Locating the range of an operator with an adjoint

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In this paper we consider the following question: *given a linear operator*[1] *on a Hilbert space, can we compute the projection on the closure of its range?*

Instead of making the notion of computation precise, we use Bishop's informal approach [1], in which 'there exists' is interpreted strictly as 'we can compute'. It turns out that the reasoning we use to capture this interpretation can be described by intuitionistic logic. This logic differs from classical logic by not recognising certain principles, such as the scheme 'P or not P', as generally valid. Since we do not adopt axioms that are classically false, all our theorems are acceptable in classical mathematics.

To answer our initial question affirmatively, it is enough to show that the range $\operatorname{ran}(T)$ of the operator T on the Hilbert space H is **located**—that is, the **distance**

$$\rho(x, \operatorname{ran}(T)) = \inf \{\|x - Ty\| : y \in H\}$$

exists (is computable) for each $x \in H$ ([2], pages 366 and 371). The locatedness of the kernel $\ker(T^*)$ of the adjoint T^* of T is easily seen to be a necessary—but according to Example 1 of [6], not sufficient—condition for $\operatorname{ran}(T)$ to be located. Theorem gives necessary and sufficient conditions under which the locatedness of $\ker(T^*)$ ensures that of $\operatorname{ran}(T)$. Proposition 9 below shows that (despite an earlier claim by Bridges–Ishihara [6]) in recursive mathematics, and hence in Bishop–style mathematics, a condition known as *well–behavedness* is not sufficient for the locatedness of $\operatorname{ran}(T)$.

As we saw in [6, 15, 16], sequential versions of boundedness and openness play an important role for linear operators; for example, the Hellinger–Toeplitz theorem [6] holds for sequential continuity.

Proposition 1. *An operator on H that has an adjoint is* ***sequentially continuous*** *in the sense that if $x_n \to 0$, then $Tx_n \to 0$.*

Moreover, in connection with the question at the start of this paper, we have the following result [6].

Proposition 2. *Let T be an operator on H with an adjoint, and suppose that T is* ***sequentially open*** *in the following sense: for each sequence (x_n) in H such that (Tx_n) converges to* 0, *there exists a sequence (y_n) in $\ker(T)$ such that $x_n + y_n \to 0$. Then ran(T) is located.*

The following definition introduces a notion related to, but weaker than, sequential openness. We say that an operator T on a Hilbert space H is **decent** if for any *bounded* sequence (x_n) such that $Tx_n \to 0$, there exists a sequence (y_n) in $\ker(T)$ such that $x_n + y_n \rightharpoonup 0$ (where, as usual, $\rightharpoonup$ denotes weak convergence—that is, $\langle x_n + y_n, z \rangle \to 0$ for all $z \in H$). Clearly, sequential openness implies decency.

[1] For Bishop, an operator is bounded, by definition; we do not require that our operators be bounded. Note that even a bounded operator on a Hilbert space need not have an adjoint (see [14] and *[11]*).

If T has an adjoint and is decent, then T^*T is also decent. For if (x_n) is a bounded sequence in H such that $T^*Tx_n \to 0$, and if $c > 0$ is a bound for the sequence $(\|x_n\|)$, then

$$\|Tx_n\|^2 = \langle T^*Tx_n, x_n\rangle \leq c\,\|T^*Tx_n\| \to 0.$$

Hence there exists a sequence (y_n) in $\ker(T) = \ker(T^*T)$ such that $x_n + y_n \rightharpoonup 0$.

A linear mapping T between normed spaces X and Y is said to be **well–behaved** if $Tx \neq 0$ [2] whenever $x \in X$ and $x \neq x'$ for all $x' \in \ker(T)$. The notion of well–behavedness was introduced in [5], where it was shown that a linear mapping onto a Banach space is well–behaved. The following proposition relates well–behavedness and decency.

