Toeplitz Operators on Semi-Simple Lie Groups

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Zusammenfassung

Eine seit den Anfängen der Operatortheorie bekannte und bis heute bedeutsame Klasse von Operatoren sind die so genannten Toeplitz-Operatoren. Klassischerweise sind dies beschränkte Operatoren auf dem Hardyraum $\mathbf{H}^2(\mathbb{B})$, der aus den auf dem Einheitskreis $\mathbb{B}$ holomorphen Funktionen mit quadratintegrierbaren Randwerten besteht. Der Toeplitz-Operator zur Symbolfunktion f ist dann $T_f = EM_fE$, wobei M_f der Multiplikationsoperator mit f sei und E den orthogonalen Projektor auf $\mathbf{H}^2(\mathbb{B})$ bezeichne.
Ist nun $\mathcal{T}$ die von den T_f erzeugte C*-Algebra, so besagt das klassische Theorem von Gohberg-Kreĭn, dass der Quotient von $\mathcal{T}$ nach dem Ideal der kompakten Operatoren die Symbolalgebra ist. Dies ist interessant, weil sich die nicht-kommutative Operatoralgebra $\mathcal{T}$ klassischen spektraltheoretischen Methoden zunächst verschließt. Anwendungen findet der Satz in der Indextheorie von Fredholmoperatoren.
Der Satz von Gohberg-Kreın ist vom Einheitskreis auf den Fall so genannter beschränkter symmetrischer Gebiete verallgemeinert worden. Der entsprechende Hardyraum ist dort auf einem minimalen Teil des Randes, dem Shilovrand, realisiert. Das Hauptresultat, das im allgemeinen Fall auf Upmeier zurückgeht, besagt, dass an den Platz der kurzen exakten Sequenz eine Kompositionsreihe tritt, deren sukzessive Quotienten bestimmt und geometrisch interpretiert werden können. Die Länge des Sequenz entspricht dabei der Länge des Facettenverbands des betrachteten (konvexen) Gebiets, welche wiederum durch eine fundamentale geometrische Invariante, den Rang, beschrieben werden kann. Im Einheitskreis kommen als Facetten nur die Extremalpunkte vor.
In dieser Arbeit wird die Rolle des Shilovrands von einer beliebigen (nicht-kompakten) Liegruppe G Hermiteschen Typs eingenommen. Das entsprechende (nicht-homogene) Gebiet ist die so genannte Ol'shanskiĭ-Halbgruppe, die im Rahmen des Gel'fand-Gindikin-Programms konstruiert wurde. Der zugeordnete Hardyraum kann als die Summe aller Darstellungen der holomorphen diskreten Reihe beschrieben werden, so dass die harmonische Analyse der Gruppe G eine bedeutende Rolle spielt.
Letztendliches Ziel ist es, die Konstruktion einer Kompositionsreihe der Toeplitz C*-Algebra in diesem Rahmen zu verallgemeinern. Dazu wird eine geometrische Stratifizierung des Ol'shanskiĭ-Gebiets entwickelt. Weiterhin wird die harmonische Analyse des Hardyraums und der Gruppe G, sowie die mikrolokale Analysis der Szegö-Projektion E im Hinblick auf den Übergang zu Randkomponenten des Gebiets untersucht. Dabei werden Resultate zur Wellenfront des Faltungskerns dieser Projektion sowie zur Einbettung reduzierter Spektren (im Sinne der abstrakten Fouriertransformation) erzielt. Schließlich werden operatortheoretische Methoden, die die Anwendung dieser Ergebnisse auf das Problem der Konstruktion einer Kompositionsreihe erlauben, in der entsprechenden Allgemeinheit bereitgestellt.

Es folgt eine ausführliche deutsche Einleitung zu dieser Arbeit.

Einleitung

Klassischerweise betrachtet man in der Theorie der Toeplitz-Operatoren den Einheitskreis $\mathbb{B}$ in der komplexen Ebene $\mathbb{C}$. Toeplitz-Operatoren sind dann beschränkte Operatoren auf dem Hardyraum $\mathbf{H}^2(\mathbb{B}) \sqsubset \mathbf{L}^2(\mathbb{T})$, der aus den $\mathbf{L}^2$-Randwerten von auf dem Einheitskreis holomorphen Funktionen auf der Kreislinie besteht. Sie werden in natürlicher Weise mit Hilfe der Szegö-Projektion auf $\mathbf{H}^2(\mathbb{B})$ definiert.

Hauptgegenstand mathematischer Analyse ist nun die C*-Algebra $\mathcal{T}$, die von der Gesamtheit aller Toeplitz-Operatoren mit stetigem Symbol erzeugt wird. Typischerweise sind in dieser Algebra enthaltene Operatoren nicht normal, so dass der gewöhliche Spektralsatz zu ihrer Untersuchung nicht anwendbar ist. Um die Spektraltheorie dieser Operatoren zu verstehen, müssen weitaus elaboriertere Methoden zur Anwendung gebracht werden.

Der Hauptsatz über die Toeplitz-C*-Algebra $\mathcal{T}$ geht auf Gohberg-Kreĭn [GK58] zurück: Es gibt eine kurze exakte Sequenz

$$0 \to \mathbb{K}(\mathbf{H}^2(\mathbb{B})) \to \mathcal{T} \to \mathcal{C}(\mathbb{T}) \to 0\,,$$

welche sich sogar an $\mathcal{T}$ aufspaltet: die Symbolabbildung definiert eine einseitige Umkehrung der kanonischen Projektion auf $\mathcal{C}(\mathbb{T})$. Insbesondere stimmt das Kommutatorideal von $\mathcal{T}$ mit dem Ideal der kompakten Operatoren überein und das Spektrum von $\mathcal{T}$ ist als der *nicht-kommutative Raum* $\mathbb{T} \cup \mathrm{pt}$ vollständig bestimmt.

Von größerer Bedeutung ist, dass diese kurze exakte Sequenz es erlaubt, mächtige operatortheoretische Werkzeuge — wie etwa die K-Theorie von Operatoralgebren — auf solche Probleme wie Fredholm-Kriterien anzuwenden, die für normale Operatoren klassischerweise mit Hilfe des Spektralsatzes angegangen werden würden. Solchen Überlegungen entstammt der Indexsatz von Gohberg-Kreĭn, ein (nicht-triviales) Korollar der obigen kurzen exakten Sequenz.

Für den Fall der oberen Halbebene (anstelle der Einheitskreisscheibe) erhält man ganz ähnliche Resultate. Den Toeplitz-Operatoren auf diesem Gebiet entsprechen die so genannten Wiener-Hopf-Operatoren auf der positiven Halbgeraden. Die Korrespondenz ist hierbei durch die euklidische Fourier-Laplace-Transformation gegeben.

Eine naheliegende Verallgemeinerung der obigen Theorie in einer Veränderlichen besteht darin, statt der Einheitskreisscheibe die euklidische Einheitskugel in $\mathbb{C}^n$ zu betrachten (die 'Hilbertkugel'), oder auch Produktgebiete wie den Polyzylinder. Die erstere Situation (die von Coburn [Cob74] betrachtet und von Raeburn, Janas und Venugopalkrishna für pseudokonvexe Gebiete mit glattem Rand verallgemeinert wurde) liefert allerdings keine neuen Ergebnisse. Dies liegt in der geometrischen Tatsache begründet, dass die euklidischen Einheitskugeln wie im Falle einer Veränderlichen einen glatten Rand besitzen. Anders gesagt haben sie als konvexe Mengen nur einen Typ von Facetten, nämlich die Extremalpunkte. Der Fall von Produktgebieten ist schwieriger, da Tensorprodukte von Fredholmoperatoren in der Regel nicht mehr Fredholm sind. Die in diesem Fall von Douglas-Howe [DH71] erzielten Ergebnisse sind eher vorläufiger Natur (wie die beiden Autoren selbst einräumen).

Eine viel tiefere Theorie ergibt sich, wenn man stattdessen die Klasse der *beschränkten symmetrischen Gebiete* betrachtet. Dies sind konvexe, zirkulare Gebiete, welche eine (nicht-triviale) Darstellung als Einheitskugeln bezüglich einer Art verallgemeinerter 'Operatornorm' besitzen. Ein typisches Beispiel ist die *Matrixkugel*, die aus komplexen $n \times n$-Matrizen u besteht, so dass der Spektralradius von u^*u echt kleiner Eins ist.

Durch Nachahmung der Konstruktion im Fall des Einheitskreises erhält man für allgemeine beschränkte symmetrische Gebiete B wiederum einen Hardyraum $\mathbf{H}^2(B)$. Dieser ist ein abgeschlossener Unterraum von $\mathbf{L}^2$, jedoch diesmal nicht des vollen Randes, sondern nur des *Shilov-Rands* S . Letzterer ist der minimale 'Rand', für den das Maximumprinzip Gültigkeit behält, d.h., Randwerte holomorpher Funktionen bestimmen diese im Inneren vollständig. Sobald der Hardyraum definiert ist, ist es ein leichtes, eine C*-Algebra von Toeplitz-Operatoren zu definieren.

Die Gebiete B haben eine reiche konvexe Geometrie: Sie besitzen r Typen konvexer Facetten, wobei r eine fundamentale Invariante des Gebiets B darstellt, den *Rang*. Der Fall $r = 1$ entspricht dabei den Gebieten mit glattem Rand. Jeder der Facetten B_j vom Typ j ist ein beschränktes symmetrisches Gebiet vom Rang $r - j$. Die Gesamtheit der Facetten gleichen Typs j bildet einen 'partiellen Rand' $\partial_j B$ des Gebiet B . Der Shilovrand entspricht, als Menge aller Extremalpunkte, gerade dem Fall $j = r$.

Da jede Facette B_j selbst wieder ein beschränktes symmetrisches Gebiet mit einem Shilovrand S_j ist, können ein Hardyraum und eine Toeplitz-C*-Algebra für jede dieser Facetten definiert werden. Natürliche Kandidaten für nicht-triviale Darstellungen sind nun dadurch gegeben, dass man dem Toeplitz-Operator vom Symbol f den Toeplitz-Operator vom Symbol $f \mid S_j$ (Einschränkung auf einen der Shilov-Ränder S_j) zuordnet. Es ist höchst nicht-trivial, dass dies eine Darstellung der C*-Algebra $\mathcal{T}$ wohl definiert.

Indes gilt sogar mehr: Jede irreduzible Darstellung von $\mathcal{T}$ ist durch *Einschränkung* auf eine der Facetten B_j gegeben und jede solche Facette *induziert* eine irreduzible Darstellung der vollen Toeplitz-C*-Algebra $\mathcal{T}$.

Indem man die Kerne der zu den Facetten B_j eines festen Typs j assoziierten Darstellungen schneidet, erhält man ein Ideal I_j von $\mathcal{T}$. Diese Ideale stehen wie folgt in Beziehung: Die aufsteigende Kette

$$0 \lhd I_0 = \mathbb{K}(\mathbf{H}^2(\mathbb{B})) \lhd I_1 \lhd \cdots \lhd I_{r-1} \lhd I_r = \left[\mathcal{T}, \mathcal{T}\right] \lhd \mathcal{T} = I_{r+1}$$

bildet eine *Kompositionsreihe* der C*-Algebra $\mathcal{T}$, deren sukzessive Quotienten gerade

$$I_{j+1}/I_j = \mathcal{C}(M_j) \otimes \mathbb{K}(\mathbf{H}^2(B_j))$$

sind, wobei M_j die kompakte Basis eines dem partiellen Rand $\partial_j B$ zugeordneten Faserbündels ist. Insbesondere ist $\mathcal{T}$ vom Typ I, genauer, auflösbar der Länge r im Sinne von Dynin. Das Spektrum kann aus der Kompositionsreihe bestimmt werden. Weiterhin kann eine Indextheorie für Toeplitzoperatoren auf der Basis dieses Satzes entwickelt werden.

Das obige Resultat wurde im Falle eines gewissen Gebiets vom Rang 2 (der *Liekugel*) von Berger, Coburn und Korányi [BC79, BCK80] erzielt, während der allgemeine Fall vollständig auf Upmeier zurückgeht [Upm84, Upm96]. In eine andere Richtung wurde die Theorie einer Veränderlichen von Boutet de Monvel verallgemeinert, der u.a. den Fall streng pseudokonvexer Gebiete und deren Beziehung zu symplektischer Geometrie betrachtet. Für den Fall von Wiener-Hopf-Operatoren wurde der klassischen Rahmen der positiven Halbgeraden von Muhly und Renault [MR82] auf polyedrische Kegel ausgedehnt, wobei Gruppoid-C*-Algebren als Werkzeug dienten. Diese Theorie wurde im Rahmen geordneter homogener Räume von Hilgert und Neeb [HN95] weiter verallgemeinert.

Die Ergebnisse für beschränkte symmetrische Gebiete beruhen wesentlich auf dem Umstand, dass das Gebiet B homogen unter der Wirkung eine halb-einfachen Liegruppe G ist, was zu einer Polarzerlegung $G = K \cdot \exp \mathfrak{p}_{\mathbb{R}}$ relativ der maximalen kompakten Untergruppe K von G führt. Folglich ist die harmonische Analyse auf der Gruppe K, insbesondere die Theorie des höchsten Gewichts, für die Untersuchung der Toeplitz C*-Algebra von entscheidender Bedeutung.

Diese Dissertation leitet die Untersuchung von Toeplitz-Operatoren und den von ihnen erzeugten C*-Algebren in dem allgemeinen Rahmen des wohl bekannten Gel'fand-Gindikin-Programms [GG77] ein. Ausgehend von einer *nicht-kompakten* Liegruppe G vom Hermiteschen Typ, wird die Rolle des symmetrischen Gebiets B nun von dem so genannten *Ol'shanskiĭ-Gebiet* Γ° übernommen, einem nicht-linearen Gebiet vom Tubentyp, das in der komplexifizierten Liegruppe $G^{\mathbb{C}}$ realisiert ist. Dabei stellt sich (neben der Nichtkompaktheit des Shilovrandes) als wesentlicher Unterschied gegenüber dem Fall beschränkter symmetrischer Gebiete heraus, dass das Ol'shanskiĭ-Gebiet nicht mehr homogen ist.

Allerdings lässt sich der Shilovrand von Γ° mit der G zu Grunde liegenden Mannigfaltigkeit (genauer, mit dem zugrundeliegenden $(G \times G)$-Raum) identifizieren. Insbesondere ist der Shilovrand noch homogen und es gibt ein Analogon der Polarzerlegung für Γ . Es gilt nämlich $\Gamma = G \cdot \exp i\Omega^-$, wobei Ω^- ein $\mathrm{Ad}(G)$-invarianter konvexer Kegel in $\mathfrak{g}_{\mathbb{R}}$ ist, der Liealgebra von G . Das Gebiet Γ° ist unter der natürlichen Wirkung von $G \times G$ invariant, wobei letztere Gruppe nun die Rolle der Gruppe K einnimmt.

