}$ is given by

$$\partial_i(e_J) = \sum_{r=1}^{i} (-1)^{r+1} \frac{m_J}{m_{J \setminus \{j_r\}}} e_{J \setminus \{j_r\}}$$

in case $J = \{j_1 < \ldots < j_i\}$. It is easy to see that $T_\bullet$ is a complex.

Proposition 2.1. *The Taylor complex is a free resolution of $S/\mathfrak{a}$ as S-module, called the* Taylor resolution*. Moreover, the Taylor resolution is a cellular resolution supported by the simplex $\Delta = \Delta(\mathrm{MinGen}(\mathfrak{a})) = \Delta([l])$.* □

In the simplex Δ each face σ has a multidegree given by the corresponding least common multiple m_σ. Let $\Delta_{\mathcal{S}} \subset \Delta$ be the subcomplex of Δ consisting of those faces σ such that no other face $\tau \in \Delta_{\mathcal{S}}$ exists with $m_\sigma = m_\tau$.
The *Scarf complex* $\mathcal{S}_\bullet$ is given by

(1) $\mathcal{S}_i$ is the free S-module with basis e_σ indexed by $\sigma \in \mathcal{S}$ with $|\sigma| = i$ for $i \geq 0$,
(2) the differential $\partial_i : \mathcal{S}_i \to \mathcal{S}_{i-1}$ is given by

$$\partial_i(e_\sigma) = \sum_{\substack{\tau \in \sigma \\ |\tau| = i-1}} \varepsilon(\sigma, \tau) \frac{m_\sigma}{m_\tau} e_\tau,$$

where $\varepsilon(\sigma, \tau) = \pm 1$ and depends on the orientation of Δ.
Again, it is easy to see that $\mathcal{S}_\bullet$ is a subcomplex of the Taylor complex, but in general it is not a resolution. Directly from the definition we get:

Proposition 2.2. *The Scarf complex is cellular (supported by the simplicial complex $\Delta_{\mathcal{S}}$) and if it is a free resolution of $S/\mathfrak{a}$ as S-module, then it is even a minimal resolution, called the* Scarf resolution*.* □

2.2. Poset Resolution for a Monomial Ordered Family. (see [**39**]) In this section we introduce a resolution which is induced by a partially ordered set. This resolution was first introduced by [**39**].

Again, let $S := k[x_1, \ldots, x_n]$ be the commutative polynomial ring and $\mathfrak{a} \trianglelefteq S$ a monomial ideal in S and $\mathcal{B} \subset S$ a set of monomials such that $\mathfrak{a} = \langle \mathcal{B} \rangle$.

Definition 2.3. We say that $\mathcal{B}$ is a monomial ordered family if there exists a partially ordered set $P := (P, \prec)$ on the ground set $[|\mathcal{B}|]$ and a bijection $f : P \to \mathcal{B}$ such that

(OM) for any two monomials $m, n \in \mathcal{B}$ there exists a monomial $w \in \mathcal{B}$ such that
  (a) $f^{-1}(w) \succ f^{-1}(m), f^{-1}(n)$ and
  (b) $w \,|\, \mathrm{lcm}(m, n)$.

Note that $\mathcal{B}$ does not have to be a minimal generating system of $\mathfrak{a}$ (for example, if $\mathrm{MinGen}(\mathfrak{a})$ is a minimal monomial generating system of $\mathfrak{a}$, then $\mathcal{B} := \{m_J \big| J \subset \mathrm{MinGen}(\mathfrak{a})\}$ - the set of all least common multiples - ordered by divisibility is a monomial ordered family).

Let $\Delta := \Delta(P)$ be the order complex of the poset P. For a chain $\sigma \in \Delta$ with $\sigma = i_1 < \ldots < i_r \in \Delta$, $i_1, \ldots, i_r \in P$, we set $m_\sigma := \operatorname{lcm}\Big\{f(i_1), \ldots, f(i_r)\Big\}$. We define the complex $C(P)_\bullet$ as follows

(1) C_i is the free S-module with basis e_σ indexed by $\sigma \in \Delta(P)$ with $|\sigma| = i$ for $i \geq 0$,

(2) the differential $\partial_i : C_i \to C_{i-1}$ is given by

$$\partial_i(e_\sigma) = \sum_{\substack{\tau\in\sigma \\ |\tau|=i-1}} \varepsilon(\sigma,\tau)\frac{m_\sigma}{m_\tau} e_\tau,$$

where $\varepsilon(\sigma, \tau) = \pm 1$ and depends on the orientation of Δ.

Proposition 2.4. *The complex $C(P)_\bullet$ is a free cellular resolution of $S/\langle \mathcal{B} \rangle$ as an S-module, called the* poset resolution.

Proof. By definition the resolution is supported by the complex $\Delta(P)$. The assertion follows then by Lemma 3.3.2 of [**5**]. □

Clearly, the resolution can only be minimal if $\mathcal{B}$ is a minimal generating system for $\langle \mathcal{B} \rangle$, but even then the resolution is in general not minimal. In Chapter 4 we minimize this resolution via Algebraic Discrete Morse theory (ADMT). A second proof of Proposition 2.2 via ADMT is also given in Chapter 4.

If $\mathcal{B}$ is the set of lcm's of the minimal generating system, ordered by inclusion, then the poset resolution coincides with the lcm-resolution, introduced by Batzies [**4**].

2.3. Koszul Complex. (see [**18**]) Let R be a standard $\mathbb{N}^n$-graded ring and $x_1, \ldots, x_r \in R$ a regular sequence in R.
The complex $K_\bullet^R(x_1, \ldots, x_r)$ is defined as follows

(1) K_i is the free R-module with basis e_I indexed by $I \subset [r]$ with $|I| = i$ for $i \geq 0$,

(2) the differential $\partial_i : K_i \to K_{i-1}$ of e_I with $I = \{j_1 < \ldots < j_i\}$ is given by

$$\partial_i(e_I) = \sum_{l=1}^{i} (-1)^{l+1}\; x_{j_l}\; e_{I\setminus\{j_l\}}.$$

The complex $K_\bullet$ is called the Koszul complex of R with respect to the sequence $x_1, \ldots, x_r$. If the sequence is given from the context, we sometimes write K^R for the Koszul complex. We will need the following simple proposition:

Proposition 2.5. *If $R = k[x_1, \ldots, x_n]/\mathfrak{a}$ is the commutative polynomial ring divided by an ideal, then the Koszul complex with respect to the sequence $x_1, \ldots, x_n$ has homology $H_0(K^R) = k$.
If $R = k[x_1, \ldots, x_n]$, then K^R with respect to the sequence $x_1, \ldots, x_n$ is a minimal multigraded R-free resolution of k.* □

2.4. Bar Resolution. (see [**47**]) Let M be an R-module. Define

$$\mathcal{B}_i := R^{\otimes_k (i+1)} \otimes_k M.$$

Then the complex

$$\cdots \mathcal{B}_i \to \mathcal{B}_{i-1} \to \cdots \to \mathcal{B}_0 = R \otimes_k M \to M \to 0$$

with differential

$$\begin{aligned} r_1 \otimes r_2 \otimes \ldots \otimes r_{i+1} \otimes m \quad \mapsto \quad & \sum_{j=1}^{i} (-1)^{j+1} r_1 \otimes \ldots \otimes r_j r_{j+1} \otimes \ldots \otimes r_{i+1} \otimes m \\ & + (-1)^i r_1 \otimes \ldots \otimes r_i \otimes r_{i+1} m \end{aligned}$$

is a free resolution of the R-module M, called the Bar resolution.
If we write $\tilde{R} = \mathrm{Coker}(k \to R)$, where $k \to R$ is the map sending 1 to 1, and define $\mathcal{NB}_i := R \otimes \left(\tilde{R}^{\,\otimes i}\right) \otimes M$, then the complex

$$\cdots \mathcal{NB}_i \to \mathcal{NB}_{i-1} \to \cdots \to \mathcal{NB}_0 = R \otimes_k M \to M \to 0$$

with differential

$$\begin{aligned} r_0 \otimes \tilde{r}_1 \otimes \ldots \otimes \tilde{r}_i \otimes m \quad \mapsto \quad & r_0 \tilde{r}_1 \otimes \tilde{r}_2 \otimes \ldots \otimes \tilde{r}_i \\ & + \sum_{j=1}^{i-1} (-1)^j \; r_0 \otimes \tilde{r}_1 \ldots \otimes \tilde{r}_j \tilde{r}_{j+1} \otimes \ldots \otimes \tilde{r}_i \otimes m \\ & + (-1)^i r_0 \otimes \tilde{r}_1 \otimes \ldots \otimes \tilde{r}_{i-1} \otimes \tilde{r}_i \; m \end{aligned}$$

is an R-free resolution of M, called the normalized Bar resolution. A proof that the normalized Bar resolution can be derived from the Bar resolution is given in Chapter 3.3.

We consider the special case $M = k$. Since $R \otimes_k k \cong R$ we get in this case for the resolutions:

$$\mathcal{B}_i = R^{\otimes (i+1)},$$

with differential

$$\begin{aligned} r_1 \otimes r_2 \otimes \ldots \otimes r_{i+1} \quad \mapsto \quad & \sum_{j=1}^{i} (-1)^{j+1} r_1 \otimes \ldots \otimes r_j r_{j+1} \otimes \ldots \otimes r_{i+1} \\ & + (-1)^i \varepsilon(r_{i+1}) r_1 \otimes \ldots \otimes r_i r_{i+1}, \end{aligned}$$

where

$$\varepsilon(r_{i+1}) := \begin{cases} 1 & , \quad r_{i+1} \in k \\ 0 & , \quad \text{else,} \end{cases}$$

and

$$\mathcal{NB}_i = R \otimes \tilde{R}^{\,\otimes i}$$

with differential

$$\begin{aligned} r_0 \otimes \tilde{r}_1 \otimes \ldots \otimes \tilde{r}_i \quad \mapsto \quad & r_0 \tilde{r}_1 \otimes \tilde{r}_2 \otimes \ldots \otimes \tilde{r}_i \\ & + \sum_{j=1}^{i-1} (-1)^j \; r_0 \otimes \tilde{r}_1 \ldots \otimes \tilde{r}_j \tilde{r}_{j+1} \otimes \ldots \otimes \tilde{r}_i. \end{aligned}$$

Finally, we consider Bar and normalized Bar resolutions for k-algebras. Let A be a k-algebra and let W be a basis of A as a k-vectorspace such that $1 \in W$.

Lemma 2.6. *The Bar resolution is in this case given by*

$$\mathcal{B}_i \;:=\; \bigoplus_{w_1,\ldots,w_i \in W} A\,[w_1|\ldots|w_i]$$

with differential

$$\begin{aligned}\partial([w_1|\ldots|w_i]) &= w_1\,[w_2|\ldots|w_i] \\ &+\sum_{j=1}^{i-1}(-1)^j \left\{ \begin{array}{c} a_0\,[w_1|\ldots|w_{j-1}|1|w_{j+2}|\ldots|w_i] + \\ \sum_l a_l\,[w_1|\ldots|w_{j-1}|w'_l|w_{j+2}|\ldots|w_i] \end{array} \right\}\end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l\, w'_l$, *with* $a_0, a_l \in k$ *and* $w'_l \in W \setminus \{1\}$.
The normalized Bar resolution is in this case given by

$$\mathcal{NB}_i \;:=\; \bigoplus_{w_1,\ldots,w_i \in W\setminus\{1\}} A\,[w_1|\ldots|w_i]$$

with differential

$$\begin{aligned}\partial([w_1|\ldots|w_i]) &= w_1\,[w_2|\ldots|w_i] \\ &+\sum_{j=1}^{i-1}(-1)^j \sum_l a_l\,[w_1|\ldots|w_{j-1}|w'_l|w_{j+2}|\ldots|w_i]\end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l\, w'_l$, *with* $a_0, a_l \in k$ *and* $w'_l \in W \setminus \{1\}$.

Proof. Identifying $[w_1|\ldots|w_i]$ with $1 \otimes w_1 \otimes \ldots \otimes w_i$ proves the assertion. □

In the case $R = k\langle x_1,\ldots,x_n\rangle/\mathfrak{a}$, where R is the (not necessarily commutative) polynomial ring divided by an ideal $\mathfrak{a} = \langle f_1,\ldots,f_k\rangle$ such that the set $\{f_1,\ldots,f_k\}$ is a reduced Gröbner basis with respect to a fixed degree-monomial order $\prec$ (for example degree-lex or degree-revlex), one can choose for the basis $W \setminus \{1\}$ the set $\mathcal{G}$ of standard monomials of degree ≥ 1.

2.5. Acyclic Hochschild Complex. (see [**7**]) Let R be a commutative ring and A an R-algebra which is projective as an R-module. The acyclic Hochschild complex is defined as follows. For $n \geq -1$ we write $S_n(A)$ for the left $A \otimes_R A^{\mathrm{op}}$-module

$$\underbrace{A \otimes_R \ldots \otimes_R A}_{n+2 \text{ copies}},$$

where $A \otimes_R A^{\mathrm{op}}$ acts via

$$(\mu \otimes \gamma^*)(\lambda_0 \otimes \ldots \otimes \lambda_{n+1}) = (\mu\lambda_0) \otimes \lambda_1 \otimes \ldots \otimes \lambda_n \otimes (\lambda_{n+1}\gamma).$$

We define the maps

$$\begin{aligned} b'_n : S_n(A) &\;\to\; S_{n-1}(A) \\ \lambda_0 \otimes \ldots \otimes \lambda_{n+1} &\;\mapsto\; \sum_{i=0}^{n}(-1)^i \lambda_0 \otimes \ldots \otimes \lambda_i\lambda_{i+1} \otimes \ldots \otimes \lambda_{n+1}, \end{aligned}$$

$$\begin{aligned} s_n : S_{n-1}(A) &\;\to\; S_n(A) \\ \lambda_0 \otimes \ldots \otimes \lambda_n &\;\mapsto\; \lambda_0 \otimes \ldots \otimes \lambda_n \otimes 1. \end{aligned}$$

Then $b'_{n-1} \circ b'_n = 0$ and $b'_{n+1} \circ s_{n+1} - s_n \circ b'_n = \mathrm{id}$ and therefore $S_\bullet(A)$ is exact. If we write $\tilde{S}_n(A)$ for the module

$$\underbrace{A \otimes_R \ldots \otimes_R A}_{n+2 \text{ copies}},$$

we have an isomorphism $S_n(A) \cong (A \otimes A^{\mathrm{op}}) \otimes \tilde{S}_n(A)$ as $(A \otimes A^{\mathrm{op}})$-modules. Since A is R-projective it follows that $S_n(A)$ is a projective $(A \otimes A^{\mathrm{op}})$-module, and therefore $S_\bullet(A)$ is a projective resolution of A as an $(A \otimes A^{\mathrm{op}})$-module. The complex $S_\bullet(A)$ is called the *acyclic Hochschild complex.*

The acyclic Hochschild complex is used to define the Hochschild (co-)homology of an $(A \otimes_k A^{\mathrm{op}})$-module M. Here, we only want to give a short definition for A-bimodules. For more details see [**7**].

Let M be a A-bimodule. We regard it as a right $A \otimes A^{\mathrm{op}}$-module via $a(\mu \otimes \gamma^*) = \gamma a \mu$. The *Hochschild homology* $HH_n(A, M)$ of A with coefficients in M is defined to be the homology of the *Hochschild complex*

$$S_n(A, M) := M \otimes_{A \otimes A^{\mathrm{op}}} S_n(A).$$

The *Hochschild cohomology* $HH^n(A, M)$ of A with coefficients in M is defined to be the cohomology of the cochain complex

$$S^n(A, M) := \mathrm{Hom}_{A \otimes A^{\mathrm{op}}}(S_n(A), M).$$

Proposition 2.7 (see [**7**])**.**

$$\begin{aligned} HH_n(A, M) &\cong \mathrm{Tor}_n^{A \otimes A^{\mathrm{op}}}(M, A) \\ HH^n(A, M) &\cong \mathrm{Ext}^n_{A \otimes A^{\mathrm{op}}}(M, A) \end{aligned}$$

This Lemma shows that it is useful to minimize the acyclic Hochschild complex in order to calculate the Hochschild homology of the k-algebra A with coefficients in k, where $A = \langle x_1, \ldots, x_n \rangle / \mathfrak{a}$ is a (non-commutative) polynomial ring. We do this in Chapter 5.3. For this special case we finally want to give another description of the acyclic Hochschild complex.

Let A be a k-algebra and let W be a basis of A as a k-vector space such that $1 \in W$. The acyclic Hochschild complex

$$\mathcal{HC}_A : \cdots \stackrel{\partial_{i+1}}{\to} C_i \stackrel{\partial_i}{\to} C_{i-1} \stackrel{\partial_{i-1}}{\to} \cdots \stackrel{\partial_2}{\to} C_1 \stackrel{\partial_1}{\to} C_0 = k$$

with respect to W is then given by:

(1) C_i is the free $(A \otimes_k A)$-module with basis $[w_1|\ldots|w_i]$, $w_1, \ldots, w_i \in W$,

(2) the differential ∂_i is given by

$$\begin{aligned} \partial_i([w_1|\ldots|w_i]) = (w_1 \otimes 1)\,[w_2|\ldots|w_i] + (-1)^i (1 \otimes w_i)\,[w_1|\ldots|w_{i-1}] \\ + \sum_{j=1}^{i-1} (-1)^j \begin{pmatrix} a_0\,[w_1|\ldots|w_{j-1}|1|w_{j+2}|\ldots|w_i] \\ + \sum_l a_l\,[w_1|\ldots|w_{j-1}|w'_l|w_{j+2}|\ldots|w_i] \end{pmatrix} \end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l\, w'_l$, with $a_0, a_l \in k$ and $w'_l \in W \setminus \{1\}$.

In this case the *normalized acyclic Hochschild complex* $\mathcal{NHC}$ is defined by

(1) C_i is the free $(A \otimes_k A)$-module with basis $[w_1|\dots|w_i]$, $w_1, \dots, w_i \in W \setminus \{1\}$,

(2) the differential ∂_i is given by

$$\partial_i([w_1|\dots|w_i]) = (w_1 \otimes 1)\,[w_2|\dots|w_i] + (-1)^i (1 \otimes w_i)\,[w_1|\dots|w_{i-1}] + \sum_{j=1}^{i-1} (-1)^j \left(\sum_l a_l\,[w_1|\dots|w_{j-1}|w_l'|w_{j+2}\dots|w_i] \right)$$

if $w_j w_{j+1} = a_0 + \sum_l a_l\, w_l'$, with $a_0, a_l \in k$ and $w_l' \in W \setminus \{1\}$.

A proof of the acyclic normalized Hochschild complex is given in Chapter 3.3.

3. Eagon Complex and the Golod Property

In the first section of this paragraph we introduce a resolution of the residue class field $k = R/\mathfrak{m}$ over R, called the Eagon resolution, discovered by Eagon (see [**26**]). In the second section we introduce the Massey operations on the Koszul homology and the Golod property of R. We show that the following three conditions are equivalent:

(1) The Eagon complex is minimal.
(2) All Massey operations vanish.
(3) The ring R is Golod.

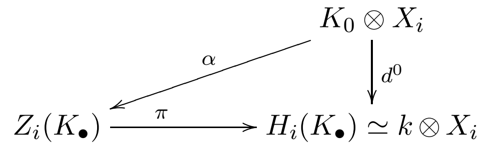
For the whole paragraph we follow the notes in [**26**].

This paragraph is a preparation for Chapter 6, where we generalize the Massey operations in order to get a more explicit description of the Eagon complex. We then define an acyclic matching on the Eagon complex and the resulting Morse complex helps us to explain our conjecture about the minimal resolution of k over $R = S/\mathfrak{a}$, where $\mathfrak{a} \subset S$ is a monomial ideal in the commutative ring of polynomials. This conjecture has interesting consequences for the Golod property of monomial rings.

3.1. The Eagon Resolution. Let $K_\bullet$ be any complex of free R-modules of finite type such that $H_i(K_\bullet)$ is a k-vector space for each $i > 0$ and $H_0(K_\bullet) \cong k$. For example, the Koszul complex $K_\bullet = K_\bullet^R$ satisfies these constraints. We denote with $Z(K_\bullet)$ the set of cycles and with $B(K_\bullet)$ the set of boundaries of $K_\bullet$.
Let X_i, $i \geq 0$, be free R-modules such that $X_i \otimes k \cong H_i(K_\bullet)$. We define a sequence of complexes inductively. $Y^0 = K_\bullet$ and d^0 is the differential of the complex $K_\bullet$. Assuming Y^n is defined, we set

$$\begin{aligned} Y_i^{n+1} &:= Y_{i+1}^n \oplus Y_0^n \otimes X_i \text{ if } i > 0, \\ Y_0^{n+1} &:= Y_1^n. \end{aligned}$$

Now we define the differential d^1 on $Y_i^1 = K_{i+1} \oplus K_0 \otimes X_i$: Since $K_0 \otimes X_i$ is free (hence projective), there exists a map $\alpha : K_0 \otimes X_i \to Z_i(K)$ making the diagram



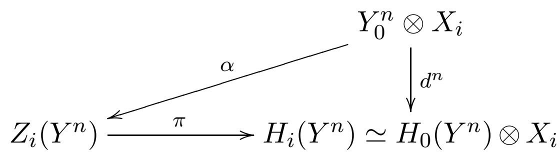
commute. Then for $x \in K_0$ and $y \in X_i$ we define

$$d^1(x \otimes y) = \alpha(x \otimes y) \in Z_i(K_\bullet).$$

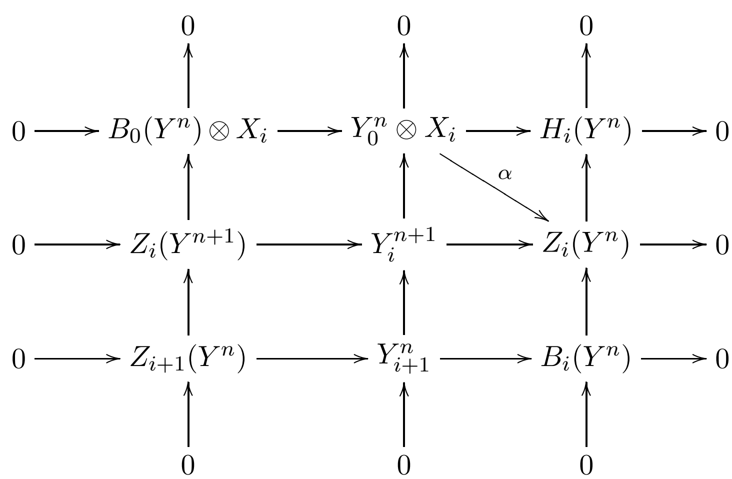
By definition we have $d^1 \circ d^1 = 0$ and $d^1(Y_i^1) = Z_i(K_\bullet)$.
We continue this process by induction:

Lemma 3.1. *Assume that d^n has been defined on Y^n such that $H_i(Y^n) \cong H_0(Y^n) \otimes X_i$. Then one can define d^{n+1} on Y^{n+1} such that $H_i(Y^{n+1}) \cong H_0(Y^{n+1}) \otimes X_i$.*

Proof. As above there exists a map $\alpha : Y_0^n \otimes X_i \to Z_i(K_\bullet)$ making the diagram



commute. This implies the following commutative diagram:



The rows are exact and the last two columns are exact. Therefore, the 9-lemma from homological algebra tells us that the first column is exact. By construction we have $Z_{i+1}(Y^n) = B_i(Y^{n+1})$. Now we have in addition the exactness of

$$0 \longrightarrow B_0(Y^{n+1}) \longrightarrow Y_0^{n+1} \longrightarrow H_0(Y^{n+1}) \longrightarrow 0$$

$$0 \longrightarrow Z_1(Y^n) \longrightarrow Y_1^n \longrightarrow B_0(Y^n) \longrightarrow 0.$$

Since $B_0(Y^{n+1}) = Z_1(Y^n)$ and $Y_0^{n+1} = Y_1^n$, we get $B_0(Y^n) \cong H_0(Y^{n+1})$ and the exactness of the first column implies $H_i(Y^{n+1}) \cong H_0(Y^{n+1}) \otimes X_i$ and we are done. □

Criterion 3.2. *To make the diagram*

$$\begin{array}{ccc} & & Y_0^s \otimes X_i \simeq Y_1^{s-1} \otimes X_i \\ & \swarrow^{\alpha} & \big\downarrow {\scriptstyle d^{s-1}} \\ Z_i(Y^s) & \xrightarrow{\ \pi\ } & H_i(Y^s) \simeq B_0(Y^{s-1}) \otimes X_i \end{array}$$

commutative, it is enough to define for $n \otimes f \in Y_0^s \otimes X_i$ the map α such that $\alpha(n \otimes f) = (m, d^{s-1}(n) \otimes f)$, with $m \in Y_{i+1}^{s-1}$ and

$$d^{s-1}(m) + d^{s-1}\big(d^{s-1}(n) \otimes f\big) = 0.$$

Proof. Since $H_i(Y^s) \simeq B_0(Y^{s-1}) \otimes X_i$, the assertion follows. □

Corollary 3.3. *The complex*

$$\cdots \longrightarrow Y_0^{n+1} \xrightarrow{d^{n+1}} Y_0^n \xrightarrow{d^n} \cdots \longrightarrow Y_0^1 \xrightarrow{d^1} Y_0^0 \xrightarrow{\pi} k$$

is an R-free resolution of k, called the Eagon resolution.

Proof. By construction we have $d^{n+1}(Y_i^{n+1}) \subset Y_i^n$, therefore the complex is well defined. Since $Y_0^n = Y_1^{n-1}$, we have $Z_0(Y^n) = Z_1(Y^{n-1})$. The exactness follows since $Z_1(Y^{n-1}) = B_0(Y^n)$. □

3.2. The Massey Operations and the Golod Property. In this section we introduce the Massey operations. From now on let $K_\bullet$ be the Koszul complex. The Massey operations are defined by induction. Let $z_1, z_2 \in Z(K_\bullet)$ be two cycles in the complex $K_\bullet$. Then their product $z_1 z_2$ is again a cycle and the class of $z_1 z_2$ in $H(K_\bullet)$ only depends on the classes of z_1 and z_2 in $H(K_\bullet)$. This product makes $H(K_\bullet)$ into a ring. Then $\gamma(z_1, z_2) := [z_1][z_2]$ is a well defined product on the homology $H(K_\bullet)$ and we call it the first Massey operation.
Now assume that the first Massey operation vanishes for all cycles $z_1, z_2 \in Z(K_\bullet)$. Then there exist elements $g(z_1, z_2) \in K_\bullet$ such that $d^0(g(z_1, z_2)) = z_1 z_2$. Let $z_1, z_2, z_3 \in Z(K_\bullet)$ be three cycles. Then it is straightforward to prove that

$$\gamma(z_1, z_2, z_3) := z_1 \ g(z_2, z_3) + (-1)^{\deg(z_1)+1} \ g(z_1, z_2) \ z_3$$

is again a cycle and the class of $\gamma(z_1, z_2, z_3)$ in $H(K_\bullet)$ only depends on the classes of the cycles z_1, z_2, z_3 in $H(K_\bullet)$. Therefore, $\gamma(z_1, z_2, z_3)$ induces a well defined operation on the homology $H(K_\bullet)$. We call $\gamma(z_1, z_2, z_3)$ the ternary Massey operation on $H(K_\bullet)$. We go on by induction. Assume that the $(n-1)$th order Massey operation is given by

$$\begin{aligned} \gamma(z_1, \ldots, z_{n-1}) \quad := \quad & z_1 \ g(z_2, \ldots, z_{n-1}) \\ & + (-1)^{\sum_{j=1}^{n-3} (\deg(z_j)+1)} \ g(z_1, \ldots, z_{n-2}) \ z_{n-1} \\ & + \sum_{j=2}^{n-3} (-1)^{\sum_{i=1}^{j-1} (\deg(z_i)+1)} \ g(z_1, \ldots, z_j) \ g(z_{j+1}, \ldots, z_{n-1}) \end{aligned}$$

and that the class of $\gamma(z_1, \ldots, z_{n-1})$ in $H(K_\bullet)$ only depends on the classes of $z_1, \ldots, z_{n-1}$ in $H(K_\bullet)$.
If the $(n-1)$th order Massey operation vanishes for all cycles $z_1, \ldots, z_{n-1}$, then there exist elements $g(z_1, \ldots, z_{n-1})$ such that $d(g(z_1, \ldots, z_{n-1})) = \gamma(z_1, \ldots, z_{n-1})$.

We then define the n-th order Massey operation for cycles $z_1, \ldots, z_n \in Z(K_\bullet)$ by

$$\begin{aligned}\gamma(z_1, \ldots, z_n) \;:=\; & z_1\; g(z_2, \ldots, z_n) \\ & +(-1)^{\sum_{j=1}^{n-2}(\deg(z_j)+1)}\; g(z_1, \ldots, z_{n-1})\; z_n \\ & +\sum_{j=2}^{n-2}(-1)^{\sum_{i=1}^{j-1}(\deg(z_i)+1)}\; g(z_1, \ldots, z_j)\; g(z_{j+1}, \ldots, z_n).\end{aligned}$$

Again, it is straightforward to prove that $\gamma(z_1, \ldots, z_n)$ is a cycle and that the class of $\gamma(z_1, \ldots, z_n)$ in $H(K_\bullet)$ only depends on the classes of $z_1, \ldots, z_n$ in $H(K_\bullet)$ and therefore the operation $\gamma(z_1, \ldots, z_n)$ on the homology $H(K_\bullet)$ is well defined.

The Massey operations helps us to give an explicit description of the Eagon complex:

Theorem 3.4. *If all Massey operations vanish, then the differential of the Eagon complex is given by*

$$\begin{aligned}d^s\big(c \otimes z_1 \otimes \ldots \otimes z_n\big) := {} & d^0(c) \otimes z_1 \otimes \ldots \otimes z_n \\ & +(-1)^{\deg(c)}\; c\; z_1 \otimes \ldots \otimes z_n \\ & +\sum_{j=2}^{n}(-1)^{\sum_{i=1}^{j-1}(\deg(z_i)+1)}\; c\; g(z_1, \ldots, z_j) \otimes z_{j+1} \otimes \ldots \otimes z_n.\end{aligned}$$

In particular, the Eagon complex is a minimal (multigraded) R-resolution of the residue class field k.

Proof. We have to define the maps α such that the diagram below commutes.

$$\begin{array}{ccc} & & Y_0^s \otimes X_i \simeq Y_1^{s-1} \otimes X_i \\ & \swarrow^{\alpha} & \big\downarrow d^{s-1} \\ Z_i(Y^s) & \xrightarrow{\;\pi\;} & H_i(Y^s) \simeq B_0(Y^{s-1}) \otimes X_i \end{array}$$

By Criterion 3.2 we can define for $n \otimes f \in Y_0^s \otimes X_i$ the map $\alpha(n \otimes f) = (m, d^{s-1}(n) \otimes f)$, with $m \in Y_{i+1}^{s-1}$ and $d^{s-1}(m) + d^{s-1}\big(d^{s-1}(n) \otimes f\big) = 0$.
Assume that for $c \otimes z_1 \otimes z_2 \otimes \ldots \otimes z_r \in Y^{s-1}$ we have

$$\begin{aligned} & d^{s-1}\big(c \otimes z_1 \otimes z_2 \otimes \ldots \otimes z_r\big) \\ = \;& d^0(c) \otimes z_1 \otimes z_2 \otimes \ldots \otimes z_r + (-1)^{\deg(c)}\; c\; z_1 \otimes z_2 \otimes \ldots \otimes z_r \\ & +\sum_{j=2}^{r}(-1)^{\sum_{i=1}^{j-1}(\deg(z_i)+1)}\; c\; g(z_1, \ldots, z_j) \otimes z_{j+1} \otimes \ldots \otimes z_r.\end{aligned}$$

For $(c \otimes z_1 \otimes z_2 \otimes \ldots \otimes z_r) \otimes z_{r+1}$ we define

$$m := \begin{cases} (-1)^{\deg(c)}(-1)^{\sum_{i=1}^{r}(\deg(z_i)+1)}\; c\; g(z_1, \ldots, z_r, z_{r+1}) & , \quad r > 0 \\ (-1)^{\deg(c)} c\; z_{r+1} & , \quad r = 0. \end{cases}$$

Now consider $d^{s-1}(m)$:

$$\begin{aligned} d^{s-1}(m) \;=\; & (-1)^{\deg(c)}(-1)^{\sum_{i=1}^{r}(\deg(z_i)+1)}\; d^0(c)\; g(z_1, \ldots, z_r, z_{r+1}) \\ + \; & (-1)^{\sum_{i=1}^{r}(\deg(z_i)+1)}\; c\; \gamma(z_1, \ldots, z_r, z_{r+1}).\end{aligned}$$

Since $d^{s-1}d^{s-1} = 0$, we have for $d^{s-1}\big(d^{s-1}(c \otimes z_1 \otimes z_2 \otimes \ldots \otimes z_r) \otimes z_{r+1}\big)$:

$$
\begin{aligned}
&d^{s-1}\big(d^{s-1}(c \otimes z_1 \otimes z_2 \otimes \ldots \otimes z_r) \otimes z_{r+1}\big)\\
&= (-1)^{\deg(d^0(c))}(-1)^{\sum_{i=1}^{r}(\deg(z_i)+1)} d^0(c)\ g(z_1,\ldots,z_r,z_{r+1})\\
&\quad +(-1)^{\deg(z_1)}(-1)^{\sum_{i=2}^{r}(\deg(z_i)+1)}\ c\ z_1\ g(z_2,\ldots,z_r,z_{r+1})\\
&\quad +\sum_{j=2}^{r-1}\left\{\begin{array}{l}(-1)^{\sum_{i=1}^{j-1}(\deg(z_i)+1)}(-1)^{\deg(g(z_1,\ldots,z_j))}(-1)^{\sum_{i=j+1}^{r}(\deg(z_i)+1)}\\ c\ g(z_1,\ldots,z_j)g(z_{j+1},\ldots,z_r,z_{r+1})\end{array}\right.\\
&\quad +(-1)^{\sum_{i=1}^{r-1}(\deg(z_i)+1)}(-1)^{\deg(g(z_1,\ldots,z_r))}\ c\ g(z_1,\ldots,z_r)\ z_{r+1}\\
&= (-1)^{\deg(d^0(c))}(-1)^{\sum_{i=1}^{r}(\deg(z_i)+1)} d^0(c)\ g(z_1,\ldots,z_r,z_{r+1})\\
&\quad +(-1)(-1)^{\sum_{i=1}^{r}(\deg(z_i)+1)}\ c\ \gamma(z_1,\ldots,z_r,z_{r+1})
\end{aligned}
$$

since $\deg(g(z_1,\ldots,z_j)) = -1+\sum_{i=1}^{j}(\deg(z_j)+1)$. Therefore, our map α is well defined and the desired form of the differential follows. The Eagon complex is in this case minimal since no coefficient of the differential lies in the field k. □

We now come to the Golod property of monomial rings. This property was introduced in order to prove that the Poincaré-Betti series is rational. This class was the first class for which one could prove the rationality.

Definition 3.5. We call a ring R a Golod ring, if all Massey operations on the Koszul homology $H_\bullet(K^R)$ vanish.

We have the following equivalence:

Theorem 3.6. *The following statements are equivalent:*

(1) *The Eagon complex is a minimal multigraded R-free resolution of k.*
(2) *All Massey operations on the Koszul homology $H_\bullet(K^R)$ vanish.*
(3) *The ring R is a Golod ring.*
(4) *The multigraded Poincaré-Betti series of R is given by*

$$P_k^R(\underline{x},t) = \frac{\mathrm{Hilb}_{K_\bullet \otimes_R k}(\underline{x},t)}{1 - t\ \sum_{i,\alpha}\beta_{i,\alpha}\ \underline{x}^\alpha\ t^i},$$

where $\beta_{i,\alpha} = \dim_k\big((H_i(K^R))_\alpha\big)$.

Proof. (2) ⇒ (1) follows from Theorem 3.4, (2) ⇔ (3) is the definition of a Golod ring, and (4) ⇔ (1) follows by counting basis elements in the Eagon resolution. Thus, we only have to proof (1) ⇒ (2):
Since this is a very long and technical, but not so difficult proof we forbear from doing it here and give just a sketch of it. The reader can find the full proof in [**26**]. With the same arguments as in Criterion 3.2 one can assume that for $y \in Y_0^n$ and $x \in X_i$ the differential (resp. the map α) is given by

$$\alpha(y \otimes x) = dy \otimes x + (-1)^n \beta(y,x),$$

where $\beta(y,x) \in Y_{i+1}^{n-1}$. Then one can prove the following statements:

- ▷ If the Eagon complex is minimal, then one can choose $\beta(y,x) \in Y^0 = K$ ([**26**], Theorem 4.1.2).

- ▷ If one can choose $\beta(y,x) \in K$ and $\mathfrak{m}^2 K \cap B(K) \subset \mathfrak{m} B(K)$, then the Eagon complex is minimal ([**26**], Theorem 4.1.3).
- ▷ The Koszul complex with respect to the sequence $x_1, \ldots, x_n$ satisfies $\mathfrak{m}^2 K \cap B(K) \subset \mathfrak{m} B(K)$.
- ▷ It is possible to choose $\beta(y,x)$ such that for $\lambda \in K$, $\beta(\lambda y, x) = \lambda\beta(y,x)$. It follows that for $\lambda \in K_i$, $w \in Y_j^n$ we have ([**26**], Lemma 4.2.1)

$$d(\lambda w) = (d\lambda)w + (-1)^i \lambda dw. \tag{3.1}$$

- ▷ Show that $d\beta(y,x) = \beta(dy,x) \in B(K)$ and identify (using (3.1) and induction) $\beta(dy,x)$ with the Massey operations ([**26**], Theorem 4.2.2).

Clearly, these statements imply the equivalence of (1) and (2). □

In general, the Eagon complex is not minimal, but it gives an upper bound for the Poincaré-Betti series:

Corollary 3.7. *With the notation from Theorem 3.6 we have for the multigraded Poincaré-Betti series:*

$$P_k^R(\underline{x},t) \leq \frac{\mathrm{Hilb}_{K_\bullet \otimes_R k}(\underline{x},t)}{1 - t\ \sum_{i,\alpha} \beta_{i,\alpha}\ \underline{x}^\alpha\ t^i}.$$

Equality holds if and only if R is a Golod ring. □

We have the following criterion for Golodness:

Proposition 3.8. *Let S be a k-basis of $H_\bullet(K)$. If for any $z_1, z_2 \in S$ we have $z_1 \cdot z_2 = 0$, then R is a Golod ring.*

Proof. We define the modules $X_i := \bigoplus_{\substack{z\in S\\ \deg(z)=i}} R\ e_z$. Remember from Criterion 3.2 that in order to calculate the differential of the Eagon complex it is enough to define the map $\alpha : Y_0^s \otimes X_i \to Z_i(Y^s)$ such that for any $n \otimes f \in Y_0^s$ we have $\alpha(n\otimes f) = (m, d^{s-1}(n)\otimes f)$, with $m \in Y_{i+1}^{s-1}$ and $d^{s-1}(m) + d^{s-1}\big(d^{s-1}(n)\otimes f\big) = 0$. We define m as follows. Let $n \otimes e_z \in Y_0^s$.

$$m := \begin{cases} (-1)^{\deg(n)} n \cdot z & , \quad \text{if } n \in Y_S^0 = K_s, \\ 0 & , \quad \text{else.} \end{cases}$$

Assume that for $c \otimes e_{z_1} \otimes e_{z_2} \otimes \ldots \otimes e_{z_r} \in Y^{s-1}$ we have

$$\begin{aligned} d^{s-1}\big(c\otimes e_{z_1}\otimes e_{z_2}\otimes\ldots\otimes e_{z_r}\big) &= d^0(c)\otimes e_{z_1}\otimes e_{z_2}\otimes\ldots\otimes e_{z_r} \\ &+ (-1)^{\deg(c)}\ c\ z_1 \otimes e_{z_2}\otimes\ldots\otimes e_{z_r}. \end{aligned}$$

Let $n\otimes f := \big(c\otimes e_{z_1}\otimes e_{z_2}\otimes\ldots\otimes e_{z_r}\big)\otimes e_z \in Y^s \otimes X_i$. We have to prove that $d^{s-1}(m) + d^{s-1}\big(d^{s-1}(n)\otimes f\big) = 0$. First case $r > 0$. Then m equals to 0 and therefore $d^{s-1}(m) = 0$. We have:

$$\begin{aligned} & d^{s-1}\Big(d^{s-1}\big(c\otimes e_{z_1}\otimes\ldots\otimes e_{z_r}\big)\otimes e_z\Big) \\ = \ & d^{s-1}\Big(d^0(c)\otimes e_{z_1}\otimes\ldots\otimes e_{z_r}\otimes e_z + (-1)^{\deg(c)}\ c\cdot z_1\otimes e_{z_2}\otimes\ldots\otimes e_{z_r}\otimes e_z\Big) \\ = \ & (-1)^{\deg(d^0(c))}\ d^0(c)\ z_1\otimes e_{z_2}\otimes\ldots\otimes e_{z_r}\otimes e_z \\ & +(-1)^{\deg(c)} d^0(c)\ z_1\otimes e_{z_2}\otimes\ldots\otimes e_{z_r}\otimes e_z \\ & +(-1)^{\deg(c)+\deg(z_1)} c\ z_1\ z_2\otimes e_{z_3}\otimes\ldots\otimes e_{z_r}\otimes e_z \end{aligned}$$

$$= \quad 0$$

since $z_1\ z_2 = 0$ and $\deg(c) = \deg(d^0(c)) + 1$.
Now assume $r = 0$. Then $n \otimes f := c \otimes e_z$. By definition we have $m := (-1)^{\deg(c)} c\ z$. It follows:

$$\begin{aligned} d^{s-1}(m) &= (-1)^{\deg(c)} d^0(c)\ z, \\ d^{s-1}(d^{s-1}(c) \otimes e_z) &= d^{s-1}(d^0(c) \otimes e_z) = (-1)^{\deg(d^0(c))}. \end{aligned}$$

This proves that the differential of the Eagon complex is given by Equation (3.2). Clearly, the differential contains no coefficient in k and therefore the resolution is minimal. By Theorem 3.6 the ring R is Golod. □

We finally consider the case where $R := S/\mathfrak{a}$ is the quotient algebra of the commutative polynomial ring $S := k[x_1, \ldots, x_n]$ and $\mathfrak{a} \subset S$ is a monomial ideal. In this case we get:

Corollary 3.9. *The ring A is a Golod ring if and only if the multigraded Poincaré-Betti series has the following form*

$$P_k^A(\underline{x}, t) := \frac{\prod_{i=1}^n (1 + t\ x_i)}{1 - t\ \sum_{i,\alpha} \beta_{i,\alpha}\ \underline{x}^\alpha\ t^i},$$

where $\beta_{i,\alpha} := \dim_k \left(\mathrm{Tor}_i^S(A, k)_\alpha \right)$. □

Algebraic Discrete Morse Theory

In this chapter we derive an algebraic version of Discrete Morse theory as developed by Forman (see [**21**], [**22**]). Our theory is a generalization of results from [**5**] and an almost identical theory has been developed independently by Sköldberg [**41**]. Our applications require a slightly more general setting than the one covered in [**41**].

The contents of this chapter can also be found in our article [**36**].

1. Algebraic Discrete Morse Theory

Let R be a ring and $\mathsf{C}_\bullet = (C_i, \partial_i)_{i\geq 0}$ be a chain complex of free R-modules C_i. We choose a basis $X = \bigcup_{i\geq 0} X_i$ such that $C_i \simeq \bigoplus_{c\in X_i} R\, c$. From now on we write the differentials ∂_i with respect to the basis X in the following form:

$$\partial_i : \begin{cases} C_i & \to \quad C_{i-1} \\ c & \mapsto \quad \partial_i(c) = \displaystyle\sum_{c'\in X_{i-1}} [c:c'] \cdot c'. \end{cases}$$

Given the complex $\mathsf{C}_\bullet$ and the basis X, we construct a directed, weighted graph $G(\mathsf{C}_\bullet) = (V, E)$. The set of vertices V of $G(\mathsf{C}_\bullet)$ is the basis $V = X$ and the set E of (weighted) edges is given by the rule

$$(c, c', [c:c']) \in E \quad :\Leftrightarrow \quad c \in X_i, c' \in X_{i-1}, \text{ and } [c:c'] \neq 0.$$

We often omit the weight and write $c \to c'$ to denote an edge in E. Also by abuse of notation we write $e \in G(\mathsf{C}_\bullet)$ to indicate that e is an edge in E.

Definition 1.1. A finite subset $\mathcal{M} \subset E$ of the set of edges is called an acyclic matching if it satisfies the following three conditions:

(1) (Matching) Each vertex $v \in V$ lies in at most one edge $e \in \mathcal{M}$.
(2) (Invertibility) For all edges $(c, c', [c:c']) \in \mathcal{M}$ the weight $[c:c']$ lies in the center of R and is a unit in R.

(3) (Acyclicity) The graph $G_{\mathcal{M}}(V, E_{\mathcal{M}})$ has no directed cycles, where $E_{\mathcal{M}}$ is given by

$$E_{\mathcal{M}} := (E \setminus \mathcal{M}) \cup \left\{ \left(c', c, \frac{-1}{[c : c']} \right) \text{ with } (c, c', [c : c']) \in \mathcal{M} \right\}.$$

For an acyclic matching $\mathcal{M}$ on the graph $G(\mathsf{C}_\bullet) = (V, E)$ we introduce the following notation, which is an adaption of the notation introduced in [**21**] to our situation.

(1) We call a vertex $c \in V$ critical with respect to $\mathcal{M}$ if c does not lie in an edge $e \in \mathcal{M}$; we write

$$X_i^{\mathcal{M}} := \{ c \in X_i \mid c \text{ critical } \}$$

for the set of all critical vertices of homological degree i.

(2) We write $c' \leq c$ if $c \in X_i$, $c' \in X_{i-1}$, and $[c : c'] \neq 0$.

(3) $\mathrm{Path}(c, c')$ is the set of paths from c to c' in the graph $G_{\mathcal{M}}(\mathsf{C}_\bullet)$.

(4) The weight $w(p)$ of a path $p = c_1 \to \cdots \to c_r \in \mathrm{Path}(c_1, c_r)$ is given by

$$\begin{aligned} w(c_1 \to \ldots \to c_r) &:= \prod_{i=1}^{r-1} w(c_i \to c_{i+1}), \\ w(c \to c') &:= \begin{cases} -\dfrac{1}{[c : c']} &, \quad c \leq c', \\[2ex] [c : c'] &, \quad c' \leq c. \end{cases} \end{aligned}$$

(5) We write $\Gamma(c, c') = \sum_{p \in \mathrm{Path}(c, c')} w(p)$ for the sum of weights of all paths from c to c'.

Now we are in position to define a new complex $\mathsf{C}_\bullet^{\mathcal{M}}$, which we call the Morse complex of $\mathsf{C}_\bullet$ with respect to $\mathcal{M}$. The complex $\mathsf{C}_\bullet^{\mathcal{M}} = (C_i^{\mathcal{M}}, \partial_i^{\mathcal{M}})_{i \geq 0}$ is defined by

$$C_i^{\mathcal{M}} := \bigoplus\nolimits_{c \in X_i^{\mathcal{M}}} R\, c,$$

$$\partial_i^{\mathcal{M}} : \begin{cases} C_i^{\mathcal{M}} & \to & C_{i-1}^{\mathcal{M}} \\ c & \mapsto & \sum\limits_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma(c, c') c', \end{cases} .$$

Theorem 1.2. *$\mathsf{C}_\bullet^{\mathcal{M}}$ is a complex of free R-modules and is homotopy-equivalent to the complex $\mathsf{C}_\bullet$; in particular, for all $i \geq 0$*

$$H_i(\mathsf{C}_\bullet) \cong H_i(\mathsf{C}_\bullet^{\mathcal{M}}).$$

The maps defined below give a chain homotopy between $\mathsf{C}_\bullet$ and $\mathsf{C}_\bullet^{\mathcal{M}}$:

$$f : \begin{cases} \mathsf{C}_\bullet & \to & \mathsf{C}_\bullet^{\mathcal{M}} \\ c \in X_i & \mapsto & f(c) := \sum\limits_{c' \in X_i^{\mathcal{M}}} \Gamma(c, c') c', \end{cases}$$

$$g : \begin{cases} \mathsf{C}_\bullet^{\mathcal{M}} & \to \quad \mathsf{C}_\bullet \\ c \in X_i^{\mathcal{M}} & \mapsto \quad g_i(c) := \sum_{c' \in X_i} \Gamma(c, c')c'. \end{cases}$$

The proof of Theorem 1.2 is given in the next paragraph. Note that if $\mathsf{C}_\bullet$ is the cellular chain complex of a regular CW-complex and X is the set of cells of the regular CW-complex, then Algebraic Discrete Morse theory is the part of Forman's [**21**] Discrete Morse theory which describes the impact of a discrete Morse matching on the cellular chain complex of the CW-complex.

Sometimes it is useful to consider the same construction for matchings which are not acyclic. Clearly, Theorem 1.2 does not hold anymore for $\mathsf{C}_\bullet^{\mathcal{M}}$ if $\mathcal{M}$ is not acyclic. In general, there is not even a good definition of the differentials $\partial^{\mathcal{M}}$. But for calculating invariants it is sometimes useful to consider $\mathsf{C}_\bullet^{\mathcal{M}}$ for matchings that are not acyclic. In these cases one considers just the graded vectorspace $\mathsf{C}_\bullet^{\mathcal{M}}$.

Finally we would like to generalize the construction of the Morse complex to infinite acyclic matchings:
Note, that the definition of an acyclic matching makes perfect sense also for infinite sets of edges. But if $\mathcal{M}$ is an infinite acyclic matching then $\Gamma(c, c')$ may no longer be well defined in case the set of paths from c to c' is infinite. Moreover, we indeed use finiteness in our proof of Theorem 1.2 since we use induction on the cardinality of the acyclic matching.

In order to be able to formulate a result similar to Theorem 1.2 for infinite acyclic matchings we have to introduce an additional finiteness condition:

Let $C_\bullet$ be a complex and $\mathcal{M}$ an infinite acyclic matching. Clearly the matching $\mathcal{M}$ induces a finite matching on each finite subcomplex $C_\bullet^f$ of $C_\bullet$. Therefore, we make the following definition:

Definition 1.3 (Finiteness)**.** Let $C_\bullet$ be a complex of free R-modules and let $\mathcal{M}$ be an infinite acyclic matching. We say that $\mathcal{M}$ defines a Morse matching if there exists a sequence of finite subcomplexes $D_i := (D_\bullet)_i$, $i \geq 0$ of $C_\bullet$ such that:

(1) D_i is a subcomplex of D_{i+1}, for all $i \geq 0$.
(2) $C_\bullet = \operatorname{colim}_{i \geq 0} D_i$.
(3) $(D_i)^{\mathcal{M}}$ is a subcomplex of $(D_{i+1})^{\mathcal{M}}$.

Note that the last condition implies $\Gamma(c, c') < \infty$ and thus the conclusion of Theorem 1.2 still holds for those infinite Morse matchings.

In our applications all complexes are multigraded by $\alpha \in \mathbb{N}^n$ and the α-graded part of $C_\bullet$ is finite. Therefore the subcomplexes D_i, defined by

$$D_i := \bigoplus_{j=0}^{i} \bigoplus_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = j}} (C_\bullet)_\alpha$$

are finite subcomplexes. It is easy to see that for multigraded complexes whose graded parts are of finite rank any acyclic matching fulfills the additional finiteness condition with the sequence D_i. This indeed holds for all complexes in our applications.

Therefore we get:

Theorem 1.4. *Let $C_\bullet$ be a $\mathbb{N}^n$-graded complex of free R-modules such that $(C_\bullet)_\alpha$ is a finite subcomplex for all $\alpha \in \mathbb{N}^n$. Then the conclusion of Theorem 1.2 still holds for infinite acyclic matchings $\mathcal{M}$.*

In the following chapters we will use the conclusions of Theorem 1.2 and 1.4 in order to construct minimal resolutions without explicitly referring to the theorems.

2. Proof of Theorem 1.2

We write $\Gamma_\downarrow(c,c')$ (resp. $\Gamma_\uparrow(c,c')$) for the sum of the weights of all those paths from c to c' for which the first step $c \to c_1$ satisfies $c \in X_i^{\mathcal{M}}$ and $c_1 \in X_{i-1}^{\mathcal{M}}$ (resp. $c' \in X_{i+1}^{\mathcal{M}}$). In most cases it will be clear from the context, e.g. if c is critical, whether the first step increases or decreases dimension. Still for the sake of readability we will always equip Γ with the respective arrow.

From now on we assume always that $\mathcal{M}$ satisfies the three conditions.

We first prove that the Morse differential satisfies $\partial_i^{\mathcal{M}} \circ \partial_{i+1}^{\mathcal{M}} = 0$.

Lemma 2.1. *Let $\mathcal{M} \subset E$ be an acyclic matching on $G(C_\bullet) = (V,E)$. Then*

(P1) *$\partial^{\mathcal{M}}$ is a differential (i.e. $\partial^{\mathcal{M}} \circ \partial^{\mathcal{M}} = 0$).*

(P2) *For $(\alpha,\beta,[\alpha:\beta]) \in \mathcal{M}$ with $\alpha \in X_{i+1}$, $\beta \in X_i$ we have for all $c \in X_{i-1}^{\mathcal{M}}$:*

$$\Gamma_\downarrow(\beta,c) = \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c).$$

Proof. The proof is by induction over the cardinality of $\mathcal{M}$. In order to prove the induction, we assume that both properties are satisfied for smaller matchings.

Let $\mathcal{M} = \{(\alpha,\beta,[\alpha:\beta])\}$ be a matching of cardinality 1.

Property (P2):

$$\begin{aligned} 0 = \partial^2(\alpha) &= \sum_{c' \in X_i^{\mathcal{M}}} [\alpha:c']\partial(c') + [\alpha:\beta]\partial(\beta) \\ &= \sum_{c \in X_{i-1}} \left(\sum_{c' \in X_i^{\mathcal{M}}} [\alpha:c'][c':c] \right) c + \sum_{c \in X_{i-1}} [\alpha:\beta][\beta:c]c \\ &= -[\alpha:\beta] \sum_{c \in X_{i-1}} \left(\sum_{c' \in X_i^{\mathcal{M}}} \left(-\frac{1}{[\alpha:\beta]} \right) [\alpha:c'][c':c] \right) c + \sum_{c \in X_{i-1}} [\alpha:\beta][\beta:c]c \end{aligned}$$

$$\begin{aligned}
&= -[\alpha:\beta]\sum_{c\in X_{i-1}}\sum_{c'\in X_i^{\mathcal{M}}}\Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c)c+\sum_{c\in X_{i-1}}[\alpha:\beta][\beta:c]c\\
&= [\alpha:\beta]\sum_{c\in X_{i-1}}\left([\beta:c]-\sum_{c'\in X_i^{\mathcal{M}}}\Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c)\right)c\\
&= [\alpha:\beta]\sum_{c\in X_{i-1}}\left(\Gamma_\downarrow(\beta,c)-\sum_{c'\in X_i^{\mathcal{M}}}\Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c)\right)c.
\end{aligned}$$

Since $[\alpha:\beta]\in Z(R)\cap R^*$ is not a zero-divisor and the critical cells are linearly independent, we get the desired result:

$$\Gamma_\downarrow(\beta,c)-\sum_{c'\in X_i^{\mathcal{M}}}\Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c)=0.$$

Property (P1): Let $c\in X_{i+1}^{\mathcal{M}}$ be a critical cell. We have to distinguish three cases. Note that the validity of property (P2) has been established above.

Case 1: $(\partial^{\mathcal{M}})^2(c)=\partial^2(c)$. Since ∂ is a differential, we are done.

Case 2: There exist elements $\beta\in X_i$ and $c\neq\alpha\in X_{i+1}$ with $[c:\beta]\neq 0$ and $\{(\alpha,\beta,[\alpha:\beta])\}=\mathcal{M}$. Then we have:

$$\begin{aligned}
(\partial^{\mathcal{M}})^2(c) &= \sum_{\beta\neq c'\leq c}[c:c']\partial^{\mathcal{M}}(c')+[c:\beta](-\frac{1}{[\alpha:\beta]})\sum_{\substack{c'\in X_i^{\mathcal{M}}\\ c'\neq\beta}}[\alpha:c']\partial^{\mathcal{M}}(c')\\
&= \sum_{\beta\neq c'\leq c}\sum_{c''\leq c'}[c:c'][c':c'']c''\\
&\quad +[c:\beta](-\frac{1}{[\alpha:\beta]})\sum_{\substack{c'\in X_i^{\mathcal{M}}\\ c'\neq\beta}}\sum_{c''\leq c'}[\alpha:c'][c':c'']c''\\
&= \sum_{c''\in X_{i-1}^{\mathcal{M}}}\left(\sum_{\beta\neq c'\leq c}[c:c'][c':c'']\right.\\
&\quad \left.+[c:\beta](-\frac{1}{[\alpha:\beta]})\sum_{\substack{c'\in X_i^{\mathcal{M}}\\ c'\neq\beta}}[\alpha:c'][c':c'']\right)c''\\
&= \sum_{c''\in X_{i-1}^{\mathcal{M}}}\left(\sum_{\beta\neq c'\leq c}[c:c'][c':c'']+[c:\beta]\sum_{\substack{c'\in X_i^{\mathcal{M}'}\\ c'\neq\beta}}\Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c'')\right)c''\\
&\overset{(P2)}{=} \sum_{c''\in X_{i-1}^{\mathcal{M}}}\left(\sum_{\beta\neq c'\leq c}[c:c'][c':c'']+[c:\beta]\Gamma_\downarrow(\beta,c'')\right)c''\\
&= \sum_{c''\in X_{i-1}}\left(\sum_{c'\leq c}[c:c'][c':c'']\right)c''=\partial^2(c)=0.
\end{aligned}$$

Case 3: There exist elements $\beta \in X_i$ and $\alpha \in X_{i-1}$ with $[c:\beta] \neq 0$ and $\{(\beta, \alpha, [\beta:\alpha])\} = \mathcal{M}$.
Since $\partial^2(c) = 0$, we have

$$\begin{aligned} 0 &= \sum_{c' \leq c} [c:c'][c':\alpha] \\ &= [c:\beta][\beta:\alpha] + \sum_{\substack{c' \leq c \\ c' \neq \beta}} [c:c'][c':\alpha] \\ &= [\beta:\alpha] \left([c:\beta] + \sum_{\substack{c' \leq c \\ c' \neq \beta}} \frac{1}{[\beta:\alpha]} [c:c'][c':\alpha] \right). \end{aligned}$$

Since $[\beta:\alpha] \in Z(R) \cap R^*$ is not a zero-divisor, it follows

$$[c:\beta] = \sum_{\substack{c' \leq c \\ c' \neq \beta}} \left(-\frac{1}{[\beta:\alpha]} \right) [c:c'][c':\alpha]. \tag{2.1}$$

This observation allows us to deduce the desired result:

$$\begin{aligned} (\partial^{\mathcal{M}})^2(c) &= \sum_{\substack{c' \leq c \\ c' \neq \beta}} [c:c'] \partial^{\mathcal{M}}(c') \\ &= \sum_{\substack{c' \leq c \\ c' \neq \beta}} \sum_{\substack{c'' \leq c' \\ c'' \neq \alpha}} [c:c'][c':c'']c'' \\ &\quad + \sum_{\substack{c'' \leq \beta \\ c'' \neq \alpha}} \underbrace{\left(\sum_{\substack{c' \leq c \\ c' \neq \beta}} [c:c'][c':\alpha] \left(-\frac{1}{[\beta:\alpha]} \right) \right)}_{=[c:\beta] \text{ by } (2.1)} [\beta:c'']c'' \\ &= \sum_{\substack{c' \leq c \\ c' \neq \beta}} \sum_{\substack{c'' \leq c' \\ c'' \neq \alpha}} [c:c'][c':c'']c'' \\ &\quad + \sum_{\substack{c'' \leq \beta \\ c'' \neq \alpha}} [c:\beta][\beta:c'']c'' \\ &= \sum_{c' \leq c} \sum_{\substack{c'' \leq c' \\ c'' \neq \alpha}} [c:c'][c':c'']c'' = 0 \text{ since } \partial^2 = 0. \end{aligned}$$

We now assume properties (P1) and (P2) for matchings of cardinality $\leq n$. Let $\mathcal{M}$ be an acyclic matching of cardinality $n+1$, and $\mathcal{M}' := \mathcal{M} \setminus \{(\alpha, \beta, [\alpha:\beta])\}$ with $\alpha \in X_{i+1}^{\mathcal{M}'}$ and $\beta \in X_i^{\mathcal{M}'}$. Then α, β are critical with respect to $\mathcal{M}'$, and by induction $\mathcal{M}'$ satisfies (P1) and (P2).
Property (P2):

$$0 = (\partial^{\mathcal{M}'})^2(\alpha) = \sum_{c' \in X_i^{\mathcal{M}'}} \sum_{c \leq \alpha} [\alpha:c] \Gamma_{\uparrow}(c,c') \partial^{\mathcal{M}'}(c')$$

$$\begin{aligned}
&= [\alpha:\beta]\partial^{\mathcal{M}'}(\beta) + \sum_{\substack{c'\in X_i^{\mathcal{M}'}\\ c'\neq\beta}} \sum_{c\leq\alpha} [\alpha:c]\Gamma_\uparrow(c,c')\partial^{\mathcal{M}'}(c') \\
&= [\alpha:\beta]\partial^{\mathcal{M}'}(\beta) + \sum_{\substack{c'\in X_i^{\mathcal{M}'}\\ c'\neq\beta}} \Gamma_\downarrow(\alpha,c')\partial^{\mathcal{M}'}(c') \\
&= [\alpha:\beta]\left(\sum_{c'\leq\beta}[\beta:c']\sum_{c\in X_{i-1}^{\mathcal{M}'}}\Gamma_\uparrow(c',c)c\right) \\
&\quad + \sum_{\substack{c'\in X_i^{\mathcal{M}'}\\ c'\neq\beta}} \Gamma_\downarrow(\alpha,c')\sum_{c\in X_{i-1}^{\mathcal{M}'}}\Gamma_\downarrow(c',c)c \\
&= [\alpha:\beta]\left(\sum_{c\in X_{i-1}^{\mathcal{M}'}}\Gamma_\downarrow(\beta,c)c\right) \\
&\quad -[\alpha:\beta]\sum_{c\in X_{i-1}^{\mathcal{M}'}}\left(-\frac{1}{[\alpha:\beta]}\right)\sum_{\substack{c'\in X_i^{\mathcal{M}'}\\ c'\neq\beta}}\Gamma_\downarrow(\alpha,c')\Gamma_\downarrow(c',c)c \\
&= [\alpha:\beta]\sum_{c\in X_{i-1}^{\mathcal{M}}}(\Gamma_\downarrow(\beta,c) - \sum_{c'\in X_i^{\mathcal{M}}}\Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c))c.
\end{aligned}$$

Since the critical cells are linearly independent and $[\alpha:\beta]$ is a unit, we get the desired result:

$$\Gamma_\downarrow(\beta,c) - \sum_{c'\in X_{\mathcal{M}}^{(i)}}\Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c) = 0$$

Property (P1): Let $c \in X_{i+1}^{\mathcal{M}}$ be a critical cell. In order to prove the first statement, we have, as in the case of cardinality 1, to distinguish three cases:

Case 1: $(\partial^{\mathcal{M}})^2(c) = (\partial^{\mathcal{M}'})^2(c)$. Since by induction $(\partial^{\mathcal{M}'})^2 = 0$, we are done.

Case 2: There exist elements $c \neq \alpha \in X_{i+1}^{\mathcal{M}'}$ and $\beta \in X_i^{\mathcal{M}'}$ with $[c:\beta]\neq 0$ and $(\alpha,\beta,[\alpha:\beta]) \in \mathcal{M}$. Then we have:

$$\partial^{\mathcal{M}}(c) = \sum_{c'\in X_i^{\mathcal{M}}}[c:\beta]\Gamma_\uparrow(\beta,c')c' + \sum_{c'\in X_i^{\mathcal{M}}}\Gamma_\downarrow(c,c')c',$$

where the last sum is over all paths which do not go through β. It follows

$$\begin{aligned}
(\partial^{\mathcal{M}})^2(c) &= \sum_{c'\in X_i^{\mathcal{M}}}[c:\beta]\Gamma_\uparrow(\beta,c')\partial^{\mathcal{M}}(c') + \sum_{c'\in X_i^{\mathcal{M}}}\Gamma_\downarrow(c,c')\partial^{\mathcal{M}}(c') \\
&= \sum_{c''\in X_{i-1}^{\mathcal{M}}}[c:\beta]\left(\sum_{c'\in X_i^{\mathcal{M}}}\Gamma_\uparrow(\beta,c')\Gamma_\downarrow(c',c'')\right)c'' + \sum_{c''\in X_{i-1}^{\mathcal{M}}}\Gamma_\downarrow(c,c'')c'' \\
&= \sum_{c''\in X_{i-1}^{\mathcal{M}}}[c:\beta]\Gamma_\downarrow(\beta,c'')c'' + \sum_{c''\in X_{i-1}^{\mathcal{M}}}\Gamma_\downarrow(c,c'')c'' \\
&= \sum_{c''\in X_{i-1}^{\mathcal{M}'}}\Gamma_\downarrow(c,c'')c'' = 0 \text{ since by induction } \partial^{\mathcal{M}'}\circ\partial^{\mathcal{M}'} = 0.
\end{aligned}$$

Case 3: There exist elements $\beta \in X_i^{\mathcal{M}'}$ and $\alpha \in X_{i-1}^{\mathcal{M}'}$ with $[c:\beta] \neq 0$ and $(\beta, \alpha, [\beta:\alpha]) \in \mathcal{M}$. Then we have

$$\begin{aligned}
(\partial^{\mathcal{M}})^2(c) = & \sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c,c')\partial^{\mathcal{M}}(c') \\
= & \sum_{c'' \neq \alpha} \left(\sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c,c')[c':\alpha] \left(-\frac{1}{[\beta:\alpha]} \right) \right) \Gamma_{\downarrow}(\beta, c'')c'' \\
& + \sum_{c'' \neq \alpha} \left(\sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c,c')\Gamma_{\downarrow}(c',c'') \right) c'', \text{ where } \Gamma_{\downarrow}(c',c'') \text{ does not go through } \alpha \\
\overset{(*)}{=} & \sum_{c'' \neq \alpha} \Gamma_{\downarrow}(c,\beta)\Gamma_{\downarrow}(\beta,c'')c'' + \sum_{c'' \neq \alpha} \left(\sum_{\substack{c' \in X_{i-1}^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c,c')\Gamma_{\downarrow}(c',c'') \right) c'' \\
= & \sum_{c'' \neq \alpha} \left(\sum_{c' \in X_i^{\mathcal{M}'}} \Gamma_{\downarrow}(c,c')\Gamma_{\downarrow}(c',c'') \right) c'' \\
= & \ 0 \text{ since } (\partial^{\mathcal{M}'})^2 = 0.
\end{aligned}$$

In $(*)$ we use the fact $\Gamma_{\downarrow}(c,\beta) = \sum_{\substack{c' \in X_i^{\mathcal{M}} \\ c' \neq \beta}} \Gamma_{\downarrow}(c,c') \left(-\frac{1}{[\beta:\alpha]} \right) [c':\alpha]$, which holds with the same argument as in (2.1). □

In the following, we show that the Morse complex is homotopy-equivalent to the original complex. Thereby, it will be possible to minimize a complex of free R-modules by means of Algebraic Discrete Morse theory.

Let $(C(X,R), \partial)$ be a complex of free R-modules, $\mathcal{M} \subset E$ a matching on the associated graph $G(C(X,R)) = (V,E)$, and $(C(X^{\mathcal{M}},R), \partial^{\mathcal{M}})$ the Morse complex. We consider the following maps:

$$(2.2) \qquad \begin{aligned} f: C(X,R) &\to C(X^{\mathcal{M}},R) \\ c \in X_i &\mapsto f(c) := \sum_{c' \in X_i^{\mathcal{M}}} \Gamma(c,c')c', \end{aligned}$$

$$(2.3) \qquad \begin{aligned} g: C(X^{\mathcal{M}},R) &\to C(X,R) \\ c \in X_i^{\mathcal{M}} &\mapsto g_i(c) := \sum_{c' \in X_i} \Gamma(c,c')c', \end{aligned}$$

$$(2.4) \qquad \begin{aligned} \chi: C(X,R) &\to C(X,R) \\ c \in X_i &\mapsto \chi_i(c) := \sum_{c' \in X_{i+1}} \Gamma(c,c')c'. \end{aligned}$$

Then:

Lemma 2.2. *The maps f and g are homomorphisms of complexes of free R-modules. In particular,*

(C1) $\partial^{\mathcal{M}} \circ f = f \circ \partial$,

(C2) $\partial \circ g = g \circ \partial^{\mathcal{M}}$.

Lemma 2.3. *The maps g and f define a chain homotopy. In particular,*

(H1) $g_i \circ f_i - \mathrm{id} = \partial \circ \chi_{i+1} + \chi_i \circ \partial$, *i.e. it is null-homotopic,*

(H2) $f_i \circ g_i - \mathrm{id} = 0$; *in particular, $f \circ g$ is null-homotopic.*

Corollary 2.4 (Thm. 1.2)**.** *$C(X^{\mathcal{M}}, R)$ is a complex of free R-modules and*

$$H_i(C(X,R),R) = H_i(C(X^{\mathcal{M}},R),R) \text{ for all } i \geq 0.$$

Proof. This is an immediate consequence of Lemma 2.3. □

Proof of Lemma 2.2: Property (C1): Let $c \in X_i$. Then:

$$\left(\partial^{\mathcal{M}} \circ f\right)(c) = \partial^{\mathcal{M}} \left(\sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(c,c')c' \right) = \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(c,c')\Gamma_{\downarrow}(c',c'')c''$$

and

$$\begin{aligned} \left(f \circ \partial\right)(c) &= f\left(\textstyle\sum_{c' \leq c}[c:c']c'\right) \\ &= \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \sum_{c' \leq c} [c:c']\Gamma_{\uparrow}(c',c'')c'' \\ &= \sum_{c'' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c,c''). \end{aligned}$$

Using Lemma 2.1 (P2) the assertion now follows.

Property (C2): Let $c \in X_i^{\mathcal{M}}$. Then:

$$\begin{aligned} \left(\partial \circ g\right)(c) &= \sum_{c'' \leq c} [c:c'']c'' + \sum_{c' \in X_i} \Gamma_{\downarrow}(c,c') \sum_{c'' \leq c'} [c':c'']c'' \\ &= \underbrace{\sum_{c'' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c,c'')c''}_{(A)} \\ &\quad + \underbrace{\sum_{c' \in X_i} \Gamma_{\downarrow}(c,c') \sum_{\substack{c'' \leq c' \\ (c'',\beta,[c'':\beta]) \in \mathcal{M}}} [c':c'']c''}_{(B)} \\ &\quad + \underbrace{\sum_{c' \in X_i} \Gamma_{\downarrow}(c,c') \sum_{\substack{c'' \leq c' \\ (\beta,c'',[\beta:c'']) \in \mathcal{M}}} [c':c'']c''}_{(C)}. \end{aligned}$$

We have $(C) = 0$: Fix $c'' \in X_{i-1}$ and $\beta \in X_i$ such that $(\beta, c'', [\beta : c'']) \in \mathcal{M}$. Then:

$$\begin{aligned}\sum_{c' \in X_i} \Gamma_{\downarrow}(c, c')[c' : c''] &= \sum_{c' \neq \beta} \Gamma_{\downarrow}(c, c')[c' : c'']c'' + \Gamma_{\downarrow}(c, \beta)[\beta : c'']c'' \\ &= \sum_{c' \neq \beta} \Gamma_{\downarrow}(c, c')[c' : c'']c'' \\ &\quad + \left(\Gamma_{\downarrow}(c, c'') \left(-\frac{1}{[\beta : c'']}\right)\right) [\beta : c'']c'' \\ &= \Gamma_{\downarrow}(c, c'')c'' - \Gamma_{\downarrow}(c, c'')c'' = 0.\end{aligned}$$

On the other hand:

$$\begin{aligned}&\left(g \circ \partial^{\mathcal{M}}\right)(c) = g\left(\sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c : c')c'\right) \\ &= \underbrace{\sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c, c')c'}_{(A)} + \underbrace{\sum_{c' \in X_{i-1}^{\mathcal{M}}} \sum_{\substack{c'' \in X_{i-1} \\ (c'', \beta, [c'' : \beta]) \in \mathcal{M}}} \Gamma_{\downarrow}(c, c')\Gamma_{\downarrow}(c', c'')c''}_{(D)}.\end{aligned}$$

We will verify $(B) = (D)$: Consider the matching $\mathcal{M}' \setminus \{(c'', \beta, [c'', \beta])\}$. Since c'' and β are critical cells in $\mathcal{M}'$, it follows by Lemma 2.1 (P1) (i.e. $(\partial^{\mathcal{M}'})^2 = 0$) that

$$0 = \sum_{\substack{c' \in X_{i-1}^{\mathcal{M}'} \\ c' \neq c''}} \Gamma_{\downarrow}(c, c')\Gamma_{\downarrow}(c', \beta) + \Gamma_{\downarrow}(c, c'')[c'' : \beta].$$

Multiplying by $\left(-\frac{1}{[c''.\beta]}\right)$ yields:

$$\sum_{c' \in X_{i-1}^{\mathcal{M}}} \Gamma_{\downarrow}(c, c')\Gamma_{\downarrow}(c', c'') = \sum_{c' \in X_i} \Gamma_{\downarrow}(c, c')[c' : c''].$$

Thus $(B) = (D)$. □

Proof of Lemma 2.3: Property (H2): Let $c \in X_i^{\mathcal{M}}$. The map g sends c to a sum over all $c' \in X_i$ that can be reached from c. Since c is critical, c' can be reached from c if either $c = c'$ or there is a $c'' \in X_{i-1}$ such that $(c', c'', [c' : c'']) \in \mathcal{M}$. Moreover,

$$f(c) = 0 \text{ if there is a } c' \in X_{i-1} \text{ such that } (c, c', [c : c']) \in \mathcal{M}.$$

Since f and g are R-linear, it follows that $\left(f_i \circ g_i\right)(c) = f_i(c)$. From $f_{X^{\mathcal{M}}} = \mathrm{id}$ we infer the assertion.