Proposition 3. *Let H be a Hilbert space, and T a decent operator on H with located kernel. Then T is well–behaved.*

Proof. Let P be the projection of H on $\ker(T)$, and consider any $x \in H$ such that $x \neq y$ for all $y \in \ker(T)$ (so, in particular, $x \neq Px$). Construct an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 &\quad\Rightarrow\quad \|Tx\| < 1/n, \\ \lambda_n = 1 &\quad\Rightarrow\quad \|Tx\| > 1/(n+1). \end{aligned}$$

We may assume that $\lambda_1 = 0$. If $\lambda_n = 0$, set $x_n = x - Px$; if $\lambda_n = 1$, set $x_n = 0$. Then $\|Tx_n\| < 1/n$ for each n, so $Tx_n \to 0$. Since T is decent, there exists a sequence (y_n) in $\ker(T)$ such that $x_n + y_n \rightharpoonup 0$. In particular,

$$\begin{aligned} \langle x_n, x - Px\rangle &= \langle (I-P)(x_n + y_n), x - Px\rangle \\ &= \langle x_n + y_n, x - Px\rangle \to 0, \end{aligned}$$

and we can find N such that $|\langle x_N, x - Px\rangle| < \|x - Px\|^2$. If $\lambda_N = 0$, then $|\langle x_N, x - Px\rangle| = \|x - Px\|^2$, a contradiction. Hence $\lambda_N = 1$ and therefore $Tx \neq 0$. ∎

Proposition 4. *Let H be a Hilbert space, and T an operator on H with an adjoint, such that* $\mathrm{ran}(T^*)$ *is located. Then T is decent.*

Proof. Let P be the projection of H onto the closure of $\mathrm{ran}(T^*)$, let (x_n) be a sequence in H such that $Tx_n \to 0$, and set $y_n = Px_n - x_n$. Then $y_n \in \mathrm{ran}(T^*)^\perp = \ker(T)$. For each $z \in H$ we have

$$\langle x_n + y_n, T^*z\rangle = \langle Px_n, T^*z\rangle = \langle x_n, PT^*z\rangle = \langle x_n, T^*z\rangle = \langle Tx_n, z\rangle \to 0,$$

so $\langle x_n + y_n, Pz\rangle \to 0$ and therefore

$$\langle x_n + y_n, z\rangle = \langle x_n + y_n, Pz\rangle + \langle Px_n, (I-P)z\rangle \to 0.$$

∎

Note that we do not require the sequence (x_n) to be bounded in the proof of the foregoing proposition.

Theorem 5. *Let H be a Hilbert space, and T a decent operator on H with an adjoint and located kernel. Then* $\mathrm{ran}(T^*)$ *is located.*

[2] We use $x \neq y$ to signify that $\|x\| > 0$.

Proof. Let P be the projection of H on $\ker(T)$. It suffices to show that for each $x \in H$, $x - Px$ is in the closure $\overline{\operatorname{ran}(T^*)}$ of $\operatorname{ran}(T^*)$; for then $\rho(x, \operatorname{ran}(T^*)) = \|Px\|$. To this end, fix a vector x in H and $\varepsilon > 0$. For convenience, for each positive integer n denote the closed ball with centre 0 and radius n by B_n. Since $T^*(B_n)$ is located in H [17], we can construct an increasing binary sequence (λ_n) such that

$$\begin{aligned} \lambda_n = 0 &\Rightarrow \rho(x - Px, T^*(B_n)) > \varepsilon/2, \\ \lambda_n = 1 &\Rightarrow \rho(x - Px, T^*(B_n)) < \varepsilon. \end{aligned}$$

Without loss of generality, $\lambda_1 = 0$. If $\lambda_n = 0$, then by the separation theorem [13] and the Riesz representation theorem, there exists a unit vector y_n such that for each $u \in B_n$,

$$\langle x - Px, y_n \rangle > |\langle T^* u, y_n \rangle| + \frac{\varepsilon}{2} = |\langle u, Ty_n \rangle| + \frac{\varepsilon}{2}.$$

Taking $u = nTy_n$, we obtain

$$\frac{\varepsilon}{2} + n\,\|Ty_n\| < \langle x - Px, y_n \rangle \leq \|x\|,$$

and so $\|Ty_n\| < \|x\|/n$. On the other hand, if $\lambda_n = 1 - \lambda_{n-1}$, we set $y_k = 0$ for all $k \geq n$. Clearly, the sequence (Ty_n) converges to 0. But T is decent, so there exists a sequence (z_n) in $\ker(T)$ such that $y_n + z_n \rightharpoonup 0$. Choosing N such that

$$|\langle x - Px, y_n \rangle| = |\langle x - Px, y_n + z_n \rangle| < \varepsilon/2$$

for all $n \geq N$, we see that $\lambda_n = 1$ for some $n \leq N$. Since $\varepsilon > 0$ is arbitrary, it follows that $x - Px \in \overline{\operatorname{ran}(T^*)}$. ■

It is shown in [18] that if T is an operator on H with an adjoint, and if both $\operatorname{ran}(I + T^*T)$ and $\operatorname{ran}(I + TT^*)$ are located, then the graph of T,

$$\mathbf{G}(T) = \{(x, Tx) : x \in H\},$$

is located in $H \times H$.