In diesem Rahmen wurde der Begriff des *Hardyraums* von Ol'shanskiĭ [Ol'82, Ol'91] und, unabhängig, von Stanton [Sta86] geklärt. Analytisch betrachtet besteht der Hardyraum aus holomorphen Funktionen auf dem Inneren Γ° von Γ, welche quadratintegrierbare Randwerte auf dem Shilovrand G besitzen. Dabei spielt in der Definition der Randwerte eine entscheidende Rolle, dass Γ eine *Halbgruppe* bezüglich der von der umgebenden komplexen Gruppe $G^{\mathbb{C}}$ induzierten Verknüpfung ist. Die Nichtlinearität von Γ ist also grundlegend.

Der für die Betrachtung dieser Art von Hardyräumen natürliche Rahmen ist der affiner symmetrischer Räume. Der Shilovrand des betreffenden Gebiets ist dann der Form H/H^σ, wobei H^σ (bis auf Fragen des Zusammenhangs) die Fixgruppe einer Involution σ auf H ist. Die in dieser Arbeit untersuchte Situation, in der der Shilovrand selbst eine Gruppe ist, entspricht dem affinen symmetrischen Raum $(G \times G)/G$, wobei man die Flip-Involution betrachtet. In voller Allgemeinheit wurde der Hardyraum von Hilgert, Ólafsson und Ørsted [HÓØ91] eingeführt und im Detail untersucht.

Neben der analytischen Definition des Hardyraums gibt es auch eine Beschreibung durch die harmonische Analyse des Shilovrandes G. Es ist der Hardyraum nämlich gerade die Summe der Darstellungen, die zur *holomorphen diskreten Reihe* gehören. Dies war, innerhalb des von Gel'fand und Gindikin initiierten Programms, sogar die ursprüngliche Motivation für die Konstruktion des Gebiets Γ° und des zugeordneten Hardyraums. Das Ziel bestand dabei darin, zu jeder der zur Plancherelformel des Raumes $\mathbf{L}^2(G)$ beitragenden 'Reihen' von Darstellungen Räume vom 'Hardy-Typ' zu finden, die aus auf Gebieten, in deren Rand G enthalten ist, definierten analytischen Data (wie Funktionen, Formen oder Schnitten) bestehen. Jüngst haben Bernstein und Reznikov [BR99] in dieser Richtung Fortschritte gemacht; ihre Arbeit ist von Krötz und Stanton [KS04, KS] auf Gruppen höheren Rangs verallgemeinert worden.

Nachdem ein geeigneter Hardyraum für Γ° definiert wurde, ist die Definition einer entsprechenden C^*-Algebra $\mathcal{T}$ von Toeplitz-Operatoren, die stetigen Symbolfunktionen $f \in \mathcal{C}_0(G)$ auf dem Shilovrand zugeordnet sind, ein naheliegendes Unterfangen. Das letztendliche Ziel ist es dann, die für den Fall beschränkter symmetrische Gebiete existente Theorie für $\mathcal{T}$ in diesem Rahmen zu verallgemeinern.

Ein wichtiger Spezialfall ist dabei $G = \mathrm{SU}(1,1)$ (isomorph zu der geläufigeren Gruppe $\mathrm{SL}(2,\mathbb{R})$), was zu den klassischerweise von Gel'fand und Gindikin betrachteten Gebieten führt. Für diesen Fall können wir eine nahezu vollständige Theorie vorlegen und die zu den verschiedenen Facetten gehörigen Darstellungen von $\mathcal{T}$ beschreiben (siehe Abschnitt IV).

Wie zu erwarten ist, stellt sich der Fall von Liegruppen G höheren Rangs als wesentlich schwieriger heraus, obwohl das (vermutete) Hauptresultat immernoch einfach zu formulieren ist: Jede irreduzible Darstellung von $\mathcal{T}$ wird von einer 'Facette' von Γ getragen und umgekehrt entspricht jeder solchen Facette in natürlicher Weise eine Darstellung. Dies könnte man treffenderweise als das *Prinzip von Restriktion und Induktion* für die

C^*-Algebra $\mathcal{T}$ bezeichnen.
Diesem ehrgeizigen Ziel entgegen ist der erste Schritt natürlich die Analyse der Geometrie des zugrundeliegenden Ol'shanskiĭ-Gebiets Γ° . Dieser widmet sich Abschnitt I, wobei das Endergebnis die Bestimmung einer Stratifizierung des Abschlusses Γ von Γ° ist, deren Strata wiederum Ol'shanskiĭ-Gebiete sind. Aufgrund der Existenz einer 'Polarzerlegung' für das Gebiet Γ° kann man dies auf die Beschreibung des Facettenverbands des Kegels Ω^- zurückführen, welche der Inhalt des ersten Teils dieser Arbeit ist.
Der nächste Schritt in unserem Programm ist es, die harmonische Analyse des Γ zugeordneten Hardyraums zu verstehen, sowie die der Gruppe G . Dieses Ziel greifen wir in Abschnitt II an, in dem unser Blickwinkel stets wesentlich lokal und mikrolokal ist: Wir studieren die mikrolokale Struktur der Szegö-Projektion, d.h., des orthogonalen Projektors auf den Hardyraum, aufgefasst als abgeschlossener Unterraum von $\mathbf{L}^2(G)$. Diese Projektion ist durch die Faltung mit einer invarianten Distribution gegeben, deren Wellenfront und singulären Träger wir beschreiben. Unser Theorem — das zweite Hauptresultat dieser Arbeit — besagt, dass die Faser der Wellenfront dieser 'Szegö-Distribution' im Dual des Kegels Ω^- liegt. Überdies beschreiben wir präzize die Lage ihrer Singularitäten, auf den Konjugationsklassen eines maximalen Torus.
Weiterhin ist es von entscheidender Bedeutung, das asymptotische Verhalten der Matrixkoeffizienten (wenn der Darstellungsparameter gegen Unendlich strebt) derjenigen irreduziblen unitären Darstellungen der Gruppe G zu verstehen, welche zur Plancherelformel von $\mathbf{L}^2(G)$ beitragen. Für die holomorphe diskrete Reihe erzielen wir genaue Abschätzungen. Desweiteren setzen wir die Plancherelformel für die Gruppe G und für die den 'Facetten' des Ol'shanskiĭ-Gebiets Γ° zugeordneten Untergruppen in Beziehung, indem wir Einbettungen der einen in die andere explizit konstruieren. Dies ist das dritte wesentliche Ergebnis dieser Dissertation.
Nachdem nun die Probleme der harmonischen Analyse und mikrolokalen Analysis abschließend behandelt wurden, widmen wir uns in Abschnitt III der Untersuchung operatortheoretischer Fragestellungen.
Im klassischen Fall des Einheitskreises war die gewöhnliche Fouriertransformation bedeutsam für das Studium der C^*-Algebra aller Toeplitz-Operatoren. Allgemeiner spielt im Falle kompakter Gruppen (beschränkte symmetrische Gebiete) die abstrakte Fouriertransformation eine Schlüsselrolle, erlaubt sie doch die Anwendung der harmonischen Analyse der Gruppe K auf die Toeplitz-C^*-Algebra. Für den Spezialfall der unitären Gruppen $\mathrm{U}(n)$ wurden die notwendigen operatortheoretischen Techniken erstmals von Wassermann [Was84] eingesetzt. Für allgemeine kompakte Gruppen wurden sie von Upmeier [Upm91, Upm96] rigoros und in voller Allgemeinheit entwickelt. Die Idee besteht hier darin, $\mathcal{T}$ als 'Ecke' eines *Cokreuzprodukts* von C^*-Algebren darzustellen, ein Objekt, das im Rahmen von Takesakis Theorie der nicht-kommutativen Dualität lokalkompakter Gruppen definiert wird, mit Hilfe der fortgeschrittenen Methode der Hopf-C^*-Algebren und ihrer Coaktionen.

Um dies auf die Toeplitz C^*-Algebra $\mathcal{T}$ im in dieser Arbeit gesteckten Rahmen anwenden zu können, bedarf es der Verallgemeinerung der entsprechenden Resultate auf nicht-kompakte Gruppen. Wir entwickeln die dazu gehörige Theorie sogar für beliebige lokal-kompakte, unimodulare Gruppen und ohne auf den Raum $\mathbf{L}^2(G)$ Bezug zu nehmen. Diese 'raumfreie' Betrachtungsweise sollte es letzten Endes erlauben, auch den Fall affiner symmetrischer Räume zu behandeln. Der Rahmen ist auch weit genug um Hardyräume zu betrachten, die durch andere Reihen von Darstellungen als die holomorphe diskrete Reihe definiert sind.
Desweiteren zeigen wir, wie die Konstruktion irreduzibler Darstellungen der Toeplitz-C^*-Algebra $\mathcal{T}$ vollständig mit Hilfe lokaler, mikrolokaler und asymptotischer Information über die Szegö-Projektion vollzogen werden kann. Die entsprechenden Informationen, die wir für den in dieser Arbeit betrachteten speziellen geometrisch definierten Rahmen erzielt haben, können daher dem Endziel entgegen, das Prinzip von Restriktion und Induktion in voller Allgemeinheit zu beweisen, benutzt werden.
Um dies weiter zu erhärten, stellen wir in Abschnitt IV, einer detaillierten Diskussion des Rang 1-Falles folgend, eine Strategie für den allgemeinen Fall vor, in der wir die zum Erreichen dieses Zieles nötigen Schritte angeben, sowie die Methoden, die die benötigten Ergebnisse am besten liefern werden.

Es gibt etliche Richtungen, in die man dieses Programm weiter verfolgen könnte. Am beachtenswertesten ist vielleicht die Untersuchung von Hardyräumen, die anderen Reihen von Darstellungen zugeordnet sind, und daher zu nicht mehr konvexen Kegeln gehören. Für kompakte Gruppen (die euklidischen Jordanalgebren zugeordnet sind) ist dieses Problem von Hagenbach [Hag99] und Hagenbach-Upmeier [HU98] behandelt worden. Für nicht-kompakte Gruppen ist es noch weithin offen.

Introduction

In the theory of Toeplitz operators, one considers, classically, the unit disc $\mathbb{B}$ in the complex plane $\mathbb{C}$. Toeplitz operators are bounded operators on the Hardy space $\mathbf{H}^2(\mathbb{B})$, which consists of all $\mathbf{L}^2$ boundary values on the circle $\mathbb{T}$ of holomorphic functions on $\mathbb{B}$. They are defined naturally in terms of the *Szegö projection*, the orthogonal projection onto $\mathbf{H}^2(\mathbb{B})$.

A principal of object of study is the C*-algebra $\mathcal{T}$, generated by all Toeplitz operators with continuous symbols. Generically, operators in this C*-algebra are non-normal, and hence, the Spectral Theorem is not applicable. To understand their Spectral Theory, more sophisticated methods have to be applied.

The fundamental theorem on the Toeplitz C*-algebra $\mathcal{T}$ is due to Gohberg-Kreĭn [GK58]: There is a short exact sequence

$$0 \to \mathbb{K}(\mathbf{H}^2(\mathbb{B})) \to \mathcal{T} \to \mathcal{C}(\mathbb{T}) \to 0$$

which, in fact, splits at $\mathcal{T}$, the symbol map giving a partial inverse to the quotient map onto $\mathcal{C}(\mathbb{T})$. In particular, the commutator ideal equals the ideal of compact operators, and the spectrum of $\mathcal{T}$ is completely determined, as the *non-commutative space* $\mathbb{T} \cup \mathrm{pt}$.

More importantly, this short exact sequence makes high-powered Operator Theoretic techniques — such as the K-theory of Operator Algebras — applicable to problems such as Fredholmness criteria, which, for normal operators, would traditionally be attacked by applying the Spectral Theorem. Along this road lies the Index Theorem of Gohberg-Kreĭn, essentially an corollary (albeit non-trivial) of the above short exact sequence.

Similar results are, of course, also valid for the case of the upper half plane, instead of the unit disc. To Toeplitz operators on this domain, there correspond the so-called Wiener-Hopf operators of the real half line, the correspondence being given by the Euclidean Fourier-Laplace transform.

An obvious extension of the one-variable theory is the treatment, in place of the unit disc, of the Euclidean unit ball in $\mathbb{C}^n$ (the 'Hilbert ball'), or of product domains such as the polydisc. The former situation (considered by Coburn [Cob74], and generalised to strongly pseudo-convex domains with smooth boundary by Raeburn, Janas, and Venugopalkrishna) however leads to no new results, the geometric reason being that these domains still have smooth boundaries. Put differently, they only have one type of convex face, namely, the extremal points, just as for the case of the unit disc. The product case is more difficult, since tensor products of Fredholm operators are not necessarily Fredholm, and the results by Douglas-Howe [DH71] are of a preliminary nature (as they themselves state).

A much deeper theory ensues if one considers, instead, the class of *bounded symmetric* *domains*. These are convex circled domains which can be shown to have a (non-trivial) representation as the unit ball with respect to a generalised type of 'operator norm'. A typical example would be the *matrix ball*, consisting of complex $n \times n$ matrices u such that the spectral radius of u^*u is strictly smaller than one.

In close analogy to the unit disc case, general bounded symmetric domains B allow for the definition of a Hardy space $\mathbf{H}^2(B)$, a subspace of $\mathbf{L}^2$, however not of the entire boundary, but only of the so-called *Shilov boundary* S. The latter is the minimal 'boundary' on which the maximum principle remains valid, i.e. boundary values determine holomorphic functions on the interior. In the presence of the Hardy space $\mathbf{H}^2(B)$, a C*-algebra of Toeplitz operators is straightforward to define.

The domains B have a rich convex geometry. Namely, they have r types of non-trivial convex faces, where r is a fundamental invariant of B, its *rank*, the case $r = 1$ corresponding to the domains with smooth boundary. Each of the faces B_j of type j is a bounded symmetric domain, of rank $r - j$. The totality of all faces of a fixed type j determines a 'partial boundary' $\partial_j B$ of the domain B, the Shilov boundary corresponding to $j = r$, the set of all extreme points.

Since each face B_j is again a bounded symmetric domain with a Shilov boundary S_j, a Hardy space and a Toeplitz C*-algebra can be defined for each of these faces. Natural candidates for non-trivial representations of the Toeplitz C*-algebra $\mathcal{T}$ are then given by associating to the Toeplitz operator with symbol f, the operator with symbol $f \mid S_j$ (restriction to one of the Shilov boundaries S_j). That this actually well-defines a representation of the C*-algebra $\mathcal{T}$, is a highly non-trivial matter.

However, even more is true: Every irreducible representation of $\mathcal{T}$ is given by *restriction* to one the faces B_j, and every such face *induces* an irreducible representation of the full Toeplitz C*-algebra $\mathcal{T}$.

By intersecting the kernels of all the representations associated to faces of the same type j, one obtains ideals I_j of $\mathcal{T}$. They are related as follows: The ascending chain

$$0 \triangleleft I_0 = \mathbb{K}(\mathbf{H}^2(\mathbb{B})) \triangleleft I_1 \triangleleft \cdots \triangleleft I_{r-1} \triangleleft I_r = \big[\mathcal{T},\mathcal{T}\big] \triangleleft \mathcal{T} = I_{r+1}$$

is a *composition series* of the C*-algebra $\mathcal{T}$, whose successive quotients are

$$I_{j+1}/I_j = \mathcal{C}(M_j) \otimes \mathbb{K}(\mathbf{H}^2(B_j)) ,$$

where M_j is the compact base of a fibre bundle associated to the partial boundary $\partial_j B$. In particular, $\mathcal{T}$ is of type I, more precisely, solvable of length r in the sense of Dynin, and its spectrum can be determined from the composition series. Moreover, an Index Theory for Toeplitz operators can be developed with the help of this theorem.