Property (H1): We distinguish three cases.

Case 1: Assume c is critical. Then

$$\left(g_i \circ f_i - \mathrm{id}\right)(c) = g_i(c) - c = \sum_{\substack{c' \in X_i \\ (c', \beta, [c' : \beta]) \in \mathcal{M}}} \Gamma_{\downarrow}(c, c')c'.$$

Moreover, $\chi_i(c) = 0$, in particular, $\Big(\partial \circ \chi_i\Big)(c) = 0$.

$$\begin{aligned}
\chi(\partial(c)) &= \chi\left(\sum_{c' \leq c} [c:c']c'\right) \\
&= \sum_{c' \leq c} [c:c'] \sum_{c'' \in X_i} \Gamma_\uparrow(c',c'')c'' \\
&= \sum_{\substack{c'' \in X_i \\ (c'',\beta,[c':\beta]) \in \mathcal{M}}} \Gamma_\downarrow(c,c'')c'' = (g_i \circ f_i - \mathrm{id})(c).
\end{aligned}$$

Case 2: There is an $\alpha \in X_{i-1}$ such that $(c,\alpha,[c:\alpha]) \in \mathcal{M}$. Then $\chi(c) = 0$ and $(g_i \circ f_i - \mathrm{id})(c) = -\mathrm{id}(c) = -c$. Moreover,

$$\begin{aligned}
\chi(\partial(c)) &= \chi\left(\sum_{c' \leq c} [c:c']c'\right) \\
&= \sum_{c' \leq c} [c:c'] \sum_{c'' \in X_i} \Gamma_\uparrow(c',c'')c'' \\
&= [c:\alpha]\left(-\frac{1}{[c:\alpha]}\right) c \\
&\quad + \sum_{\substack{c' \leq c \\ c' \neq \alpha}} [c:c'] \sum_{\substack{c'' \in X_i \\ c'' \neq \alpha}} \Gamma_\uparrow(c',c'')c'' + [c:\alpha] \sum_{c'' \in X_i} \Gamma_\uparrow(\alpha,c'')c''.
\end{aligned}$$

Since

$$\Gamma_\uparrow(\alpha,c'') = \left(-\frac{1}{[c:\alpha]}\right) \sum_{\substack{c' \leq c \\ c' \neq \alpha}} [c:c']\Gamma_\uparrow(c',c''),$$

the assertion follows.

Case 3: There is an $\alpha \in X_{i+1}$ such that $(\alpha,c,[\alpha:c]) \in \mathcal{M}$. Then:

$$\begin{aligned}
\Big(g_i \circ f_i - \mathrm{id}\Big)(c) &= -c + \underbrace{\sum_{c' \in X_i^{\mathcal{M}}} \Gamma_\uparrow(c,c')c'}_{(A)} \\
&\quad + \underbrace{\sum_{\substack{c'' \in X_i \\ (c'',\beta,[c'':\beta]) \in \mathcal{M}}} \sum_{c' \in X_i^{\mathcal{M}}} \Gamma_\uparrow(c,c')\Gamma_\downarrow(c',c'')c''}_{(B)}.
\end{aligned}$$

On the other hand:

$$\begin{aligned}
\partial\chi(c) &= \partial\left(\sum_{c' \in X_{i+1}} \Gamma_\uparrow(c,c')c'\right) \\
&= \sum_{c' \in X_{i+1}} \Gamma_\uparrow(c,c') \sum_{c'' \leq c'} [c':c'']c''
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{c''\leq\alpha}\left(-\frac{1}{[\alpha:c]}\right)[\alpha:c'']c'' \\
&\quad +\sum_{c'\neq\alpha}\Gamma_{\uparrow}(c,c')\sum_{c''\leq c'}[c':c'']c'' \\
&= -c+(A)+\underbrace{\sum_{\substack{c''\in X_i\\(c'',\beta,[c'':\beta])\in\mathcal{M}}}\sum_{c'\neq\alpha}\Gamma_{\uparrow}(c,c')\sum_{c''\leq c'}[c':c'']c''}_{(C)} \\
&\quad +\underbrace{\sum_{\substack{c''\in X_i\\(\beta,c'',[\beta:c''])\in\mathcal{M}}}\sum_{c'\neq\alpha}\Gamma_{\uparrow}(c,c')\sum_{c''\leq c'}[c':c'']c''}_{(D)}
\end{aligned}
$$

and

$$
\begin{aligned}
\chi\partial(c) &= \chi\left(\sum_{c'\leq c}[c:c']c'\right) \\
&= \sum_{\substack{c''\in X_i\\(c'',\beta,[c'':\beta])\in\mathcal{M}}}\sum_{c'\leq c}[c:c']\Gamma_{\uparrow}(c',c'')c'' \\
&= \underbrace{\sum_{\substack{c''\in X_i\\(c'',\beta,[c'':\beta])\in\mathcal{M}}}\Gamma_{\downarrow}(c,c'')c''}_{(E)}.
\end{aligned}
$$

We show:

(a) $(D)=0$,

(b) $(E)+(C)=(B)$.

Assertion (a); Fix $c''\in X_i$ and $\beta\in X_{i+1}$ such that $(\beta,c'',[\beta:c''])\in\mathcal{M}$. Then:

$$
\begin{aligned}
&\sum_{c'\in X_{i+1}}\Gamma_{\uparrow}(c,c')[c':c''] \\
&\qquad = \sum_{c'\neq\beta}\Gamma_{\uparrow}(c,c')[c':c'']c''+\Gamma_{\uparrow}(c,\beta)[\beta:c''] \\
&\qquad = \sum_{c'\neq\beta}\Gamma_{\uparrow}(c,c')[c':c'']c'' \\
&\qquad\quad +\left(\Gamma_{\uparrow}(c,c'')\left(-\tfrac{1}{[\beta:c'']}\right)\right)[\beta:c'']c'' \\
&\qquad = \Gamma_{\uparrow}(c,c'')c''-\Gamma_{\uparrow}(c,c'')c''=0.
\end{aligned}
$$

Assertion (b); Let $c''\in X_i$ and $\beta\in X_{i-1}$ such that $(c'',\beta,[c'':\beta])\in\mathcal{M}$. Consider the matching $\mathcal{M}'=\mathcal{M}\setminus\{(c'',\beta,[c'':\beta])\}$. Then by Lemma 2.1 (P2)

$$
\sum_{c'\in X_i^{\mathcal{M}'}}\Gamma_{\uparrow}(c,c')\Gamma_{\downarrow}(c',\beta)=\Gamma_{\downarrow}(c,\beta).
$$

Since c'' is critical with respect to $\mathcal{M}'$, it follows that

$$\sum_{\substack{c' \in X_i^{\mathcal{M}'} \\ c' \neq c''}} \Gamma_{\uparrow}(c,c')\Gamma_{\downarrow}(c',\beta) + \Gamma_{\uparrow}(c,c'')[c'':\beta] = \Gamma_{\downarrow}(c,\beta).$$

Multiplying the equation with $\left(-\frac{1}{[c'':\beta]}\right)$ yields

$$\sum_{c' \in X_i^{\mathcal{M}}} \Gamma_{\uparrow}(c,c')\Gamma_{\downarrow}(c',c'') = \sum_{c' \in X_{i+1}} \Gamma_{\uparrow}(c,c')[c':c''] + \Gamma_{\downarrow}(c,c''),$$

where paths are taken with respect to the matching $\mathcal{M}$. Hence $(B) = (C) + (E)$. □

3. Normalized Bar and Hochschild Resolution via ADMT

In this paragraph we give a proof of the normalized Bar and Hochschild resolution in the case where A is a k-Algebra and W a basis of A as a k-vector space such that $1 \in W$ (the proofs still holds if A is an R-algebra, which is projective as an R-module, where R is a commutative ring). Let M be an R-module. Remember that the Bar resolution of M is given by

(1) $\mathcal{B}_i$ is the free $(A \otimes_k M)$-module with basis $[w_1|\dots|w_i]$, $w_1,\dots,w_i \in W$.

(2) The differential ∂_i is given by

$$\begin{aligned}\partial([w_1|\dots|w_i]) &= (w_1 \otimes 1)\ [w_2|\dots|w_i] \\ &+ \sum_{j=1}^{i-1}(-1)^j \begin{pmatrix} (a_0 \otimes 1)\ [w_1|\dots|w_{j-1}|1|w_{j+2}|\dots|w_i] \\ + \sum_l (a_l \otimes 1)\ [w_1|\dots|w_{j-1}|w_l'|w_{j+2}|\dots|w_i] \end{pmatrix} \\ &+ (-1)^i\ (1 \otimes w_i)\ [w_1|\dots|w_{i-1}],\end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l\, w_l'$ with $a_0, a_l \in k$ and $w_l' \in W$.

Proposition 3.1 (Normalized Bar Resolution)**.** *There is an acyclic matching $\mathcal{M}$ on the Bar resolution $\mathcal{B}_A$ with respect to W such that the corresponding Morse complex $\mathcal{B}^{\mathcal{M}}$ is given by:*

(1) *$\mathcal{B}_i^{\mathcal{M}}$ is the free $(A \otimes M)$-module with basis $[w_1|\dots|w_i]$, $w_1,\dots,w_i \in W \setminus \{1\}$.*

(2) *The Mores differential $\partial_i^{\mathcal{M}}$ is given by*

$$\begin{aligned}\partial^{\mathcal{M}}([w_1|\dots|w_i]) &= (w_1 \otimes 1)\ [w_2|\dots|w_i] \\ &+ \sum_{j=1}^{i-1}(-1)^j \sum_l (a_l \otimes 1)\ [w_1|\dots|w_{j-1}|w_l'|w_{j+2}|\dots|w_i] \\ &+ (-1)^i (1 \otimes w_i)\ [w_1|\dots|w_{i-1}]\end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l\, w_l'$ with $a_0, a_l \in k$ and $w_l' \in W \setminus \{1\}$.

In particular, $\mathcal{B}^{\mathcal{M}} = \mathcal{N}\mathcal{B}$ is the normalized Bar resolution.

Proof. We define the matching $\mathcal{M}$ by

$$[w_1|\dots|w_l|w_{l+1}|\dots|w_i] \to [w_1|\dots|w_l w_{l+1}|\dots|w_i] \in \mathcal{M}$$

if $w_l := \min(j \mid w_j = 1)$, $w_{l'} := \max(j \mid w_r = 1 \text{ for all } l \leq r \leq j)$, and $l' - l$ is odd. The invertibility is given since in both cases the coefficient in the differential is ± 1:

$$\partial([w_1|\ldots|w_l|w_{l+1}|\ldots|w_i] = \pm 1\,[w_1|\ldots|w_l w_{l+1}|\ldots|w_i].$$

It is easy to see that the other conditions of an acyclic matching are satisfied as well. The critical cells are exactly the desired basis elements and an element $[w_1|\ldots|w_i]$ for which $w_j = 1$ for some j is never mapped to an element $[w_1|\ldots|w_i]$, with $w_j \neq 1$ for all j. This implies the formula for the Morse differential. □

Proposition 3.2 (Normalized Acyclic Hochschild Complex)**.** *There is an acyclic matching $\mathcal{M}$ on the acyclic Hochschild complex $\mathcal{HC}_A$ of A such that the corresponding Morse complex $\mathcal{HC}^{\mathcal{M}}$ is given by:*

(1) *$C_i^{\mathcal{M}}$ is the free $(A \otimes_k A)$-module with basis $[w_1|\ldots|w_i]$, $w_1, \ldots, w_i \in W \setminus \{1\}$.*

(2) *The Morse differential $\partial_i^{\mathcal{M}}$ is given by*

$$\begin{aligned}\partial_i([w_1|\ldots|w_i]) &= (w_1 \otimes 1)\,[w_2|\ldots|w_i] + (-1)^i (1 \otimes w_i)\,[w_1|\ldots|w_{i-1}] \\ &\quad + \sum_{j=1}^{i-1} (-1)^j \left(\sum_l a_l\,[w_1|\ldots|w_{j-1}|w_l'|w_{j+2}|\ldots|w_i] \right)\end{aligned}$$

if $w_j w_{j+1} = a_0 + \sum_l a_l\, w_l'$ with $a_0, a_l \in k$ and $w_l' \in W \setminus \{1\}$.

In particular, $\mathcal{HC}^{\mathcal{M}}$ is the normalized acyclic Hochschild complex.

Proof. The proof is essentially identical to the proof of Proposition 3.1. □

Free Resolutions of Monomial Ideals

1. Algebraic Discrete Morse Theory on the Taylor Resolution

In this paragraph we consider acyclic matchings on the Taylor resolution. First, we introduce a *standard matching*, which we use in Chapter 6 in order to formulate and prove our conjecture about the minimal multigraded free resolution of the residue class field and to calculate the multigraded Poincaré-Betti series $P_A(\underline{x}, t)$. Then Section 1.2 considers the Taylor resolution for monomial ideals which are generated in degree two. The resolutions of those ideals are important for the proof of our conjecture in the case where A is Koszul (see Chapter 6.4). Next, we give a matching on the Taylor resolution of Stanley Reisner ideals of the order complex of a partially ordered set, which we use in Chapter 6 in order to calculate the multigraded Hilbert and Poincaré-Betti series.
Finally, we introduce the (strong) gcd-condition for monomial ideals and give a special acyclic matching on the Taylor resolution for this type of ideals, which are in connection with the Golod property of monomial rings (see Chapter 6.5).

1.1. Standard Matching on the Taylor Resolution. Let $S = k[x_1, \dots, x_n]$ be the commutative polynomial ring over a field k of arbitrary characteristic and $\mathfrak{a} \trianglelefteq S$ a monomial ideal.

The basis of the Taylor resolution is given by the subsets $I \subset \text{MinGen}(\mathfrak{a})$ of the minimal monomial generating system $\text{MinGen}(\mathfrak{a})$ of the ideal $\mathfrak{a}$. For a subset $I \subset \text{MinGen}(\mathfrak{a})$ we denote by m_I the least common multiple of the monomials in I, $m_I := \text{lcm}\big(m \in I\big)$.

On this basis we introduce an equivalence relation: We say that two monomials $m, n \in I$ with $I \subset \text{MinGen}(\mathfrak{a})$ are equivalent if $\gcd(m, n) \neq 1$ and write $m \sim n$. The transitive closure of $\sim$ gives us an equivalence relation on each subset I. We denote by $cl(I) := \#I/\sim$ the number of equivalence classes of I.

Based on the Taylor resolution, we define a product by

$$I \cdot J = \begin{cases} 0 & , \quad \gcd(m_I, m_J) \neq 1 \\ I \cup J & , \quad \gcd(m_I, m_J) = 1. \end{cases}$$

Then the number $cl(I)$ counts the factors of I with respect to the product defined above.

The aim of this section is to introduce an acyclic matching on the Taylor resolution which preserves this product.

We call two subset $I, J \subset \text{MinGen}(\mathfrak{a})$ a *matchable pair* and write $I \to J$ if $|J|+1 = |I|$, $m_J = m_I$, and the differential of the Taylor complex maps I to J with coefficient $[I, J] \neq 0$.

Let $I \to J$ be a matchable pair in the Taylor resolution with $cl(I) = cl(J) = 1$ such that no subset of J is matchable. Then define

$$\mathcal{M}_{11} := \{I \,\dot{\cup}\, K \to J \,\dot{\cup}\, K \text{ for each } K \text{ with } \gcd(m_K, m_I) = \gcd(m_K, m_J) = 1\}.$$

For simplification we write $I \in \mathcal{M}_{11}$ if there exists a subset J with $I \to J \in \mathcal{M}_{11}$ or $J \to I \in \mathcal{M}_{11}$. It is clear that this is an acyclic matching. Furthermore, the differential changes in each homological degree in the same way and for two subsets I, K with $\gcd(m_I, m_K) = 1$ we have $I \,\dot{\cup}\, K \in \mathcal{M}_{11} \iff I \in \mathcal{M}_{11}$ or $K \in \mathcal{M}_{11}$. Because of these facts, we can repeat this matching $\mathcal{M}_{11}$ on the resulting Morse complex. This gives us a sequence of acyclic matchings, which we denote by $\mathcal{M}_1 := \bigcup_{i \geq 1} \mathcal{M}_{1i}$. If no repetition is possible, we reach a resolution with basis given by some subsets $I \subset \text{MinGen}(\mathfrak{a})$ with the following property: If we have a matchable pair $I \to J$ where I has a higher homological degree than J, then $cl(I) \geq 1$ and $cl(J) \geq 2$. We now construct the second sequence:
Let $I \to J$ be a matchable pair in the resulting Morse complex with $cl(I) = 1, cl(J) = 2$ such that no subset of J is matchable. Then define

$$\mathcal{M}_2 := \{I \,\dot{\cup}\, K \to J \,\dot{\cup}\, K \text{ for each } K \text{ with } \gcd(m_K, m_I) = \gcd(m_K, m_J) = 1\}.$$

With the same arguments as before this defines an acyclic matching, and a repetition is possible. The third sequence starts if no repetition of $\mathcal{M}_2$ is possible and is given by a matchable pair $I \to J$ in the resulting Morse complex with $cl(I) = 1, cl(J) = 3$ such that no subset of J is matchable. Then define

$$\mathcal{M}_3 := \{I \,\dot{\cup}\, K \to J \,\dot{\cup}\, K \text{ for each } K \text{ with } \gcd(m_K, m_I) = \gcd(m_K, m_J) = 1\}.$$

Since every matchable pair is of the form $I \,\dot{\cup}\, K \to J \,\dot{\cup}\, K$ with $m_I = m_J$, $\gcd(m_I, m_K) = 1$, and $cl(I) = 1, cl(J) \geq 1$, we finally reach with this procedure a minimal resolution of the ideal $\mathfrak{a}$ as S-module. Let $\mathcal{M}$ be the union of all matchings. As before we write $I \in \mathcal{M}$ if there exists a subset J with $I \to J \in \mathcal{M}$ or $J \to I \in \mathcal{M}$. Then the minimal resolution has a basis given by $\text{MinGen}(\mathfrak{a}) \setminus \mathcal{M}$.
We give a matching of this type a special name:

Definition 1.1 (standard matching)**.** A sequence of matchings $\mathcal{M} := \bigcup_{i \geq 1} \mathcal{M}_i$ is called a standard matching on the Taylor resolution if all the following holds:

(1) $\mathcal{M}$ is graded, i.e. for all edges $I \to J$ in $\mathcal{M}$ we have $m_I = m_J$,

(2) $T_\bullet^{\mathcal{M}}$ is minimal, i.e. for all edges $I \to J$ in $T_\bullet^{\mathcal{M}}$ we have $m_I \neq m_J$,

(3) $\mathcal{M}_i$ is a sequence of acyclic matchings on the Morse complex $T_\bullet^{\mathcal{M}_{<i}}$ ($\mathcal{M}_{<i} := \bigcup_{j=1}^{i-1} \mathcal{M}_j$, $T_\bullet^{\mathcal{M}_{<1}} = T_\bullet$),

(4) for all $I \to J \in \mathcal{M}_i$ we have

$$\begin{aligned} cl(J) - cl(I) &= i - 1, \\ |J| + 1 &= |I|, \end{aligned}$$

(5) there exists a set $\mathcal{B}_i \subset \mathcal{M}_i$ such that

(a) $\mathcal{M}_i = \mathcal{B}_i \cup \left\{ I \cup K \to J \cup K \;\middle|\; \begin{array}{c} K \text{ with } \gcd(m_I, m_K) = 1 \\ \text{and } I \to J \in \mathcal{B}_i \end{array} \right\}$ and

(b) for all $I \to J \in \mathcal{B}_i$ we have $cl(I) = 1$ and $cl(J) = i$.

The construction above shows that a standard matching always exists. For a standard matching we have two easy properties, which we will need in Paragraph 2 of Chapter 6:

Lemma 1.2. *Let $\mathcal{M}$ and $\mathcal{M}'$ be two different standard matchings. Then*

(1) *for all $i \geq 1$ we have*

$$1 + \sum_{I \notin \mathcal{M}_{<i}} (-1)^{cl(I)} m_I t^{cl(I)+|I|} = 1 + \sum_{I \notin \mathcal{M}'_{<i}} (-1)^{cl(I)} m_I t^{cl(I)+|I|},$$

(2) *if $I, J \notin \mathcal{M}$, $\gcd(m_I, m_J) = 1$, and $I \cup J \in \mathcal{M}$, then there exists a set K with $|K| = |I| + |J| + 1$, $cl(K) = 1$, and $(I \cup J \to K) \in \mathcal{M}$.*

Proof. The result follows directly from the definition of a standard matching. $\square$

If the ideal is generated in degree two, every standard matching ends after the second sequence: Assume that we have a matchable pair $I \to J$ such that $cl(I) = 1$ and $cl(J) \geq 3$. Then J has at least three subsets $J = J_1 \cup J_2 \cup J_3$ such that $\gcd(m_{J_i}, m_{J_{i'}}) = 1$, $i, i' = 1, 2, 3$. Since I and J have the same multidegree and $cl(I) = 1$, there would exist a generator $u \in \mathrm{MinGen}(\mathfrak{a})$ such that $\gcd(m_{J_i}, u) \neq 1$ for $i = 1, 2, 3$. But u is a monomial of degree two, which makes such a situation impossible.
In this case we have

Lemma 1.3. *If every standard matching ends after the second sequence, i.e. $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$, then*

$$\sum_{I \notin \mathcal{M}_1} (-1)^{cl(I)} m_I t^{cl(I)+|I|} = \sum_{I \notin \mathcal{M}} (-1)^{cl(I)} m_I t^{cl(I)+|I|}.$$

Proof. By definition an edge $I \to J$ matched by the second sequence has the property $|I| = |J| + 1$ and $cl(I) = cl(J) - 1$ and $m_I = m_J$. Therefore, $(-1)^{cl(I)} m_I t^{cl(I)+|I|} = -\left((-1)^{cl(J)} m_J t^{cl(J)+|J|}\right)$, which proves the assertion. $\square$

1.2. Resolutions of Monomial Ideals Generated in Degree Two. Let $\mathfrak{a} \trianglelefteq S$ be a monomial ideal with minimal monomial generating system $\mathrm{MinGen}(\mathfrak{a})$ such that for all monomials $m \in \mathrm{MinGen}(\mathfrak{a})$ we have $deg(m) = 2$. We assume, in addition, that $\mathfrak{a}$ is squarefree. This is no restriction since via polarization we get similar results for the general case.

First we fix a monomial order $\prec$. We introduce the following notation: To each subset $I \subset \mathrm{MinGen}(\mathfrak{a})$ we associate an undirected graph $G_I = (V, E)$ on the ground set $V = [n]$, by setting $\{i, j\} \in E$ if the monomial $x_i x_j$ lies in I.

We call a subset I an **nbc**-set if the associated graph $G_I = (V, E)$ contains no broken circuit, i.e. there exists no edge $\{i, j\}$ such that

(1) $E \cup \{\{i, j\}\}$ contains a circuit c and
(2) $x_i x_j = \max_{\prec} \{x_{i'} x_{j'} \mid \{i', j'\} \in c\}$.

Proposition 1.4. *There exists an acyclic matching $\mathcal{M}_1$ on the Taylor resolution such that*

(1) *$\mathcal{M}_1$ is the first sequence of a standard matching,*
(2) *the resulting Morse complex $T_\bullet^{\mathcal{M}_1}$ is a subcomplex of the Taylor resolution and*
(3) *$T_\bullet^{\mathcal{M}_1}$ has a basis indexed by the* **nbc***-sets.*

Proof. Let Z be a circuit in $T_\bullet$ of maximal cardinality. Let $x_i x_j := \max_{\prec}\{Z\}$. We then define

$$\mathcal{M}_{1,0} := \Big\{(Z \cup I) \to ((Z \setminus \{x_i x_j\}) \cup I) \Bigm| I \in T_\bullet \text{ with } Z \cap I = \emptyset\Big\}.$$

It is clear that I is an acyclic matching and the resulting Morse complex $T^{\mathcal{M}_{1,0}}$ is a subcomplex of the Taylor resolution.
Now let Z_1 be a maximal circuit in $T^{\mathcal{M}_{1,0}}$ and let $x_\nu x_l := \max_{\prec}\{Z_1\}$. We then define

$$\mathcal{M}_{1,1} := \Big\{(Z_1 \cup I) \to ((Z_1 \setminus \{x_\nu x_l\}) \cup I) \Bigm| I \in T^{\mathcal{M}_{1,0}} \text{ with } Z_1 \cap I = \emptyset\Big\}.$$

We only have to guarantee that $(Z_1 \cup I) \notin \mathcal{M}_{1,0}$.
Assume $(Z_1 \cup I) \in \mathcal{M}_{1,0}$. Since $(Z_1 \setminus \{x_\nu x_l\}) \cup I \notin \mathcal{M}_{1,0}$, we see that $x_\nu x_l \neq x_i x_j$ and $x_\nu x_l \in Z$. But then $W := Z \cup (Z_1 \setminus \{x_\nu x_l\})$ is a circuit, which is a contradiction to the maximality of Z. Therefore, $\mathcal{M}_{1,1}$ is a well defined acyclic matching and the resulting Morse complex is a subcomplex of the Taylor resolution.
If we continue this process, we reach a subcomplex $T^{\mathcal{M}_1}$ of the Taylor resolution with a basis indexed by all **nbc**-sets. It is clear that $\mathcal{M}_1 := \bigcup_i \mathcal{M}_{1,i}$ satisfies all conditions of the first sequence of a standard matching. Furthermore, if I is an **nbc**-set and $m_I = m_{I \setminus \{m\}}$, then it follows that $cl(I) = cl(I \setminus \{m\}) - 1$ (otherwise we would have a circuit). This implies that $\mathcal{M}_1$ is exactly the first sequence of a standard matching. □

We denote by $T_{\mathbf{nbc}}$ the resulting Morse complex.

Corollary 1.5. *Let $\mathfrak{a} \trianglelefteq S$ be a monomial ideal generated in degree two. We denote with $\mathbf{nbc}_i$ the number of* **nbc***-sets of cardinality $i - 1$. Then for the Betti number of $\mathfrak{a}$ we have the inequality $\beta_i \leq \mathbf{nbc}_i$.*

1.3. Resolution of Stanley Reisner Ideals of a Partially Ordered Set. In this section we give a (not acyclic) matching on the subcomplex $T_{\mathbf{nbc}}$ in the case where $\mathfrak{a} = J_{\Delta(P)}$ is the Stanley Reisner ideal of the order complex of a partially ordered set $(P, \prec)$. In this case $\mathfrak{a}$ is generated in degree two by monomials $x_i x_j$ where $\{i, j\}$ is an antichain in P. For simplification we assume that $P = [p] = \{1, \ldots, p\}$ and the order $\prec$ preserves the natural order, i.e. $i \prec j \Rightarrow i < j$, where $<$ is the natural order on the natural numbers $\mathbb{N}$. Then the minimal monomial generating system MinGen($\mathfrak{a}$) of the Stanley Reisner ideal is given by

$$\mathrm{MinGen}(\mathfrak{a}) := \left\{ \mathrm{x_i x_j} \;\middle|\; i < j \text{ and } i \not\prec j \right\}.$$

Since MinGen($\mathfrak{a}$) consists of monomials of degree two, we can work on the subcomplex $T_{\mathbf{nbc}}$ of the Taylor resolution, where $T_{\mathbf{nbc}}$ is constructed with respect to the lexicographic order such that $x_1 \succ x_2 \succ \ldots \succ x_n$.

First we introduce some notation:

Definition 1.6. A subset $I \subset \mathrm{MinGen}(\mathfrak{a})$ is called a sting-chain if there exists a sequence of monomials $x_{i_1}x_{i_2}, x_{i_2}x_{i_3}, \ldots, x_{i_{\nu-1}}x_{i_\nu} \in I$ with

(1) $1 \leq i_1 < \ldots < i_\nu \leq n$,
(2) $i_1 = \min\{j \text{ with } x_j \text{ divides } \mathrm{lcm}(m_I)\}$,
(3) $i_\nu = \max\{j \text{ with } x_j \text{ divides } \mathrm{lcm}(m_I)\}$,
(4) for all monomials $x_r x_s \in I$ with $r < s$ exists an index $1 \leq j \leq \nu - 1$ such that either
  (a) $x_r x_s = x_{i_j} x_{i_{j+1}}$ or
  (b) $r = i_j$, $s < i_{j+1}$, and $x_s x_{i_{j+1}} \notin I$ or
  (c) $r > i_j$, $s = i_{j+1}$, and $i_j \prec r$ (i.e. $x_{i_j} x_r \notin \mathfrak{a}$).

Let $\mathcal{B}$ be the set of all chains of sting-chains:

$$\mathcal{B} := \left\{ (I_1, \ldots, I_l) \;\middle|\; \begin{array}{c} I_j \text{ sting-chain for all } j = 1, \ldots, l \text{ and} \\ \max(I_j) < \min(I_{j+1}) \text{ for all } j = 1, \ldots, l-1 \end{array} \right\},$$

where

$$\begin{aligned} \max(I) &:= \max\{i \mid x_i \text{ divides } \mathrm{lcm}(m_I)\} \\ \min(I) &:= \min\{i \mid x_i \text{ divides } \mathrm{lcm}(m_I)\}. \end{aligned}$$

Note that a sting-chain is not necessarily an **nbc**-set. For example, the set $\{x_i x_l, x_\nu x_l, x_j x_l\}$ with $i < \nu < j < l$ is a sting-chain, if $x_i x_\nu, x_i x_j \notin \mathfrak{a}$, but it contains a broken circuit if $x_\nu x_j \in \mathfrak{a}$. But with an identification of those sets we get the following Proposition:

Proposition 1.7. *There exists a matching $\mathcal{M}_2$ (not necessary acyclic) on the complex $T_{\mathbf{nbc}}$ such that*

(1) *there exists a bijection between the sets $I \in T_{\mathbf{nbc}}^{\mathcal{M}_2}$ and the chains of sting-chains $I \in \mathcal{B}$,*
(2) *for $I \to I' \in \mathcal{M}_2$ we have*
  (a) *$\mathrm{lcm}(m_I) = \mathrm{lcm}(m_{I'})$ and*
  (b) *$cl(I) = cl(I') - 1$ and $|I| = |I'| + 1$.*

Proof. For a set $I \in T_{\mathbf{nbc}} \setminus \mathcal{B}$ let $x_i x_\nu x_j x_l$ be the maximal monomial with respect to the lexicographic order such that $i < \nu < j < l$ and at least one of the following conditions is satisfied:

(1) $x_i x_j, x_\nu x_l \in I$ and $x_i x_l \notin I$,
(2) $x_i x_l, x_\nu x_j \in I$.

Case $x_i x_j, x_\nu x_l \in I$: Because of the transitivity of the order $\prec$ on P we have either $x_i x_\nu \in \mathfrak{a}$ or $x_\nu x_j \in \mathfrak{a}$.

- ▷ Assume $x_i x_\nu \in \mathfrak{a}$. Since $x_i x_\nu x_j x_l$ is the maximal monomial satisfying one of the conditions above, it follows that if $I \cup \{x_i x_\nu\}$ contains a broken circuit, then $I \setminus \{x_i x_\nu\}$ contains a broken circuit as well. We set

$$\Big((I \setminus \{x_i x_\nu\}) \,\dot\cup\, J\Big) \to \Big((I \cup \{x_i x_\nu\}) \,\dot\cup\, J\Big) \in \mathcal{M}_2$$
$$\text{for all } J \text{ with } \gcd(\operatorname{lcm}(I), \operatorname{lcm}(J)) = 1.$$

- ▷ If $x_i x_\nu \notin \mathfrak{a}$, then $x_\nu x_j \in \mathfrak{a}$. Again, we have that if $I \cup \{x_\nu x_j\}$ contains a broken circuit, then $I \setminus \{x_\nu x_j\}$ contains a broken circuit as well. In this case we set

$$\Big((I \setminus \{x_\nu x_j\}) \,\dot\cup\, J\Big) \to \Big((I \cup \{x_\nu x_j\}) \,\dot\cup\, J\Big) \in \mathcal{M}_2$$
$$\text{for all } J \text{ with } \gcd(\operatorname{lcm}(I), \operatorname{lcm}(J)) = 1.$$

Case $x_i x_l, x_\nu x_j \in I$: Again, the transitivity implies $x_i x_\nu \in \mathfrak{a}$ or $x_\nu x_l \in \mathfrak{a}$ and $x_i x_j \in \mathfrak{a}$ or $x_j x_l \in \mathfrak{a}$:

- ▷ Assume $x_i x_\nu \in \mathfrak{a}$. As above we have that if $I \cup \{x_i x_\nu\}$ contains a broken circuit, then $I \setminus \{x_i x_\nu\}$ contains a broken circuit as well. We set

$$\Big((I \setminus \{x_i x_\nu\}) \,\dot\cup\, J\Big) \to \Big((I \cup \{x_i x_\nu\}) \,\dot\cup\, J\Big) \in \mathcal{M}_2$$
$$\text{for all } J \text{ with } \gcd(\operatorname{lcm}(I), \operatorname{lcm}(J)) = 1.$$

- ▷ If $x_i x_\nu \notin \mathfrak{a}$, then $x_\nu x_l \in \mathfrak{a}$. Assume $x_i x_j \in \mathfrak{a}$. Then again we have that if $I \cup \{x_i x_j\}$ contains a broken circuit, then $I \setminus \{x_i x_j\}$ also contains a broken circuit. In this case we set

$$\Big((I \cup \{x_i x_j\}) \,\dot\cup\, J\Big) \to \Big((I \setminus \{x_i x_j\}) \,\dot\cup\, J\Big) \in \mathcal{M}_2$$
$$\text{for all } J \text{ with } \gcd(\operatorname{lcm}(I), \operatorname{lcm}(J)) = 1.$$

- ▷ Now assume $x_i x_\nu, x_i x_j \notin \mathfrak{a}$, then $x_\nu x_l, x_j x_l \in \mathfrak{a}$. Assume further that $x_j x_l \notin I$. Then we set

$$\Big((I \cup \{x_\nu x_l\}) \,\dot\cup\, J\Big) \to \Big((I \setminus \{x_\nu x_l\}) \,\dot\cup\, J\Big) \in \mathcal{M}_2$$
$$\text{for all } J \text{ with } \gcd(\operatorname{lcm}(I), \operatorname{lcm}(J)) = 1.$$

- ▷ Finally, we have to discuss the case $x_i x_\nu, x_i x_j \notin \mathfrak{a}$ and $x_j x_l \in I$. Then the set I cannot be matched because adding $x_\nu x_l$ would give a circuit and by removing $x_j x_l$ we get a set which is already matched. We identify these sets with the sets containing $x_i x_l, x_\nu x_l, x_j x_l$ instead of $x_i x_l, x_\nu x_j, x_j x_l$. Therefore, this case gives us all sets which are sting-chains but not **nbc**-sets.

With the identification we can say that an **nbc**-set $I \notin \mathcal{M}$ satisfies the following two properties, which are exactly the properties of $I \in \mathcal{B}$:

(1) If there exist $i < \nu < j < l$ such that $x_i x_j, x_\nu x_l \in I$, then $x_i x_l \in I$ and $x_\nu x_j, x_j x_l \notin I$ and $x_i x_\nu \notin \mathfrak{a}$.

(2) There exist no $i < \nu < j < l$ such that $x_i x_l, x_\nu x_j \in I$.

□

Note that $T^{\mathcal{M}_2}$ is not a resolution (not even a complex), but we need it because of the following corollary, which will be important in Paragraph 3 of Chapter 6.

Corollary 1.8. *Let $\mathfrak{a}$ be a monomial ideal generated in degree two and $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ a standard matching on the Taylor resolution. With the notation above we get:*

$$\begin{aligned} \sum_{I \notin \mathcal{M}_1} (-1)^{cl(I)} m_I t^{cl(I)+|I|} &= \sum_{I \notin \mathcal{M}} (-1)^{cl(I)} m_I t^{cl(I)+|I|} \\ &= \sum_{I \ \mathbf{nbc}\text{-}set} (-1)^{cl(I)} m_I t^{cl(I)+|I|}. \end{aligned} \tag{1.1}$$

If $\mathfrak{a}$ is the Stanley Reisner ideal of the order complex of a partially ordered set P, then

$$(1.1) = \sum_{I \notin \mathcal{B}} (-1)^{cl(I)} m_I t^{cl(I)+|I|}. \tag{1.2}$$

Proof. Lemma 1.3 implies the first equality and the second equality follows by Proposition 1.4. If $\mathfrak{a}$ is the Stanley Reisner ideal of the order complex of a partially ordered set P, then Proposition 1.7 together with the proof of Lemma 1.3 imply Equation (1.2). □

1.4. The gcd-Condition. In this section we introduce the gcd-condition. Let $\mathfrak{a} \trianglelefteq S$ be a monomial ideal in the commutative polynomial ring and $\mathrm{MinGen}(\mathfrak{a})$ a minimal monomial generating system.

Definition 1.9 (gcd-condition).
(1) We say that $\mathfrak{a}$ satisfies the gcd-condition, if for any two monomials $m, n \in \mathrm{MinGen}(\mathfrak{a})$ with $\gcd(m, n) = 1$ there exists a monomial $m, n \neq u \in \mathrm{MinGen}(\mathfrak{a})$ with $u \mid \mathrm{lcm}(m, n)$;
(2) We say that $\mathfrak{a}$ satisfies the strong gcd-condition if there exists a linear order $\prec$ on $\mathrm{MinGen}(\mathfrak{a})$ such that for any two monomials $m \prec n \in \mathrm{MinGen}(\mathfrak{a})$ with $\gcd(m, n) = 1$ there exists a monomial $m, n \neq u \in \mathrm{MinGen}(\mathfrak{a})$ with $m \prec u$ and $u \mid \mathrm{lcm}(m, n)$.

Example 1.10. Let $\mathfrak{a} = \langle x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_1x_5 \rangle$ be the Stanley Reisner ideal of the triangulation of the 5-gon. Then $\mathfrak{a}$ satisfies the gcd-condition, but not the strong gcd-condition.

Proposition 1.11. *Let $\mathfrak{a}$ be a monomial ideal which satisfies the strong* gcd-*condition. Then there exists an acyclic matching $\mathcal{M}$ on the Taylor resolution such that for all* $\mathrm{MinGen}(\mathfrak{a}) \supset \mathrm{I} \notin \mathcal{M}$ *we have $cl(I) = 1$. We call the resulting Morse complex T_{gcd}.*

Proof. Assume $\mathrm{MinGen}(\mathfrak{a}) = \{\mathrm{m}_1 \prec \mathrm{m}_2 \prec \ldots \prec \mathrm{m}_l\}$. We start with m_1. Let $m_{i_0} \in \mathrm{MinGen}(\mathfrak{a})$ be the smallest monomial such that $\gcd(m_1, m_{i_0}) = 1$. Then there exists a monomial $m_1 \prec u_0 \in \mathrm{MinGen}(\mathfrak{a})$ with $u_0 \mid \mathrm{lcm}(m_1, m_{i_0})$. Then we define

$$\mathcal{M}_0 := \Big\{ \Big(\{m_1, m_{i_0}, u_0\} \cup I \Big) \to \Big(\{m_1, m_{i_0}\} \cup I \Big) \ \Big| \ I \subset \mathrm{MinGen}(\mathfrak{a}) \Big\}.$$

It is clear that this is an acyclic matching and that the Morse complex $T_\bullet^{\mathcal{M}_0}$ is a subcomplex of the Taylor resolution.

Now let m_{i_1} be the smallest monomial $\neq m_{i_0}$ such that $\gcd(m_1, m_{i_1}) = 1$. Then there exists a monomial $m_1 \prec u_1 \in \mathrm{MinGen}(\mathfrak{a})$ with $u_1 \mid \mathrm{lcm}(m_1, m_{i_1})$ and we define

$$\mathcal{M}_1 := \Big\{ \Big(\{m_1, m_{i_1}, u_1\} \cup I \Big) \to \Big(\{m_1, m_{i_1}\} \cup I \Big) \;\Big|\; I \subset \mathrm{MinGen}(\mathfrak{a}) \Big\}.$$

Again, it is straightforward to prove that $\mathcal{M}_1$ is an acyclic matching on $T^{\mathcal{M}_0}$ and that the Morse complex is a subcomplex of the Taylor resolution. We repeat this process for all $m_1 \prec m_i$ with $\gcd(m_1, m_i) = 1$ and we reach a subcomplex $T^{\mathcal{M}_{m_1}}$, $\mathcal{M}_{m_1} = \bigcup_i \mathcal{M}_i$, of the Taylor resolution which satisfies the following condition: For all remaining subsets $I \subset \mathrm{MinGen}(\mathfrak{a}) \setminus \mathcal{M}_{\mathrm{m}_1}$ we have:

(1) $m_1 \in I \Rightarrow cl(I) = 1$,

(2) $m_1 \notin I \Rightarrow cl(I) \geq 1$.

We repeat now this process with the monomial m_2. Here we have to guarantee that for a set $\{m_2, m_i\} \cup I$ the corresponding set $\{m_2, m_i, u_i\} \cup I$, with $\gcd(m_2, m_i) = 1$ and $m_2 \prec u_i$ and $u_i \mid \mathrm{lcm}(m_2, m_i)$, is not matched by the first sequence $\mathcal{M}_{m_1}$. Since all sets $J \in \mathcal{M}_{m_1}$ satisfy $m_1 \in J$, this would be the case if either $u_i = m_1$ or $m_1 \in I$. The first case is impossible since $m_1 \prec m_2 \prec u_i$. In second case we have $cl\big(\{m_2, m_i\} \cup I\big) = 1$. We define:

$$\mathcal{M}_2 := \left\{ \Big(\{m_2, m_i, u_2\} \cup I \Big) \to \Big(\{m_2, m_i\} \cup I \Big) \;\middle|\; \begin{array}{c} I \subset \mathrm{MinGen}(\mathfrak{a}) \setminus \mathcal{M}_{\mathrm{m}_1} \\ \text{and } cl\big(\{m_2, m_i\} \cup I\big) \geq 2 \end{array} \right\}.$$

Condition (1) implies then that $\mathcal{M}_2$ is a well defined sequence of acyclic matchings. Since we make this restriction, the resulting Morse complex is not anymore a subcomplex of the Taylor resolution, but we have still the following fact: For all remaining subsets $I \subset \mathrm{MinGen}(\mathfrak{a}) \setminus \big(\mathcal{M}_{\mathrm{m}_1} \cup \mathcal{M}_{\mathrm{m}_2}\big)$ we have:

(1) $m_1 \in I \Rightarrow cl(I) = 1$,

(2) $m_2 \in I \Rightarrow cl(I) = 1$,

(3) $m_1, m_2 \notin I \Rightarrow cl(I) \geq 1$.

We apply this process to all monomials. Then we finally reach a complex with the desired properties. $\square$

2. Algebraic Discrete Morse Theory for the Poset Resolution

In this paragraph we consider a monomial ideal $\mathfrak{a} = \langle \mathcal{B} \rangle \trianglelefteq S = k[x_1, \ldots, x_n]$ in the commutative polynomial ring generated by a monomial ordered family $\mathcal{B}$. In the first section we prove that the poset resolution can be obtained by an acyclic matching from the Taylor resolution. Then we develop several acyclic matchings on the poset resolution in order to minimize it. We define two algorithms which produce from the poset resolution a rather small - and sometimes minimal - cellular resolution. The quality of these resolutions depends strongly on the underlying partially ordered set. We discuss the properties of a "good" underlying partially ordered set in the second subsection. We consider the special case where $\mathcal{B}$ is the set of lcm's of a minimal monomial generating system and the underlying poset is the lcm-lattice ordered by divisibility. We show how one can optimize the lcm-lattice in order to get better results.

2.1. ADMT for the Poset Resolution. Let $\mathcal{B}$ be a monomial ordered family, $P = (P, \prec)$ the partially ordered set corresponding to $\mathcal{B}$, and $\Delta(P)$ the order complex of P. Recall the poset resolution from Chapter 2.2.2:
We define the complex $C(P)_\bullet$ as follows:

(1) For $i \geq 0$, let C_i be the free S-module with basis e_σ indexed by $\sigma \in \Delta(P)$ with $|\sigma| = i$,
(2) the differential $\partial_i : C_i \to C_{i-1}$ is given by

$$\partial_i(e_\sigma) = \sum_{\substack{\tau \in \sigma \\ |\tau| = i-1}} \varepsilon(\sigma, \tau) \frac{m_\sigma}{m_\tau} e_\tau,$$

where $\varepsilon(\sigma, \tau) = \pm 1$ depends on the orientation of $\Delta(P)$.

Proposition 2.1 (see [**39**]). *The complex $C(P)_\bullet$ is a free cellular resolution of $S/\langle \mathcal{B} \rangle$ as an S-module, called the* poset resolution.

We want to show that the complex $C(P)_\bullet$ can be obtained from the Taylor resolution by an acyclic matching. This will give a new proof of Proposition 2.1. In order to do so, we have to introduce some notation for partially ordered sets:

For any partially ordered set $P = (P, \prec)$ we have a rank-function defined by

$$\mathrm{rank}(m) := \max \Big\{ j \geq 0 \Big| \text{ there exist } n_1, \ldots, n_j \in P \text{ with } n_1 \prec \ldots \prec n_j \prec m \Big\}.$$

We write $P_i \subset P$ for the set of elements of rank i.

Furthermore, we need a total order on the powerset $\mathbb{P}(P)$ of P. In order to define this, we first define a total order on the powerset $\mathbb{P}(P_i)$ of P_i:
For this we fix bijections

$$g_i : P_i \to [|P_i|]. \tag{2.1}$$

To each $U \subset P_i$ we associate a monomial $m_U := \prod_{j \in g_i(U)} x_j$ in the polynomial ring $k[x_1, \ldots, x_{|P_i|}]$. Set $m_\emptyset := 1$. For two subsets $U_i, V_i \subset P_i$ we say that

$$U_i \prec_{\mathbb{P}_i} V_i \Leftrightarrow m_{U_i} <_{\deg\,\mathrm{lex}} m_{V_i}.$$

It is clear that $\prec_{\mathbb{P}_i}$ is a total order on $\mathbb{P}(P_i)$. We use this order to define a total order on the powerset of P: For two subsets $U = \uplus_i U \cap P_i$ and $V = \uplus_i V \cap P_i$ we say that

$$U \prec_{\mathbb{P}} V :\Leftrightarrow U \cap P_j \prec_{\mathbb{P}_j} V \cap P_j \text{ where } j = \min \Big\{ i \geq 0 \;\Big|\; U \cap P_i \neq V \cap P_i \Big\},$$

Since $\prec_{\mathbb{P}_i}$ is a total order for all i, it is clear that $\prec_{\mathbb{P}}$ is a total order as well.

The fact that we have a total order on the powerset of P implies that there exists a unique maximal (with respect to $\prec_{\mathbb{P}}$) antichain A_U in any subset $U \subset P$, namely

$$A_U := \max_{\prec_{\mathbb{P}}} \left\{ V \subset U \;\middle|\; \begin{array}{l} V \text{ antichain of } P \text{ with respect} \\ \text{to the order of } P \text{ and } |V| \geq 2 \end{array} \right\}.$$

Note that this definition of A_U implies the fact that U is a chain if and only if A_U is empty.

From now on let $\mathcal{B}$ be a monomial ordered family and $P = (P, \prec)$, with $f : P \to \mathcal{B}$, the corresponding partial order. In this case, we assume that the bijections (2.1) preserve the lexicographic order:

$$g_i(w) < g_i(w') \iff f(w) <_{\text{lex}} f(w') \text{ for all } w, w' \in P_i.$$

Lemma 2.2. *For any nonchain $U \subset P$ there exists an element $w_U \in P$ such that*

(1) $w_U \succ v$ *for all* $v \in A_U$ *and*
(2) $f(w_U)$ *divides* $\operatorname{lcm}\big(f(A_U)\big)$.

Let $w_{A_U} \in P$ be the minimum, with respect to the lexicographic order on the corresponding elements in $\mathcal{B}$, of all w_U.

Proof. The proof is by induction on the cardinality of A_U: For $|A_U| = 2$ it is the definition of a monomial ordered family. Assume $|A_U| \geq 3$. Fix any element $v_0 \in A_U$. By induction there exists an element $\tilde{w} \in P$ with $\tilde{w} \succ v$ for all $v \in A_U \setminus \{v_0\}$ and $\tilde{w} \mid \operatorname{lcm}(A_U \setminus \{v_0\})$. By the definition of a monomial ordered family there exists an element $w_U \in P$ with $w_U > v_0, \tilde{w}$ and $w_U \mid \operatorname{lcm}(v_0, \tilde{w})$. Clearly, w_U satisfies the desired properties. □

We are now in position to prove Proposition 2.1:

Proof of Proposition 2.1. Let $T_\bullet$ be the Taylor resolution of the ideal $\langle \mathcal{B} \rangle$. Note that $\mathcal{B}$ does not have to be a minimal generating system. The Taylor resolution is taken over the set $\mathcal{B}$ and is supported by the simplicial complex $\Delta(\mathcal{B})$. We identify the basis of $T_\bullet$ with the subsets $U \subset P$. We then define the acyclic matching on the Taylor resolution by

$$\mathcal{M} := \Big\{ \Big(U \cup \{w_{A_U}\}\Big) \to \Big(U \setminus \{w_{A_U}\Big)\} \;\Big|\; U \subset P \text{ such that } A_U \neq \emptyset \Big\}.$$

Since $f(w_{A_U})$ divides $\operatorname{lcm}\big(f(A_U)\big)$, we have

$$\operatorname{lcm}\big(f(U \cup \{w_{A_U}\})\big) = \operatorname{lcm}\big(f(U \setminus \{w_{A_U}\})\big),$$

which provides invertibility. For the matching property it is sufficient to prove that for a subset U in P with $\emptyset \neq A_U$ we have $A_{U \cup \{w_{A_U}\}} = A_{U \setminus \{w_{A_U}\}}$.
Let B be the maximal antichain of $U \cup \{w_{A_U}\}$. If $w_{A_U} \notin B$, we have $B = A_U$. Assume now that w_{A_U} lies in B: It follows that $B \cap A_U = \emptyset$. Since A_U is also an antichain of $U \cup \{w_{A_U}\}$, the maximality of B implies $B \succ_{\mathbb{P}} A_U$. Therefore, there exists an index i such that the following holds:

(1) $B \cap P_i \neq \emptyset$,
(2) $A_U \cap P_j = \emptyset$ and $B \cap P_j = \emptyset$ for all $j < i$,
(3) $B_i \succ_{\mathbb{P}_i} A_i$,
(4) $w_{A_U} \notin B \cap P_i$.

Let b be an element of $B \cap P_i$. Since $\{b, w_{A_U}\}$ is an antichain and $w_{A_U} \succ v$ for all $v \in A_U$, it follows that $C := \{b\} \cup A_U$ is an antichain of $U \cup \{w_{A_U}\}$ and an antichain of $U \setminus \{w_{A_U}\}$. By construction we have $C \succ_{\mathbb{P}} A_U$, which is a contradiction to the maximality of A_U.

Finally we have to prove acyclicity: Let $\sigma_n := U_n \cup \{w_{A_{U_n}}\}$ and $\tau_n := U_n$, with $w_{A_{U_n}} \notin U_n$. Assume there exists a directed cycle in the Morse graph $G_{\mathcal{M}}$:

$$\sigma_n \to \tau_1 \to \sigma_1 \cdots \tau_{n-1} \to \sigma_{n-1} \to \tau_n \to \sigma_n$$

For simplification we write A_i instead of A_{U_i}. It follows that

$$\tau_1 = U_1 = \sigma_n \setminus \{u\} = \Big(U_n \cup \{w_{A_n}\}\Big) \setminus \{u\}. \tag{2.2}$$

(Case 1) $u \notin A_n$. Then clearly $A_1 = A_n$, and the minimality of w_{A_n} implies $w_{A_n} = w_{A_1}$. Since $w_{A_1} \notin U_1$, Equation (2.2) implies $u = w_{A_n} = w_{A_1}$ and we have $\sigma_1 = \sigma_n$ and $\tau_1 = \tau_n$.

(Case 2) $u \in A_n$. Assume that $w_{A_n} \neq w_{A_1}$ (otherwise go to (Case 1)). It follows that $\sigma_1 = \Big(U_n \cup \{w_{A_n}, w_{A_1}\}\Big) \setminus \{u\}$. Since we have a cycle, there exists a position i with $\sigma_i \setminus \tau_i = \{u\}$. It follows $u \succ v$ for all $v \in A_i$ and $u|\operatorname{lcm}\big(f(A_i)\big)$. Since A_n is the maximal antichain, it follows $u \notin A_n$, which is a contradiction.

Note that in the Taylor complex the differential maps a subset $U \in P$ which is a chain only to chains $U' \in P$. Therefore, the Morse differential equals the original differential, and we are done. □

Now we apply ADMT again to this complex in order to minimize it. For simplification we identify the monomial set $\mathcal{B}$ with the corresponding poset $P = (P, \prec)$ and write $m \in P$ for a monomial $m \in \mathcal{B}$. We denote with $\prec_t$ any linear extension of the order $\prec$. We introduce the following definition:

Definition 2.3. A chain $\sigma = m_1 \prec \cdots \prec m_k \in P$ is called

(1) minimal at m_i (or minimal at rank i) if
  (a) m_i divides $\operatorname{lcm}(\sigma \setminus \{m_i\})$ and
  (b) for all monomials $n \in P$ with $m_{i-1} \prec n \prec m_{i+1}$ and $n|\operatorname{lcm}(\sigma \setminus \{m_i\})$ we have $m_i \prec_t n$,
(2) minimal if there exists $1 \leq i \leq k$ such that σ is minimal at rank i.

A monomial n is called minimal with respect to σ and the monomial m_{i+1} if

(1) $m_i|\operatorname{lcm}(\sigma \setminus \{m_i\})$ and $n|\operatorname{lcm}(\sigma \setminus \{m_i\})$ and
(2) the chain $\sigma[^n/_{m_i}] := m_1 \prec \cdots \prec m_{i-1} \prec n \prec m_{i+1} \prec \cdots \prec m_k$ is minimal at rank i.

We define an acyclic matching $\mathcal{M}_i$ for $i = 1, \ldots, \operatorname{rank}(P)$:

Proposition 2.4. *The set*

$$\mathcal{M}_i := \left\{ \sigma \to \sigma \setminus \{m_i\} \;\middle|\; \begin{array}{c} \sigma = m_1 \prec \ldots \prec m_i \prec \ldots \prec m_k,\ i \leq k \leq \operatorname{rank}(P), \\ \text{minimal at } m_i \end{array} \right\}$$

is for all $1 \leq i \leq \operatorname{rank}(P)$ an acyclic matching. The Morse complex is cellular, supported by a regular CW-complex, and for the remaining chains $\sigma_1 = m_1 \prec \cdots \prec m_k$ we have

(1) *σ is not minimal at m_i,*
(2) *there exists no monomial $n \in P$ with $m_{i-1} \prec n \prec m_i$ and $n|lcm(\sigma)$.*

Proof. The invertibility follows from Definition 2.3. The minimality at m_i proves the matching property: Assume we have $\sigma \to \sigma \setminus \{m_i\} \in \mathcal{M}_i$.

(Case 1) $\sigma \setminus \{m_i\} \to \sigma \setminus \{m_i, m_{i+1}\} \in \mathcal{M}_i$. Then m_{i+1} is minimal with respect to $\sigma \setminus \{m_i\}$ and m_{i+2}. But since $m_i \prec m_{i+1}$, it follows that $\sigma \setminus \{m_i\}$ is not minimal at m_{i+1}.

(Case 2) $\sigma \cup n \to \sigma \in \mathcal{M}_i$. Then $m_{i-1} \prec n \prec m_i$ is minimal with respect to $\sigma \cup n$ and m_i. With analogous arguments we have that n is minimal with respect to σ and m_{i+1}. This is a contradiction to the minimality of m_i.

For the acyclicity we again assume a directed cycle in the kth homological degree:

$$\sigma_1 \to \tau_1 \to \cdots \to \tau_{n-1} \to \sigma_n \to \tau_n \to \sigma_1,$$

with $|\sigma_i| = k$ and $|\tau_{i-1}| = k-1$. Assume further that $\sigma_1 = m_1 \prec \cdots \prec m_k$, $\tau_n = \sigma_1 \setminus \{m_i\}$ and $\tau_1 = \sigma_1 \setminus \{m_j\}$, $j \neq i$. It is obvious that the minimality of m_i and the fact that, by passing to a higher homological degree cell, the only changing monomial is the monomial at the ith position imply that such a cycle is not possible.

It is clear that the remaining chains satisfy the desired condition. Since $C_\bullet(P)$ is supported by a regular CW-complex, the Morse complex is also supported by a CW-complex (see [**4**]). For the regularity of the CW-complex we only have to prove that all coefficients of the Morse differential $\partial^{\mathcal{M}}$ are ± 1. This follows, since there is a unique directed path from each chain to another chain, which again follows by the minimality of the monomial m_i and the fact that, by passing to a higher homological degree cell, only the monomial at the ith position changes. □

In order to give an explicit description of the Morse complex with respect to $\mathcal{M}_i$, we make the following definition:

Definition 2.5. A chain σ is called critical with respect to $\mathcal{M}_i$ if

(1) σ is not minimal at position i and
(2) there is no monomial $n \in P$ with $m_{i-1} \prec n \prec m_i$ and $n | lcm(\sigma)$.

Theorem 2.6. *The Morse complex with respect to $\mathcal{M}_i$ is given by:*

$$C_n^{\mathcal{M}_i} := \bigoplus_{\substack{\sigma \ \text{critical} \\ |\sigma| = n}} S\ \sigma$$

with the differential

$$d_j(\sigma) := \begin{cases} \frac{\mathrm{lcm}(\sigma)}{\mathrm{lcm}(\sigma \setminus \{m_j\})}\ \sigma \setminus \{m_j\} & , \quad \sigma \setminus \{m_j\} \ \textit{critical} \\ (-1)^{i+1} \sum\limits_{\substack{k=1 \\ k \neq i}}^{i+1} (-1)^k d_k(\sigma[{}^{n}/_{m_i}]) & , \quad j = i,\ m_i | \,\mathrm{lcm}(\sigma \setminus \{m_i\}) \ \textit{and} \\ & \quad\ n \ \textit{minimal w.r.t.}\ \sigma \ \textit{and}\ m_{i+1} \\ 0 & , \quad \textit{else}, \end{cases}$$

$$\partial_n(\sigma) := \sum_{j=1}^{n} (-1)^j \ d_j(\sigma). \tag{2.3}$$

Here $\sigma[{}^{n}/_{m_i}]$ denotes the chain $\sigma \setminus \{m_i\} \cup \{n\}$.

Proof. The critical chains are exactly the remaining basis elements after applying the acyclic matching $\mathcal{M}_i$ from Proposition 2.1. The differential coincides with the definition of the Morse differential. □

Remark 2.7. Note that the second part of the definition of critical chains (Definition 2.5) implies that redundant monomials in $\mathcal{B}$ are not critical 0-chains with respect to $\mathcal{M}_1$. Therefore, the 0-chains critical with respect to the acyclic matching $\mathcal{M}_1$ are in one-to-one correspondence with the minimal generating system.

Clearly, in general the Morse complex with respect to $\mathcal{M}_i$ is still far away from being minimal. We define two algorithms in order to get a resolution which is close to the minimal resolution. We proceed as follows:

- If the underlying poset P has rank N, then apply the acyclic matching $\mathcal{M}_i$ to the Morse complex $C_\bullet^{\mathcal{M}_1\cup\ldots\cup\mathcal{M}_{i-1}}$ for all $i = 1, \ldots, N$. Here we run into problems with the matching property. In order to keep the matching property, we match in $\mathcal{M}_i$ only those chains which are critical with respect to the matching $\mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{i-1}$.
- As in the first case, we apply the acyclic matchings iteratively, but we slightly change the definition of the matchings $\mathcal{M}_i$ to
$$\mathcal{M}_i' := \Big\{\sigma \to \sigma \setminus \{m_i\} \;\Big|\; \sigma \text{ minimal at } m_i \text{ w.r.t. } \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{i-1}\Big\}.$$

Algorithm 1:
For a partially ordered set of rank N we define the acyclic matching as follows:

$$\mathcal{M}_i := \left\{\sigma \to \sigma \setminus \{m_i\} \;\middle|\; \begin{array}{l} \sigma \text{ minimal at } m_i \\ \sigma,\ \sigma \setminus \{m_i\} \text{ critical w.r.t. } \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{i-1} \end{array}\right\},$$
$$\mathcal{M} := \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_N.$$

Let $\sigma = m_1 \prec \ldots \prec m_n$ be minimal at m_i. Then we want to match $\sigma \to \sigma \setminus \{m_i\}$. Assume $\sigma \setminus \{m_i\}$ is matched by an acyclic matching $\mathcal{M}_j$ with $j < i$. If $\sigma \setminus \{m_i\}$ is minimal at m_j, then σ is also minimal at m_j and therefore not critical. If there exists a monomial $m_{j-1} \prec n \prec m_j$ such that $\big(\sigma \setminus \{m_i\}\big) \cup \{n\} \to \big(\sigma \setminus \{m_i\}\big) \in \mathcal{M}_j$, then it follows $\sigma \cup \{n\} \to \sigma \in \mathcal{M}_j$. Thus, the matching property holds.
Now assume there exists a monomial $m_{i-1} \prec n \prec m_i$ such that n is minimal w.r.t. $\sigma \cup \{n\}$ and m_i. In this case we want to match $\sigma \cup \{n\} \to \sigma$. Assume $\sigma \cup \{n\}$ is already matched by an acyclic matching $\mathcal{M}_j$ with $j < i$.

(Case 1) $j < i - 1$ and $\sigma \cup \{n\}$ is minimal at m_j. Then σ is also minimal at m_j and therefore matched by $\mathcal{M}_j$.

(Case 2) There exists a monomial $m_{j-1} \prec u \prec m_j$ with $\sigma \cup \{n, u\} \to \sigma \cup \{n\} \in \mathcal{M}_j$. Then we have $\sigma \cup \{u\} \to \sigma \in \mathcal{M}_j$.

In both cases the matching property holds. The only case where we get a problem is if $\sigma \cup \{n\}$ is matched by the matching $\mathcal{M}_{i-1}$ and $\sigma \cup \{n\}$ is minimal at m_{i-1}, hence m_{i-1} is minimal w.r.t. $\sigma \cup \{n\}$ and n.
These facts give rise to the following definition of critical chains:

Definition 2.8. A chain $\sigma = m_1 \prec \cdots \prec m_k$ is called critical if

(1) σ is not minimal and
(2) for an index $1 \leq i \leq k-1$ and a monomial $m_i \prec n \prec m_{i+1}$ such that $\sigma \cup \{n\}$ is minimal at n, we have $\sigma \cup \{n\}$ is minimal at m_i.

We get the following resolution:

Theorem 2.9. *The complex* $(C_\bullet^{\mathcal{M}}, \partial)$ *with*

$$C_n = \bigoplus_{\substack{\sigma \ \text{critical} \\ |\sigma|=n}} S\,\sigma$$

with differential as defined in Equation (2.3) defines a free cellular resolution of the ideal $\mathfrak{a} := \langle\ \mathcal{B}\ \rangle$ *which is supported by a regular CW-complex.*

Proof. We only have to prove that the matching $\mathcal{M}$ defined above is a well defined acyclic matching. The matching property as well as the invertibility is given by definition. By Proposition 2.4 each matching $\mathcal{M}_i$ is acyclic.
We prove that $\mathcal{M}_1 \cup \ldots \cup \mathcal{M}_i$ is acyclic for all $i = 1, \ldots, N$. Consider the following path:

$$\sigma_1 \to \tau_1 \to \sigma_2 \to \tau_2 \to \ldots \to \sigma_{n-1} \to \tau_{n-1} \to \sigma_n$$

with σ_j in homological degree k and τ_j in homological degree $k-1$. We prove the following fact:

($*$) For all $j = 1, \ldots, n$ we have:
If $\sigma_j = m_1 \prec \ldots \prec m_k$, then there exists an index l such that m_l divides $\operatorname{lcm}\big(\sigma_j \setminus \{m_l\}\big)$ and $\sigma_{j+1} = m_1 \prec \ldots \prec m_{l-1} \prec u \prec m_{l+1} \prec \ldots \prec m_k$, where u is the smallest monomial between m_{l-1} and m_{l+1} dividing $\operatorname{lcm}(\sigma_j)$ and m_l, u is an antichain in P with $u \prec_t m_l$.

Clearly, this proves acyclicity since by passing from the chain σ_1 to the chain τ_1 we leave out one monomial m, which can never be inserted again along the path. Therefore, we get $\sigma_n \neq \sigma_1$.

In order to prove property ($*$), we consider a step in the path.
Let $\sigma_j := m_1 \prec \ldots \prec m_l \prec \ldots \prec m_k$ and $\tau_j = m_1 \prec \ldots \prec m_{l-1} \prec m_{l+1} \prec \ldots \prec m_k$. By passing to the chain σ_{j+1} we insert at position $\tilde{l}$ a monomial u. Assume $\tilde{l} < l$. In this case we can also insert the monomial u in the chain σ_j, which is impossible, since σ_j is matched with τ_{j-1}. Therefore, we get $\tilde{l} \geq l$. Since we apply the matchings $\mathcal{M}_i$ inductively, we have to apply $\mathcal{M}_l$. This means that we insert a monomial u with $m_{l-1} \prec u \prec m_{l+1}$ and $u \prec_t m_l$. If $u \prec m_l$, then the chain σ_j is matched with $\sigma_j \cup \{u\}$, which is impossible. Therefore, $\{m_l, u\}$ is an antichain, and we are done. □

Algorithm 2:
This algorithm proceeds similar to Algorithm 1 but defines matchings $\mathcal{M}'_i$ with respect to the Morse complex of the preceding matchings. We have to introduce a new notion of minimal chains:

Definition 2.10. Let $\sigma = m_1 \prec \cdots \prec m_k \in \Delta(P)^{\mathcal{M}'_{<i}}$ be a chain in P and $\Delta(P)^{\mathcal{M}'_{<i}}$ the set of critical chains with respect to $\mathcal{M}'_{<i} := \mathcal{M}'_1 \cup \ldots \cup \mathcal{M}'_{i-1}$.

(1) σ is called minimal at m_i with respect to $\mathcal{M}'_{<i}$ if
   (a) $m_i | \operatorname{lcm}(\sigma \setminus \{m_i\})$ and
   (b) for all $n \in P$ with $m_{i-1} \prec n \prec m_{i+1}$ and $n | \operatorname{lcm}(\sigma \setminus \{m_i\})$ and $\sigma[{}^n/_{m_i}] \in \Delta(P)^{\mathcal{M}'_{<i}}$ we have $m_i \prec_t n$.

(2) σ is called relatively minimal if there exists an index $1 \leq i \leq k$ such that σ is minimal at m_i w.r.t. $\mathcal{M}'_{<i}$.

(3) A monomial n is called relatively minimal with respect to σ and m_{i+1} if
   (a) $m_i | \operatorname{lcm}(\sigma \setminus \{m_i\})$, $n | \operatorname{lcm}(\sigma \setminus \{m_i\})$ and
   (b) $\sigma[{}^n/_{m_i}] \in \Delta(P)^{\mathcal{M}'_{<i}}$ and
   (c) $\sigma[{}^n/_{m_i}]$ is minimal at n with respect to $\mathcal{M}'_{<i}$.

We construct the following matching:

$$\mathcal{M}'_i := \left\{ \sigma \to \sigma \setminus \{m_i\} \;\middle|\; \sigma \text{ relatively minimal at } m_i \right\}$$

and set $\mathcal{M}' := \mathcal{M}'_1 \cup \ldots \cup \mathcal{M}'_N$. By construction the matching property as well as the invertibility is satisfied. In order to describe the critical chains, we introduce the following definition:

Definition 2.11. A chain $\sigma = m_1 \prec \cdots \prec m_k$ is called relatively critical if

(1) σ is not relatively minimal and

(2) for an index $1 \leq i \leq k$ and a monomial $m_i \prec n \prec m_{i+1}$, $n \in \mathcal{B}$ such that $\sigma \cup \{n\}$ is minimal at n w.r.t. $\mathcal{M}'_{<i+1}$, we have $\sigma \cup \{n\}$ is minimal at m_i with respect to $\mathcal{M}'_{<i}$.

With the same proof as for Theorem 2.9 we get

Theorem 2.12. *The complex $(C_\bullet^{\mathcal{M}'}, \partial)$ defined by*

$$C_n = \bigoplus_{\substack{\sigma \text{ relatively critical} \\ |\sigma|=n}} S\,\sigma,$$

$$d_j(\sigma) := \begin{cases} \frac{\operatorname{lcm}(\sigma)}{\operatorname{lcm}(\sigma\setminus\{m_j\})}\, \sigma \setminus \{m_j\} & , \quad \sigma \setminus \{m_j\} \text{ rel. critical} \\ 0 & , \quad \sigma \setminus \{m_j\} \text{ rel. minimal} \\ (-1)^{i+1} \sum\limits_{\substack{k=1 \\ k \neq i}}^{i+1} (-1)^k\, d_k(\sigma[{}^n/_{m_i}]) & , \quad j = i,\ m_i | \operatorname{lcm}(\sigma \setminus \{m_i\}), \text{ and } \\ & \quad\ n \text{ rel. min. w.r.t. } \sigma \text{ and } m_{i+1}, \end{cases}$$

$$\partial(\sigma) := \sum_{j=1}^{n} (-1)^j\, d_j(\sigma)$$

is a free cellular resolution of the ideal $\mathfrak{a} = \langle \mathcal{B} \rangle$ supported by a regular CW-complex. □

The disadvantage of this resolution is that the relatively critical chains cannot be calculated directly from the partially ordered set P. But the resolution is much smaller than the resolution constructed by Algorithm 1.

Example 2.13. Consider the Stanley Reisner ideal $\mathfrak{a}$ of a triangulation of the real projective plane:

$$\mathfrak{a} := \left\langle \begin{array}{c} x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, \\ x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6 \end{array} \right\rangle$$

The computer algebra system "Macaulay2" [**25**] calculates the following Betti numbers:

in characteristic 2: $(1, 7, 15, 10, 1)$
in characteristic 0: $(0, 6, 15, 10, 1)$

We choose as underlying partially ordered set the lcm-lattice ordered by divisibility. As linear extension we take the opposite lexicographic order, i.e. $m \prec n$ iff $m \succ_{\text{lex}} n$. We get resolutions with the following ranks:

Algorithm 1: $(3, 14, 20, 10, 1)$
Algorithm 2: $(2, 10, 17, 10, 1)$

If we choose the lcm-lattice ordered by divisibility as underlying partially ordered set, we can apply the following matching on the Morse complexes of the algorithms above.

Since we have chosen the lcm-lattice, the multidegree of a chain is given by its top element. Suppose we have in the resulting Morse complex two chains σ, τ with the same top element such that σ maps to τ. Then it follows that the coefficient is ± 1. It is easy to see that in this situation the chain τ is non-saturated, i.e. there exists a monomial $n \mid \text{lcm}(\tau)$ such that $\tau \cup \{n\}$ is again a chain.
Consider the Morse complex constructed by Algorithm 1 or 2. Assume σ is a critical cell which is non-saturated and there exists no non-saturated chain σ' such that $\text{lcm}(\sigma) = \text{lcm}(\sigma')$. Then there exists a chain τ such that $\tau \to \sigma$, with coefficient ± 1. Let τ_σ be the smallest (w.r.t $\prec$) minimal chain τ with the desired properties and define

$$\mathcal{M}'' := \{\tau_\sigma \to \sigma\}.$$

We do this for all non-saturated chains σ such that there is no other non-saturated chain σ' with $\text{lcm}(\sigma) = \text{lcm}(\sigma')$. This defines an acyclic matching since for each multidegree α we have at most one matched pair $\tau_\sigma \to \sigma$ with $\deg(\sigma) = \alpha$.
Note that the resulting Morse complex is still cellular.