Lemma 6. *Let T be an operator with an adjoint. Then $\mathbf{G}(T)$ is located in $H \times H$.*

Proof. By the foregoing remark, it suffices to show that $\operatorname{ran}(I + T^*T)$ and $\operatorname{ran}(I + TT^*)$ are located. Clearly $\ker(I + T^*T)$ is $\{0\}$ and is therefore located. As

$$\|(I + T^*T)x\|^2 = \|x\|^2 + \|T^*Tx\|^2 + 2\|Tx\|^2,$$

it follows that $\|(I + T^*T)x\| \geq \|x\|$; whence $I + T^*T$ is decent. So, by Theorem 5, $\operatorname{ran}(I + T^*T)$ is located. Interchanging the roles of T and T^*, we see that $\operatorname{ran}(I + TT^*)$ is located. ■

Before applying Lemma 6, we note some results found on pages 250–252 of [11]. If T is an operator with an adjoint, then its absolute value $|T|$ exists, and is uniquely defined by the equation $|T|^2 = T^*T$. If also $\operatorname{ran}(T)$ is located, then T has an exact polar decomposition $T = U|T|$ where U is an isometry from $\overline{\operatorname{ran}(|T|)}$ onto $\overline{\operatorname{ran}(T)}$ and U is 0 on the orthogonal complement of $\operatorname{ran}(T)$. Such a mapping U is said to be a **partial isometry** with **initial space** $\overline{\operatorname{ran}(|T|)}$ and **final space** $\overline{\operatorname{ran}(T)}$.

Lemma 7. *Let T be an operator with an adjoint; then* $\operatorname{ran}(T)$ *is located if and only if* $\operatorname{ran}(T^*)$ *is located.*

Proof. If $\operatorname{ran}(T)$ is located, then by Lemma 2 of [8], so is $\operatorname{ran}(TT^*)$. Since T^* has an adjoint, $|T^*|$ exists. The range of $|T^*|$ is located, because it contains $\operatorname{ran}(TT^*)$ as a located dense subset. So T^* has an exact polar decomposition $T^* = U|T^*|$, where U is a partial isometry whose initial space is the closure of $\operatorname{ran}(|T^*|)$ and whose final space is $\overline{\operatorname{ran}(T^*)}$. Since $\overline{\operatorname{ran}(T^*)}$ is the range of the projection UU^*, it is located; hence $\operatorname{ran}(T^*)$ itself is located. Interchanging the roles of T and T^* completes the proof. ■

Let T be a operator with an adjoint, then the following four statements are equivalent:

Theorem 8. (i) $\operatorname{ran}(T)$ *is located.*

(ii) $\operatorname{ran}(T^*)$ *is located.*

(iii) $\ker(T)$ *is located and T is decent.*

(iv) $\ker(T^*)$ *is located and T^* is decent.*

Proof. Since $\langle T^*x, y\rangle = \langle x, Ty\rangle$, we have $\operatorname{ran}(T^*)^{\perp} = \ker(T)$. If $\operatorname{ran}(T^*)$ is located, then the projection P on $\overline{\operatorname{ran}(T^*)}$ exists; since $I - P$ is the projection of H onto $\operatorname{ran}(T^*)^{\perp}$, we see that $\ker(T)$ is located. Moreover, by Proposition 4, T is decent. Thus (ii) $\Rightarrow$ (iii). It follows from Theorem 5 that (ii) $\Leftrightarrow$ (iii). Interchanging T and T^*, we now see that (i) $\Leftrightarrow$ (iv). Since (i) $\Leftrightarrow$ (ii) by Lemma 7, we conclude that (i)–(iv) are equivalent. ■

In [6], Bridges and Ishihara claimed to have a constructive proof that a bounded operator T with an adjoint on H has a located range if and only if $\ker(T^*)$ is located and T is well–behaved. The following theorem shows that, although their argument is valid for operators on a finite–dimensional Hilbert space, their conclusion cannot be obtained constructively if H is infinite–dimensional and we assume the Church–Markov–Turing thesis (for more on which, see [10, 19]).

Note that when we refer to an operator T on a Hilbert space H as **injective** we mean that $\|x\| > 0$ entails $\|Tx\| > 0$. Since $\ker(T) = \{0\}$ in that case, T has located kernel and is well–behaved.