The above result was established for the case of a certain rank 2 domain (the *Lie ball*) by Berger, Coburn, and Korányi [BC79, BCK80], and the general case was settled completely by Upmeier [Upm84, Upm96]. A different road to generalisation was pursued by Boutet de Monvel, who treats e.g. the case of strictly pseudo-convex domains, and their connection to symplectic geometry. For Wiener-Hopf operators, the classical real half-line setup was generalised to polyhedral cones by Muhly and Renault [MR82], using the technique of groupoid C*-algebras. This theory has been further extended, to ordered homogeneous spaces, by Hilgert and Neeb [HN95].

The results for bounded symmetric domains rely heavily on the fact that the domain B is homogeneous under the action of a semi-simple Lie group G, which leads to a polar decomposition $G = K \cdot \exp \mathfrak{p}_{\mathbb{R}}$ with respect to its maximal compact subgroup K. Therefore, the harmonic analysis of the compact group K, in particular, the Theory of the Highest Weight, is crucial for the analysis of the Toeplitz C*-algebra.

In this thesis, we initiate the study of Toeplitz operators, and the C*-algebra generated by them, within the general framework of the well-known Gel'fand-Gindikin programme [GG77]. Starting with a *non-compact* Lie group G of Hermitian type, the role of the symmetric domain B is now played by the so-called *Ol'shanskiĭ domain* Γ°, a non-linear tube type domain realised in the complexified Lie group $G^{\mathbb{C}}$. The essential new feature of this domain (besides the non-compactness of its Shilov boundary) is that it is no longer homogeneous.

However, its Shilov boundary can be identified with the underlying manifold (more precisely, the underlying $(G \times G)$-space) of the group G. In particular, the Shilov boundary is still homogeneous, and there is an analogue of the polar decomposition for Γ. Namely, $\Gamma = G \cdot \exp i\Omega^-$ where Ω^- is an $\mathrm{Ad}(G)$-invariant convex cone in $\mathfrak{g}_{\mathbb{R}}$, the Lie algebra of G. The domain Γ° is invariant under the natural action of $G \times G$, which now plays the role of the group K.

In this setting, the appropriate notion of *Hardy space* has been clarified, independently, by Ol'shanskiĭ [Ol'82, Ol'91] and Stanton [Sta86]. Analytically, the Hardy space consists of holomorphic functions on the interior Γ° of Γ, which have $\mathbf{L}^2$ boundary values on the Shilov boundary G. In the definition of these boundary values, an important role is played by the fact that Γ is a *semigroup* for the composition induced by the ambient complex group $G^{\mathbb{C}}$. Hence, the non-linear nature of Γ is fundamental.

The natural framework in which to treat this type of Hardy spaces is in fact that of Affine Symmetric Spaces. Here, the Shilov boundary of the respective domain is of the form H/H^σ where H^σ is (up to connectivity issues) the group of fixed points for some involution σ defined on the group H. Our situation, where the Shilov boundary is itself a group, corresponds to $(G \times G)/G$, for the flip involution. In the general case, the Hardy space was introduced and thoroughly analysed by Hilgert, Ólafsson, and Ørsted [HÓØ91].

Besides the analytic definition of the Hardy space, it can be given a description in terms of the harmonic analysis of the Shilov boundary G. Namely, it is the sum of all *holomorphic discrete series representations* of the group G. In fact, this was the original motivation for the construction of the domain Γ° and the associated Hardy space, within the programme initiated by Gel'fand and Gindikin. Here, the aim was to find 'Hardy type' spaces of analytic data (functions, forms, or sections) on domains containing G in their boundary, for each of the 'series' of representations belonging to the Plancherel decomposition of $\mathbf{L}^2(G)$. Recently, progress in this direction has been made by Bernstein and Reznikov [BR99], work which has been extended higher rank by Krötz and Stanton [KS04, KS].

Having defined an appropriate Hardy space, it is straightforward to define a corresponding C*-algebra $\mathcal{T}$ of Toeplitz operators, associated to continuous symbol functions $f \in \mathcal{C}_0(G)$ defined on the Shilov boundary G . The ultimate goal is then to generalise to this setup the theory for the C*-algebra $\mathcal{T}$, in particular, the construction of a composition series.

An important special case is $G = \mathrm{SU}(1,1)$ (isomorphic to the more familiar $\mathrm{SL}(2,\mathbb{R})$), which leads to the domains classically considered by Gel'fand and Gindikin. In this case, we present a more or less complete theory, describing (all) representations of $\mathcal{T}$ corresponding to the different 'faces' of the domain Γ° (see part IV).

As is to be expected, the case of semi-simple Lie groups G of higher rank turns out to be much more difficult, although the principal (conjectural) result is still easy to formulate: Every irreducible representation of $\mathcal{T}$ is supported by a unique 'face' of Γ , and conversely, every such face gives rise to such a representation in a natural way. This could be called the *Principle of Restriction-Induction* for the C*-algebra $\mathcal{T}$.

Towards this ambitious goal, the first step is of course the analysis of the geometry of the underlying Ol'shanskiĭ domain Γ° . This is the content of part I, and the final result is the determination of a stratification of the closure Γ of the domain Γ° into strata which are again Ol'shanskiĭ domains. Due to the existence of a 'polar decomposition' for the domain Γ° , this reduces to thedescription of the face lattice of the cone Ω^- , our first main endeavour in this thesis.

The next step in our programme is to understand the harmonic analysis of the Hardy space associated to Γ , and of the group G . We tackle this objective in part II, where our perspective is, throughout, local and micro-local in a crucial way. Namely, we study the micro-local structure of the Szegö projection, the orthogonal projection onto Hardy space, realised as a closed subspace of $\mathbf{L}^2(G)$. This projection is given by convolution with an invariant distribution, whose wave front and singular support we are able to describe. Our theorem — the second main result of this work — states that the fibre of the wave front set of this 'Szegö distribution' is contained in the dual of the cone Ω^- . Moreover, we give precise information of the location of singularities, on the conjugacy classes of a maximal torus.

Furthermore, it is decisive to understand the asymptotic behaviour of matrix coefficients of the irreducible unitary representations of the group G which contribute to the Plancherel formula of $\mathbf{L}^2(G)$, as the representation parameter tends to infinity. For the holomorphic discrete series, we obtain precise bounds. In addition, we relate the Plancherel formulae for the group G and the subgroups associated to the 'faces' of the Ol'shanskiĭ domain Γ° , by explicitly constructing embeddings of one into the other. This is the third essential result of this thesis.

The harmonic analysis and micro-local analysis issues having been settled, we begin, in part III, the investigation of Operator Theoretic questions.

For the classical case of the unit disc, the classical Fourier transform plays an important role in the study of the C^*-algebra of Toeplitz operators. More generally, in the case of compact groups (bounded symmetric domains), the abstract Fourier transform is a key tool, allowing for the application of the harmonic analysis on K to the Toeplitz C^*-algebra. For the case of the unitary groups $U(n)$, the necessary Operator Theoretic machinery was first employed by Wassermann [Was84]. For general compact Lie groups, it was rigorously developed, in full generality, by Upmeier [Upm91, Upm96]. The idea is to represent $\mathcal{T}$ as the 'corner' of a C^*-algebra *co-crossed product*, an object defined in the framework of Takesaki's theory of non-commutative duality of locally compact groups, using the advanced technique of Hopf C^*-algebras and their coactions.

To treat the C^*-algebra $\mathcal{T}$ in our present setting, it is necessary to generalise this result to non-compact groups. In fact, we develop the theory, in part III, for arbitrary locally compact unimodular groups, and without reference to the space $\mathbf{L}^2(G)$, our fourth main result. The 'space-free' viewpoint should eventually allow for the treatment of Hardy spaces defined on Affine Symmetric Spaces, and is also sufficiently general to accommodate Hardy spaces defined in terms of other series of representations, rather than the holomorphic discrete series.

Moreover, we show how the construction and irreducibility of representations of $\mathcal{T}$ can be completed solely in terms of local, micro-local, and asymptotic information on the Szegö projection. The corresponding information we have gathered in our particular geometric setup can thus be applied towards the ultimate goal of establishing the Principle of Restriction-Induction in full generality.

To make this more substantial, we present in part IV, after a detailed discussion of the rank one case, a strategy for the general case, indicating the precise steps that should be taken to achieve this goal, and presenting the methods which are most likely to produce the required results.

There are various directions in which this programme could be generalised, most notably, the treatment of Hardy spaces associated to other series of representations, and hence, to non-convex cones. For compact groups (associated to Euclidean Jordan algebras), this problem has been treated by Hagenbach [Hag99] and Hagenbach-Upmeier [HU98]. For non-compact groups, it is, as yet, wide open.

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Geometry of Ol'shanskiĭ domains

IN THIS PART, we introduce two types of geometric objects fundamental to our work and study their structure in detail. Namely, the objects we consider are certain, intimately related, convex cones and complex domains.

The investigation proceeds on two levels, the first being the study of *symmetric* domains and cones. On the second level, we examine geometric objects associated to automorphisms of these. More precisely, the matter of interest is the (minimal) invariant convex cone lying in the Lie algebra of all complete holomorphic vector fields on the underlying symmetric domain B, and the bi-invariant so-called Ol'shanskiĭ domain it defines in the complexification of the automorphism group of B.

Throughout, our emphasis is not on the interior of these geometric objects, which is well-known, but on their boundary. Of course, the boundaries of symmetric domains and symmetric cones have already been studied conclusively, and our account in 1.1–1.3 of this theory is largely expository, seeing that there are already several well-established and generally accessible monographs developing its principal results. Nevertheless, we have found this reconsideration of known material useful for the purpose of reference and fixing our notation.

In 2.1, we introduce the minimal and maximal cone in the Lie algebra of infinitesimal automorphisms of B, and present the basic theory of these cones. The content is mostly known and even classical, nonetheless, the presentation in terms of Jordan theory appears to be new. This elementary approach allows for a concise, complete and largely self-contained development of both the fundamental results on these cones and the classification of nilpotent orbits of convex type.

Subsection 2.2 contains our main result in this part, the description of the faces of the minimal cone. The most important realisation is that they are not classified by the faces of the associated polyhedral cone in a compact Cartan subalgebra. In particular, the length of the face lattice exceeds the rank of B. Moreover, the Lie algebras generated (as vector spaces) by the faces are not all simple Hermitian, but of a more general type.

We describe the faces in five alternative forms: as intersection with a supporting hyperplane, with parabolic subalgebras, or a generalised Jacobi algebra, as a union of adjoint orbits, and by projection onto a compact Cartan subalgebra. Of course, in themselves, these faces are known as cones, and form part of the classification of Lie algebras with (pointed, elliptic, or weakly elliptic) generating invariant convex cones, as settled in general by Hilgert, Hofmann and Neeb [HH88, Nee94]. Their appearance as faces of an invariant cone in a simple Hermitian Lie algebra, however, is apparently new. Other to the author's knowledge novel features include the description the of conjugacy classes of these faces, and their relation to nilpotent orbits of convex type.

Finally, in 2.3, we discuss the Ol'shanskiĭ domains associated to the minimal cone and its faces. We obtain a stratification of the closure of the minimal Ol'shanskiĭ domain into fibre bundles whose fibres are lower-dimensional Ol'shanskiĭ domains in generalised Jacobi groups.

1 Symmetric domains and cones, and their boundaries

1.1 Bounded symmetric domains and Jordan triple systems

1.1.1. Let Z be an n-dimensional complex vector space, and $B \subset Z$ a bounded domain. The *holomorphic vector fields* are of the form

$$h(z)\,\frac{\partial}{\partial z} \quad \text{for some } h \in \mathcal{O}(B,Z)$$

where

$$\Big(h(z)\,\frac{\partial}{\partial z}\Big) f(z) = f'(z)h(z) \quad \text{for all } f \in \mathcal{O}(B,Z)\,.$$

The set of all holomorphic vector fields forms a complex Lie algebra via

$$\Big[h(z)\,\frac{\partial}{\partial z}, k(z)\,\frac{\partial}{\partial z}\Big] = \big(h'(z)k(z) - k'(z)h(z)\big)\,\frac{\partial}{\partial z}\,.$$

A basic result due to H. Cartan states that since B is bounded, the set $\mathrm{Aut}(B)$ of all holomorphic automorphisms of B, endowed with the compact-open topology, is a (finite-dimensional) real Lie group. Moreover, any $g \in \mathrm{Aut}(B)$ is uniquely determined by $g(z)$ and $g'(z)$, if $z \in B$ is arbitrary, but fixed. Hence, the action of the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ of $\mathrm{Aut}(B)$ by holomorphic vector fields is faithful.

So, $\mathfrak{g}_{\mathbb{R}}$ identifies with a subalgebra of all holomorphic vector fields. In fact, it coincides with the set $\mathrm{aut}(B)$ of all *complete* holomorphic vector fields, for which the local flow is defined for all times. Moreover, Liouville's theorem shows that $\mathfrak{g}_{\mathbb{R}} \cap i\mathfrak{g}_{\mathbb{R}} = 0$ (since the vector fields in the intersection define a bounded, entire complex flow). So $\mathfrak{g}_{\mathbb{R}}$, in contrast to the set of all holomorphic vector fields, is totally real.

Under the identification of $\mathfrak{g}_{\mathbb{R}}$ with the set of complete holomorphic vector fields, the adjoint action of $\mathrm{Aut}(B)$ corresponds to the pullback of vector fields,

$$\mathrm{Ad}\big(g^{-1}\big)(\xi)(z) = g'(z)^{-1}\xi\big(g(z)\big) \quad \text{for all } \xi \in \mathfrak{g}_{\mathbb{R}}\,, g \in \mathrm{Aut}(B)\,,\ z \in B\,.$$

1.1.2. The domain B is called *symmetric*, if for all $z \in B$, there is an involutive automorphism $s_z \in \mathrm{Aut}(B)$, such that z is an isolated fixed point of s_z. This amounts to

$$s_z(z) = z \quad \text{and} \quad s_z'(z) = -1 \in \mathrm{End}(Z) .$$

By Cartan's theorem, this determines s_z uniquely.

If B is symmetric, then it is biholomorphically equivalent to a circular domain [Loo75, th. 1.6]. Here, *circular* means that the domain contains 0 and is $\mathrm{U}(1)$-invariant. We assume w.l.o.g. that B be circular.

Let $\mathfrak{k}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ be the subalgebra of all complete linear vector fields. The circularity of B implies that

$$iz\frac{\partial}{\partial z} \in \mathfrak{z}\big(\mathfrak{k}_{\mathbb{R}}\big) \subset \mathfrak{k}_{\mathbb{R}} ,$$

so $\mathfrak{k}_{\mathbb{R}}$ has non-trivial centre. This will be of fundamental importance in all that follows.

1.1.3. Let $G = \mathrm{Aut}_0(B)$ be the connected component of $\mathrm{id}_B \in \mathrm{Aut}(B)$, and let $K \sqsubset G$ be the fixed group of $0 \in B$. (The full automorphism group may have up to two connected components if B is irreducible, cf. [Tak64].)