Example 2.14 (Continuation of Example 2.13)**.** If we apply this matching to the Morse complex constructed by Algorithm 2, we get a resolution with ranks $(2, 8, 15, 10, 1)$.

Remark 2.15.

(1) Both algorithms construct cellular free resolutions of the ideal $\mathfrak{a}$ which are supported by regular CW complexes. Since we start in both cases with the acyclic matching $\mathcal{M}_1$, the 0-cells of the CW complexes are in one-to-one correspondence with the minimal monomial generating system.
(2) The quality of the resolutions constructed by Algorithm 1 or 2 depends on the chosen underlying partially ordered set. If $\mathfrak{a} \trianglelefteq S$ is any monomial ideal with minimal monomial generating system $\text{MinGen}(\mathfrak{a})$, then

one can always take the lcm-lattice ordered by divisibility. In this case the poset resolution coincides with the lcm-resolution introduced by Batzies [**4**].
Here our algorithms produce resolutions "close" to the minimal resolution.

If one chooses for P a totally ordered set (i.e. one chain), then the poset resolution coincides with the Taylor resolution. In this case the quality of the resolution depends on the position of the monomials in the chain. For example, if $\mathfrak{a}$ is generated by monomials m_1, m_2, m_3, m_4, then the algorithms on $P := \{m_1 \prec m_2 \prec m_3 \prec m_4\}$ might give better resolutions than on $P' := \{m_2 \prec m_4 \prec m_1 \prec m_3\}$.

(3) The poset resolution can be slightly generalized to the following resolution:

Let P be a partially ordered set and $\mathcal{B} \subset S$ a set of monomials such that there is a bijection $f : P \to \mathcal{B}$. Let $\Delta := \Delta(P)$ be the order complex of P. Define

$$\Delta_{\leq\alpha} := \Big\{\sigma = p_1 \prec \ldots \prec p_r \;\Big|\; \deg\big(f(p_i)\big) \leq \alpha,\ i = 1, \ldots, r\Big\}.$$

By [**4**] we have that Δ supports a cellular resolution of the ideal $\langle\mathcal{B}\rangle \trianglelefteq S$ if and only if $H_i(\Delta_{\leq\alpha}) = 0$ for all $i \geq 1$ and $\alpha \in \mathbb{N}^n$.
On this resolution we can also apply our algorithm.

We will discuss the question of a "good" underlying partially ordered set for the general case in the next section.

2.2. What is a "good" underlying partially ordered set P ?

The quality of the resolutions constructed by our algorithms depends heavily on the chosen partial order on the generating set $\mathcal{B}$ and its linear extension. Clearly, the best result would be a minimal resolution.

From now on we consider a partially ordered set P such that for any chain $\sigma = m_1 \prec \ldots \prec m_k$ we have $lcm(\sigma) \neq \mathrm{lcm}(\sigma \setminus \{m_k\})$. For example, the lcm-lattice satisfies this property.

In this case, Algorithm 1 constructs a minimal resolution if all critical chains are saturated. We introduce the following property:

Definition 2.16. We say that a partially ordered set $(P, \prec)$ with linear extension $\prec_t$ satisfies property $(*)$ if for any three monomials $m_1 \prec m_2 \prec m_3 \in P$ satisfying

$$m_1 \prec m_3 \text{ not minimal,}$$
$$m_1 \prec m_2 \prec m_3 \text{ minimal at } m_2$$

we have $m_1 \prec m_2 \prec m_3$ not minimal at m_1.

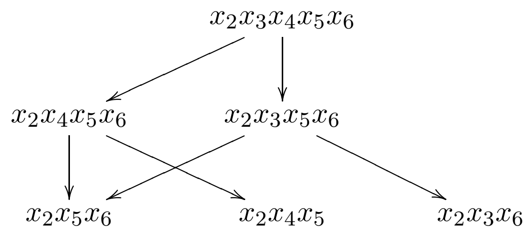
Proposition 2.17. *Assume the underlying partially ordered set P satisfies property $(*)$. Then the Morse complex constructed by Algorithm 1 is a minimal free resolution of the ideal $\mathfrak{a} = \langle\mathcal{B}\rangle$.*

Proof. Property $(*)$ implies that all critical chains are saturated. Therefore, a critical chain of length j has a top element different from that of a critical chain of length $j-1$. Since the multidegree is given by the top element, it follows that the resolution is minimal. □

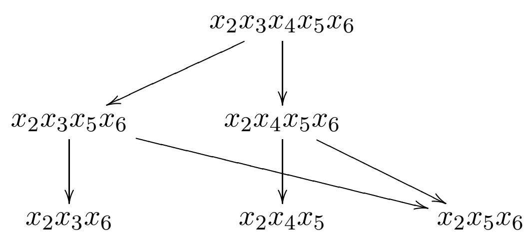
Remark 2.18. In the cases where the Taylor resolution is minimal or the Scarf complex is a resolution, property $(*)$ implies that Algorithm 1, applied to the poset resolution with respect to the lcm-lattice, constructs a minimal resolution.

The following example shows the dependence on the linear extension:

Example 2.19. Consider the ideal $\mathfrak{a} := \langle x_2x_3x_6, x_2x_4x_5, x_2x_5x_6\rangle$. Let P be the lcm-lattice. We choose as linear extension the lexicographic order. If we write elements of the same rank in increasing order from left to right, we get the following Hasse diagram:



In this case, Algorithm 1 constructs a minimal resolution. But if we take for the linear extension the opposite order, we get the following Hasse diagram:



In this case, the chain $x_2x_3x_4x_5x_6 \succ x_2x_4x_5$ is critical and we have

$$x_2x_3x_4x_5x_6 \succ x_2x_4x_5x_6 \succ x_2x_5x_6 \xrightarrow{\pm 1} x_2x_3x_4x_5x_6 \succ x_2x_4x_5.$$

Therefore, the resolution constructed by Algorithm 1 is not minimal.

By removing superfluous elements in P we get the next improvement of the results of our algorithms:

Definition 2.20. Let P be the underlying partially ordered set and $u \in P$ a redundant monomial, i.e. $\langle P\rangle = \langle P \setminus \{u\}\rangle$. If

$$H_i\Big(\Delta_{\leq\alpha}\big(P \setminus \{u\}\big), k\Big) = 0$$

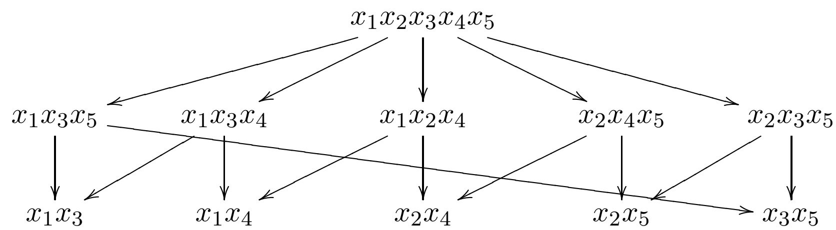
for all $i \geq 1$ and all $\alpha \in \mathbb{N}^n$, then we pass over to $P := P \setminus \{u\}$. We repeat this for all redundant elements in P.
Note that the order of the removed elements might be important. In order to define a unique *reduced poset* P_r, we always remove the largest (with respect to the linear extension) redundant monomial satisfying the above properties.

By Remark 2.15 (3) we can apply our algorithms to the poset resolution with respect to the reduced poset P_r.

Example 2.21. Consider the Stanley Reisner ideal of the 5-gon

$$\mathfrak{a} := \langle x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5\rangle.$$

If we take the reduced lcm-lattice, with linear extension given by extending the rank function by ordering each rank from left to right,



then Algorithm 1 constructs a minimal resolution.

In general, the problem of finding a “good” underlying partially ordered set with a ”good” linear extension is not easier than finding a minimal resolution. But the advantage of our algorithms is that they construct cellular resolutions.

We think that this gives a new approach for the proof of the conjecture that any monomial ideal admits a minimal cellular resolution:

Given a monomial ideal $\mathfrak{a} \trianglelefteq S$ with minimal generating system $\mathrm{MinGen}(\mathfrak{a})$.

- ▷ Find a partial order on $P := \mathrm{MinGen}(\mathfrak{a})$ satisfying the property (OM) (i.e. for any two monomials $m, n \in P$ there exists a monomial $w \in P$ such that $w \succ m, n$ and $w | \mathrm{lcm}(m, n)$).
  If such an order does not exist, add redundant monomials to P until there exists an order $\prec$ satisfying (OM).
- ▷ Pass to the reduced partially ordered set P_r. Note that in the definition of the reduced partially ordered set P_r the characteristic of k plays a role.
- ▷ Find a linear extension such that P satisfies property $(*)$.

Then the Morse complex constructed by Algorithm 1 is a minimal cellular resolution of $\mathfrak{a}$.

We think that there is a large field for further research in this direction. For example:

- ▷ For which classes of ideals does there exist an underlying partially ordered set P such that Algorithm 1 or 2 constructs a minimal resolution ?
- ▷ Does there exist further criteria on the partially ordered set P, such that Algorithm 1 or 2 constructs a minimal resolution ?
- ▷ Given two monomial ideals $\mathfrak{a} \trianglelefteq S$ and $\mathfrak{b} \trianglelefteq S$ with underlying partially ordered sets $P_\mathfrak{a}$ and $P_\mathfrak{b}$, does there exist a poset operation depending on $P_\mathfrak{a}$ and $P_\mathfrak{b}$ which constructs a “good” underlying partially ordered set P for
  - ▷ the sum $\mathfrak{a} + \mathfrak{b} \trianglelefteq S$?
  - ▷ the intersection $\mathfrak{a} \cap \mathfrak{b} \trianglelefteq S$?
  - ▷ the product ideal $\langle \mathrm{MinGen}(\mathfrak{a}) \cdot \mathrm{MinGen}(\mathfrak{b}) \rangle \trianglelefteq \mathrm{S}$?
  - ▷ the union $\langle \mathrm{MinGen}(\mathfrak{a}) \cup \mathrm{MinGen}(\mathfrak{b}) \rangle \trianglelefteq \mathrm{S}$?

3. Minimal Resolution and Regularity of Principal (p-)Borel Fixed Ideals

In this paragraph we develop minimal resolutions for principal Borel and p-Borel fixed ideals. The resolutions are derived with a single acyclic matching from the Taylor resolution and therefore supported by a CW-complex (see [**4**]). In addition we calculate the regularity of these type of ideals.

3.1. Cellular Minimal Resolution for Principal Borel Fixed Ideals. Throughout this paragraph let $S := k[x_1, \ldots, x_n]$ be the commutative ring of polynomials and $\mathfrak{a} \trianglelefteq S$ a monomial ideal. In this paragraph we develop a minimal cellular resolution for principal Borel fixed ideals. A minimal resolution and the regularity is known for general Borel fixed ideals (see for example [**19**]). Even a cellular minimal resolution is known (the Lyubeznik resolution is in this case a minimal cellular resolution [**5**]). We give another minimal cellular resolution for this type of ideals, which is a generalization of the *hypersimplex resolution of powers of the maximal homogeneous ideal* developed by Batzies [**4**].

Definition 3.1. A monomial ideal $\mathfrak{a} \trianglelefteq S$ is called Borel fixed, or stable, if for any monomial $m \in \mathfrak{a}$ and any $x_i \mid m$ we have

$$S_{ji}(m) = \frac{x_j\ m}{x_i} \in \mathfrak{a} \text{ for all } j < i. \tag{3.1}$$

Remark 3.2. For a Borel fixed ideal $\mathfrak{a} \trianglelefteq S$ there exists a unique set $\mathcal{G} := \{m_1, \ldots, m_l\}$ of monomials such that $\mathcal{G}$ is a minimal Borel-generating System of the ideal $\mathfrak{a}$, in the sense that any monomial $m \in \mathfrak{a}$ has a decomposition $m = u\ v$ such that v is constructed from a monomial m_r in $\mathcal{G}$ by iterative applications of rule (3.1), i.e. there exist indices $i_1, \ldots, i_l$ and $j_1, \ldots, j_l$ with $j_k < i_k$ for $k = 1, \ldots, l$ such that $v = S_{j_1 i_1}(\ldots(S_{j_l i_l}(m_r))\ldots)$. The minimality of $\mathcal{G}$ requires that the ideal $\mathfrak{a}'$ Borel-generated by a proper subset $\mathcal{G}'$ of $\mathcal{G}$ is a proper subideal of $\mathfrak{a}$.
We call $\mathcal{G}$ the Borel-generating system of $\mathfrak{a}$.

Definition 3.3. A Borel fixed ideal $\mathfrak{a} \trianglelefteq S$ is called principal Borel fixed if the Borel-generating system consists of one monomial, i.e. $\mathcal{G} = \{m\}$ for $m \in S$.

Lemma 3.4. *Let $\mathfrak{a} \trianglelefteq S$ be a principal Borel fixed ideal with $\mathcal{G} := \{x_1^{i_1} \cdots x_n^{i_n}\}$. Then*

$$\mathfrak{a} = \prod_{j=1}^{n} I_j$$

where

$$I_j := \begin{cases} \langle x_1, \ldots, x_j \rangle^{i_j} & , \quad i_j > 0 \\ 1 & , \quad i_j = 0. \end{cases}$$

Proof. The lemma is a direct consequence of Remark 3.2. □

We now give a generalization of the hypersimplex resolution introduced by Batzies [**4**]. This resolution will be used to construct a minimal cellular resolution for principal Borel fixed ideals.

We first recall the definition of a hypersimplicial complex from [**4**]:

Definition 3.5 (see [**4**]). Let C_d^n be the polytopal CW-complex with

$$\Delta_n := n \cdot \Delta^{d-1} = \{(y_1, \ldots, y_d) \in \mathbb{R}^d \mid \sum_{i=1}^{d} y_i = n, y_i \geq 0, i = 1, \ldots, d\}$$

as underlying space and CW-complex structure induced by intersection with the cubical CW-complex structure on $\mathbb{R}^d$ given by the integer lattice $\mathbb{Z}^d$. That is the closed cells of C_d^n are given by all hypersimplices

$$C_{\alpha,J} := \operatorname{conv}\left(\alpha + \sum_{j \in J} \varepsilon_j e_j \;\middle|\; \varepsilon_j \in \{0,1\},\ \sum_{j \in J} \varepsilon_j = n - |\alpha| \right)$$

with $\alpha \in \mathbb{N}^d, J \subset \{1, \ldots, d\}, |\alpha| = \sum_{i=1}^{d} \alpha_i e_i$ the ith unit vector in $\mathbb{R}^d$, either subject to the conditions $|\alpha| = n$ and $J = \emptyset$ or the condition $1 \leq n - |\alpha| \leq |J| - 1$. The CW-complex is multigraded by setting $\operatorname{lcm}(C_{\alpha,J}) := \alpha + \sum_{j \in J} e_j$. We refer to C_d^n as the hypersimplicial complex.

Example 3.6. Consider the ideal $\langle x_1, x_2 \rangle^2 = \langle x_1^2, x_1 x_2, x_2^2 \rangle$. The hypersimplicial complex consists of the following cells:

$$\begin{array}{cc} C_{(2,0),\emptyset} & |\alpha| = 2, J = \emptyset \\ C_{(1,1),\emptyset} & |\alpha| = 2, J = \emptyset \\ C_{(0,2),\emptyset} & |\alpha| = 2, J = \emptyset \\ C_{(1,0),\{1,2\}} & 1 \leq 2 - |\alpha| \leq |J| - 1 \\ C_{(0,1),\{1,2\}} & 1 \leq 2 - |\alpha| \leq |J| - 1 \end{array}$$

It is easy to see that in this case the hypersimplicial complex defines a minimal free resolution of the ideal $\langle x_1, x_2 \rangle^2$.

The unit vectors e_i, $i = 1, \ldots, n$, of R^n induce an orientation on the hypersimplicies by setting $e_1 < \ldots < e_n$. We call this orientation the canonical orientation.

Lemma 3.7 (see [**4**]). *Considering canonical orientations of these hypersimplices and denoting $J = \{j_0 < \ldots < j_r\}$, $J_\nu := J \setminus \{j_\nu\}$ the differential of C_d^n is given by*

$$\begin{array}{rll} \partial C_{\alpha,J} := \sum_{\nu=0}^{r} (-1)^\nu (C_{\alpha,J_\nu} - C_{\alpha+e_{j_\nu},J_\nu}) & \textit{if} & 2 \leq n - |\alpha| \leq |J| - 2, \\ \partial C_{\alpha,J} := \sum_{\nu=0}^{r} (-1)^\nu C_{\alpha,J_\nu} & \textit{if} & 1 = n - |\alpha| \leq |J| - 2, \\ \partial C_{\alpha,J} := \sum_{\nu=0}^{r} (-1)^{\nu+1} C_{\alpha+e_{j_\nu},J_\nu} & \textit{if} & 2 \leq n - |\alpha| = |J| - 1, \\ \partial C_{\alpha,\{j_0,j_1\}} := C_{\alpha+e_{j_1},\emptyset} - C_{\alpha+e_{j_0},\emptyset} & \textit{if} & 1 = n - |\alpha|, \\ \partial C_{\alpha,\emptyset} = 0 & \textit{if} & |\alpha| = n. \end{array}$$

Proposition 3.8 (see [**4**]). *C_d^n defines a multigraded cellular free resolution of $\langle x_1, \ldots, x_d \rangle^n$.* □

Now consider the Borel fixed ideal

$$\mathfrak{a} := \langle x_1, \ldots, x_{d_1} \rangle^{a_1} \cdots \langle x_1, \ldots, x_{d_k} \rangle^{a_k} \trianglelefteq S$$

with $d_1 < \ldots < d_k$ and $a_j \in \mathbb{N} \setminus \{0\}$. We set for all $j = 1, \ldots, d_k$

$$\beta_j := \sum_{\substack{l=1 \\ d_l \geq j}}^{k} a_l.$$

We consider the subcomplex of the hypersimplicial complex $C_{d_k}^{a_1+\ldots+a_k}$ consisting of the hypersimplices $C_{\alpha,J}$ satisfying one of the following conditions:

(1) $|\alpha| = a_1 + \ldots + a_k$, $J = \emptyset$, and $\sum_{j=i}^{d_k} \alpha_j \leq \beta_i$ for all $i = 1, \ldots, d_k$,

(2) $1 \leq a_1 + \ldots + a_k - |\alpha| \leq |J| - 1$ and for $i = 1, \ldots, d_k$ we have

$$\sum_{j=i}^{d_k} \gamma_j \leq \beta_i$$

where $\gamma = \alpha + \sum_{j \in J} e_j$.

We denote this subcomplex with $SC_{d_k}^{\beta_1}$.

Lemma 3.9. *The complex $SC_{d_k}^{\beta_1}$ with the differential inherited from the hypersimplicial complex $C_{d_k}^{\beta_1}$ is a subcomplex of $C_{d_k}^{\beta_1}$.*

Proof. It is easy to see that if $C_{\alpha,J}$ is a hypersimplex satisfying Condition (1) or Condition (2), then any face of $C_{\alpha,J}$ satisfies one of the Conditions (1), (2). □

We want to prove that the subcomplex $SC_{d_k}^{\beta_1}$ defines a cellular resolution for the ideal $\mathfrak{a} := \langle x_1, \ldots, x_{d_1} \rangle^{a_1} \cdots \langle x_1, \ldots, x_{d_k} \rangle^{a_k} \trianglelefteq S$. In order to do so, we define the following intersection of halfspaces

$$H_{\leq \beta} := \left\{ (y_1, \ldots, y_{d_k}) \in \mathbb{R}^{d_k} \;\middle|\; \sum_{j=i}^{d_k} y_j \leq \beta_i, \; i = 1, \ldots, d_k \right\}.$$

Clearly, every hypersimplex $C_{\alpha,J}$ of the subcomplex $SC_{d_k}^{\beta_1}$ lies in $H_{\leq \beta}$. Furthermore, it is easy to see that no other hypersimplex $C_{\alpha,J}$ of the complex $C_{d_k}^{\beta_1}$ lies in $H_{\leq \beta}$. Therefore, we have

$$SC_{d_k}^{\beta_1} = C_{d_k}^{\beta_1} \cap H_{\leq \beta}.$$

We are now in position to prove the following proposition:

Proposition 3.10. *The subcomplex $SC_{d_k}^{\beta_1}$ defines a multigraded cellular free resolution of*

$$\mathfrak{a} = \langle x_1, \ldots, x_{d_1} \rangle^{a_1} \langle x_1, \ldots, x_{d_2} \rangle^{a_2} \cdots \langle x_1, \ldots, x_{d_k} \rangle^{a_k}.$$

Proof. First we calculate the zero-cells. These cells are given by

$$C_{\alpha,\emptyset} = \{\alpha\}$$

satisfying $\sum_{j=i}^{d_k} \alpha_j \leq \beta_i$ for all $i = 1, \ldots, d_k$. Clearly, the vectors α satisfying these properties are in one-to-one correspondence with the minimal generators of $\mathfrak{a}$.

By Proposition 2.2.3 of [**5**], $SC_{d_k}^{\beta_1} = C_{d_k}^{\beta_1} \cap H_{\leq \beta}$ supports a cellular resolution of

$\mathfrak{a}$ if and only if $H_i\Big((C_{d_k}^{\beta_1} \cap H_{\leq\beta})_{\leq\gamma}, k\Big) = 0$ for all $i \geq 1$ and $\gamma \in \mathbb{R}^{d_k}$.
Note that

$$(C_{d_k}^{\beta_1})_{\leq\gamma} = \Delta_n \cap \big\{(y_1, \ldots, y_{d_k}) \in \mathbb{R}^{d_k} \mid y_i \leq \gamma_i,\ i = 1, \ldots, d_k\big\}$$

is contractible or empty for all $\gamma \in \mathbb{Z}^{d_k}$ (see [**4**]). Furthermore, $(C_{d_k}^{\beta_1})_{\leq\gamma}$ is convex. Therefore, the intersection of $(C_{d_k}^{\beta_1})_{\leq\gamma}$ with the convex set $H_{\leq\beta}$ does not change homology. It follows

$$H_i\Big((SC_{d_k}^{\beta_1})_{\leq\gamma}\Big) = 0 \text{ for all } \gamma \in \mathbb{R}^{d_k}.$$

Proposition 2.2.3 of [**5**] implies the assertion. □

Using Formans theory, Batzies defined in [**4**] a Morse-matching for the complex C_d^n such that the resulting Morse complex is a minimal cellular multigraded resolution of $\langle x_1, \ldots, x_d\rangle^n$ (see Proposition 4.3.1 of [**4**]). The acyclic matching $\mathcal{M}$ given in [**4**] is defined by

$$C_{\alpha,J} \to C_{\alpha+e_{\max J}, J\setminus\{\max J\}} \in \mathcal{M} \tag{3.2}$$

for all $\alpha \in \mathbb{N}^{d_\kappa}$, $J \subset [d_\kappa]$, such that $2 \leq n - |\alpha| \leq |J| - 1$ and $\max J \geq \max\alpha := \max\{i \in [d_\kappa] \mid \alpha_i \neq 0\}$.

From the definition of $SC_{d_k}^{\beta_1}$ it follows that if $C_{\alpha,J}$ (resp. $C_{\alpha+e_{\max J}, J\setminus\{\max J\}}$) lies in $SC_{d_k}^{\beta_1}$, then $C_{\alpha+e_{\max J}, J\setminus\{\max J\}}$ (resp. $C_{\alpha,J}$) lies in $SC_{d_k}^{\beta_1}$. Therefore, the matching $\mathcal{M}$ described by (3.2) is a well defined acyclic matching for our subcomplex $SC_{d_k}^{\beta_1}$.

Denote with $\tilde{C}_{d_k}^{\beta_1}$ (resp. $\widetilde{SC}_{d_k}^{\beta_1}$) the resulting Morse complex. With the same proof as for Proposition 3.10 of [**4**] we get the following theorem:

Theorem 3.11. *The complex $\widetilde{SC}_{d_k}^{\beta_1}$ defines a minimal multigraded cellular free resolution of*

$$\mathfrak{a} = \langle x_1, \ldots, x_{d_1}\rangle^{a_1}\langle x_1, \ldots, x_{d_2}\rangle^{a_2} \cdots \langle x_1, \ldots, x_{d_k}\rangle^{a_k}.$$

□

3.2. Cellular Minimal Resolution for a Class of p-Borel Fixed Ideals.

In this section we develop a minimal cellular resolution for a class of p-Borel fixed ideals. This class includes the class of principal Cohen-Macaulay p-Borel fixed ideals. A minimal free resolution for Cohen-Macaulay p-Borel fixed ideals was first given by Aramova and Herzog [**2**]. Batzies proved in [**4**] that the resolution from [**2**] is even a cellular resolution.
In addition, we calculate the regularity of our class of p-Borel fixed ideals and give a formula for the multigraded Poincaré-Betti series $P^S_{S/\mathfrak{a}}(\underline{x}, t)$. The formula for the regularity is in terms of the minimal monomial generating system and hence can be applied to any p-Borel fixed ideal. We conjecture that this formula gives a lower bound for the regularity for general p-Borel fixed ideals. Our formula is a generalization of results obtained in [**2**] and [**20**].

Let $p \in \mathbb{N}$ be a prime. Define the partial order $\leq_p$ on $\mathbb{N}$ by setting $a \leq_p b$ if and only if the p-adic expansions $a = \sum_{i=0}^s a_i\, p^i$, $0 \leq a_i \leq p-1$, and $b = \sum_{i=0}^s b_i\, p^i$, $0 \leq b_i \leq p-1$, fulfill $a_i \leq b_i$ for all $i = 1, \ldots, s$.

Definition 3.12. A monomial ideal $\mathfrak{a} \trianglelefteq S$ is called p-Borel fixed if for all monomials $m \in \mathfrak{a}$ and all variables $x_i \in S$ we have

$$x_i^l \mid m,\ x_i^{l+1} \nmid m,\ j < i, \text{ and } \kappa \leq_p l \ \Rightarrow\ S_{ji}^{\kappa}(m) := \left(\frac{x_j}{x_i}\right)^{\kappa} m \in \mathfrak{a}. \tag{3.3}$$

It is known that this property only needs to be checked for the minimal monomial generating system of $\mathfrak{a}$ (see [**2**]).

Remark 3.13. As for Borel fixed ideals, there exists for each p-Borel fixed ideal a unique set $\mathcal{G} = \{m_1, \ldots, m_r\}$ such that $\mathcal{G}$ generates the ideal $\mathfrak{a}$ minimally, in the sense that any monomial $m \in \mathfrak{a}$ has a decomposition $m = u\ v$ such that v is constructed from a monomial in $\mathcal{G}$ by iterative applications of rule (3.3) to a monomial $m_r \in \mathcal{G}$ (i.e. there exist numbers $i_1, \ldots, i_l$, $j_1, \ldots, j_l$, and $\kappa_1, \ldots, \kappa_l$ with $j_t < i_t$, $t = 1, \ldots, l$ such that $v = S_{j_1 i_1}^{\kappa_1}(\ldots(S_{j_l i_l}^{\kappa_l}(m_r))\ldots))$ and such that each proper subset $\mathcal{G}'$ of $\mathcal{G}$ generates a proper subideal $\mathfrak{a}'$ of $\mathfrak{a}$. Again, we call $\mathcal{G}$ the Borel-generating system of $\mathfrak{a}$.

Definition 3.14. A p-Borel fixed ideal $\mathfrak{a}$ is called principal p-Borel fixed if $\mathcal{G}$ consists of only one monomial, i.e. $\mathcal{G} = \{m\}$.

Lemma 3.15. *If $\mathfrak{a} \trianglelefteq S$ is a principal p-Borel fixed ideal, then there exists a family of principal Borel fixed ideals I_t, $t = 1, \ldots, T$, and numbers $0 \leq r_1 < r_2 < \ldots < r_T$ such that*

$$\mathfrak{a} = \prod_{j=1}^{T} I_j^{[p^{r_j}]}$$

where $I^{[q]}$ is the Frobenius power of an ideal, that is, if I is generated by $m_1, \ldots, m_r$, then $I^{[q]}$ is generated by $m_1^q, \ldots, m_r^q$.

Proof. Assume $\mathcal{G} = \{x_1^{a_1} \cdots x_n^{a_n}\}$. Let $a_i = \sum_{j=0}^{l_i} a_{ij}\ p^j$ be the p-adic expansion of a_i. We define ideals I_j for $j = 1, \ldots, n$:

$$I_j := \prod_{i=0}^{n} \langle x_1, \ldots, x_i \rangle^{a_{ij}},$$

where we set $\langle x_1, \ldots, x_i \rangle^{a_{ij}} := \langle 1 \rangle$ if $a_{ij} = 0$. Then it is straightforward to check (see [**2**]) that

$$\mathfrak{a} = \prod_{j=1}^{\max_{i=1}^{n}(l_i)} I_j^{[p^j]}.$$

Deleting the factors $I_j^{[p^j]} = \langle 1 \rangle^{[p^j]} = \langle 1 \rangle = S$ and relabeling the ideals I_j gives the numbers T and $0 \leq r_1 < \ldots < r_T$. By Lemma 3.4 the ideals I_j are principal Borel fixed. □

In general, the ideal $\mathfrak{a} = \prod_{j=1}^{T} I_j^{[p^{r_j}]}$ with $0 \leq r_1 < \ldots < r_T$ and I_j Borel fixed for $j = 1, \ldots, T$ is clearly p-Borel fixed. But it should be mentioned that not any p-Borel fixed ideal has such a decomposition.
From now on we consider the following class of p-Borel fixed ideals:

$$\mathfrak{a} = \prod_{j=1}^{T} I_j^{[p^{r_j}]} \text{ with } \begin{cases} 0 \leq r_1 < \ldots < r_T \\ \deg(I_j) < p^{r_{j+1} - r_j},\ j = 1, \ldots, T-1 \\ I_j \text{ Borel fixed},\ j = 1, \ldots, T \end{cases}. \tag{3.4}$$

For an ideal $\mathfrak{a} \trianglelefteq S$ with minimal monomial generating system $\mathrm{MinGen}(\mathfrak{a})$ we define the total degree of $\mathfrak{a}$ by

$$\deg(\mathfrak{a}) := \max\big\{ \deg(m) \bigm| m \in \mathrm{MinGen}(\mathfrak{a}) \big\}.$$

We further define for all $i = 1, \ldots, n$

$$\deg_i(\mathfrak{a}) := \max\big\{ l \bigm| x_i^l \text{ divides a generator of } \mathfrak{a} \big\}.$$

Note that if $\mathfrak{a}$ is of type (3.4), we have for all $i = 1, \ldots, T$

$$\sum_{j=1}^{i} \deg(I_j)\; p^{r_j} \le \sum_{j=1}^{i} (p^{r_{j+1}-r_j} - 1) p^{r_j} = p^{r_{i+1}} - p^{r_1} < p^{r_{i+1}}.$$

Aramova and Herzog find in [**2**] explicit minimal free resolutions of principal p-Borel fixed ideals which are Cohen-Macaulay, i.e. $\mathfrak{a}$ is generated by the power of a single variable $\mathcal{G} = \{x_i^d\}$. Batzies proved in [**4**] that the resolution given in [**2**] is even cellular.

We develop a minimal cellular resolution for p-Borel fixed ideals $\mathfrak{a}$ of type (3.4) such that the factors I_j are principal Borel fixed for all $j = 1, \ldots, T$. Clearly, this includes the case considered in [**2**] and [**4**]. In addition, we calculate regularity and the Poincaré-Betti series $P^S_{S/\mathfrak{a}}(\underline{x}, t)$ for these ideals.

We proceed by defining an acyclic matching on the Taylor resolution of $\mathfrak{a}$ such that the Morse complex is minimal. Since the Taylor resolution is cellular, it follows by [**5**] that the Morse complex is cellular, too. In order to do so, we have to introduce the following notation:

Notation:

(1) $I_{<t} := \prod_{i=1}^{t-1} I_i, \qquad I_{>t} := \prod_{i=t+1}^{T} I_i,$

$I_{<t}^{[p]} := \prod_{i=1}^{t-1} I_i^{[p^{r_i}]}, \quad I_{>t}^{[p]} := \prod_{i=t+1}^{T} I_i^{[p^{r_i}]}.$

(2) $M(u) := \max\big\{ i \bigm| x_i \text{ divides } u \big\}.$

(3) For a monomial $u \in S$ we define $m_i(u) := \min\{ l > i \mid x_l \text{ divides } u\}.$

(4) For all $j = 1, \ldots, T$ let

$$\mathrm{supp}(I_j) := \big\{ i \bigm| x_i \text{ divides } u \text{ for some } u \in I_j \big\}.$$

(5) For a set $W := \{j_1 < \ldots < j_\kappa\} \subset \{1, \ldots, n\}$ we define the monomials $x_{t,\max}(W), x_{t,\min}(W) \in k[x_{j_1}, \ldots, x_{j_\kappa}] =: k[W]$ such that $x_{t,\max}(W)$ and $x_{t,\min}(W)$ divide a generator of $I_{<t}^{[p]}$ and for all monomials $m \in k[W]$ dividing a generator of $I_{<t}^{[p]}$ we have either
   - $\deg(m) < \deg(x_{t,\max}(W))$ (resp. $\deg(m) < \deg(x_{t,\min}(W))$) or
   - $\deg(m) = \deg(x_{t,\max}(W))$ (resp. $\deg(m) = \deg(x_{t,\min}(W))$) and $m \preceq x_{t,\max}(W)$ (resp. $m \succeq x_{t,\min}(W)$),

   where $\prec$ is the reverse lexicographic order.

(6) For $j \in \{1, \ldots, n\}$ we define the p-Borel fixed ideal

$$I_{<t}^{[p]}(j) := \langle u \in k[x_1, \ldots, x_{j-1}] \text{ with } u\, x_{t,\max}(\{j, \ldots, n\}) \in I_{<t}^{[p]} \rangle.$$

(7) We introduce the p-reverse lexicographic order:

$$m = \prod_{i=1}^{T} u_i^{p^{r_i}} \prec_t n = \prod_{i=1}^{T} v_i^{p^{r_i}}$$

if and only if there exists an index j such that $u_i = v_i$ for all $i > j$ and $u_j \prec v_j$.

Remark 3.16. If $\mathfrak{a}$ is Cohen-Macaulay, i.e. $\mathfrak{a} = \prod_{i=1}^{T} \left(\langle x_1, \ldots, x_d \rangle^{a_i}\right)^{[p^{r_i}]}$, then we have always

$$\begin{aligned} x_{t,\max}(\{j_1 < \ldots < j_\kappa\}) &= x_{j_1}^b, \\ x_{t,\min}(\{j_1 < \ldots < j_\kappa\}) &= x_{j_l}^b \end{aligned}$$

where $b = \sum_{i=1}^{t-1} a_i \, p^{r_i}$ and $l = \max\left\{j \in \{j_1, \ldots, j_\kappa\} \mid j \leq d\right\}$.
If $\mathfrak{a}$ is of type (3.4) such that the ideals I_j are principal Borel fixed we have

$$x_{t,\max}(\{j_1 < \ldots < j_\kappa\}) = x_{j_1}^b \text{ with } b = \sum_{\substack{i=1 \\ j_1 \in \mathrm{supp}(I_i)}}^{t-1} \deg_i(I_i) p^{r_i}.$$

From now on we assume that $\mathfrak{a}$ is of type (3.4) such that in addition the ideals I_j are principal Borel fixed. This implies that each generator in I_j has the same degree. We have the following decomposition:

Lemma 3.17. *Let $v \in I_{<t}^{[p]}$ be a monomial generator and $W := \{j_1 < \ldots < j_\kappa\} \subset \{1, \ldots, n\}$. Then there exists a monomial v' dividing v such that*

(1) *$M(v') < j_1$ and*

(2) *$v' x_{t,\max}(W)$ and $v' x_{t,\min}(W)$ are generators of $I_{<t}^{[p]}$,*

For a monomial $v \in I_{<t}^{[p]}$ we let $\tilde{v}$ be the maximal monomial with respect to the order $\prec_t$ satisfying the properties (1) and (2).

Proof. By definition the monomial v decomposes uniquely into a product $v = v_1 \cdots v_{t-1}$ such that $v_j \in I_j^{[p^{r_j}]}$. For all ideals I_j such that $\mathrm{supp}(I_j) \cap W = \emptyset$, the rule (3.3) implies $M(v_j) < j_1$. In the other case, we construct new monomials v_i' such that $M(v_i') < j_1$.
Assume $\mathrm{supp}(I_j) \cap W \neq \emptyset$. By definition I_j has the form

$$I_j = \langle x_1, \ldots, x_{l_1} \rangle^{a_1} \cdots \langle x_1, \ldots, x_{l_\kappa} \rangle^{a_\kappa}.$$

Therefore, v_j admits a decomposition $v_j = v_{j1} \cdots v_{j\kappa}$ with $v_{js} \in \langle x_1, \ldots, x_{l_s} \rangle^{a_s}$. If $l_s < j_1$, we set $v_{js}' := v_{js}$, else we set $v_{js}' := 1$. We then define $v_j' := v_{j1}' \cdots v_{js}'$ and $v' := v_1' \cdots v_{t-1}'$. By construction we have $v' \mid v$ and $M(v') < j_1$. Finally, the definition of $x_{t,\max}$ and $x_{t,\min}$ implies that $v' x_{t,\min}$ and $v' x_{t,\max}$ are generators of the ideal $I_{<t}^{[p]}$. □

3.2.1. *Construction of the acyclic matching.*
We construct the matching on the Taylor resolution of $\mathfrak{a}$.

The construction of the matching is technical and proceed in six steps. In order to allow the reader to keep tract of the critical cells we provide a description of the remaining critical cells after each construction step.

Let $B := \left\{ v_0 \, u_0^{p^{rt}} \, u, \ldots, v_\kappa \, u_\kappa^{p^{rt}} \, u \right\}$ be a basis element of the Taylor resolution such that $v_i \in I_{<t}^{[p]}$, $u_i \in I_t$ for $i = 1, \ldots, \kappa$, and $u \in I_{>t}^{[p]}$. We assume that the elements $v_i \, u_i^{p^{rt}} \, u$ are in increasing order with respect to the order $\prec_t$.

Now we define a monomial $w \in \mathfrak{a}$ and set

$$B \cup \{w\} \;\rightarrow\; B \setminus \{w\} \in \mathcal{M}.$$

Step 1: Let $0 < R \leq \kappa$ be the smallest index such that there exists an index $j < M(u_R)$ with either $\alpha_j > \beta_j + 1$ or $\beta_j > \alpha_j + 1$, where $u_0 = x^\alpha$ and $u_R = x^\beta$. In order to fulfill the matching property, we take the smallest index j with the desired properties.

(1) If $\alpha_j > \beta_j + 1$, we set $w := v_R \left(\frac{x_j \, u_R}{x_{M(u_R)}} \right)^{p^{rt}} \, u$. Then w divides $\operatorname{lcm}(B)$ since the degree of x_j in w is the degree of x_j in v_R plus $p^{rt} + \beta_j \, p^{rt}$. Since $\deg(v_R) < p^{rt}$, we finally get that the degree of x_j in w is smaller than $p^{rt} + p^{rt} + \beta_j \, p^{rt} \leq \alpha_j p^{rt}$.

(2) If $\beta_j > \alpha_j + 1$, we set $w := v_0 \left(\frac{x_j \, u_0}{x_{M(u_0)}} \right)^{p^{rt}} \, u$ and with the same argumentation we get $w \mid \operatorname{lcm}(B)$.

The critical cells in homological degree $\kappa + 1 \geq 1$ are given by

$$\mathcal{B}_{\kappa+1} := \left\{ v_0 \, u_0^{p^{rt}} \, u, \ldots, v_\kappa \, u_\kappa^{p^{rt}} \, u \right\}$$

such that either $u_i = u_0$ or $u_i = u_0 \frac{x_{i_1} \cdots x_{i_r}}{x_{j_1} \cdots x_{j_r}}$ for some $i_1 < i_2 < \ldots < i_r$ and $j_1, \ldots, j_r$. Note that the indices $j_1, \ldots, j_r$ are not necessarily pairwise different, consider for example $p = 11$, $u_0 = x_1^2 x_2^2 x_3^2 x_4^2 x_5$ and $u_1 = x_1^3 x_2^3 x_3^3$. But the increasing order implies $i_r < j_r \leq M(u_0)$.

Step 2: Let $B := \left\{ v_0 \, u_0^{p^{rt}} \, u, \ldots, v_\kappa \, u_\kappa^{p^{rt}} \, u \right\}$ be a critical cell. We define $\alpha := \deg\Big(\operatorname{lcm} \left\{ u_0, \ldots, u_\kappa \right\} \Big)$. Let i_0 be the smallest index such that the exponent of x_{i_0} in u_0 is smaller than α_{i_0}. Let j be the smallest index such that $u_j = u_0$ and $v_j \neq \tilde{v}_j \; x_{t,\max}(\{i_0\})$. We set

$$w := \tilde{v}_j \; x_{t,\max}(\{i_0\}) \; u_j^{p^{rt}} \; u.$$

Since $M(\tilde{v}_j) < i_0$, $\tilde{v}_j \mid v_j$, and $x_{t,\max}(\{i_0\}) = x_{i_0}^b$ with $b < p^{rt}$, we have $w \mid \operatorname{lcm}(B)$.

The critical cells in homological degree $l_0 + \kappa \geq 1$ are then given by

$$\mathcal{B}_{l_0+\kappa} := \left\{ \begin{array}{c} v_{01} \, x_{t,\max}(\{i_0\}) \, u_0^{p^t} \, u, \ldots, v_{0l_0} \, x_{t,\max}(\{i_0\}) \, u_0^{p^t} \, u, \\ v_1 \, u_1^{p^{rt}} \, u, \ldots, v_\kappa \, u_\kappa^{p^{rt}} \, u \end{array} \right\}$$

with $v_{01}, \ldots, v_{0l_0} \in I_{<t}^{[p]}(i_0)$, i_0 is the smallest index such that the exponent of x_{i_0} in u_0 is smaller than α_{i_0}, where $\alpha := \deg\Big(\operatorname{lcm} \left\{ u_0, \ldots, u_\kappa \right\} \Big)$. For all $j = 1, \ldots, \kappa$ we have $u_j = u_0 \, \frac{x_{\nu_1} \cdots x_{\nu_r}}{x_{\mu_1} \cdots x_{\mu_r}}$, for some $\nu_1 < \ldots < \nu_r$ and $\mu_1, \ldots, \mu_r \leq M(u_0)$ with $\nu_r < \mu_r$.

Step 3: Let $1 \leq R \leq \kappa$ be the smallest index such that $u_R = u_0 \frac{x_{i_1} \cdots x_{i_r}}{x_{j_1} \cdots x_{j_r}}$ for some $i_1 < i_2 < \ldots < i_r$, $j_1, \ldots, j_r$, and $r > 2$. Since $r > 2$, $i_r < j_r \leq M(u_0)$, and $i_0 \leq i_1$, there exists an index l such that $i_0 < i_l < j_r \leq M(u_0)$. Therefore, we can define $w := v_{01}\ x_{t,\max}(\{i_0\})\ \left(\frac{x_{i_l} u_0}{x_{M(u_0)}}\right)^{p^r t}\ u$. By construction we have $w \mid \operatorname{lcm}(B)$.

The critical cells in homological degree $l_0 + \kappa \geq 1$ are now given by

$$\mathcal{B}_{l_0+\kappa} := \left\{ \begin{array}{c} v_{01}\, x_{t,\max}(\{i_0\})\ u_0^{p^r t}\ u, \ldots, v_{0l_0}\, x_{t,\max}(\{i_0\})\ u_0^{p^r t}\ u, \\ v_1\ u_1^{p^r t}\ u, \ldots, v_\kappa\ u_\kappa^{p^r t}\ u \end{array} \right\}$$

with $v_{01}, \ldots, v_{0l_0} \in I_{<t}^{[p]}(i_0)$ such that for all $j = 1, \ldots, \kappa$ we have $u_j = u_0 \frac{x_{\nu_j}}{x_{\mu_j}}$ for some $\nu_j < \mu_j \leq M(u_0)$ and we have $i_0 = \min\left\{\nu_j \mid u_j = u_0 \frac{x_{\nu_j}}{x_{\mu_j}}, j = 1, \ldots, \kappa\right\}$.

Step 4: Let B be a critical cell from Step 3 such that $u_j \in \left\{\frac{x_{\nu_j} u_0}{x_{\mu_j}} \mid \nu_j < \mu_j\right\}$ and $i_0 := \min\left\{\nu_j \mid 1 \leq j \leq \kappa\right\}$. We set $R := \min\left\{\nu_j > i_0 \mid 1 \leq j \leq \kappa\right\}$, where we set $R := 0$ and $x_R := 1$ if all ν_j equal.
Case $R > 0$.

(1) Let $u_j = \frac{x_{\nu_j} u_0}{x_{\mu_j}}$ such that $\nu_j = i_0$ and $v_j \neq \tilde{v}_j x_{t,\min}(\{R, \mu_j\})$. We define the monomial by $w := \tilde{v}_j\, x_{t,\min}(\{R, \mu_j\})\ u_j^{p^r t}\ u$. Since $\tilde{v}_j \mid v_j$, $M(\tilde{v}_j) < R$, $\deg(x_{t,\min}(\{R, \mu_j\}) < p^{rt}$, and $u_j = u_0 \frac{x_{\nu_j} u_0}{x_{\mu_j}}$, we have clearly $w \mid \operatorname{lcm}(B)$.

(2) If $u_j = \frac{x_{\nu_j} u_0}{x_{\mu_j}}$ with $\nu_j > i_0$ and $v_j \neq \tilde{v}_j\ x_{t,\min}(\{i_0, \mu_j\})$, we define $w := \tilde{v}_j\ x_{t,\min}(\{i_0, \mu_j\})\ u_j^{p^r t}\ u$, with $M(\tilde{v}_j) < i_0$. Since $\tilde{v}_j \mid v_j$, $M(\tilde{v}_j) < i_0$, $i_0 < \nu_j$, and $u_j = u_0 \frac{x_{\nu_j} u_0}{x_{\mu_j}}$, we have clearly $w \mid \operatorname{lcm}(B)$.

(3) If neither (1) nor (2) is satisfied, we know that $v_j = \tilde{v}_j\ x_{t,\min}(\{i_0, \mu_j\})$. If there exists an index j such that $\mu_j < M(u_0)$, we define

$$w := \tilde{v}_j\ x_{t,\min}(\{i_0, M(u_0)\})\ \left(\frac{x_{\nu_j} u_0}{x_{M(u_0)}}\right)^{p^r t}\ u.$$

We always take the smallest index j with the desired properties. Clearly, we have $w \mid \operatorname{lcm}(B)$.

Case $R = 0$.

(1) Let $u_j = \frac{x_{i_0} u_0}{x_{\mu_j}}$ such that $v_j \neq \tilde{v}_j\ x_{t,\min}(\{\mu_j\})$. We define the monomial w by $w := \tilde{v}_j\, x_{t,\min}(\{\mu_j\})\ u_j^{p^r t}\ u$. Since $\tilde{v}_j \mid v_j$, $M(\tilde{v}_j) < \mu_j$, $\deg(x_{t,\min}(\{\mu_j\}) < p^{rt}$, and $u_j = u_0 \frac{x_{\nu_j} u_0}{x_{\mu_j}}$, we have clearly $w \mid \operatorname{lcm}(B)$.

(2) If (1) is not satisfied, we have $v_j = \tilde{v}_j\ x_{t,\min}(\{\mu_j\})$ for all j. By definition of $m_{i_0}(u_0)$ we have $\mu_j \geq m_{i_0}(u_0)$ for all μ_j. If there exists an index j such that $\mu_j > m_{i_0}(u_0)$, we define

$$w := \tilde{\tilde{v}}_j\, x_{t,\min}(\{m_{i_0}(u_0)\})\ \left(\frac{x_{i_0} u_0}{x_{m_{i_0}(u_0)}}\right)^{p^r t}\ u$$

with $\tilde{\tilde{v}}_j \mid \tilde{v}_j$ and $M(\tilde{\tilde{v}}_j) < m_{i_0}(u_0)$.
Again, it is easy to see that $w \mid \mathrm{lcm}(B)$.

The critical cells in homological degree $l_0 + l_1 \geq 1$ (resp. $l_0 + \ldots + l_\kappa \geq 1$) are now given by
Type I:

$$\mathcal{B}_{l_0+l_1} = \left\{ \begin{array}{cc} v_{0\kappa}\, x_{t,\max}(\{i_1\}) u_0^{p^{rt}}\ u, & \kappa = 1, \ldots, l_0, \\ v_{1\kappa}\, x_{t,\min}(\{m_{i_1}(u_0)\}) \left(\frac{x_{i_1}\, u_0}{x_{m_{i_1}(u_0)}}\right)^{p^{rt}} u, & \kappa = 1, \ldots, l_1 \end{array} \right\}$$

with $v_{01}, \ldots, v_{0l_0} \in I^{[p]}_{<t}(i_1)$ and $v_{11}, \ldots, v_{1l_1} \in I^{[p]}_{<t}(m_{i_1}(u_0))$, $1 \leq i_1 < M(u_0)$.
Type II:

$$\mathcal{B}_{l_0+\ldots+l_\kappa} = \left\{ \begin{array}{cc} v_{0j}\, x_{t,\max}(\{i_1\})\, u_0^{p^{rt}}\ u, & j = 1, \ldots, l_0, \\ v_{1j}\, x_{t,\min}(\{i_2, M(u_0)\}) \left(\frac{x_{i_1}\, u_0}{x_{M(u_0)}}\right)^{p^{rt}} u, & j = 1, \ldots, l_1, \\ v_{2j}\, x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_2}\, u_0}{x_{M(u_0)}}\right)^{p^{rt}} u, & j = 1, \ldots, l_2, \\ \vdots & \\ v_{\kappa j}\, x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_\kappa}\, u_0}{x_{M(u_0)}}\right)^{p^{rt}} u, & j = 1, \ldots, l_\kappa \end{array} \right\}$$

with $1 \leq i_1 < \ldots < i_\kappa < M(u_0)$, $v_{11}, \ldots, v_{1l_1} \in I^{[p]}_{<t}(i_2)$, and $v_{j1}, \ldots, v_{jl_j} \in I^{[p]}_{<t}(i_1)$ for all $j = 0, 2, \ldots, \kappa$.
For a fixed index i we assume that the monomials $v_{i1}, \ldots, v_{il_i}$ are in increasing order with respect to $\prec_t$.

Step 5: Consider a critical cell B of Step 4. For the monomial v_{11} we construct the monomial $\tilde{v}_{11}\ x_{t,\max}(\{i_1\})$ with $M(v'_{11}) < i_1$. If either $l_0 > 1$ or $l_0 = 1$ and $v_{01} \neq \tilde{v}_{11}$, we define $w := \tilde{v}_{11}\ x_{t,\max}(\{i_1\}) u_0^{p^{rt}}\ u$. By construction we have $w \mid \mathrm{lcm}(B)$.

The critical cells in homological degree $l_1 + 1 \geq 1$ (resp. $1 + l_1 + \ldots + l_\kappa \geq 1$) are now given by
Type I:

$$\mathcal{B}_{1+l_1} = \left\{ \begin{array}{cc} v\ x_{t,\max}(\{i_1\}) u_0^{p^{rt}}\ u, & \\ v_{1j}\, x_{t,\min}(\{m_{i_1}(u_0)\}) \left(\frac{x_{i_1}\, u_0}{x_{m_{i_1}(u_0)}}\right)^{p^{rt}} u, & j = 1, \ldots, l_1 \end{array} \right\},$$

$1 \leq i_1 < M(u_0)$ and $v_{11}, \ldots, v_{1l_1} \in I^{[p]}_{<t}(m_{i_1}(u_0))$, and
Type II:

$$\mathcal{B}_{1+l_1+\ldots+l_\kappa} = \left\{ \begin{array}{cc} v\ x_{t,\max}(\{i_1\})\, u_0^{p^{rt}}\ u, & \\ v_{1j}\, x_{t,\min}(\{i_2, M(u_0)\}) \left(\frac{x_{i_1}\, u_0}{x_{M(u_0)}}\right)^{p^{rt}} u, & j = 1, \ldots, l_1, \\ v_{2j}\, x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_2}\, u_0}{x_{M(u_0)}}\right)^{p^{rt}} u, & j = 1, \ldots, l_2, \\ \vdots & \\ v_{\kappa j}\, x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_\kappa}\, u_0}{x_{M(u_0)}}\right)^{p^{rt}} u, & j = 1, \ldots, l_\kappa \end{array} \right\}$$

with $1 \leq i_1 < \ldots < i_\kappa < M(u_0)$, $v_{11}, \ldots, v_{1l_1} \in I^{[p]}_{<t}(i_2)$, and $v_{j1}, \ldots, v_{jl_j} \in I^{[p]}_{<t}(i_1)$ for all $j = 2, \ldots, \kappa$.
In both cases $v = \min\left\{v' \mid v_{11} \;\middle|\; v' x_{t,\min}(i_1) \text{ is a generator of } I^{[p]}_{<t}\right\}$.
For a fixed index i we assume that the monomials $v_{i1}, \ldots, v_{il_i}$ are in increasing order with respect to $\prec_t$.

Step 6: Consider a critical cell B of Step 5. Let $j_1 \in \{2, \ldots, \kappa\}$ be the smallest index such that either $l_{j_1} > 1$ or $l_{j_1} = 1$ and $v_{j_1 1} \neq v$.
We define $w := v\, x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_{j_1}} u_0}{x_{M(u_0)}}\right)^{p^r t} u$. Clearly, $w \mid \operatorname{lcm}(B)$.

The critical cells in homological degree $l_1 + 1 \geq 1$ (resp. $\kappa + l_1 \geq 1$) are now given by
Type I:

$$\mathcal{B}_{1+l_1} = \left\{ \begin{array}{c} v\, x_{t,\max}(\{i_1\}) u_0^{p^r t}\, u, \\ v_j\, x_{t,\min}(\{m_{i_1}(u_0)\}) \left(\frac{x_{i_1} u_0}{x_{m_{i_1}(u_0)}}\right)^{p^r t} u, \quad j = 1, \ldots, l_1 \end{array} \right\}$$

with $1 \leq i_1 < M(u_0)$ and $v_1, \ldots, v_{l_1} \in I^{[p]}_{<t}(m_{i_1}(u_0))$ and
Type II:

$$\mathcal{B}_{\kappa+l_1} = \left\{ \begin{array}{c} v\, x_{t,\max}(\{i_1\})\, u_0^{p^r t}\, u, \\ v_j\, x_{t,\min}(\{i_2, M(u_0)\}) \left(\frac{x_{i_1} u_0}{x_{M(u_0)}}\right)^{p^r t} u, \quad j = 1, \ldots, l_1, \\ v\, x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_2} u_0}{x_{M(u_0)}}\right)^{p^r t} u, \\ \vdots \\ v\, x_{t,\min}(\{i_1, M(u_0)\}) \left(\frac{x_{i_\kappa} u_0}{x_{M(u_0)}}\right)^{p^r t} u \end{array} \right\}$$

with $1 \leq i_1 < \ldots < i_\kappa < M(u_0)$, $\kappa \geq 2$, and $v_1, \ldots, v_{l_1} \in I^{[p]}_{<t}(i_2)$. In both cases $v = \min\left\{v' \mid v_1 \;\middle|\; v' x_{t,\min}(\{i_1\}) \text{ is a generator of } I^{[p]}_{<t}\right\}$.
The monomials $v_1, \ldots, v_{l_1}$ are in increasing order with respect to $\prec_t$.

Proposition 3.18. *The matching $\mathcal{M}$ defined by the six steps above is a well defined acyclic matching.*

Proof. The matching property as well as the invertibility are satisfied by construction. Now we prove acyclicity. We start with a set $\mathcal{B} = \left\{v_0 u_0^{p^r t} u, \ldots, v_\kappa u_\kappa^{p^r t} u\right\}$ in homological degree $\kappa + 1$. We assume that the elements $v_i u_i^{p^r r} u$ are in increasing order with respect to the order $\prec_t$.
Assume, that we have a directed cycle with elements in homological degree $\kappa+1$ and $\kappa + 2$ starting with the element $\mathcal{B}$. In the first step we add a monomial w_1 constructed in Step 1,..., Step 6. In the second step we have to remove a monomial $v_i u_i^{p^r r} u$ from $\mathcal{B}$. In order to have a cycle we have to add this element again. Since $v_0 u_0^{p^r t} u$ is the smallest monomial of $\mathcal{B}$, the construction of the monomials w in Step 1,..., Step 6 implies that it can never be added in the cycle. Hence all sets occurring in the cycle have smallest element $v_0 u_0^{p^r t} u$.
The construction of each monomial w in Step 1,..., Step 6 depends either on

the monomial u_0 or on the monomials u_0 and u_j for some j. Now we look at the cycle:

$$\mathcal{B} \to \mathcal{B} \cup \{w_1\} \to \mathcal{B} \cup \{w_1\} \setminus \{m_1\} \to \cdots .$$

Assume that w_1 is constructed with respect to u_0 and u_i for some i. If $m_1 \neq v_i u_i^{p^{r_r}} u$ then $\mathcal{B} \cup \{w_1\} \setminus \{m_1\}$ is matched with $\mathcal{B} \setminus \{m_1\}$. This contradicts the existence of the cycle. This reasoning shows that the existence of a cycle implies the existence of a sequence of added monomials $v_1, \ldots, v_s$ such that v_i is constructed with respect to u_0 and v_{i+1} for all $i = 1, \ldots, s-1$ and v_s is constructed with respect to u_0 and v_1.

We prove that those sequences cannot exist: Let $v_1, \ldots, v_s$ be a sequence of added monomials such that v_i is constructed with respect to v_{i+1}. The following table lists the possible situations for these sequences, except Step 1, (1):

v_3	v_2	v_1
Step 5		
Step 6		
Step 4, $R = 0$, (ii)		
Step 4, $R = 0$, (i)	Step 4 $R = 0$, (ii)	
Step 4, $R > 0$, (i),(ii)	Step 4, $R = 0$, (ii)	
Step 4, $R > 0$, (iii)	Step 6	
Step 3	Step 4, $R > 0$, (iii)	Step 6
Step 2	Step 5	
Step 1 (2)	Step 4, $R > 0$, (iii)	Step 6
Step 1 (2)	Step 4, $R > 0$, (i),(ii)	Step 4, $R = 0$, (ii)

In none of these cases the sequence can consist of more than three monomials. If the sequence has exactly three monomials, the third monomial differs from the first monomial. The only step which can be applied more than once is Step 1 (1). But in this case the monomials are in increasing order with respect to $\prec_t$. Therefore, v_s can never be constructed with respect to v_1. This makes such a cycle impossible and acyclicity follows. □

3.2.2. *Cellular resolutions and Poincaré-Betti series for p-Borel fixed ideals.* In general, the Morse complex with respect to the acyclic matching from Proposition 3.18 is not minimal since the critical cells depend on the monomials $v_1, \ldots, v_{l_1} \in I^{[p]}_{<t}(i_1)$. But since the matching is constructed on the Taylor resolution, the Morse complex defines a cellular multigraded resolution of the ideal $\mathfrak{a}$. In the following case, which includes the principal Cohen-Macaulay p-Borel fixed ideals, the Morse complex is even minimal:

Theorem 3.19. *Let* $\mathfrak{a} := \prod_{j=1}^{T} I_j^{[p^{r_j}]}$ *be a p-Borel fixed ideal such that* $I_j = \langle x_1, \ldots, x_{l_j} \rangle^{a_j}$ *with* $a_j < p^{r_{j+1} - r_j}$ *and* $l_1 \geq l_2 \ldots \geq l_T$.
Then the minimal free resolution of $\mathfrak{a}$ *has the following basis in the ith* ($i \geq 2$) *homological degree:*

$$S_i := \Big\{ (u_t, \ldots, u_T, \{j_1 < \ldots < j_{i-1}\}) \;\Big|\; t = 1, \ldots, T, u_j \in I_j, j_{i-1} < M(u_t) \Big\}.$$

Moreover the minimal resolution is cellular.

Note that if $l_1 = \ldots = l_T$, the ideal $\mathfrak{a}$ is principal Cohen-Macaulay p-Borel fixed and the sets S_i are exactly the basis of the minimal resolution given by Aramova and Herzog in [**2**].

Proof. Consider a critical cell B from the matching of Proposition 3.18. In this case we have

$$\begin{aligned} x_{t,\min}(\{i_1, M(u_0)\}) &= x_{t,\min}(\{i_2, M(u_0)\}) = x^b_{M(u_0)}, \\ x_{t,\min}(\{m_{i_1}(u_0)\}) &= x^b_{m_{i_1}(u_0)}, \\ x_{t,\max}(\{i_1\}) &= x^b_{i_1} \end{aligned}$$

with $b = \sum_{j=1}^{t-1} a_j\, p^{r_j}$. Therefore, it follows $v = v_1 = \ldots = v_{l_1} = 1$. This implies that the Morse complex has basis

Type I:

$$\mathcal{B}_2 = \left\{ x^b_{i_1}\ u_0^{p^{r_t}}\ u, x^b_{m_{i_1}(u_0)} \left(\frac{x_{i_1}\, u_0}{x_{m_{i_1}(u_0)}}\right)^{p^{r_t}} u \right\}$$

and

Type II:

$$\mathcal{B}_{\kappa+1} = \left\{ x^b_{i_1}\ u_0^{p^{r_t}}\ u, x^b_{M(u_0)} \left(\frac{x_{i_1}\, u_0}{x_{M(u_0)}}\right)^{p^{r_t}} u, \ldots, x^b_{M(u_0)} \left(\frac{x_{i_\kappa}\, u_0}{x_{M(u_0)}}\right)^{p^{r_t}} u \right\}$$

with $i_1 < \ldots < i_\kappa < M(u_0)$. A multidegree argument then implies that the Morse complex is minimal.

The basis elements depend only on $u_0 \in I_t^{[p]}$, $u \in I_{>t}^{[p]}$, and $i_1 < \ldots < i_\kappa < M(u_0)$. Since u decomposes uniquely into a product $u = u_{t+1}^{p^{r_{t+1}}} \cdots u_T^{p^{r_T}}$ with $u_j \in I_j$, it follows that they are in bijection to the tuples $(u_t, \ldots, u_T, \{i_1 < \ldots < i_\kappa\})$ with the desired properties. This proves the assertion. □

For general principle Borel fixed ideals I_j the Morse complex from 3.18 is not minimal, but we can still calculate the multigraded Poincaré-Betti series

$$P_{S/\mathfrak{a}}(\underline{x}, \tau) := \sum_{i,\alpha} \dim_\kappa \left(Tor_i^S(S/\mathfrak{a}, \kappa)_\alpha\right)\, \tau^i\, \underline{x}^\alpha.$$

For this we define

$$\overline{P}_{S/\mathfrak{a}}(\underline{x}, \tau) := \begin{cases} P_{S/\mathfrak{a}}(\underline{x}, \tau) - 1 & , \quad \mathfrak{a} \neq \langle\emptyset\rangle \\ 1 & , \quad \mathfrak{a} = \langle\emptyset\rangle. \end{cases}$$

We have the following formula to calculate the Poincaré-Betti series:

Theorem 3.20. *Let $\mathfrak{a} = \prod_{j=1}^T I_j^{[p^{r_j}]}$ be a p-Borel fixed ideal of type (3.4) such that the ideals I_j are principal Borel fixed. Then the multigraded Poincaré-Betti series is given by*

$$P_{S/\mathfrak{a}}(\underline{x}, \tau) = 1 + \sum_{u \in \mathrm{MinGen}(\mathfrak{a})} u\ \tau + \sum_{t=1}^{T} \sum_{u \in G\left(I_{>t}^{[p]}\right)} \sum_{u_t \in G(I_t)} u\ F(u_t)$$

with

$$F(u_t) \quad = \quad \sum_{1 \leq i < M(u_t)} (x_i\, u_t)^{p^t}\ \tau\ \overline{P}_{S/I_{<t}^{[p]}(m_i(u_t))}(\underline{x}, \tau)$$

$$+ \sum_{\substack{1\leq i_1<\ldots<i_\kappa<M(u_t)\\ \kappa\geq 2}} (x_{i_1}\cdots x_{i_\kappa}\, u_t)^{p^t}\ \ \tau^\kappa\ \overline{P}_{S/I^{[p]}_{<t}(i_2)}(\underline{x},\tau).$$

In particular, the minimal resolution is independent of the characteristic of k.

Proof. For the proof, we apply the matching 3.18 inductively. For that we have to argue that after a single application of the matching we essentially end up with a situation that allows to apply the matching again. In general, the Morse complex is not minimal after a single application, since the basis elements depend on the monomials $v_1,\ldots,v_{l_1}\in I^{[p]}_{<t}(i_0)$. It is easy to see that $I^{[p]}_{<t}(i_0)$ is again a p-Borel fixed ideal. Thus, we can apply the same acyclic matching on the Taylor resolution of $I^{[p]}_{<t}(i_0)$ and consider in the original complex only monomials $v_1,\ldots,v_{l_1}$ such that $\{v_1,\ldots,v_{l_1}\}$ is a critical cell with respect to the acyclic matching on $I^{[p]}_{<t}(i_0)$. Since these critical cells depend again on some monomials in $I^{[p]}_{<t'}(i'_0)$, we can go on by induction. The induction is finite since $\operatorname{supp}\left(I^{[p]}_{<t'}(i'_0)\right)\subsetneq \operatorname{supp}\left(I^{[p]}_{<t}(i_0)\right)$. Therefore, by applying the acyclic matching inductively, we reach a minimal cellular multigraded resolution of $\mathfrak{a}$. Because of the induction we are not able to give an explicit form of the resolution, but it proves that the resolution does not depend on the characteristic of k. Therefore, we can calculate the Poincaré-Betti series by the desired way. □

The preceding proof admits the following corollary to Theorem 3.20:

Corollary 3.21. *The ideal $\mathfrak{a}=\prod_{j=1}^{T} I_j^{[p^{r_j}]}$ of Theorem 3.20 admits a minimal multigraded cellular resolution.* □

We explain the formula in two examples:

Example 3.22. Let $p=2$ and $\mathfrak{a}:=\langle x_1^2,x_2^2\rangle\langle x_1^4,x_2^4,x_3^4\rangle$. Using the computer algebra system CoCoA [**16**], we calculate the following minimal resolution:

$$\longrightarrow S^3(-12)\longrightarrow S^4(-8)\oplus S^4(-10)\longrightarrow S^6(-6).$$

Our formula gives the following Poincaré-Betti series:

$$\begin{aligned} P_{S/\mathfrak{a}}(\underline{x},\tau) &= 1+\sum_{u\in G(I)} u\ \tau + x_1^2\,x_2^2\,x_1^4\,\tau^2 + x_1^2\,x_2^2\,x_2^4\,\tau^2 + x_1^2\,x_2^2\,x_3^4\,\tau^2 + x_1^4\,x_2^4\,\tau^2\\ &\quad + x_1^4\,x_2^4\,x_3^4\,\tau^3 + \ x_1^4x_3^4\ \tau\ \overline{P}_{S/\langle x_1^2,x_2^2\rangle}(\underline{x},\tau) + \ x_2^4x_3^4\ \tau\ \overline{P}_{S/\langle x_1^2,x_2^2\rangle}(\underline{x},\tau) \end{aligned}$$

Thus, we obtain the following Poincaré-Betti series:

$$\begin{aligned} P_{S/\mathfrak{a}}(\underline{x},\tau) &= 1+\left(\underline{x}^{(6,0,0)}+\underline{x}^{(4,2,0)}+\underline{x}^{(2,4,0)}+\underline{x}^{(2,0,4)}+\underline{x}^{(0,6,0)}+\underline{x}^{(0,2,4)}\right)\ \tau\\ &\quad +\left(\underline{x}^{(6,2,0)}+\underline{x}^{(2,6,0)}+\underline{x}^{(2,2,4)}+\underline{x}^{(4,4,0)}\right)\ \tau^2\\ &\quad +\left(\underline{x}^{(6,0,4)}+\underline{x}^{(4,2,4)}+\underline{x}^{(2,4,4)}+\underline{x}^{(0,6,4)}\right)\ \tau^2\\ &\quad +\left(\underline{x}^{(6,2,4)}+\underline{x}^{(2,6,4)}+\underline{x}^{(4,4,4)}\right)\ \tau^3. \end{aligned}$$

Let $p = 3$ and $\mathfrak{a} := \langle x_1^3, x_2^3 \rangle \left(\langle x_1^2, x_1x_2, x_1x_3, x_2^2, x_2x_3, x_3^2 \rangle\right)^9$. With CoCoA [16] we calculate the following minimal resolution:

$$\longrightarrow S^3(-36) \oplus S^5(-33) \longrightarrow S^{10}(-30) \oplus S^3(-27) \oplus S^6(-24) \longrightarrow S^{12}(-21).$$

Our algorithm gives the following basis elements of the minimal resolution:

$t = 1, u_t = x_2$:
$i = 1, \quad m_i(u) = 2: \quad \{x_2^3 w, x_1^3 w\}, \quad w \in \{\ x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^2,\ x_2x_3,\ x_3^3\ \}$,

$t = 2, u_t = x_1x_2$:
$i = 1, \quad m_i(u) = 2: \quad \{\ x_1^3x_1^9x_2^9,\ x_2^3x_1^{18}\ \}$,

$t = 2, u_t = x_2^2$:
$i = 1, \quad m_i(u) = 2: \quad \{\ x_1^3x_2^{18},\ x_2^3x_1^9x_2^9\ \}$,

$t = 2, u_t = x_1x_3$:

$i = 1,$	$m_i(u) = 3:$	$\{x_1^3x_1^9x_3^9, wx_1^{18}\},$	$w \in \{x_1^3, x_2^3\},$
$i = 2,$	$m_i(u) = 3:$	$\{x_2^3x_1^9x_3^9, wx_1^9x_2^9\},$	$w \in \{x_1^3, x_2^3\},$
$i = 1,$	$m_i(u) = 3:$	$\{x_1^3x_1^9x_3^9, w_1x_1^{18}, w_2x_1^{18}\},$	$w_{1,2} \in \{x_1^3, x_2^3\},$
$i = 2,$	$m_i(u) = 3:$	$\{x_2^3x_1^9x_3^9, w_1x_1^9x_2^9, w_2x_1^9x_2^9\},$	$w_{1,2} \in \{x_1^3, x_2^3\},$
$i_1 = 1, i_2 = 2,$	$m_i(u) = 3:$	$\{x_2^3x_1^9x_3^9, x_2^3x_1^{18}, x_1^3x_1^9x_2^9\},$	

$t = 2, u_t = x_3^2$:

$i = 1,$	$m_i(u) = 3:$	$\{x_1^3x_3^{18}, wx_1^9x_3^9\},$	$w \in \{x_1^3, x_2^3\},$
$i = 2,$	$m_i(u) = 3:$	$\{x_2^3x_3^{18}, wx_2^9x_3^9\},$	$w \in \{x_1^3, x_2^3\},$
$i = 1,$	$m_i(u) = 3:$	$\{x_1^3x_3^{18}, w_1x_1^9x_3^9, w_2x_1^9x_3^9\},$	$w_{1,2} \in \{x_1^3, x_2^3\},$
$i = 2,$	$m_i(u) = 3:$	$\{x_2^3x_3^{18}, w_1x_2^9x_3^9, w_2x_2^9x_3^9\},$	$w_{1,2} \in \{x_1^3, x_2^3\},$
$i_1 = 1, i_2 = 2,$	$m_i(u) = 3:$	$\{x_1^3x_3^{18}, x_2^3x_1^9x_3^9, x_1^3x_2^9x_3^9\},$	

$t = 2, u_t = x_2x_3$:

$i = 1,$	$m_i(u) = 2:$	$\{x_1^3x_2^9x_3^9, x_2^3x_1^9x_3^9\},$	
$i = 2,$	$m_i(u) = 3:$	$\{x_2^3x_2^9x_3^9, wx_2^{18}\},$	$w \in \{x_1^3, x_2^3\},$
$i = 2,$	$m_i(u) = 3:$	$\{x_2^3x_2^9x_3^9, w_1x_2^{18}, w_2x_2^{18}\},$	$w_{1,2} \in \{x_1^3, x_2^3\},$
$i_1 = 1, i_2 = 2,$	$m_i(u) = 3:$	$\{x_1^3x_2^9x_3^9, x_2^3x_1^9x_2^9, x_1^3x_2^{18}\}.$	

Counting basis elements gives the desired Poincaré-Betti series.