Proposition 9. *Assume the Church–Markov–Turing thesis, and let H be a separable infinite–dimensional Hilbert space. Then there exists a bounded positive operator T on H that is injective (and hence is well behaved and has located kernel) but whose range is not located.*

Proof. It follows from the Church–Markov–Turing thesis that we can construct a sequence $(I_n)_{n=1}^{\infty}$ of non–overlapping closed intervals such that $[0,1] \subset \bigcup_{n=1}^{\infty} I_n$ and such that $\sum_{n=1}^{N} |I_n| < 1/4$ for each N (see [10], Chapter 3). Let $f_n : \mathbf{R} \to \mathbf{R}$ be the uniformly continuous mapping that vanishes outside I_n, takes the value 1 at the midpoint of I_n, and is linear on each half of I_n. By Theorem 2 of [4], the function

$$f = \sum_{n=1}^{\infty} n^{-2} f_n,$$

which is strictly positive almost everywhere on $[0,1]$, is Lebesgue integrable over $[0,1]$. Let $H = L_2[0,1]$ (relative to Lebesgue measure), and define a linear operator T on H by

$$Tg = gf.$$

This operator is easily seen to be bounded (by 1), selfadjoint, and positive. It is also injective: for if $\|Tg\|_2 > 0$, then $\int (gf)^2 > 0$, and so $g^2 > 0$ on a set of positive measure; whence, by [2] (page 244, (4.13)), $\|g\|_2 > 0$. Thus $\ker(T)$ is trivially located and T is well–behaved.

Let $(e_n)_{n=0}^{\infty}$ be an orthonormal basis of polynomial functions for H, with $e_0 = 1$. Let $\phi \mapsto \phi(T)$ denote the functional calculus for the selfadjoint operator T, and let μ denote the corresponding functional calculus measure on $[0,1]$, given by

$$\mu(\phi) = \sum_{n=0}^{\infty} 2^{-n} \langle \phi(T) e_n, e_n \rangle$$

([2], page 378, (8.22)). Denote Lebesgue measure by λ. It is relatively straightforward to prove that

$$\mu(\phi) = \sum_{n=0}^{\infty} 2^{-n} \int_0^1 (\phi \circ f) |e_n|^2 \, d\lambda = \int_0^1 (\phi \circ f) g d\lambda,$$

where

$$g = \sum_{n=0}^{\infty} 2^{-n} |e_n|^2 \in L_2[0,1].$$

Note that $g(x) \geq 1$ for each $x \in [0,1]$. Choose a strictly decreasing sequence (r_n) of positive numbers converging to 0 such that $(r_n, 1]$ is μ–integrable for each n, and let E_n be the complemented set

$$\left(E_n^1, E_n^0\right) = (\{x : f(x) > r_n\}, \ \{x : f(x) \leq r_n\}).$$

The first set in the ordered pair defining E_n is the classical counterpart of E_n; the characteristic function of E_n is the mapping

$$\chi_{E_n} : E_n^1 \cup E_n^0 \to \{0,1\}$$

defined to equal 1 on E_n^1, and 0 on E_n^0. Suppose that $\operatorname{ran}(T)$ is located. Then the proof of [3] (Theorem 4.6) shows that

$$\int g \, \chi_{E_n} \mathrm{d}\lambda = \mu((r_n, 1]) \to \mu([0,1]) = \int_0^1 g \, \mathrm{d}\lambda.$$

(The locatedness of the range of T is essential for this step in our proof.) By the monotone convergence theorem ([2], page 267), $\left(\chi_{E_n} g\right)_{n=1}^{\infty}$ converges λ–almost everywhere to g on $[0,1]$. Since $g \geq 1$, $\left(\chi_{E_n}\right)_{n=1}^{\infty}$ converges λ–almost everywhere to 1 on $[0,1]$; whence, again by the monotone convergence theorem, $\lambda((r_n, 1]) \to 1$ as $n \to \infty$. Choose ν such that $\lambda((r_\nu, 1]) > 1/4$. Then choose N such that $N^{-2} < r_\nu$. Since the intervals I_n are non–overlapping,

$$(r_\nu, 1] \subset \bigcup_{n=1}^{N} I_n$$

and therefore $\mu((r_\nu, 1]) < 1/4$, a contradiction. Thus ran (T) is not located. ■

It remains an open and interesting problem to find new conditions equivalent to the decency of a bounded operator on a Hilbert space.

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