G is transitive on B, and hence $B = G/K$. By [Upm82, lem. 1.7], the elements of the subgroup K are linear, in particular its Lie algebra is $\mathfrak{k}_{\mathbb{R}}$. The same lemma implies that G has trivial centre. Moreover, the Bergman metric h_z on B (cf. [Loo75, 1.2]) is Kähler, so K is a closed subgroup of the unitary group $\mathrm{U}(Z)$ (with respect to h_0) and hence compact.

Further, define $\vartheta = \mathrm{Ad}(-\mathrm{id}_B)$. Then ϑ is an involutive automorphism of G, and K is its fixed group. On the Lie algebra level, we have the decomposition into eigenspaces of ϑ with respect to the eigenvalues ± 1,

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}} , \quad \text{where} \quad \big[\mathfrak{k}_{\mathbb{R}}, \mathfrak{k}_{\mathbb{R}}\big] \subset \mathfrak{k}_{\mathbb{R}} , \ \big[\mathfrak{k}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}\big] \subset \mathfrak{p}_{\mathbb{R}} , \ \big[\mathfrak{p}_{\mathbb{R}}, \mathfrak{p}_{\mathbb{R}}\big] \subset \mathfrak{k}_{\mathbb{R}} .$$

By Proposition 2.1.11 below, G is semi-simple and ϑ is a Cartan involution. Therefore, K is maximally compact, by [Hel78, ch. III, § 7, prop. 7.4].

1.1.4. Since $B = G/K$, the map

$$\mathfrak{p}_{\mathbb{R}} \to Z \otimes \mathbb{R} : \xi \mapsto \xi(0)$$

is an $\mathbb{R}$-linear isomorphism.

For $z \in Z$, let $\xi_z^- \in \mathfrak{p}_{\mathbb{R}}$ be uniquely determined by $\xi_z^-(0) = z$. Then Q_z, defined by $Q_z(w) = z - \xi_z^-(w)$, is a homogeneous quadratic polynomial in z and conjugate linear in w by [Loo75, lem. 2.3]. This allows us to polarise

$$2Q_{u,w} = Q_{u+w} - Q_u - Q_w \quad \text{for all } \ u, w \in Z .$$

Then $\{uv^*w\} = Q_{u,w}(v)$ is linear in u and w, and conjugate linear in v. Moreover, if we define the *box operator* $u \,\square\, v^* \in \mathrm{End}(Z)$ by

$$\big(u \,\square\, v^*\big)(w) = \{uv^*w\} \quad \text{for all } \ u, v, w \in Z \,,$$

then

$$\{uv^*w\} = \{wv^*u\} \,, \tag{JTS1}$$
$$\big[u \,\square\, v^*, z \,\square\, w^*\big] = \{uv^*z\} \,\square\, w^* - z \,\square\, \{wu^*v\}^* \,, \tag{JTS2}$$

by [Loo75, lem. 2.6]. We have $Q_z(w) = \{zw^*z\}$, and an equivalent expression for the rules (JTS1-2) is given by the equations

$$\begin{aligned} \big(u \,\square\, v^*\big) Q_u &= Q_u \big(v \,\square\, u^*\big) \,, \\ \big(Q_u v\big) \,\square\, v^* &= u \,\square\, \big(Q_v u\big)^* \,, \end{aligned}$$

by [Upm85, prop. 18.8].

Any complex vector space Z, endowed with a triple product

$$\{\sqcup\sqcup^*\sqcup\} : Z \times Z \times Z \to Z : (u, v, w) \mapsto \{uv^*w\}$$

linear in u and w, and conjugate linear in v, such that the identities (JTS1-2) are satisfied, is called a *Jordan triple system* (JTS).

It is clear that the Lie algebraic properties of $\mathfrak{g}_{\mathbb{R}}$ can be analysed in terms of the Jordan triple Z. Note that

$$\mathfrak{p}_{\mathbb{R}} = \big\{\xi_u^- \ \big| \ u \in Z\big\} \quad \text{where} \quad \xi_u^- = \big(u - \{zu^*z\}\big) \frac{\partial}{\partial z} \,.$$

The fundamental identities for these vector fields are

$$\begin{aligned} \big[\xi_u^-, \xi_v^-\big] &= 2 \cdot \big(u \,\square\, v^* - v \,\square\, u^*\big) \,, \\ \big[\big[\xi_u^-, \xi_v^-\big], \xi_w^-\big] &= 2 \cdot \xi^-_{\{uv^*w\} - \{vu^*w\}} \,, \end{aligned}$$

for all $u, v, w \in Z$, cf. [Loo75, lem. 2.6.].

1.1.5. Returning to our setting with $B \subset Z$ a circular bounded symmetric domain, the Bergman metric h_z at $z = 0$ defines a positive Hermitian inner product on the holomorphic tangent space $Z = T_0(B)$. With respect to h_0,

$$\big(u \,\square\, v^*\big)^* = v \,\square\, u^* \quad \text{for all } u, v \in Z \,,$$

by [Loo75, lem. 2.6]. Moreover,

$$h_0(u, v) = \mathrm{tr}_Z\big(u \,\square\, v^*\big) \quad \text{for all } u, v \in Z \,,$$

by [Loo75, th. 2.10], so the trace form

$$Z \times Z \to \mathbb{C} : (u,v) \mapsto \mathrm{tr}_Z(u \square v^*)$$

is positive Hermitian.

A JTS such that the trace form is positive Hermitian is called *positive Hermitian* or a *JB*-triple*, cf. [Upm85].

Define the *spectral norm* on the JB*-triple Z by

$$\|z\| = \|z \square z^*\|_{op}^{1/2} = \sup_{\mathrm{tr}_Z(w \square w^*) \leqslant 1} \sqrt[4]{\mathrm{tr}_Z\left(\{zz^*w\} \square w^*\right)} .$$

The fundamental theorem [Loo75, th. 4.1] on JB*-triples states that the unit ball of Z, $B = \{z \mid \|z\| < 1\}$, is a circular bounded symmetric domain such that Z is the associated JB*-triple. Conversely, the bounded circular domain B is the unit ball of its associated JB*-triple.

In particular, any circular bounded symmetric domain is convex. By the long exact sequence in homotopy for $K \to G \to B$ where B is simply connected, K is connected.

1.1.6. Any $k \in \mathrm{GL}(Z)$ such that

$$k\{uv^*w\} = \{(ku)(kv)^*(kw)\} \quad \text{for all } u,v,w \in Z ,$$

or, equivalently,

$$k\big(u \square v^*\big)k^{-1} = (ku) \square (kv)^* \quad \text{for all } u,v \in Z ,$$

is called a *triple automorphism*. The set $\mathrm{Aut}(Z) \subset \mathrm{Aut}(B)$ of all triple automorphisms is the fixed group of 0, so $K = \mathrm{Aut}_0(Z)$ is its connected component, cf. [Loo75, cor. 4.9].

The Lie algebra $\mathfrak{k}_{\mathbb{R}}$ of $\mathrm{Aut}(Z)$ is the set $\mathrm{aut}(Z)$ of all *triple derivations* $\delta \in \mathrm{End}(Z)$, i.e.

$$\delta\{uv^*w\} = \{(\delta u)v^*w\} + \{u(\delta v)^*w\} + \{uv^*(\delta w)\} \quad \text{for all } u,v,w \in Z ,$$

or, equivalently,

$$\big[\delta, u \square v^*\big] = (\delta u) \square v^* + u \square (\delta v)^* \quad \text{for all } u,v \in Z .$$

Since Z is finite-dimensional, all triple derivations are inner, i.e.

$$\mathfrak{k}_{\mathbb{R}} = \mathrm{aut}(Z) = \langle u \square v^* - v \square u^* \mid u,v \in Z \rangle .$$

1.1.7. A non-zero JB*-triple Z is said to be *simple* if it has only trivial ideals. Here, an ideal is a triple $W \subset Z$ such that $\{WZ^*Z\} + \{ZW^*Z\} \subset W$. The triple Z associated to B is simple if and only if B is irreducible, i.e. not the product of manifolds of positive dimension, cf. [Loo75, 4.11]. Equivalently, G is simple, cf. Proposition 2.1.11 below.

The classification of irreducible bounded symmetric domains, substantially due to E. Cartan [Car35], can be stated (and proved) in terms of the classification of simple JB*-triples — although this was not Cartan's approach. We summarise it as follows

Classification of irreducible bounded symmetric domains

Cartan	Helgason	$B = G/K$	Z
$\mathrm{I}_{p,q}$	A III	$\frac{\mathrm{SU}(p,q)}{\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))}$	$\mathbb{C}^{p\times q}$
II_n	D III	$\frac{\mathrm{SO}^*(2n)}{\mathrm{U}(n)}$	$\mathbb{C}^{n\times n}_-$
III_n	C I	$\frac{\mathrm{Sp}(2n,\mathbb{R})}{\mathrm{U}(n)}$	$\mathbb{C}^{n\times n}_+$
IV_n	BD I	$\frac{\mathrm{SO}(2,q)}{\mathrm{SO}(2)\times\mathrm{SO}(q)}$	$V_{\lfloor q\rfloor+2}$
V	E III	$\frac{E_{6(-14)}}{\mathrm{SO}(2)\times\mathrm{SO}(10)}$	$\mathbb{O}_{\mathbb{C}}^{1\times 2}$
VI	E VII	$\frac{E_{7(-25)}}{\mathrm{SO}(2)\times E_6}$	$\mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$

There are four infinite series of irreducible bounded symmetric domains, usually termed *classical*, and two types which do not not belong to infinite series, termed *exceptional*. The rank (= dimension of a maximally flat totally geodesic submanifold) of the domains of type I-III is arbitrarily large, whereas the domains of type IV always have rank 2. We review the domains and their associated JB*-triples, as given in [Loo75, 4.14, 4.17].

type $\mathbf{I}_{p,q}$ The triple $Z = \mathbb{C}^{p\times q}$ consists of all complex $p\times q$ matrices. The triple product is given by

$$\{uv^*w\} = \tfrac{1}{2}\big(uv^*w + wv^*u\big) \quad \text{for all } u,v,w \in \mathbb{C}^{p\times q}\,.$$

The domain B is the *matrix ball*

$$B = \big\{z \in \mathbb{C}^{p\times q} \;\big|\; 1 - zz^* \gg 0\big\}\,.$$

type $\mathbf{II}_n$ The triple $Z = \mathbb{C}^{n\times n}_-$ consists of all skew-symmetric $n\times n$ complex matrices, with the same triple product as for type $I_{n,n}$. Then B is

$$B = \big\{z \in \mathbb{C}^{n\times n}_- \;\big|\; 1 + z\bar{z} \gg 0\big\}\,.$$

type $\mathbf{III}_n$ In this case, $Z = \mathbb{C}^{n\times n}_+$ consists of all symmetric complex matrices, with the product induced by type $I_{n,n}$. The domain B is the *Siegel ball*, given by

$$B = \big\{z \in \mathbb{C}^{n\times n}_+ \;\big|\; 1 - z\bar{z} \gg 0\big\}\,.$$

type $\mathbf{IV}_n$ Here, $Z = V_n = \mathbb{C}^n$, $n \neq 2$, consists of column vectors. The triple product is given by

$$2 \cdot \{uv^*w\} = u \cdot \bar{v}^t w - \bar{v} \cdot w^t u + w \cdot u^t \bar{v} \quad \text{for all } u, v, w \in \mathbb{C}^n .$$

For $n = 1, 3, 4$, Z is isomorphic to $\mathbb{C} = \mathbb{C}^{1\times 1}$, $\mathbb{C}_+^{2\times 2}$, and $\mathbb{C}^{2\times 2}$, respectively. For $n \geqslant 5$, $Z = V_n$ is called the *complex spin factor* of dimension n. The domain B is the *Lie ball*, given as

$$B = \left\{ z \in \mathbb{C}^n \;\middle|\; z^*z < 1\,,\, 1 - 2z^*z + \left|z^t z\right|^2 > 0 \right\} .$$

type V In this case, $Z = \mathbb{O}_{\mathbb{C}}^{1\times 2}$ consists of 1×2 matrices with entries in the complexified octonions $\mathbb{O}_{\mathbb{C}} = \mathbb{O} \otimes \mathbb{C}$. The triple product is defined as for type I. The domain B has complex dimension 16 and is given a quadratic and a quartic inequality.

type VI For this exceptional domain, $Z = \mathcal{H}_3(\mathbb{O}_{\mathbb{C}})$ is the set of Hermitian 3×3 matrices with entries in $\mathbb{O}_{\mathbb{C}}$. The product is formally the same as for type I. The domain B has complex dimension 27 and is given by two quadratic inequalities and a quartic inequality.

1.2 Peirce decomposition and boundary faces

1.2.1. An element $e \in Z$ is called a *tripotent* (triple idempotent) if $\{ee^*e\} = e$. Geometrically, the tripotents are the limit points of the geodesic rays emanating from $0 \in B$ [Loo75, cor. 4.8]. For a tripotent $e \in Z$ and $\lambda \in \mathbb{R}$, one considers the *Peirce λ-space*

$$Z_\lambda(e) = \ker\big(e \,\square\, e^* - \lambda\big) .$$

Then $Z_\lambda(e) = 0$, unless $\lambda \in \left\{0, \frac{1}{2}, 1\right\}$. Moreover, $Z = Z_0(e) \oplus Z_{1/2}(e) \oplus Z_1(e)$ and this sum is orthogonal with respect to the trace form, by [Loo75, th. 3.13].

By [Upm85, prop. 21.9], we have the *Peirce rules*

$$\left\{Z_\alpha(e) Z_\beta(e)^* Z_\gamma(e)\right\} \subset Z_{\alpha-\beta+\gamma}(e) \quad \text{for all } \alpha, \beta, \gamma \in \mathbb{R}$$

and

$$\left\{Z_0(e) Z_1(e)^* Z\right\} = \left\{Z_1(e) Z_0(e)^* Z\right\} = 0 .$$

In particular, $Z_1(e)$ and $Z_0(e)$ are subtriples of Z.

1.2.2. A tripotent e is said to be *unitary* if $e \,\square\, e^* = \mathrm{id}_Z$, i.e. $Z = Z_1(e)$. If e is unitary, define for all $z, w \in Z$

$$z \circ w = \{ze^*w\} \quad \text{and} \quad z^* = \{ez^*e\} .$$

Then $\sqcup^*$ is an involution, and $\circ$ is bilinear, commutative, with unit e, and moreover, satisfies the *Jordan identity*

$$z^2 \circ (z \circ w) = z \circ (z^2 \circ w) \quad \text{for all } z, w \in Z,$$

by [Upm85, prop. 13]. Thus, in this case, Z is a *complex Jordan algebra*. The triple product of Z is given in terms of the Jordan algebra product by

$$\{uv^*w\} = u \circ (v^* \circ w) - v^* \circ (w \circ u) + w \circ (u \circ v^*) \quad \text{for all } u, v, w \in Z\,.$$

Furthermore, $X = \{x \in Z \mid x^* = x\}$ is a real form of the complex vector space Z, $\circ$-closed and hence a *real Jordan algebra*. Since Z is a JB*-triple, we have the relation

$$x^2 + y^2 = 0 \;\Rightarrow\; x = y = 0 \quad \text{for all } x, y \in X\,,$$

i.e. X is *formally real*, by [Loo75, th. 3.13]. Conversely, for a formally real Jordan algebra X (sometimes also called *Euclidean*, cf. [FK94, prop. VIII.4.2]), the vector space complexification $Z = X \otimes \mathbb{C}$ is naturally a complex Jordan algebra whose underlying Jordan triple is a JB*-triple.