3.2.3. *Regularity of p-Borel fixed ideals.*
Finally, we study the regularity of p-Borel fixed ideals.
Recall that the regularity of an ideal $\mathfrak{a}$ is the maximal number j such that

$$Tor_i^S(S/\mathfrak{a}, k)_{i+j} \neq 0 \text{ for some } i \geq 0.$$

Let $u = \prod_{i=1}^n x^{\mu_i}$ be a monomial in S and $\mu_i := \sum_j \mu_{ij} p^j$ the p-adic expansion of μ_i. For a real number $x \in \mathbb{R}$ denote by $\lfloor x \rfloor$ the largest integer $n \leq x$. For $1 \leq \kappa \leq n$ and $j \geq 0$ we define

$$d_{\kappa j}(\mu) = \sum_{i=1}^{\kappa} \lfloor \frac{\mu_i}{p^j} \rfloor.$$

For each κ with $\mu_\kappa \neq 0$ we set $s_\kappa := \lfloor \log_p \ \mu_\kappa \rfloor$ and set

$$D_\kappa := d_{\kappa,s_\kappa}(\mu)p^{s_\kappa} + (\kappa - 1)(p^{s_\kappa} - 1).$$

Finally, we define the following functions:

Definition 3.23. (1) For a monomial $u \in S$ which is not divisible by x_1 we set

$$pa(u) := \max_{\kappa:\mu_\kappa \neq 0} \{D_\kappa\}.$$

(2) For a monomial $u = x_1^{\mu_1}\tilde{u} \in S$ such that $\tilde{u}$ is not divisible by x_1 we set

$$pa(u) := \mu_1 + pa(\tilde{u}).$$

(3) For a p-Borel fixed ideal of type (3.4) we define the following function:

$$\begin{aligned} pa_t(\mathfrak{a}) &=: \sum_{j=t+1}^{T} \deg(I_j)p^{r_j} + \max_{u \in G(I_t)} \{\deg(u) + (M(u)-1)(p^{r_t}-1)\}, \\ pa(\mathfrak{a}) &:= \max_{t=1,\dots,T} \{pa_t(\mathfrak{a})\}. \end{aligned}$$

The function $pa(\mathfrak{a})$ was first introduced by Pardue [**38**], who conjectured that if $\mathfrak{a}$ is p-Borel fixed, Borel-generated by $\mathcal{G} := \{x^\mu\}$ such that x_1 does not divide x^μ, then

$$\operatorname{reg}(\mathfrak{a}) = pa(x^\mu).$$

Herzog and Popescu proved this conjecture in [**30**]. In addition, they proved an upper bound for p-Borel ideals $\mathfrak{a}$ Borel-generated by $\mathcal{G} := \{u_1, \dots, u_r\}$.

Theorem 3.24 (see [**30**]). *If $\mathfrak{a}$ is p-Borel fixed, Borel-generated by $\mathcal{G} := \{u_1, \dots, u_r\}$, then*

$$\operatorname{reg}(\mathfrak{a}) \leq \max\{pa(u_1), \dots, pa(u_r)\}$$

and equality holds if $\mathfrak{a}$ is principal p-Borel fixed.

In [**20**], Ene, Pfister and Popescu calculate the regularity of p-Borel fixed ideals in the following case:

Theorem 3.25 (see [**20**]). *Let $\mathfrak{a}$ be a p-Borel fixed ideal of type (3.4) such that I_j contains $x_m^{p^{r_{j+1}-r_j}-1}$ for all $j = 1, \dots, T$ where $m := \max(\operatorname{supp}(I_{j+1}))$. Then*

$$\operatorname{reg}(\mathfrak{a}) = pa(\mathfrak{a}).$$

We now give a generalization of Theorem 3.25.

Theorem 3.26. *Let $\mathfrak{a} := \prod_{j=1}^{T} I_j^{[p^{r_j}]}$ be a p-Borel fixed ideal such that $I_j = \langle x_1, \dots, x_{l_j}\rangle^{a_j}$ with $a_j < p^{r_{j+1}-r_j}$ for all $j = 1, \dots, T$, and $l_1 \geq l_2 \dots \geq l_T$. Then*

$$\operatorname{reg}(\mathfrak{a}) = pa(\mathfrak{a}).$$

Proof. By Theorem 3.19 a basis of $Tor_i^S(S/\mathfrak{a}, k)$ is given by

$$S_{i-1} := \Big\{(u_t, \dots, u_T, \{j_1 < \dots < j_{i-1}\}) \Bigm| t = 1, \dots T, u_j \in I_j, j_{i-1} < M(u_t)\Big\},$$

where the multidegree is given by

$$\deg\big((u_t,\dots,u_T,\{j_1<\dots<j_{i-1}\})\big)=\sum_{j=t+1}^{T}\deg(I_j)\,p^{r_j}+\big(\deg(u_t)p^{r_t}+(i-1)p^{r_t}\big).$$

Therefore, the basis element $(u_t,\dots,u_T,\{j_1<\dots<j_{i-1}\})$ gives a contribution to $Tor^S_{i-1}(S/\mathfrak{a},k)_{i-1+j}$ with

$$j=\sum_{l=t+1}^{T}\deg(I_l)\,p^{r_l}\;+\;\big(\deg(u_t)p^{r_t}+(i-1)(p^{r_t}-1)\big).$$

Clearly, this becomes maximal if $i=M(u_t)-1$ and if $\deg(u_t)$ is maximal. Taking the maximum over $t=1,\dots,T$ proves the assertion. □

If $\mathfrak{a}$ is a p-Borel fixed ideal of type (3.4) such that I_j is principal Borel fixed, we have the following regularity:

Theorem 3.27. *Let $\mathfrak{a}=\prod_{j=1}^{T}I_j^{[p^{r_j}]}$ be a principal p-Borel fixed ideal of type (3.4) such that the factors I_j are principal Borel fixed. Then*

$$reg(\mathfrak{a})=\max_{t=1,\dots,T}\Big\{pa_t(\mathfrak{a})+\operatorname{reg}\big(I^{[p]}_{<t}(2)\big)\Big\}\geq pa(\mathfrak{a}).$$

Note that it is actually possible to deduce Theorem 3.26 and the equality in Theorem 3.24 from Theorem 3.27: In the situation of Theorem 3.26 we have $I^{[p]}_{<t}(2)=\emptyset$ and therefore $\operatorname{reg}\big(I^{[p]}_{<t}(2)\big)=0$.
If $\mathcal{G}=\{x^\mu\}$ is the Borel-generating system of $\mathfrak{a}$, Lemma 3.15 implies that $\operatorname{reg}\big(I^{[p]}_{<t}(2)\big)=\mu_1$. Therefore, this reproves the equality of Theorem 3.24:

Corollary 3.28. *If $\mathfrak{a}$ is Borel-generated by $\mathcal{G}=\{x^\mu\}$, we have:*

$$\operatorname{reg}(\mathfrak{a})=\max_{t=1,\dots,T}\Big\{pa_t(\mathfrak{a})+\operatorname{reg}\big(I^{[p]}_{<t}(2)\big)\Big\}=pa(\mathfrak{a})+\mu_1.$$

In particular, if $\mathfrak{a}$ is Borel-generated by $\mathcal{G}=\{x^\mu\}$ and x_1 does not divide x^μ, we have $\operatorname{reg}(\mathfrak{a})=pa(\mathfrak{a})=pa(x^\mu)$*, which reproves Pardue's conjecture.* □

Proof of Theorem 3.27. The set of critical cells with respect to the matching constructed in Proposition 3.18 give a basis for $Tor^S_i(S/\mathfrak{a},k)_{i+l}$. It follows that we get the following possible l's:

$$\sum_{j=t+1}^{T}\deg(I_j)p^{r_j}+\max_{u_t\in G(I_t)}\Big(\deg(u_t)p^{r_t}+\max_{\substack{1\leq i_1<\dots<i_\kappa<M(u_t)\\ \kappa\geq 2}}\big\{\kappa(p^{r_t}-1)+\operatorname{reg}(I^{[p]}_{<t}(i_2)\big\}\Big),$$

$$\sum_{j=t+1}^{T}\deg(I_j)p^{r_j}+\max_{u_t\in G(I_t)}\Big(\deg(u_t)p^{r_t}+\max_{1\leq i_1<M(u_t)}\big\{(p^{r_t}-1)+\operatorname{reg}(I^{[p]}_{<t}(m_{i_1}(u_t))\big\}\Big).$$

Since $\operatorname{reg}\big((I^{[p]}_{<t}(i_2)\big)<p^{r_t}$, maximality is achieved for $\kappa=M(u)-1$. Then $i_2=2$ and therefore $I^{[p]}_{<t}(i_2)$ has only one generator, namely x_1^b for some b. It follows that the regularity of $I^{[p]}_{<t}$ is b. Thus we have

$$\sum_{j=t+1}^{T}\deg(I_j)p^{r_j}+\max_{u_t\in G(I_t)}\Big(\deg(u_t)p^{r_t}+(M(u_t)-1)(p^{r_t}-1)+\operatorname{reg}(I^{[p]}_{<t}(2)\Big).$$

Taking the maximum over $t=1,\dots,T$ proves the assertion. □

Example 3.29. Let $p = 2$ and $\mathfrak{a} := \langle x_1\rangle^{p^0} \langle x_1, x_2, x_3\rangle^{p^1}$. Then it follows: $pa(\mathfrak{a}) = 4 = pa_2(\mathfrak{a})$. The program Macaulay 2 [**25**] calculates $\text{reg}(\mathfrak{a}) = 5$. Our formula gives:

$$\begin{aligned} \text{reg}(\mathfrak{a}) &= \max_{t=1,2} \left\{ pa_t(\mathfrak{a}) + \text{reg}\left(I^{[p]}_{<t}(2)\right)\right\} \\ &= 4 + \text{reg}\left(\langle x_1\rangle^{[p^0]}\right) \\ &= 4 + 1 = 5. \end{aligned}$$

Note that the acyclic matching $\mathcal{M}$ from Proposition 3.18 can be applied in a slightly more general setting. Let $\mathfrak{a}$ be any p-Borel fixed ideal of type (3.4). Then Lemma 3.17 does not hold anymore, but one can prove that for each monomial v there exists a generator v_1 such that v_1 divides $\tilde{v}\ x_{t,\min}(W)$ (resp. $\tilde{v}\ x_{t,\max}(W)$). One can construct the matching in the same way, but instead of $B \cup \{w\} \to B \setminus \{w\}$ one defines $B \cup \{\tilde{w}\} \to B \setminus \{\tilde{w}\}$ where $\tilde{w}$ is the smallest generator of $\mathfrak{a}$ dividing w. The same arguments as used in the proof of Proposition 3.18 imply that the matching is still acyclic. In this case, we cannot anymore describe the critical cells. But we get the same multidegrees since the smallest generator (with respect to a chosen monomial order) dividing $\frac{x_j u_0}{x_{M(u_0)}}$ must have the same exponent for x_j, otherwise it would divide u_0, which is impossible since u_0 is a generator. Furthermore, the induction principle is still valid, hence we finally reach a minimal resolution, which is cellular since it comes from the Taylor resolution by a single matching.. Therefore, Theorem 3.20, Corollary 3.21 and Theorem 3.27 are still true in this situation:

Theorem 3.30. *If $\mathfrak{a}$ is any p-Borel fixed ideal of type (3.4), then:*

(1) *$\mathfrak{a}$ admits a minimal cellular multigraded resolution.*
(2) *The multigraded Poincaré-Betti series $P_{S/\mathfrak{a}}(\underline{x}, \tau)$ is given by the formula of Theorem 3.20.*
(3) *The regularity is given by*

$$\text{reg}(\mathfrak{a}) = \max_{t=1,\dots,T} \left\{ pa_t(\mathfrak{a}) + \text{reg}\left(I^{[p]}_{<t}(2)\right)\right\}.$$

In particular, the minimal resolution does not depend on the characteristic of k. ☐

In general, we would like to conjecture the following:

Conjecture 3.31. *If $\mathfrak{a} = \prod_{j=1}^{T} I_j^{[p^{r_j}]}$ is p-Borel fixed such that there exists an index j with $\deg(I_j) \geq p^{r_{j+1}-r_j}$, then*

$$\max_{t=1,\dots,T} \left\{ pa_t(\mathfrak{a}) + \text{reg}\left(I^{[p]}_{<t}(2)\right)\right\} \leq \text{reg}(\mathfrak{a}).$$

Free Resolution of the Residue Class Field k

In this chapter, which is submitted under the title "Resolution of the Residue Class Field via Algebraic Discrete Morse Theory" (see [**36**]), we provide three applications of our theory:

In Paragraph 1 we consider resolutions of the field k over a quotient $A = S/\mathfrak{a}$ of the commutative polynomial ring $S = k[x_1, \dots, x_n]$ in n variables by an ideal $\mathfrak{a}$. We construct a free resolution of k as an A-module, which can be seen as a generalization of the Anick resolution to the commutative case. Our resolution is minimal if $\mathfrak{a}$ admits a quadratic Gröbner basis. Also we give an explicit description of the minimal resolution of k if the initial ideal of $\mathfrak{a}$ is a complete intersection.

Paragraph 2 considers the same situation in the non-commutative case. We apply Algebraic Discrete Morse theory in order to obtain the Anick resolution of the residue field k over $A = k\langle x_1, \dots, x_n\rangle/\mathfrak{a}$ from the normalized Bar resolution, where $k\langle x_1, \dots, x_n\rangle$ is the polynomial ring in n non-commuting indeterminates and $\mathfrak{a}$ is a two-sided ideal with a finite Gröbner basis. This result has also been obtained by Sköldberg [**41**]. In addition to his results, we prove the minimality of this resolution when $\mathfrak{a}$ is monomial or the Gröbner basis consists of homogeneous polynomials which all have the same degree. In these cases it follows from our results that the Poincaré-Betti series is rational. In particular, we get the rationality of the Hilbert series if $\mathfrak{a}$ admits a quadratic Gröbner basis.

In Paragraph 3 we give a projective resolution of A as an $A \otimes A^{op}$-module, where again $A = k\langle x_1, \dots, x_n\rangle/\mathfrak{a}$. Using this resolution we obtain the minimal resolution of $A = k[x_1, \dots, x_n]/\langle f_1, \dots f_s\rangle$ as an $A \otimes A^{op}$-module when the initial ideal of $\langle f_1, \dots, f_s\rangle$ is a complete intersection. In case $\mathfrak{a} = \langle f\rangle$, such a construction was first given by BACH in [**9**].

1. Resolution of the Residue Field in the Commutative Case

Let $A = S/\mathfrak{a}$ be the quotient algebra of the commutative polynomial ring $S = k[x_1, \ldots, x_n]$ in n indeterminates by the ideal $\mathfrak{a} \trianglelefteq S$.

The aim of this paragraph is to deduce via Algebraic Discrete Morse theory a new free resolution of the residue field $k \cong A/\langle x_1, \ldots, x_n \rangle$ as an A-module from the normalized Bar resolution. We write $\mathsf{NB}_\bullet^A = (B_i, \partial_i)_{i\geq}$ for the normalized Bar resolution of k over A (see Chapter 2.2.4 or [**47**]).

From now on let $\mathfrak{a} = \langle f_1, \ldots, f_s \rangle \trianglelefteq S$ be an ideal, such that the set $\{f_1, \ldots, f_s\}$ is a reduced Gröbner basis with respect to a fixed degree-monomial order '$\prec$' (for example degree-lex or degree-revlex). We assume that $x_1 \succ x_2 \succ \ldots \succ x_n$ and we write $\mathcal{G}$ for the corresponding set of standard monomials of degree ≥ 1.

It is well known that $\mathcal{G} \cup \{1\}$ is a basis of A as k-vectorspace. Thus, for any monomial w in S there is a unique representation

$$w = a_1 + \sum_{v \in \mathcal{G}} a_v v, \;\; a_1, a_v \in k, \tag{1.1}$$

as a linear combination of standard monomials in A.

Since we assume that our monomial order is a refinement of the degree order on monomials, it follows that $a_v = 0$ for $|v| > |w|$. Here we denote with $|v|$ the total degree of the monomial v. In this situation we say that v is reducible to $-\sum_{v \in \mathcal{G}} a_v v$. Note that since we use the normalized Bar resolution, the summand a_1 can be omitted.

Using the described reduction process we write the normalized Bar resolution $\mathsf{NB}_\bullet^A = (B_i, \partial_i)$ as

$$\begin{aligned} B_0 &:= A, \\ B_i &:= \bigoplus_{w_1, \ldots, w_i \in \mathcal{G}} A\,[w_1|\ldots|w_i], \;\; i \geq 1 \end{aligned}$$

with differential

$$\begin{aligned} \partial_i([w_1|\ldots|w_i]) &= w_1\,[w_2|\ldots|w_i] \\ &+ \sum_{j=1}^{i-1}(-1)^j \sum_{v \in \mathcal{G}} a_{jv}\,[w_1|\ldots|w_{j-1}|v|w_{j+2}|\ldots|w_i], \end{aligned}$$

for $w_j w_{j+1} = a_{j,1} + \sum_{v \in \mathcal{G}} a_{jv}\,\nu$, with $a_{jv} \in k, v \in \mathcal{G}$.

The following convention will be convenient. For a monomial $w \in S$ we set $m(w) := \min\{i \mid x_i \text{ divides } w\}$. Finally, we think of $[w_1|\ldots|w_i]$ as a vector, and we speak of w_j as the entry in the jth coordinate position.

Now we describe the acyclic matching on the normalized Bar resolution, which will be crucial for the proof of Theorem 1.6. Since all coefficients in the normalized Bar resolutions are ± 1, condition (Invertibility) of Definition 5.1.1 is automatically fulfilled. Thus, we only have to take care of the conditions (Matching) and (Acyclicity):

We inductively define acyclic matchings $\mathcal{M}_j$, $j \geq 1$, that are constructed with respect to the jth coordinate position. We start with the leftmost coordinate position $j = 1$. We set

$$\mathcal{M}_1 := \left\{ \begin{array}{c} [x_{m(w_1)}|w_1'|w_2|\dots|w_l] \\ \downarrow \\ [w_1|w_2|\dots|w_l] \end{array} \in G(\mathsf{NB}^A_\bullet) \;\Big|\; w_1 = x_{m(w_1)}w_1' \right\}.$$

The set of critical cells $B_l^{\mathcal{M}_1}$ in homological degree $l \geq 1$ is given by:

(1) $B_1^{\mathcal{M}_1} := \left\{[x_i] \;\Big|\; 1 \leq i \leq n\right\}$, $l = 1$,

(2) $B_l^{\mathcal{M}_1}$ is the set of all $[x_i|w_2|w_3|\dots|w_l]$, $w_2, \dots, w_l \in \mathcal{G}$, that satisfy either
  $\to$ $i \leq m(w_2)$ and $x_i w_2$ is reducible or
  $\to$ $i > m(w_2)$.

Assume now $j \geq 2$ and $\mathcal{M}_{j-1}$ is defined. Let $\mathcal{B}^{\mathcal{M}_{j-1}}$ be the set of critical cells left after applying $\mathcal{M}_1 \cup \dots \cup \mathcal{M}_{j-1}$.

Let $\mathcal{E}_j$ denote the set of edges in $G(\mathsf{NB}^A_\bullet)$ that connect critical cells in $\mathcal{B}^{\mathcal{M}_{j-1}}$.

The following condition on an edge in $\mathcal{E}_j$ will define the matching $\mathcal{M}_j$.

Definition 1.1 (Matching Condition)**.** Let

$$\begin{array}{c} [x_{i_1}|w_2|\dots|w_{j-1}|u_1|u_2|w_{j+1}|\dots|w_l] \\ \downarrow \\ [x_{i_1}|w_2|\dots|w_{j-1}|w_j|w_{j+1}|\dots|w_l] \end{array}$$

be an edge in $\mathcal{E}_j$. In particular, $w_j = u_1u_2$. We say that the edge satisfies the matching condition if u_1 is the maximal monomial with respect to '$\prec$' such that

(i) u_1 divides w_j,

(ii) $[x_{i_1}|w_2|\dots|w_{j-1}|u_1|u_2|w_{j+1}|\dots|w_l] \in \mathcal{B}^{\mathcal{M}_{j-1}}$,

(iii) $[x_{i_1}|w_2|\dots|w_{j-1}|v_1|v_2|w_{j+1}|\dots|w_l] \notin \mathcal{B}^{\mathcal{M}_{j-1}}$ for each $v_1 \mid u_1$, $v_1 \neq u_1$ and $v_1v_2 = w_j$.

$$\mathcal{M}_j := \left\{ \begin{array}{c} [x_{i_1}|w_2|\dots|w_{j-1}|u_1|u_2|w_{j+1}|\dots|w_l] \\ \downarrow \\ [x_{i_1}|w_2|\dots|w_{j-1}|w_j|w_{j+1}|\dots|w_l] \end{array} \in \mathcal{E}_j \text{ satisfying 1.1} \right\}.$$

We write $\mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$ for the minimal, monomial generating system of the initial ideal of $\mathfrak{a}$ with respect to the chosen monomial order $\prec$. The set of critical cells $B_l^{\mathcal{M}_j}$ in homological degree $l \geq 1$ is given by

(1) $B_1^{\mathcal{M}_j} := \left\{[x_i] \;\Big|\; 1 \leq i \leq n\right\}$,

(2) $B_2^{\mathcal{M}_j}$ consists of elements $[x_i|w_2]$ such that either $w_2 = x_{i'}$ for some i' and $i > i'$ or $x_iw_2 \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$,

(3) $\mathcal{B}_l^{\mathcal{M}_j}$ consists of elements $[x_i|w_2|\dots|w_j|\dots|w_l] \in \mathcal{B}_l^{\mathcal{M}_{j-1}}$, such that for each divisor $u \mid w_j$ we have $[x_i|w_2|\dots|w_{j-1}|u|\dots|w_l] \notin \mathcal{B}_l^{\mathcal{M}_{j-1}}$ and one of the following conditions is satisfied:
  $\to$ w_jw_{j+1} is reducible or
  $\to$ $w_jw_{j+1} = uv \in \mathcal{G}$ and
    • $[x_i|w_2|\dots|w_{j-1}|u|v|w_{j+2}|\dots|w_l] \in \mathcal{B}_l^{\mathcal{M}_{j-1}}$,

- $u \succ w_j$,
- $[x_i|w_2|\dots|w_{j-1}|u'|v'|w_{j+2}|\dots|w_l] \notin \mathcal{B}_l^{\mathcal{M}_{j-1}}$ for each divisor $u' \mid u$, $u' \neq u$ and $u'v' = w_j w_{j+1}$.

We finally set $\mathcal{M} := \bigcup_{j\geq 1} \mathcal{M}_j$ and we write $\mathcal{B}^{\mathcal{M}}$ for the set of critical cells with respect to $\mathcal{M}$.

Lemma 1.2. *$\mathcal{M}$ is an acyclic matching.*

Proof. We have already seen that since all coefficients are ± 1, the condition (Invertibility) of Definition 5.1.1 is automatic. Property (Matching) is satisfied by definition of $\mathcal{M}$. Now consider an edge in the matching. Then there exists a coordinate where the degree of the monomial decreases by passing to the higher homological degree cell. Now since we have chosen a degree-monomial order along any edge in the graph and for any coordinate, the degree of the monomial in this positions decreases weakly. Since any cycle must contain a matched edge, this shows that there cannot be any directed cycles and (Acyclicity) is satisfied as well. □

1.1. An Anick Resolution for the Commutative Polynomial Ring. In this section we look closer into the Morse complex corresponding to the acyclic matching $\mathcal{M}$ from Lemma 1.2. For this we choose the degree-lex order as our fixed monomial order. We write $\mathrm{MinGen}(\mathrm{in}_{\prec}(\mathfrak{a}))$ for the minimal, monomial generating system of the initial ideal of $\mathfrak{a}$ with respect to degree-lex.

In order to describe the critical cells for the chosen term order, we first define the concept of a minimal fully attached tuple. Note that the notation "fully attached" was introduced by Sturmfels (see Example 2.10 and [**42**]).

Definition 1.3. A pair $[w_1|w_2]$ is called minimal fully attached if $w_1 = x_{m(w_1 w_2)}$ and $w_1 w_2 \in \mathrm{MinGen}(\mathrm{in}_{\prec}(\mathfrak{a}))$.

Assume $l > 2$. An l-tuple $[w_1|\dots|w_{l-1}|w_l]$ is called minimal fully attached if $[w_1|\dots|w_{l-1}]$ is minimal fully attached, $m(w_1) \leq m(w_j)$ for $j = 3, \dots, l$, and one of the following conditions is satisfied:

(1) $w_{l-1}w_l$ is reducible or

(2) $w_{l-1}w_l = uv \in \mathcal{G}$ with $u \succ w_{l-1}$ and $[w_1|\dots|w_{l-2}|u]$ is a minimal fully attached $(l-1)$-tuple,

and w_l is the minimal monomial such that no divisor $w_l' \mid w_l$, $w_l' \neq w_l$, satisfies one of the two conditions above.

It is easy to see that the basis of the free modules in the Morse complex $\mathsf{NB}_\bullet^{\mathcal{M}}$ is given as the set $\mathcal{B}$ of words over the alphabet

$$\begin{aligned}\Sigma \;=\; & \Big\{[x_{i_1}|x_{i_2}|\dots|x_{i_r}] \;\Big|\; 1 \leq i_r < i_{r-1} < \dots < i_1 \leq n \Big\} \cup \\ & \Big\{[x_{w_2}|w_2|\dots|w_l] \;\Big|\; [x_{w_2}|w_2|\dots|w_l] \text{ minimal fully attached } \Big\}\end{aligned}$$

that contain none of the words:

$$[x_{i_1}|\dots|x_{i_r}][x_{w_2}|w_2|\dots|w_l], \quad x_{w_2} \preceq x_{i_r}, \tag{1.2}$$

$$(1.3)\qquad [x_{i_1}|\dots|x_{i_r}][x_{j_1}|\dots|x_{j_s}],\quad x_{j_1}\preceq x_{i_r},$$

$$(1.4)\qquad [x_{w_2}|w_2|\dots|w_l][x_{i_1}|\dots|x_{i_r}],\quad x_{i_1}\prec x_{w_2},$$

$$(1.5)\qquad [x_{w_2}|w_2|\dots|w_l][x_{v_2}|v_2|\dots|v_l],\quad x_{v_2}\prec x_{w_2}.$$

In order to be able to identify elements of $\mathcal{B}$ as basis elements of the Bar resolution, we read in a word from $\mathcal{B}$ the sequence of letters '][' as '|'. If this convention is applied, then any element of $\mathcal{B}$ can be read as some $[w_1|\dots|w_j]$ and corresponds to a basis element in homological degree j. We collect the elements from $\mathcal{B}$ which are of homological degree j in $\mathcal{B}_j$ and call an element of $\mathcal{B}$ a fully attached tuple. We claim that there is a bijection between $\mathcal{B}^{\mathcal{M}}$ and $\mathcal{B}$ preserving the homological degree. To see this, consider a fully attached tuple $[x_{i_1}|w_2|\dots|w_i]$. Then the definition of a fully attached tuple implies that either $w_2 = x_s$ with $x_s \succ x_{i_1}$ (resp. $i_1 > s$) or $x_{i_1}w_2 \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$. In the first case we cut the tuple to $[x_{i_1}][x_s|w_3|\dots|w_i]$. If we continue this process, we obtain

$$[x_{i_1}|x_{i_2}|\dots|x_{i_r}][x_{v_2}|v_2|\dots|v_s]$$

with $i_1 > \dots > i_r$, $x_{i_r} \prec x_{v_2}$, and $x_{v_2}v_2 \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$. This explains the rules (1.2) and (1.3). Now consider $[x_{v_2}|v_2|\dots|v_s]$. Then the definition of a fully attached tuple implies that either $v_3 = x_j$ with $x_j \succeq x_{v_2}$ or $x_{m(v_3)} \prec x_{v_2}$. In the first case we cut the tuple to

$$[x_{v_2}|v_2][x_j|v_4|\dots|v_i],$$

otherwise we consider the monomial v_4. Then v_4 satisfy the same conditions as v_3, so we cut if necessary to

$$[x_{v_2}|v_2|v_3][x_j|v_5|\dots|v_i].$$

By construction $[x_{v_2}|v_2|v_3]$ is a minimal fully attached tuple and the conditions for v_3 and x_j explain the rules (1.4) and (1.5). If we continue this process, we obtain exactly the words in $\mathcal{B}$.

Remark 1.4. Let $\mathcal{L}$ be the language over the alphabet

$$\Big\{[x_{w_2}|w_2|\dots|w_l]\;\Big|\;[x_{w_2}|w_2|\dots|w_l] \text{ minimal fully attached }\Big\}$$

that contains none of the words (1.5). To a letter $[x_{i_1}|x_{i_2}|\dots|x_{i_r}] \in \Sigma$ with $1 \le i_r < i_{r-1} < \dots < i_1 \le n$, we associate the symbol $e_{\{i_r<i_{r-1}<\dots<i_1\}}$.
For $w \in \mathcal{B}^{\mathcal{M}}$, such that $w = e_{I_1}\cdots e_{I_s}$, rule (1.3) shows that this word is considered as a basis element of $\mathsf{NB}_\bullet^{\mathcal{M}}$, equivalent to the symbol $e_{I_1\cup\dots\cup I_s}$.
To an arbitrary word $w \in \mathcal{B}^{\mathcal{M}}$ we first associate the word

$$w_1\; e_{I_1}\; w_2\; e_{I_2}\cdots w_s\; e_{I_s}.$$

The rules (1.2) and (1.4) imply that the sets I_i are pairwise disjoint and in a decreasing order. Therefore, as a basis element of $\mathsf{NB}_\bullet^{\mathcal{M}}$ the word w is equivalent to

$$e_{I_1\cup\dots\cup I_s}\; w_1w_2\cdots w_s.$$

It follows that we have a degree-preserving bijection between $\mathcal{B}^{\mathcal{M}}$ and the set

$$\Big\{e_I\mathbf{w}\;\Big|\;I\subset\{1,\dots n\} \text{ and } \mathbf{w}\in\mathcal{L}\Big\}.$$

We will use this fact later in order to calculate the multigraded Poincaré-Betti series of k over A (see Corollary 1.8).

In order to describe the differential, we introduce three reduction rules for fully attached tuples. These reduction rules will be based on the unique Gröbner representation (1.1) which will play the role of the basic set of rules:

$$\mathcal{R} := \left\{ v_1 v_2 \xrightarrow{a_w} w \;\middle|\; \begin{array}{l} v_1, v_2 \in \mathcal{G} \\ v_1 v_2 \notin \mathcal{G} \end{array} \text{ and } \begin{array}{l} v_1 \cdot v_2 = a_0 + \sum_{w \in \mathcal{G}} a_w w, \\ a_w \in k \end{array} \right\}.$$

Note that $w \xrightarrow{0} 0 \in \mathcal{R}$ is allowed (it happens if one of the generators f_i is a monomial).

Definition 1.5. Let $e_1 := [w_1|\dots|w_{i-1}|w_i|w_{i+1}|w_{i+2}|\dots|w_l]$ be an l-tuple of standard monomials.

Type I: Assume $[w_1|\dots|w_i]$ is fully attached. We say e_1 can be reduced to $e_2 := [w_1|\dots|w_{i-1}|v_i|v_{i+1}|w_{i+2}|\dots|w_l]$ if
- (i) $[w_1|\dots|w_{i-1}|v_i]$ is fully attached,
- (ii) $v_i v_{i+1} \in \mathcal{G}$,
- (iii) $w_i w_{i+1} \xrightarrow{a} v_i v_{i+1} \in \mathcal{R}$ with $a \neq 0$.

In this case we write $e_1 \xrightarrow{-a}_1 e_2$.

Type II: We say that e_1 can be reduced to $e_2 := [w_1|\dots|w_{i-1}|v|w_{i+2}|\dots|w_l]$ if
- (i) $w_i w_{i+1} \xrightarrow{a} v \in \mathcal{R}$ with $a \neq 0$ and
- (ii) e_2 is a fully attached $(l-1)$-tuple.

In this case we write $e_1 \xrightarrow{(-1)^i a}_2 e_2$.

Type III: We say that e_1 can be reduced to e_2 with coefficient $c := w_1$ (we write $e_1 \xrightarrow{w_1}_3 e_2$) if $|w_2| \geq 2$ and $e_2 := [x_{m(w_2)}|w_2/x_{m(w_2)}|w_3|\dots|w_l]$.

Now let $e = [w_1|\dots|w_l]$ and $f = [v_1|\dots|v_{l-1}]$ be fully attached l- and $(l-1)$-tuples. We say that e can be reduced to f with coefficient c ($e \xrightarrow{c} f$) if there exists a sequence $e = e_0, e_1, \dots, e_{r-1}$ and either

(1) an e_r with $e_r = [u|v_1|\dots|v_{l-1}] = [u|f]$, such that e_0 can be reduced to e_r with reductions of Type I and III, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r;$$

in this case we set $c := \left((-1)^r \prod_{i=1}^r a_i\right) u$, or

(2) an e_r, such that e_0 can be reduced to e_r with reductions of Type I and III and e_r can be reduced to f with the reduction of Type II, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r \xrightarrow{(-1)^j b} f;$$

in this case we set $c := (-1)^{r+j} \cdot b \cdot \prod_{i=1}^r a_i$.

There may be several possible reduction sequences leading from e to f and the reduction coefficient may depend on the chosen sequence. Therefore, we define *the* reduction coefficient $[e : f]$ to be the sum over all possible sequences. If there exists no sequence, we set $[e : f] := 0$.

The complex $\mathsf{F}_\bullet$ is then given by

$$F_j := \bigoplus_{e \in \mathcal{B}_j} A\, e,$$

$$\begin{aligned} \partial : F_i &\rightarrow F_{i-1} \\ e &\mapsto \sum_{f \in \mathcal{B}_{i-1}} [e:f]\, f. \end{aligned}$$

Now we have:

Theorem 1.6. $\mathsf{F}_\bullet = (F_\bullet, \partial)$ *is an A-free resolution of the residue field k, which is minimal if and only if no reduction of Type II is possible.*

Proof. The fully attached tuples are exactly the critical cells. The reduction rules describe the Morse differential: As seen before, we have

$$\partial^{\mathcal{M}}([w_1|\dots|w_l]) := w_1[w_2|\dots|w_l] + \sum_{i=1}^{l-1}(-1)^i[w_1|\dots|w_iw_{i+1}|\dots|w_l]$$

If $[w_2|\dots|w_l] \notin \mathcal{B}$, we have $[w_2|\dots|w_l] = \partial([x_{i_2}|w_2'|w_3\dots|w_l])$, which is described by the reduction of Type III.

For $[w_1|\dots|w_iw_{i+1}|\dots|w_l]$ we have to distinguish three cases:

(Case 1) $[w_1|\dots|v_{ij}|\dots|w_l]$ is critical. Then we have $w_{i-1}v_{ij}$, $v_{ij}w_{i+2}$ reducible and $w_{i-1}u_1\ v_{ij}u_2 \in \mathcal{G}$ for all divisors u_1 of v_{ij} and u_2 of w_{i+2}. This situation is described by the reduction of Type II.

(Case 2) $[w_1|\dots|v_{ij}|\dots|w_l]$ is matched by a higher degree cell. Then we have $w_{i-1}u_1$ reducible for $v_{ij} = u_1u_2$, and for all divisors u' of u_1 the monomial $w_{i-1}u'$ lies in $\mathcal{G}$. Then we have

$$[w_1|\dots|v_{ij}|\dots|w_l] = (-1)^{i+1}[w_1|\dots|w_{i-1}|u_1|u_2|w_{i+2}|\dots|w_l],$$

which is a reduction of Type I.

(Case 3) $[w_1|\dots|v_{ij}|\dots|w_l]$ is matched by a lower degree cell. In this case we have $[w_1|\dots|v_{ij}|\dots|w_l] = 0$.

The coefficients of the reductions are exactly the coefficients of the Morse differential. Hence the Morse differential induces a sequence of reductions of Type I and III with either a reduction of Type II or the map $e_r = [v_1|\dots|v_l] \xrightarrow{v_1} [v_2|\dots|v_l]$ at the end, which gives our definition of the reduction coefficient. □

Remark 1.7. In Paragraph 2, we will see that in the non-commutative case our matching on the normalized Bar resolution gives the Anick resolution (for the definition, see [**1**]). Therefore, one can understand the resolution $F_\bullet$ as a generalization of the Anick resolution to the commutative polynomial ring.

If A is endowed with the natural multigrading $\deg(x_i) = e_i \in \mathbb{N}^n$, the multigraded Poincaré-Betti series of k over A is defined to be

$$P_k^A(\underline{x}, t) := \sum_{\substack{i \geq 0 \\ \alpha \in \mathbb{N}^n}} \dim_k(\mathrm{Tor}_i^A(k,k)_\alpha)\, \underline{x}^\alpha\, t^i.$$

Remark 1.4 implies:

Corollary 1.8. *The Poincaré-Betti series of A satisfies*

$$P_k^A(\underline{x}, t) \leq \prod_{i=1}^{n}(1 + x_i\, t)\ F(\underline{x}, t),$$

where $F(\underline{x}, t) := \sum_{w\in\mathcal{L}} w\, t^{|w|}$ *counts the words* $w \in \mathcal{L}$*. Here* w *is treated as the monomial in* $x_1, \ldots, x_n$ *and* $|w|$ *denotes the length of* w*.*
The inequality is an inequality between the coefficients of the power series expansion. □

1.2. Two Special Cases. First, we consider a subclass of the class of Koszul algebras. It is well known that $A = S/\mathfrak{a}$ is Koszul if $\mathfrak{a}$ has a quadratic Gröbner basis. It is easy to see that in this case the minimal fully attached tuples have the following form: $[x_{i_1}|x_{i_2}|\ldots|x_{i_r}]$. Therefore, a reduction of Type II is not possible and we get:

Corollary 1.9. *If* $A = S/\mathfrak{a}$ *and* $\mathfrak{a}$ *admits a quadratic Gröbner basis, then the resolution* $F_\bullet$ *is minimal.* □

To get an explicit form of the multigraded Poincaré-Betti series in this case, one only has to calculate the word-counting function $F(\underline{x}, t)$ of the language $\mathcal{L}$. In this case, the multigraded Poincaré-Betti series coincides with the multigraded Poincaré-Betti series of $S/\operatorname{in}_\prec(\mathfrak{a})$. Since the Poincaré-Betti series of monomial rings are studied by us in a larger context in Chapter 6, we do not give the explicit form here.

The second case, we would like to discuss, is the following:
Let $\mathfrak{a} = \langle f_1, \ldots, f_s\rangle \trianglelefteq S$ be an ideal, such that $f_1, \ldots, f_s$ is a reduced Gröbner basis with respect to the degree-lex order and such that the initial ideal $\operatorname{in}_\prec(\mathfrak{a})$ is a complete intersection. Assume $f_j = m_j + \sum_{\alpha\in\mathbb{N}^n} f_{j\alpha}x^\alpha$ with leading monomial m_j. Since $\operatorname{in}_\prec(\mathfrak{a})$ is a complete intersection, there exist exactly s minimal fully attached tuples, namely $t_i := \left[x_m(m_i)\middle|\frac{m_i}{x_m(m_i)}\right]$ for $i = 1, \ldots, s$ and $m_i \in \operatorname{MinGen}(\operatorname{in}_\prec(\mathfrak{a}))$. The rule (1.5) implies $t_i t_j \in \mathcal{B}$ iff $m(m_i) \geq m(m_j)$. It follows from Remark 1.4 that the set of fully attached i-tuples is in bijection with the set

$$\mathcal{B}_i := \left\{ e_{i_r}\ldots e_{i_1} t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)} \;\middle|\; \begin{array}{l} 1 \leq i_1 < \ldots < i_r \leq n \\ 1 \leq j_1 < \ldots < j_q \leq s \\ l_1, \ldots, l_q \in \mathbb{N} \text{ and } i = r + 2\sum_{t=1}^{q} l_t \end{array} \right\}.$$

For $f_j = m_j + \sum_{\alpha\in\mathbb{N}^n} f_{j\alpha}x^\alpha$ we define

$$T_p(f_j) := \sum_{\substack{\alpha\in\mathbb{N}^n \\ p=\max(\operatorname{supp}(\alpha))}} f_{j\alpha}\,\frac{x^\alpha}{x_p}.$$

We have the following theorem:

Theorem 1.10. *Let* $\mathfrak{a} = \langle f_1, \ldots, f_s\rangle \trianglelefteq S$ *be an ideal, such that* $f_1, \ldots, f_s$ *is a reduced Gröbner basis with respect to the degree-lex order and such that the initial ideal* $\operatorname{in}_\prec(\mathfrak{a})$ *is a complete intersection, and* $A := S/\mathfrak{a}$ *be the quotient algebra.*
Then the following complex is a minimal A*-free resolution of the residue class*

field k and carries the structure of a differential-graded algebra:

$$F_i := \bigoplus_{\substack{1 \le i_1 < \ldots < i_r \le n \\ 1 \le j_1 < \ldots < j_q \le s \\ l_1, \ldots, l_q \in \mathbb{N} \\ i = r + 2\sum_{j=1}^{q} l_j}} A\, e_{i_r} \ldots e_{i_1} t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)}$$

$$\begin{aligned} e_{i_r} \ldots e_{i_1} &\overset{\partial}{\mapsto} \sum_{m=1}^{r} (-1)^{\#\{i_j > i_m\}} x_{i_m}\, e_{i_r} \ldots \widehat{e_{i_m}} \ldots e_{i_1} \\ t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)} &\overset{\partial}{\mapsto} \sum_{m=1}^{s} \sum_{p=1}^{n} T_p(f_{j_m})\, e_p t_{j_1}^{(l_1)} \ldots t_{j_m}^{(l_{j_m}-1)} \ldots t_{j_q}^{(l_q)}, \end{aligned}$$

where $t_{i_j}^{(0)} := 1$*,* $e_i e_j = -e_j e_i$*, and* $e_i e_i = 0$*. The differential is given by*

$$\begin{aligned} \partial(e_{i_r} \ldots e_{i_1} t_{j_1}^{(l_1)} \ldots t_{j_s}^{(l_s)}) &= \partial(e_{i_r} \ldots e_{i_1}) t_{j_1}^{(l_1)} \ldots t_{j_s}^{(l_s)} \\ &\quad + (-1)^r\, e_{i_r} \ldots e_{i_1} \partial(t_{j_1}^{(l_1)} \ldots t_{j_s}^{(l_s)}). \end{aligned}$$

In particular, we have

$$P_k^A(\underline{x}, t) = \frac{\prod_{i=1}^{n} (1 + x_i\, t)}{\prod_{i=1}^{k} (1 - m_i\, t^2)}.$$

Proof. We only have to calculate the differential: Let $[w_1|\ldots|w_l]$ be a fully attached tuple, such that w_j is either a variable or a minimal fully attached tuple.

First assume that w_j is a variable, i.e. $w_j = x_{r_j}$. We prove that $w_i w_j$ can be permuted to $w_j w_i$ for all $i \neq j$. If w_i is a variable, say $w_i = x_{j_i}$, we have by (1.5) $j_i > r_j$ it follows $|x_{j_i}|x_{j_r}| \to |x_{j_i} x_{r_j}| \to |x_{r_j}|x_{j_i}|$. If w_i is a minimal fully attached tuple, i.e. $w_i = \left| x_{m(m_i)} \left| \frac{m_i}{x_{m(m_i)}} \right.\right|$, we have

$$\begin{aligned} \left| x_{m(m_i)} \left| \frac{m_i}{x_{m(m_i)}} \right| x_{r_j} \right| &\to \left| x_{m(m_i)} \left| x_{r_j} \frac{m_i}{x_{m(m_i)}} \right.\right| \to \left| x_{m(m_i)} \left| x_{r_j} \left| \frac{m_i}{x_{m(m_i)}} \right.\right.\right| \\ &\to \left| x_{r_j} x_{m(m_i)} \left| \frac{m_i}{x_{m(m_i)}} \right.\right| \to \left| x_{r_j} \left| x_{m(m_i)} \left| \frac{m_i}{x_{m(m_i)}} \right.\right.\right|. \end{aligned}$$

In the first case we have a reduction with coefficient -1 and in the second case with coefficient $+1$. Therefore, it is enough to consider the number of $w_i's$, $i < j$, which are variables. It follows that w_j can be permuted to the left with coefficient $(-1)^{\#\{w_i \,\mid\, w_i \text{ variable and } w_i <_{\text{lex}} x_{r_j}\}}$.

Now let w_j be a minimal fully attached tuple, i.e. $w_j = \left[x_{m(m_j)} \left| \frac{m_j}{x_{m(m_j)}} \right.\right]$. Then we have

$$\left[x_{m(m_j)} \left| \frac{m_j}{x_{m(m_j)}} \right.\right] \to -\sum_{\alpha} f_{j\alpha} [x^\alpha] \to \sum_{\alpha} f_{j\alpha} \left[x_\alpha \left| \frac{x^\alpha}{x_\alpha} \right.\right],$$

where $x_\alpha := x_{m(x^\alpha)}$. Since $\left[\frac{x^\alpha}{x_\alpha}\right]$ is matched with $\left[x_\beta \middle| \frac{x^\alpha}{x_\beta x_\alpha}\right]$ (where $x_\beta = x_{m(x^\beta)}$ with $x^\beta := \frac{x^\alpha}{x_\alpha}$) the exponent α decreases successively up to the element $[x_p]$ with $p = \max(\mathrm{supp}(\alpha))$. Therefore, we get

$$\left[x_{m(m_j)} \middle| \frac{m_j}{x_{m(m_j)}}\right] \to \sum_{p=1}^{n} T_p(f_j)e_p. \tag{1.6}$$

We now consider the tuple $[w_1|\ldots|w_l]$. With the same argument as before, one can check that the minimal fully attached tuple w_j can be permuted with coefficient $+1$ to the right. After a chain of reductions, we reach the tuple $[w_j|w_1|\ldots|w_{j-1}|w_{j+1}|\ldots|w_l]$. Applying Equation (1.6) we get

$$[w_1|\ldots|w_l] \to \sum_{p=1}^{n} T_p(f_j)[x_p|w_1|\ldots|\widehat{w_j}|\ldots|w_l].$$

In order to reach a fully attached tuple we have to permute the variable x_p to the correct position. This permutation yields a coefficient

$$(-1)^{\#\{w_i \mid w_i \text{ variable and } w_i <_{\text{lex}} x_p\}}.$$

The bijection between the elements $e_{i_r}\ldots e_{i_1} t_{j_1}^{(l_1)}\ldots t_{j_q}^{(l_q)}$ and the fully attached tuples finally implies the coefficient

$$\left((-1)^{\#\{w_i \mid w_i \text{ variable and } w_i <_{\text{lex}} x_p\}}\right)^2 (-1)^r = (-1)^r.$$

Therefore, our differential has the desired form

$$\begin{aligned}
&\partial(e_{i_r}\ldots e_{i_1} t_{j_1}^{(l_1)}\ldots t_{j_q}^{(l_q)})\\
&= \sum_{m=1}^{r} (-1)^{\#\{i_j > i_m\}} x_{i_m}\, e_{i_r}\ldots \widehat{e_{i_m}}\ldots e_{i_1} t_{j_1}^{(l_1)}\ldots t_{j_q}^{(l_q)}\\
&\quad + \sum_{m=1}^{q} \sum_{\substack{p=1\\ p\neq i_1,\ldots,i_r}}^{n} (-1)^r\, T_p(f_{j_m})\, e_{i_r}\cdots e_{i_1} e_p t_{j_1}^{(l_1)}\ldots t_{j_m}^{(l_{j_m}-1)}\ldots t_{j_q}^{(l_q)}.
\end{aligned}$$

It is easy to see that these are all possible reductions. □

If $\mathrm{in}_\prec(\mathfrak{a}) = \mathfrak{a}$, then the preceding result about the Poincaré-Betti series can be found in [**26**].

2. Resolution of the Residue Field in the Non-Commutative Case

In this paragraph we study the same situation as in Paragraph 1 over the polynomial ring in n non-commuting indeterminates. In this case, the acyclic matching on the normalized Bar resolution is slightly different to the acyclic matching in Paragraph 1, and the resulting Morse complex will be isomorphic to the Anick resolution. These results were independently obtained by Sköldberg [**41**]. In addition to Sköldberg's results, we prove minimality of this resolution in special cases which give information about the Poincaré-Betti series, and we give an explicit description of the complex if the two-sided ideal $\mathfrak{a}$ admits a (finite) quadratic Gröbner basis, which proves a conjecture by Sturmfels [**42**].

Let $A = k\langle x_1, \ldots, x_n\rangle/\mathfrak{a}$ be the quotient algebra of the polynomial ring in n non-commuting indeterminates by a two-sided ideal

$$\mathfrak{a} \trianglelefteq k\langle x_1, \ldots, x_n\rangle.$$

As before, we assume that $\mathfrak{a} = \langle f_1, \ldots, f_s\rangle$, such that $\{f_1, \ldots, f_s\}$ is a finite reduced Gröbner basis with respect to a fixed degree-monomial order $\prec$. For an introduction to the theory of Gröbner basis in the non-commutative case, see [**32**].

Again, we have for the product of any two standard monomials a unique (Gröbner) representation of the form:

$$w \cdot v := \sum_i a_i\, w_i \text{ with } a_i \in k,\ w_i \in \mathcal{G}, \text{ and } |w \cdot v| \geq |w_i| \text{ for all } i,$$

where $\mathcal{G}$ is the corresponding set of standard monomials of degree ≥ 1 and $|m|$ is the total degree of the monomial m.

The acyclic matching on the normalized Bar resolution is defined as follows: As in the commutative case, we define $\mathcal{M}_j$ by induction on the coordinate $1 \leq j \leq n$: For $j = 1$ we set

$$\mathcal{M}_1 := \left\{ \begin{array}{c} [x_i|w_1'|w_2|\ldots|w_l] \\ \downarrow \\ [w_1|\ldots|w_l] \end{array} \in G(\mathsf{NB}_\bullet^A) \;\Big|\; w_1 = x_i w_1' \right\}.$$

The critical cells with respect to $\mathcal{M}_1$ are given by

(1) $\mathcal{B}_1^{\mathcal{M}_1} := \left\{ [x_i] \;\Big|\; 1 \leq i \leq n \right\}$, $l = 1$,

(2) $\mathcal{B}_l^{\mathcal{M}_1}$ is the set of all $[x_i|w_2|w_3|\ldots|w_l]$, $w_2, w_3, \ldots, w_l \in \mathcal{G}$, such that $x_i w_2$ is reducible.

Assume now $j \geq 2$ and $\mathcal{M}_{j-1}$ is defined. Let $\mathcal{B}^{\mathcal{M}_{j-1}}$ be the set of critical cells left after applying $\mathcal{M}_1 \cup \ldots \cup \mathcal{M}_{j-1}$.

Let $\mathcal{E}_j$ denote the set of edges in $G(\mathsf{NB}_\bullet^A)$ that connect critical cells in $\mathcal{B}^{\mathcal{M}_{j-1}}$.

The following condition on an edge in $\mathcal{E}_j$ will define the matching $\mathcal{M}_j$.

Definition 2.1 (Matching Condition)**.** Let

$$\begin{array}{c} [x_{i_1}|w_2|\ldots|w_{j-1}|u_1|u_2|w_{j+1}|\ldots|w_l] \\ \downarrow \\ [x_{i_1}|w_2|\ldots|w_{j-1}|w_j|w_{j+1}|\ldots|w_l] \end{array}$$

be an edge in $\mathcal{E}_j$. In particular, $w_j = u_1 u_2$. We say that the edge satisfies the matching condition if

(i) u_1 is a prefix of w_j,

(ii) $[x_{i_1}|w_2|\ldots|w_{j-1}|u_1|u_2|w_{j+1}|\ldots|w_l] \in \mathcal{B}^{\mathcal{M}_{j-1}}$,

(iii) $[x_{i_1}|w_2|\ldots|w_{j-1}|v_1|v_2|w_{j+1}|\ldots|w_l] \notin \mathcal{B}^{\mathcal{M}_{j-1}}$ for each prefix v_1 of u_1 and $v_1 v_2 = w_j$.

$$\mathcal{M}_j := \left\{ \begin{array}{c} [x_{i_1}|w_2|\ldots|w_{j-1}|u_1|u_2|w_{j+1}|\ldots|w_l] \\ \downarrow \\ [x_{i_1}|w_2|\ldots|w_{j-1}|w_j|w_{j+1}|\ldots|w_l] \end{array} \in \mathcal{E}_j \text{ satisfying 2.1} \right\}.$$

The set of critical cells $\mathcal{B}_l^{\mathcal{M}_j}$ in homological degree $l \geq 1$ is given by

(1) $\mathcal{B}_1^{\mathcal{M}_j} := \left\{ [x_i] \;\middle|\; 1 \leq i \leq n \right\}$,

(2) $\mathcal{B}_2^{\mathcal{M}_j}$ consists of elements $[x_{i_1}|w_2]$ with $x_{i_1}w_2 \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$,

(3) $\mathcal{B}_l^{\mathcal{M}_j}$ consists of elements $[x_{i_1}|w_2|w_3|\dots|w_l] \in \mathcal{B}_l^{\mathcal{M}_{j-1}}$ such that for each prefix u of w_j we have $[x_{i_1}|w_2|\dots|w_{j-1}|u|\dots|w_l] \notin \mathcal{B}_l^{\mathcal{M}_{j-1}}$ and w_jw_{j+1} is reducible.

We finally set $\mathcal{M} := \bigcup_{j\geq 1} \mathcal{M}_j$ and we write $\mathcal{B}^{\mathcal{M}}$ for the set of critical cells with respect to $\mathcal{M}$.

With the same proof as in Paragraph 1 we get

Lemma 2.2. *$\mathcal{M}$ defines an acyclic matching.* □

2.1. The Anick Resolution. As in the commutative case, we give a second description of the Morse complex with respect to the acyclic matching from Lemma 2.2. In this case, this description shows that it is isomorphic to the Anick resolution [**1**].

Definition 2.3. Let $m_{i_1}, \dots, m_{i_{l-1}} \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$ be monomials, such that for $j = 1, \dots, l-1$ we have $m_{i_j} = u_{i_j}v_{i_j}w_{i_j}$ with $u_{i_{j+1}} = w_{i_j}$ and $|u_{i_1}| = 1$. Then we call the l-tuple

$$[u_{i_1}|v_{i_1}w_{i_1}|v_{i_2}w_{i_2}|\dots|v_{i_{l-1}}w_{i_{l-1}}]$$

fully attached if for all $1 \leq i \leq l-2$ and each prefix u of $v_{i_{j+1}}w_{i_{j+1}}$ the monomial $v_{i_j}w_{i_j}u$ lies in $\mathcal{G}$. We write $\mathcal{B}_j := \{[w_1|\dots|w_j]\}$ for the set of fully attached j-tuples ($j \geq 2$) and $\mathcal{B}_1 := \{[x_1], \dots, [x_n]\}$.

We define the reduction types (Type I, Type II, and Type III) and the reduction coefficient $[e : f]$ for two fully attached tuples e, f in a similar way as in the commutative case (see Definition 1.5). Now we are able to define the following complex:

$$F_j := \bigoplus_{e \in \mathcal{B}_j} A\, e,$$

$$\begin{aligned} \partial : F_i &\to F_{i-1} \\ e &\mapsto \sum_{f \in \mathcal{B}_{i-1}} [e : f]\, f. \end{aligned}$$

Note that the basis elements of F_j are exactly the basis elements in the Anick resolution (see [**1**]), therefore, the complex $F_\bullet$ is isomorphic to the Anick resolution. Again, we have:

Theorem 2.4. *$(F_\bullet, \partial)$ is an A-free resolution of the residue field k over A. If no reduction of Type II is possible, the resolution $(F_\bullet, \partial)$ is minimal.*

Proof. The fully attached tuples are exactly the critical cells. The rest is analogous to the commutative case. □

If one applies Theorem 2.4 to the ideal $\langle x_i x_j - x_j x_i, \mathfrak{a}\rangle$, one reaches the commutative case. But in general, the Morse complex with respect to the acyclic matching from Lemma 2.2 is much larger (with respect to the rank) than the Morse complex of the acyclic matching developed in Paragraph 1 for commutative polynomial rings.

Since only by reductions of Type II coefficient $[e : f] \in k$ can enter the resolution, we have:

Proposition 2.5. *The following conditions are equivalent:*

(1) $(F_\bullet, \partial)$ *is not minimal.*

(2) *There exist standard monomials* $w_1, \ldots, w_4$ *and minimal generators* $m_{i_1}, m_{i_2}, m_{i_3} \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$*, such that* $w_1 w_2 = u_1 m_{i_1}$*,* $w_2 w_3 = u_2 m_{i_2}$*,* $w_1 w_4 = u_1' m_{i_3}$ *with* u_1, u_1' *suffixes of* w_1*,* u_2 *suffix of* w_2*, and* $w_2 w_3 \to w_4 \in \mathcal{R}$

Proof. $(F_\bullet, \partial)$ is minimal iff no reduction of Type II is possible, which is equivalent to the second condition. □

Corollary 2.6. *In the following two cases, the resolution* $(F_\bullet, \partial)$ *is a minimal* A*-free resolution of* k *and independent of the characteristic of* k*.*

(1) $\mathfrak{a}$ *admits a monomial Gröbner basis.*

(2) *The Gröbner basis of* $\mathfrak{a}$ *consists of homogeneous polynomials, all of the same degree.*

Proof. If the Gröbner basis consists of monomials, the situation of Proposition 2.5 is not possible. In the other case, there exists a constant l, such that for all $w \to v \in \mathcal{R}$ we have $|w| = |v| = l$. Assume there exist standard monomials $w_1, \ldots, w_4$ and minimal generators $m_{i_1}, m_{i_2}, m_{i_3} \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$, such that $w_1 w_2 = u_1 m_{i_1}$, $w_2 w_3 = u_2 m_{i_2}$, $w_1 w_4 = u_1' m_{i_3}$ with u_1, u_1' suffixes of w_1, u_2 suffix of w_2, and $w_2 w_3 \to w_4 \in \mathcal{R}$. Then we get $|w_i| < l$ for $i = 2, 3, 4$. On the other hand, we have $w_2 w_3 \to w_4 \in \mathcal{R}$ and therefore $|w_4| = l$. This is a contradiction. □

2.2. The Poincaré-Betti Series of k. In this section we draw some corollaries on the Poincaré-Betti series of k.

Recall the definition of a fully attached l-tuple: There exist leading monomials $m_{i_1}, \ldots, m_{i_{l-1}} \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$, such that for all $j = 1, \ldots, l-1$ there exist monomials $u_{i_j}, v_{i_j}, w_{i_j} \in \mathcal{G}$ with $m_{i_j} = u_{i_j} v_{i_j} w_{i_j}$ and $u_{i_{j+1}} = w_{i_j}$. It follows that the fully attached l-tuples are in one-to-one correspondence with $l-1$ chains of monomials $(m_{i_1}, \ldots, m_{i_{l-1}})$ with the condition before. We write again $\mathcal{B}$ for the set of all those chains. Now consider the subset

$$E := \left\{ (m_{i_1}, \ldots, m_{i_l}) \in \mathcal{B} \;\middle|\; m_{i_1}, \ldots, m_{i_l} \text{ pairwise different} \right\} \subset \mathcal{B}.$$

Since we consider only finite Gröbner bases, it is clear that E is finite. We construct a DFA (deterministic finite automaton, see for example [**31**]) over the alphabet E, which accepts $\mathcal{B}$. For each letter $f \in E$ we define a state f. Each state f is a final state. Let S be the initial state and Q be an error state. From the state S we go to state f if we read the letter $f \in E$. Let $f_1 = (m_{i_1}, \ldots, m_{i_l}), f_2 = (m'_{j_1}, \ldots, m'_{j_{l'}}) \in E$ be two chains of monomials with corresponding fully attached tuples $[w_{i_1}|\ldots|w_{i_{l+1}}]$ and $[w'_{j_1}|\ldots|w'_{j_{l'+1}}]$. Then we have $(f_1, f_2) \in \mathcal{B}$ iff there exists a monomial $n \in \mathrm{MinGen}(\mathrm{in}_\prec(\mathfrak{a}))$ with $n = uw'_{j_1}$ and u suffix of $w_{i_{l+1}}$. In this case, we change by reading f_2 from state f_1 to f_2. If such a monomial does not exist, we change by reading f_2 from state f_1 to the error state Q. The language of this DFA is exactly the set $\mathcal{B}$. This proves that the basis of our resolution $F_\bullet$ is a regular language. Since the word-counting function of a regular language is always a rational function (see [**31**]), we get in particular the following theorem:

Theorem 2.7. *For the Poincaré-Betti series of k over the ring A we have*

$$P_k^A(\underline{x}, t) \leq F(\underline{x}, t),$$

where $F(\underline{x}, t)$ is a rational function. Equality holds iff $F_\bullet$ is minimal. □

Corollary 2.8. *For the following two cases the Poincaré-Betti series of k over the ring A is a rational function:*

(1) $\mathfrak{a}$ *admits a Gröbner basis consisting of monomials.*
(2) *The Gröbner basis of $\mathfrak{a}$ consists of homogeneous polynomials, all of the same degree.*

Proof. The result is a direct consequence of the Theorem 2.7 and Corollary 2.6. □

Corollary 2.9. *If $\mathfrak{a}$ has a quadratic Gröbner basis, then $F_\bullet$ is an A-free minimal linear resolution. Hence A is Koszul and its Hilbert and Poincaré-Betti series are rational functions.* □

2.3. Examples. We finally want to give some examples of the Morse complex and we verify a conjecture by Sturmfels:

Example 2.10 (Conjecture of Sturmfels (see [**42**])). Let Λ be a graded sub-semigroup of $\mathbb{N}^d$ with n generators. We write its semigroup algebra over a field k as a quotient of the free associative algebra

$$k\langle y_1, y_2, \ldots, y_n\rangle / J_\Lambda \quad = \quad k[\Lambda].$$

Suppose that the two-sided ideal J_Λ possesses a quadratic Gröbner basis $\mathcal{G}$. The elements in the non-commutative Gröbner basis $\mathcal{G}$ are quadratic reduction relations of the form $y_i y_j \to y_{i'} y_{j'}$. If w and w' are words in $y_1, \ldots, y_n$, then we write $w \xrightarrow{j} w'$ if there exists a reduction sequence of length j from w to w'. A word $w = y_{i_1} y_{i_2} \cdots y_{i_r}$ is called *fully attached* if every quadratic subword $y_{i_j} y_{i_{j+1}}$ can be reduced with respect to $\mathcal{G}$. Let $\mathbf{F}_r$ be the free $k[\Lambda]$-module with basis $\{\, E_w : w \text{ fully attached word of length } r\}$. Let $\mathbf{F} = \bigoplus_{r \geq 0} \mathbf{F}_r$ and define a differential ∂ on $\mathbf{F}$ as follows:

$$\partial : \mathbf{F}_r \to \mathbf{F}_{r-1}\,, \quad E_w \quad \mapsto \quad \sum (-1)^j \, x_i \, E_{w'},$$

where the sum is over all fully attached words w' of length $r-1$ such that $w \xrightarrow{j} x_i \, w'$ for some i, j. Note that this sum includes the trivial reduction

$w \xrightarrow{0} w$. Then Theorem 2.4 together with Proposition 2.5 implies that $(\mathbf{F}, \partial)$ is a minimal free resolution of k over $k[\Lambda]$. □

Example 1 (The twisted cubic curve): The Gröbner basis consists of nine binomials:

$$\mathcal{G} = \left\{ \begin{array}{l} ac \to bb,\ ca \to bb,\ ad \to cb,\ da \to cb,\ bd \to cc, \\ db \to cc,\ ba \to ab,\ bc \to cb, dc \to cd \end{array} \right\}.$$

The minimal free resolution $(\mathbf{F}, \partial)$ has the format

$$\cdots\cdots \xrightarrow{\partial} k[\Lambda]^{72} \xrightarrow{\partial} k[\Lambda]^{36} \xrightarrow{\partial} k[\Lambda]^{18} \xrightarrow{\partial} k[\Lambda]^{9} \xrightarrow{\partial} k[\Lambda]^{4} \xrightarrow{\partial} k.$$

One of the 36 fully attached monomials of degree four is $adad$. It admits three relevant reductions $adad \xrightarrow{0} adad$, $adad \xrightarrow{1} cbad$ and $adad \xrightarrow{3} bbdb$. This implies

$$\partial(E_{adad}) \;=\; a \cdot E_{dad} \;-\; c \cdot E_{bad} \;-\; b \cdot E_{bdb}.$$

Example 2 (The Koszul complex): Let $\Lambda = \mathbb{N}^d$. The Gröbner basis $\mathcal{G}$ consists of the relations $y_i y_j \to y_j y_i$ for $1 \le j < i \le n$. A word w is fully attached if and only if $w = y_{i_1} y_{i_2} \cdots y_{i_r}$ for $i_1 > i_2 > \cdots > i_r$. In this case, $\partial(E_w) = \sum_{j=1}^{r} (-1)^{r-j} y_{i_j} E_{w_j}$ where $w_j = y_{i_1} \cdots y_{i_{j-1}} y_{i_{j+1}} \cdots y_{i_r}$. Hence $(\mathbf{F}, \partial)$ is the Koszul complex on n indeterminates.

Example 2.11 (The Cartan complex)**.** If A is the exterior algebra, then $F_\bullet$ with

$$F_i \;:=\; \bigoplus_{\substack{1 \le j_1 < \ldots < j_r \\ l_1, \ldots, l_r \in \mathbb{N} \\ i = \sum_{t=1}^{r} l_t}} A\, e_{i_1}^{(l_1)} \ldots e_{i_r}^{(l_r)}$$

$$e_{i_1}^{(l_1)} \ldots e_{i_r}^{(l_r)} \;\to\; \sum_{t=1}^{r} x_{i_t}\, e_{i_1}^{(l_1)} \ldots e_{i_t}^{(l_t - 1)} \ldots e_{i_r}^{(l_r)}$$

defines a minimal resolution of k as A-module, called the Cartan complex.

Proof. For the exterior algebra $A = k(x_1, \ldots, x_n)/\langle x_i x_j + x_j x_i \rangle$ the resolution $F_\bullet$ is by Corollary 2.6 minimal. The set of reduction rules is given by $\mathcal{R} := \{x_i^2 \to 0, x_i x_j \xrightarrow{-1} x_j x_i \text{ for } i < j\}$. Then the fully attached tuples are exactly the words

$$(x_{i_1}, \ldots, x_{i_1}, x_{i_2}, \ldots, x_{i_2}, \ldots, x_{i_r}, \ldots, x_{i_r}) \text{ with } 1 \le i_1 < \ldots < i_r \le n.$$

Since $x_i x_j$ is reduced to $-x_j x_i$, if $i \neq j$, and each reduction has factor (-1), we get for each reduction the coefficient $(-1)(-1) = 1$. Since $x_i x_i$ is reduced to 0, the differential follows. □

The following example shows that even in the case where the Gröbner basis is not finite one can apply our theory:

Example 2.12. Consider the two-side ideal $\mathfrak{a} = \langle x^2 - xy \rangle$. By [**32**] there does not exist a finite Gröbner basis with respect to degree-lex for $\mathfrak{a}$. One can show that $\mathfrak{a} = \langle xy^n x - xy^{n+1} \mid n \in \mathbb{N} \rangle$ and that $\{xy^n x - xy^{n+1} \mid n \in \mathbb{N}\}$ is an infinite Gröbner basis with respect to degree-lex.

If one applies our matching from Lemma 2.2, it is easy to see that the critical cells are given by tuples of the form

$$[x|y^{n_1}|x|y^{n_2}|x|\dots|x|y^{n_l}|x] \text{ and } [x|y^{n_1}|x|y^{n_2}|x|\dots|x|y^{n_l}]$$

with $n_1, \dots, n_l \in \mathbb{N}$.
A degree argument implies that the Morse complex is even a minimal resolution. Therefore, we get a minimal resolution $F_\bullet$ of k over $A = k\langle x_1, \dots x_n\rangle/\mathfrak{a}$.

In this case, this proves that k does not admit a linear resolution and hence A is not Koszul.

3. Application to the Acyclic Hochschild Complex

Now, let $A = k\langle x_1, \dots, x_n\rangle/\langle f_1, \dots, f_s\rangle$ be the non-commutative (resp. commutative) polynomial ring in n indeterminates divided by a two-sided ideal, where $f_1, \dots, f_s$ is a finite reduced Gröbner basis of $\mathfrak{a} = \langle f_1, \dots, f_s\rangle$ with respect to the degree-lex order. We now give an acyclic matching on the acyclic Hochschild complex, which is minimal in special cases. Let $\mathcal{G}$ be the set of standard monomials of degree ≥ 1 with respect to the degree-lex order. In this case, the normalized acyclic Hochschild complex is given by

$$HC_i \quad := \bigoplus_{w_1,\dots,w_i \in \mathcal{G}} (A \otimes A^{\mathrm{op}})\,[w_1|\dots|w_i]$$

with differential

$$\begin{aligned}\partial([w_1|\dots|w_i]) &:= (w_1 \otimes 1)\,[w_2|\dots|w_i] \\ &+(-1)^i(1\otimes w_i)[w_1|\dots|w_{i-1}] \\ &+\sum_{j=1}^{i-1}(-1)^j\left(\sum_r a_r[w_1|\dots|w_{j-1}|v_r^j|w_{j+2}|\dots|w_i]\right)\end{aligned}$$

if w_jw_{j+1} is reducible to $a_0 + \sum_r a_r v_r^j$ (if $w_jw_{j+1} \in \mathcal{G}$, we set $v_r^j = w_jw_{j+1}$).

We apply the same acyclic matching as in Paragraph 2 (resp. Paragraph 1).
Since in addition in this case the differential maps the element $[w_1|\dots|w_i]$ also to $(-1)^i(1\otimes w_i)[w_1|\dots|w_{i-1}]$ we have to modify the differential:
The reduction rules are the same as in Paragraph 1, except that the reduction coefficient in Definition 1.5 is $(c\otimes 1)$ instead of c. In order to define the coefficient, we say e can be reduced to f with coefficient c (we write $e \xrightarrow{c} f$), where $e = (w_1, \dots, w_l)$ and $f = (v_1, \dots, v_{l-1})$ are two fully attached l (resp. $l-1$)-tuples, if there exists a sequence of l-tuples $e = e_0, e_1, \dots, e_{r-1}$ such that either there exists:

(1) an l-tuple $e_r = (u, f)$, such that e_0 can be reduced to e_r with reductions of Type I and III, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r;$$

in this case we set $c := \left((-1)^r \prod_{i=1}^r a_i\right)(u\otimes 1)$, or

(2) an l-tuple $e_r = (f, u)$, such that e_0 can be reduced to e_r with reductions of Type I and III, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r;$$

in this case we set $c := \left((-1)^{r+k} \prod_{i=1}^{r} a_i\right) (1 \otimes u)$, or

(3) an l-tuple e_r, such that e_0 can be reduced to e_r with reductions of Type I and III and e_r can be reduced to f with a reduction of Type II, i.e.

$$e_0 \xrightarrow{-a_1} e_1 \xrightarrow{-a_2} e_2 \xrightarrow{-a_3} \dots \xrightarrow{-a_r} e_r \xrightarrow{(-1)^j b} f;$$

in this case we set $c := (-1)^{r+j}\, b \prod_{i=1}^{r} a_i$.

We define the reduction coefficient $[e : f]$ and the complex $F_\bullet$ as in Paragraph 2 (resp. Paragraph 1). With the same proof as in Paragraph 2 (resp. Paragraph 1) we obtain the following theorem:

Theorem 3.1. *$(F_\bullet, \partial)$ is a free resolution of A as an $A \otimes A^{\mathrm{op}}$-module. If no reduction of Type II is possible, then $(F_\bullet, \partial)$ is minimal.* □

Moreover, we get similar results to the results from Paragraph 1, 2 about minimality of $F_\bullet$ and rationality of the Poincaré-Betti series

$$P_k^{A \otimes A^{\mathrm{op}}}(\underline{x}, t) = \sum_{i,\alpha} \dim_k \left((\mathrm{Tor}_i^{(A \otimes A^{\mathrm{op}})}(k, A))_\alpha \right) \underline{x}^\alpha t^i$$

from Paragraph 2 (resp. Paragraph 1).
As in Paragraph 1 we can give an explicit description of the minimal resolution $F_\bullet$ in the following cases:

(1) $A = S/\langle f_1, \dots, f_s \rangle$, where $S = k[x_1, \dots, x_n]$ is the commutative polynomial ring in n indeterminates and f_i a reduced Gröbner basis with respect to the degree-lex order, such that the initial ideal is a complete intersection (note that in case $s = 1$ this resolution was first given by BACH (see [**9**])).

(2) $A = E$, where E is the exterior algebra.

Let $A = k[x_1, \dots, x_n]/\langle f_1, \dots, f_s \rangle$ be the commutative polynomial ring in n indeterminates with $f_i = x^{\gamma_i} + \sum_{\alpha_i \neq 0} f_{i,\alpha_i} x^{\alpha_i}$, $1 \leq i \leq s$, a reduced Gröbner basis with respect to the degree-lex order, such that x^{γ_i} is the leading term (since we start with the normalized Hochschild resolution, the condition $\alpha \neq 0$ is no restriction).