Z contains a unitary tripotent (i.e. is the Jordan triple defined by a Jordan algebra) if and only if B is of tube type (i.e. biholomorphic to a tube domain over a symmetric cone). For a simple JB*-triple Z, B is of not of tube type for type $\mathrm{I}_{p,q}$, $p \neq q$, type II_n, $n \equiv 1 \pmod 2$, and type V.

If the tripotent e is arbitrary, then e is unitary in $Z_1(e)$ and $Z_1(e)$ is a complex Jordan algebra. Its canonical real form is denoted $X_1(e)$. Note that if $e \in Z$ is a tripotent, then so is ie, and $Z_\lambda(ie) = Z_\lambda(e)$ since $(ie) \,\square\, (ie)^* = e \,\square\, e^*$. However, the real forms differ:

$$X_1(ie) = iX_1(e) = \{x \in Z_1(e) \mid x^* = -x\}\,.$$

Moreover, the product with respect to *ie* is

$$\{u(ie)^*v\} = -i\{ue^*v\} = i \cdot (-iu) \circ (-iv) \quad \text{for all } u, v \in Z_1(e)\,,$$

where on the right hand side, $\circ$ denotes the product with respect to e. Hence,

$$x \mapsto ix : X_1(e) \to X_1(ie) = iX_1(e)$$

is an isomorphism of real Jordan algebras.

1.2.3. For two tripotents $e, c \in Z$,

$$e \,\square\, c^* = 0 \;\Leftrightarrow\; \{ee^*c\} = 0\,,$$

by [Loo75, lem. 3.9]. If this the case, we write $e \perp c$ and say that e and c are *orthogonal*.

Since $\left(e \,\square\, c^*\right)^* = c \,\square\, e^*$, this relation is symmetric.

Further, we define an order on the set of non-zero tripotents by

$$c \leqslant e \;:\Leftrightarrow\; \{(e-c)(e-c)^*(e-c)\} = e - c\,,\; c \perp e - c\,.$$

A non-zero tripotent $e \in Z$ is *primitive* if it is minimal, and *maximal* if it is maximal with respect to this order. e is primitive if and only if $Z_1(e) = \mathbb{C} \cdot e$, and maximal if and only if $Z_0(e) = 0$. If e is unitary, then it is maximal, and the converse is true if and only if Z is a Jordan algebra.

A maximal set $e_1, \ldots, e_r$ of mutually orthogonal primitive tripotents is called a *frame* of Z. The length r of a frame is unique and coincides with the rank of B. We define $\operatorname{rk} Z = r$, the *rank of the Jordan triple* Z. By definition, the *rank of a tripotent* $e \in Z$ is the rank of $Z_1(e)$. Any sum of mutually orthogonal tripotents is a tripotent, and any tripotent can be written as the sum $e_1 + \cdots + e_k$ of the initial segment of a frame, [Loo75, 5.1, th. 3.11].

Given a frame $e_1, \ldots, e_r$ of Z, define for $0 \leqslant i \leqslant j \leqslant r$ the *joint Peirce spaces*

$$Z_{ij} = Z_{ji} = \left\{ z \in Z \;\middle|\; \{e_k e_k^* z\} = \frac{\delta_{ik} + \delta_{jk}}{2} \cdot z\,,\; k = 1, \ldots, r \right\}.$$

Then $Z = \sum^{\oplus}_{0 \leqslant i \leqslant j \leqslant r} Z_{ij}$ is an orthogonal direct sum with respect to the trace form, we have $Z_{00} = 0$, and $Z_{ii} = \mathbb{C} \cdot e_i$, $1 \leqslant i \leqslant r$.

If Z is simple, $a = \dim Z_{ij}$, $1 \leqslant i < j \leqslant r$, and $b = \dim Z_{0j}$, $1 \leqslant j \leqslant r$, are fixed. The tuple (r, a, b) is called the *signature* of Z. Then $b = 0$ if and only if Z is a Jordan algebra.

The *canonical inner product* $(\sqcup \mid \sqcup)$ of Z is the unique positive Hermitian inner product on Z which is K-invariant, associative, i.e.

$$(u \,\square\, v^*)^* = v \,\square\, u^* \quad \text{for all } \; u, v \in Z\,,$$

and for which $(e \mid e) = 1$ for every primitive tripotent $e \in Z$. The restriction of $(\sqcup \mid \sqcup)$ to any subtriple is the canonical inner product of that triple.

For simple Z, the canonical inner product is expressed in terms of the signature as

$$(u \mid v) = \frac{2r}{2n - rb} \cdot \operatorname{tr}_Z(u \,\square\, v^*) \quad \text{for all } \; u, v \in Z\,.$$

1.2.4. Consider the topological closure $\overline{B}$ of B in Z. The *faces* or *holomorphic arc components* of B are the equivalence classes of $\overline{B}$ under the equivalence relation

$$z \sim w \;:\Leftrightarrow\; z \in \mathbb{B}_0\,,\; \mathbb{B}_j \cap \mathbb{B}_{j+1} \neq \varnothing\,,\; \mathbb{B}_m \ni w \text{ for some } \; \mathbb{B}_j \subset \overline{B}\,,$$

where $\mathbb{B}_j = f_j(\mathbb{B})$, $f_j \in \mathcal{O}(\mathbb{B}, Z)$. Here $\mathbb{B} \subset \mathbb{C}$ denotes the unit disc and the $\mathbb{B}_j$ are called *holomorphic arcs*.

The faces of B can described quite effectively in terms of tripotents. For any tripotent, define $B_0(e) = B \cap Z_0(e)$. Then $B_0(e)$ is the circular bounded symmetric domain of the JB*-triple $Z_0(e)$. Moreover, the set $e + B_0(e)$ is a face of B, and

$$e \mapsto e + B_0(e) : E_Z = \{e \in Z \mid \{ee^*e\} = e\} \to \{F \subset \overline{B} \mid F \text{ face}\}$$

is a bijection, by [Loo75, th. 6.3]. (The trivial tripotent 0 gives the interior B.) The set of faces coincides with the set of (exposed) convex faces (i.e. intersections with supporting hyperplanes).

To describe the closure $\overline{B}$ as a whole, define the smooth vector bundle

$$\mathbb{E}_Z = \big\{(e,z) \in E_Z \times Z \bigm| z \in Z_0(e)\big\} \to E_Z$$

with projection $(e,z) \mapsto z$. Consider further the disc bundle

$$\mathbb{B}_Z = \big\{(e,z) \in \mathbb{E}_Z \bigm| z \in B_0(e)\big\} \to E_Z$$

which is an open sub-fibre bundle of $\mathbb{E}_Z$, and, set-theoretically, the disjoint union of the faces $e + B_0(e)$. Then, by [Loo75, prop. 6.8], the bijective map

$$(e,z) \mapsto e + z : \mathbb{B}_Z \to \overline{B}$$

is an immersion which restricts to an embedding on each connected component. If Z is simple, K acts transitively on the sets E_Z^k of rank k tripotents, by [Loo75, cor. 5.12]. Hence, in this case, $\overline{B}$ has $r+1$ connected strata

$$B_k = \bigcup\nolimits_{e \in E_Z^k} \big(e + B_0(e)\big) \ , \ k = 0, \dots, r \ .$$

Moreover, by [Loo75, cor. 9.17],

$$B_k = G.e^k = K.\big(e^k + B_0(e^k)\big) \quad \text{for } e^k \in E_Z^k \ .$$

Hence, B_k can be viewed as a G-homogeneous space or as a K-fibre bundle.

Definition 1.2.5. To each face $e + B_0(e)$ of B, we associate the *facial subgroup*

$$G_e = \mathrm{Aut}_0\big(B_0(e)\big) \ .$$

Then $G_e = K_e \cdot \exp \mathfrak{p}_{\mathbb{R},0}(e)$ where

$$K_e = \mathrm{Aut}_0\big(Z_0(e)\big) \quad \text{and} \quad \mathfrak{p}_{\mathbb{R},0}(e) = \big\{\xi_u^- \bigm| u \in Z_0(e)\big\} \ ,$$

by [Loo75, cor. 4.9]. Moreover, K_e is connected and therefore generated by the exponential of its Lie algebra

$$\mathfrak{k}_{\mathbb{R},0}(e) = \mathrm{aut}\big(Z_0(e)\big) = \langle u \,\square\, v^* - v \,\square\, u^* \mid u,v \in Z_0(e)\rangle \subset \mathfrak{k}_{\mathbb{R}} \,.$$

Here, the equality holds because all triple derivations of $Z_0(e)$ are inner. We conclude that $G_e \sqsubset G$, i.e. G_e is indeed a closed subgroup of G.

If $c \in Z$ is a tripotent such that $c \leqslant e$, then $Z_0(e) \subset Z_0(c)$ and e is a tripotent in the JB*-triple $Z_0(c)$. Hence, $G_e \sqsubset G_c = \mathrm{Aut}_0(B_0(c))$.

1.3 Symmetric cones and formally real Jordan algebras

1.3.1. Let X be an n-dimensional real vector space, endowed with a symmetric bilinear form $(\sqcup : \sqcup)$. A convex cone $\Omega \subset X$ (with vertex 0) is said to be *pointed* if it contains no affine line. If Ω is closed, this means that $-\Omega \cap \Omega = 0$.

Define the (closed) *dual cone* by

$$\Omega^* = \big\{ x \in X \,\big|\, (x : y) \geqslant 0 \ \text{ for all } \ y \in \Omega \big\} \,.$$

Then Ω^* has non-trivial interior $\Omega^{*\circ}$ if and only if Ω is pointed.

Let $\Omega \subset X$ be a closed cone with $\Omega^\circ \neq \varnothing$. Then

$$\mathrm{GL}(\Omega) = \big\{ g \in \mathrm{GL}(X) \,\big|\, g\Omega = \Omega \big\}$$

is a closed subgroup of $\mathrm{GL}(X)$ and hence a Lie group. Ω is called *symmetric* if it is *self-dual*, i.e. $\Omega^* = \Omega$, and Ω° is homogeneous, i.e. $\mathrm{GL}(\Omega)$ acts transitively on the interior Ω°. In particular, a symmetric cone is pointed.

1.3.2. Assume Ω is symmetric. Then $\mathrm{GL}(\Omega)^t = \mathrm{GL}(\Omega)$, if g^t denotes the transpose with respect to $(\sqcup : \sqcup)$. Hence, $\vartheta(g) = \big(g^{-1}\big)^t$ defines an involutive automorphism of $\mathrm{GL}(\Omega)$ whose fixed group $\mathrm{O}(\Omega) = \mathrm{O}(X) \cap \mathrm{GL}(\Omega)$ is compact. Therefore, the group $\mathrm{GL}(\Omega)$ is reductive, and the ± 1-eigenspaces of ϑ give a Cartan decomposition of its Lie algebra $\mathfrak{gl}(\Omega)$,

$$\mathfrak{gl}(\Omega) = \mathfrak{o}(\Omega) \oplus \mathfrak{p}_{\mathbb{R}}(\Omega) \,.$$

By [FK94, prop. I.1.8], for any $x \in \Omega^\circ$, the stabiliser $\mathrm{GL}(\Omega)_x$ of x in $\mathrm{GL}(\Omega)$ is compact, and any compact subgroup of $\mathrm{GL}(\Omega)$ is contained in some $\mathrm{GL}(\Omega)_x$. Fix $e \in \Omega^\circ$ such that $\mathrm{GL}(\Omega)_e = \mathrm{O}(\Omega)$.

By definition, $\mathfrak{gl}(\Omega)$ is the set of complete linear vector fields on Ω°. Since the interior $\Omega^\circ = \mathrm{GL}(\Omega)/\mathrm{O}(\Omega)$, the linear map

$$\xi \mapsto \xi(e) : \mathfrak{p}_{\mathbb{R}}(\Omega) \to X$$

is a bijection. For $x \in X$, define $M_x \in \mathfrak{p}_{\mathbb{R}}(\Omega)$ uniquely by $M_x(e) = x$, so that we have

$M_X = \mathfrak{p}_{\mathbb{R}}(\Omega)$. Then $M_x \in \operatorname{End}(X)$, and if we define

$$x \circ y = M_x y \quad \text{for all } x, y \in X\,,$$

then X is a formally real Jordan algebra with identity e , cf. [FK94, th. III.3.1]. Moreover, we have $\Omega = \{x^2 \mid x \in X\}$.

Conversely, by [FK94, th. III.2.1], for any formally real Jordan algebra X , the cone of squares $\Omega = \{x^2 \mid x \in X\}$ is symmetric, and the Jordan algebra structure of X is induced by the cone Ω .

1.3.3. Similarly as for Jordan triple systems, any idempotent $c \in X$ (i.e. $c^2 = c$) gives rise to a *Peirce decomposition*

$$X = X_0(c) \oplus X_{1/2}(c) \oplus X_1(c)$$

orthogonal with respect to the trace form $(x, y) \mapsto \operatorname{tr}_X(M_{x \circ y})$. Here, the *Peirce λ-space* is $X_\lambda(c) = \ker(M_c - \lambda)$. The trace form on X is positive symmetric, $O(\Omega)$-invariant, and is hence proportional to $(\sqcup : \sqcup)$.

As for Jordan triples, we define the *canonical inner product* $(\sqcup \mid \sqcup)$ by the requirements of $O(\Omega)$-invariance, associativity (i.e. $(u \circ v \mid w) = (v \mid u \circ w)$) and that the norm of any primitive idempotent be 1 . If X is simple,

$$(x \mid y) = \frac{r}{n} \cdot \operatorname{tr}_X(M_{x \circ y}) \quad \text{for all } x, y \in X\,,$$

where r is the rank of X .

1.3.4. It is clear that the connected component $GL_+(\Omega)$ of $GL(\Omega)$ is transitive on Ω° , and $SO(\Omega) = GL_+(\Omega) \cap O(\Omega)$ is connected since Ω° is simply connected.

An element $k \in GL(X)$ is called a *Jordan algebra automorphism* if

$$k(x \circ y) = (kx) \circ (ky) \quad \text{for all } x, y \in X\,.$$

By [FK94, th. III.5.1], $\operatorname{Aut}(X) = O(\Omega)$ and the connected component is $SO(\Omega)$. Moreover, the Lie algebra $\mathfrak{o}(\Omega)$ coincides with the set of all *Jordan algebra derivations* $\delta \in \operatorname{aut}(X)$, i.e. $\delta \in \operatorname{End}(X)$,

$$\delta(x \circ y) = (\delta x) \circ y + x \circ (\delta y) \quad \text{for all } x, y \in X\,.$$

It can be seen that $\operatorname{Aut}(X)$ is the set of those Jordan triple automorphisms k of $X \otimes \mathbb{C}$ such that $ke = e$, and that $\operatorname{aut}(X)$ consists of all Jordan triple derivations δ such that $\delta(e) = 0$.

1.3.5. A subset $F \subset \Omega$ is called a (convex) *face* of Ω, if

$$\tfrac{1}{2} \cdot (x+y) \in F \;\Rightarrow\; x, y \in F \quad \text{for all } \; x, y \in \Omega\,.$$

Denote the set of faces of Ω by $\mathcal{F}(\Omega)$. The faces F of Ω are closed pointed convex cones. The *relative interior* F° of $F \in \mathcal{F}(\Omega)$ is defined as the interior of F in the subspace $F - F$ of X generated by F.

A proper face $F \in \mathcal{F}(\Omega)$ is called *exposed* if it is the intersection of a hyperplane with Ω. General cones may have non-exposed proper faces, an example is given in [Rup88, ex. 2.7.(5)].

As for Jordan triples, we can define an order on the set E_X of idempotents. A non-zero minimal element of E_X is called *primitive*. The rank of $c \in E_X$, defined as the rank of $X_1(c)$, is the number k of summands in the decomposition $c = c_1 + \cdots + c_k$ as a sum of mutually orthogonal primitive idempotents.

If X is simple, by [FK94, prop. IV.3.1], the closed symmetric cone Ω decomposes into exactly $r+1$ orbits $\mathrm{GL}_+(\Omega).c^k$ where $c^k \in E_X^k$, i.e. c^k is a rank k tripotent. The orbit $\mathrm{GL}_+(\Omega).c^k$ consists of the elements of rank k.

To any idempotent $c \in E_X$, we associate the closed cone $\Omega_0(c)$ of squares in $X_0(c) = X_1(e-c)$. Then $\Omega_0(c) \subset \overline{G.(e-c)} \subset \Omega$.

Proposition 1.3.6. Let X be simple. The set $\mathcal{F}(\Omega)$ of faces consists of

$$\Omega_0(c) = X_0(c) \cap \Omega = c^\perp \cap \Omega = \left\{ x^2 \;\middle|\; x \in X_0(c) \right\}, \; c \in E_X\,.$$

In particular, all the faces of Ω are exposed. The dual face of $\Omega_0(c)$ is $\Omega_0(e-c)$. Two faces $\Omega_0(c)$ and $\Omega_0(c')$ are $\mathrm{GL}_+(\Omega)$-conjugate if and only if $\operatorname{rk} c = \operatorname{rk} c'$.

Proof. The conjugacy follows from the above, and from [FK94, prop. IV.3.1]. We prove that $\Omega_0(c)$ is a face of Ω. Since Ω is the set of $x \in X$ for which M_x is positive semi-definite, by [FK94, prop. III.2.2], we see that $\Omega \cap X_0(c) = \Omega_0(c)$. Moreover, $c \in \Omega = \Omega^*$, so $c^\perp \cap \Omega$ is a face. We have $c^\perp \supset \Omega_0(c)$. If $x \in \Omega$, $(x \mid c) = 0$, then $x \circ c = 0$, i.e. $x \in X_0(c)$, by [FK94, ex. III.3]. This proves that $\Omega_0(c)$ is a face.

More generally, $\Omega_0(e-c) \subset \Omega \cap \Omega_0(c)^\perp$, and we have already proved the converse inclusion. Hence these faces are dual to each other.

By [FK94, prop. IV.3.2], the extremal rays of Ω are all of the form $\Omega_0(c)$ where $c < e$ is maximal. The orthogonal face $\Omega \cap \Omega_0(c)^\perp = \Omega_0(e-c)$, as seen above. Since Ω is self-dual, any proper face $F \subsetneq \Omega$ has a non-trivial dual face. Hence, $\Omega_0(e-c)$ is a maximal proper face. Since any face of Ω contained in $\Omega_0(e-c)$ is a face thereof, and vice versa, it follows by induction by that all faces are of the form $\Omega_0(c)$ where $c \in E_Z$. ∎

2 Ol'shanskiĭ domains and their boundaries

2.1 Invariant cones

2.1.1. We return to our setting where $G = \mathrm{Aut}_0(B)$, $B \subset Z$ a circular bounded symmetric domain. The Cartan decomposition $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}}$ gives rise to a decomposition of the complexified Lie algebra, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. The Cartan involution ϑ extends naturally to the conjugation of $\mathfrak{g}$ with respect to the (compact) real form $\mathfrak{u}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$. We denote it by the same letter.

Lemma 2.1.2. The space $\mathfrak{p}$ can be decomposed as $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ where

$$\mathfrak{p}^+ = \left\{ u \,\frac{\partial}{\partial z} \;\middle|\; u \in Z \right\} \quad \text{and} \quad \mathfrak{p}^- = \left\{ \{zu^*z\} \,\frac{\partial}{\partial z} \;\middle|\; u \in Z \right\} .$$

Then

$$\left[\mathfrak{k}, \mathfrak{p}^\pm\right] \subset \mathfrak{p}^\pm \quad \text{and} \quad \left[\mathfrak{p}^\pm, \mathfrak{p}^\pm\right] = 0 .$$

Proof. The decomposition is trivial. We have

$$\vartheta\Big(u \,\frac{\partial}{\partial z}\Big) = \frac{1}{2} \cdot \vartheta(\xi_u^- - i\xi_{iu}^-) = -\frac{1}{2} \cdot \left(\xi_u^- + i\xi_{iu}^-\right) = \{zu^*z\} \,\frac{\partial}{\partial z} .$$

Since the vector fields in $\mathfrak{p}^+$ are constant, $\left[\mathfrak{p}^+, \mathfrak{p}^+\right] = 0$. Applying ϑ, we find that $\left[\mathfrak{p}^-, \mathfrak{p}^-\right] = 0$.

Since $\delta \in \mathfrak{k}$ is linear,

$$\left[\delta, u \,\frac{\partial}{\partial z}\right] = \delta u \,\frac{\partial}{\partial z} \quad \text{for all } u \in Z .$$

Since $\mathfrak{k}_{\mathbb{R}}$ leaves $\mathfrak{u}_{\mathbb{R}}$ invariant and hence commutes with ϑ, and $\mathfrak{p}^\pm$ are complex vector spaces, the assertion follows. □

2.1.3. Fix a frame $e_1, \dots, e_r$ of Z. Then

$$i \cdot e_j \,\square\, e_j^* \in \mathfrak{k}_{\mathbb{R}} = \mathrm{aut}(Z) , \; j = 1, \dots, r ,$$

is a commutative family of triple derivations. Consequently, there exists a maximal commutative subalgebra $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}$ containing

$$\mathfrak{t}_{\mathbb{R}}^- = i\langle e_j \,\square\, e_j^* \mid j = 1, \dots, r \rangle .$$

By [Upm86, lem. 1.1-2], $\mathfrak{t}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}}^+ \oplus \mathfrak{t}_{\mathbb{R}}^-$, where

$$\mathfrak{t}_{\mathbb{R}}^+ = \left\{ \delta \in \mathfrak{t}_{\mathbb{R}} \;\middle|\; \delta e_j = 0 \text{ for all } j = 1, \dots, r \right\} .$$

By maximality of $\mathfrak{t}_{\mathbb{R}}$, $\mathfrak{z}(\mathfrak{k}_{\mathbb{R}}) \subset \mathfrak{t}_{\mathbb{R}}$, in particular $iz\frac{\partial}{\partial z} \in \mathfrak{t}_{\mathbb{R}}$.

Lemma 2.1.4. The centraliser of $iz\frac{\partial}{\partial z}$ in $\mathfrak{g}$ is $\mathfrak{k}$. More precisely,

$$\operatorname{ad}\Big(iz\,\frac{\partial}{\partial z}\Big) = \pm i \quad \text{on } \mathfrak{p}^{\pm}\,.$$

In particular, $\mathfrak{t}_{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$, and $\operatorname{rk}\mathfrak{g}_{\mathbb{R}} = \operatorname{rk}\mathfrak{k}_{\mathbb{R}}$.

Proof. Clearly, $iz\frac{\partial}{\partial z}$ acts by multiplication with i on $\mathfrak{p}^{+}$. Since $iz\frac{\partial}{\partial z} \in \mathfrak{k}_{\mathbb{R}}$ and hence commutes with ϑ,

$$\Big[iz\,\frac{\partial}{\partial z}, \{zu^*z\}\,\frac{\partial}{\partial z}\Big] = \vartheta\Big[iz\,\frac{\partial}{\partial z}, u\,\frac{\partial}{\partial z}\Big] = \vartheta\Big(iu\,\frac{\partial}{\partial z}\Big) = -i\{zu^*z\}\,\frac{\partial}{\partial z}\,.$$

Hence, the first statement follows.

As for the second, any element of $\mathfrak{g}_{\mathbb{R}}$ centralising $\mathfrak{t}_{\mathbb{R}}$ is contained in $\mathfrak{k}_{\mathbb{R}}$ by the first part. But $\mathfrak{t}_{\mathbb{R}}$ is its own centraliser in $\mathfrak{k}_{\mathbb{R}}$. This proves the remaining assertions. □

2.1.5. Since the complexification $\mathfrak{t}$ of $\mathfrak{t}_{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{g}$, we can consider for $\alpha \in i\mathfrak{t}_{\mathbb{R}}^*$ the root spaces

$$\mathfrak{g}^{\alpha} = \big\{\xi \in \mathfrak{g}_{\mathbb{R}} \bigm| [\delta,\xi] = \alpha(\delta)\cdot\xi \text{ for all } \delta \in \mathfrak{t}\big\}\,.$$

Then the set

$$\Delta = \Delta(\mathfrak{g}:\mathfrak{t}) = \big\{\alpha \in i\mathfrak{t}_{\mathbb{R}}^* \setminus 0 \bigm| \mathfrak{g}^{\alpha} \neq 0\big\}$$

is a reduced root system in the subspace of $i\mathfrak{t}_{\mathbb{R}}^*$ it generates, by [Bou68, ch. VIII, § 2.2, th. 2]. By lemma 2.1.4, for $\alpha \in \Delta$,

$$\mathfrak{g}^{\alpha} \subset \mathfrak{p} \;\Leftrightarrow\; \alpha\Big(iz\,\frac{\partial}{\partial z}\Big) \neq 0 \quad\text{and}\quad \mathfrak{g}^{\alpha} \subset \mathfrak{k} \;\Leftrightarrow\; \alpha\Big(iz\,\frac{\partial}{\partial z}\Big) = 0\,.$$

In the first case, we say that α is *non-compact*, in the second, that it is *compact*. We denote the set of non-compact resp. compact roots by Δ_n resp. Δ_c. Then $\Delta = \Delta_c \cup \Delta_n$ is a $\mathbb{Z}_2$-grading of Δ in the sense of Loos-Neher [LN02], namely

$$\Delta \cap \big(\Delta_c + \Delta_c\big) \subset \Delta_c\,,\ \Delta \cap \big(\Delta_c + \Delta_n\big) \subset \Delta_n\,,\ \Delta \cap \big(\Delta_n + \Delta_n\big) \subset \Delta_c\,.$$

In particular, Δ_c is a root system in the subspace of $i\mathfrak{t}_{\mathbb{R}}^*$ it generates.

We consider the Weyl groups

$$W = \langle s_{\alpha} \mid \alpha \in \Delta\rangle \quad\text{and}\quad W_c = \langle s_{\alpha} \mid \alpha \in \Delta_c\rangle$$

generated by the reflections

$$s_{\alpha}\big(\beta\big) = \beta - \beta\big(H_{\alpha}\big)\cdot\alpha \quad\text{for all } \beta \in i\mathfrak{t}_{\mathbb{R}}^*\,.$$

Here $H_\alpha \in i\mathfrak{t}_{\mathbb{R}}$ is determined by $\alpha\big(H_\alpha\big) = 2$, cf. [Bou68], [Kna86, ch. IV].

Lemma 2.1.6. Let $\Delta_c^{++} \subset \Delta_c$ be any positive system. The set

$$\Delta_n^{++} = \Big\{\alpha \in \Delta \,\Big|\, -i \cdot \alpha\Big(iz\,\frac{\partial}{\partial z}\Big) > 0\Big\}$$

is W_c-invariant, and $\Delta^{++} = \Delta_c^{++} \cup \Delta_n^{++}$ is a positive system of Δ such that the sum of two positive non-compact roots is never a root. Moreover,

$$\mathfrak{p}^\pm = \textstyle\sum^\oplus_{\alpha \in \pm\Delta_n^{++}} \mathfrak{g}^\alpha \,.$$

Proof. The last statement follows from lemma 2.1.4. Now, $\big[\mathfrak{p}^+, \mathfrak{p}^+\big] = 0$ implies that the sum of $\alpha, \beta \in \Delta_n^{++}$ is never a root. Because all compact roots annihilate $iz\,\frac{\partial}{\partial z}$,

$$\Delta \cap (\Delta_c^{++} + \Delta_n^{++}) \subset \Delta_n^{++} \,.$$

Since Δ_c^{++} is closed, so is Δ^{++}. But

$$\Delta = -\Delta^{++} \cup \Delta^{++} \quad \text{and} \quad -\Delta^{++} \cap \Delta^{++} = \varnothing$$

because $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{k} \oplus \mathfrak{p}^-$ by lemma 2.1.2. Thus, Δ^{++} is a positive system, by [Bou68, ch. VI, § 1.7, cor. 1]. The statement about W_c-invariance now follows from [Nee00a, prop. VII.2.12]. □

Remark 2.1.7. The positive system constructed in the lemma is of a somewhat special type (which merits the notation Δ^{++}, in contrast to a general positive system Δ^+). Namely, it is Δ_c-*adapted*.

This means that the sum of two positive non-compact roots is never a root. Equivalently, the set of positive non-compact roots is W_c-invariant; any Δ_c^{++}-simple compact root is Δ^{++}-simple; the non-compact positive roots are strictly larger than all compact roots for any (some) total vector space order defining Δ^{++}; or, the subset $\Delta_c \cup \Delta_n^{++}$ of Δ is closed (and hence parabolic).

If Z is simple, the adapted positive system Δ^{++} is, up to sign and W_c-conjugacy, uniquely determined, by [Nee00a, lem. VII.2.16]. Moreover, the property that there exists a Δ_c-adapted positive system singles out the class of non-compact simple Lie algebras (Hermitian Lie algebras) which occur as the set of complete holomorphic vector fields of a bounded symmetric domain, by [Nee00a, prop. VII.2.14].

2.1.8. Recall that the Killing form B of $\mathfrak{g}_{\mathbb{R}}$ is given by

$$B(\xi, \eta) = -\operatorname{tr}\big(\operatorname{ad}\xi\ \operatorname{ad}\eta\big) \quad \text{for all } \ \xi, \eta \in \mathfrak{g}_{\mathbb{R}} \,.$$

B is symmetric, and the adjoint action is skew-adjoint (Jacobi identity). It extends by complex bilinearity to $\mathfrak{g}$, and coincides with the Killing form of $\mathfrak{g}$. Moreover, we define

$$(\xi : \eta) = -B(\xi, \vartheta\eta) \quad \text{for all } \xi, \eta \in \mathfrak{g} .$$

This form is symmetric, and conjugate bilinear with respect to the complex structure of $\mathfrak{g}$ induced by the compact real form $\mathfrak{u}_{\mathbb{R}}$.