Let $\mathcal{G} = \{x^\alpha \mid x^\alpha \notin \langle x^{\gamma_1}, \dots, x^{\gamma_s} \rangle\}$ be the set of standard monomials of degree ≥ 1. We assume that the initial ideal $\mathrm{in}_\prec(\mathfrak{a}) = \langle x^{\gamma_1}, \dots, x^{\gamma_s} \rangle$ is a complete intersection. With the same arguments as in Theorem 1.10 it follows that $F_\bullet$ is minimal. We use the same notation as [**9**] and write

$$\begin{aligned} T(x_i) &= (x_i \otimes 1) - (1 \otimes x_i), \\ \frac{T_i(f)}{T(x_i)} &= \sum_{\alpha \in \mathbb{N}^n} f_\alpha \sum_{j=0}^{\alpha_i - 1} (x^{\alpha_1} \cdots x^{\alpha_{i-1}} x^j \otimes x^{\alpha_i - 1 - j} x^{\alpha_{i+1}} \cdots x^{\alpha_n}). \end{aligned}$$

Under these conditions, we get the following result:

Theorem 3.2. *Let $A = S/\langle f_1, \ldots, f_s\rangle$ such that the initial ideal $\mathrm{in}_\prec(\langle f_1, \ldots, f_s\rangle)$ is a complete intersection. Then the following complex is a multigraded minimal resolution of A as an $A \otimes A^{\mathrm{op}}$-module and carries the structure of a differential-graded algebra:*

$$F_i := \bigoplus_{\substack{1 \le i_1 < \ldots < i_r \le n \\ 1 \le j_1 < \ldots < j_q \le s \\ l_1, \ldots, l_q \in \mathbb{N} \\ i = r + 2\sum_{j=1}^{q} l_j}} A \otimes A^{\mathrm{op}}\, e_{i_r} \ldots e_{i_1} t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)},$$

$$e_{i_r} \ldots e_{i_1} \overset{\partial}{\mapsto} \sum_{m=1}^{r} (-1)^{\#\{i_j > i_m\}} T(x_{i_m})\, e_{i_r} \ldots \widehat{e_{i_m}} \ldots e_{i_1},$$

$$t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)} \overset{\partial}{\mapsto} \sum_{m=1}^{q} \sum_{p=1}^{n} \frac{T_p(f_{j_m})}{T(x_p)}\, e_p t_{j_1}^{(l_1)} \ldots t_{j_m}^{(l_{j_m}-1)} \ldots t_{j_q}^{(l_q)},$$

where $t_{i_j}^{(0)} := 1$, $e_i e_j = -e_j e_i$, and $e_i e_i = 0$. For the differential we have:

$$\begin{aligned} \partial(e_{i_r} \ldots e_{i_1} t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)}) &= \partial(e_{i_r} \ldots e_{i_1}) t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)} \\ &\quad + (-1)^r\, e_{i_r} \ldots e_{i_1} \partial(t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)}). \end{aligned}$$

Note that in case $A = S/\langle f\rangle$ this result was first obtained in [**9**] and our complex coincides with the complex given in [**9**].

Corollary 3.3. *Under the assumptions of Theorem 3.2 the Hilbert series of the Hochschild homology of A with coefficients in k has the form:*

$$\begin{aligned} \mathrm{Hilb}_{HH(A,k)}(\underline{x}, t) &= \sum_{i,\alpha} \dim_k \left((\mathrm{Tor}_i^{A \otimes A^{\mathrm{op}}}(k, A)_\alpha \right) x^\alpha\, t^i \\ &= \frac{\prod_{i=1}^{n} (1 + x_i\, t)}{\prod_{i=1}^{k} (1 - \underline{x}^{\gamma_i} t^2)}. \end{aligned}$$

If $\mathfrak{a}$ is the zero-ideal, we get with the same arguments the following special case:

Corollary 3.4. *Let $A = k[x_1, \ldots, x_n]$, then the following complex is a minimal resolution of A as an $A \otimes A^{\mathrm{op}}$-module:*

$$\begin{aligned} F_i &:= \bigoplus_{1 \le i_1 < \ldots < i_r \le n} A \otimes A^{\mathrm{op}}\, e_{i_1} \ldots e_{i_r}, \\ e_{i_1} \ldots e_{i_r} &\overset{\partial}{\mapsto} \sum_{m=1}^{r} (-1)^{\#\{i_j < i_m\}} T(x_{i_m})\, e_{i_1} \ldots \widehat{e_{i_m}} \ldots e_{i_r}. \end{aligned}$$

In particular, we have:

$$\begin{aligned} \mathrm{Hilb}_{HH(A,k)}(\underline{x}, t) &= \sum_{i,\alpha} \dim_k \left((\mathrm{Tor}_i^{A \otimes A^{\mathrm{op}}})(k, A)_\alpha \right) \underline{x}^\alpha\, t^i \\ &= \prod_{i=1}^{n} (1 + x_i\, t). \end{aligned}$$

Proof of Theorem 3.2. The description of the basis of F_i follows with exactly the same arguments as for the proof of Theorem 1.10. Since no constant term appears in the differential it suffices, to verify that the differential has the given form.

First, we consider a variable $[x_i]$. Clearly, it maps to $(x_i \otimes 1) - (1 \otimes x_i)$.

Next, we consider a minimal fully attached tuple $w_j = \left[x_{\gamma_i} \middle| \frac{x^{\gamma_i}}{x_{\gamma_i}}\right]$, where $x_\gamma := x_{m(x^\gamma)}$. Then we have:

$$\left[x_{\gamma_i} \middle| \frac{x^{\gamma_i}}{x_{\gamma_i}}\right] \to -\sum_\alpha f_{i\alpha}[x^\alpha] \to \sum_\alpha f_{i\alpha} \left[x_\alpha \middle| \frac{x^\alpha}{x_\alpha}\right].$$

As in the commutative case, the multi-index α decreases successively, but here $\left[x_\beta \middle| \frac{x^\alpha}{x_\beta x_\alpha}\right]$, for $x_\beta = x_{m(x^\beta)}$ with $x^\beta := \frac{x^\alpha}{x_\alpha}$, maps in addition to $\left(1 \otimes \frac{x^\alpha}{x_{a'} x_\alpha}\right)[x_{a'}]$, hence in this case we get:

$$\left[x_{\gamma_i} \middle| \frac{x^{\gamma_i}}{x_{\gamma_i}}\right] \to \sum_{j=1}^n \frac{T_j(f_i)}{T(x_j)} \; e_j.$$

For a fully attached tuple $[w_1|\ldots|w_l]$, we have to calculate the sign of the permutations. This calculation is similar to the calculation of the sign in the commutative case (see proof of Theorem 1.10) and is left to the reader.

With the bijection between the elements $e_{i_r} \ldots e_{i_1} t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)}$ and the fully attached tuples, we finally get the following differential:

$$\begin{aligned} &\partial(e_{i_r} \ldots e_{i_1} t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)}) \\ &= \sum_{m=1}^r (-1)^{\#\{i_j > i_m\}} T(x_{i_m})\, e_{i_r} \ldots \widehat{e_{i_m}} \ldots e_{i_1} t_{j_1}^{(l_1)} \ldots t_{j_q}^{(l_q)} \\ &\quad + \sum_{m=1}^q \sum_{\substack{p=1 \\ p \neq i_1, \ldots, i_r}}^n (-1)^r \, \frac{T_p(f_{j_m})}{T(x_p)} \; e_{i_r} \cdots e_{i_1} e_p t_{j_1}^{(l_1)} \ldots t_{j_m}^{(l_{j_m}-1)} \ldots t_{j_q}^{(l_q)}, \end{aligned}$$

and the desired result follows. □

We now consider the exterior algebra:

Theorem 3.5. *Let $E = k[x_1, \ldots, x_n]/\langle x_i^2, x_i x_j + x_j x_i\rangle$ be the exterior algebra. The following complex is a minimal resolution of E as $E \otimes E^{\mathrm{op}}$-module:*

$$F_i := \bigoplus_{\substack{1 \le i_1 < \ldots < i_r \le n \\ l_1, \ldots, l_r \in \mathbb{N}^n}} E \otimes E^{\mathrm{op}} \; e_{i_1}^{(l_1)} \ldots e_{i_r}^{(l_r)}$$

with

$$e_{i_1}^{(l_1)} \ldots e_{i_r}^{(l_r)} \overset{\mapsto}{\sum}{}_{j=1}^{r} \; (x_{i_j} \otimes 1) + (1 \otimes x_{i_j})\, e_{i_1}^{(l_1)} \ldots e_{i_j}^{(l_j - 1)} \ldots e_{i_r}^{(l_r)}.$$

In particular, we have :

$$\begin{aligned}\mathrm{Hilb}_{HH(E,k)}(\underline{x},t) &= \sum_{i,\alpha} \dim_k \left((\mathrm{Tor}_i^{E\otimes E}(k,E))_\alpha \right) \underline{x}^\alpha\, t^i \\ &= \prod_{i=1}^{n} \frac{1}{1-x_i\, t}.\end{aligned}$$

Let S be the commutative polynomial ring in n indeterminates, then we have the following duality:

$$\begin{aligned}\mathrm{Hilb}_{HH(E,k)}(\underline{x},t) &= \mathrm{Hilb}_S(\underline{x},t), \\ \mathrm{Hilb}_{HH(S,k)}(\underline{x},t) &= \mathrm{Hilb}_E(\underline{x},t).\end{aligned}$$

Proof. The proof is the same as in Example 2.11 from Paragraph 2, only with the modified differential. □

Chapter 6

The Multigraded Hilbert and Poincaré-Betti Series and the Golod Property

In this chapter, which is submitted under the title "On the Multigraded Hilbert and Poincaré-Betti Series and the Golod Property of Monomial Rings" (see [**35**]), we study the multigraded Hilbert and Poincaré-Betti series of algebras $A = S/\mathfrak{a}$, where S is the commutative polynomial ring in n indeterminates and $\mathfrak{a}$ is a monomial ideal with minimal monomial generating system $\mathrm{MinGen}(\mathfrak{a}) := \{\mathrm{m}_1, \ldots, \mathrm{m}_\mathrm{l}\}$.

Recall that the multigraded Poincaré-Betti series $P_k^A(\underline{x}, t)$ and $\mathrm{Hilb}_A(\underline{x}, t)$ of A are defined as

$$\begin{aligned} P_k^A(\underline{x}, t) &:= \sum_{i=0}^{\infty} \sum_{\alpha \in \mathbb{N}^n} \dim_k(Tor_i^A(k,k)_\alpha)\ \underline{x}^\alpha\ t^i, \\ \mathrm{Hilb}_A(\underline{x}, t) &:= \sum_{i=0}^{\infty} \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = i}} \dim_k(A_\alpha)\ \underline{x}^\alpha\ t^i. \end{aligned}$$

In [**13**] Charalambous and Reeves proved that in the case where the Taylor resolution of $\mathfrak{a}$ over S is minimal the Poincaré-Betti series takes the following form:

$$P_k^A(\underline{x}, t) = \frac{\prod_{i=1}^{n} (1 + x_i\ t)}{1 + \sum_{I \subset \{1, \ldots, l\}} (-1)^{cl(I)} m_I\ t^{cl(I) + |I|}},$$

where $cl(I)$ is the number of equivalence classes of I with respect to the relation defined as the transitive closure of $i \sim j :\Leftrightarrow \gcd(m_i, m_j) \neq 1$ and $m_I := \operatorname{lcm}(m_i \mid i \in I)$ is the least common multiple.
In the general case, they conjecture that

$$P_k^A(\underline{x}, t) = \frac{\prod_{i=1}^{n}(1 + x_i\, t)}{1 + \sum_{\substack{I \subset [l] \\ I \in U}} (-1)^{cl(I)} m_I\, t^{cl(I)+|I|}},$$

where $[l] = \{1, \dots, l\}$ and $U \subset 2^{[l]}$ is the "basis"-set. However, the conjecture does not include a description of the basis-set U.

Using our standard matching from Chapter 4, we are able to specify the basis-set U and prove the conjecture in several cases. In fact, we give a general conjecture about the multigraded minimal A-free resolution of k over A. This conjecture implies in these cases an explicit description of the multigraded Hilbert and Poincaré-Betti series, hence it implies the conjecture by Charalambous and Reeves.

In Paragraph 1 we formulate our conjecture on the multigraded minimal resolution of k as an A-module and we show that our conjecture gives an explicit form of the multigraded Hilbert and Poincaré-Betti series. This generalizes the conjecture by Charalambous and Reeves. We say that an algebra A has property **(P)** (resp. **(H)**) if the multigraded Poincaré-Betti series (resp. multigraded Hilbert series) has the conjectured form.

In Paragraph 2 we give a description of the Koszul homology $H_\bullet(K^A)$ of the Koszul complex over A with respect to the sequence $x_1, \dots, x_n$ in terms of a standard matching on the Taylor resolution. We need this description later in the proof of our conjecture.

In Paragraph 3 we prove that the Stanley Reisner ring $A = k[\Delta]$, where $\Delta = \Delta(P)$ is the order complex of a partially ordered set P, satisfies property **(P)** and property **(H)**.

In the first section of Paragraph 4 we prove our conjecture for algebras for which $H_\bullet(K^A)$ is an M-ring, a notion introduced by Fröberg [**23**]. Using a theorem of Fröberg, we also prove property **(P)** for algebras $A = S/\mathfrak{a}$ for which in addition the minimal free resolution of $\mathfrak{a}$ carries the structure of a differential-graded algebra. In the second part we prove our conjecture for all Koszul algebras A. Note that this, as a particular case, gives another proof that $A = k[\Delta]$ satisfies property **(P)** and **(H)**.
Finally, we explain why our conjecture makes sense in general. We generalize the Massey operation in order to get an explicit description of the Eagon complex. On this complex we define an acyclic matching. If the resulting Morse complex is minimal, one has to find an isomorphism to the conjectured complex. We give some ideas on how to construct this isomorphism. This construction justifies our conjecture.

Since an algebra is Golod if and only if

$$P_k^A(\underline{x},t) = \frac{\displaystyle\prod_{i=1}^{n}(1+x_i\ t)}{1-t\displaystyle\sum_{\beta_{\alpha,i}\neq 0}\beta_{\alpha,i}\underline{x}^{\alpha}\ t^i},$$

where $\beta_{i,\alpha} := \dim_k\left(\mathrm{Tor}_i^S(A,k)_\alpha\right)$, we can give some applications to the Golod property of monomial rings in the last paragraph of this chapter. We prove, under the assumption of property **(P)**, that A is Golod if and only if the first Massey operation is trivial. In addition we give, again under the assumption of property **(P)**, a very simple, purely combinatorial condition on the minimal monomial generating system MinGen($\mathfrak{a}$) which implies Golodness. We conjecture that this is an equivalence. This would imply that, in the monomial case, Golodness is independent of the characteristic of the residue class field k.

Recently, Charalambous proved in [**14**] that if

$$P_k^A(\underline{x},t) = \frac{\displaystyle\prod_{i=1}^{n}(1+x_i\ t)}{Q_R(\underline{x},t)} \text{ with } Q_R(\underline{x},t) = \sum\Big(\sum_{\alpha} c_\alpha \underline{x}^{\alpha}\Big)t^i,$$

then $\underline{x}^\alpha$ equals to a least common multiple of a subset of the minimal monomial generating system MinGen($\mathfrak{a}$). However an explicit form of $Q_R(\underline{x},t)$ in terms of subsets of MinGen($\mathfrak{a}$) is still not known.
In addition, Charalambous proves a new criterion for generic ideals to be Golod. In Section 5 we reprove this criterion using our approach.

In another recent paper, Berglund gives an explicit form of the denominator $Q_R(\underline{x},t)$ in terms of the homology of certain simplicial complexes. Since there seems to be no obvious connection of the approach taken in [**8**] and our approach, it is an interesting problem to link these two methods.

1. The Multigraded Hilbert and Poincaré-Betti Series

Let $\mathfrak{a} \trianglelefteq S$ be a monomial ideal and $\mathcal{M} = \mathcal{M}_1 \cup \bigcup_{i\geq 2}\mathcal{M}_i$ a standard matching on the Taylor resolution. We introduce a new non-commutative polynomial ring $\tilde{R}$, defined by

$$\tilde{R} := k\langle Y_I \text{ for } \mathrm{MinGen}(\mathfrak{a}) \supset \mathrm{I} \notin \mathcal{M}_1 \text{ and } \mathrm{cl(I)} = 1\rangle.$$

On this ring, we define three gradings:

$$\begin{aligned} |Y_I| &:= |I|+1, \\ \deg(Y_I) &:= \alpha, \text{ with } \underline{x}^\alpha = m_I, \\ \deg_t(Y_I) &:= ||\alpha||, \text{ with } \underline{x}^\alpha = m_I, \end{aligned}$$

where $||\alpha|| = \sum_i \alpha_i$ is the absolute value of α. This makes $\tilde{R}$ into a multigraded ring:

$$\tilde{R} = \bigoplus_{\alpha\in\mathbb{N}^n}\bigoplus_{i\geq 0}\tilde{R}_{i,\alpha}$$

with $\tilde{R}_{i,\alpha} := \left\{u \in \tilde{R}\ \middle|\ \deg(u) = \alpha \text{ and } |u| = i\right\}$.

Let $[Y_I, Y_J] := Y_I Y_J - (-1)^{|Y_I||Y_J|} Y_J Y_I$ be the graded commutator of Y_I and Y_J. We define the following multigraded two-sided ideal

$$\mathfrak{r} := \langle [Y_I, Y_J] \text{ for } \gcd(m_I, m_J) = 1 \rangle,$$

and set

$$R := \tilde{R}/\mathfrak{r}.$$

Let $\operatorname{Hilb}_R(\underline{x}, t, z) := \sum_{\alpha \in \mathbb{N}^n} \sum_{i \geq 0} \dim_k(R_{i,\alpha})\ \underline{x}^\alpha\ t^{||\alpha||}\ z^i$ be the multigraded Hilbert series of R. We have the following fact:

Proposition 1.1. *The multigraded Hilbert series* $\operatorname{Hilb}_R(\underline{x}, t, z)$ *of* R *is given by*

$$\operatorname{Hilb}_R(\underline{x}, t, z) = \frac{1}{1 + \sum\limits_{\substack{I \subset \mathrm{MinGen}(\mathfrak{a}) \\ I \notin \mathcal{M}_1}} (-1)^{cl(I)}\ m_I\ t^{m_I}\ z^{cl(I)+|I|}},$$

where $t^{m_I} := t^{|\alpha|}$ *with* $\underline{x}^\alpha = m_I$.

Proof. In [**12**], Cartier and Foata prove that the Hilbert series of an arbitrary non-commutative polynomial ring divided by an ideal, which is generated by some (graded) commutators, is given by

$$\operatorname{Hilb}_R(\underline{x}, t, z) := \frac{1}{1 + \sum\limits_F (-1)^{|F|}\ \underline{x}^{\deg(y_F)}\ t^{\deg_t(y_F)}\ z^{|Y_F|}},$$

where $F \subset \{Y_I \text{ with } I \notin \mathcal{M}_1, cl(I) = 1\}$ is a commutative part (i.e. $Y_I Y_J = (-1)^{|J||I|} Y_J Y_I$ for all $Y_I, Y_J \in F$) and $Y_F = \prod_{Y_I \in F} Y_I$.
Therefore, we only have to calculate the commutative parts. Since $\mathfrak{r}$ is generated by the relations $Y_I Y_J = (-1)^{|J||I|} Y_J Y_I$, if $\gcd(m_I, m_J) = 1$, we see that the commutative parts are given by

$$F := \left\{ Y_{I_{i_1}}, \ldots, Y_{I_{i_r}} \;\middle|\; \gcd(m_{I_{i_j}}, m_{I_{i_{j'}}}) = 1 \text{ for all } j \neq j' \right\}.$$

But the fact that $Y_{I_{i_1}}, \ldots, Y_{I_{i_r}}$ is a commutative part is equivalent to $I_{i_1} \cup \ldots \cup I_{i_r} \notin \mathcal{M}_1$. Therefore, we can identify the commutative parts F with the elements $I \notin \mathcal{M}_1$ and sum over all $I \notin \mathcal{M}_1$. It is clear that the cardinality of a commutative part equals to the number $cl(I)$. If $I = I_1 \dot\cup \ldots \dot\cup I_r$ with $cl(I_j) = 1$ is a commutative part, it follows that $Y_I = Y_{I_1} \cdots Y_{I_r}$, which implies the exponents of $t, z, \underline{x}$. □

We formulate the following conjecture:

Conjecture 1.2. *Let* $F_\bullet$ *be a multigraded minimal* A*-free resolution of* k *as an* A*-module with* $F_i := \bigoplus_{\alpha \in \mathbb{N}^n} A(-\alpha)^{\beta_{i,\alpha}}$ *for* $i \geq 0$*. Then we have the following isomorphism as* k*-vectorspaces:*

$$F_i \cong \bigoplus_{\substack{J \subset \{1,\ldots,n\} \\ |J| = l}} \bigoplus_{\substack{u \in \mathcal{G}(R) \\ |u| = i-l}} A\big(-(\alpha_J + \deg(u))\big),$$

where $\mathcal{G}(R)$ *is the set of monomials in* R *and* α_J *is the characteristic vector of* J*, defined by*

$$(\alpha_J)_i = \begin{cases} 0 & , \quad i \notin J, \\ 1 & , \quad i \in J. \end{cases}$$

This conjecture gives a precise formulation of the conjecture by Charalambous and Reeves on the multigraded Poincaré-Betti series. In addition, we get an explicit form of the multigraded Hilbert series of $S/\mathfrak{a}$ for monomial ideals $\mathfrak{a}$.

Proposition 1.3. *Let $A = S/\mathfrak{a}$ be the quotient of the commutative polynomial ring by a monomial ideal $\mathfrak{a}$, and let $\mathcal{M} := \mathcal{M}_1 \cup \bigcup_{i\geq 2} \mathcal{M}_i$ be a standard matching on the Taylor resolution. If Conjecture 1.2 holds, then the multigraded Poincaré-Betti and Hilbert series have the following form:*

$$\begin{aligned} P_k^A(\underline{x},t) &= \prod_{i=1}^{n}(1+x_i\ t)\ \mathrm{Hilb}_R(\underline{x},1,t) \\ &= \frac{\prod_{i=1}^{n}(1+x_i\ t)}{1+\sum_{\substack{I\subset \mathrm{MinGen}(\mathfrak{a})\\ \mathrm{I}\notin\mathcal{M}_1}}(-1)^{cl(I)}\ m_I\ t^{cl(I)+|I|}}, \end{aligned} \tag{1.1}$$

$$\begin{aligned} \mathrm{Hilb}_A(\underline{x},t) &= \Big(\prod_{i=1}^{n}(1-x_i\ t)\ \mathrm{Hilb}_R(\underline{x},t,-1)\Big)^{-1} \\ &= \frac{1+\sum_{\substack{I\subset \mathrm{MinGen}(\mathfrak{a})\\ \mathrm{I}\notin\mathcal{M}_1}}(-1)^{|I|}\ m_I\ t^{m_I}}{\prod_{i=1}^{n}(1-x_i\ t)}. \end{aligned} \tag{1.2}$$

Note that Equation (1.1) is a reformulation of the conjecture by Charalambous and Reeves.

Proof. The form of the Poincaré-Betti series follows directly from the conjecture, by counting basis elements of F_i.
For the Hilbert series we consider the complex $F_\bullet \to k \to 0$, which is exact since $F_\bullet$ is a minimal free resolution of k. Since the Hilbert series of k is 1, the Euler characteristic implies:

$$\sum_{i\geq 0}(-1)^i\ \mathrm{Hilb}_{F_i}(\underline{x},t) = 1.$$

Conjecture 1.2 implies

$$\mathrm{Hilb}_{F_i}(\underline{x},t) = \sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=l}}\ \sum_{\substack{u\in R\\ |u|=i-l}} \underline{x}^{\alpha_J}\ t^{|J|}\ \underline{x}^{\deg(u)}\ t^{\deg_t(u)}\ \mathrm{Hilb}_A(\underline{x},t).$$

The Cauchy product finally implies:

$$\begin{aligned} \sum_{i\geq 0}(-1)^i\,\mathrm{Hilb}_{F_i}(\underline{x},t) &= \mathrm{Hilb}_A(\underline{x},t)\sum_{i\geq 0}\sum_{\substack{J\subset\{1,\ldots,n\}\\ |J|=l}}(-1)^l\ \underline{x}^{\alpha_J}\ t^{|J|} \\ &\qquad \sum_{\substack{u\in R\\ |u|=i-l}}(-1)^{i-l}\ \underline{x}^{\deg(u)}\ t^{\deg_t(u)} \\ &= \mathrm{Hilb}_A(\underline{x},t)\left(\sum_{J\subset\{1,\ldots,n\}}\underline{x}^{\alpha_J}\ (-t)^{|J|}\right) \end{aligned}$$

$$\left(\sum_{u\in R} \underline{x}^{\deg(u)}\ t^{\deg_t(u)}\ (-1)^{|u|}\right)$$
$$= \quad \mathrm{Hilb}_A(\underline{x},t)\ \prod_{i=1}^{n}(1-t\ x_i)\ \mathrm{Hilb}_R(\underline{x},t,-1).$$

□

It is known that if A is Koszul, then $\mathrm{Hilb}_A(\underline{x},t) = 1/P_k^A(\underline{x},-t)$. In our case, this means:

Proposition 1.4. *If A is Koszul, then* $\mathrm{Hilb}_R(\underline{x},t,-1) = \mathrm{Hilb}_R(\underline{x},1,-t)$.

Proof. In the monomial case, the Koszul property is equivalent to the fact that $\mathfrak{a}$ is generated in degree two. We prove that a subset $I \in \mathrm{MinGen}(\mathfrak{a})$ which is not matched by $\mathcal{M}_1$ satisfies $cl(I) + |I| = \deg_t(Y_I)$. It is clear that this proves the assertion.
It is enough to prove it for subsets $I \subset \mathrm{MinGen}(\mathfrak{a})$ with $cl(I) = 1$. Let $m_I = \underline{x}^{\alpha}$ be the least common multiple of the generators in I. Since all generators have degree two, it follows $||\alpha|| \leq 2 + |I| - 1 = |I| + 1 = |I| + cl(I)$. Since $\mathrm{Tor}_i^S(S/\mathfrak{a},k)_i = 0$, we get $||\alpha|| = |I| + 1 = |I| + cl(I)$. □

We introduce some notation for rings A satisfying the consequences of Conjecture 1.2.

Definition 1.5. We say that A has property

(P) if $P_k^A(\underline{x},t) = \prod_{i=1}^n (1 + x_i\ t)\ \mathrm{Hilb}_R(\underline{x},1,t)$ and has property

(H) if $\mathrm{Hilb}_A(\underline{x},t) = \left(\prod_{i=1}^n (1 - x_i\ t)\ \mathrm{Hilb}_R(\underline{x},t,-1)\right)^{-1}$.

2. The Homology of the Koszul Complex K^A

Let $\mathcal{M}$ be a standard matching on the Taylor resolution of $\mathfrak{a}$. The basis of the k-vectorspace $T_\bullet^{\mathcal{M}} \otimes_S k$ is then given by the sets $I \subset \mathrm{MinGen}(\mathfrak{a})$ with $I \notin \mathcal{M}$.

We denote with $K_\bullet^A$ the Koszul complex of A with respect to the sequence $x_1,\ldots,x_n$, i.e.

$$K_i := \bigoplus_{\{j_1<\ldots<j_i\}} A\ e_{\{j_1<\ldots<j_i\}}$$

with differential

$$\partial_i : \begin{cases} K_i & \to & K_{i-1} \\ e_{\{j_1<\ldots<j_i\}} & \mapsto & \sum_{l=1}^i (-1)^{l+1}\ x_{j_l}\ e_{\{j_1<\ldots<j_{l-1}<j_{l+1}<\ldots j_i\}} \end{cases}$$

We denote further by $Z(K_\bullet)$ (resp. $B(K_\bullet)$) the set of cycles (resp. boundaries) of the complex $K_\bullet$. Finally, we denote with $H(K_\bullet)$ the homology of the Koszul complex.

Proposition 2.1. *If $\mathcal{M}$ is a standard matching, then there exists a homogeneous homomorphism*

$$\phi : \begin{cases} T_\bullet^{\mathcal{M}} \otimes_S k & \to & K_\bullet^A \\ I & \mapsto & \phi(I) \end{cases}$$

such that for all $I, J \notin \mathcal{M}$ with $\gcd(m_I, m_J) = 1$ *we have*

(1) $\phi(I)$ *is a cycle,*
(2) $\phi(I)\phi(J) = \phi(I \cup J)$ *if* $I \cup J \notin \mathcal{M}$,
(3) *if* $I \cup J \in \mathcal{M}$,

$$\phi(I)\phi(J) = \partial(c) + \sum_{\substack{L \notin \mathcal{M} \\ cl(L) \geq cl(I)+cl(J)}} a_L \phi(L) \qquad \text{for some } a_L \in k,$$

for some $c \in K_\bullet^A$.

Note that $\phi(I)\phi(J) \in B(K_\bullet)$ *might happen if all coefficients* a_L *are zero.*

Proof. We consider the following double complex:

$$\begin{array}{ccccccccccc}
 & & 0 & & & & 0 & & 0 & & \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & T_n^{\mathcal{M}} \otimes_S k & \to & \dots & \to & T_0^{\mathcal{M}} \otimes_S k & \to & S/I \otimes_S k & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & T_n^{\mathcal{M}} \otimes_S K_0^S & \to & \dots & \to & T_0^{\mathcal{M}} \otimes_S K_0^S & \to & S/I \otimes_S K_0^S & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & \vdots & \to & \dots & \to & \vdots & \to & \vdots & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
0 & \to & T_n^{\mathcal{M}} \otimes_S K_n^S & \to & \dots & \to & T_0^{\mathcal{M}} \otimes_S K_n^S & \to & S/I \otimes_S K_n^S & \to & 0 \\
 & & \uparrow & & & & \uparrow & & \uparrow & & \\
 & & 0 & & & & 0 & & 0 & &
\end{array}$$

Since every row and every column, except the first row and the right column, are exact, we get by diagram chasing a homogeneous homomorphism

$$\phi : \begin{cases} T_\bullet^{\mathcal{M}} \otimes_S k & \to & K_\bullet \\ I & \mapsto & \phi(I). \end{cases}$$

By construction it is clear that $\phi(I)$ is a cycle. The second condition of a standard matching is: if $(I \to J) \in \mathcal{M}$, then $(I \cup K \to J \cup K) \in \mathcal{M}$ for all K with $\gcd(m_K, m_I) = 1$. This condition implies that one can chose the homomorphism ϕ such that $\phi(I)\phi(J) = \phi(I \cup J)$ if $I \cup J \notin \mathcal{M}$.
Now let $I \cup J \in \mathcal{M}$. Since $I, J \notin \mathcal{M}$, it follows from the standard matching that $I \cup J$ is matched with a set $\hat{I}$ of higher homological degree. We now consider $\mathcal{M}' := \mathcal{M} \setminus \{\hat{I} \to I \cup J\}$. We then have

$$0 = \partial^{\mathcal{M}'} \partial^{\mathcal{M}'}(\hat{I}).$$

Hence we get:

$$\partial^{\mathcal{M}'}(I \cup J) = \sum_{L \notin \mathcal{M}} a_L \partial^{\mathcal{M}}(L).$$

Since we take the tensor product $\otimes_S k$ with k, all summands with $a_L \notin k$ cancel out. Hence $\phi(I)\phi(J) \in B(K_\bullet^A)$ or, again with diagram chasing:

$$\phi(I)\phi(J) - \sum_{\substack{L \notin \mathcal{M} \\ cl(L) \geq cl(I)+cl(J)}} a_L \phi(L) \in B(K_\bullet^A).$$

From the construction of the standard matching it follows, in addition, that $cl(L) \geq cl(I) + cl(J)$ (otherwise L would have been matched before). □

We define the following new k-algebra:
For each $I \notin \mathcal{M}$ with $cl(I) = 1$ we define one indeterminate Y_I with total degree $\deg_t(Y_I) := |I|$ and multidegree $\deg_m(Y_I) := x^\alpha$, if $x^\alpha = m_I$. Let $R' := k(Y_I, I \notin \mathcal{M}, cl(I) = 1)/\mathfrak{r}'$ be the quotient algebra of the graded commutative polynomial ring $k(Y_I, I \notin \mathcal{M}, cl(I) = 1)$ (i.e. $Y_I Y_J = (-1)^{|I||J|} Y_J Y_I$) and the multigraded ideal $\mathfrak{r}'$ that is generated by the relations given by Proposition 2.1, i.e.:

(1) $Y_I Y_J = 0$ if $\gcd(m_I, m_J) \neq 1$,
(2) $Y_{I_{i_1}} \cdots Y_{I_{i_r}} = \sum a_L Y_L$ if $\phi(I_{i_1}) \cdots \phi(I_{i_r}) = \sum a_L \phi(L) + \mathit{boundary}$,
(3) $Y_{I_{i_1}} \cdots Y_{I_{i_r}} = 0$ if $[\phi(I_{i_1}) \cdots \phi(I_{i_r})] = 0$.

Theorem 2.2. *If $\mathcal{M}$ is a standard matching, then R' is isomorphic to $H(K_\bullet)$.*

Proof. The isomorphism is given by Proposition 2.1. We only have to prove that $[\phi(I)][\phi(J)] = 0$ if $\gcd(m_I, m_J) \neq 1$. This follows from the next lemma and the next corollary. □

Lemma 2.3. *Let $c = \sum_I \alpha_I \frac{m}{x_I} e_I$ be a homogeneous cycle with multidegree $\deg(c) = m$. We fix an $x_0 \mid m$. Then there exists a cycle $c' = \sum_{I'} \alpha_{I'} \frac{m}{x_{I'}} e_{I'}$, homologic to c, such that $x_0 \mid x_{I'}$ for all I'.*

Proof. Let I be an index set such that $\alpha_I \neq 0$ in the expansion of c with $x_0 \nmid x_I$. Then

$$\text{(2.1)} \quad \frac{m}{x_I} e_I = \sum_{i \in I} (-1)^{pos(i)+1} \frac{m\, x_i}{x_0\, x_I} e_{x_0} \wedge e_{I \setminus \{i\}} + \partial \left(\frac{m_I}{x_0\, x_I} e_{x_0} \wedge e_I \right).$$

If we replace each index set I with respect to (2.1), we finally reach a cycle c' with the desired property. By construction there exists an element d with $c - c' = \partial(d) \in B(K_\bullet)$. □

Corollary 2.4. *Let c_1, c_2 be two homogeneous cycles with multidegrees $\deg(c_1) = m$ and $\deg(c_2) = n$. If $\gcd(m, n) \neq 1$, we have $[c_1][c_2] = 0$.*

Proof. Let $c_1 := \sum_I \alpha_I \frac{m}{x_I} e_I$ and $c_2 := \sum_J \beta_J \frac{n}{x_J} e_J$ with $\gcd(m, n) \neq 1$. We fix a $j \in \mathrm{supp}(\gcd(m, n))$. By Lemma 2.3 we can assume that $j \in I \cap J$ for all I, J. This implies $[c_1][c_2] = 0$. □

Corollary 2.5. *$H(K_\bullet)$ is generated by $I \notin \mathcal{M}$ with $cl(I) = 1$.*

3. Hilbert and Poincaré-Betti Series of the Algebra $A = k[\Delta]$

In this paragraph we prove property **(P)** and **(H)** for $A = S/\mathfrak{a}$ where $\mathfrak{a} = I_{\Delta(P)}$ is the Stanley Reisner ideal of the order complex $\Delta(P)$ of a partially ordered set P.

Let $P := (\{1, \ldots, n\}, \prec)$ be a partially ordered set, where $i \prec j$ implies $i < j$. The Stanley Reisner ring of the order complex $\Delta = \Delta(P)$ is given by

$$A := k[\Delta] = k[x_i,\, i \in P]/\langle x_i x_j \text{ with } i < j \text{ and } i \not\prec j \rangle.$$

We now define a sequence of regular languages L_i over the alphabet $\Gamma_i := \{x_i, \ldots, x_n\}$:

(1) $x_i x_j \in L_i$ for all $i < j$ and $i \not\prec j$,

(2) $x_i x_{j_1} \cdots x_{j_l} \in L_i$ if $x_i x_{j_1} \cdots x_{j_{l-1}} \in L_i$ and $i < j_r$ for all $r = 1, \dots l$ and either

(a) $j_{l-1} \not\prec j_l$ or

(b) $x_i x_{j_1} \cdots x_{j_{l-2}} x_{j_l} \in L_i$ and $j_l < j_{l-1}$.

Let $f_i(x,t) := \sum_{w \in L_i} t^{|w|}\, w$ be the word counting function of L_i.

Corollary 1.8 and Corollary 1.9 of Chapter 5 imply the following theorem:

Theorem 3.1. *The Poincaré-Betti series of A is given by:*

$$P_k^A(\underline{x}, t) := \prod_{i=1}^{n} (1 + t\, x_i) \quad \prod_{i=1}^{n} (1 + F_i(x,t)) = \prod_{i=1}^{n} \frac{1 + t\, x_i}{1 - f_i(x,t)},$$

where $F_i(x,t) := \frac{f_i}{1 - f_i(x,t)}$.

We only have to calculate the word counting functions f_i. Since the language L_n is empty, it follows that $f_n := 0$. We construct recursively non-deterministic finite automata A_i such that the language $L(A_i)$ accepted by A_i is L_i (for the basic facts on deterministic finite automata we use here [**31**]). We assume that A_j is defined for all $j > i$. Let A_j^+ be the automaton which accepts the language $L_j^+ \cup \{w\, x_j \text{ with } w \in L_j^*\}$, where

$$\begin{aligned} L^+ &:= \{w_1 \circ \ldots \circ w_i \mid i \in \mathbb{N} \setminus \{0\} \text{ and } w_j \in L,\ j = 1, \ldots, i\}, \\ L^* &:= L^+ \cup \{\varepsilon\} = \{w_1 \circ \ldots \circ w_i \mid i \in \mathbb{N} \text{ and } w_j \in L,\ j = 1, \ldots, i\}, \end{aligned}$$

where $\circ$ denotes the concatenation and ε is the empty word. It follows that the word counting function of $L(A_j^+)$ is given by $\frac{f_j + t\, x_j}{1 - f_j}$.
We now construct A_i:

- ▷ From the starting state we go to the state i if we read the letter x_i, otherwise we reject the input word.
- ▷ From the state i we can switch by reading the empty word to the state j, which represents the automaton A_j^+, if $i < j$ and $i \not\prec j$. We then accept if A_j^+ accepts.
- ▷ Now assume we have the transitions $i \to j_1$ and $i \to j_2$ with $j_1 < j_2$. Because of condition (2b) we can switch by reading the empty word from state j_2 to state j_1.
- ▷ Assume that we have the transition $i \to j_2$ and we do not have the transition $i \to j_1$, with $j_1 < j_2$. This means $i \prec j_1$ and $i \not\prec j_2$. Therefore, we must have $j_1 \prec j_2$, otherwise we get a contradiction to the transitivity of the order in P. It follows by condition (1) that we can switch by reading the empty word from state j_2 to j_1.

It is clear that A_i accepts the language L_i. Since the state j represents the automaton A_j^+, we get a recursion for the word counting functions:

Lemma 3.2. *For the word counting functions f_i we get the following recursion:*

$$f_n := 0,$$

$$f_i \quad := \quad t\,x_i \sum_{\substack{i<j \\ i\not\to j}} \frac{f_j + t\,x_j}{1-f_j} \prod_{r=i+1}^{j-1} \frac{1+t\,x_j}{1-f_j}.$$

Proof. The state j represents the automaton A_j^+ with word counting function $\frac{f_j+t\,x_j}{1-f_j}$. By the argumentation above we have $j \to \nu$ for all $\nu = i+1,\ldots,j-1$ if we have $i \to j$. Since we accept when the automaton A_j^+ accepts, we get the desired recursion. □

By standard facts on regular languages the functions f_i are rational functions, but we want to have an expression of the Poincaré-Betti series by polynomials:

Lemma 3.3. *For the rational functions f_i we have:*

$$f_i := \frac{w_i}{1 - \sum_{r=i+1}^{n} w_r},$$

where w_i are polynomials and $w_n = 0$.

Proof. We prove it by induction: w_n is a polynomial and we have $f_n = \frac{w_n}{1-0}$. We now assume that f_j satisfies the desired condition for all $j > i$. Then

$$\begin{aligned}
f_i = t\,x_i & \sum_{\substack{i<j \\ x_ix_j\in\mathfrak{a}}} \frac{t\,x_j + f_j}{1-f_j} \prod_{r=i+1}^{j-1} \frac{1+t\,x_r}{1-f_r} \\
= \quad t\,x_i & \sum_{\substack{i<j \\ x_ix_j\in\mathfrak{a}}} \frac{t\,x_j + \frac{w_j}{1-\sum_{r>j} w_r}}{1 - \frac{w_j}{1-\sum_{r>j} w_r}} \prod_{r=i+1}^{j-1} \frac{1+t\,x_r}{1-\frac{w_r}{1-\sum_{l>r} w_l}} \\
= \quad t\,x_i & \sum_{\substack{i<j \\ x_ix_j\in\mathfrak{a}}} \frac{t\,x_j\left(1-\sum_{r>j} w_r\right) + w_j}{1-\sum_{r\geq j} w_r} \left(\prod_{r=i+1}^{j-1} (1+t\,x_r)\right) \left(\prod_{r=i+1}^{j-1} \frac{1-\sum_{l>r} w_l}{1-\sum_{l\geq r} w_l}\right) \\
= \quad t\,x_i & \sum_{\substack{i<j \\ x_ix_j\in\mathfrak{a}}} \frac{t\,x_j\left(1-\sum_{r>j} w_r\right) + w_j}{1-\sum_{r\geq j} w_r} \left(\prod_{r=i+1}^{j-1} (1+t\,x_r)\right) \frac{1-\sum_{l>j-1} w_l}{1-\sum_{l\geq i+1} w_l} \\
= \quad t\,x_i & \sum_{\substack{i<j \\ x_ix_j\in\mathfrak{a}}} \left(w_j + t\,x_j - t\,x_j \sum_{r>j} w_r\right) \left(\prod_{r=i+1}^{j-1} (1+t\,x_r)\right) \frac{1}{1-\sum_{l\geq i+1} w_l} \\
= \quad & \frac{w_i}{1-\sum_{l\geq i+1} w_l}
\end{aligned}$$

with

$$w_i := t\,x_i \sum_{\substack{i<j \\ x_i x_j \in \mathfrak{a}}} \left(w_j + t\,x_j - t\,x_j \sum_{r>j} w_r \right) \left(\prod_{r=i+1}^{j-1} (1 + t\,x_r) \right).$$

By induction, w_r is for $r > i$ a polynomial and therefore w_i is a polynomial. $\square$

Corollary 3.4. *The Poincaré-Betti series of A is given by:*

$$P_k^A(\underline{x}, t) := \prod_{i=1}^{n} (1 + t\,x_i) \frac{1}{1 - w_1 - \ldots - w_n}$$

with

$$\begin{aligned} w_n &:= 0, \\ w_i &:= t\,x_i \sum_{\substack{i<j \\ x_i x_j \in \mathfrak{a}}} \left(w_j + t\,x_j - t\,x_j \sum_{r>j} w_r \right) \left(\prod_{r=i+1}^{j-1} (1 + t\,x_r) \right). \end{aligned}$$

Proof. The result is a direct consequence of Lemma 3.3 and Theorem 3.1. $\square$

We now solve the recursion of w_i. For this, we introduce a directed graph $G = (V, E)$ with vertex set $V = \{1, \ldots, n\}$ and two vertices i, j are joined (i.e. $i \mapsto j$) if $i < j$ and $i \not\prec j$. We write $G\big|_{i_1,\ldots,i_\nu}$ for the induced subgraph on the vertices $i_1, \ldots, i_\nu$.
For a sequence $1 \leq i_1 < \ldots < i_\nu \leq n$ we define

$$\begin{aligned} d(i_1, \ldots, i_\nu) &:= \#\{\text{paths from } i_1 \text{ to } i_\nu \text{ in } G\big|_{i_1,\ldots,i_\nu}\}, \\ c(i_1, \ldots, i_\nu) &:= \sum_{\substack{0=a_0<a_1<\ldots<a_r=\nu \\ a_{i+1}-a_i \geq 2 \\ r \geq 1}} (-1)^r\, d(i_{a_0+1}, \ldots, i_{a_1}) \cdots d(i_{a_{r-1}+1}, \ldots, i_{a_r}). \end{aligned}$$

Note that a path counted by $d(i_1, \ldots, i_\nu)$ does not have to pass through all vertices $i_1, \ldots, i_\nu$.

With this notation we get

Corollary 3.5. *The Poincaré-Betti series of A is given by:*

$$P_k^A(\underline{x}, t) := \prod_{i=1}^{n} (1 + t\,x_i) \frac{1}{W(t, \underline{x})}$$

with

$$W(t, \underline{x}) = 1 + \sum_{\substack{1 \leq i_1 < \ldots < i_\nu \leq n \\ \nu \geq 2}} c(i_1, \ldots, i_\nu)\, t^\nu\, x_{i_1} \cdots x_{i_\nu}.$$

Proof. The result follows if one solves the recursion of the w_i's and collects the coefficients of the monomials $x_{i_1} \cdots x_{i_\nu}$. $\square$

In order to prove property **(P)** , we give a bijection between the paths in $G\big|_{i_1,\ldots,i_\nu}$ and the sting-chains:

Lemma 3.6. *For any sequence $1 \leq i_1 < \ldots < i_\nu \leq$ there exists a bijection between the paths from i_1 to i_ν in $G\big|_{i_1,\ldots,i_\nu}$ and the sting-chains I with* $\operatorname{lcm}(I) = x_{i_1} \cdots x_{i_\nu}$.

Proof. We consider the path $i_1 \to j_2 \to j_3 \to \ldots \to j_r \to i_\nu$. To this path, we associate the set $I := \{x_{i_1}x_{j_2}, x_{j_2}x_{j_3}, \ldots, x_{j_r}x_{i_\nu}\}$. Now we define the stings: Assume $j_r < i_{l_0}, \ldots, i_{l_1} < j_{r+1}$. Then we must have either $j_r \not\prec i_s$ or $i_s \not\prec j_{r+1}$ for all $s = l_0, \ldots, l_1$ (otherwise we would have a contradiction to $j_r \not\prec j_{r+1}$). This implies

$$\{x_{j_r}x_{i_s}, x_{i_s}x_{j_{r+1}}\} \cap \mathfrak{a} \neq \emptyset \text{ for all } s = l_0, \ldots, l_1.$$

If $x_{j_r}x_{i_s} \in \{x_{j_r}x_{i_s}, x_{i_s}x_{j_{r+1}}\} \cap \mathfrak{a}$, we choose $x_{j_r}x_{i_s}$, otherwise we choose $x_{i_s}x_{j_{r+1}}$. With this choice we get that I satisfies condition (4b) and (4c) of Definition 1.6 of Chapter 4. By construction we have $\operatorname{lcm}(I) = x_{i_1} \cdots x_{i_\nu}$.

If we start with a sting-chain I with $\operatorname{lcm}(I) = x_{i_1} \cdots x_{i_\nu}$, then by definition there exist monomials $x_{i_1}x_{j_2}, x_{j_2}x_{j_3}, \ldots, x_{j_r}x_{i_\nu} \in I$. This sequence defines a path $i_1 \mapsto j_2 \mapsto \ldots \mapsto j_r \mapsto i_\nu$. Since both constructions are inverse to each other, the assertion follows. □

It follows:

$$W(t, \underline{x}) := 1 + \sum_{I \in \mathcal{B}} (-1)^{cl(I)} m_I t^{cl(I)+|I|}, \tag{3.1}$$

where $\mathcal{B}$ is the set of chains of sting-chains, defined in Paragraph 1 of Chapter 4.

We now can prove property **(P)** and **(H)** for the ring $A = k[\Delta]$:

Theorem 3.7. *Let P be a partially ordered set and Δ the order complex of P. The multigraded Poincaré-Betti and Hilbert series of the Stanley Reisner ring $A = k[\Delta] = S/\mathfrak{a}$ are given by:*

$$P_k^A(\underline{x}, t) := \frac{\prod_{i \in P} (1 + t\, x_i)}{W(t, \underline{x})},$$

$$\operatorname{Hilb}_A(\underline{x}, t) := \frac{W(-t, \underline{x})}{\prod_{i \in P} (1 - t\, x_i)},$$

where

$$\begin{aligned} W(t, \underline{x}) &= 1 + \sum_{I \notin \mathcal{M}} (-1)^{cl(I)}\, m_I\, t^{cl(I)+|I|} \\ &= 1 + \sum_{I \notin \mathcal{M}_1} (-1)^{cl(I)}\, m_I\, t^{cl(I)+|I|} \\ &= 1 + \sum_{I \in \mathcal{B}} (-1)^{cl(I)}\, m_I\, t^{cl(I)+|I|} \\ &= 1 + \sum_{I\, \textbf{nbc}\text{-set}} (-1)^{cl(I)}\, m_I\, t^{cl(I)+|I|} \end{aligned}$$

with $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ a standard matching on the Taylor resolution $T_\bullet$ of $\mathfrak{a}$.

Proof. The assertion is a direct consequence of Corollary 1.8 of Chapter 4, Corollary 3.5 and Equation (3.1). □

4. Proof of Conjecture 1.2 for Several Classes of Algebras A

In this paragraph we prove Conjecture 1.2 in some special cases. In the first section, we prove the conjecture for algebras A for which the Koszul homology is an M-ring - a notion introduced by Fröberg [**23**]. If in addition the minimal resolution of $\mathfrak{a}$ has the structure of a differential-graded algebra, we prove property **(P)** for A.

In the second section, we prove Conjecture 1.2 for all Koszul algebras. Note that this gives another proof that for a partially ordered set P the Stanley Reisner ring $A = k[\Delta(P)]$ satisfies property **(P)** and **(H)**.

In the last section, we outline an idea for a proof of Conjecture 1.2 in general.

4.1. Proof for Algebras A for which $H_\bullet(K^A)$ is an M-ring.

The first class for which we can prove Conjecture 1.2 uses a theorem by Fröberg [**23**]. We use the notation of Fröberg:

Definition 4.1. A k-algebra R isomorphic to a (non-commutative) polynomial ring $k\langle X_1, \dots, X_r\rangle$ divided by an ideal $\mathfrak{r}$ of relations is called

(1) a weak M-ring if $\mathfrak{r}$ is generated by relations of the following types:
  (a) the (graded) commutator $[X_i, X_j] = 0$,
  (b) $m = 0$, where m is a monomial in X_i.
(2) an M-ring if if $\mathfrak{r}$ is generated by relations of the following types:
  (a) the (graded) commutator $[X_i, X_j] = 0$,
  (b) $m = 0$ with m a quadratic-monomial in X_i.

Now we assume that $H(K_\bullet)$ is an M-ring and $\mathcal{M}$ is a standard matching. Let $R'' := k\langle Y_I, I \notin \mathcal{M}, cl(I) = 1\rangle/\mathfrak{r}''$ be the non-commutative polynomial ring divided by an ideal $\mathfrak{r}''$, where $\mathfrak{r}''$ is generated by the following relations:

$$Y_I Y_J = (-1)^{\deg_t(Y_I Y_J)} Y_J Y_I, \text{ if } \begin{cases} \gcd(m_I, m_J) = 1 \text{ and } I \cup J \notin \mathcal{M} \\ \text{for all } I, J \notin \mathcal{M} \text{ with } cl(I) = cl(J) = 1. \end{cases}$$

In the notion of Fröberg, $R'' \otimes R'$ is the MM-ring belonging to the M-ring $R' \simeq H(K_\bullet)$. Each literal Y_I has two degrees: the total degree $|Y_I| := |I| + 1$ and the multidegree $\deg(Y_I) := \alpha$, with $x^\alpha = m_I$.

We define $F_\bullet := R'' \otimes_k K_\bullet^A$. Since $K_\bullet^A$ is an A-module, $F_\bullet$ is a free graded A-module with $\deg(m \otimes n) := \deg_t^{R''}(m) + \deg_t^{K_\bullet^A}(n)$. Let F_i be the homogeneous part of degree i. The next theorem proves Conjecture 1.2 in our situation.

Theorem 4.2. *Let $\mathcal{M}$ be a standard matching. Assume $H(K_\bullet)$ an M-ring. If there exists a homomorphism $s : H_\bullet(K^A) \to Z_\bullet(K^A)$, such that $\pi \circ s = \mathrm{id}_{H_\bullet(K^A)}$, then A satisfies Conjecture 1.2.*

Corollary 4.3. *Under the assumptions of Theorem 4.2 the algebra A has properties* **(P)** *and* **(H)**. □

Proof of Theorem 4.2. Theorem 2.2 verifies the conditions for Theorem 3 in [**23**]. In the proof of this theorem, Fröberg shows that $F_\bullet$ defines a minimal free resolution of k as an A-module. By Theorem 2.2 the homology of the Koszul complex is isomorphic to the ring $R'/\mathfrak{r}'$. Since $H_\bullet(K^A)$ is an M-ring, it follows that the ideal $\mathfrak{r}'$ is generated in degree two. The construction of the ideal $\mathfrak{r}'$ implies that every standard matching ends after the second sequence. In the second sequence of $\mathcal{M}$, we have that $I \to J \in \mathcal{M}_2$ satisfies $cl(I) = cl(J) - 1$ and $|I| = |J| + 1$. Now let $I \to J \in \mathcal{M}_2$ with $cl(I) = 1$ and $cl(J) = cl(J_1) + cl(J_2) = 2$. The difference between the ring R'' and the ring R is that in R we have a variable Y_I and the variables Y_{J_1}, Y_{J_2} commute. In the ring R'' the variables Y_{J_1}, Y_{J_2} do not commute and the variable Y_I is omitted. Identifying $Y_{J_1}Y_{J_2} \in R''$ with $Y_{J_1}Y_{J_2} \in R$ and $Y_{J_2}Y_{J_1} \in R''$ with $Y_I \in R$ gives an isomorphism as k-vectorspaces of R and R''. The property $cl(I) = cl(J) - 1$ and $|I| = |J| + 1$ proves that this isomorphism preserves the degrees, and we are done. □

The theorem includes the theorem by Charalambous and Reeves since in their case every standard matching is empty and Charalambous and Reeves proved the existence of the map $s : H_\bullet(K^A) \to Z_\bullet(K^A)$:

Corollary 4.4 ([**13**]). *If the Taylor resolution of $\mathfrak{a}$ is minimal, then $A = S/\mathfrak{a}$ satisfies Conjecture 1.2.* □

Note that $H_\bullet(K^A) \cong R'$ carries three gradings. Let $u \in R'$ with $u = Y_{I_1} \cdots Y_{I_r}$. Then we have $\gcd(m_{I_j}, m_{I_{j'}}) = 1$, for $j \neq j'$, and $I_1 \cup \ldots \cup I_r \notin \mathcal{M}$ (otherwise $u \in \mathfrak{r}'$). We set

$$\begin{aligned}
\deg(u) &= \alpha \text{ if } \underline{x}^\alpha = m_{I_1} \cdots m_{I_r} = m_{I_1 \cup \ldots \cup I_r},\\
\deg_t(u) &= r = cl(I_1 \cup \ldots \cup I_r),\\
|u| &= |I_1| + \ldots + |I_r| = |I_1 \cup \ldots \cup I_r|.
\end{aligned}$$

It follows:

$$H_\bullet(K^A) \cong R' = \bigoplus_{\substack{\alpha \in \mathbb{N}^n \\ i,j \geq 0}} R'_{\alpha,i,j} = \bigoplus_{\substack{I \notin \mathcal{M} \\ \deg_t(I) = i \\ |I| = j}} k\, Y_I,$$

where $Y_I = Y_{I_1} \cdots Y_{I_r}$ if $cl(I) = r$ and $\gcd(m_{I_j}, m_{I_{j'}}) = 1$, for $j \neq j'$.
Fröberg proved that in the case where $H_\bullet(K^A)$ is an M-ring and the minimal resolution of $\mathfrak{a}$ has the structure of a differential-graded algebra we have:

$$P_k^A(\underline{x}, t) = \frac{\mathrm{Hilb}_{K_\bullet \otimes_A k}(\underline{x}, t)}{\mathrm{Hilb}_{H_\bullet(K^A)}(x, -t, t)} = \prod_{i=1}^n (1 + t\, x_i) \frac{1}{\mathrm{Hilb}_{H_\bullet(K^A)}(x, -t, t)}.$$

Therefore, we only have to calculate the Hilbert series $\mathrm{Hilb}_{H_\bullet(K^A)}(x, -t, t)$:

$$\begin{aligned}
\mathrm{Hilb}_{H_\bullet(K^A)}(x, -t, t) &= \sum_{\substack{\alpha \in \mathbb{N}^n \\ i,j \geq 0}} \dim_k(R'_{\alpha,i,j})\ \underline{x}^\alpha\ (-t)^i\ t^j\\
&= \sum_{I \notin \mathcal{M}} m_I\ (-t)^{cl(I)}\ t^{|I|}\\
&= 1\ /\ \mathrm{Hilb}_R(\underline{x}, 1, t).
\end{aligned}$$

The last equation follows from Lemma 1.3 of Chapter 4 since if $H_\bullet(K^A)$ is an M-ring, every standard matching ends after the second sequence. It follows:

Corollary 4.5. *If $H_\bullet(K^A)$ is an M-ring and the minimal resolution of $\mathfrak{a}$ has the structure of a differential-graded algebra, then A has property* **(P)**. □

4.2. Proof for Koszul Algebras. In this section we give the proof of Conjecture 1.2 for Koszul algebras $A = S/\mathfrak{a}$. Note that since $\mathfrak{a}$ is monomial, this is equivalent to the fact that $\mathfrak{a}$ is generated in degree two. We assume in addition that $\mathfrak{a}$ is squarefree. This is no restriction since via polarization we can reduce the calculation of the Hilbert and Poincaré-Betti series of $S/\mathfrak{a}$ to the calculation of the series for $S/\mathfrak{b}$ for a squarefree ideal $\mathfrak{b} \trianglelefteq S$.

Theorem 4.6. *Let $A = S/\mathfrak{a}$ be the quotient algebra of the polynomial ring and a squarefree monomial ideal $\mathfrak{a}$ generated by monomials of degree two and $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$ a standard matching of $\mathfrak{a}$. Then A satisfies Conjecture 1.2.*

Corollary 4.7. *The multigraded Poincaré-Betti and Hilbert series of Koszul algebras $A = S/\mathfrak{a}$ for a squarefree monomial ideal $\mathfrak{a} \trianglelefteq S$ are given by:*

$$\begin{aligned} P_k^A(\underline{x}, t) &:= \frac{\prod\limits_{i \in P} (1 + t\, x_i)}{W(t, \underline{x})}, \\ \mathrm{Hilb}_A(\underline{x}, t) &:= \frac{W(-t, \underline{x})}{\prod\limits_{i \in P} (1 - t\, x_i)}, \end{aligned}$$

where

$$\begin{aligned} W(t, \underline{x}) &= 1 + \sum_{I \notin \mathcal{M}} (-1)^{cl(I)}\, m_I\, t^{cl(I)+|I|} \\ &= 1 + \sum_{I \notin \mathcal{M}_1} (-1)^{cl(I)}\, m_I\, t^{cl(I)+|I|} \\ &= 1 + \sum_{I\ \mathbf{nbc}\text{-set}} (-1)^{cl(I)}\, m_I\, t^{cl(I)+|I|}. \end{aligned}$$

Proof. The assertion follows directly from Theorem 4.6, the standard matching for ideals generated in degree two given in Paragraph 1 of Chapter 4 and the fact that, in this case, every standard matching ends after the second sequence. □

Note that if $\mathfrak{a} \trianglelefteq S$ is any ideal with a quadratic Gröbner basis, this corollary gives a form of the multigraded Hilbert and Poincaré-Betti series of $A = S/\mathfrak{a}$ since, in this case, the series coincide with the series of $S/\operatorname{in}_\prec(\mathfrak{a})$.

Proof of Theorem 4.6. In this proof we sometimes consider the variables $x_1, \ldots, x_n$ as elements of the polynomial ring S and sometimes as letters. In the second case the variables do not commute and we consider words over the alphabet $\Gamma := \{x_1, \ldots, x_n\}$. It will be clear from the context if we consider w as a monomial in S or as a word over Γ. For example, if we write $w \in \mathfrak{a}$ or $x_i \mid w$, we see w as a monomial.

For $j = 1, \ldots, n$, let $\mathcal{L}_j$ be the sets of words $x_{i_1} x_{i_2} \cdots x_{i_r}$, $r \geq 2$, over the alphabet $\{x_1, \ldots, x_n\}$, such that

(1) $i_1 = j < i_2, \ldots, i_r$,

(2) for all $2 \leq l \leq r$ there exists an $1 \leq l' < l$ such that $x_{i_{l'}} x_{i_l} \in \mathfrak{a}$ and $i_t > i_l$ for all $l' < t < l$.

We define

$$\mathcal{L} := \left\{ w_{i_1} \cdots w_{i_r} \;\middle|\; \begin{array}{c} i_1 > \ldots > i_r \\ w_{i_j} \in \mathcal{L}_{i_j}, j = 1, \ldots, r \end{array} \right\}.$$

Note that here the variables x_i are considered as letters and do not commute. In Chapter 5 we construct for Koszul algebras A a minimal free resolution of k. The basis in homological degree i in this resolution is given by the following set (see Corollary 1.9 of Chapter 5):

$$\mathcal{B}_i = \left\{ e_I \ \mathbf{w} \;\middle|\; \begin{array}{c} I \subset \{1, \ldots, n\} \\ \mathbf{w} \in \mathcal{L} \\ |J| + |\mathbf{w}| = i \end{array} \right\},$$

where $|\mathbf{w}|$ is the length of the word $\mathbf{w}$.

Thus in order to prove the theorem, we have to find a bijection between the words $\mathbf{w} \in \mathcal{L}$ of length i and the monomials $u \in R$ with degree $|u| = i$. Remember that in our case the subsets $I \notin \mathcal{M}_1$ are exactly the **nbc**-sets (see Paragraph 1.2 of Chapter 4) and therefore the ring R has the following form:

$$R = \frac{k\langle Y_I, I \text{ is an } \mathbf{nbc}\text{-set }, cl(I) = 1\rangle}{\langle [Y_I, Y_J] \mid \gcd(m_I, m_J) = 1\rangle}.$$

We assume that the monomials $u \in R$ are ordered, i.e. if $u = Y_{I_1} \cdots Y_{I_r}$ and Y_{I_j} commute with $Y_{I_{j+1}}$, then $\min(I_j) > \min(I_{j+1})$.

Clearly, it is enough to construct a bijection between the sets $\mathcal{L}_j$ and the ordered monomials $u = Y_{I_1} \cdots Y_{I_r}$, with $cl(I_1 \cup \ldots \cup I_r) = 1$ and $j = \min(I_1) < \min(I_i)$, for $i = 2, \ldots, r$.

For a word w over the alphabet $\{x_1, \ldots, x_n\}$ we denote by $x_{f(w)}$ (resp. $x_{l(w)}$) the first (resp. the last) letter of w, i.e. $w = x_{f(w)} w'$ (resp. $w = w' x_{f(w)}$).

We call a word w over the alphabet $\{x_1, \ldots, x_n\}$ an **nbc**-word if there exists an index j such that $w \in \mathcal{L}_j$ and each variable x_i, $i = 1, \ldots, n$, appears at most once in the word w.

The existence of the bijection follows from the following four claims.

Claim 1: For each j and each word $w \in \mathcal{L}_j$ which is not an **nbc**-word there exists a unique subdivision of the word w,

$$\phi_1(w) := u_1 || v_1 || u_2 || v_2 || \ldots || u_r || v_r,$$

such that

(i) $u_1 v_1 \cdots u_r v_r = w$.

(ii) The subword u_i is either a variable or an **nbc**-word in the language $\mathcal{L}_{f(u_i)}$.

(iii) The words v_i are either the empty word ε or a descending chain of variables, i.e. $v_i = x_{j_1} \cdots x_{j_{v_i}}$ with $j_1 > \ldots > j_{v_i}$.

(iv) If $v_i \neq \varepsilon$ and u_i is an **nbc**-word, then

$$f(u_i) \geq f(v_i) > l(v_i) > f(u_{i+1}).$$

(v) If $v_i \neq \varepsilon$ and u_i is a variable, then
$$f(u_i) < f(v_i) > l(v_i) > f(u_{i+1}).$$
(vi) If $v_i = \varepsilon$ and u_i is an **nbc**-word, then
$$f(u_i) \geq f(u_{i+1}).$$
(vii) If $v_i = \varepsilon$ and u_i is a variable, then
$$f(u_i) < f(u_{i+1}).$$

Claim 2: There exists an injective map ϕ_2 on the subdivisions of Claim 1 such that
$$\phi_2\big(\phi_1(w)\big) := w_1||w_2||\ldots||w_s$$
and for each w_i, $i = 1, \ldots, s$, we have the following properties:

(i) If $w_i = x_{j_1} \cdots x_{j_t}$, then for all $1 \leq l \leq t$ there exists an index $0 \leq l' < l$ with $x_{j_{l'}} x_{j_l} \in \mathfrak{a}$ and $j_\nu > j_l$ for all $l' < \nu < l$.

(ii) In each word w_i, each variable $x_1, \ldots, x_n$ appears at most once.

(iii) w_i is not a variable.

(iv) There exists an index t such that $x_t \mid w_1 \cdots w_{i-1}$ and $x_t x_{f(w_i)} \in \mathfrak{a}$ and either $x_{f(w_i)} \mid w_1 \cdots w_{i-1}$ or $t > f(w_i)$.

(v) For all $x_j \mid w_i$, $j < f(w_i)$, and $x_t \mid w_1 \cdots w_{i-1}$ with $x_t x_j \in \mathfrak{a}$, we have $t < j$.

(vi) If $\gcd(w_i, w_{i+1}) = 1$, then $f(w_i) > f(w_{i+1})$.

Claim 3: There exists an injection ϕ_3 between the sequences $\phi_2\phi_1\big(\mathcal{L}_j\big)$ from Claim 2 and the sequences $w_1||w_2||\ldots||w_s$,, satisfying, in addition to the conditions from Claim 2, the following properties:

(i) There exists an $j < i$ such that $\gcd(w_i, w_j) \neq 1$.

Claim 4: For each j there is a bijection
$$\phi_4 : \phi_3\phi_2\phi_1\Big(\mathcal{L}_j\Big) \to \left\{ Y_{I_1} \cdots Y_{I_r} \;\middle|\; \begin{array}{c} cl(I_1 \cup \ldots \cup I_r) = 1 \text{ and} \\ j = \min(I_1) < \min(I_i) \text{ , for } i = 2, \ldots, r \\ Y_{I_1} \cdots Y_{I_r} \text{ ordered} \end{array} \right\}$$

Since $\phi_1, \ldots, \phi_3$ are injections and ϕ_4 is a bijection, the composition $\phi_4\phi_3\phi_2\phi_1$ is the desired map.

Proof of Claim 1. Let $x_{j_1} \cdots x_{j_r} \in \mathcal{L}_j$, for some j, which is not an **nbc**-word. Then we have the following uniquely defined subdivision:
$$\underbrace{x_{i_1}x_{i_2}\cdots x_{i_{j_0-1}}}_{i_2>\ldots>i_{j_0-1}} \;||\; \underbrace{x_{i_{j_0}}\cdots x_{i_{j_1-1}}}_{\substack{\in\mathcal{L}_{j_0}\\ i_{j_0-1}>i_{j_0}}} \;||\; \underbrace{x_{i_{j_1}}\cdots x_{i_{j_2-1}}}_{\substack{i_{j_1}>\ldots>i_{j_2-1}\\ i_{j_0}\geq i_{j_1}}} \;||\; \underbrace{x_{i_{j_2}}\cdots x_{i_{j_3-1}}}_{\substack{\in\mathcal{L}_{j_2}\\ i_{j_2-1}>i_{j_2}}} \;||\; \cdots .$$
The first part $x_{i_1}x_{i_2}\cdots x_{i_{j_0-1}}$ we split again into
$$u_1||v_1 := x_{i_1}||x_{i_2}\cdots x_{i_{j_0-1}}.$$
Thus, we get the subdivision
$$u_1 \;||\; v_1 \;||\; u_2 \;||\; v_2 \;||\ldots||\; u_{s_1} \;||\; v_{s_1},$$

where u_1 is a variable, v_i are the monomials of the descending chains of variables (note that $v_i = \varepsilon$ is possible) and the words u_i, $i \geq 2$, are words in $\mathcal{L}_{f(u_i)}$. If all u_i are **nbc**-words, we are done. But in general, it is not the case. Therefore, we define the following map φ: For an **nbc**-word w we set $\varphi(w) := w$. If w is not an **nbc**-word, we construct the above subdivision and set

$$\varphi(w) := u_1 \mid\mid v_1 \mid\mid \varphi(u_2) \mid\mid v_2 \mid\mid \ldots \mid\mid \varphi(u_{s_1}) \mid\mid v_{s_1}.$$

Since the word w is of finite length the recursion, is finite and $\varphi(w)$ produces a subdivison of the word w.
Since each $\varphi(w)$ ends with a word v, which is possibly the empty word ε, the u's and v's do not always alternate in $\varphi(w)$. In order to define the desired subdivision, we therefore have to modify $\varphi(w)$:

- ▷ If we have the situation $v_i||v_{i+1}$ such that v_i, v_{i+1} are descending chains of variables, possibly ε, then by construction we have that the word v_iv_{i+1} is a descending chain of variables. We replace the subdivison $v_i||v_{i+1}$ by the word v_iv_{i+1}.

The construction implies that the resulting subdivison fulfills all desired properties. Let ϕ_1 be the map which associates to each word w the corresponding subdivison. Clearly, this subdivision is unique and therefore ϕ_1 is an injection.

Proof of Claim 2. Let $\phi_1(w) = u_1 \mid\mid v_1 \mid\mid u_2 \mid\mid v_2 \mid\mid \ldots \mid\mid u_s \mid\mid v_s$ be a subdivision of Claim 1. We construct the image under ϕ_2 by induction.

(R) If $f(v_s) \leq f(u_s)$ and there exists a variable $x_t \mid u_1v_1 \cdots u_{s-1}v_{s-1}$ with $x_tx_{f(v_s)} \in \mathfrak{a}$, we replace v_{s-1} by $v'_{s-1} := v_{s-1}x_{f(v_s)}$, else we replace u_s by $u'_s := u_sx_{f(v_s)}$. Finally, we replace v_s by the v'_s such that $v_s = x_{f(v_s)}v'_s$.

We repeat this process until $v'_s = \varepsilon$. We get a word

$$u_1||v_1||\ldots||u_{s-1}||v'_{s-1}||u'_s,$$

such that u_i, v_i, for $i = 1, \ldots, s-2$, and u_{s-1} are as before, v'_{s-1} is a descending chain of variables and for u'_s we have:

(∗) If there exist variables $x_i \mid u'_s$ with $i < f(u'_s)$ and $x_j \mid u_1v_1 \cdots u_{s-1}v'_{s-1}$ such that $x_ix_j \in \mathfrak{a}$, then $j < i$.

Now we repeat the same process for $u_{s-1}||v'_{s-1}$. We get a word

$$u_1||v_1||\ldots||u_{s-2}||v'_{s-2}||u'_{s-1}||u'_s,$$

such that u_i, v_i are from the original decomposition and u'_s, u'_{s-1} have property (∗).
We repeat this process for all words $u_i||v_i$ and we reach a sequence of words

$$\phi_{2,1}\big(\phi_1(w)\big) := u'_1||u'_2||\ldots||u'_{s-1}||u'_s.$$

By construction this sequence satisfies the conditions (i), (ii), and (v).
Note that our construction implies that each word u'_i has a unique decomposition $u'_i = u''_iv''_i$ such that u''_i is either a variable or an **nbc**-word in $\mathcal{L}_{f(u''_i)}$ and v''_i is descending chain of variables. Now we begin with v''_1 and permute the variables with respect to the rule (R) to the right, if necessary, and go on by induction. It is clear that these two algorithms are inverse to each other and

therefore $\phi_{2,1}$ is an injection onto its image.
In order to satisfy conditions (iii), (iv), and (vi), we define an injective map $\phi_{2,2}$ on the image of $\phi_{2,1}$. The composition $\phi_2 := \phi_{2,2}\phi_{2,1}$ gives then the desired map.
Let $\phi_{2,1}\big(\phi_1(w)\big) = u_1||u_2||\dots||u_{s-1}||u_s$. Let i be the smallest index such that $\gcd(u_i, u_{i+1}) = 1$ and $f(u_i) < f(u_{i+1})$. By construction the word $u_i = u_i'v_i$ has a decomposition such that v_i is a descending chain of variables and $f(v_i) < f(u_{i+1})$ (v_i was constructed by the map $\phi_{2,1}$). The word u_{i+1} has a decomposition $u_{i+1} = u_{i+1}'v_{i+1}$ such that u_{i+1}' is either a variable or an **nbc**-word and v_{i+1} a descending chain of variables. We replace $u_i||u_{i+1}$ by the new word $\varphi(u_i||u_{i+1}) := u_i'u_{i+1}'c(v_iv_{i+1})$ where $c(v_i, v_{i+1})$ is the descending chain of variables consisting of the variables of v_i and v_{i+1}.
We repeat this procedure until there are no words u_i, u_{i+1} with $\gcd(u_i, u_{i+1}) = 1$ and $f(u_i) < f(u_{i+1})$.
It is straightforward to check that the resulting sequence

$$\phi_{2,2}\phi_{2,1}\big(\phi_1(w)\big) := \tilde{u}_1||\tilde{u}_2||\dots||\tilde{u}_{\tilde{s}-1}||u_{\tilde{s}}$$

satisfies all desired conditions.
To reverse the map $\phi_{2,2}$, we apply to each word u_i the maps ϕ_1 and $\phi_{2,1}$. Then it is easy to see that the sequence

$$\phi_{2,1}\phi_1(u_1)||\phi_{2,1}\phi_1(u_2)||\dots||\phi_{2,1}\phi_1(u_{s-1})||\phi_{2,1}\phi_1(u_s)$$

is the preimage of $\phi_{2,2}$. Therefore, $\phi_{2,2}$ is an injection and the map $\phi_2 := \phi_{2,2}\phi_{2,1}$ is the desired injection.