Lemma 2.1.9. The decomposition $\mathfrak{g} = \mathfrak{p}^+ \oplus \mathfrak{k} \oplus \mathfrak{p}^-$ is B-orthogonal. We have

$$B\big(\delta, u \,\Box\, v^*\big) = 2 \operatorname{tr}_Z\big((\delta u) \,\Box\, v^*\big) \quad \text{for all } \delta \in \operatorname{aut}(Z) ,\ u, v \in Z ,$$

and

$$B\Big(u \frac{\partial}{\partial z}, \{zv^*z\} \frac{\partial}{\partial z}\Big) = -4 \operatorname{tr}_Z\big(u \,\Box\, v^*\big) \quad \text{for all } u, v \in Z .$$

Moreover, $\mathfrak{p}^\pm$ are isotropic, and

$$B\big(\xi_u^-, \xi_v^-\big) = 4 \cdot \operatorname{tr}_Z\big(u \,\Box\, v^* + v \,\Box\, u^*\big) \quad \text{for all } u, v \in Z .$$

Proof. The orthogonality follows since the spaces $\mathfrak{p}^+$, $\mathfrak{k}$, and $\mathfrak{p}^-$ are eigenspaces of $\operatorname{ad}\big(iz \frac{\partial}{\partial z}\big)$ for the eigenvalues i, 0, and $-i$, respectively.

Note

$$\Big[u \frac{\partial}{\partial z}, \{zv^*z\} \frac{\partial}{\partial z}\Big] = -2 \cdot u \,\Box\, v^* \quad \text{for all } u, v \in Z .$$

Hence

$$B(\delta, u \,\Box\, v^*) = -\frac{1}{2} \cdot B\Big(\Big[\delta, u \frac{\partial}{\partial z}\Big], \{zv^*z\} \frac{\partial}{\partial z}\Big) = -\frac{1}{2} \cdot B\Big((\delta u) \frac{\partial}{\partial z}, \{zv^*z\} \frac{\partial}{\partial z}\Big) ,$$

the first equation follows from the second, and to prove the second, we need only prove the first for $\delta = iz \frac{\partial}{\partial z}$. But

$$B\Big(iz \frac{\partial}{\partial z}, u \,\Box\, v^*\Big) = i \operatorname{tr}_{\mathfrak{p}^+} \operatorname{ad}\big(u \,\Box\, v^*\big) - i \operatorname{tr}_{\mathfrak{p}^-} \operatorname{ad}\big(u \,\Box\, v^*\big)$$

by lemma 2.1.4. Moreover,

$$\Big[u \,\Box\, v^*, w \frac{\partial}{\partial z}\Big] = \{uv^*w\} \frac{\partial}{\partial z} ,$$

and by (JTS2),

$$\Big[u \,\Box\, v^*, \{zw^*z\} \frac{\partial}{\partial z}\Big] = -\{z\{vu^*w\}^*z\} \frac{\partial}{\partial z} .$$

We conclude

$$B\Big(iz \frac{\partial}{\partial z}, u \,\Box\, v^*\Big) = 2i \operatorname{tr}_Z\big(u \,\Box\, v^*\big) .$$

This proves the first two formulae.

The subspace $\mathfrak{p}^+$ is isotropic, because

$$B\Big(u\,\frac{\partial}{\partial z},v\,\frac{\partial}{\partial z}\Big)=-i\cdot B\Big(\Big[iz\,\frac{\partial}{\partial z},u\,\frac{\partial}{\partial z}\Big],v\,\frac{\partial}{\partial z}\Big)=-i\cdot B\Big(iz\,\frac{\partial}{\partial z},\Big[u\,\frac{\partial}{\partial z},v\,\frac{\partial}{\partial z}\Big]\Big)=0$$

for all $u,v\in Z$. By ϑ-invariance of B, $\mathfrak{p}^-$ is isotropic, too.

This implies the third formula, as

$$\begin{aligned}B(\xi_u^-,\xi_v^-)&=B\Big(\big(u-\{zu^*z\}\big)\,\frac{\partial}{\partial z},\big(v-\{zv^*z\}\big)\,\frac{\partial}{\partial z}\Big)\\&=-B\Big(u\,\frac{\partial}{\partial z},\{zv^*z\}\,\frac{\partial}{\partial z}\Big)-B\Big(\{zu^*z\}\,\frac{\partial}{\partial z},v\,\frac{\partial}{\partial z}\Big)\\&=4\cdot\operatorname{tr}_Z\big(u\,\square\,v^*+v\,\square\,u^*\big)\end{aligned}$$

for all $u,v\in Z$. $\square$

Remark 2.1.10. For an alternative proof of lemma 2.1.9, we refer to [Koe69, lem. 4.2] and [Upm82, lem. 6.1].

Proposition 2.1.11. The form $(\sqcup:\sqcup)$ is positive symmetric. In particular, B is non-degenerate, $\mathfrak{g}_{\mathbb{R}}$ is semi-simple, and $\mathfrak{g}_{\mathbb{R}}=\mathfrak{k}_{\mathbb{R}}\oplus\mathfrak{p}_{\mathbb{R}}$ is a Cartan decomposition. If Z is simple, then so is $\mathfrak{g}_{\mathbb{R}}$.

Proof. Since $\mathfrak{k}_{\mathbb{R}}\perp\mathfrak{p}_{\mathbb{R}}$, it suffices to check positivity for each component individually. By lemma 2.1.9, for all $u,v\in Z$,

$$\big(\xi_u^-:\xi_v^-\big)=B(\xi_u^-,\xi_v^-)=4\cdot\operatorname{tr}_Z\big(u\,\square\,v^*+v\,\square\,u^*\big)=8\operatorname{Re}\operatorname{tr}_Z\big(u\,\square\,v^*\big)$$

Since the trace form is positive Hermitian, $(\xi_u^-:\xi_u^-)>0$ for $u\neq 0$.

By lemma 2.1.9, if $\delta\in\mathfrak{k}_{\mathbb{R}}\setminus 0$, then

$$(\delta:u\,\square\,v^*)\neq 0\quad\text{for some }\ u,v\in Z\,.$$

In particular, $(\sqcup:\sqcup)$ and B are non-degenerate. Since K is compact and the centre of G is trivial, B is negative on $\mathfrak{k}_{\mathbb{R}}$, by [Hel78, ch. II, § 6, prop. 6.8]. Hence $\mathfrak{g}_{\mathbb{R}}$ is semi-simple, and ϑ is a Cartan involution by [Hel78, ch. III, § 7, prop. 7.4]. Finally, if Z is simple, then $\mathfrak{g}_{\mathbb{R}}$ is simple by [Koe69, th. 4.4]. $\square$

In the following, we shall always understand orthogonality in $\mathfrak{g}_{\mathbb{R}}$ in terms of the positive symmetric form $(\sqcup:\sqcup)$.

2.1.12. Consider the following polyhedral cones in $\mathfrak{t}_{\mathbb{R}}$:

$$\omega^-=\operatorname{cone}\langle iH_\alpha\mid\alpha\in\Delta_n^{++}\rangle$$

and its dual cone

$$\omega^+=\big(\Delta_n^{++}\big)^*=\Big\{H\in\mathfrak{t}_{\mathbb{R}}\;\Big|\;-i\alpha(H)\geqslant 0\ \text{ for all }\ \alpha\in\Delta_n^{++}\Big\}\,.$$

These cones are closed and W_c-invariant by lemma 2.1.6. By [HC56, lem. 10], $\alpha(H_\beta) \geqslant 0$ for all $\alpha, \beta \in \Delta_n^{++}$, so $\omega^- \subset \omega^+$.

For $k = 1, \ldots, r$, define $\gamma_k \in i\mathfrak{t}_\mathbb{R}^*$ by

$$\gamma_k(e_\ell \square e_\ell^*) = \delta_{k,\ell} \quad \text{and} \quad \gamma_k\big|_{\mathfrak{t}_\mathbb{R}^+} = 0\,.$$

Then, by [Upm86, lem. 1.3], the γ_k are mutually strongly orthogonal roots, i.e.

$$\gamma_k \pm \gamma_\ell \notin \Delta \quad \text{for all } 1 \leqslant k \neq \ell \leqslant r\,.$$

Note that

$$iz\frac{\partial}{\partial z} = \sum_{k=1}^r i \cdot e_k \square e_k^* + \delta \quad \text{for some } \delta \in \mathfrak{t}_\mathbb{R}^+\,.$$

Therefore, it is clear that $\gamma_k \in \Delta_n^{++}$. There is a total vector space order on $i\mathfrak{t}_\mathbb{R}^*$ defining Δ^{++}, such that $0 < \gamma_1 < \cdots < \gamma_r$. Consequently, $\gamma_1, \ldots, \gamma_r$ is the *Harish-Chandra fundamental sequence*, cf. [HC56, II.6]. In particular, by [HC56, lem. 8, cor.], the cardinality of $\gamma_1, \ldots, \gamma_r$ is maximal.

By [Moo64, th. 2] and [Pan83, lem. 1], all the γ_k are long. Here we say that α is long if $|\beta| \leqslant |\alpha|$ for all roots β contained in the irreducible factor of Δ containing α. (An irreducible, reduced root system has at most two root lengths, by [Bou68, ch. VI, § 1.4, prop. 12].)

Lemma 2.1.13. The generators of the extremal rays of ω^- are iH_α, $\alpha \in \Delta_n^{++}$, α long. In particular,

$$\omega^- = \operatorname{cone}\left\langle \sigma(i \cdot e_j \square e_j^*) \;\middle|\; \sigma \in W_c\,,\, j = 1, \ldots, r \right\rangle.$$

Proof. It is clear by definition of ω^- that the generators of extremal rays are among the H_α where $\alpha \in \Delta_n^{++}$. Since ω^- decomposes into a direct product according to the decomposition of $\mathfrak{g}_\mathbb{R}$ into simple factors, we may assume w.l.o.g. that $\mathfrak{g}_\mathbb{R}$ be simple.

Then [Pan83, lem. 1] shows that for short $\gamma \in \Delta_n^{++}$,

$$\gamma = \frac{\gamma_k + \gamma_\ell}{2} \quad \text{for some } k \neq \ell\,.$$

This implies

$$4\,|\gamma|^2 = |\gamma_k|^2 + |\gamma_\ell|^2 = 2\,|\gamma_k|^2\,,$$

and

$$(H_\gamma : \xi) = \frac{2\gamma(\xi)}{|\gamma|^2} = \frac{2\gamma_k(\xi)}{|\gamma_k|^2} + \frac{2\gamma_\ell(\xi)}{|\gamma_\ell|^2} = (H_{\gamma_k} + H_{\gamma_\ell} : \xi) \quad \text{for all } \xi \in \mathfrak{t}\,.$$

Hence, $H_\gamma = H_{\gamma_k} + H_{\gamma_\ell}$ lies in the interior of a face of dimension at least 2.

On the other hand, because ω^- is polyhedral, there is $\alpha \in \Delta_n^{++}$ such that $i\mathbb{R}_+ \cdot H_\alpha$ is extremal. α is necessarily long. By [Pan83, lem. 2], all such iH_α generate extremal rays.

Returning to the general (semi-simple) case, any irreducible factor of Δ contains some γ_k, by maximality of this set. Moreover, [Pan83, lem. 2] shows that any long non-compact $\alpha \in \Delta_n^{++}$ is W_c-conjugate to any γ_k contained in the same irreducible factor. Finally, note that $H_{\gamma_j} = 2 \cdot e_j \Box e_j^*$ by definition of γ_j. Since a polyhedral cone is generated by its extremal rays, we have proved the lemma. □

Lemma 2.1.14. Let $\gamma \in \Delta_n^{++}$ be long. There exist $1 \leqslant \ell \leqslant r$ and a frame $c_1, \dots, c_r$ of Z such that

$$\mathfrak{t}_{\mathbb{R}} = \langle i \cdot c_k \Box c_k^* \mid k = 1, \dots, r \rangle \oplus \{\delta \in \mathfrak{t}_{\mathbb{R}} \mid \delta c_k = 0 \text{ for all } k = 1, \dots, r\} ,$$

and γ is determined by

$$\gamma(c_k \Box c_k^*) = \delta_{\ell,k} \quad \text{and} \quad \gamma(\delta) = 0 \quad \text{whenever} \quad \delta c_k = 0 , \, k = 1, \dots, r .$$

Proof. There exists $1 \leqslant \ell \leqslant r$ such that γ_ℓ and γ lie in the same irreducible factor of Δ. By [Pan83, lem. 2], there exists $\sigma \in W_c$ such that $\sigma\gamma_\ell = \gamma$. By [Kna02, th. 4.54], $\sigma = \mathrm{Ad}(k)$ for some $k \in N_K(\mathfrak{t}_{\mathbb{R}})$. Since $k \in \mathrm{Aut}(Z)$, the ke_j, $j = 1, \dots, r$, are mutually orthogonal primitive tripotents, and

$$\begin{aligned} \mathrm{Ad}(k)\Big(e_j \Box e_j^* \frac{\partial}{\partial z}\Big) &= \big(k^{-1\prime}(z)\big)^{-1} \{e_j e_j^* k^{-1}(z)\} \frac{\partial}{\partial z} \\ &= k(e_j \Box e_j^*)k^{-1} = (ke_j) \ \Box (ke_j)^* . \end{aligned}$$

Moreover, since $\mathrm{Ad}(k)$ normalises $\mathfrak{t}_{\mathbb{R}}$, we have a decomposition as stated. By the definition of γ_ℓ, the lemma follows. □

Remark 2.1.15. By lemma 2.1.13 and lemma 2.1.14, the extremal rays of ω^- are generated by $i \cdot e \Box e^*$ where e is a primitive tripotent W_c-conjugate to an element of the frame $e_1, \dots, e_r$.

2.1.16. At this point, it seems appropriate to point out the relation between the cone ω^+ and the Weyl chambers of $\mathfrak{t}_{\mathbb{R}}$.

Consider for $\xi \in \mathfrak{g}_{\mathbb{R}}$ the expansion $\det(t - \mathrm{ad}\,\xi) = \sum_{j=1}^n a_j(\xi) \cdot t^j$. The polynomial coefficients a_j are independent of ξ. The lowest index j for which a_j is non-trivial is the rank $R = \mathrm{rk}\,\mathfrak{g}_{\mathbb{R}}$, by [Bou68, ch. VII, § 3.3, th. 2]. Then ξ is called *regular* if $a_R(\xi) \neq 0$. If particular, ξ is regular if and only if its centraliser $\mathfrak{g}_{\mathbb{R}}^\xi = \ker(\mathrm{ad}\,\xi)$ is a Cartan subalgebra (loc.cit.). The set $\mathfrak{g}_{\mathbb{R},*}$ of regular elements in $\mathfrak{g}_{\mathbb{R}}$ is open and dense both in the Hausdorff vector space and the Zariski topology. Moreover, it is Ad-invariant.

The intersection $\mathfrak{t}_{\mathbb{R},*} = \mathfrak{g}_{\mathbb{R},*} \cap \mathfrak{t}_{\mathbb{R}}$ coincides with the complement of the union of all $\ker \alpha$, where $\alpha \in \Delta$, cf. [Bou68, ch. VII,§ 3.2, lem. 2]. The connected components of $\mathfrak{t}_{\mathbb{R},*}$ are called *Weyl chambers*. They are polyhedral cones, and the Weyl group W acts simply transitively on their totality. Hence, they are in one-to-one correspondence with the set of positive systems for Δ. The Weyl chamber associated to Δ^{++} is

$$c_+ = \left\{ H \in \mathfrak{t}_{\mathbb{R}} \;\middle|\; -i\alpha(H) > 0 \text{ for all } \alpha \in \Delta^{++} \right\} .$$

By definition, it is obvious that $c_+ \subset \omega^{+\circ}$.