Proof of Claim 3: Let $\phi_2\phi_1(w) = u_1||u_2||\dots||u_{s-1}||u_s$ be a sequence from Claim 2. In order to satisfy the desired condition, we construct a map ϕ_3 similar to $\phi_{2,2}$. Let i be the largest index such that $\gcd(\mathrm{lcm}(u_1, \dots, u_i), u_{i+1}) = 1$. Then it follows from Claim 2 that $f(u_i) > f(u_{i+1})$. If we replace $u_i||u_{i+1}$ by a new word which is constructed in a similar way as in the map $\phi_{2,2}$, we risk to violate condition (v) from Claim 2. Therefore, we first have to permute the word u_{i+1} in the correct position. Let $l < i+1$ be the smallest index such that there exists an index $t > f(u_{i+1})$ with $x_t \mid u_l$ and $x_tx_{f(u_{i+1})} \in \mathfrak{a}$. By Condition (iv) from Claim 2, such an index always exists. We replace the sequence $u_1||u_2||\dots||u_{s-1}||u_s$ by the sequence

$$u_1||\dots||u_{l-1}||\varphi(u_l||u_{i+1})||u_{l+1}||\dots||u_i||u_{i+2}||\dots||u_s,$$

where $\varphi(u_l||u_{i+1})$ is the map from the construction of $\phi_{2,2}$ of Claim 2. Now the construction implies that all conditions of Claim 2 are still satisfied.
We repeat this procedure until the sequence satisfies the desired condition.
To reverse this procedure we reverse the map φ with the maps ϕ_1 and ϕ_2 and permute the words to the right until Condition (vi) from Claim 2 is satisfied.
It follows that ϕ_3 is an injection onto its image.

Proof of Claim 4. Let $\phi_3\phi_2\phi_1(w) = w_1||w_2||\dots||w_s$ be a sequence from Claim 3. We now construct a bijection between these sequences of words and the ordered monomials $Y_{I_1}\cdots Y_{I_r}$ with $cl(I_1 \cup \dots \cup I_r) = 1$ and $\min(I_1) < \min(I_j)$ for all $j = 2, \dots, r$. We now assume:

Assumption A:

(a) For each **nbc**-set I and each index i with $x_i \mid m_I = \mathrm{lcm}(I)$, there exists a unique word $\psi(I) := w$ such that $w = x_i w'$ and w satisfies conditions (i) - (iii) from Claim 2.

(b) For each word w satisfying conditions (i) - (iii) from Claim 2, there exists a unique **nbc**-set $\varphi(w) := I$.

In addition, the maps ψ and φ are inverse to each other.

We now prove Claim 4:
Let $Y_{I_1} \cdots Y_{I_s}$ be an ordered monomial with $cl(I_1 \cup \ldots \cup I_s) = 1$ and $\min(I_1) < \min(I_j)$, for $j = 2, \ldots, s$. Let j_{I_l} be the smallest index i such that $x_i | \mathrm{lcm}(I_l)$ and either

- there exists a variable $x_t \mid w_1 w_2 \cdots w_{l-1}$ with $t > i$ and $x_i x_t \in \mathfrak{a}$
- or $x_i \mid \mathrm{lcm}(I_1, I_2, \ldots, I_{l-1})$.

Such an index always exists since $\gcd(m_{I_1 \cup I_2 \cup \ldots \cup I_{l-1}}, m_{I_l}) \neq 1$. By definition the variables Y_I, Y_J commute if $\gcd(m_I, m_J) = 1$. It is easy to see that one can reorder the monomial $Y_{I_1} \cdots Y_{I_s}$, such that if $\gcd(m_{I_i}, m_{I_{i+1}}) = 1$, we have $j_{I_i} > j_{I_{i+1}}$. We now construct a bijection between monomials $Y_{I_1} \cdots Y_{I_s}$ ordered in that way and the sequences of Claim 3.

Let $\phi_3\phi_2\phi_1(w) = w_1||w_2||\ldots||w_s$ be a sequence of Claim 3 and I_j be the **nbc**-sets corresponding to the words w_j. Then we associate to the sequence the following monomial

$$\phi_4(w_1||w_2||\ldots||w_s) := Y_{I_1} \cdots Y_{I_s}.$$

Condition (i) from Claim 3 and Condition (vi) from Claim 2 imply that we get an ordered monomial.
On the other hand, consider an ordered monomial $Y_{I_1} \cdots Y_{I_s}$. We associate to Y_{I_1} the corresponding **nbc**-word w_1 whose front letter is $x_{\min(I_1)}$.
For $l = 2, \ldots s$ let w_l be the word corresponding to I_l whose front letter is $x_{j_{I_l}}$.
It follows directly from the construction that the sequence $w_1||w_2||\ldots||w_s$ satisfies all desired conditions.
Conditions (iv) and (v) of Claim 2 imply that both constructions are inverse to each other and therefore ϕ_4 is a bijection.

In order to finish our proof, we have to verify Assumption A.
To a word $w = x_{j_1} \cdots x_{j_s}$ satisfying Conditions (i) - (iii) we associate a graph on the vertex set $V = [n]$. The edges are constructed in the following way: We set $E := \big\{\{j_1, j_2\}\big\}$. For j_s there exists an index $0 \leq l < s$ such that $x_{j_l} x_{j_s} \in \mathfrak{a}$. Let P_{j_s} be the set of those indices. Now let l_2 be the maximum of P_{j_2}. If $E \cup \big\{\{j_{l_2}, j_2\}\big\}$ contains no broken circuit (with respect to the lexicographic order), we set $E := E \cup \big\{\{j_{l_2}, j_2\}\big\}$. Else we set $P_{j_2} := P_{j_2} \setminus \{l_2\}$ and repeat the process. It is clear that there exists at least one index in P_{j_2} such that the constructed graph contains no broken circuit. We repeat this for $P_{j_3}, P_{j_4}, \ldots, P_{j_r}$. By construction we obtain a graph which contains no broken circuit. Now graphs without broken circuits are in bijection with the **nbc**-sets (define $I := \{x_i x_j \mid \{i,j\} \in E\}$).
Given an **nbc**- graph and a vertex i such that there exist $j \in V$ with $\{i,j\} \in E$,

we construct a word w satisfying Conditions (i) - (iii) by induction: Assume we can construct to each graph of length ν and each vertex i a word w which satisfies the desired conditions.
Given a graph of length $\nu+1$ and a vertex i. Let $P_i := \{i < j \mid \{i,j\} \in E\}$ and $E_1 := E \setminus \big\{\{i,j\} \in E \;\big|\; j \in P_i\big\}$. Then $E \setminus E_1$ decomposes in $|P_i|+1$ connected components. One component is the vertex i and for each $j > i$ we have exactly one component G_j with $j \in G_j$. By induction we can construct words w_j corresponding to G_j. Now assume $P_i = \{j_1 < \ldots < j_r\}$. We set $w := iw_{j_r} \cdots w_{j_1}$. Finally, we permute $x_t \in w_{j_l}$, with $t < j_{l+1}$ to the right until it is in the correct position.
Let w be a word constructed from a graph. Assume there is $x_t \in w_j$ which was permuted to the right in the word $w_{j'}$, $j < j'$. If there exists an index l such that $x_l \in w_{j'}$, $x_l x_t \in \mathfrak{a}$, and $l > t$, then we would add an edge $\{l,t\}$. But since $x_t \in w_j$ and the original graph was connected, this leads to a broken circuit for the constructed graph. Therefore, the edge for the vertex t has to be constructed with the corresponding index in w_j. This proves that both constructions are inverse to each other. □

4.3. Idea for a Proof in the General Case. In this paragraph we outline a program which we expect to yield a proof of Conjecture 1.2 in general.

The only way to prove the conjecture is to find a minimal A-free resolution of the field k, which in general is a very hard problem. With the Algebraic Discrete Morse theory one can minimize a given free resolution, but one still needs a free resolution to start. The next problem is the connection to the minimized Taylor resolution of the ideal $\mathfrak{a}$.
The Eagon complex is an A-free resolution of the field k which has a natural connection to the Taylor resolution of the $\mathfrak{a}$ since the modules in this complex are tensor products of $H_\bullet(K^A) \simeq T^{\mathcal{M}} \otimes_S k$. The problem with the Eagon complex is that the differential is defined recursively.
In the first part of this paragraph, we define a generalization of the Massey operations which gives us an explicit description of the differential of the Eagon complex. We apply Algebraic Discrete Morse theory to the Eagon complex. The resulting Morse complex is not minimal in general, but it is minimal if for example $H_\bullet(K^A)$ is an M-ring. In order to prove our conjecture in general, one has to find an isomorphism between the minimized Eagon complex and the conjectured minimal resolution. We can not give this isomorphism in general, but with this Morse complex we can explain our conjecture.
For the general case, we think that one way to prove the conjecture is the following:

- calculate the Eagon complex,
- minimize it with the given acyclic matching,
- find a degree-preserving k-vectorspaces-isomorphism to the ring $K_\bullet \otimes_k R$.

As before we fix one standard matching $\mathcal{M}$ on the Taylor resolution of $\mathfrak{a}$. The set of cycles $\{\phi(I) \mid I \notin \mathcal{M}\}$ is a system of representatives for the Koszul homology. With the product on the homology, we can define the following operation:

For two sets $J, I \notin \mathcal{M}$ we define:

$$I \wedge J := \begin{cases} 0 & , \quad \gcd(m_I, m_J) \neq 1 \\ 0 & , \quad \gcd(m_I, m_J) = 1, I \cup J \in \mathcal{M} \text{ and } [\phi(I)][\phi(J)] = 0 \\ I \cup J & , \quad [\phi(I)][\phi(J)] = [\phi(I \cup J)] \text{ and } I \cup J \notin \mathcal{M} \\ \sum_{L \notin \mathcal{M}} a_L L & , \quad [\phi(I)][\phi(J)] = \sum_{L \notin \mathcal{L}} a_L [\phi(L)] \text{ and } I \cup J \in \mathcal{M}. \end{cases}$$

Now we can define the function $(I, J) \mapsto g(I, J) \in K_\bullet^A$ such that

$$\partial(g(I, J)) := \phi(I)\phi(J) - \frac{m_I m_J}{m_{I \cup J}} \phi(I \wedge J).$$

By Proposition 2.1 this function is well defined.
We now define a function for three sets $\gamma(I_1, I_2, I_3)$ by:

$$\gamma(I_1, I_2, I_3) := \phi(I_1) g(I_2, I_3) + (-1)^{|I_1|+1} g(I_1, I_2)\phi(I_3)$$
$$+(-1)^{|I_1|+1} \frac{m_{I_1} m_{I_2}}{m_{I_1 \cup I_2}} g(I_1 \wedge I_2, I_3) - (-1)^{|I_1|+1} \frac{m_{I_2} m_{I_3}}{m_{I_2 \cup I_3}} g(I_1, I_2 \wedge I_3).$$

It is straightforward to prove that $\partial(\gamma(I_1, I_2, I_3)) = 0$. If $\gamma(I_1, I_2, I_3)$ is a boundary for all sets I_1, I_2, I_3, we can define $g(I_1, I_2, I_3)$ such that $\partial(g(I_1, I_2, I_3)) = \gamma(I_1, I_2, I_3)$.
Similar to the Massey operations we go on by induction:
Assume $\gamma(I_1, \ldots, I_l)$ vanishes for all l-tuples $I_1, \ldots, I_l$, with $l \geq \nu - 1$. Then there exist cycles $g(I_1, \ldots, I_l)$ such that $\partial(g(I_1, \ldots, I_l)) = \gamma(I_1, \ldots, I_l)$. We then define:

$$\gamma(I_1, \ldots, I_\nu) := \phi(I_1) g(I_2, , \ldots, I_\nu) + (-1)^{\sum_{j=1}^{\nu-2} |I_j|+1} g(I_1, \ldots, I_{\nu-1})\phi(I_\nu)$$
$$+ \sum_{i=2}^{\nu-2} (-1)^{\sum_{j=1}^{i-1} |I_j|+1} g(I_1, \ldots, I_i) g(I_{i+1}, \ldots, I_\nu)$$
$$+ \sum_{i=1}^{\nu-2} (-1)^{\sum_{j=1}^{i} |I_j|+1} \frac{m_{I_j} m_{I_{j+1}}}{m_{I_j \cup I_{j+1}}} g(I_1, \ldots, I_{j-1}, I_j \wedge I_{j+1}, I_{j+2}, \ldots, I_\nu)$$
$$-(-1)^{\sum_{j=1}^{\nu-2} |I_j|+1} \frac{m_{I_{\nu-1}} m_{I_\nu}}{m_{I_{\nu-1} \cup I_\nu}} g(I_1, \ldots, I_{\nu-2}, I_{\nu-1} \wedge I_\nu).$$

It is straightforward to prove that $\gamma(I_1, \ldots, I_\nu)$ is a cycle. Therefore, we get an induced operation on the Koszul homology. Since the first three summands are exactly the summands of the Massey operations, we call $\gamma(I_1, \ldots, I_\nu)$ the ν-th generalized Massey operations.

From now on we assume that all generalized Massey operations vanish. We then can give an explicit description of the Eagon complex:
We define free modules X_i to be the free A-modules over $I \notin \mathcal{M}$ with $|I| = i$. It is clear that we have $X_i \otimes_A k \simeq H_i(K^A)$. The Eagon complex is defined by a sequence of complexes Y^i, with $Y^0 = K_\bullet^A$ and Y^n is defined by

$$\begin{aligned} Y_i^{n+1} &:= Y_{i+1}^n \oplus Y_0^n \otimes X_i, \qquad i > 0, \\ Y_0^{n+1} &= Y_1^n. \end{aligned}$$

Let $Z_i(Y_\bullet^s)$ and $B_i(Y_\bullet^s)$ denote cycles and boundaries, respectively. The differentials d^s on Y^s are defined by induction. d^0 is the differential on the Koszul complex. Assume d^{s-1} is defined. One has to find a map α that makes the diagram in Figure 1 commutative: One can then define $d^s := (d^{s-1}, \alpha)$.
The map d^s satisfies $H_i(Y^s) = H_0(Y^s) \otimes X_i$ and $B_{i-1}(Y^s) = d^s(Y_1^s) = Z_i(Y^{s-1})$.

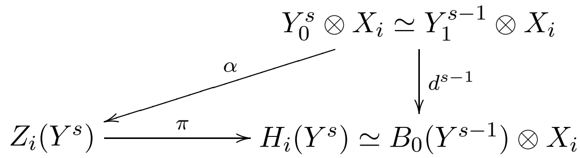


Figure 1

The first property allows us to continue this procedure for $s+1$ and the second gives us exactness of the following complex:

$$F_\bullet : \cdots Y_0^{s+1} \xrightarrow{d^s} Y_0^s \xrightarrow{d^{s-1}} Y_0^{s-1} \longrightarrow \cdots \longrightarrow Y_0^0 \longrightarrow k.$$

Note that to make the diagram commutative, it is enough to define $\alpha(n \otimes f)$ for all generators $n \otimes f$ of $Y_0^s \otimes X_i$ such that $\alpha(n \otimes f) = (m, d^{s-1}(n) \otimes f)$, with $m \in Y_{i+1}^{s-1}$ and the property that $d^{s-1}(m) + d^{s-1}(d^{s-1}(n) \otimes f) = 0$.

The ν-th module of the complex $Y_\bullet^s$ is given by $Y_\nu^s = K_j \otimes X_{i_1} \otimes \ldots \otimes X_{i_r}$ with $j + r + \sum_{j=1}^r i_j = \nu + s$. We fix an R-basis of Y_ν^s, by $e_L \otimes I_1 \otimes \ldots \otimes I_r$ with $I_j \notin \mathcal{M}$ and $e_L = e_{l_1} \wedge \ldots \wedge e_{l_t}$. We are now able to define the maps α: Since all generalized Massey operations vanish, there exists elements $g(I_1, \ldots I_r)$ such that $\partial(g(I_1, \ldots I_r)) = \gamma(I_1, \ldots I_r)$

Lemma 4.8. *Suppose that $d^{s-1} : Y_\bullet^{s-1} \to Y_\bullet^{s-1}$ is such that*

$$\begin{aligned} d^{s-1}(e_L \otimes I_1 \otimes \ldots \otimes I_r) &= \partial^K(e_L) \otimes I_1 \otimes \ldots \otimes I_r \\ &+(-1)^{|L|} e_L \phi(I_1) \otimes I_2 \otimes \ldots \otimes I_r \\ &+(-1)^{|L|} \sum_{j=1}^{r-1} (-1)^{\sum_{i=1}^j |I_j|+1} \frac{m_{I_j} m_{I_{j+1}}}{m_{I_j \cup I_{j+1}}} e_L \otimes I_1 \otimes \ldots \otimes I_j \wedge I_{j+1} \otimes \ldots \otimes I_r \\ &+(-1)^{|L|} \sum_{j=1}^{r-1} (-1)^{\sum_{i=1}^j |I_j|+1} e_L \; g(I_1, \ldots, I_{j+1}) \otimes I_{j+2} \otimes \ldots \otimes I_r. \end{aligned}$$

If $n := e_L \otimes I_1 \otimes \ldots \otimes I_r \in Y_0^s$ and J is a generator of X_i, we define $\alpha(n \otimes J)$ to be the map that sends $n \otimes J$ to $(m, d^{s-1}(n) \otimes J)$ with

$$\begin{aligned} m = (-1)^{|L|}(-1)^{\sum_{i=1}^r |I_j|+1} &\frac{m_{I_r} m_J}{m_{I_j \cup J}} e_L \otimes I_1 \otimes \ldots \otimes I_{r-1} \otimes I_r \wedge J \\ &+(-1)^{|L|}(-1)^{\sum_{i=1}^r |I_j|+1} e_L \; g(I_1, \ldots, I_r, J). \end{aligned}$$

Then α makes the diagram in Figure 1 commutative.

Proof. We only have to check that $d^{s-1}(m) + d^{s-1}(d^{s-1}(n) \otimes f) = 0$. This is a straightforward calculation and is left to the reader. □

Corollary 4.9. *The map d^s can be defined as follows:*

$$\begin{aligned} d^s(e_L \otimes I_1 \otimes \ldots \otimes I_r) &= \partial^K(e_L) \otimes I_1 \otimes \ldots \otimes I_r \\ &+(-1)^{|L|} e_L \phi(I_1) \otimes I_2 \otimes \ldots \otimes I_r \\ &+(-1)^{|L|} \sum_{j=1}^{r-1} (-1)^{\sum_{i=1}^j |I_j|+1} \frac{m_{I_j} m_{I_{j+1}}}{m_{I_j \cup I_{j+1}}} e_L \otimes I_1 \otimes \ldots \otimes I_j \wedge I_{j+1} \otimes \ldots \otimes I_r \end{aligned}$$

$$+(-1)^{|L|}\sum_{j=1}^{r-1}(-1)^{\sum_{i=1}^{j}|I_j|+1}e_L\ g(I_1,\ldots,I_{j+1})\otimes I_{j+2}\otimes\ldots\otimes I_r.$$

With this corollary we get an explicit description of the Eagon resolution of k over A.

In order to define the acyclic matching, we first use Theorem 2.2 to define the Eagon complex with the ring $H_\bullet(K^A)\cong R' = k[Y_I \mid cl(I)=1, I\notin\mathcal{M}]/\mathfrak{r}'$ instead of $H_\bullet$. The operation $I\wedge J$ then is nothing but the multiplication Y_IY_J in R'. We write y_I for the class of Y_I in R'.

It is clear that this complex is not minimal in general. The idea now is to minimize this complex via Algebraic Discrete Morse theory. It is easy to see, that the only invertible coefficient occurs by mapping $\ldots\otimes y_I\otimes y_J\otimes\ldots$ to the element $\ldots\otimes y_Iy_J\otimes\ldots$, with $\gcd(m_I,m_J)=1$. The idea is to match all such basis elements, with $I\wedge J = I\cup J$ and $I\cup J\notin\mathcal{M}$. In order to do this, we have to define an order on the variables y_I with $I\notin\mathcal{M}$: We order the sets I by cardinality and if two sets have the same cardinality by the lexicographic order on the multidegrees m_I, m_J. The monomials in R' are ordered by the degree-lexicographic order. The acyclic matching is similar to the Morse matching on the normalized Bar resolution (see Chapter 5, Lemma 1.2). Since $\mathcal{M}$ is a standard matching on the Taylor resolution, we know that if $I_1\cup I_2\cup\ldots\cup I_r\notin\mathcal{M}$ with $cl(I_j)=1$ and $\gcd(m_{I_j},m_{I_{j'}})=1$ for all $j\neq j'$, then it follows that $I_2\cup\ldots\cup I_r\notin\mathcal{M}$. Therefore, the following matching is well defined:

$$e_L\otimes y_{I_1}\otimes y_{I_2}\cdots y_{I_r}\otimes\ldots\mapsto e_L\otimes y_{I_1}y_{I_2}\cdots y_{I_r}\otimes\ldots,$$

where $I_1<I_2<\ldots<I_r$ and $I_1\cup I_2\cup\ldots\cup I_r\notin\mathcal{M}$ and $cl(I_j)=1$ and $\gcd(m_{I_j},m_{I_{j'}})=1$ for all $j\neq j'$. On the remaining basis elements we do the same matching on the second coordinate, and so on. The exact definition of the acyclic matching and the proof is given in Definition 1.1 of Chapter 5.

We describe the remaining basis elements, as in Chapter 5, by induction. $[y_I|u_1]$ with $u_1 = y_{J_1}\cdots y_{J_r}$ is called fully attached (see Definition 1.3 of Chapter 5) if one of the following conditions is satisfied:

(1) $r=1$ and $\gcd(m_I,m_{J_1})\neq 1$ or $y_I>y_{J_1}$,
(2) $\gcd(m_I,m_{J_i})=1$ for all i and $I\cup J_1\cup\ldots\cup J_r\in\mathcal{M}$, and for all $1\le i\le r$ we have $I\cup J_1\cup\ldots\cup\widehat{J_i}\cup\ldots\cup J_r\notin\mathcal{M}$.

A tuple $[y_J|u_1|\ldots|u_r]$ is called fully attached if $[y_J|u_1|\ldots|u_{r-1}]$ is fully attached, one of the following properties is satisfied and u_r is minimal in the sense that there is no proper divisor $v_r\mid u_r$ satisfying one of the conditions below:

(1) u_r is a variable and $\gcd(m_{u_{r-1}},m_{u_r})\neq 1$,
(2) u_r, u_{r-1} are both variables and $u_{r-1}>u_r$,
(3) $[y_J|u_1|\ldots|u_{r-2}|u_r]$ is a fully attached tuple and $u_{r-1}>u_r$,
(4) $u_{r-1}=y_{I_1}\cdots y_{I_t}$, $u_r=y_{J_1}\cdots y_{J_s}$ such that $\gcd(m_{u_{r-1}},m_{u_r})=1$ and $I_1\cup\ldots\cup I_t\cup J_1\cup\ldots\cup J_s\in\mathcal{M}$.

Here $m_u := \operatorname{lcm}(I_1 \cup \ldots \cup I_r)$ if $u = y_{I_1} \cdots y_{I_r}$.

The basis of the Morse complex is given by elements $e_L|\mathbf{w}$, where $\mathbf{w}$ is a fully attached tuple. If $H_\bullet(K^A)$ is an M-ring, the Morse complex is minimal since in this case the fully attached tuple has the form $[y_{I_1}|y_{I_2}|\cdots|y_{I_r}]$. In order to prove Conjecture 1.2 one has to find an isomorphism between the fully attached tuples and the monomials in R.

We can not give this isomorphism in general, but we think that this Morse complex helps for the understanding of our conjecture:

Let $[y_{I_1}|y_{I_2}|\ldots|y_{I_r}]$ be a fully attached tuple, with $y_{I_1} > \ldots > y_{I_r}$. We map such a tuple to the monomial $Y_{I_1} \cdots Y_{I_r} \in R$. Clearly, this map preserves the degree. We get a problem if $[y_J|u_1|\ldots|u_r]$ is a fully attached tuple and $u_1 = I_1 \cup \ldots \cup I_r$ with $r > 1$. For example, assume $J \mapsto I_1 \cup \ldots \cup I_r \in \mathcal{M}_r$, with $cl(J) = cl(I_1) = \ldots = cl(I_r) = 1$ and $\gcd(m_{I_j}, m_{I_{j'}}) = 1$ for $j \neq j'$, is matched. Assume further $y_{I_1} < \ldots < y_{I_r}$. Then $[y_{I_1}|y_{I_2}\cdots y_{I_r}]$ is a fully attached tuple. We cannot map $[y_{I_1}|y_{I_2}\cdots y_{I_r}]$ to $Y_{I_1}Y_{I_2}\cdots Y_{I_r}$, since in R the variables commute, i.e. $Y_{I_1}Y_{I_2}\cdots Y_{I_r} = Y_{I_r}Y_{I_{r-1}}\cdots Y_{I_1}$ and the tuple $[y_{I_r}|y_{I_{r-1}}|\ldots|y_{I_1}]$ maps already to this element. But we can define

$$[y_{I_1}|y_{I_2}\cdots y_{I_r}] \mapsto Y_J \in R.$$

The degree of $Y_J \in R$ is $|J| + 1$ and the homological degree of $[y_{I_1}|y_{I_2}\cdots y_{I_r}]$ is

$$|I_1| + 1 + (|I_2| + \ldots + |I_r|) + 1 = (|I_1| + \ldots + |I_r| + 1) + 1 = |J| + 1,$$

therefore this map preserves the degree.

These facts show that the variables Y_I, with $I \in \mathcal{M}$, $cl(I) = 1$, but $I \notin \mathcal{M}_1$, are necessary and this justifies our conjecture.

5. Applications to the Golod Property of Monomial Rings

In this paragraph we give some applications to the Golod property. Remember that a ring A is Golod if and only if one of the following conditions is satisfied (see [**26**]):

(5.1)
$$P_k^A(\underline{x}, t) = \frac{\prod_{i=1}^{n}(1 + x_i\, t)}{1 - t \sum_{\alpha \in \mathbb{N}^n, i \geq 0} \dim_k(Tor_i^S(A,k)_\alpha) x^\alpha\, t^i}.$$

(5.2) All Massey operations on the Koszul homology vanish.

If an algebra satisfies property **(P)**, then we get in the monomial case the following equivalence:

Theorem 5.1. *If $A = S/\mathfrak{a}$ satisfies property* **(P)** *, then A is Golod if and only if one of the following conditions is satisfied:*

(1) *For all subsets $I \subset \operatorname{MinGen}(\mathfrak{a})$ with $cl(I) \geq 2$ we have $I \in \mathcal{M}$ for any standard matching $\mathcal{M}$.*
(2) *The product (i.e. the first Massey operation) on the Koszul homology is trivial.*

Proof. Property **(P)** implies the equivalence of (5.1) and the first condition. Theorem 2.2 implies the equivalence of the first and the second condition. □

Corollary 5.2. *If $A = S/\mathfrak{a}$ satisfies one of the following conditions, then A is Golod if and only if the first Massey operation vanishes.*

(1) *$\mathfrak{a}$ is generated in degree two,*
(2) *$H_\bullet(K^A)$ is an M-ring and either there exists a homomorphism $s : H_\bullet(K^A) \to Z_\bullet(K^A)$ such that $\pi \circ s = \mathrm{id}_{H_\bullet(K^A)}$ or the minimal resolution of $\mathfrak{a}$ has the structure of a differential graded algebra.*

Proof. In the previous paragraph we proved property **(P)** in these cases, therefore the result follows from the theorem above. □

Recently, Charalambous proved in [**14**] a criterion for generic ideals to be Golod. Remember that a monomial ideal $\mathfrak{a}$ is generic if the multidegree of two minimal monomial generators of $\mathfrak{a}$ are equal for some variable, then there is a third monomial generator of $\mathfrak{a}$ whose multidegree is strictly smaller than the multidegree of the least common multiple of the other two. It is known that for generic ideals $\mathfrak{a}$ the Scarf resolution is minimal. Charalambous proved the following proposition:

Proposition 5.3 (see [**14**])**.** *Let $\mathfrak{a} \trianglelefteq S$ be a generic ideal. $A = S/\mathfrak{a}$ is Golod if and only if $m_I m_J \neq m_{I\cup J}$ whenever $I \cup J \in \Delta_S$ for $I, J \subset \mathrm{MinGen}(\mathfrak{a})$. Here Δ_S denotes the Scarf complex.*

Assuming property **(P)**, our Theorem 5.1 gives a second proof of this fact:

Proof. It is easy to see that the condition

$$m_I m_J \neq m_{I\cup J} \text{ whenever } I \cup J \in \Delta_S$$

is equivalent to fact that the product on the Koszul homology is trivial. Thus, Theorem 5.1 implies the assertion. □

We have the following criterion:

Lemma 5.4. *Let $A = S/\mathfrak{a}$ with $\mathfrak{a} = \langle m_1, \ldots, m_l\rangle$.*

(1) *If $\gcd(m_i, m_j) \neq 1$ for all $i \neq j$, then A is Golod (see [**13**], [**29**]).*
(2) *If $A = S/\mathfrak{a}$ is Golod, then $\mathfrak{a}$ satisfies the gcd-condition.*

Proof. If a ring A is Golod, then the product on $H_\bullet(K^A)$ is trivial. This implies $Y_I Y_J = 0$ if $\gcd(m_I, m_J) = 1$. With Theorem 2.2 it follows that all sets $I \cup J$ with $\gcd(m_I, m_J) = 1$ are matched. In particular, all sets $\{m_i, m_j\}$ with $\gcd(m_i, m_j) = 1$. Such a set can only be matched with a set $\{m_{i_1}, m_{i_1}, m_{i_1}\}$ with the same lcm. But this implies that there must exist a third generator m_r with $m_r | m_i\, m_j$. □

The following counterexample shows that the converse of the second statement is false: Let $\mathfrak{a} := \langle xy, yz, zw, wt, xt\rangle$ be the Stanley Reisner ideal of the triangulation of the 5-gon. It is easy to see that $\mathfrak{a}$ satisfies the gcd-condition. But $\mathfrak{a}$ is Gorenstein and therefore not Golod. But we have:

Theorem 5.5. *If $A = S/\mathfrak{a}$ has property* **(P)** *and $\mathfrak{a}$ satisfies the strong* gcd-*condition, then A is Golod.*

Proof. We prove that $H_\bullet(K^A)$ is an M-ring and isomorphic as an algebra to the ring

$$R := k(Y_I \mid I \notin \mathcal{M}, cl(I) = 1)/\langle Y_I Y_J \text{ for all } I, J \notin \mathcal{M}_0 \cup \mathcal{M}\rangle,$$

where $\mathcal{M}_0$ is the sequence of matchings constructed in Proposition 1.11 in order to obtain the complex T_{gcd} and $\mathcal{M}$ is a standard matching on the complex T_{gcd}. It follows that the first Massey operation is trivial and then Theorem 5.1 implies the assertion.
The idea is to make the same process as in Paragraph 2 with the complex T_{gcd} from Proposition 1.11 from Chapter 4 instead of the Taylor resolution $T_\bullet$. Since all sets I in T_{gcd} satisfy $cl(I) = 1$, the result follows directly from property (**P**). Note that $\mathcal{M}_0$ satisfies all conditions required in the proof of Proposition 2.1 except the following: Assume $I \cup J \in \mathcal{M}_0$ with $\gcd(m_I, m_J) = 1$ and $I, J \notin \mathcal{M}_0$. Then there exists a set $\hat{I}$ such that $\hat{I} \to I \cup J \in \mathcal{M}_0$. It follows

$$0 = \partial^2(\hat{I}) = \partial(I \cup J) + \sum_{L \notin \mathcal{M}_0} a_L \, L$$

and therefore as in the proof of Proposition 2.1

$$\phi(I \cup J) = \sum_{L \notin \mathcal{M}_0} a_L \, \phi(L) \qquad \text{for some } a_L \in k.$$

In the case of Proposition 2.1 we could guarantee that $cl(L) \geq cl(I \cup J)$. We can not deduce this fact here, but this is the only difference between $\mathcal{M}_0 \cup \mathcal{M}$ and a standard matching on the Taylor resolution. Since all sets L with $cl(L) \geq 2$ are matched, we only could have

$$\phi(I \cup J) = \sum_{\substack{L \notin \mathcal{M}_0 \\ cl(L)=1}} a_L \, \phi(L) \qquad \text{for some } a_L \in k.$$

We prove that this cannot happen. If $I \cup J$ is matched, then there exists a monomial m with $I \cup J \cup \{m\} \to I \cup J \in \mathcal{M}_0$. But then, since $cl(I \cup J \setminus \{n\}) \geq cl(I \cup J) \geq 2$, by the definition of $\mathcal{M}_0$ any image $I \cup J \cup \{m\} \setminus \{n\}$ is also matched:

$$I \cup J \cup \{m\} \setminus \{n\} \to I \cup J \setminus \{n\} \in \mathcal{M}_0.$$

This proves that the situation above is not possible and we are done. □

Corollary 5.6. *Suppose that $A = S/\mathfrak{a}$ has property* **(P)**. *Then A is Golod if*

(1) $\mathfrak{a}$ *is shellable (for the definition see* [**4**]*),*

(2) $\text{MinGen}(\mathfrak{a})$ *is a monomial ordered family (for the definition see* [**39**]*),*

(3) $\mathfrak{a}$ *is stable and* $\#\,\text{supp}(m) \geq 2$ *for all* $m \in \text{MinGen}(\mathfrak{a})$*,*

(4) $\mathfrak{a}$ *is p-Borel fixed and* $\#\,\text{supp}(m) \geq 2$ *for all* $m \in \text{MinGen}(\mathfrak{a})$.

Here $\text{supp}(m) := \{1 \leq i \leq n \mid x_i \text{ divides } m\}$.

Proof. We order $\text{MinGen}(\mathfrak{a})$ with the lexicographic order. Then it follows directly from the definitions of the ideals that $\mathfrak{a}$ satisfies the strong gcd-condition. The assertion follows then from Theorem 5.5. □

Theorem 5.5 and the preceding Lemma give rise to the following conjecture:

Conjecture 5.7. *Let* $\mathfrak{a} = \langle m_1, \ldots, m_l \rangle \subset S$ *be a monomial ideal and* $A = S/\mathfrak{a}$. *Then* A *is Golod if and only if* $\mathfrak{a}$ *satisfies the strong* gcd*-condition.*
In particular: Golodness is independent of the characteristic of k.

It is known that if $\mathfrak{a}$ is componentwise linear, then A is Golod (see [**29**]). One can generalize this result to the following:

Corollary 5.8. *Let* $\mathfrak{a}$ *be generated by monomials with degree* l.

(1) *If* $\dim_k \left(Tor_i^S(S/\mathfrak{a}, k)_{i+j}\right) = 0$ *for all* $j \geq 2(l-1)$, *then* $A = S/\mathfrak{a}$ *is Golod,*
(2) *if* A *is Golod, then* $\dim_k \left(Tor_i^S(S/\mathfrak{a}, k)_{i+j}\right) = 0$ *for all* $j \geq i(l-2)+2$.

In particular: If A *is Koszul, then* A *is Golod if and only if the minimal free resolution of* $\mathfrak{a}$ *is linear.*

Proof. Let $I \subset \{m_1, \ldots, m_l\}$ with $cl(I) = 1$ and $\text{lcm}(I) \neq \text{lcm}(I \setminus \{m\})$ for all $m \in I$. Then $l + |I| - 1 \leq \deg(I) \leq (l-1)|I| + 1$. Now assume that $L = I \cup J \notin \mathcal{M}$ with $\gcd(m_I, m_J) = 1$, then $\deg(L) \geq 2l - 2 + |I \cup J|$, which is a contradiction to $\dim_k \left(Tor_i^S(S/\mathfrak{a}, k)_{i+j}\right) = 0$ for all $j \geq 2l-2$. Therefore, the product on the Koszul homology is trivial. By the same multidegree reasons it follows that all Massey operations have to vanish, hence A is Golod.
If A is Golod, then the product on $H_\bullet(K^A)$ is trivial, hence (by theorem 2.2) $I \notin \mathcal{M}$ implies $cl(I) = 1$. But for those subsets we have $l + |I| - 1 \leq \deg(I) \leq (l-1)|I| + 1$. Therefore, it follows that $\dim_k \left(Tor_i^S(S/\mathfrak{a}, k)_{i+j}\right) = 0$ for all $j \geq i(l-2) + 2$. □

Part 2

Two Problems in Algebraic Combinatorics

Introduction

This part of the thesis treats two loosely related problems in combinatorics. It is separated from the first part since the results were obtained with other combinatorial methods than Algebraic Discrete Morse theory.

The first problem considers combinatorial questions in Lie algebra homology. We give a short overview of the theory and the problems; and we present some results. A detailed introduction is given at the beginning of Chapter 2.

The second problem we study is the Neggers-Stanley conjecture also known as poset conjecture. We give an introduction to the problem and present known results as well as still open questions. Finally, we present our work on the Neggers-Stanley conjecture. As for the first problem we give a detailed introduction at the beginning of Chapter 3.

Homology of Nilpotent Lie Algebras of Finite Type

In this chapter we discuss some problems about the homology of nilpotent Lie algebras that are of combinatorial nature. A survey article about combinatorial problems in Lie algebra homology in general can be found in [**27**].

In the first paragraph we give a very short introduction to the theory of Lie algebras and the definition of their homology. We introduce the theory of root systems and Lie algebras associated to root systems. We list the classical examples: We introduce the root systems A_n, B_n, C_n, D_n and their corresponding reflection groups as well as their corresponding Lie algebras. For more details see for example [**33**] and [**34**].
At the end of this paragraph we present a list of known results and still open questions and conjectures about the homology of (nilpotent) Lie algebras.

The second paragraph contains our work. We define a new type of isomorphism for nilpotent Lie algebras called quasi-isomorphism, and we prove that the homology groups $H_\bullet(L)$ and $H_\bullet(L')$ of two quasi-isomorphic Lie algebras L and L' are isomorphic. The surprising fact is that

$$H_\bullet(L) = \bigoplus_{i\geq 0} H_i(L) \;\cong\; H_\bullet(L') = \bigoplus_{i\geq 0} H_i(L'),$$

but there may be an index i such that $H_i(L) \ncong H_i(L')$.

In Paragraph 3 we draw some corollaries for subalgebras of the nilpotent part of Lie algebras associated to root systems. For the root system A_n our subalgebras are in one-to-one correspondence with partially ordered sets. We introduce a new type of isomorphism for partially ordered sets, which we also call quasi-isomorphism. The definition only depends on the corresponding order complexes. We prove that in this setting two Lie algebras $L(P)$ and $L(P')$ are quasi-isomorphic if the underlying partially ordered sets P and P' are quasi-isomorphic.

1. General Theory

Lie algebras arise in nature as vectorspaces of linear transformations endowed with a new operation, which is in general neither commutative nor associative: $[x, y] = xy - yx$. It is possible to describe these kind of systems abstractly by a few axioms.

Definition 1.1. A vectorspace over a field k, with an operation $L \times L \to L$ denoted $(x, y) \to [x, y]$ and called the Lie-bracket or commutator of x and y, is called a Lie algebra over k if the following axioms are satisfied:

(L1) The bracket operation is bilinear.

(L2) $[x, x] = 0$ for all $x \in L$.

(L3) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$.

Axiom (L3) is called the Jacobi identity.

We say that two Lie algebras L, L' over k are isomorphic if there exists a vectorspace isomorphism $\phi : L \to L'$ satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$.

An ideal I of a Lie algebra L is a subvectorspace such that for all $x \in L$ and $y \in I$ we have $[x, y] \in I$.

Given a Lie algebra L, we define two sequences of Lie algebras:

(1) The derived series $L^{(0)} := L$, $L^{(1)} := [L, L]$, ..., $L^{(i)} := [L^{(i-1)}, L^{(i-1)}]$.

(2) The descending central series $L^0 := L$, $L^1 := [L, L]$, $L^2 := [L, L^1]$, ..., $L^i := [L, L^{i-1}]$.

We call a Lie algebra nilpotent if there exists a number $n \geq 1$ such that $L^n = 0$, and we call it solvable if there exists a number $n \geq 1$ such that $L^{(n)} = 0$.

We denote with $\mathrm{Rad}(L)$ the unique maximal solvable ideal of a Lie algebra L (existence and uniqueness is proved in [**33**]) and call it the radical of the Lie algebra L. If $L \neq 0$ and $\mathrm{Rad}(L) = 0$, we call L semi-simple.

We now list the classical examples of Lie algebras.

Example 1.2. (1) Let $\mathfrak{gl}(n, k)$ be the set of all $n \times n$ matrices over k. We use the standard basis consisting of the matrices e_{ij} having 1 in the (i, j) position and 0 elsewhere. It follows that

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}.$$

Then $\mathfrak{gl}(n, k)$ is a Lie algebra, called the *general linear Lie algebra.*

(2) Let $\mathfrak{t}(n, k) \subset \mathfrak{gl}(n, k)$ be the subalgebra of all upper triangular $n \times n$ matrices over k with standard basis e_{ij} with $1 \leq i \leq j \leq n$. Then $\mathfrak{t}(n, k)$ is a Lie algebra.

(3) Let $\mathfrak{d}(n, k)$ be the subspace of all diagonal matrices with standard basis e_{ii} with $1 \leq i \leq n$. Then $\mathfrak{d}(n, k)$ is a Lie algebra.

(4) Let $\mathfrak{n}(n, k)$ be the subspace of all strictly upper triangular matrices with standard basis e_{ij} with $1 \leq i < j \leq n$. Then $\mathfrak{n}(n, k)$ is a Lie algebra.

(5) Let $\mathfrak{sl}(n + 1, k)$ be the set of all $(n + 1) \times (n + 1)$ matrices over k having trace zero. Then $\mathfrak{sl}(n + 1, k)$ is a Lie algebra, called the *special*

linear algebra. A standard basis is given by e_{ij} for $i \neq j$ and $h_i := e_{ii} - e_{i+1}e_{i+1}$ for $1 \leq i \leq n$, hence the dimension of $\mathfrak{sl}(n,k)$ is $(n+1)^2 - 1$.

(6) Let $s := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ be a non-degenerate skew-symmetric form. The *symplectic algebra* $\mathfrak{sp}(2n,k)$ is given by all $2n \times 2n$ matrices x satisfying $sx = -x^t s$.

(7) Let $s := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$ be a non-degenerate symmetric bilinear form. The *orthogonal algebra* $\mathfrak{o}(2n+1,k)$ is given by all $(2n+1) \times (2n+1)$ matrices x satisfying $sx = -x^t s$.

(8) Let $s := \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ be a non-degenerate symmetric bilinear form. The *orthogonal algebra* $\mathfrak{o}(2n,k)$ is given by all $2n \times 2n$ matrices x satisfying $sx = -x^t s$.

The algebra $\mathfrak{t}(n,k)$ is solvable, $\mathfrak{n}(n,k)$ is nilpotent and $\mathfrak{sl}(n,k), \mathfrak{sp}(2n,k)$, $\mathfrak{o}(2n+1,k)$, and $\mathfrak{o}(2n,k)$ are semi-simple.

1.1. Root Space Decomposition. We call an endomorphism $x \in \operatorname{End}(V)$ semi-simple if the roots of its minimal polynomial over k are all distinct. The Jordan-Chevalley decomposition says that for each $x \in \operatorname{End}(V)$ there exist unique elements $x_s, x_n \in \operatorname{End}(V)$, such that x_s is semi-simple and x_n is nilpotent and $x = x_s + x_n$.
Let L be a semi-simple Lie algebra. A toral subalgebra of L is defined to be the span of the semi-simple elements $x_s \in L$. We denote with H a maximal toral subalgebra of L.

For example, for the Lie algebra $L = \mathfrak{sl}(n,k)$ the subalgebra consisting of all diagonal matrices having trace zero is the maximal toral subalgebra of L.

For $\alpha \in H^*$ we define

$$L_\alpha := \{x \in L \mid \operatorname{ad} h(x) = \alpha(h)x \text{ for all } h \in H\},$$

where $\operatorname{ad} h : L \to L$ maps an element x to $[h,x]$ and $\alpha(h)$ is the eigenvalue of $\operatorname{ad} h$ corresponding to the eigenvector x.
We set $\Phi := \{\alpha \in H^* \mid L_\alpha \neq 0\}$. It is known that Φ is a finite set and it is called *the root system related to* H.

Theorem 1.3 (Root Space Decomposition)**.** *Let L be a semi-simple Lie algebra, H a maximal toral subalgebra with root system* Φ*. Then we have:*

(1) *H is abelian ($[H,H] = 0$).*
(2) $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$.
(3) $[L_\alpha, L_\beta] \subset L_{\alpha+\beta}$.
(4) *If $\alpha \in \Phi$, then $-\alpha \in \Phi$.*

In fact, it can be shown that the root system Φ characterizes the algebra L completely.

The root system of a semi-simple Lie algebra L satisfies the following conditions:

Proposition 1.4. (1) *If $\alpha \in \Phi$, then $-\alpha \in \Phi$, but no other multiple of α is a root.*

(2) *If $\alpha, \beta \in \Phi$, then $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$, where $(\cdot,\cdot)$ is a positive definite symmetric bilinear form.*

(3) *If $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.*

From this proposition it follows that there exists a decomposition of Φ into a negative and a positive part of roots, say $\Phi = \Pi^+ \cup \Pi^-$, such that the following algebras are nilpotent Lie algebras:

$$NL^+ := \bigoplus_{\alpha \in \Pi^+} L_\alpha \quad \text{and} \quad NL^- := \bigoplus_{\alpha \in \Pi^-} L_\alpha.$$

It follows:

Corollary 1.5. *Let L be a semi-simple Lie algebra with root system Φ. Then L decomposes into*

$$L = H \oplus NL^+ \oplus NL^-.$$

The algebras NL^+ and NL^- are nilpotent.

We call the subalgebra NL^+ the *nilpotent part* of the Lie algebra L.

Example 1.6. The nilpotent part of the Lie algebra $\mathfrak{sl}(n, k)$ is given by $\mathfrak{n}(n, k)$.

It is possible to characterize root systems axiomatically and to associate to a given root system a semi-simple Lie algebra. It turns out that this construction and the one from Theorem 1.3 are inverse to each other (see Theorem 1.9).

1.2. Root Systems and Reflection Groups. In this section let V be a real vectorspace with a positive definite symmetric bilinear form $(\cdot,\cdot)$.
A *reflection* is a linear operator s on V which sends some nonzero vector α to its negative while fixing pointwise the hyperplane H_α orthogonal to α. There is a simple formula for s_α:

$$s_\alpha\, \lambda = \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha.$$

It is easy to see that s_α is an orthogonal transformation, i.e. $s_\alpha \in O(V)$. Hence a finite group generated by some reflection is a finite subgroup of $O(V)$.

Now let $\Phi \subset V$ be a set of vectors in V satisfying

(R1) $\Phi \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}$ for all $\alpha \in \Phi$,

(R2) $s_\alpha \Phi = \Phi$ for all $\alpha \in \Phi$.

Define W to be the group (the Weyl group) generated by all reflections s_α, $\alpha \in \Phi$. Call Φ a root system with associated reflection group W. The first condition implies that Φ decomposes into a positive and a negative part, $\Phi = \Pi^+ \cup \Pi^-$.

A subset $\Delta \subset \Phi$ of the root system is called a *simple system* if Δ is a vectorspace basis for the $\mathbb{R}$-span of Φ in V and if moreover each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign (all nonnegative or all nonpositive). The following theorem assures that simple systems always exists.

Theorem 1.7. (1) *If Δ is a simple system in Φ, then there is a unique positive system containing Δ.*

(2) *Every positive system Π^+ in Φ contains a unique simple system; in particular, simple systems exist.*

Example 1.8. (1) (A_{n-1}, $n \geq 2$) Consider the symmetric group S_n. It can be thought of as a subgroup of the group $O(n, \mathbb{R})$ of $n \times n$ matrices in the following way. Make a permutation act on R^n by permuting the standard basis vectors $\varepsilon_1, \ldots, \varepsilon_n$. The transpositions (ij) acts as a reflection, sending $\varepsilon_i - \varepsilon_j$ to its negative and fixing pointwise the orthogonal complement, which consists of all vectors in $\mathbb{R}^n$ having equal ith and jth components. Since S_n is generated by transpositions, it is a reflection group.
A root system Φ and a simple system Δ for S_n is given by

$$\Phi := \Big\{\varepsilon_i - \varepsilon_j \;\Big|\; 1 \leq i, j \leq n, i \neq j\Big\},$$

$$\Delta := \Big\{\varepsilon_i - \varepsilon_{i+1} \;\Big|\; 1 \leq i \leq n-1\Big\}.$$

The root system to the reflection group S_n is called A_n.

(2) (B_n, $n \geq 2$) Again let $V = \mathbb{R}^n$, so S_n acts on V as above. Other reflections can be defined by sending ε_i to its negative and fixing all other ε_j. These sign changes generate a group of order 2^n isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Taking the semidirect product with S_n gives a reflection group. A root system Φ and a simple system Δ for $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ are given by

$$\Phi := \Big\{ \pm\,\varepsilon_i,\ \pm\varepsilon_i \pm \varepsilon_j \;\Big|\; 1 \leq i, j \leq n, i \neq j\Big\},$$

$$\Delta := \Big\{\varepsilon_n,\ \varepsilon_i - \varepsilon_{i+1} \;\Big|\; 1 \leq i \leq n-1\Big\}.$$

The root system to the reflection group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ is called B_n.

(3) (C_n, $n \geq 2$) Starting with B_n, one can define C_n to be its inverse root system. This means that we replace each root α by $\frac{2\alpha}{(\alpha,\alpha)}$. The root system Φ and a simple system Δ for C_n are given by

$$\Phi := \Big\{ \pm\,2\varepsilon_i,\ \pm\varepsilon_i \pm \varepsilon_j \;\Big|\; 1 \leq i, j \leq n, i \neq j\Big\},$$

$$\Delta := \Big\{2\varepsilon_n,\ \varepsilon_i - \varepsilon_{i+1} \;\Big|\; 1 \leq i \leq n-1\Big\}.$$

(4) (D_n, $n \geq 4$) Consider the reflection group of type B_n. Since S_n normalizes the subgroup consisting of sign changes which involve an even number of signs, generated by the reflections $\varepsilon_i + \varepsilon_j \mapsto -(\varepsilon_i + \varepsilon_j)$, $i \neq j$, the semidirect product is also a reflection group. A root system Φ and a simple system Δ for $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ are given by

$$\Phi := \Big\{ \pm\,\varepsilon_i \pm \varepsilon_j \;\Big|\; 1 \leq i < j \leq n\Big\},$$

$$\Delta := \Big\{\varepsilon_{n-1} + \varepsilon_n,\ \varepsilon_i - \varepsilon_{i+1} \;\Big|\; 1 \leq i \leq n-1\Big\}.$$

The root system to the reflection group $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ is called D_n.

To a root system one can associate a semi-simple Lie algebra if the root systems satisfies a third axiom:

(R3) $\langle \alpha, \beta \rangle := \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

We say that a subset $\mathcal{B} \in L$ generates L as Lie algebra if every element $x \in L$ can be obtained from elements in $\mathcal{B}$ in the following way:

$$x = [y_1, [y_2, [y_3, \ldots, [y_{l-1}, y_l] \ldots]]], \text{ with } y_1, \ldots, y_l \in \mathcal{B}.$$

Theorem 1.9 (Serre, see [**33**])**.** *Fix a root system Φ satisfying (R1)-(R3), with base $\Delta = \{\alpha_1, \ldots, \alpha_l\}$. Let L be the Lie algebra generated by $3l$ elements $\{x_i, y_i, h_i \mid 1 \leq i \leq l\}$ subject to the relations*

(S1) $[h_i, h_j] = 0$ *for* $1 \leq i, j \leq l$,

(S2) $[x_i, y_i] = h_i$, $[x_i, y_j] = 0$ *if* $i \neq j$,

(S3) $[h_i, x_j] = \langle \alpha_j, \alpha_i \rangle x_j$, $[h_i, y_j] = -\langle \alpha_j, \alpha_i \rangle y_j$,

(S^+_{ij}) $(\operatorname{ad} x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0$ *for* $i \neq j$,

(S^-_{ij}) $(\operatorname{ad} y_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(y_j) = 0$ *for* $i \neq j$.

Then L is a (finite dimensional) semi-simple Lie algebra with maximal toral subgroup H spanned by the h_i and with corresponding root system Φ. The nilpotent parts NL^+ and NL^- are generated as Lie algebra by the elements x_i and y_i.

Example 1.10. (1) The Lie algebra constructed with respect to the root system A_n is isomorphic to $\mathfrak{sl}(n+1, k)$. The nilpotent part NL^+ is isomorphic to $\mathfrak{n}(n+1, k)$, and NL^- is isomorphic to the subspace $\mathfrak{n}^-(n+1, k)$ consisting of all strictly lower triangular $(n+1) \times (n+1)$ matrices, spanned by the standard basis e_{ij}, $1 \leq j < i \leq n+1$.

(2) The Lie algebra constructed with respect to the root system B_n is isomorphic to $\mathfrak{o}(2n+1, k)$. The algebra $NL^+ \oplus NL^-$ is isomorphic to

$$\Big(\mathfrak{n}(2n+1, k) \oplus \mathfrak{n}^-(2n+1, k)\Big) \cap \mathfrak{o}(2n+1).$$

(3) The Lie algebra constructed with respect to the root system C_n is isomorphic to $\mathfrak{sp}(2n, k)$. The algebra $NL^+ \oplus NL^-$ is isomorphic to

$$\Big(\mathfrak{n}(2n, k) \oplus \mathfrak{n}^-(2n, k)\Big) \cap \mathfrak{sp}(2n).$$

(4) The Lie algebra constructed with respect to the root system D_n is isomorphic to $\mathfrak{o}(2n, k)$. The algebra $NL^+ \oplus NL^-$ is isomorphic to

$$\Big(\mathfrak{n}(2n, k) \oplus \mathfrak{n}^-(2n, k)\Big) \cap \mathfrak{o}(2n).$$

1.3. Homology of Lie Algebras. In this section we follow the notes of Hanlon [**27**]. Consider the exterior algebra ΛL over the Lie algebra L. On the exterior algebra $\Lambda L = \bigoplus_{r \geq 0} \Lambda^{(r)} L$ we have two differentials:
Let $\mathcal{B} = \{z_1, \ldots, z_d\}$ be a basis for L and c_{ijl} denote the coefficients that describe the bracket in L, i.e.

$$[z_i, z_j] = \sum_l c_{ijl} z_l.$$

A basis for $\Lambda^{(r)}$ is given by

$$\mathcal{B}_r = \{z_{i_1} \wedge \ldots \wedge z_{i_r} \mid 1 \leq i_1 < \ldots < i_r \leq d\}.$$

The differential on ΛL is given by

$$\begin{aligned} \partial_r : \Lambda^{(r)} L &\rightarrow \Lambda^{(r-1)} L \\ z_{i_1} \wedge \ldots \wedge z_{i_r} &\mapsto \sum_{1 \leq l < j \leq r} (-1)^{l+j+1} [z_{i_l}, z_{i_j}] \wedge z_{i_1} \wedge \ldots \wedge \widehat{z}_{i_l} \wedge \ldots \wedge \widehat{z}_{i_j} \wedge \ldots \wedge z_{i_r}, \end{aligned}$$

where $\widehat{z}$ means that we omit the element z.

Lemma 1.11. *For any r we have $\partial_r \circ \partial_{r+1} = 0$.*

Proof. Consider $\partial^2(x \wedge y \wedge z)$:

$$\begin{aligned} &\partial^2(x \wedge y \wedge z) \\ &= \partial\Big((-1)^{1+2+1}[x,y] \wedge z + (-1)^{1+3+1}[x,z] \wedge y + (-1)^{2+3+1}[y,z] \wedge x\Big) \\ &= [[x,y],z] - [[x,z],y] + [[y,z],x] \\ &= -[z,[x,y]] - [y,[z,x]] - [x,[y,z]] \\ &= 0. \end{aligned}$$

The last equation follows from the Jacobi identity. Using this fact and the alternating sign, the general proof is straightforward. □

The transpose $\partial_r^t : \Lambda^{(r-1)} L \rightarrow \Lambda^{(r)} L$ of ∂ is given by

$$\begin{aligned} &\partial_r^t\big(z_{i_1} \wedge \ldots \wedge z_{i_{r-1}}\big) := \\ &\quad \sum_{j=1}^{r-1} (-1)^{j+1} z_{i_1} \wedge \ldots \wedge z_{i_{j-1}} \wedge \Big(\sum_{a<b} c_{abi_j} z_a \wedge z_b\Big) \wedge z_{i_{j+1}} \wedge \ldots \wedge z_{i_{r-1}}. \end{aligned}$$

Again we have

Lemma 1.12. *For any r we have $\partial_r^t \circ \partial_{r-1}^t = 0$.* □

The homology of a Lie algebra L is defined to be the homology of the complex ΛL:

$$H_i(L) := H_i(\Lambda L) = \frac{\ker(\partial_i)}{\operatorname{Im}(\partial_{i+1})}.$$

Finally, we define the Laplacian:

$$\begin{aligned} \Lambda_r : \Lambda^{(r)} &\rightarrow \Lambda^{(r)} \\ z &\mapsto \big(\partial_{r+1}\partial_{r+1}^t + \partial_r^t \partial_r\big)(z). \end{aligned}$$

The following elegant theorem is proved by Kostant in Section 2 of [**37**] using nothing more than standards facts from linear algebra.

Theorem 1.13 (Kostant)**.** *For each r*

$$\dim_{\mathbb{C}}(H_r) = \dim_{\mathbb{C}}\big(\ker(\Lambda_r)\big).$$

Example 1.14 (Kostant)**.** Consider the Lie algebra $\mathfrak{n}(n,k)$. Then Theorem 1.13 implies

$$\dim_{\mathbb{C}}(H_r) = \#\big\{\sigma \in S_n \bigm| \# \operatorname{Inv}(\sigma) = r\big\},$$

where $\mathrm{Inv}(\sigma) = \{(i,j) \mid i < j,\ \sigma(i) > \sigma(j)\}$. In fact, if one associates to each permutation $\sigma \in S_n$ the following element in ΛL

$$\bigwedge_{(i,j)\in \mathrm{Inv}(\sigma)} e_{ij},$$

one can show that Theorem 1.13 implies the assertion.

This result shows that the homology of $\mathfrak{n}(n,k)$ is of combinatorial nature.

We have seen that $\Lambda^{(r)}L$ has a $\mathbb{C}$-vector space basis $\mathcal{B}_r$. Let R be any ring with 1. Then we consider the free R-modules generated by $\mathcal{B}_r$ instead of the modules $\Lambda^{(r)}L$. With the same differential we obtain a well defined complex and we can ask for the homology. We call this homology the *homology of L with coefficients in R* and write $H_r(L,R)$.

If we consider in Example 1.14 the homology with coefficients in $R = \mathbb{Z}_2$, the dimension $\dim_{\mathbb{Z}_2}(H_r(L,\mathbb{Z}_2))$ is still an open question. We come back to it in Section 1.4.

Remark 1.15. The calculation of the homology of a Lie algebra L can be approached with Algebraic Discrete Mores theory since all coefficients are ± 1. We tried to find good acyclic matchings on the complex ΛL in the case where $L = \mathfrak{n}(n,k)$ is the Lie algebra of strictly upper triangular matrices or a subalgebra of it. But we were not able to define good acyclic matchings for the general case. Computer algorithms gave us some conjectures (see Section 1.4).

Nevertheless, we are convinced that there are Lie algebras L where Algebraic Discrete Morse theory produces good results. Sködberg, for example, calculates in [**41**] - using Algebraic Discrete Morse theory - the homology of the nilpotent Lie algebra, generated by $\{x_1,\dots,x_n,y_1,\dots,y_n,z\}$ with the only nonvanishing Lie bracket being $[z,x_i] = y_i$ over a field of characteristic 2.
Note that the Lie algebra studied by Sköedberg is quasi-isomorphic (see Definition 2.5) to the Heisenberg Lie algebra.

1.4. Conjectures and Open Questions. In this section we present some conjectures and open questions on the homology of Lie algebras.

Consider the homology of $L = \mathfrak{n}(n,k)$. Then Kostant theorem implies that

$$\dim_{\mathbb{C}} H_i(L) = \#\{\sigma \in S_n \mid \#\,\mathrm{Inv}(\sigma) = i\}. \tag{1.1}$$

The first question one can ask is: what is the dimension of the homology with coefficients in fields of other characteristic or with coefficients in $\mathbb{Z}$? It turns out that this question is a very hard problem. The only known result about this question is the following, proved by Dwyer [**17**]:

$$\dim_{\mathbb{Z}_p} H_i(\mathfrak{n}(n,k)) = \dim_{\mathbb{C}} H_i(\mathfrak{n}(n,k)) \text{ for all } p \geq n-1.$$

Computer experiments up to $n \leq 7$ show that for small prime numbers p the homology of $\mathfrak{n}(n,k)$ depends strongly on p (see Table 1.4).

We tried to find some number sequences which describe the torsion of the Lie algebra $\mathfrak{n}(n,k)$. Using Algebraic Discrete Morse theory and Computer experiments We get the following conjectures:

1.4 Homology of $\mathfrak{n}(n,k)$ with coefficients in $\mathbb{Z}$

i/n	2	3	4	5	6
0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
1	$\mathbb{Z}$	$\mathbb{Z}^2$	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$
2		$\mathbb{Z}^2$	$\mathbb{Z}^5 \oplus \mathbb{Z}_2$	$\mathbb{Z}^9 \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^{14} \oplus \mathbb{Z}_2^3$
3		$\mathbb{Z}$	$\mathbb{Z}^6 \oplus \mathbb{Z}_2$	$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^6 \oplus \mathbb{Z}_6^2$	$\mathbb{Z}^{29} \oplus \mathbb{Z}_2^{16} \oplus \mathbb{Z}_6^4$
4			$\mathbb{Z}^5$	$\mathbb{Z}^{20} \oplus \mathbb{Z}_2^7 \oplus \mathbb{Z}_6^3$	$\mathbb{Z}^{49} \oplus \mathbb{Z}_2^{37} \oplus \mathbb{Z}_6^{10} \oplus \mathbb{Z}_{12}^3$
5			$\mathbb{Z}^3$	$\mathbb{Z}^{22} \oplus \mathbb{Z}_2^7 \oplus \mathbb{Z}_6^3$	$\mathbb{Z}^{71} \oplus \mathbb{Z}_2^{62} \oplus \mathbb{Z}_6^{17} \oplus \mathbb{Z}_{12}^9$
6			$\mathbb{Z}$	$\mathbb{Z}^{20} \oplus \mathbb{Z}_2^6 \oplus \mathbb{Z}_6^2$	$\mathbb{Z}^{90} \oplus \mathbb{Z}_2^{95} \oplus \mathbb{Z}_6^{23} \oplus \mathbb{Z}_{12}^{12}$
7				$\mathbb{Z}^{15} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^{101} \oplus \mathbb{Z}_2^{114} \oplus \mathbb{Z}_6^{24} \oplus \mathbb{Z}_{12}^{12}$
8				$\mathbb{Z}^9$	$\mathbb{Z}^{101} \oplus \mathbb{Z}_2^{95} \oplus \mathbb{Z}_6^{23} \oplus \mathbb{Z}_{12}^{12}$
9				$\mathbb{Z}^4$	$\mathbb{Z}^{90} \oplus \mathbb{Z}_2^{62} \oplus \mathbb{Z}_6^{17} \oplus \mathbb{Z}_{12}^9$
10				$\mathbb{Z}$	$\mathbb{Z}^{71} \oplus \mathbb{Z}_2^{37} \oplus \mathbb{Z}_6^{10} \oplus \mathbb{Z}_{12}^3$
11					$\mathbb{Z}^{49} \oplus \mathbb{Z}_2^{16} \oplus \mathbb{Z}_6^4$
12					$\mathbb{Z}^{29} \oplus \mathbb{Z}_2^3$
13					$\mathbb{Z}^{14}$
14					$\mathbb{Z}^5$
15					$\mathbb{Z}$

Conjecture 1.16. (1) *Every torsion $p^i \leq n-2$, with p prime, appears in $H_\bullet(\mathfrak{n}(n,k))$.*

(2) *If r is the largest number such that $\mathbb{Z}_r$ appears in $H_\bullet(\mathfrak{n}(n,k))$, then it appears in $2 \cdot \frac{n!}{24}$ copies.*

The next question one can ask is about the homology of subalgebras of $\mathfrak{n}(n,k)$. Does there exist a similar combinatorial description of the dimension of the ith homology?

Equation (1.1) is still true for the nilpotent part of Lie algebras of other root systems. The number of the ith homology is then given by the number of elements in the Weyl group of length i. Therefore, one can ask the same question for subalgebras of the nilpotent part of Lie algebras associated to other root systems.

A lot of other combinatorial problems in Lie algebra homology can be found in the articles “A Survey of Combinatorial Problems in Lie Algebra Homology” and “Some Conjectures and Results Concerning the Homology of Nilpotent Lie Algebras” both written by Hanlon ([**27**] and [**28**]).

2. New Invariance Theorem for Nilpotent Lie Algebras of Finite Type

In this paragraph we develop for nilpotent Lie algebras a new type of isomorphism and prove a new invariance theorem for the homology of a Lie algebra.

Let L be a Lie algebra and $\mathcal{B}$ a basis of L as a k-vectorspace. We call L a Lie algebra of finite type if the set of Lie relations

$$\big\{(a,b,[a,b]) \;\big|\; a,b \in \mathcal{B} \text{ with } [a,b] \neq 0\big\}$$

is finite. Since L is nilpotent, it follows that $[a,b] \neq a, b$ for all $a, b \in \mathcal{B}$.
Let $\mathcal{R} \subset \{(a, b, [a,b]) \mid a, b \in \mathcal{B} \text{ with } [a,b] \neq 0\}$ be the subset of the Lie relations such that for all $a, b \in \mathcal{B}$ with $[a,b] \neq 0$ we have either $(a, b, [a,b]) \in \mathcal{R}$ or $(b, a, [b,a]) \in \mathcal{R}$. We call $\mathcal{R}$ *the set of positive Lie relations.*

We introduce the following additional condition:

(2.1) For each generator $e_\alpha \in \mathcal{B}$ there exists a generator $e_\beta \in \mathcal{B}$ such that $(e_\alpha, e_\beta, [e_\alpha, e_\beta]) \in \mathcal{R}$.

From now on we only consider Lie algebras of finite type satisfying the additional condition (2.1).

First we define a new type of isomorphisms, namely quasi-isomorphism, depending on the set of positive Lie relations. Before we are able to give the definition of a quasi-isomorphism, we have to introduce the notion of a two-colored simplicial complex:

Definition 2.1. Let Δ be a pure simplicial complex of dimension d with n facets. We denote by $\mathcal{F}_i := \{F \in \Delta \mid \dim(F) = i\}$ the set of faces of dimension i, for $i = 0, \ldots, d$. On the set of facets we fix an order and write:

$$\mathcal{F}_d := \{F_1 < F_2 < \ldots < F_n\}.$$

(1) A two-coloring of a facet F is a map f_F which associates to each zero-dimensional face of F a color from $\{r, g\}$ ($r =$red, $g =$green).

(2) Given a two-coloring f_{F_i}, $i = 1, \ldots, n$, the pair (Δ, f) is a two-colored simplicial complex if f is a map defined by

$$\begin{array}{rcl} f : \mathcal{F}_0 & \to & \{r, g, -\}^n \\ v & \mapsto & (f_1(v), \ldots, f_n(v)), \end{array}$$

where $n := \#\mathcal{F}_d$ is the number of facets of Δ and

$$f_i(v) := \begin{cases} f_{F_i}(v) & , \quad v \in F_i \\ - & , \quad v \notin F_i. \end{cases}$$

For a color vector $c := (c_1, \ldots, c_n) \in \{r, g, -\}^n$ we define the complement vector $\overline{c} := (\overline{c_1}, \ldots, \overline{c_n})$ by

$$\overline{c_i} := \begin{cases} r & , \quad c_i = g \\ g & , \quad c_i = r \\ - & , \quad c_i = - \end{cases}$$

Definition 2.2. Let (Δ, f) and (Δ', f') be two two-colored simplicial complexes, with n facets.
We say that (Δ, f) is isomorphic to (Δ', f'), if

(1) Δ and Δ' are isomorphic as simplicial complexes and

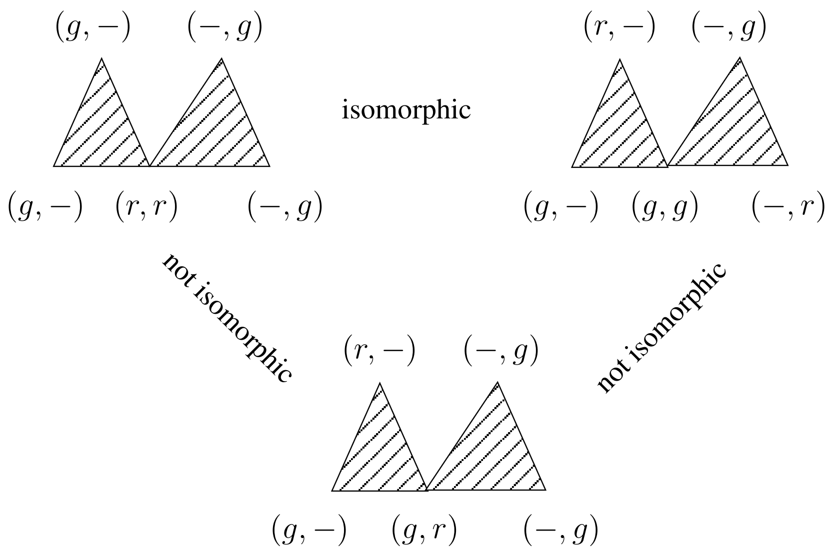
(2) there exists a permutation $\sigma \in S_n$ such that for all vertices v we have

$$f'(v) \in \left\{\sigma(f(v)),\ \overline{\sigma(f(v))}\right\},$$

where σ acts by permuting coordinates.

Since the color vector is an ordered tuple and depends on the chosen order on the facets $\mathcal{F}_d := \{F_1 < F_2 < \ldots < F_n\}$, we need the permutation σ in order to define a non-trivial isomorphism. By relabeling the facets one can always assume that the ith facet of Δ is mapped to the ith facet of Δ'. In this case we can omit the permutation in Definition 2.2.

Example 2.3.



Now let L be a nilpotent Lie algebra of finite type and $\mathcal{R}$ the set of positive Lie relations. To this set of Lie relations we associate a two-colored simplicial complex.

Definition 2.4. Let L be a finitely generated Lie algebra with Lie relations $\mathcal{R}$. The two-dimensional two-colored simplicial complex (Δ, f) defined by

$$\Delta = \Delta(\mathcal{R}) := \bigcup_{(a,b,c)\in\mathcal{R}} \Delta\big((a,b,c)\big),$$

$$\Delta\big((a,b,c)\big) := \big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\big\},$$

and coloring

$$f_{(a,b,c)}(v) := \begin{cases} g & , \quad v = a,b \\ r & , \quad v = c = [a,b]. \end{cases}$$

is called the *Lie relation complex.*

Clearly, the Lie-bracket $[\cdot,\cdot]$ of L is - up to a sign - uniquely determined by the Lie relation complex.

We are now in position to define the new type of isomorphism:

Definition 2.5. Two nilpotent Lie algebras L and L' of finite type are *quasi-isomorphic* if there is a choice of positive Lie relations $\mathcal{R}$ and $\mathcal{R}'$ such that the corresponding two-colored simplicial complexes (Δ, f) and (Δ', f') are isomorphic.

Now we can formulate our main theorem:

Theorem 2.6. *Let L and L' be two quasi-isomorphic nilpotent Lie algebras of finite type. Then*

$$\bigoplus_{i\geq 0} H_i(L) \cong \bigoplus_{i\geq 0} H_i(L').$$

Proof. The homology of the Lie algebras is calculated by the complexes ΛL and $\Lambda L'$. As seen in the previous paragraph, there are two differentials:

$$\begin{aligned} \partial : \Lambda^{(i)} L &\rightarrow \Lambda^{(i-1)} L \\ \partial^t : \Lambda^{(i)} L &\rightarrow \Lambda^{(i+1)} L \end{aligned}$$

We construct an isomorphism $\psi : \Lambda L \rightarrow \Lambda L'$ which commutes with both differentials. Not that ϕ is not necessarily a map of graded vectorspaces, so $\phi(a)$ may have a different homological degree than a. Therefore, we will only be able to deduce an isomorphism

$$\bigoplus_{i \geq 0} H_i(L) \cong \bigoplus_{i \geq 0} H_i(L').$$

We consider the corresponding two-colored simplicial complexes (Δ, f) and (Δ', f'). Let $\phi : (\Delta, f) \rightarrow (\Delta', f')$ be the isomorphism of complexes. Let

$$\begin{aligned} \mathcal{F}_d &:= \{F_1 < \ldots < F_n\}, \\ \mathcal{F}'_d &:= \{F'_1 < \ldots < F'_n\} \end{aligned}$$

be the sets of facets, ordered by an order fulfilling $\phi(F_i) = F'_i$. The map ϕ induces an isomorphism between the sets of color vectors, which we again denote with ϕ:

$$\{f(v) \mid v \in \mathcal{F}_0\} \xrightarrow{\phi} \{f'(v) \mid v \in \mathcal{F}'_0\}.$$

The isomorphism ψ is constructed by the following algorithm. We start with an element $e = v_1 \wedge \ldots \wedge v_r \in \Lambda L$, where the v_i are elements of the chosen basis of L. In the simplicial complex (Δ, f) we mark each point v_j, $1 \leq j \leq r$. Now consider a facet F_i of (Δ, f), for $i = 1, \ldots, n$. We denote by p_1, p_2, p_3 the vertices of F_i and by q_1, q_2, q_3 the vertices of the corresponding facet $F'_i \in \Delta(P')$, where p_j maps to q_j, $j = 1, 2, 3$. For the image of the vertices we have to distinguish two cases. We only consider the ith coordinate of the corresponding color vectors:

$F_i \in (\Delta, f)$	$\phi(F_i) \in (\Delta', f')$
$f(p_1) := (\ldots, r, \ldots)$	$f(q_1) := (\ldots, r, \ldots)$
$f(p_2) := (\ldots, g, \ldots)$	$f(q_2) := (\ldots, g, \ldots)$
$f(p_3) := (\ldots, g, \ldots)$	$f(q_3) := (\ldots, g, \ldots)$

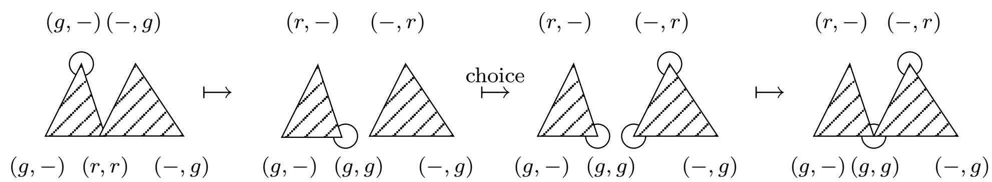
$F_i \in (\Delta, f)$	$\phi(F_i) \in (\Delta', f')$
$f(p_1) := (\ldots, r, \ldots)$	$f(q_1) := (\ldots, g, \ldots)$
$f(p_2) := (\ldots, g, \ldots)$	$f(q_2) := (\ldots, r, \ldots)$
$f(p_3) := (\ldots, g, \ldots)$	$f(q_3) := (\ldots, g, \ldots)$

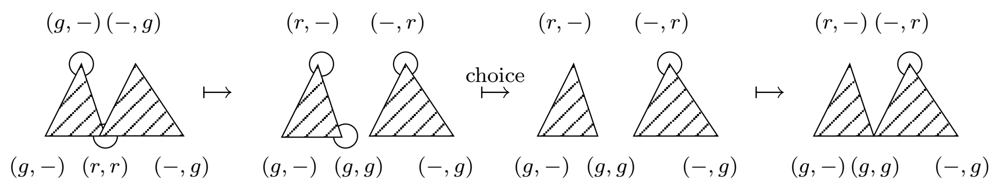
We mark q_3 if and only if p_3 is marked. In the first case we mark the points q_i if and only if the points p_i are marked. In the second case we have to distinguish four cases:

$F_i \in (\Delta, f)$	$\phi(F_i) \in (\Delta', f')$
p_1 is marked, p_2 is not	mark q_2
p_2 is marked, p_1 is not	mark q_1
p_1, p_2 not marked	mark either no vertex or q_1, q_2
p_1, p_2 are both marked	mark either q_1, q_2 or no vertex

In the last two cases we have a choice. In a first step we always choose the first possibility (p_1, p_2 not marked, do not mark q_1, q_2, and p_1, p_2 both marked, mark q_1, q_2). We do this for all facets in (Δ, f). This gives us some marked points for each facet in (Δ', f'). By glueing the facets of (Δ', f') it can happen that marked (resp. unmarked) points are glued to unmarked (resp. marked) points. The choice in the last two cases allows us to avoid such a situation. We explain this by two examples:

Example 2.7.





The aim is to mark the vertices such that marked (resp. unmarked) points are glued to marked (resp. unmarked) points. In order to construct an isomorphism, this process has to be shown reversible. The image of e - up to a sign - is then given by the marked points in the complex (Δ', f'). Finally, we have to show that this process commutes - up to a sign - with both differentials.

We prove this by induction on the number of facets. We assume that both simplicial complexes are connected (otherwise consider the connected components). The first interesting case is if the simplicial complex Δ has two facets. If the two facets have a common one-dimensional face, then the only possible images of Δ are permutations of the labeling of the two vertices of the common edge. Therefore, the map is reversible and commutes - up to a sign - with both differentials.

If the two facets have a common vertex, then Example 2.7 explains that one can always find a marking such that marked (resp. unmarked) points are glued to unmarked (resp. marked) points: Consider the first example. We are not allowed to make the choice if $(-, r)$ is already marked. But in this case the point $(r, -)$ in the image and the point (r, r) in the preimage have to be marked as well, which is a contradiction. Similarly in the second example we are not allowed to make the choice if the point $(r, -)$ in the image is not marked, but then the point (r, r) in the preimage cannot be marked.

Note that for this process it does not matter if the third point is marked or not. In all other cases the argumentation is similar to the above argumentation for Example 2.7.

Since this process is symmetric, it follows that applying the same algorithm on the image produces the preimage, hence we have an isomorphism.

Note that here the condition that a color vector only maps to itself or to its complement plays an important role.

We say that a Lie relation $[a,b]=c$ is applicable in (Δ, f) if either a, b are marked and c is not marked or a, b are not marked and c is marked.

In order to prove that the construction commutes with both differentials, we make the following observation:

- A Lie relation is applicable in the preimage if and only if the corresponding Lie relation is applicable in the image: If a Lie relation is applicable in a facet F of the preimage (resp. image), then for the marking of the vertices in the image (resp. preimage) of F we have no choice. Since marked (resp. unmarked) points are glued to marked (resp. unmarked) points, the corresponding relation is applicable in the image (resp. preimage).
- In the case where we have the choice (mark either q_1 and q_2 or none of them), the set of applicable Lie relations does not change.
- Since a color vector maps either to itself or to its complement and marked (resp. unmarked) points are glued to marked (resp. unmarked) points, we see that if after applying a Lie relation in the preimage (resp. image) there is a new applicable Lie relation in the preimage (resp. image), then we have the same situation in the image (resp. preimage).

These three facts prove that the assignment commutes - up to a sign - with both differentials.

Now assume that Δ has n facets. We start the algorithm with two connected facets. Then the image is already well defined. We choose a facet which is connected to the two facets. With the same arguments as before we get a well defined image. The assignment is reversible and commutes with both differentials. Then choose a facet connected to the first three and choose the correct marking. We continue in this way step by step. With the same arguments as in the case $n=2$, we finally reach a marking in the image and the assignment is reversible and commutes with both differentials. This completes our proof. $\square$

3. Applications to Lie Algebras of Root Systems

In this paragraph we draw some corollaries of Theorem 2.6, in the case where L is the nilpotent part of a Lie algebra associated to root systems.

Let V be any vectorspace over a field k and $\Phi \subset V$ a root system. Recall from the first paragraph that Φ decomposes into a positive and a negative part,

$$\Phi = \Pi^+ \cup \Pi^-.$$

Let $\Delta := \{\alpha_1, \ldots, \alpha_l\} \subset \Pi^+$ be a basis for Φ. The nilpotent part of the Lie algebra associated to the root system Φ is generated as Lie algebra by elements x_i, $1 \le i \le l$, with respect to the condition

(S^+_{ij}) $(\operatorname{ad} x_i)^{-\langle \alpha_j, \alpha_i \rangle + 1}(x_j) = 0$ for $i \neq j$.

On the other hand, we know that $L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$, where L_α is a one-dimensional space, i.e. $L_\alpha = \mathbb{C}\, X_\alpha$. In order to construct the nilpotent part of the Lie algebra associated to a root system $\Phi = \Pi^+ \cup \Pi^-$, we have to define basis elements X_α, for $\alpha \in \Pi^+$, and relations

$$[X_\alpha, X_\beta] = \begin{cases} 0 & , \quad \alpha+\beta \notin \Pi^+ \\ c_{\alpha,\beta}\; X_{\alpha+\beta} & , \quad \alpha+\beta \in \Pi^+, c_{\alpha,\beta} \in \mathbb{C}. \end{cases}$$

The nilpotent part is then given by

$$NL^+ := \bigoplus_{\alpha \in \Pi^+} \mathbb{C}\; X_\alpha.$$

Example 3.1. (1) Let $V = \mathbb{R}^n$ be the Euclidean space and $\Phi := \{\varepsilon_i - \varepsilon_j \mid 1 \le i, j \le n\}$ the root system A_{n-1}, where e_i is the ith unit vector in V. Then Φ decomposes in

$$\begin{aligned} \Pi^+ &:= \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}, \\ \Pi^- &:= \{\varepsilon_i - \varepsilon_j \mid 1 \le i > j \le n\}. \end{aligned}$$

To the vector $\alpha = \varepsilon_i - \varepsilon_j$ we associate the matrix $X_\alpha := e_{ij}$. The nilpotent part of the Lie algebra associated to A_{n-1} is given by

$$L := \bigoplus_{1 \le i < j \le n} \mathbb{C}\; e_{ij},$$

with Lie relations

$$[e_{ij}, e_{kl}] := \delta_{jk} e_{il} - \delta_{il} e_{kj}.$$

Clearly, this is exactly the algebra $\mathfrak{n}(n,k)$ of strictly upper triangular matrices.

(2) Consider the root system $B_n = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}$. We choose a positive system $\Pi^+ := \{\varepsilon_i, \varepsilon_i \pm \varepsilon_j\}$. Now we choose matrices X_α, for each $\alpha \in \Pi^+$:

$$\begin{aligned} X_{\varepsilon_i - \varepsilon_j} &:= e_{i+1,j+1} - e_{l+j+1,l+i+1} \\ X_{\varepsilon_i + \varepsilon_j} &:= e_{i+1,l+j+1} - e_{j+1,l+i+1} \\ X_{\varepsilon_i} &:= e_{1,l+i+1} - e_{i+1,1} \end{aligned}$$

The nilpotent part is then given by

$$L := \bigoplus_{\alpha \in \Pi^+} \mathbb{C}\; X_\alpha.$$

We now consider closed subsets of the positive part of the root system Φ.

Definition 3.2. A subset $S \subset \Pi^+$ of the set of positive roots is called *closed* iff for any $\alpha, \beta \in S$ we have

$$\alpha + \beta \in \Pi^+ \implies \alpha + \beta \in S.$$

In the same way as before, we can associate to S a nilpotent Lie algebra:

$$NL_S := \bigoplus_{\alpha \in S} \mathbb{C}\; X_\alpha.$$

Clearly, NL_S is a subalgebra of the nilpotent part NL^+ of the Lie algebra associated to the root system Φ.

We are now interested in finding conditions on the closed subsets of a root system such that the corresponding nilpotent Lie algebras are quasi-isomorphic.

3.1. Homology of Lie Algebras Associated to A_n. The first application of our Theorem 2.6 we get for Lie algebras associated to closed subsets of the root system A_{n-1}.
The nilpotent part of the Lie algebra corresponding to A_{n-1} is the Lie algebra $\mathfrak{n}(n,k)$.

We consider closed subsets of the root system A_{n-1}. Note that they are in one-to-one correspondence with partially ordered sets P, with $\#P \leq n$:

Let P be a finite partially ordered set.

Definition 3.3 (Lie algebra associated to P)**.** For any $p < q \in P$ we define the matrix e_{pq} to be the matrix having 1 in the (p,q) coordinate and 0 elsewhere. Let $\mathcal{B}$ be the set of all those matrices. The Lie algebra L associated to the partially ordered set P is given by

$$L(P) := \bigoplus_{e \in \mathcal{B}} \mathbb{C}\ e.$$

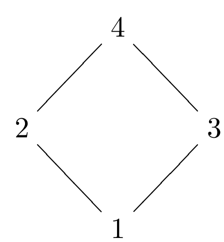
It is the subalgebra of $\mathfrak{n}(n,k)$ consisting of those matrices with $a_{ij} = 0$ if $i \not<_P j$.

Lemma 3.4. *The vectorspace L from Definition 3.3 with the Lie-bracket inherited from the Lie algebra $\mathfrak{n}(n,k)$ is a Lie algebra.*

Proof. For a partially ordered set P we have: $p < q$ and $q < u$ implies $p < u$. Thus, it is straightforward to check that this fact proves the Jacobi identity and the fact that the relation $[e_{pq}, e_{uv}]$ is either zero or an element of $\mathcal{B}$. □

It is easy to see that the Lie algebras $L(P)$ are in one-to-one correspondence with the nilpotent part of Lie algebras corresponding to closed subsets of A_n.

Example 3.5. Let P be the partially ordered set given by the Hasse diagram



Then the Lie algebra L associated to P is given by

$$L := V\ e_{12} \oplus V\ e_{13} \oplus V\ e_{14} \oplus V\ e_{24} \oplus V\ e_{34}.$$

Note that the Lie relations are in one-to-one correspondence with two-chains in $\Delta(P)$ (see Figure 1). In order to translate the notion of quasi-isomorphism to partially ordered sets, we introduce two types of pairs of two-chains (see Figure 2 and Figure 3).

Definition 3.6. We call two partially ordered sets P and P' strongly isomorphic if their corresponding order complexes $\Delta(P)$ and $\Delta(P')$ are isomorphic as simplicial complexes and if there exists an isomorphism

$$\phi : \Delta(P) \to \Delta(P')$$

that maps pairs of two-chains of Type 1 to pairs of two-chains of Type 1 and pairs of two-chains of Type 2 to pairs of two-chains of Type 2.

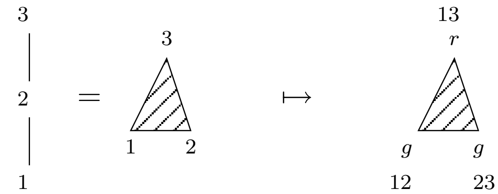


Figure 1.

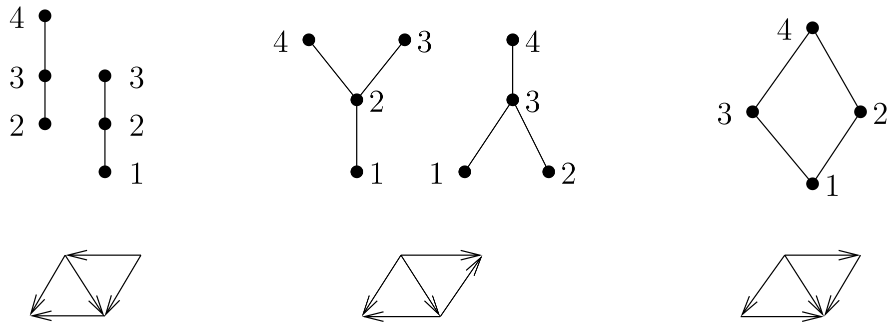


Figure 2. Pairs of two-chains of Type 1

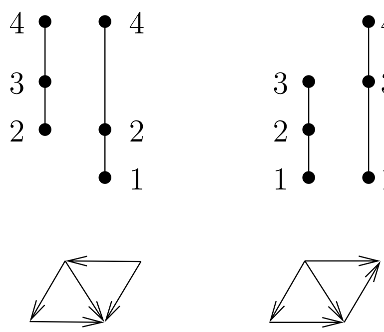


Figure 3. Pairs of two-chains of Type 2

With this notation we get the following proposition:

Proposition 3.7. *Let P and P' be two partially ordered sets and $L(P)$ and $L(P')$ their corresponding Lie algebras.*

(1) *If P and P' are strongly isomorphic, then $L(P)$ and $L(P')$ are quasi-isomorphic.*
(2) *If $L(P)$ and $L(P')$ are quasi-isomorphic with respect to the standard basis, then P and P' are strongly isomorphic.*

Proof. If P and P' are strongly isomorphic, then by definition there is an isomorphism ϕ between the order complexes. The map ϕ induces an isomorphism between the Lie relation complexes. The fact that ϕ maps pairs of two-chains of Type i to pairs of two-chains of Type i, $i = 1, 2$, implies that the induced map ϕ maps color vectors only to themselves or to their complements. It follows that the Lie algebras $L(P)$ and $L(P')$ are quasi-isomorphic.
On the other hand, if the Lie algebras $L(P)$ and $L(P')$ are quasi-isomorphic

with respect to the standard basis, then there is an isomorphism ϕ between the Lie relation complex. Since the Lie relations are in one-to-one correspondence with two-chains in $\Delta(P)$, the map ϕ induces an isomorphism between the two-skeletons. Since order complexes are flag complexes (i.e. every minimal non-face is one-dimensional), ϕ induces an isomorphism between the whole order complexes. The fact that ϕ maps color vectors only to themselves or to their complements implies that the induced map ϕ maps pairs of two-chains of Type i to pairs of two-chains of Type i, $i = 1, 2$. □

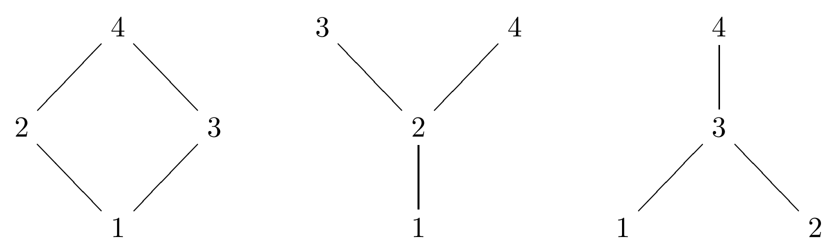
Corollary 3.8. *Let P, P' be two strongly isomorphic partially ordered sets. Then*

$$\bigoplus_{i\geq 0} H_i(L) \cong \bigoplus_{i\geq 0} H_i(L').$$

Proof. The assertion is a direct consequence of Proposition 3.7. □

In Example 3.5 we get:

Example 3.9. The homology of the Lie algebras associated to the following three partially ordered sets are isomorphic:



Since there is no three-chain, every isomorphism ϕ between the order complexes maps pairs of two-chains of Type i to pairs of two-chains of Type i, $i = 1, 2$. Therefore, Corollary 3.8 implies the assertion.

For all poset transformations which preserve the order complexes we only have to check if there exists an isomorphism between the order complexes with the desired conditions. We present one example:

Let P, P' be two partially ordered sets. We define the new poset $P\#P'$ by

$$p < q \Longleftrightarrow \begin{cases} p <_P q & , \quad p, q \in P \\ p <_{P'} q & , \quad p, q \in P' \\ \text{true} & , \quad p \in P, q \in P'. \end{cases}$$

Corollary 3.10. *If P, P' are two partially ordered sets, then*

$$\bigoplus_i H_i\big(L(P\#P')\big) = \bigoplus_i H_i\big(L(P'\#P)\big).$$

Proof. Clearly, $\Delta(P\#P') \cong \Delta(P'\#P)$. We define the isomorphism between the order complexes on the maximal chains:
Let $p_1 < \ldots < p_l$ be a maximal chain in $P\#P'$. By the definition of $P\#P'$ there exists an index j such that $p_1 < \ldots < p_j$ is a maximal chain in P and $p_{j+1} < \ldots < p_l$ is a maximal chain in P'. We define ϕ as follows:

$$\phi\big(p_1 < \ldots < p_l\big) := p_{j+1} < \ldots < p_l < p_1 < \ldots < p_j.$$

It follows directly from the definition of ϕ that ϕ maps pairs of two-chains of Type i to pairs of two-chains of Type i, $i = 1, 2$. Therefore, $P\#P'$ and $P'\#P$ are strongly isomorphic. The assertion follows then from Corollary 3.8. □

3.2. Homology of Lie Algebras Associated to other Root Systems. If we consider subalgebras of the nilpotent part of other root systems, then clearly Theorem 2.6 is still applicable since it holds for any finitely generated nilpotent Lie algebra. But in general it is harder to find nice conditions on the closed subsets of the root systems since we do not have a nice combinatorial description as in the case A_n. Reiner gave in [**43**] and [**44**] combinatorial interpretations for the root system B_n. Therefore, we think that Type B_n results analogue to the ones in Section 3.1 are within reach. We leave this as a project for further research.

The Neggers-Stanley Conjecture

In this chapter we study the Neggers-Stanley conjecture, also known as the poset conjecture. In the first paragraph we introduce the conjecture and we present several equivalent conjectures. Here we follow the notes of Brenti [**11**]. In the second paragraph we study the special case where (P, ω) is naturally labeled. Reiner and Welker proved unimodality of the W-polynomial of graded posets. In fact, they proved that the W-polynomial coincides with the h-vector of a certain simplicial complex Δ_{eq}, which is a polytopal sphere and therefore has a unimodal h-vector. We explain their results and sketch their proof.
Paragraph 3 contains of our results. It is known that the W-polynomial appears in the nominator of the Hilbert series of the Hibi ring. Using the results of Part 1 of this thesis we can prove a recursion formula for the W-polynomial.
In the second part we generalize the idea of Reiner and Welker to arbitrary naturally labeled posets: Given any naturally labeled poset P, we construct an analogous simplicial complex Δ_{eq} and show in the same way that the h-vector of Δ_{eq} coincides with the W-polynomial. In general, Δ_{eq} is not a polytopal sphere. But we prove that in general Δ_{eq} is isomorphic to the intersection of a polytopal sphere $\hat{\Delta}_{eq}$ with an intersection of coordinate and diagonal hyperplanes.

1. The Poset Conjecture

In this paragraph we explain the Neggers-Stanley conjecture. We present some elementary results and some equivalent formulations of it. Finally, we list some special cases in which the conjecture is known to be true and we present the counterexample for the general case, which was found recently by Brändén [**10**]. We follow in this paragraph the notes of Brenti [**11**].

Let P be a (finite) partially set. A *labeling* of P is a bijection $\omega : P \to \{1, \ldots, \#P\}$. The pair (P, ω) is called a *labeled poset*. We call a labeling natural if $x, y \in P$ and $x \leq y$ implies $\omega(x) \leq \omega(y)$ in the natural order.

If (P,ω) is a labeled poset and $s \in \mathbb{N}$ then a (P,ω)-partition with largest part $\leq s$ is an order-reversing map $\sigma : P \to [s]$ such that $x < y$ and $\omega(x) > \omega(y)$ implies $\sigma(x) > \sigma(y)$.

We denote with $\Omega(P,\omega;s)$ (respectively, $e_s(P,\omega)$) the number of (P,ω)-partitions (respectively, surjective (P,ω)-partitions) with largest part $\leq s$. It is easy to see that

$$\Omega(P,\omega;x) = \sum_{s=1}^{\#P} e_s(P,\omega) \binom{x}{s}$$

for all $x \in N$. This shows that $\Omega(P,\omega;x)$ is a polynomial function of x of degree $\#P$. The polynomial $\Omega(P,\omega;x)$ is called the order polynomial of (P,ω).

From the theory of generating functions it follows that there exists a polynomial $W(P,\omega;z) \in \mathbb{R}[x]$ of degree $\leq \#P$ such that

$$\sum_{n\geq 0} \Omega(P,\omega;n)\ z^n = \frac{W(P,\omega;z)}{(1-z)^{\#P+1}}, \tag{1.1}$$

as formal power series in z. We are now in position to formulate the conjecture:

Conjecture 1.1 (poset conjecture)**.** *For all labeled posets (P,ω) the polynomial $W(P,\omega;z)$ defined by Equation (1.1) has only real zeros.*

In the case where ω is a natural labeling of P, the poset conjecture was first stated in 1978 by Neggers. In the present form it has been first conjectured by Stanley in 1986.

Recently, P. Brändén [**10**] has found a labeled poset (P,ω) for which the poset conjecture does not hold. We present it at the end of this paragraph. Since his labeling is not natural, the following weaker conjecture is still open:

Conjecture 1.2 (poset conjecture)**.** *For all naturally labeled posets (P,ω) the polynomial $W(P,\omega;z)$ defined by Equation (1.1) has only real zeros.*

The polynomial $W(P,\omega;z)$ has a combinatorial interpretation. Before we can state it, we need some additional definitions:

Let P be a poset. A *linear extension* is an order-preserving bijection $\tau : P \to \{1,\ldots,\#P\}$, i.e. a natural labeling of P. We write $\mathcal{L}(P)$ for the set of all linear extensions. If (P,ω) is a labeled poset and τ is a linear extension of P, then we let

$$D(\tau,\omega) := \big\{ i \in [\#P-1] \bigm| \omega(\tau^{-1}(i)) > \omega(\tau^{-1}(i+1)) \big\}$$

be the set of descents and denote with $d(\tau,\omega) := \#D(\tau,\omega)$ the cardinality of it. Now it is possible to show that

$$W(P,\omega;z) = \sum_{\tau\in\mathcal{L}(P)} z^{d(\tau,\omega)+1}. \tag{1.2}$$

Sometimes the polynomial $W(P,\omega;z)$ is defined by the following equation

$$W(P,\omega;z) = \sum_{\tau\in\mathcal{L}(P)} z^{d(\tau,\omega)}.$$

Since both polynomials only differ by a factor z, one of them has only real zeros if and only if the other has.

From the definition of the W-polynomial and Equation (1.2) we obtain

$$\begin{aligned} W(P,\omega;z) &= (1-z)^{\#P+1}\sum_{n\geq 0}\sum_{s=1}^{\#P} e_s(P,\omega)\binom{n}{s}z^n \\ &= (1-z)^{\#P+1}\sum_{s=1}^{\#P} e_s(P,\omega)\frac{z^s}{(1-z)^{s+1}} \\ &= (1-z)^{\#P}E\left(P,\omega;\frac{z}{(1-z)}\right), \end{aligned}$$

where $E(P,\omega;z) := \sum_{s=1}^{\#P} e_s(P,\omega)z^s$. Therefore, we can reformulate the poset conjecture in the following form:

Conjecture 1.3. *For any labeled poset (P,ω) the polynomial $E(P,\omega;z)$ defined above has only real zeros.*

There are some weaker conjectures about the polynomials E and W, which follow from the poset conjecture. We need some more definitions.

A sequence $\{a_0, a_1, \ldots, a_d\}$ of real numbers is called *log-concave* if $a_i^2 \geq a_{i-1}a_{i+1}$ for $i = 1, \ldots, d-1$. It is said to be *unimodal* if there exists an index $0 \leq j \leq d$ such that $a_i \leq a_{i+1}$ for $i = 0, \ldots, j-1$ and $a_i \geq a_{i+1}$ for $i = j, \ldots, d-1$ and is said to have internal zeros if there are not three indices $0 \leq i < j < k \leq d$ such that $a_i, a_k \neq 0$ and $a_j = 0$. We say that a polynomial $\sum_{i=0}^{d} a_i x^i$ is log-concave with no internal zeros (resp. unimodal) if the sequence $\{a_0, a_1, \ldots, a_d\}$ has the corresponding property. The following theorem gives the connection to the poset conjecture:

Theorem 1.4. *Let $\sum_{i=0}^{d} a_i x^i$ be a polynomial with nonnegative coefficients and with only real zeros. Then the sequence $\{a_0, a_1, \ldots, a_d\}$ is log-concave with no internal zeros; in particular, it is unimodal.* □

We are now able to formulate weaker conjectures:

Conjecture 1.5. *For any labeled poset (P,ω) the polynomials $W(P,\omega;x)$ and $E(P,\omega;x)$ are*

(1) *log-concave with no internal zeros and*
(2) *unimodal.*

We finish this section by presenting one general class of posets and labelings for which the poset conjecture is true.

Theorem 1.6. *Let P be a disjoint union of chains. Then the polynomial $W(P;z)$ has only real zeros.*

But if one adds one more order relation, the result turns out to be false. The example was found by Brändén [**10**]:

From now on we define the W-polynomial by the equation

$$W(P,\omega;z) = \sum_{\tau\in\mathcal{L}(P)} z^{d(\tau,\omega)}.$$

Let $\mathbf{m} \sqcup \mathbf{n}$ be the disjoint union of the chains $1 < 2 < 3 \ldots < m$ and $m+1 < m+2 < \ldots < m+n$. Theorem 1.6 implies that $W(\mathbf{m} \sqcup \mathbf{n}, z)$ has only real zeros.

Now we define $P_{m,n}$ to be the partially ordered set $\mathbf{m} \sqcup \mathbf{n}$ together with the additional relation $m+1 < m$.

Theorem 1.7 (Brändén, [**10**]). *If m, n are large enough, then the W-polynomial with respect to the labeled poset $P_{m,n}$ has non-real zeros.*

Here are two examples: Consider $P_{11,11}$. Then $W(P,z)$ has two non-real zeros, which are approximately $z = -0.10902 \pm 0.01308i$.
Another example is

$$W(P_{36,6}, z) = 216t + 9450t^2 + 142800t^3 + 883575t^4 + 2261952t^5 + 1947792t^6.$$

This polynomial has two non-real zeros.

Just before finishing this thesis, the author learned that there is also a counterexample to the real-rootness of the W-polynomial in the naturally labeled case, which is still unpublished and therefore we cannot give a reference.

For the counterexamples found by Brändén, the polynomial $W(P, \omega, z)$ is still unimodal. Therefore, Conjecture 1.5 is still open.

2. The Naturally Labeled Case for Graded Posets

If (P, ω) is a naturally labeled poset, then the combinatorial interpretation of the polynomials $W(P, \omega; z)$ and $E(P, \omega; z)$ becomes particularly nice and simple. In fact, for $s \in \mathbb{N}$, $e_s(P, \omega)$ is the number of surjective order-reversing (equivalently preserving) maps $P \to \mathbf{s}$ (where $\mathbf{s}$ is a chain of s elements) while the coefficient of z^s in $W(P, \omega; z)$ equals the number of linear extensions of P with exactly s descents.

In the naturally labeled case, we get another description of the W-polynomial via a Hilbert series of a polynomial ring, namely the Hibi ring. In order to give this description, we have to introduce the lattice of order ideals:

Definition 2.1. Let (P, ω) be a naturally labeled partially ordered set. An order ideal $I \subset P$ is a subset of P satisfying the condition

$$p \in I,\ q \prec p \ \Rightarrow q \in I.$$

We denote with $\mathcal{J}(P)$ the set of all order ideals ordered by inclusion.

We have the following property:

Theorem 2.2. (1) *$\mathcal{J}(P)$ is distributive lattice.*
(2) *Every finite distributive lattice is of the form $\mathcal{J}(P)$ for some poset P.*

Now, if $\sigma : P \to \mathbf{s}$ is a surjective order-preserving map, then $\sigma^{-1}([1]) \subset \sigma^{-1}([2]) \subset \ldots \subset \sigma^{-1}([s-1])$ is a chain of $s-1$ order ideals in $\mathcal{J}(P)$ and this correspondence is a bijection. Therefore, we may think of $e_s(P, \omega)$ as the number of chains of length s from $\emptyset$ to P in $\mathcal{J}(P)$. Therefore, we can restate the poset conjecture as follows:

Conjecture 2.3. *Let D be a finite distributive lattice and let, for $s \in \mathbb{N}$, $c_s(D)$ be the number of chains of length s from $\hat{0}$ to $\hat{1}$ in D. Then the polynomial $C(D,z) := \sum_{s=0}^{|P|} c_s(D)z^s$ has only real zeros.*

To the set of order ideals $\mathcal{J}(P)$ one can associate a polynomial ring, called the Hibi ring:

Definition 2.4. Let (P,ω) be a naturally labeled partially ordered set and $\mathcal{J}(P)$ the set of order ideals. The polynomial ring

$$R(P) := \frac{k[x_I,\ I \in \mathcal{J}(P)]}{\langle x_I x_J - x_{I\cap J} x_{I\cup J}\rangle}$$

with the bigrading

$$\begin{aligned} |x_I| &:= |I|, \\ \deg(x_I) &:= \alpha_I \in \mathbb{N}^{\#P} \end{aligned}$$

is called the Hibi ring.
Here $(\alpha_I)_i = \begin{cases} 1 & , \ \omega^{-1}(i) \in I \\ 0 & , \ \omega^{-1}(i) \notin I \end{cases}$ is the characteristic vector of the order ideal.

The following theorem gives one more description of the W-polynomial.

Theorem 2.5. *Let $R(P)$ be the Hibi ring. Then the Hilbert series* $\mathrm{Hilb}_{R(P)}(\underline{x},t)$ *is given by*

$$\mathrm{Hilb}_{R(P)}(1,t) = \frac{W(P,\omega,t)}{(1-t)^{\#P+1}}.$$

Since we are able to calculate the multigraded Hilbert series of this kind of polynomial rings (see Part 1, Chapter 6), we will get a description of the W-polynomial in terms of **nbc**-sets of an undirected graph (see Paragraph 3).

With this characterization of the W-polynomial, Reiner and Welker were able to prove unimodality for graded posets P:

Theorem 2.6 (Reiner, Welker, [**45**])**.** *Let (P,ω) be a graded naturally labeled partially ordered set. Then the W-polynomial is unimodal.*

The proof uses polytope theory. Since we want to generalize the idea to arbitrary naturally labeled poset, we give in the following section an overview of the proof and its methods.

2.1. Proof of Theorem 2.6. We first describe the topological background of the W-polynomial. For this we have to recall some definitions:
Given an abstract simplicial complex Δ, one can collate the face numbers f_i, which count the number of i-dimensional faces, into its f-vector and f-polynomial

$$\begin{aligned} f(\Delta) &:= (f_{-1}, f_0, f_1, \ldots, f_{d-1}), \\ f(\Delta,t) &:= \sum_{i=0}^{d} f_{i-1}t^i. \end{aligned}$$

The h-polynomial and h-vector are easily seen to encode the same information:

$$h(\Delta) := (h_0, h_1, \ldots, h_d), \text{ where}$$

$$h(\Delta, t) = \sum_{i=0}^{d} h_i t^i \text{ satisfies}$$
$$t^d h(\Delta, t^{-1}) = \left[t^d f(\Delta, t^{-1})\right]_{t \mapsto t-1}.$$

Given a naturally labeled poset P on $[n]$, the vector space of functions $f = (f(1), \ldots, f(n)) : P \to \mathbb{R}$ will be identified with $\mathbb{R}^n$. One says that f is a P-partition if $f(i) \geq 0$ for all i and $f(i) \geq f(j)$ for all $i <_P j$. Denote by $\mathcal{A}(P)$ the cone of all P-partitions in R^n. The convex polytope

$$\mathcal{O}(P) = \mathcal{A}(P) \cap [0,1]^n$$

is called the order polytope of P. It is known that the order polytope is the convex hull of the characteristic vectors $\chi_I \in [0,1]^n$ of the order ideals in P.

To each permutation $\omega = (\omega_1, \ldots, \omega_n) \in S_n$ we define a cone

$$\mathcal{A}(\omega) := \left\{ f \in \mathbb{R}^n \;\middle|\; \begin{array}{c} f(\omega_i) \geq f(\omega_{i+1}), \text{ for } i \in [n-1] \\ f(\omega_i) > f(\omega_{i+1}), \text{ if } i \in \mathrm{Des}(\omega) \end{array} \right\},$$

where $\mathrm{Des}(\omega) = \{i \mid \omega_i > \omega_{i+1}\}$ is the set of descents of ω.

Proposition 2.7 (see Prop. 2.1 in [**45**]). (1) *The cone of P-partitions decomposes into a disjoint union as follows:*

$$\mathcal{A}(P) = \sqcup_{\omega \in \mathcal{L}(P)} \mathcal{A}(\omega).$$

The closure of the cones $\mathcal{A}(\omega)$ for $\omega \in \mathcal{L}(P)$ gives a unimodular triangulation of $\mathcal{A}(P)$.

(2) *The unimodular triangulation of $\mathcal{A}(P)$ described in (1) restricts to a unimodular triangulation of the order polytope*

$$\mathcal{O}(P) = \sqcup_{\omega \in \mathcal{L}(P)} \mathcal{A}(\omega) \cap [0,1]^n.$$

We call the triangulation of $\mathcal{A}(P)$ and $\mathcal{O}(P)$ from Proposition 2.13 their canonical triangulations.

The combinatorics of these triangulations is closely related to the distributive lattice $\mathcal{J}(P)$ of all order ideals:
Given a set of vectors $V \subset R^n$, define their positive span to be the (relatively open) cone

$$\mathrm{pos}(V) := \left\{ \sum_{v \in V} c_v \cdot v \;\mid c_v \in \mathbb{R}, c_v > 0 \right\}.$$

Proposition 2.8 (see Prop. 2.2 in [**45**]). (1) *Every P-partition $f \in \mathcal{A}(P)$ can be uniquely expressed in the form*

$$f = \sum_{i=1}^{t} c_i \chi_{I_i},$$

where the c_i are positive reals and $I_1 \subset \ldots \subset I_t$ is a chain of ideals in P. In other words,

$$\mathcal{A}(P) = \bigsqcup_{\substack{I_1 \subset \ldots \subset I_t \subset P \\ I_j \text{ order ideals}}} \mathrm{pos}\left(\{\chi_{I_i}\}_{i=1}^{t}\right).$$

(2) *The canonical triangulation of the order polytope $\mathcal{O}(P)$ is isomorphic (as an abstract simplicial complex) to $\Delta(\mathcal{J}(P))$, via an isomorphism sending an ideal I to its characteristic vector χ_I.*

(3) *The lexicographic order of permutations in $\mathcal{L}(P)$ gives rise to a shelling order on $\Delta(\mathcal{J}(P))$.*

(4) *In this shelling, for each $\omega \in \mathcal{L}(P)$, the minimal faces of its corresponding simplex in $\Delta(\mathcal{J}(P))$, which is not contained in a lexicographically earlier simplex, are spanned by the ideals $\{\omega_1, \dots, \omega_i\}$ where $i \in \mathrm{Des}(\omega)$.*

Part (4) of the preceding proposition implies the following identity:

$$W(P,t) := \sum_{\omega \in \mathcal{L}(P)} t^{\#\,\mathrm{Des}(\omega)} = h(\Delta(\mathcal{J}(P)), t).$$

In order to prove the unimodality for graded naturally labeled posets P, Reiner and Welker go on as follows:

They exhibit an alternative triangulation of the order polytope $\mathcal{O}(P)$, which they call the equatorial triangulation. Then they show the following properties:

- ▷ It is a unimodular triangulation.
- ▷ It is isomorphic as an abstract simplicial complex, to the join of an r-simplex with a simplicial $(\#P - r - 1)$-sphere, which they denote $\Delta_{eq}(P)$ and call the equatorial sphere.
- ▷ Δ_{eq} is a subcomplex of $\Delta(\mathcal{J}(P))$.
- ▷ The equatorial sphere $\Delta_{eq}(P)$ is polytopal and hence shellable and a PL-sphere.
- ▷ $h(\Delta_{eq}(P), t) = h(\Delta(\mathcal{J}(P)), t) = W(p,t)$.

The last two properties imply the unimodality for the W-polynomial for all graded naturally labeled posets P.

In Section 3.2 we generalize this idea for all naturally labeled posets P. We define a similar triangulation, for which we prove the following properties:

- ▷ It is a unimodular triangulation.
- ▷ It is isomorphic, as an abstract simplicial complex, to the join of an r-simplex with a space $\Delta_{eq}(P)$, which we call the equatorial space.
- ▷ Δ_{eq} is a subcomplex of $\Delta(\mathcal{J}(P))$.
- ▷ $h(\Delta_{eq}(P), t) = h(\Delta(\mathcal{J}(P)), t) = W(p,t)$.

But in the general case, the equatorial space Δ_{eq} is not polytopal and therefore we cannot make conclusions about the h-vector as in [**45**].

Finally, we give the definitions from [**45**] of the r-simplex and the equatorial sphere:
Let P be a graded naturally labeled poset of rank r.

Definition 2.9. A P-partition f will be called *rank-constant* if it is constant along ranks, i.e. $f(p) = f(q)$ whenever $p, q \in P_j$ for some j.

A P-partition f will be called *equatorial* if $\min_{p\in P} f(p) = 0$ and for every $j \in [2, r]$ there exists a covering relation between ranks $j-1, j$ in P along which f is constant, i.e. there exist $p_{j-1} <_P p_j$ with

$$p_{j-1} \in P_{j-1}, p_j \in P_j, \text{ and } f(p_{j-1}) = f(p_j).$$

An order ideal I in P will be called *rank-constant* (resp. *equatorial*) if its characteristic vector χ_I is rank-constant (resp. equatorial).

A collection of order ideals $\{I_1, \ldots, I_t\}$ forming a chain $I_1 \subset \ldots \subset I_t$ will be called *rank-constant* (resp. *equatorial*) if the sum $\chi_{I_1} + \chi_{I_2} + \ldots + \chi_{I_t}$ (or equivalently, any vector in the cone $\text{pos}(\{\chi_{I_i}\}_{i=1}^t)$) is rank-constant (resp. equatorial).

The equatorial sphere $\Delta_{eq}(P)$ is defined to be the subcomplex of the order complex $\Delta(\mathcal{J}(P))$ whose faces are indexed by the equatorial chains of non-empty ideals.

3. The Naturally Labeled Case for General Posets

In this paragraph we present our results. In the first part we calculate the multigraded Hilbert series of the Hibi ring in terms of **nbc**-sets of an undirected graph. This result gives us a reformulation of the poset conjecture in graph-theoretic terms. We develop a recursion formula for the W-polynomial.
In the second section we generalize the construction of Reiner and Welker [**45**] to arbitrary naturally labeled posets P:
Similar to [**45**], we construct a unimodular triangulation of the order polytope, which is isomorphic (as abstract simplicial complex) to the join of a simplex and an "equatorial space" Δ_{eq}. It follows that the W-polynomial coincides with the h-polynomial of Δ_{eq}. In general, Δ_{eq} is not a polytopal sphere, but we prove that it is isomorphic to the intersection of a polytopal sphere $\hat{\Delta}_{eq}$ with an intersection of coordinate and diagonal hyperplanes. Finally, we study the set of possible configurations. Hence our results give a new topological interpretation of the W-polynomial.

3.1. W-Polynomial in Graph Theory.

3.1.1. *The Multigraded Hilbert Series of the Hibi Ring.* In Part 1, Chapter 6, we calculated the multigraded Hilbert series of $k[\Delta(P)]$, where $\Delta(P)$ is the order complex of a partially ordered set P. Remember that the Hibi ring of a poset P is defined by

$$R(P) := k[x_i,\, i \in \mathcal{J}(P)]/\langle x_i x_j - x_{i\cap j} x_{i\cup j}\rangle,$$

with the multigrading defined by $\deg(x_i) := \alpha$, with $t^\alpha = \prod_{j\in w_0^{-1}(i)} t_j =: t^{w_0^{-1}(i)}$. If we order the indeterminates by $x_0 > x_1 > \ldots > x_{\mathcal{J}(P)-1}$, it is easy to prove that $\langle x_i x_j$ with $i \cap j \neq i, j\rangle$ is the initial ideal of $\langle x_i x_j - x_{i\cap j} x_{i\cup j}\rangle$ with respect to the reverse lexicographic order. We write

$$R(P)_{\text{rlex}} := k[x_i \mid i \in \mathcal{J}(P)]/\langle x_i x_j \mid i \cap j \neq i, j\rangle.$$

By standard Gröbner basis arguments we have the following identity:

$$\text{Hilb}_{R(P)}(t, \underline{x}) \quad = \quad \text{Hilb}_{R(P)_{\text{rlex}}}(t, \underline{x}). \tag{3.1}$$

Recall the definition of an **nbc**-set:

Definition 3.1. Let $G = (V, E)$ be an undirected graph and $\prec$ a total order on E. A subgraph $I = (V_I, E_I) \subset G$ is called an **nbc**-set if it contains no broken circuit, i.e.

(1) The graph I contains no circuit and
(2) there exists no edge $c \in E$ such that $E_I \cup \{c\}$ contains a circuit $Z \subset E_I$ and $c = \min_\prec Z$.

Let $\mathfrak{a} \trianglelefteq S$ be an ideal generated in degree two with minimal monomial generating system $\mathrm{MinGen}(\mathfrak{a})$. To $\mathrm{MinGen}(\mathfrak{a})$ we associate an undirected graph $G = (V, E)$ on the vertex set $V = [n]$ by setting $\{i, j\} \in E$ if $x_i x_j \in \mathrm{MinGen}(\mathfrak{a})$. To a subset $I \subset \mathrm{MinGen}(\mathfrak{a})$ we get a corresponding subgraph G_I of G. We call a subset $I \subset \mathrm{MinGen}(\mathfrak{a})$ an **nbc**-set, if G_I contains no broken circuit.

As a direct consequence of Theorem 3.7 of Part 1, Chapter 6, we get a formula for the multigraded Hilbert series of the Hibi ring.

Corollary 3.2. *The multigraded Hilbert series of the Hibi ring is given by*

$$\mathrm{Hilb}_A(t_0, t_1, \ldots, t_{\#P}) := \frac{W(t_0, t_1, \ldots, t_{\#P})}{\prod_{i=0}^{n} (1 - t_0\, t^{w_0^{-1}(i)})},$$

with

$$\begin{aligned} W(t_0, t_1, \ldots, t_{\#P}) &:= 1 + \sum_{I \notin \mathcal{M}} (-1)^{|I|} t_0^{cl(I)+|I|} \underline{t}^{\deg(I)}, \\ &= 1 + \sum_{I \in \mathcal{B}} (-1)^{|I|} t_0^{cl(I)+|I|} \underline{t}^{\deg(I)} \\ &= 1 + \sum_{I \ \mathbf{nbc}\text{-set}} (-1)^{|I|} t_0^{cl(I)+|I|} \underline{t}^{\deg(I)}, \end{aligned}$$

where $\mathcal{M}$ is a standard matching on the Taylor resolution of the ideal $\mathfrak{a} := \langle x_i x_j \mid i \cap j \neq i, j \rangle$ and $\mathcal{B}$ is the set of chains of sting-chains (see Part 1 Chapter 4).

Proof. By Theorem 3.7 of Chapter 6 of Part 1 we get

$$\mathrm{Hilb}_{A_{\mathrm{rlex}}}(t, \underline{x}) = \frac{1 + \sum_{I \notin \mathcal{M}} (-1)^{cl(I)}\, m_I\, (-t)^{cl(I)+|I|}}{\prod_{i \in P} (1 - t\, x_i)}$$

Since we have here a different grading, we have to transform the degrees with the map:

$$\begin{aligned} \phi : \mathbb{N}^{\#O(P)} &\to \mathbb{N}^{\#P} \\ e_i &\mapsto \sum_{j \in w_0^{-1}(i)} \delta_j, \end{aligned}$$

where e_i (resp. δ_i) is the ith unit vector in $\mathbb{N}^{\#O(P)}$ (resp. ith unit vector in $\mathbb{N}^{\#P}$).
The assertion follows then from the identity (3.1). $\square$

3.1.2. *W-Polynomial in Graph Theory.* Given a lattice $\mathcal{L}$ with a natural labeling $\omega : \mathcal{L} \to [\#\mathcal{L}]$, we associate an undirected graph $G(\mathcal{L}) = (V, E)$ on the vertex set $V := [\#\mathcal{L}]$ by setting $\{i, j\} \in E$ if $\omega^{-1}(i)$ is incomparable to $\omega^{-1}(j)$ with respect to the order of $\mathcal{L}$.

Using the notation from Corollary 3.2, Theorem 2.5 implies the following equation:

$$W(t, 1, \ldots, 1) = (1-t)^{\#\mathcal{J}(P)-\#P-1}\, W(P, \omega, t).$$

Therefore, we can reformulate the poset conjecture as follows:

Conjecture 3.3. *Let $\mathcal{L}$ be a distributive lattice and $G(\mathcal{L})$ the corresponding graph. Then the following polynomial has only real zeros:*

$$W(G(\mathcal{L}), t) := 1 + \sum_{\substack{I \subset G(\mathcal{L}) \\ I\ \textbf{nbc}\text{-set}}} (-1)^{|I|} t^{cl(I)+|I|}.$$

The Polynomial $W(G(\mathcal{L}), t) * (1-t)^{\#P+1-\#\mathcal{J}(P)}$ is unimodal and log-concave with no internal zeros.

For the polynomial $W(G(\mathcal{L}), t)$ we have the following formula:

Proposition 3.4. *Let $\mathcal{L}$ be a distributive lattice, $G(\mathcal{L})$ the corresponding graph and $\{p, q\}$ an edge in $G(\mathcal{L})$ (or equivalently an anti-chain in $\mathcal{L}$). Then*

$$\begin{aligned} W(G(\mathcal{L}), t) &= (1-t)\,\Big(W\big(G(\mathcal{L} \setminus \{p\}), t\big) + W\big(G(\mathcal{L} \setminus \{q\}), t\big)\Big) \\ &\quad -(1-t)^2\, W\big(G(\mathcal{L} \setminus \{p, q\}), t\big). \end{aligned}$$

Proof. We split the sum $\displaystyle\sum_{\substack{I \subset G(\mathcal{L}) \\ I\ \textbf{nbc}\text{-set}}}$ into eight sums. We write $p \notin I$, if there is no edge e in I with $p \in e$. Let e_{pq} be the edge joining p and q. We get the following equation:

$$\begin{aligned} W(G(\mathcal{L}), t) &= \sum_{I\ \textbf{nbc}} (-1)^{|I|}\, t^{cl(I)+|I|} \\ &= \sum_{\substack{I\ \textbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I \setminus \{e_{pq}\}}} + \sum_{\substack{I\ \textbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}} + \sum_{\substack{I\ \textbf{nbc} \\ e_{pq} \notin I \\ p,q \in I \setminus \{e_{pq}\}}} + \sum_{\substack{I\ \textbf{nbc} \\ e_{pq} \notin I \\ p,q \in I}} \\ &\quad + \sum_{\substack{I\ \textbf{nbc} \\ e_{pq} \notin I \\ p \in I \setminus \{e_{pq}\} \\ q \notin I \setminus \{e_{pq}\}}} + \sum_{\substack{I\ \textbf{nbc} \\ e_{pq} \notin I \\ p \in I \\ q \notin I}} + \sum_{\substack{I\ \textbf{nbc} \\ e_{pq} \notin I \\ p \notin I \setminus \{e_{pq}\} \\ q \in I \setminus \{e_{pq}\}}} + \sum_{\substack{I\ \textbf{nbc} \\ e_{pq} \notin I \\ p \notin I \\ q \in I}} \end{aligned}$$

where the summands are always given by $g_I := (-1)^{|I|}\, t^{cl(I)+|I|}$.
The **nbc**-property depends on the chosen linear order on the edges of $G(P)$. We fix a linear order such that the edge e_{pq} is the smallest edge. Hence we get

the following fact: $I \subset G(P)$ with $e_{pq} \notin I$ is an **nbc**-set if and only if $I \cup \{e_{pq}\}$ is an **nbc**-set. If we compare the exponents $|I|$ and $cl(I)$, we get the following equations

$$\sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \in I \\ p,q \notin I \setminus \{e_{pq}\}}} + \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}} = (1-t^2) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}} ,$$

$$\sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \in I \\ p \in I \setminus \{e_{pq}\} \\ q \notin I \setminus \{e_{pq}\}}} + \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p \in I \\ q \notin I}} = (1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p \in I \\ q \notin I}} ,$$

$$\sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \in I \\ p \notin I \setminus \{e_{pq}\} \\ q \in I \setminus \{e_{pq}\}}} + \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p \notin I \\ q \in I}} = (1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p \notin I \\ q \in I}} ,$$

$$\sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \in I \\ p,q \in I \setminus \{e_{pq}\}}} + \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \in I}} = 0.$$

If we add $(1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}} - (1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}}$, we get

$$\begin{aligned} W(G(\mathcal{L}),t) = {} & (1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p \in I \\ q \notin I}} +(1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}} \\ & +(1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p \notin I \\ q \in I}} +(1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}} \\ & +t\,(1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}} -(1-t) \sum_{\substack{I \ \mathbf{nbc} \\ e_{pq} \notin I \\ p,q \notin I}} \end{aligned}$$

$$\begin{aligned} = \quad & (1-t)\ \Big(W\big(G(\mathcal{L}\setminus\{p\}),t\big)+W\big(G(\mathcal{L}\setminus\{q\}),t\big)\Big) \\ & -(1-t)^2\ W\big(G(\mathcal{L}\setminus\{p,q\}),t\big). \end{aligned}$$

□

This proposition allows to calculate the W-polynomial recursively.

3.2. Unimodality for Naturally Labeled Posets. The main idea of this section is to associate to each naturally labeled poset P in a unique way a graded naturally labeled poset $\hat{P}$. Using the construction of [**45**] for $\hat{P}$, we construct a unimodular triangulation of the order polytope $\mathcal{O}(P)$. The aim is to show that this triangulation is isomorphic to the simplicial join $\sigma^r * \Delta_{eq}(P)$, where σ^r is the interior r-simplex spanned by the chain of (generalized) rank-constant ideals.
As a consequence we get the following identity:

$$W(P,t)=h(\Delta_{eq},t)=h(\hat{\Delta}_{eq}\cap H,t),$$

where $\hat{\Delta}_{eq}$ is the equatorial sphere of $\hat{P}$ and H is an intersection of coordinate and diagonal hyperplanes.
These facts give a new possibility to approach unimodality of the W-polynomial: Study the h-vector of the intersection of the equatorial sphere of $\hat{P}$ with an intersection of coordinate and diagonal hyperplanes.
We conjecture that $\Delta_{eq}(P)$ is a shellable ball (remember that in the graded case $\Delta_{eq}(P)$ was a polytopal sphere).

From now on let (P,ω) be any naturally labeled poset.

We first associate to each element p of P a rank $\rho(p)$ defined by

$$\rho(p):=\max\big\{i\in\mathbb{N}\ \big|\ \text{there exist } p_1,\dots,p_{i-1}\in P \text{ s.t. } p_1\prec p_2\prec\dots\prec p_{i-1}\prec p\big\}.$$

The rank of the poset P is defined as follows:

$$\rho(P):=\max\{\rho(p)\mid p\in P\}.$$

If P is not graded, then there exist maximal chains $p_1\prec\dots\prec p_i$ such that $i<\rho(P)$. The following algorithm extends P to a graded partially ordered set $\hat{P}$ with $\rho(P)=\rho(\hat{P})$.

Algorithm: Let $p_1\prec\dots\prec p_i$ be a maximal chain, with $i<\rho(P)$.
Let j be the smallest number such that $\rho(p_j)=j$ and $\rho(p_{j+1})=j+k$ with $k\geq 2$. Then we add new elements $p_{j,1},\dots,p_{j,k-1}$ to the poset P with order relations

$$p\prec p_j\prec p_{j,1}\prec\dots\prec p_{j,k-1}\prec p_{j+1}\prec q \text{ for all } p\prec p_j \text{ and } p_{j+1}\prec q.$$

All other elements of P are incomparable to the elements $p_{j,1},\dots,p_{j,k-1}$. By construction the element p_{j+1} has still rank $j+k$ and the elements $p_{j,l}$ have rank $j+l$.

The natural labeling ω for P is constructed as follows: For all $q\in P$ with $\omega(q)>\omega(p_j)$ we shift the labeling to $\hat{\omega}(q):=\omega(q)+k-1$. The labeling of the new elements is then given by $\hat{\omega}(p_{j,l}):=\omega(p_j)+l$.

Now we repeat the algorithm with the poset $P:=P\cup\{p_{j,1},\dots,p_{j,k-1}\}$ until P satisfies the following property:

For every maximal chain $p_1 \prec \ldots \prec p_i$ we have $\rho(p_j) = j$. In particular, P is graded.
We denote with $(\hat{P}, \hat{\omega})$ the resulting graded poset P.

In Figure 1 and Figure 2 we present two examples to clarify the construction.

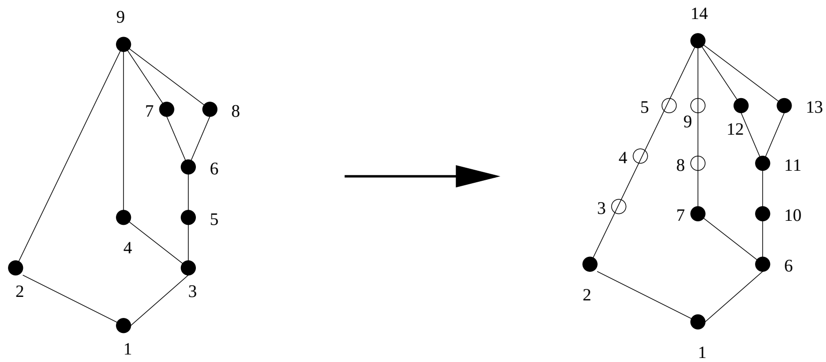


Figure 1

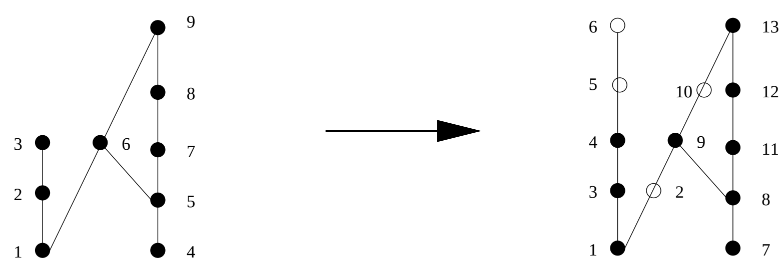


Figure 2

From now on we denote with $\prec_P$ the order of P and with $\prec_{\hat{P}}$ the order of $\hat{P}$. We write $i \in P$ (resp. $i \in \hat{P} \setminus P$) if $\hat{\omega}^{-1}(i) \in P$ (resp. $\hat{\omega}^{-1}(i) \in \hat{P} \setminus P$) and $i \prec_P j$ (resp. $i \prec_{\hat{P}} j$) if $\omega^{-1}(i) \prec \omega^{-1}(j)$ (resp. $\omega^{-1}(i) \prec_{\hat{P}} \omega^{-1}(j)$).
Define

$$m(i) := \begin{cases} i & , \quad i \in P \\ \min \left\{ j \in P \;\middle|\; i \prec_{\hat{P}} j \right\} & , \quad i \in \hat{P} \setminus P, \min \text{ exists} \\ 0 & , \quad i \in \hat{P} \setminus P, \min \text{ does not exist.} \end{cases}$$

We denote with $n = \#P$ and $\hat{n} = \#\hat{P}$ the sizes of P and $\hat{P}$.

By definition the order polytope $\mathcal{O}(P)$ lives in $\mathbb{R}^n$ and the order polytope $\mathcal{O}(\hat{P})$ lives in $\mathbb{R}^{\hat{n}}$. We now define an embedding of $\mathcal{O}(P)$ into $\mathcal{O}(\hat{P})$.
In order to do this, we see the space $\mathbb{R}^n$ as a subspace of $\mathbb{R}^{\hat{n}}$, via

$$x_i := \begin{cases} x_{\omega(\hat{\omega}^{-1}(i))} & , \quad i \in P \\ 0 & , \quad i \in \hat{P} \setminus P. \end{cases}$$

(1) Let $f = (f_1, \ldots, f_n) \in R^n$ be a P-partition. We define the image of f as

$$\hat{f}_i := \begin{cases} f_i & , \quad i \in P \\ f_{m(i)} & , \quad i \in \hat{P} \setminus P, m(i) \neq 0 \\ 0 & , \quad i \in \hat{P} \setminus P, m(i) = 0. \end{cases}$$

It is straightforward to check that $\hat{f}$ is a $\hat{P}$-partition.

(2) Let $I \subset \mathcal{J}(P)$ be an order ideal. Then clearly $I \subset \hat{P}$. Now denote with $\hat{I}$ the smallest (with respect to the inclusion) order ideal in $\mathcal{J}(\hat{P})$ containing I.

Hence we get an embedding of the lattice of order ideals of P to the lattice of order ideals of $\hat{P}$ and an embedding of the order polytope $\mathcal{O}(P)$ to the order polytope $\mathcal{O}(\hat{P})$:

$$\phi : \begin{array}{ccc} \mathcal{J}(P) & \hookrightarrow & \mathcal{J}(\hat{P}) \\ I & \mapsto & \hat{I} \\ & & \\ \mathcal{O}(P) & \hookrightarrow & \mathcal{O}(\hat{P}) \\ f & \mapsto & \hat{f}. \end{array}$$

Since ϕ is left-invertible ($\phi^{-1}(\hat{I}) := I \cap P$ and $\phi^{-1}(\hat{f}) = f \cap R^n$), we have an embedding.

Let $H \subset \mathbb{R}^{\hat{n}}$ be the following intersection of coordinate and diagonal hyperplanes

$$H := \left\{ x \in \mathbb{R}^{\hat{n}} \;\middle|\; \begin{array}{lcl} x_i = x_{m(i)} & , & i \in \hat{P} \setminus P, m(i) \neq 0, \\ x_i = 0 & , & i \in \hat{P} \setminus P, m(i) = 0. \end{array} \right\}.$$

Then it follows directly from the definitions that H is a linear subspace and hence convex. The image of $\mathcal{O}(P)$ under ϕ equals the intersection of $\mathcal{O}(\hat{P})$ and H:

$$\phi\big(\mathcal{O}(P)\big) = \mathcal{O}(\hat{P}) \cap H.$$

We are now in position to give the definitions of equatorial and rank-constant in the general case:

Definition 3.5. Let P be a naturally labeled partially ordered set.

(1) A P-partition $f : P \to \mathbb{R}^n$ will be called *rank-constant* (resp. *equatorial*) if $\phi(f) = \hat{f} : \hat{P} \to \mathbb{R}^{\hat{n}}$ is rank-constant (resp. equatorial).

(2) An order ideal I in P will be called *rank-constant* (resp. *equatorial*) if the ideal $\phi(I) = \hat{I}$ in $\hat{P}$ is rank-constant (resp. equatorial).

(3) A collection of ideals $\{I_1, \ldots, I_t\}$ forming a chain $I_1 \subset I_2 \subset \ldots \subset I_t$ will be called *rank-constant* (resp. *equatorial*) if the collection $\{\hat{I}_1, \ldots, \hat{I}_t\}$ in $\hat{P}$ is rank-constant (resp. equatorial).

Definition 3.6. The *equatorial complex* $\Delta_{eq}(P)$ is defined to be the subcomplex of the order complex $\Delta(\mathcal{J}(P))$ whose faces are indexed by the equatorial chains of non-empty ideals.

The simplex σ^r is defined to be the subcomplex of the order complex $\Delta(\mathcal{J}(P))$ whose faces are indexed by the rank-constant chains of ideals.

With the same arguments as for the graded case, we exhibit now an alternative triangulation of the order polytope $\mathcal{O}(P)$, which we call the equatorial triangulation. Then with exactly the same proofs as for the graded case, we show the following properties:

▷ It is a unimodular triangulation.

▷ It is isomorphic, as an abstract simplicial complex, to the join of an r-simplex with a space $\Delta_{eq}(P)$, which we call the equatorial space.

▷ The equatorial space $\Delta_{eq}(P)$ is a subcomplex of $\Delta(\mathcal{J}(P))$.

▷ $h(\Delta_{eq}(P), t) = h(\Delta(\mathcal{J}(P)), t) = W(p, t)$.

In addition, we get - by definition - the following isomorphism (as abstract simplicial complexes):

$$\Delta_{eq}(P) \cong \Delta_{eq}(\hat{P}) \cap H.$$

As corollary we get the following useful fact:

Corollary 3.7. *Let P be a naturally labeled poset and $\hat{P}$ the above embedding. Then*

$$W(P, t) = h(\Delta_{eq}, t) = h(\hat{\Delta}_{eq} \cap H, t).$$

Hence in order to prove the unimodality of the W-polynomial for all naturally labeled posets, one has to study the h-vector of the intersection of a polytopal sphere with an intersection H of coordinate and diagonal hyperplanes.

Before we come to the proofs, we discuss an example.

Example 3.8. Consider the naturally labeled poset $P = \{1, 2, 3, 4\}$ with order relation given by the Hasse diagram in Figure 3. The lattice of order ideals $\mathcal{J}(P)$, the equatorial space Δ_{eq}, and the simplex σ^r are also given in Figure 3. The corresponding graded lattice $\hat{P}$ with his lattice of ordered ideals $\mathcal{J}(\hat{P})$, the equatorial sphere $\hat{\Delta}_{eq}$, and the simplex $\hat{\sigma}^r$ are given in Figure 4.

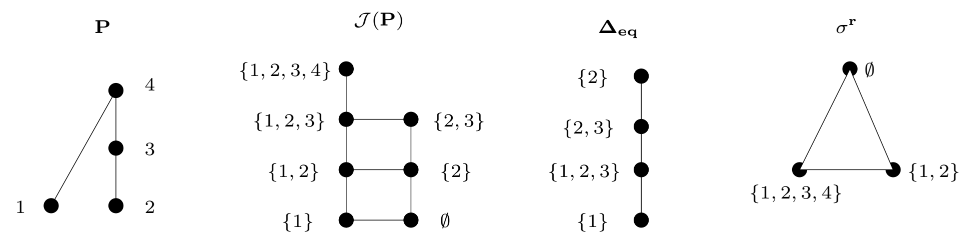


Figure 3

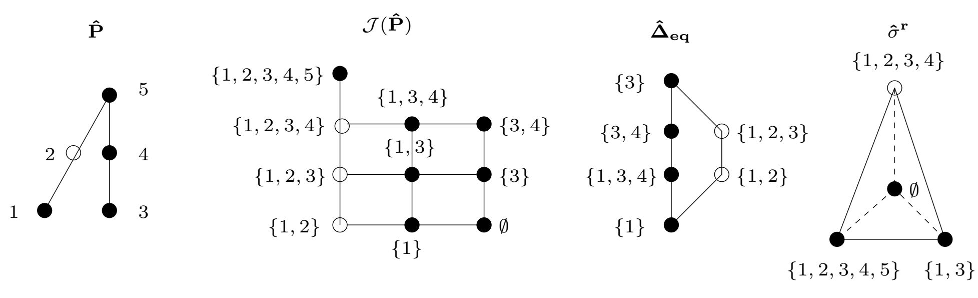


Figure 4

The W-polynomials, the f-vectors, and the h-vectors are as follows:

$$W(P, t) \quad = \quad 1 + 2t$$

$$\begin{aligned} f(\Delta_{eq}) &= (1,4,3) \\ h(\Delta_{eq}) &= (1,2) \end{aligned}$$

$$\begin{aligned} W(\hat{P},t) &= 1+4t+t^2 \\ f(\hat{\Delta}_{eq}) &= (1,6,6) \\ h(\hat{\Delta}_{eq}) &= (1,4,1) \end{aligned}$$

The convex set $H \subset \mathbb{R}^5$ is given by

$$H := \Big\{ x \in \mathbb{R}^5 \;\Big|\; x_2 = x_5 \Big\}.$$

If we identify each order ideal I with the corresponding characteristic vector χ_I, we have

$$\Delta_{eq} = \hat{\Delta}_{eq} \cap H,$$

$$\sigma^r = \hat{\sigma}^r \cap H.$$

The proofs of

- ▹ $\sigma^r * \Delta_{eq}$ indexes a unimodular triangulation of $\mathcal{O}(P)$,
- ▹ the equatorial complex $\Delta_{eq}(P)$ is a subcomplex of $\Delta(\mathcal{J}(P))$,
- ▹ $h(\Delta_{eq}(P),t) = h(\Delta(\mathcal{J}(P)),t) = W(p,t)$

follow directly from the embedding $\phi : \mathcal{O}(P) \to \mathcal{O}(\hat{P})$ and Proposition 3.3, Proposition 3.4, Proposition 3.6, Corollary 3.8, and Proposition 3.10 of [**45**]:

Proposition 3.9 (see Prop. 3.3. of [**45**])**.** *Every non-zero P-partition f can be uniquely expressed as*

$$f = f^{rc} + f^{eq},$$

where f^{rc}, f^{eq} are rank-constant and equatorial P-partitions, respectively.

Proof. Proposition 3.3 in [**45**] shows the assertion for graded posets. Therefore, the assertion holds for all $\hat{P}$-partitions $\hat{f}$, in particular, the assertion holds for all $\hat{P}$-partitions $\hat{f}$ living in $\mathcal{O}(\hat{P}) \cap H$. Since we have $\mathcal{O}(P) \cong \phi(\mathcal{O}(P)) = \mathcal{O}(\hat{P}) \cap H$, the assertion follows. □

Proposition 3.10 (see Prop. 3.4. of [**45**])**.** *The rank-constant subcone of $\mathcal{A}(P)$ is interior that is, it does not lie in the boundary subcomplex of the cone $\mathcal{A}(P)$.*

Proof. With the same arguments as in the proof of Proposition 3.9, the result follows directly for Proposition 3.4. of [**45**]. □

Proposition 3.11 (see Prop. 3.6. of [**45**])**.** *The collection of all cones*

$$\operatorname{pos}\Big(\{\chi_I \mid I \in \mathcal{R} \cup \mathcal{E}\}\Big),$$

where $\mathcal{R}$ (resp. $\mathcal{E}$) is a chain of non-empty rank-constant (resp. equatorial) ideals in P, gives a unimodular triangulation of the cone of P-partitions $\mathcal{A}(P)$.

Proof. With the same arguments as in the proof of Proposition 3.9, the result follows directly for Proposition 3.6. of [**45**]. □

Corollary 3.12 (see Cor. 3.8. of [**45**]). *The equatorial triangulation of the order polytope $\mathcal{O}(P)$ is abstractly isomorphic to the simplicial join $\sigma^r * \Delta_{eq}$. As a consequence of its unimodularity, one has*

$$h(\Delta_{eq}, t) = h(\Delta(\mathcal{J}(P)), t) = W(P, t).$$

Proof. Recall that for a convex polytope Q in $\mathbb{R}^n$ having vertices in $\mathbb{Z}^n$ the number of lattice points contained in an integer dilation dQ grows as a polynomial in the dilation factor $d \in \mathbb{N}$. This polynomial in d is called the *Erhart polynomial*:

$$\text{Erhart}(Q, d) := \#\Big(dQ \cap \mathbb{N}^n\Big).$$

Whenever Q has a unimodular triangulation abstractly isomorphic to a simplicial complex Δ, there is the following relationship:

$$\sum_{d \geq 0} \text{Erhart}(Q, d)\, t^d = \frac{h(\Delta, t)}{(1-t)^n}. \tag{3.2}$$

The first assertion follows directly from Proposition 3.11. For the second, note that both $\sigma^r * \Delta_{eq}$ and $\Delta(\mathcal{J}(P))$ index unimodular triangulations of the order polytope, so (3.2) implies

$$h(\Delta_{eq}, t) = h(\Delta(\mathcal{J}(P)), t).$$

On the other hand, the definition of the h- and f-vector shows that

$$\begin{aligned} f(\Delta_1 * \Delta_2, t) &= f(\Delta_1, t) * f(\Delta_2, t), \\ h(\Delta_1 * \Delta_2, t) &= h(\Delta_1, t) * h(\Delta_2, t), \\ h(\sigma, t) &= 1, \end{aligned}$$

and hence $h(\sigma^r * \Delta_{eq}, t) = h(\Delta_{eq}, t)$. □

Corollary 3.13.

$$h(\Delta_{eq}, t) = h(\hat{\Delta}_{eq} \cap H, t) = W(P, t).$$

Proof. The result follows from the preceding corollary and the fact that Δ_{eq} is isomorphic to $\hat{\Delta}_{eq} \cap H$. □

The fact that $\Delta_{eq}(P)$ is a subcomplex of $\Delta(\mathcal{J}(P))$ follows from the definition of $\Delta_{eq}(P)$.

Finally, we can ask which subspaces H are possible:

Let P be a graded naturally labeled poset on the set $[n]$ of rank r and $\Delta_{eq} \in \mathbb{R}^n$ its equatorial sphere.

First we define the set of possible coordinates:

Definition 3.14. A set $C \subset [n]$ is called a set of possible coordinates if each $i \in C$ satisfies the following properties:

(1) i has a unique maximal predecessor, i.e. there exists exactly one $j \prec_P i$ such that there is no $k \in P$ with $j \prec_P k \prec_P i$,

(2) i has a unique minimal successor or is maximal, i.e. there exists either no $j \in P$ with $i \prec_P j$ or there exists exactly one $i \prec_P j$ such that there is no $k \in P$ with $i \prec_P k \prec_P j$,

(3) if for each $i \in P \setminus C$ the rank of i does not change if one removes all elements lying in C, i.e.

$$\rho_P(i) = \rho_{P \setminus C}(i) \text{ for all } i \in P \setminus C.$$

We denote with $Co(P)$ the maximal subset (with respect to inclusion) satisfying these properties and call it *the set of possible coordinates*.

Definition 3.15. Let $A \subset Co(P)$ be a set of possible coordinates.

(1) We call a set of equations

$$E_0^A = \Big\{ \{x_i = 0\} \;\Big|\; i \in A \Big\}$$

a *P-valid set of equations* if it satisfies the following property:

$$\{x_i = 0\} \in E_0^A,\; i \prec_P j \Rightarrow j \in A \text{ and } \{x_j = 0\} \in E_0^A.$$

(2) We call a set of equations

$$E_1^A := \Big\{ \{x_i = x_j\} \;\Big|\; i \prec_P j \text{ and } i \in A \Big\}$$

a *P-valid set of equations*, if it satisfies the following property:

$$\{x_i = x_j\} \in E_1^A, i \prec_P j \Rightarrow \begin{cases} \text{There exists an } l \in P \setminus A \text{ such that } i \prec_P l \\ \text{and for all } i \prec_p k \prec_P l \text{ we have } k \in A \text{ and} \\ \{x_i = x_k\} \in E_1^A. \end{cases}$$

(3) We call a set of equations $E(P)$ a *P-valid set of equations* if there exists a set of possible coordinates $A \subset Co(P)$ such that E_0^A and E_1^A are P-valid set of equations and $E(P) = E_0^A(P) \cup E_1^A(P)$.

For a P-valid set $E(P)$ of equations we define the convex set $H(P) := H(E(P)) \in \mathbb{R}^n$ to be the intersection of coordinate and diagonal hyperplanes:

$$H(P) := \Big\{ x \in \mathbb{R}^n \;\Big|\; x \text{ satisfies all equations in } E(P) \Big\}.$$

With this setting we get the following theorem:

Theorem 3.16. *The following statements are equivalent:*

(1) *For all naturally labeled posets, the W-polynomial is unimodal (resp. log-concave).*

(2) *For all graded, naturally labeled posets P on $[n]$ with equatorial sphere Δ_{eq} and for all P-valid sets of equations $E(P)$ with corresponding set of coordinate and diagonal hyperplanes $H(P)$, the h-polynomial*

$$h\big(\Delta_{eq} \cap H(P), t\big)$$

is unimodal (resp. log-concave).

Proof. The result follows from Corollary 3.13 and the fact that to each naturally labeled posets P there exists a unique graded naturally labeled posets $\hat{P}$ and a unique $\hat{P}$-valid set of equations $E(\hat{P})$ such that $h(\Delta_{eq}, t) = h(\hat{\Delta}_{eq} \cap H(P), t)$. □

Theorem 3.16 gives a new approach to prove the unimodality of the W-polynomial for all naturally labeled posets P.

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Part 3

Appendix

German Abstract (Deutsche Zusammenfassung)

1. Struktur der Arbeit

Die vorliegende Arbeit gliedert sich in zwei Teile mit den Titeln "Algebraische Diskrete Morse-Theorie und Anwendungen in der kommutativen Algebra" und "Zwei Probleme der Algebraischen Kombinatorik".

Der erste Teil der Arbeit ist ein Beitrag zur kombinatorischen kommutativen Algebra. Hier verallgemeinern wir die Diskrete Morse-Theorie von Forman auf eine algebraische Version. Mit Hilfe dieser Verallgemeinerung ist es nun möglich, verschiedene Probleme der kommutativen Algebra zu lösen: Wir können minimale Auflösungen für neue Klassen von Idealen konstruieren sowie Fragestellungen über die multigraduierte Poincaré-Betti-Reihe beantworten.

Im zweiten Teil der Arbeit diskutieren wir die "Homologie von nilpotenten Lie-Algebren endlichen Typs" und die "Neggers-Stanley-Vermutung". Nach einer kurzen Vorstellung beider Probleme präsentieren wir hier neue Resultate: Wir beweisen Aussagen über die Homologie nilpotenter Lie-Algebren zu Wurzelsystemen - speziell zum Wurzelsystem A_n - und reduzieren die Neggers-Stanley-Vermutung auf die Berechnung des h-Vektors des Schnittes einer polytopalen Sphäre mit einer Menge von Hyperebenen.

2. Algebraische Diskrete Morse-Theorie und Anwendungen

2.1. Einführung.

Die Theorie der minimalen freien Auflösungen befasst sich mit Abhängigkeiten von Polynomen $p_1, \ldots, p_n \in S = k[x_1, \ldots, x_n]$ über S. Solche Abhängigkeiten sind deutlich komplexer als Abhängigkeiten von Vektoren $v_1, \ldots, v_n$ eines Vektorraums, da zum Beispiel die beiden Begriffe maximal linear unabhängig und minimal erzeugend nicht äquivalent sind.

Ein Maß für Abhängigkeiten von Polynomen sind minimale freie Auflösungen des von den Polynomen erzeugten Ideals $\langle p_1, \ldots, p_n \rangle \trianglelefteq S$; allerdings sind solche Auflösungen im Allgemeinen schwierig zu konstruieren. Im monomialen Fall gibt es zumindest Verfahren (z.B. Taylor-Auflösung), eine Auflösung zu konstruieren, die aber in den seltensten Fällen minimal ist.

In der kommutativen Algebra werden, neben minimalen Auflösungen, viele wichtige Invarianten, wie zum Beispiel Regularität, Poincaré-Betti-Reihe, Tor und Ext, über die Homologie von algebraischen Kettenkomplexen berechnet. Oftmals ist der Kettenkomplex viel zu groß, in dem Sinne, dass es azyklische Unterkomplexe gibt, die herausdividiert werden können. Zum Beispiel ist die Taylor-Auflösung eines monomialen Moduls ein Komplex, der - tensoriert mit k - die Betti-Zahlen des Moduls berechnet, aber im Allgemeinen besitzt er einen relativ großen azyklischen Unterkomplex.

"Optimale" Komplexe zur Berechnung von Invarianten, wie zum Beispiel minimale freie Auflösungen, existieren zwar theoretisch, sind aber im Allgemeinen nicht effektiv konstruierbar. Die Theorie besagt zwar, dass man zu einem beliebigen Komplex einen Homotopie-äquivalenten Kettenkomplex konstruieren kann, von dem sich kein azyklischer Komplex abspalten lässt, jedoch gibt es bisher keinen effizienten Algorithmus, der dies leistet.

2.2. Bisherige Lösungsansätze.
Zur Berechnung von minimalen freien Auflösungen monomialer Ideale liefert die Diskrete Morse-Theorie von Forman (vgl. [**21**], [**22**]) einen vielversprechenden Lösungsansatz. Die Diskrete Morse-Theorie von Forman ist eigentlich eine topologische Theorie: Sie ordnet einem regulären CW-Komplex X einen Homotopie-äquivalenten CW-Komplex $X_{\mathcal{M}}$ mit weniger Zellen zu. Die Homologie von CW-Komplexen berechnet sich durch die Homologie des zugeordneten Kettenkomplexes der zellulären Homologie. Formans Theorie besagt insbesondere, dass die Kettenkomplexe von X und $X_{\mathcal{M}}$ zueinander homotop sind. Damit ergibt sich der Zusammenhang zur kommutativen Algebra: Finde zu einem algebraischen Kettenkomplex einen passenden CW-Komplex (das heißt der Komplex der zellulären Homologie stimmt mit dem gegebenen überein) und wende dann Formans Theorie an. Damit erhält man einen zum Ausgangskomplex Homotopie-äquivalenten Kettenkomplex mit kleineren Rängen.

Leider ist diese Methode nicht auf beliebige Kettenkomplexe anwendbar, sondern nur auf solche, zu denen ein passender CW-Komplex existiert. Auflösungen monomialer Ideale, zu denen ein passender CW-Komplex existiert, heißen zelluläre Auflösungen (vgl. [**6**]). Batzies studierte in seiner Arbeit [**4**] die Anwendung der Diskreten-Morse Theorie von Forman auf zelluläre Auflösungen monomialer Ideale. Hier werden sogenannte azyklische Matchings auf zellulären Auflösungen definiert, um anschließend kleinere - bestenfalls minimale - Auflösungen zu konstruieren. Es zeigt sich in [**4**], dass diese Theorie gute Resultate liefert.

Wie bereits erwähnt, setzt die Anwendung der Diskreten Morse-Theorie von Forman stets die Existenz eines regulären CW-Komplexes voraus. Es gibt aber Auflösungen, die nicht von einem regulären CW-Komplex unterstüzt werden.

Ein weiteres Problem ist die iterative Anwendung dieses Verfahrens, da das Verfahren nur angewendet werden kann, wenn der CW-Komplex regulär ist. Es ist aber möglich, dass nach einmaliger Anwendung die Auflösung nicht minimal und der unterstützende CW-Komplex nicht mehr regulär ist. Daher kann man mit diesem Verfahren nicht jede zelluläre Auflösung minimieren. Es ist bis heute ein ungelöstes Problem, ob jeder monomiale Modul über dem Polynomring eine minimale zelluläre Auflösung besitzt.

Die Diskrete Morse-Theorie auf zellulären Auflösungen erlaubt zwar in vielen Fällen die Minimierung der Auflösung und ist somit eine wichtige Methode in der kommutativen Algebra zur Berechnung der Betti-Zahlen, sie hat aber, wie oben gezeigt, ihre Grenzen.

Somit stellt sich nun die Frage, ob es eine ähnliche Methode gibt, die erstens die Diskrete Morse-Theorie verbessert - also zum Beispiel eine Anwendung auf alle Auflösungen monomialer Ideale zulässt - und zweitens eine iterative Anwendung ermöglicht. Damit hätte man ein Konstruktionsverfahren zur Verfügung, um eine beliebige Auflösung schrittweise zu minimieren. Schließlich wäre es wünschenswert, dieselbe Methode auf beliebige algebraische Kettenkomplexe anwenden zu können, um deren Homologie anzugeben.

Der grundlegende Ansatz dieser Arbeit, ist die Diskrete Morse-Theorie von Forman auf eine algebraische Version zu verallgemeinern, die alle oben aufgeführten Forderungen erfüllt. Wir entwickeln eine solche Methode und nennen sie "Algebraische Diskrete Morse-Theorie". Anschließend wenden wir die Theorie auf verschiedene Fragestellungen in der kommutativen Algebra an.

Während der Entstehung der Arbeit ist es unabhängig von uns auch Sköldberg gelungen, die Diskrete Morse-Theorie auf algebraische Komplexe zu verallgemeinern [**41**].

In den folgenden beiden Abschnitten erklären wir die Algebraische Diskrete Morse-Theorie und geben eine detaillierte Zusammenfassung der Anwendungen in der kommutativen Algebra und unserer Resultate.

2.3. Die Algebraische Diskrete Morse-Theorie.
Die Algebraische Diskrete Morse-Theorie behält im Prinzip das Verfahren der Diskreten Morse-Theorie bei, allerdings wird kein CW-Komplex mehr benötigt, sondern nur noch der Kettenkomplex. Damit ermöglichen wir die Anwendung der Formanschen Theorie auf beliebige algebraische Kettenkomplexe und machen eine iterative Anwendung möglich. Dies erlaubt uns - rein theoretisch - jeden beliebigen algebraischen Kettenkomplex von freien R-Moduln zu minimieren und somit die Berechnung von Invarianten in der kommutativen Algebra zu vereinfachen.

Die Algebraische Diskrete Morse-Theorie wird in Kapitel 3 definiert und hergeleitet. Wir geben hier die Definition und die Aussage der Algebraischen Diskreten Morse-Theorie wieder:

Sei R ein Ring und $\mathsf{C}_\bullet = (C_i, \partial_i)_{i\geq 0}$ ein Kettenkomplex freier R-Moduln C_i. Wir fixieren eine Basis $X = \bigcup_{i=0}^n X_i$, so dass $C_i \simeq \bigoplus_{c\in X_i} R\ c$. Von nun an schreiben wir das Differential ∂_i bezüglich der Basis X in der folgenden Form:

$$\partial_i : \begin{cases} C_i & \to & C_{i-1} \\ c & \mapsto & \partial_i(c) = \displaystyle\sum_{c' \in X_{i-1}} [c : c'] \cdot c'. \end{cases}$$

Zu einem gegebenen Komplex $\mathsf{C}_\bullet$ mit Basis X konstruieren wir einen gewichteten, gerichteten Graphen $G(\mathsf{C}_\bullet) = (V, E)$. Dabei entspricht die Eckenmenge V der gewählten Basis $V = X$, und die Kantenmenge E des Graphen $G(\mathsf{C}_\bullet)$ ist durch folgende Regel definiert:

$$(c, c', [c : c']) \in E \quad :\Leftrightarrow \quad c \in X_i, c' \in X_{i-1} \text{ und } [c : c'] \neq 0.$$

Für eine Kante $(c, c', [c : c'])$ in dem Graphen schreiben wir oftmals nur $c \to c'$ und vernachlässigen das Gewicht. Mit $e \in G(\mathsf{C}_\bullet)$ meinen wir, dass e eine Kante aus E ist.

Definition 2.1. Eine endliche Teilmenge $\mathcal{M} \subset E$ der Eckenmenge heißt azyklisches Matching, falls die folgenden drei Bedingungen erfüllt sind:

(1) (Matching) Jede Ecke $v \in V$ liegt in höchstens einer Kante $e \in \mathcal{M}$.

(2) (Invertierbarkeit) Für jede Kante $(c, c', [c : c']) \in \mathcal{M}$ ist das Gewicht $[c : c']$ invertierbar und liegt im Zentrum von R.

(3) (Azyklizität) Der Graph $G_\mathcal{M}(C_\bullet) = (V, E_\mathcal{M})$ enthält keine gerichteten Zyklen, wobei $E_\mathcal{M}$ definiert ist durch

$$E_\mathcal{M} := (E \setminus \mathcal{M}) \cup \left\{ \left(c', c, \frac{-1}{[c : c']} \right) \text{ mit } (c, c', [c : c']) \in \mathcal{M} \right\}.$$

Für das azyklische Matching $\mathcal{M}$ auf dem Graphen $G(\mathsf{C}_\bullet) = (V, E)$ führen wir folgende Notation ein, die wir von Forman übernommen haben:

(1) Eine Ecke $c \in V$ heißt kritisch bezüglich $\mathcal{M}$, falls c in keiner Kante $e \in \mathcal{M}$ vorkommt; wir schreiben

$$X_i^\mathcal{M} := \{ c \in X_i \mid c \text{ kritisch } \}$$

für die Menge der kritischen Ecken im homologischen Grad i.

(2) Wir schreiben $c' \leq c$, falls $c \in X_i$, $c' \in X_{i-1}$ und $[c : c'] \neq 0$.

(3) $\mathrm{Path}(c, c')$ bezeichnet die Menge der gerichteten Pfade von c nach c' in dem Graphen $G_\mathcal{M}(\mathsf{C}_\bullet)$.

(4) Das Gewicht $w(p)$ eines Pfads $p = c_1 \to \cdots \to c_r \in \mathrm{Path}(c_1, c_r)$ ist gegeben durch

$$\begin{aligned} w(c_1 \to \ldots \to c_r) &:= \prod_{i=1}^{r-1} w(c_i \to c_{i+1}), \\ w(c \to c') &:= \begin{cases} -\dfrac{1}{[c : c']} & , \quad c \leq c' \\ [c : c'] & , \quad c' \leq c. \end{cases} \end{aligned}$$

(5) Wir schreiben $\Gamma(c, c') = \displaystyle\sum_{p \in \mathrm{Path}(c, c')} w(p)$ für die Summe der Gewichte aller Pfade von c nach c'.

Der Morse-Komplex $\mathsf{C}_\bullet^{\mathcal{M}} = (C_i^{\mathcal{M}}, \partial_i^{\mathcal{M}})_{i\geq 0}$ von $\mathsf{C}_\bullet$ bezüglich $\mathcal{M}$ ist wie folgt definiert:

$$C_i^{\mathcal{M}} := \bigoplus_{c\in X_i^{\mathcal{M}}} R\, c,$$

$$\partial_i^{\mathcal{M}} : \begin{cases} C_i^{\mathcal{M}} & \to & C_{i-1}^{\mathcal{M}} \\ c & \mapsto & \displaystyle\sum_{c'\in X_{i-1}^{\mathcal{M}}} \Gamma(c,c')c', \end{cases} .$$

Der folgende Satz ermöglicht das Minimieren von Kettenkomplexen:

Theorem 2.2. *Der Komplex $\mathsf{C}_\bullet^{\mathcal{M}}$ ist ein Komplex freier R-Moduln und ist Homotopie-äquivalent zum Komplex $\mathsf{C}_\bullet$. Insbesondere gilt für alle $i \geq 0$*

$$H_i(\mathsf{C}_\bullet) \cong H_i(\mathsf{C}_\bullet^{\mathcal{M}}).$$

Die folgenden Abbildungen definieren eine Kettenhomotopie zwischen $\mathsf{C}_\bullet$ und $\mathsf{C}_\bullet^{\mathcal{M}}$.

$$f : \begin{cases} \mathsf{C}_\bullet & \to & \mathsf{C}_\bullet^{\mathcal{M}} \\ c \in X_i & \mapsto & f(c) := \displaystyle\sum_{c'\in X_i^{\mathcal{M}}} \Gamma(c,c')c', \end{cases}$$

$$g : \begin{cases} \mathsf{C}_\bullet^{\mathcal{M}} & \to & \mathsf{C}_\bullet \\ c \in X_i^{\mathcal{M}} & \mapsto & g_i(c) := \displaystyle\sum_{c'\in X_i} \Gamma(c,c')c'. \end{cases}$$

In manchen Anwendungen ist es nützlich unendliche azyklische Matchings zu betrachten. Die Definition eines azyklischen Matchings benutzt die Endlichkeit nicht; jedoch wird die Endlichkeit in der Definition von $\Gamma(c,c')$, sowie im Beweis des Theorems 2.2 benutzt. Um die Theorie auf unendliche Matchings zu erweitern benötigen wir eine "lokale" Endlichkeit.

Wenn $C_\bullet$ ein Komplex freier R-Moduln und $\mathcal{M}$ ein unendliches azyklisches Matching, so induziert $\mathcal{M}$ auf jedem endlichen Unterkomplex $C_\bullet^f$ ein endliches azyklisches Matching. Daher definieren wir unendliche Morse-Matchings wie folgt:

Definition 2.3 (Endlichkeit)**.** Sei $C_\bullet$ ein Komplex freier R-Moduln und $\mathcal{M}$ ein unendliches azyklisches Matching. Wir sagen $\mathcal{M}$ definiert ein Morse-Matching falls eine Folge von endlichen Unterkomplexen $D_i := (D_\bullet)_i$, $i \geq 0$ von $C_\bullet$ existiert, so dass Folgendes gilt:

(1) Für alle $i \geq 0$ ist D_i ein Unterkomplex von D_{i+1}.
(2) $C_\bullet = \operatorname{colim}_{i\geq 0} D_i$.
(3) Für alle $i \geq 0$ ist $(D_i)^{\mathcal{M}}$ ein Unterkomplex von $(D_{i+1})^{\mathcal{M}}$.

Die letzte Bedingung impliziert $\Gamma(c,c') < \infty$ und es ist leicht zu sehen, dass die Aussage von Theorem 2.2 nun auch für unendliche Morse-Matchings gültig ist.

In unseren Anwendungen sind alle Komplexe durch $\alpha \in \mathbb{N}^n$ multigraduiert und der α-graduierte Teil von $C_\bullet$ ist endlich. Daher sind die folgenden Unterkomplexe alle endlich:

$$D_i := \bigoplus_{j=0}^{i} \bigoplus_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = j}} (C_\bullet)_\alpha$$

Es ist leicht zu sehen, dass auf multigraduierten Komplexen deren α-graduierte Teile endlich sind, jedes unendliche Morse-Matching mit der Folge D_i die zusätzliche Endlichkeitsbedigung erfüllt. Daher bekommen wir für diese Klasse von Komplexen folgende allgemeine Aussage:

Theorem 2.4. *Sei $C_\bullet$ ein $\mathbb{N}^n$-graduierter Komplex freier R-Moduln, so dass für alle $\alpha \in \mathbb{N}^n$ der Komplex $(C_\bullet)_\alpha$ endlich ist. Dann gilt die Aussage von Theorem 2.2 für alle unendlichen azyklischen Matchings $\mathcal{M}$.*

Als erste Anwendung zeigen wir, dass die normalisierte Bar-Auflösung, sowie die normalisierte Hochschild-Auflösung durch ein azyklisches Matching aus der Bar- bzw. Hochschild-Auflösung hervorgehen (vgl. Proposition 3.3.1 und Proposition 3.3.2).

2.4. Anwendungen in der Kommutativen Algebra.

2.4.1. *Auflösungen Monomialer Ideale.*
Wir konstruieren minimale Auflösungen von monomialen Idealen $\mathfrak{a}$ in dem Polynomring $S = k[x_1, \ldots, x_n]$. Wir entwickeln ein sogenanntes "Standard-Matching" $\mathcal{M} := \bigcup_{i \geq 1} \mathcal{M}_i$ auf der Taylor-Auflösung $T_\bullet$ eines monomialen Ideals, so dass der resultierende Morse-Komplex $T_\bullet^{\mathcal{M}}$ eine minimale Auflösung definiert. Das Standard-Matching existiert in jedem Fall und liefert zusätzlich zu einer minimalen freien Auflösung des monomialen Ideals ein Produkt auf der Basis der minimalen Auflösung. Damit können wir insbesondere zeigen, dass die Homologie des Koszul-Komplexes von $S/\mathfrak{a}$ bezüglich der Sequenz $x_1, \ldots, x_n$ als Algebra isomorph ist zum Quotienten eines graduiert-kommutativen Polynomrings:

Proposition 2.5.

$$H(K_\bullet^{S/\mathfrak{a}}) \simeq k(Y_I,\ I \notin \mathcal{M},\ cl(I) = 1)/\mathfrak{r}.$$

Dies ist ein wichtiges Resultat der Arbeit, mit dem wir Fragestellungen über die multigraduierte Poincaré-Betti-Reihe beantworten können.

Speziell studieren wir Auflösungen von in Grad zwei erzeugten monomialen Idealen. Für solche Ideale zeigen wir, dass ein "kleinerer" Unterkomplex der Taylor-Auflösung bereits eine Auflösung definiert. Zunächst geben wir eine graphische Interpretation der Basiselemente der Taylor-Auflösung. Der Unterkomplex der Taylor-Auflösung, der von denjenigen Basiselementen erzeugt wird, die keinen Broken Circuit enthalten, sogenannte **nbc**-Mengen, definiert dann bereits eine Auflösung:

Theorem 2.6. *Sei $\mathfrak{a} \trianglelefteq S$ ein quadratfreies monomiales Ideal im Polynomring S und* MinGen($\mathfrak{a}$) *ein minmales monomiales Erzeugendensystem. Dann definiert der Unterkomplex $T_{\mathbf{nbc}}$, dessen Basis gegeben ist aus $I \subset$* MinGen($\mathfrak{a}$) *mit I* **nbc***-Menge, eine freie Auflösung des Ideals $\mathfrak{a}$.*

Als Spezialfall betrachten wir Stanley-Reisner-Ideale $\mathfrak{a} = I_{\Delta(P)}$ von Ordnungskomplexen $\Delta(P)$ einer partiell geordneten Menge P. Für solche Ideale konstruieren wir auf der Auflösung $T_{\mathbf{nbc}}$ ein weiteres nicht-azyklisches Matching $\mathcal{M}$. Betrachtet man nun den Vektorraum $T_{\mathbf{nbc}}^{\mathcal{M}}$, so liefert dieser zwar keine Auflösung (für nicht azyklische Matchings ist $\partial^{\mathcal{M}}$ nicht definiert), aber die Basis von $T_{\mathbf{nbc}}^{\mathcal{M}}$ ermöglicht in diesem Fall die Berechnung der multigraduierten Poincaré-Betti-Reihe von $k \simeq (S/\mathfrak{a})/\mathfrak{m}$.

Das Standard-Matching und die Ergebnisse über Auflösungen von in Grad zwei erzeugten Idealen verwenden wir später, um die multigraduierte Poincaré-Betti-Reihe spezieller Restklassenkörper auszurechnen (siehe Abschnitt 2.4.3).

Wir definieren zwei neue Klassen monomialer Ideale über eine kombinatorische Bedingung auf dem minimalen Erzeugendensystem.

Definition 2.7. Ein monomiales Ideal $\mathfrak{a} \trianglelefteq S$ mit minimalem monomialen Erzeugendensystem $\mathrm{MinGen}(\mathfrak{a})$ erfüllt die

(1) gcd-Bedingung, falls für je zwei teilerfremde Erzeuger $m, n \in \mathrm{MinGen}(\mathfrak{a})$ ein dritter Erzeuger $m, n \neq u \in \mathrm{MinGen}(\mathfrak{a})$ existiert, so dass $u \mid m\,n$.

(2) starke gcd-Bedingung, falls auf $\mathrm{MinGen}(\mathfrak{a})$ eine lineare Ordnung $\prec$ existiert und für je zwei Erzeuger $m \prec n \in \mathrm{MinGen}(\mathfrak{a})$ mit $\gcd(m, n) = 1$ ein dritter Erzeuger $m, n \neq u \in \mathrm{MinGen}(\mathfrak{a})$ existiert, so dass $u \mid m\,n$ und $m \prec u$.

Für monomiale Ideale, die die starke gcd-Bedingung erfüllen, konstruieren wir dann ein azyklisches Matching $\mathcal{M}$ auf der Taylor-Auflösung. Den resultierenden Morse-Komplex $T^{\mathcal{M}}$ bezeichnen wir mit T_{gcd}.

Proposition 2.8. *Sei $\mathfrak{a} \trianglelefteq S$ ein monomiales Ideal, das die starke* gcd*-Bedingung erfüllt. Der Komplex T_{gcd} geht durch ein azyklisches Matching aus der Taylor-Auflösung hervor und definiert somit eine freie Auflösung des Ideals $\mathfrak{a}$.*

Mit Hilfe des Komplexes T_{gcd} erhalten wir für monomiale Ideale $\mathfrak{a} \trianglelefteq S$ interessante Zusammenhänge und neue Resultate für die Golod-Eigenschaft von $A := S/\mathfrak{a}$. Wir gehen in Abschnitt 2.4.3 näher darauf ein. Anschließend studieren wir die sogenannte Poset-Auflösung (vgl. [**39**]). Diese Auflösung löst das Ideal, das von einer monomial geordneten Familie erzeugt wird, frei auf. Eine monomial geordnete Familie ist eine partiell geordnete Menge P, deren Elemente in eindeutiger Weise Monome zugeordnet sind. Die Poset-Auflösung ist dann die zelluläre Auflösung, die von dem Ordnungskomplex der partiell geordneten Menge unterstüzt wird. Für diese Auflösung definieren wir mit Hilfe der Algebraischen Diskreten Morse-Theorie zwei Algorithmen, die die Auflösung deutlich verkleinern und in manchen Fällen minimieren. Die Effektivität unserer Algorithmen hängt wesentlich von der geordneten Menge P und einer vorher gewählten linearen Erweiterung der Ordnung ab. Wir diskutieren die Vor- und Nachteile und geben Kriterien für die partiell geordnete Menge P und deren lineare Erweiterung an, welche implizieren, dass die Algorithmen minimale Auflösungen produzieren.

Schließlich wenden wir die Algebraische Diskrete Morse-Theorie auf die Taylor-Auflösung von Borel- bzw. p-Borel-fixed-Idealen an. Für Borel-Ideale gibt es bereits explizite Darstellungen minimaler Auflösungen (vgl. z.B. [**19**]).

Hier geben wir eine neue zelluläre Auflösung einer Unterklasse dieser Ideale an, die eine direkte Verallgemeinerung der hypersimplizialen Auflösung ist, welche Batzies in seiner Dissertation [**4**] einführte, um minimale zelluläre Auflösungen von Potenzen des maximalen homogenen Ideals zu berechnen.
Eine minimale Auflösung für p-Borel-fixed-Ideale gibt es bisher nur für solche, die Cohen-Macaulay sind (vgl. [**2**]). In [**4**] wurde gezeigt, dass diese sogar zellulär sind. Wir konstruieren in diesem Abschnitt minimale zelluläre Auflösungen für eine deutlich größere Klasse von p-Borel-fixed-Idealen, die die Cohen-Macaulay Ideale umfassen.
Zusätzlich geben wir eine rekursive Formel zur Berechnung der multigraduierten Betti-Zahlen sowie eine Formel zur Berechnung der Regularität solcher Ideale an. Die Ergebnisse über die Regularität verallgemeinern bisher bekannte Resultate (vgl. [**20**],[**30**]); unter Anderem können wir Pardues [**38**] Vermutung über die Regularität von p-Borel-fixed-Idealen teilweise neu beweisen.

2.4.2. *Auflösungen des Restklassenkörpers.*
Ein Großteil der Arbeit beschäftigt sich mit der minimalen Auflösung des Körpers k als A-Modul, wobei A der Quotient aus dem (nicht notwendigerweise kommutativen) Polynomring S und einem (nicht notwendigerweise monomialen) Ideal $\mathfrak{a} \trianglelefteq S$ ist.

Zunächst betrachten wir den Restklassenkörper $k = A/\mathfrak{m}$, wobei A der Quotientenring aus dem kommutativen Polynomring $R := k[x_1, \ldots, x_n]$ und einem Ideal $\mathfrak{a} \trianglelefteq R$ ist. Auf der normalisierten Bar-Auflösung definieren wir dann ein azyklisches Matching. Um das Differential des resultierenden Morse-Komplexes angeben zu können, fixieren wir eine Gröbnerbasis des Ideals $\mathfrak{a}$ und geben dann sogenannte Reduktionsregeln an. Wir erhalten eine Auflösung des Körpers k, die man als "kommutative Version der Anick-Auflösung" verstehen kann.
Wir geben Kriterien für das monomiale Ideal $\mathfrak{a} \trianglelefteq R$ an, so dass der oben konstruierte Komplex eine minimale Auflösung definiert. Des Weiteren zeigen wir für den Fall, dass unsere Auflösung minimal ist, dass die Poincaré-Betti-Reihe gleich dem Produkt $\left(\prod_{i=1}^{n}(1+t\,x_i)\right)F(x,t)$ ist, wobei $F(x,t)$ die Wort-zählende Funktion einer regulären Sprache $\mathcal{L}$ ist.
Schließlich geben wir eine explizite minimale Auflösung des Restklassenkörpers k für den Fall an, dass $\mathfrak{a}$ ein vollständiger Durschnitt ist.

Anschließend übertragen wir die oben aufgeführten Ergebnisse auf den nichtkommutativen Fall, das heißt, wir betrachten den Restklassenkörper $k = A/\mathfrak{m}$, wobei A der Quotientenring aus dem nicht-kommutativen Polynomring $R := k\langle x_1, \ldots, x_n\rangle$ und einem beidseitigem Ideal $\mathfrak{a} \trianglelefteq R$ ist. Der Morse-Komplex des übertragenen Matchings ist nun isomorph zur Anick-Auflösung. Auch hier geben wir Kriterien für das Ideal $\mathfrak{a}$, so dass der Morse-Komplex eine minimale Auflösung des Körpers k ist. Im Falle der Minimalität können wir außerdem beweisen, dass die Poincaré-Betti-Reihe eine rationale Funktion ist. Wir zeigen unter Anderem, dass der Morse-Komplex minimal ist, falls $\mathfrak{a}$ eine quadratische Gröbnerbasis besitzt. In diesem Fall folgt zudem die Rationalität der Hilbert-Reihe.
Schließlich diskutieren wir einige Beispiele von $A = R/\mathfrak{a}$ und beweisen damit unter Anderem eine Vermutung von B. Sturmfels (vgl. [**42**]).

Mit einem ähnlichen azyklischen Matching konstruieren wir außerdem neue Auflösungen des Körpers k, aufgefasst als $A \otimes A^{\text{op}}$-Modul, wobei wieder A der Quotient aus dem (nicht notwendigerweise kommutativen) Polynomring S und einem (nicht notwendigerweise monomialen) Ideal $\mathfrak{a} \trianglelefteq S$ ist. Für den Fall, dass R kommutativ und $\mathfrak{a}$ ein vollständiger Durchschnitt ist, geben wir eine explizite Gestalt der minimalen Auflösung an. Damit lässt sich die Hochschild-Homologie von A mit Koeffizienten in k berechnen. Dieses liefert eine Verallgemeinerung eines Resultates über die Hochschild-Homologie von Bach [**9**].

Die in diesem Abschnitt vorgestellten Ergebnisse wurden von uns bereits in [**36**] vorgestellt.

2.4.3. *Multigraduierte Hilbert- und Poincaré-Betti-Reihe und Golod-Eigenschaft.* Eine weitere Invariante, die sich durch Anwendung der Algebraischen Diskreten Morse-Theorie berechnen lässt, ist die multigraduierte Poincaré-Betti-Reihe von $k \cong A/\mathfrak{m}$, wobei $A = S/\mathfrak{a}$ der Quotientenring aus dem kommutativen Polynomring $S = k[x_1, \dots, x_n]$ und einem monomialen Ideal $\mathfrak{a} \trianglelefteq S$ ist.

Backelin bewies 1982 [**3**], dass in diesem Fall die multigraduierte Poincaré-Betti-Reihe rational ist, jedoch ist bis heute keine explizite Gestalt der Reihe bekannt. Charalambous und Reeves [**13**] bewiesen 1995 eine explizite Gestalt für den Extremfall, dass die Taylor-Auflösung des monomialen Ideals minimal ist. Sie schlussfolgerten, dass im Allgemeinen die Poincaré-Betti-Reihe eine "ähnliche Gestalt" hat, waren jedoch nicht in der Lage, eine konkrete Vermutung zu formulieren. Mit Hilfe des von uns entwickelten Standard-Matchings auf der Taylor-Auflösung können wir eine Vermutung über die Basis der minimalen multigraduierten Auflösung des Körpers $k = A/\mathfrak{m}$ formulieren. Als Folgerung daraus ergibt sich sofort eine explizite Gestalt der Poincaré-Betti-Reihe, die die Vermutung von Charalambous und Reeves präzisiert und bestätigt. Des Weiteren erhalten wir über die Euler-Charakteristik eine explizite Gestalt der multigraduierten Hilbert-Reihe und somit einen allgemeinen Zusammenhang zwischen Hilbert- und Poincaré-Betti-Reihe:

Sei $\mathfrak{a} \trianglelefteq S$ ein monomiales Ideal, $A := S/\mathfrak{a}$ die Quotientenalgebra und $\mathcal{M} = \bigcup_{i \geq 1} \mathcal{M}_i$ ein Standard-Matching. Für eine Teilmenge $I \subset \text{MinGen}(\mathfrak{a})$ eines minimalen monomialen Erzeugendensystems sei m_I das kleinste gemeinsame Vielfache und $cl(I)$ die Anzahl der Äquivalenzklassen bezüglich des transitiven Abschlusses der Relation $m \equiv n \Leftrightarrow \gcd(m,n) \neq 1$, wobei $m, n \in I$. Wir konstruieren einen neuen nichtkommutativen Ring:

$$R := \frac{k\langle Y_I,\ cl(I) = 1\ , I \in \mathcal{M}_1 \rangle}{\langle Y_I Y_J - (-1)^{(|I|+1)(|J|+1)} Y_J Y_I \text{ falls } \gcd(m_I, m_J) = 1 \rangle}.$$

Der Ring R hat drei Graduierungen:

$$\begin{array}{rcll} |Y_I| & := & |I| + 1, & \\ \deg(Y_I) & := & \alpha, & \text{falls } m_I = x^\alpha, \\ \deg_t(Y_I) & := & ||\alpha||, & \text{falls } m_I = x^\alpha. \end{array}$$

Wir können nun unsere Vermutung über die minimale freie Auflösung des Restklassenkörpers formulieren:

Vermutung 2.9. *Sei $F_\bullet$ eine minmale freie graduierte Auflösung des Restklassenkörpers k über A. Sei $F_i := \bigoplus_\alpha A(-\alpha)^{\beta_{i,\alpha}}$ der Modul im i-ten homologischen Grad. Dann gilt*

$$F_i \cong \bigoplus_{I \subset [n]} \bigoplus_{u \in \mathcal{G}(R)} A(-(\alpha_I + |u|)),$$

wobei $\alpha_I \in \{0,1\}^n$ der charakteristische Vektor von I ist und $\mathcal{G}(R)$ die Menge der Monome in R bezeichnet.

Mit Hilfe der Cartier-Foata-Theorie [**12**] erhalten wir eine präzise Darstellung der multgraduierten Hilbert- und Poincaré-Betti-Reihe, die die Vermutung von Charalambous und Reeves präzisiert und bestätigt.

Proposition 2.10. *Gilt Vermutung 2.9, so haben die multigraduierte Hilbert und Poincaré-Betti Reihe folgende Gestalt:*

$$\text{(2.1)} \qquad P_k^A(\underline{x},t) = \prod_{i=1}^{n}(1+x_i\ t)\ \operatorname{Hilb}_R(\underline{x},1,t)$$

$$= \frac{\prod_{i=1}^{n}(1+x_i\ t)}{1 + \sum_{\substack{I \subset \operatorname{MinGen}(\mathfrak{a}) \\ I \notin \mathcal{M}_1}} (-1)^{cl(I)}\ m_I\ t^{cl(I)+|I|}},$$

$$\text{(2.2)} \qquad \operatorname{Hilb}_A(\underline{x},t) = \left(\prod_{i=1}^{n}(1-x_i\ t)\ \operatorname{Hilb}_R(\underline{x},t,-1)\right)^{-1}$$

$$= \frac{1 + \sum_{\substack{I \subset \operatorname{MinGen}(\mathfrak{a}) \\ I \notin \mathcal{M}_1}} (-1)^{|I|}\ m_I\ t^{m_I}}{\prod_{i=1}^{n}(1-x_i\ t)}.$$

Wir führen nun folgende Notation ein:

Definition 2.11. Wir sagen, A hat Eigenschaft

(P) ,falls $P_k^A(\underline{x},t) = \prod_{i=1}^n(1+x_i\ t)\ \operatorname{Hilb}_R(\underline{x},1,t)$, und hat Eigenschaft

(H) ,falls $\operatorname{Hilb}_A(\underline{x},t) = \left(\prod_{i=1}^n(1-x_i\ t)\ \operatorname{Hilb}_R(\underline{x},t,-1)\right)^{-1}$.

Im Folgenden beschäftigen wir uns damit, unsere Vermutung 2.9 in Spezialfällen zu beweisen. Wir zeigen unsere Vermutung für verschiedene Klassen von Algebren A:

Theorem 2.12. *Sei $A = S/\mathfrak{a}$ der Quotientenring und $\mathfrak{a} \trianglelefteq S$ ein monomiales Ideal.*

(1) *Ist $\mathfrak{a}$ in Grad zwei erzeugt, so gilt Vermutung 2.9.*

(2) *Ist die Koszul-Homologie $H_\bullet(K^A)$ ein M-Ring (vgl. [**23**]) und existiert ein Homomorphismus $\phi : H(K^A) \to Z(K^A)$, so dass $\pi\phi = \operatorname{id}_{H(K^A)}$, so gilt Vermutung 2.9, wobei $\pi : Z(K^A) \to H(K^A)$ die kanonische Projektion ist.*

(3) *Ist die Koszul-Homologie $H_\bullet(K^A)$ ein M-Ring und hat die minimale Auflösung von $\mathfrak{a}$ über S eine differentiell graduierte Struktur, so hat A Eigenschaft* **(P)**.

Für den allgemeinen Fall geben wir mit Hilfe des Eagon-Komplexes und einer Verallgemeinerung der Massey-Operationen eine Beweisidee, die zudem unsere Vermutung rechtfertigt.

Die Golod-Eigenschaft eines monomialen Ringes $A = S/\mathfrak{a}$ ist äquivalent zu einer konkreten Darstellung der Poincaré-Betti-Reihe:

$$A \text{ Golod } \Leftrightarrow P_k^A(\underline{x}, t) = \frac{\prod_{i=1}^n (1 + t\, x_i)}{1 - t \sum_{i,\alpha} \beta_{i,\alpha}\, \underline{x}^\alpha\, t^i},$$

wobei $\beta_{i,\alpha} = \dim\big(\operatorname{Tor}_i^S(A,k)_\alpha\big)$ die multigraduierten Betti-Zahlen sind.

Daher liefert unsere Vermutung interessante Folgerungen sowie Kriterien für die Golod-Eigenschaft. Unter Annahme unserer Vermutung bekommen wir folgende interessante Resultate:

Theorem 2.13. *Sei $A = S/\mathfrak{a}$ eine Quotientenalgebra mit Eigenschaft* **(P)**. *Dann ist A genau dann Golod, wenn das Produkt auf der Koszul-Homologie (die erste Massey-Operation) trivial ist.*

Diese Äquivalenz ist eine wesentliche Vereinfachung gegenüber der Definition von Golod.

Mit unserer gcd-Bedingung erhalten wir sogar noch einfachere, rein kombinatorische Kriterien für die Golod-Eigenschaft:

Theorem 2.14. *Sei $A = S/\mathfrak{a}$ eine Quotientenalgebra mit Eigenschaft* **(P)**. *Erfüllt $\mathfrak{a}$ die starke* gcd-*Bedingung, so ist A Golod.*

Hier vermuten wir sogar eine Äquivalenz:

Vermutung 2.15. *Sei $A = S/\mathfrak{a}$ eine Quotientenalgebra mit Eigenschaft* **(P)**. *Dann ist A genau dann Golod, falls $\mathfrak{a}$ die starke* gcd-*Bedingung erfüllt.*

Herzog, Reiner und Welker beweisen in [**29**], dass wenn $\mathfrak{a} \trianglelefteq S$ komponentenweise linear ist, der Ring $A = S/\mathfrak{a}$ Golod ist. Mit Hilfe der Algebraischen Diskreten Morse-Theorie können wir dieses Resultat verallgemeinern:

Theorem 2.16. *Sei $\mathfrak{a} \trianglelefteq S$ in Grad l erzeugt.*

(1) *Falls $\dim_k\big(Tor_i^S(S/\mathfrak{a},k)_{i+j}\big) = 0$ für alle $j \geq 2(l-1)$, dann ist die Algebra $A = S/\mathfrak{a}$ Golod.*
(2) *Ist A Golod, so gilt $\dim_k\big(Tor_i^S(S/\mathfrak{a},k)_{i+j}\big) = 0$ für alle $j \geq i(l-2)+2$.*

Insbesondere gilt: Ist A Koszul, so ist A genau dann Golod, wenn die minimale freie Auflösung von $\mathfrak{a}$ linear ist.

Die in diesem Abschnitt präsentierten Resultate wurden von uns bereits in [**35**] vorgestellt.

2.5. Struktur des ersten Teils.

- ▷ *Kapitel 1 enthält die Einleitung.*
- ▷ *Kapitel 2 enthält einige Definitionen, elementare Tatsachen über und Beispiele von Kettenkomplexen bzw. Auflösungen, die in der Arbeit verwendet werden:*
  - ▷ *Im ersten Paragraphen werden folgende Begriffe definiert:*
    - *Multigraduierte freie Auflösungen von R-Moduln*
    - *Zelluläre multigraduierte freie Auflösungen von R-Moduln*
    - *Multigraduierte Hilbert- und Poincaré-Betti-Reihe von Moduln*
    - *Homologie von Komplexen*

    Des Weiteren werden grundlegende Zusammenhänge erklärt.
  - ▷ *Im zweiten Paragraphen werden spezielle Kettenkomplexe definiert und ihre Anwendungen erklärt. Es werden die folgenden Komplexe definiert:*
    - *Taylor- und Scarf- Auflösung monomialer Moduln*
    - *Poset-Auflösung monomialer Moduln*
    - *Koszul-Komplex*
    - *Bar- und normalisierte Bar-Auflösung*
    - *Azyklische und normalisierte azyklische Hochschild-Auflösung*
  - ▷ *Im dritten Paragraphen definieren wir den sogenannten Eagon-Komplex, der eine freie Auflösung des Körpers k über dem Quotientenring A ist, wobei A der Quotient aus dem Polynomring S und einem Ideal $\mathfrak{a}$ in S ist. Es werden die Massey-Operation auf der Koszul-Homologie und die Golod-Eigenschaft von k-Algebren erklärt sowie deren Zusammenhänge untereinander und deren Folgerungen für die Poincaré-Betti-Reihe erläutert.*
- ▷ *Kapitel 3 enthält die Resultate, die im Abschnitt 2.3 erläutert sind.*
- ▷ *Kapitel 4 enthält die Resultate aus Abschnitt 2.4.1.*
- ▷ *Kapitel 5 enthält die Resultate aus Abschnitt 2.4.2.*
- ▷ *Kapitel 6 enthält die in Abschnitt 2.4.3 vorgestellten Ergebnisse.*

3. Zwei Probleme aus der Algebraischen Kombinatorik

3.1. Einführung.

In diesem Teil der Doktorarbeit werden zwei Probleme aus der algebraischen Kombinatorik diskutiert: “Die Homologie von Nilpotenten Lie-Algebren Endlichen Typs” und die “Neggers-Stanley-Vermutung”, auch bekannt als “Poset-Vermutung”. Für beide Probleme wird eine kurze Einleitung in die Theorie gegeben und die grundlegenden Fragestellungen erklärt. Anschließend präsentieren wir unsere Resultate.

Der Grund für die separate Behandlung dieser Probleme ist, dass die Resultate nicht mit Hilfe der Algebraischen Diskreten Morse-Theorie erzielt wurden.

3.2. Homologie von nilpotenten Lie-Algebren endlichen Typs.

In diesem Kapitel diskutieren wir die Homologie von nilpotenten Lie-Algebren. Es zeigt sich in [**27**], dass viele Fragestellungen über die Homologie von nilpotenten Lie-Algebren kombinatorischer Natur sind. Eine klassische Verbindung zwischen der Kombinatorik und den Lie-Algebren ergibt sich aus dem engen Zusammenhang zwischen halb-einfachen Lie-Algebren und endlichen Spiegelungsgruppen im $\mathbb{R}^n$.

Im ersten Paragraphen geben wir eine grundlegende Einführung in die Theorie. Wir folgen hier dem Buch “Introduction to Lie algebras and representation theory“ von Humphreys [**33**]. Wir geben die klassischen Beispiele von Lie-Algebren $\mathfrak{n}(n,k), \mathfrak{sp}(n,k), \mathfrak{sl}(n,k), \mathfrak{o}(n,k)$ und geben deren Wurzelsysteme (A_n, B_n, C_n, D_n) an. Weiter geben wir, dem Buch “Reflection groups and Coxeter groups” von Humphreys [**34**] folgend, eine knappe Einführung in die Theorie der Spiegelungsgruppen und den dazu assoziierten Wurzelsystemen. Wir erklären die Spiegelungsgruppen zu den oben angeführten Beispielen. Schließlich definieren wir den nilpotenten Teil einer Lie-Algebra, assoziiert zu einem Wurzelsystem.
Im nächsten Teil geben wir die Definition der Homologie von Lie-Algebren. Hier folgen wir dem Übersichtsartikel über kombinatorische Probleme in der Homologie von nilpotenten Lie-Algebren von Hanlon [**27**]. Wir schließen diesen Paragraphen mit ein paar Beispielen von ungelösten Problemen und interessanten Vermutungen bezüglich der Homologie von nilpotenten Lie-Algebren (vgl [**28**]).

Paragraph 2 besteht aus unseren Ergebnissen in der Theorie der nilpotenten Lie-Algebren: Wir charakterisieren nilpotente Lie-Algebren durch sogenannte “Zwei-gefärbte simpliziale Komplexe”:

Definition 3.1. Sei Δ ein d-dimensionaler simplizialer Komplex mit n Facetten. Wir bezeichnen mit $\mathcal{F}_i := \{F \in \Delta \mid \dim(F) = i\}$ die Menge der i-dimensionalen Seiten. Auf der Menge der Facetten fixieren wir eine lineare Ordnung:

$$\mathcal{F}_d := \{F_1 < F_2 < \ldots < F_n\}.$$

(1) Eine Zwei-Färbung einer Facette F ist eine Abbildung f_F, die zu jeder Ecke von F eine Farbe $\{r, g\}$ (r =rot, g =grün) zuordnet.

(2) Sei für jede Facette F_i, $i = 1, \ldots, n$, eine Zwei-Färbung f_{F_i} gegeben. Dann heißt das Paar (Δ, f) ein zwei-gefärbter Komplex, falls f definiert

ist durch

$$\begin{aligned} f : \mathcal{F}_0 &\rightarrow \{r, g, -\}^n, \\ v &\mapsto (f_1(v), \dots, f_n(v)), \end{aligned}$$

wobei σ durch Permutieren der Koordinaten agiert und $n := \#\mathcal{F}_d$ die Anzahl der Facetten von Δ ist. Die Abbildungen f_i sind definiert durch

$$f_i(v) := \begin{cases} f_{F_i}(v) & , \quad v \in F_i \\ - & , \quad v \notin F_i. \end{cases}$$

Für einen Farb-Vektor $c := (c_1, \dots, c_n) \in \{r, g, -\}^n$ definieren wir den komplementären Vektor $\overline{c} := (\overline{c_1}, \dots, \overline{c_n})$ durch

$$\overline{c_i} := \begin{cases} r & , \quad c_i = g \\ g & , \quad c_i = r \\ - & , \quad c_i = -. \end{cases}$$

Definition 3.2. Seien (Δ, f) und (Δ', f') zwei zwei-gefärbte simpliziale Komplexe mit n Facetten.
Wir nennen (Δ, f) und (Δ', f') isomorph, falls

(1) Δ und Δ' als simpliziale Komplexe isomorph sind, und
(2) eine Permutation $\sigma \in S_n$ existiert, so dass für alle Ecken v gilt

$$f'(v) \in \left\{ \sigma(f(v)),\ \overline{\sigma(f(v))} \right\},$$

wobei σ die Koordinaten permutiert.

Da der Farb-Vektor ein geordnetes Tupel ist und somit von der gewählten Ordnung abhängt, benötigen wir die Permutation, um den Isomorphiebegriff nicht-trivial zu machen. Durch Neuordnen kann man jedoch immer annehmen, dass die i-te Facette von Δ auf die i-te Facette von Δ' abgebildet wird. In diesem Fall kann man in der Definition die Permutation weglassen.

Sei nun L eine nilpotente Lie-Algebra endlichen Typs, also ein k-Vektorraum versehen mit dem Lie-produkt. Sei $\mathcal{B}$ eine k-Basis von L. Wir betrachten nun die Menge der Lie-Relationen

$$\left\{ (a, b, [a, b]) \;\middle|\; a, b \in \mathcal{B},\ [a, b] \neq 0 \right\},$$

wobei $[\cdot, \cdot]$ die Lie-Klammer ist. Da die Lie-Algebra endlichen Typs ist, ist die Menge der Lie-Relationen endlich und charakterisiert die Algebra L in eindeutiger Weise. Wir bezeichnen mit $\mathcal{R}$ die Teilmenge der Lie-Relationen, die für alle $a, b \in \mathcal{B}$ mit $[a, b] \neq 0$ genau eine der Relationen $(a, b, [a, b])$, $(b, a, -[a, b])$ enthält und nennen sie die Menge der positiven Lie-Relationen.

Zu $\mathcal{R}$ assoziieren wir einen zwei-gefärbten simplizialen Komplex.

Definition 3.3. Seit L eine endlich erzeugte nilpotente Lie-Algebra mit Lie-Relation $\mathcal{R}$. Der zweidimensionale zwei-gefärbte simpliziale Komplex (Δ, f), definiert durch

$$\Delta = \Delta(\mathcal{R}) := \bigcup_{(a,b,c) \in \mathcal{R}} \Delta\big((a, b, c)\big),$$

$$\Delta\big((a, b, c)\big) := \big\{ \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \big\},$$

mit Färbung

$$f_{(a,b,c)}(v) := \begin{cases} g \quad , & v = a, b \\ r \quad , & v = c = [a,b] \end{cases}$$

heißt der *Lie-Relationen-Komplex*.

Es ist leicht zu sehen, dass die Lie-Klammer durch den Lie-Relationen-Komplex bis auf Vorzeichen eindeutig definiert ist.

Auf dieser Ebene sind wir in der Lage einen neuen (schwächeren) Isomorphietyp (Quasi-Isomorphismus) zu definieren.

Definition 3.4. Zwei nilpotente Lie-Algebren L and L' endlichen Typs heißen *quasi-isomorph*, falls es eine Wahl von Lie-Relationen $\mathcal{R}$ und $\mathcal{R}'$ gibt, so dass die zugehörigen zwei-gefärbten simplizialen Komplexe (Δ, f) und (Δ', f') zueinander isomorph sind.

Natürlich sind zwei zueinander isomorphe Lie-Algebren insbesondere quasi-isomorph.

Diese schwache Isomorphie hat zur Folge, dass die Summe der Homologiegruppen über alle homologischen Grade invariant bleibt:

Theorem 3.5. *Seien L und L' zwei zueinander quasi-isomorphe nilpotente Lie-Algebren endlichen Typs. Dann gilt:*

$$\bigoplus_{i \geq 0} H(L, \mathbb{Z}) \cong \bigoplus_{i \geq 0} H(L', \mathbb{Z}).$$

Hier zeigt sich der Unterschied zum bisherigen Isomorphiebegriff, in dessen Folge die Homologiegruppen in jedem Grad isomorph sind. Bei unsere schwachen Isomorphie kann es passieren, dass die Gruppen in einem festen homologischem Grad nicht isomorph sind.

In Paragraph 3 suchen wir dann Kriterien, wann zwei nilpotente Lie-Algebren zueinander quasi-isomorph sind. Wir studieren im speziellen Unter-Algebren der nilpotenten Teile von Lie-Algebren assoziiert zu Wurzelsystemen. Betrachten wir das Wurzelsystem A_{n-1}, so ist der nilpotente Teil gegeben durch die Lie-Algebra, bestehend aus allen oberen Dreiecksmatritzen $\mathfrak{n}(n,k)$. Unteralgebren von $\mathfrak{n}(n,k)$ können einerseits durch sogenannte abgeschlossene Teilmengen von A_{n-1} charakterisiert werden oder durch partiell geordnete Mengen: Wenn $P = \big(\{1,\ldots,l\}, \prec\big)$ mit $l \leq n$ eine partiell geordnete Menge ist, so ist die zugehörige Unteralgebra von $\mathfrak{n}(n,k)$ gegeben durch den Span der Matrizen E_{ij} mit $i \prec j$, deren Eintrag an der i-ten Zeile und j-ten Spalte gleich 1 ist und sonst 0.

Es ist leicht zu sehen, dass die Lie-Relationen eins zu eins den zwei-Ketten im Ordnungskomplex entsprechen (vgl. Figure 1). Um den Begriff der Quasi-Isomorphie auf partiell geordnete Mengen übertragen zu können, müssen wir zwei Typen von Paaren von Zwei-Ketten einführen (vgl. Figure 2 and 3). Damit können wir folgende Definition machen:

Definition 3.6. Zwei partiell geordnete Mengen P und P' heißen stark-isomorph, falls es einen Isomorphismus $\phi : \Delta(P) \to \Delta(P')$ gibt, der Paare von Zwei-Ketten vom Typ i auf Paare von Zwei-Ketten vom Typ i abbildet, für $i = 1, 2$.

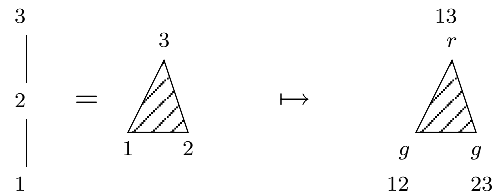

Figure 1.

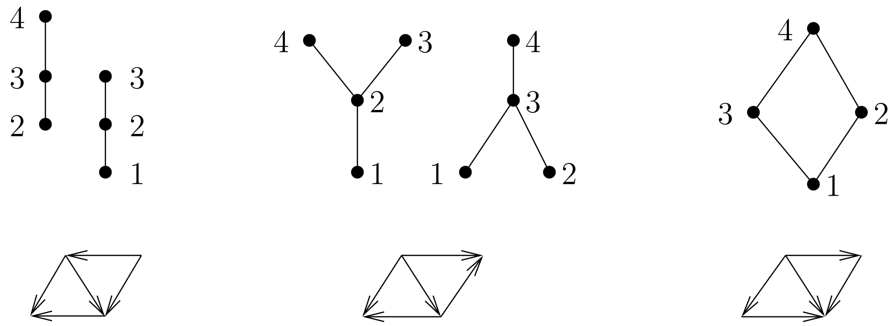

Figure 2. Paare von Zwei-Ketten vom Typ 1

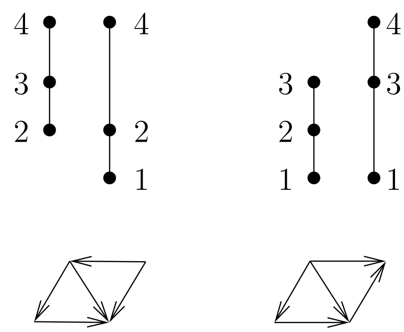

Figure 3. Paare von Zwei-Ketten vom Typ 2

Wir können dann folgende Äquivalenz beweisen:

Proposition 3.7. *Zwei partiell geordnete Mengen P und P' sind genau dann stark-isomorph, falls die zugehörigen Lie-Algebren $L(P)$ und $L(P')$ zueinander quasi-isomorph sind.*

Mit dieser Charakterisierung bekommen wir folgendes Resultat:

Corollary 3.8. *Seien P und P' zwei stark-isomorphe partiell geordnete Mengen und $L(P)$, bzw. $L(P')$ die zugehörigen nilpotenten Lie-Algebren. Dann gilt*

$$\bigoplus_{i \geq 0} H(L(P), \mathbb{Z}) \cong \bigoplus_{i \geq 0} H(L(P'), \mathbb{Z}).$$

Für Transformationen einer partiell geordneten Menge P, die den Ordnungskomplex invariant lassen, ist lediglich zu prüfen, ob ein gegebener Isomorphismus zwischen den Ordnungskomplexen die Zusatzbedingung erfüllt. Wir geben hierzu Beispiele an.

Für Unteralgebren des nilpotenten Teils anderer Wurzelsysteme macht der Begriff Quasi-Isomorphismus ebenfalls Sinn, da er für alle nilpotenten Lie-Algebren definiert ist. Allerdings muss man bei anderen Wurzelsystemen mit abgeschlossenen Teilmengen der Wurzelsysteme arbeiten, da kein dem Poset äquivalenter Begriff zur Verfügung steht. Daher lassen sich keine zum A_n-Fall ähnlichen Kriterien finden. Für das Wurzelsystem B_n hat Reiner [**43**], [**44**] eine kombinatorische Charakterisierung gefunden, die wir aber noch nicht studiert haben. Hier gibt es unserer Meinung nach eine Chance, ähnliche Kriterien zu finden.

3.3. Neggers-Stanley-Vermutung.

Die Neggers-Stanley-Vermutung, auch bekannt als "Poset-Vermutung", bezieht sich auf ein spezielles Polynom, das einer beliebigen partiell geordneten Menge P zugeordnet wird. Die Vermutung wurde erstmalig 1978 von Neggers formuliert: Einer partiell geordneten Menge P wird zunächst eine Etikettierung zugeordnet, das heißt, jedem Element in P wird eine Zahl zwischen 1 und $\#P$ zugeordnet. Wir bezeichnen diese Zuordnung im Folgenden mit $\omega : P \to \{1, \ldots, \#P\}$. Dabei unterscheidet man zwischen "natürlichen" Etikettierungen, dass heißt, gilt $p <_P q$, so folgt $\omega(p) < \omega(q)$, und beliebigen Etikettierungen. Anschließend wird der Menge P ein Polynom $W(P, \omega, t)$ zugeordnet, das wesentlich von der Etikettierung abhängt. Neggers vermutete, dass falls ω eine natürliche Etikettierung ist, das Polynom $W(P, \omega, t)$ nur reelle Nullstellen hat.
Ein Theorem aus der Analysis besagt, falls ein beliebiges Polynom $f(t) := \sum_i a_i t^i$ mit nichtnegativen Koeffizienten a_i ausschließlich reelle Nullstellen hat, dass dann die Koeffizientenfolge von f unimodal ist, das heißt, dass die Koeffizientenfolge $a_0, a_1, \ldots, a_d$ erst ansteigt und anschließend fällt:

$$a_0 \leq a_1 \leq a_2 \ldots \leq a_j \geq a_{j+1} \geq a_{j+2} \geq \ldots \geq a_d.$$

Daraus ergibt sich eine schwächere Vermutung: Das Polynom $W(P, \omega, t)$ ist unimodal.

1986 formulierte Stanley dieselbe Vermutung für eine beliebige Etikettierung. Die Neggers-Stanley-Vermutung läßt sich in vier Vermutungen aufteilen:

Vermutung 3.9. *[Stanley] Sei P eine partiell geordnete Menge mit einer beliebigen Etikettierung ω. Dann hat das Polynom $W(P, \omega, t)$ nur reelle Nullstellen.*

Vermutung 3.10. *[Stanley] Sei P eine partiell geordnete Menge mit einer beliebigen Etikettierung ω. Dann hat das Polynom $W(P, \omega, t)$ eine unimodale Koeffizientenfolge.*

Vermutung 3.11. *[Neggers] Sei P eine partiell geordnete Menge mit einer natürlichen Etikettierung ω. Dann hat das Polynom $W(P, \omega, t)$ nur reelle Nullstellen.*

Vermutung 3.12. *[Neggers] Sei P eine partiell geordnete Menge mit einer natürlichen Etikettierung ω. Dann hat das Polynom $W(P, \omega, t)$ eine unimodale Koeffizientenfolge.*

Dabei gilt:

$$\begin{array}{ccc} \text{Vermutung 3.9} & \Rightarrow & \text{Vermutung 3.10} \\ \Uparrow & & \Uparrow \\ \text{Vermutung 3.11} & \Rightarrow & \text{Vermutung 3.12} \end{array}$$

Die vier Vermutungen sind in manchen Spezialfällen mittlerweile bewiesen und für die ganz allgemeinen Vermutungen 3.9 und Vermutung 3.11 existieren seit Ende 2004 Gegenbeispiele: Für beliebig etikettierte partiell geordnete Mengen existiert ein Gegenbeispiel von Brändén [**10**], das wir auch in der Arbeit vorstellen. Von dem Gegenbeispiel für Vermutung 3.11 haben wir erst kurz vor Fertigstellung der Arbeit erfahren, daher sei hier nur auf die Existenz des Gegenbeispiels hingewiesen.

In der Arbeit beschäftigen wir uns mit Vermutung 3.12. Da wir nur partiell geordnete Mengen P mit einer natürlichen Etikettierung ω behandeln und in diesem Fall das Polynom $W(P, \omega, t)$ unabhängig von der Etikettierung ist, gehen wir im Folgenden immer davon aus, dass $P = ([n], \prec) = (\{1, \dots, n\}, \prec)$ und die Ordnung $\prec$ die natürliche Ordnung erhält: $i \prec j \Rightarrow i < j$.

Zu einer partiell geordneten Menge $P = [n]$ definiert man den sogenannten Verband der Ordnungsideale $\mathcal{J}(P)$. Ein Ordnungsideal I ist eine Teilmenge von P, so dass für $i \in I$ und $j \prec i$ auch $j \in I$ gilt. $\mathcal{J}(P)$ ist dann die Menge aller Ordnungideale, geordnet durch Inklusion. Ein klassisches Theorem besagt, dass $\mathcal{J}(P)$ ein distributiver Verband ist und umgekehrt zu jedem distributiven Verband $\mathcal{L}$ eine partiell geordnete Menge P existiert mit $\mathcal{J}(P) = \mathcal{L}$.

Zu einem distributivem Verband $\mathcal{L}$ kann man den sogenannten Hibi-Ring assoziieren:

$$R(\mathcal{L}) := \frac{k[x_i, \ i \in \mathcal{L}]}{\langle x_i x_j - x_{i \wedge j} x_{i \vee j} \rangle}.$$

Der Zusammenhang zwischen dem Hibi-Ring und der Neggers-Stanley-Vermutung ergibt sich aus der folgenden bekannten Tatsache:

$$\mathrm{Hilb}_{R(\mathcal{J}(P))}(t) = \frac{W(P, t)}{(1-t)^{\#P+1}}.$$

Da wir im ersten Teil der Arbeit multigraduierte Hilbert- und Poincaré-Betti-Reihen berechnen bekommen wir durch diese Darstellung des Polynoms eine weitere Formulierung der Neggers-Stanley-Vermutung und erhalten eine Rekursionsformel für das W-Polynom:

Dazu assoziieren wir zu einem distributivem Verband $\mathcal{L}$ einen Graphen $G(\mathcal{L})$, dessen Eckenmenge $V = \mathcal{L}$ der Verband ist, und zwei Ecken i, j sind miteinander verbunden, falls i und j eine Antikette in $\mathcal{L}$ sind, das heißt $i \not\prec j$ und $j \not\prec i$.
Eine Teilmenge $I \subset E$ der Kantenmenge nennen wir **nbc**-Menge, falls I keinen Broken Circuit enthält. Mit $cl(I)$ bezeichnen wir die Anzahl der Zusammenhangskomponenten des von I induzierten Teilgraphen von $G(\mathcal{L})$.

Damit bekommen wir folgende Darstellung der Neggers-Stanley-Vermutung:

Vermutung 3.13. *Sei $\mathcal{L}$ ein distributiver Verband und $G(\mathcal{L})$ der zugehörige Graph. Dann hat das folgende Polynom nur reelle Nullstellen:*

$$W(G(\mathcal{L}),t) := 1 + \sum_{\substack{I \subset G(\mathcal{L}) \\ I \ \mathbf{nbc}\text{-set}}} (-1)^{|I|} t^{cl(I)+|I|}.$$

Das Polynom $W(G(\mathcal{L}),t)(1-t)^{\#P+1-\#\mathcal{J}(P)}$ ist unimodal.

Des Weiteren erhalten wir folgende Rekursionsformel:

Proposition 3.14. *Sei $\mathcal{L}$ ein distributiver Verband, $G(\mathcal{L})$ der zugehörige Graph und $\{p,q\}$ eine Kante in $G(\mathcal{L})$. Dann gilt*

$$\begin{aligned} W(G(\mathcal{L}),t) &= (1-t)\Big(W\big(G(\mathcal{L}\setminus\{p\}),t\big) + W\big(G(\mathcal{L}\setminus\{q\}),t\big)\Big) \\ &\quad -(1-t)^2\, W\big(G(\mathcal{L}\setminus\{p,q\}),t\big). \end{aligned}$$

Das Polynom $W(P,t)$ einer partiell geordneten Menge $P=[n]$ hat auch eine topologische Struktur:
Dazu ordnet man jedem Ordnungsideal $I \in \mathcal{J}(P)$ durch seinen charakteristischen Vektor $\alpha_I \in \mathbb{Z}^n$, definiert durch

$$(\alpha_I)_i := \begin{cases} 0 & , \quad i \notin I \\ 1 & , \quad i \in I \end{cases}$$

einen Punkt im $\mathbb{R}^n$ zu. Die konvexe Hülle dieser Punkte bildet das sogenannte Ordnungspolytop $\mathcal{O}(P)$. Der folgende Satz gibt erklärt die topologische Interpretation des Polynoms $W(P,t)$:

Theorem 3.15. *Sei Δ eine beliebige unimodulare Triangulierung (eine simpliziale Triangulierung, bei der jeder vorkommende maximale Simplex das Volumen $1/(n!)$ hat) des Ordnungspolytops $\mathcal{O}(P)$. Dann stimmt das h-Polynom der Triangulierung mit dem W-Polynom überein:*

$$h(\Delta,t) = W(P,t).$$

Mit Hilfe dieser Interpretation ist es Reiner und Welker in [**45**] gelungen, Vermutung 3.12 für graduierte partiell geordnete Mengen zu beweisen, das sind solche, in denen alle maximalen Ketten dieselbe Länge haben.
Dazu konstruieren sie eine spezielle unimodulare Triangulierung und zeigen, dass diese, als abstrakter simplizialer Komplex, isomorph zum topologischen Verbund eines Simplex mit einem Komplex Δ_{eq} ist, den sie “equatorial complex” nennen. Daraus folgt dann, dass das W-Polynom mit dem h-Polynom des “equatorial Komplexes” Δ_{eq} übereinstimmt. Schließlich beweisen sie, dass Δ_{eq} eine polytopale Sphäre ist, und somit das h-Polynom $h(\Delta_{eq},t)$ unimodal ist.

Wir zeigen, dass diese Konstruktionen im Allgemeinen gültig ist: Wir konstruieren analog zu [**45**] eine unimodulare Triangulierung des Ordnungspolytops $\mathcal{O}(P)$, wobei nun $P=[n]$ eine beliebige partiell geordnete Menge ist. Unsere Triangulierung ist ebenfalls isomorph zum Verbund eines Simplex mit einem Komplex Δ_{eq} und analog folgt, dass im Allgemeinen das W-Polynom mit dem h-Polynom übereinstimmt.
Im Allgemeinen jedoch ist Δ_{eq} keine polytopale Sphäre und daher kann zunächst keine Aussage über die Gestalt des h-Polynoms getroffen werden. Aus unserer

Konstruktion folgt jedoch, dass Δ_{eq} isomorph (als abstrakter simplizialer Komplex) zum Schnitt eines Komplexes $\hat{\Delta}_{eq}$ mit einem Scnitt von Koordinaten- und Diagonal-Hyperebenen $H \in \mathbb{R}^n$ ist und es gilt, dass $\hat{\Delta}_{eq}$ eine polytopale Sphäre ist. Die Gleichungen, die die Hyperebenen in H bestimmen, können außerdem direkt von der partiell geordneten Menge P "abgelesen" werden.
Studiert man diesen Zusammenhang von der anderen Seite, so kann sich fragen, welche Mengen von Hyperebenen auftreten können. Wir starten dazu mit einer graduierten partiell geordneten Menge P und definieren über relativ einfache Kriterien sogenannte P-gültigen Mengen von Hyperebenen. Mit dieser Charakterisierung können wir folgende Äquivalenz beweisen:

Theorem 3.16. *Die folgenden Aussagen sind äquivalent*

(1) *Für alle partiell geordneten Mengen $P = [n]$ ist das W-Polynom unimodal.*

(2) *Für alle graduierten partiell geordneten Mengen $P = [n]$ mit zugehöriger "equatorial sphere" Δ_{eq} und für alle P-gültigen Mengen von Hyperebenen mit Schnittmenge $H(P)$ ist das h-Polynom*

$$h\big(\Delta_{eq} \cap H(P), t\big)$$

unimodal.

Damit reduzieren wir Vermutung 3.12 auf die Berechnung von h-Vektoren. Dieses Resultat eröffnet somit einen neuen Blickwinkel auf Vermutung 3.12.

Das Kapitel ist wie folgt strukturiert:

- ▷ *Im ersten Paragraphen definieren wir die Vermutung und erläutern bisher bekannte Zusammenhänge und Resultate. Außerdem bringen wir das Gegenbeispiel zur Vermutung 3.9 von Brändén* [**10**]. *Wir halten uns dabei an das Buch "Unimodal, log-concave and Pölya frequency sequences in combinatorics" von Brenti* [**11**].
- ▷ *Der zweite Paragraph beschäftigt sich mit der Vermutung 3.12. Wir erklären hier die Ergebnisse von Reiner und Welker* [**45**] *und geben eine knappe Beweisskizze ihrer Resultate.*
- ▷ *Paragraph 3 besteht aus unseren Resultaten. Im ersten Teil befindet sich die Darstellung des W-Polynoms durch die* **nbc***-Mengen und die oben erläuterte Rekursionsformel.*
  Im zweitem Teil sind die oben angesprochene unimodulare Triangulierung des Ordnungspolytops $\mathcal{O}(P)$, für beliebige, natürlich etikettierte, partiell geordnete Mengen P und die oben aufgeführten Eigenschaften und Folgerungen zu finden.

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2. Erklärung

Hiermit erkläre ich, dass ich die vorliegende Dissertationsschrift selbständig verfasst und keine anderen Hilfsmittel als die angegebenen verwendet habe.
Die Dissertation wurde in der jetzigen oder einer ähnlichen Form noch bei keiner anderen Hochschule eingereicht und hat noch keinen sonstigen Prüfungszwecken gedient.

Marburg, den 25.01.2005

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