Proposition 2.1.17. We have $\omega^+ = W_c \cdot \overline{c_+} = \bigcup_{\sigma \in W_c} \sigma(\overline{c_+})$.

Proof. Let Π_c be the set of Δ_c^{++}-simple roots. Since Δ^{++} is adapted, they are also Δ^{++}-simple. Let $H \in \omega^+$. Let

$$n_H = \#\left\{ \alpha \in \Pi_c \;\middle|\; -i\alpha(H) < 0 \right\} .$$

If $n_H = 0$, then $H \in \overline{c_+}$, because $-i\beta(H) \geqslant 0$ for all $\beta \in \Delta_n^{++}$, and $\Delta_c^{++} \subset \mathbb{N}\langle \Pi_c \rangle$.

If $n_H > 0$, let $\alpha \in \Pi_c$ such that $-i\alpha(H) < 0$. Then $-i\alpha(s_\alpha(H)) > 0$. Let $\beta \in \Pi_c$ such that $-i\beta(s_\alpha(H)) < 0$. By [Bou68, ch. VI, § 1.6, prop. 17, cor. 1], $s_\alpha(\beta)$ is a positive root, so $\beta(H_\alpha) < 0$. Now

$$0 > -i\beta(s_\alpha(H)) = -i\beta(H) - i\alpha(H) \cdot \beta(H_\alpha)$$

implies

$$-i\beta(H) \leqslant i \cdot \alpha(H) \cdot \beta(H_\alpha) = -\big(-i\alpha(H) \cdot \beta(H_\alpha)\big) < 0 .$$

Hence, $n_{s_\alpha(H)} < n_H$, and by induction, there exists $\sigma \in W_c$ such that $\sigma(H) \in \overline{c_+}$. This proves $\omega^+ \subset W_c \cdot \overline{c_+}$. The converse is clear from W_c-invariance. □

2.1.18. From now on, we assume that Z be simple. Then the centre $\mathfrak{z}(\mathfrak{k}_{\mathbb{R}})$ of $\mathfrak{k}_{\mathbb{R}}$ is one-dimensional and hence generated by $iz\,\frac{\partial}{\partial z}$. By [Pan83, lem. 3], we have $iz\,\frac{\partial}{\partial z} \in \omega^-$. Let Ω^- be the closed G-invariant cone generated by $iz\,\frac{\partial}{\partial z}$,

$$\Omega^- = \overline{\operatorname{cone}} \left\langle \operatorname{Ad}(g)\Big(iz\,\frac{\partial}{\partial z}\Big) \;\middle|\; g \in G \right\rangle .$$

Since all invariant cones in $\mathfrak{g}_{\mathbb{R}}$ with non-void interior contain a K-fixed vector by [Vin80, § 2], Ω^- is a minimal such cone. (The interior of Ω^- is non-trivial by [Pan83, lem. 3].)

For $u \in Z$, define the *Cayley vector field* $\xi_u^+ \in i\mathfrak{p}_{\mathbb{R}}$ by

$$\xi_u^+ = -i\xi_{iu}^- = \big(u + \{zu^*z\}\big) \cdot \frac{\partial}{\partial z} .$$

Moreover, consider the following standard basis for $\mathfrak{sl}(2, \mathbb{R})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \; X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \; X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} .$$

The complex structure of the upper half plane (corresponding to $iz\,\frac{\partial}{\partial z}$ on the unit disc) is given by $X^+ - X^-$, up to scalar multiples. Hence, the $\mathfrak{p}^\pm$ spaces for $\mathfrak{sl}(2, \mathbb{C})$ are generated by $H \pm i(X^+ + X^-)$.

We shall call call a triple $(\xi, \eta^\pm)$ a *disc embedding* if $\mathfrak{s}_\mathbb{R} = \langle \xi, \eta^\pm \rangle$ is a subalgebra of $\mathfrak{g}_\mathbb{R}$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ via

$$H \mapsto \xi \quad \text{and} \quad X^\pm \mapsto \eta^\pm \ ,$$

and if for $\mathfrak{s} = \mathfrak{s}_\mathbb{R} \otimes \mathbb{C}$,

$$\mathfrak{s} \cap \mathfrak{k} = \mathbb{C} \cdot (\eta^+ - \eta^-) \quad \text{and} \quad \mathfrak{s} \cap \mathfrak{p}^\pm = \mathbb{C} \cdot \big(\xi \pm i(\eta^+ + \eta^-)\big) \ .$$

The terminology is justified because such a triple corresponds to an embedding of a (poly-) disc into B .

Lie algebraically, the above conditions can be described by the condition that the embedding $\mathfrak{s}_\mathbb{R} \subset \mathfrak{g}_\mathbb{R}$ respects the Cartan involution ϑ and the choice of complex structure on $\mathfrak{p}$. Satake [Sat80, ch. II, § 8, ch. III] calls such an embedding of an Hermitian Lie algebra an (H$_1$)-homomorphism.

Proposition 2.1.19. For $u \in Z$, define

$$X_u^\pm = \frac{1}{2} \cdot \Big(\xi_{-iu}^- \pm \frac{1}{2} \cdot [\xi_u^-, \xi_{-iu}^-]\Big) = \frac{1}{2} \cdot \Big(\xi_{-iu}^- \pm 2i \cdot u \,\square\, u^*\Big) \ .$$

Let $e \in Z$ be a non-zero tripotent. Then

$$[\xi_e^-, X_e^\pm] = \pm 2 X_e^\pm \quad \text{and} \quad [X_e^+, X_e^-] = \xi_e^- \ .$$

Moreover, if $\mathfrak{s}_\mathbb{R}^e = \langle \xi_e^-, X_e^\pm \rangle$ and $\mathfrak{s}^e = \mathfrak{s}_\mathbb{R}^e \otimes \mathbb{C}$, then

$$\mathfrak{s}^e \cap \mathfrak{k} = \mathbb{C} \cdot (X_e^+ - X_e^-) = \mathbb{C} \cdot e \,\square\, e^* \ ,$$

and

$$\mathfrak{s}^e \cap \mathfrak{p}^\pm = \mathbb{C} \cdot \big(\xi_e^- \pm i(X_e^+ + X_e^-)\big) = \mathbb{C} \cdot \big(\xi_e^- \pm \xi_e^+\big) \ .$$

Hence, $(\xi_e^-, X_e^\pm)$ is a disc embedding. In addition, $\mathfrak{s}^e$ and $\mathfrak{s}^c$ commute if only if e and c are orthogonal.

Proof. First, note the formula

$$\big[[\xi_{ae}^-, \xi_{be}^-], \xi_{ce}^-\big] = 4\,\mathrm{Im}(a\bar{b}) \cdot \xi_{ic\cdot e}^- \quad \text{for all } a, b, c \in \mathbb{C} \ ,$$

whence

$$[\xi_e^-, X_e^\pm] = \frac{1}{2} \cdot [\xi_e^-, \xi_{-ie}^-] \mp \xi_{ie}^- = \pm 2 X_e^\pm \ .$$

Furthermore,

$$[X_e^+, X_e^-] = \frac{1}{4} \cdot \big[[\xi_e^-, \xi_{-ie}^-], \xi_{-ie}^-\big] = \xi_e^- \ .$$

It is immediate that $\mathfrak{s}_\mathbb{R}^e$ is ϑ-stable. Hence, the intersections $\mathfrak{s}^e \cap \mathfrak{k}$ and $\mathfrak{s}^e \cap \mathfrak{p}^\pm$ are at most 1-dimensional. But

$$X_e^+ - X_e^- = \frac{1}{4} \cdot [\xi_e^-, \xi_{-ie}^-] = 2i \cdot e \,\square\, e^*$$

and

$$\xi_e^- \pm i(X_e^+ + X_e^-) = \xi_e^- \pm \xi_e^+ = 2 \cdot \begin{Bmatrix} e \\ -\{ze^*z\} \end{Bmatrix} \frac{\partial}{\partial z} \in \mathfrak{p}^\pm \,.$$

Note $\xi_e^\pm \in \mathfrak{s}^e$ and $\xi_c^\pm \in \mathfrak{s}^c$. Since

$$\left[\xi_e^-, \xi_c^-\right] - \left[\xi_e^-, \xi_c^+\right] = \left[\xi_e^-, \xi_c^- + i\xi_{ic}^-\right] = 4\, e \square c^* \,,$$

e and c are orthogonal if $\left[\mathfrak{s}^e, \mathfrak{s}^c\right] = 0$. Conversely,

$$\left[\xi_e^-, \xi_c^-\right] = \left[e \,\square\, e^*, c \,\square\, c^*\right] = 0$$

whenever e and c are orthogonal, whence the assertion. □

Corollary 2.1.20. Let $e \in Z$ be a non-zero tripotent. We have

$$\operatorname{Ad}(\exp tX_e^\pm)\left(iz\frac{\partial}{\partial z}\right) = iz\frac{\partial}{\partial z} - \frac{t}{2} \cdot \xi_e^- \pm \frac{t^2}{2} \cdot X_e^\pm$$

for all $t \in \mathbb{R}$, and

$$\operatorname{Ad}(\exp \xi_{-ie}^-)(X_e^+) = -X_e^- \,.$$

In particular, $\pm X_e^\pm \in \Omega^-$, $-X_e^- \in \operatorname{Ad}(\exp(\mathfrak{p}_\mathbb{R}))(X_e^+)$ and $\omega^- \subset \Omega^- \cap \mathfrak{t}_\mathbb{R}$.

Proof. We note the formula $\exp \operatorname{Ad}(\xi) = e^{\operatorname{ad}\xi}$ for all $\xi \in \mathfrak{g}_\mathbb{R}$, cf. [Kna02, prop. 1.9]. Now, since $iz\frac{\partial}{\partial z} \in \mathfrak{z}(\mathfrak{k}_\mathbb{R})$,

$$\operatorname{ad}(X_e^\pm)\left(iz\frac{\partial}{\partial z}\right) = -\frac{1}{2} \cdot \left[iz\frac{\partial}{\partial z}, \xi_{-ie}^-\right] = -\frac{1}{2} \cdot \xi_e^- \,,$$

and by proposition 2.1.19,

$$\operatorname{ad}(X_e^\pm)^2\left(iz\frac{\partial}{\partial z}\right) = \frac{1}{2} \cdot [\xi_e^-, X_e^\pm] = \pm X_e^\pm \,,$$

so the higher powers vanish. Hence the first formula. By the definition of Ω^-,

$$\pm X_e^\pm = \lim_{t\to\infty} \frac{2}{t^2} \cdot \operatorname{Ad}(\exp tX_e^\pm)\left(iz\frac{\partial}{\partial z}\right) \in \Omega^- \,.$$

As for the second formula,

$$[\xi_{-ie}^-, X_e^+] = -\frac{1}{4}[[\xi_e^-, \xi_{-ie}^-], \xi_{-ie}^-] = -\xi_{-ie}^- \,,$$

and all higher powers are zero. Hence

$$\mathrm{Ad}\big(\exp \xi^-_{-ie}\big)\big(X^+_e\big) = X^+_e - \xi^-_{-ie} = -X^-_e \ .$$

This proves the formula and the conjugacy of $\pm X^\pm_e$.

We have $i \cdot e \Box e^* = X^+_e - X^-_e \in \Omega^-$ by proposition 2.1.19. Thus, $\omega^- \subset \Omega^-$, by lemma 2.1.13. □

Remark 2.1.21. For an alternative, but similar proof that $X^+_e \in \Omega^-$ (for e primitive), see [Pan83, lem. 4].

Proposition 2.1.22. Let $(\xi, \eta^\pm)$ be a disc embedding. Then there exists a non-zero tripotent $e \in Z$ such that $\xi = \xi^-_e$ and $\eta^\pm = X^\pm_e$.

Proof. Note the relation

$$\frac{1}{2} \cdot \big[\xi, \eta^+ \pm \eta^-\big] = \eta^+ \mp \eta^- \ .$$

Hence

$$B(\xi, \eta^+ + \eta^-) = \frac{1}{2} \cdot B\big(\xi, [\xi, \eta^+ - \eta^-]\big) = \frac{1}{2} \cdot B\big([\xi, \xi], \eta^+ - \eta^-\big) = 0 \ .$$

This shows that $\xi \perp \mathfrak{k}_{\mathbb{R}}$, so $\xi \in \mathfrak{p}_{\mathbb{R}}$ and there exists $u \in Z \setminus 0$ such that $\xi = \xi^-_u$. Since $\eta^+ - \eta^- \in \mathfrak{k}_{\mathbb{R}} = \mathrm{aut}(Z)$, the element $v = -\frac{1}{2} \cdot (\eta^+ - \eta^-)(u) \in Z$ makes sense. Now

$$\xi^-_v = \frac{1}{2} \cdot \big[\xi, \eta^+ - \eta^-\big] = \eta^+ + \eta^- \neq 0 \ .$$

By assumption, $\frac{1}{2} \cdot (\eta^+ - \eta^-)$ acts on $\mathfrak{s}$ just as $iz\,\frac{\partial}{\partial z}$ does, namely, by multiplication with $\pm i$ on $\mathfrak{p}^\pm$. Hence,

$$-\xi^-_v + i\xi^-_u = \frac{1}{2}\big[\eta^+ - \eta^-, \xi^-_u + i\xi^-_v\big] = \left[iz\frac{\partial}{\partial z}, \xi^-_u + i\xi^-_v\right] = \xi^-_{iu} + i\xi^-_{iv} \ .$$

This implies $v = -iu$. Moreover,

$$\xi^-_{i\{uu^*u\}} = \frac{1}{4} \cdot \big[\big[\xi^-_u, \xi^-_{-iu}\big], \xi^-_u\big] = \frac{1}{4}\big[\big[\xi, \eta^+ + \eta^-\big], \xi\big] = \frac{1}{2} \cdot \big[\eta^+ - \eta^-, \xi\big] = \xi^-_{iu} \ .$$

Therefore, $u = \{uu^*u\}$ is a non-zero tripotent. □

Remark 2.1.23. By proposition 2.1.22, disc embeddings — or, equivalently, the (H_1)-homomorphisms from $\mathfrak{sl}(2, \mathbb{R})$ to $\mathfrak{g}_{\mathbb{R}}$ in the sense of Satake — can be rephrased in terms of tripotents and Jordan theory.

2.1.24. It is clear that

$$\mathfrak{a}_{\mathbb{R}} = \big\langle \xi^-_{e_k} \bigm| k = 1, \dots, r \big\rangle \subset \mathfrak{p}_{\mathbb{R}}$$

is an Abelian subalgebra, and by [HC56, lem. 8, cor.], it is maximally so. By [Kna02, prop. 6.40], $\mathfrak{g}_{\mathbb{R}}$ decomposes as

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{m}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \sum\nolimits_{\alpha\in\Delta_{\mathfrak{a}}}^{\oplus} \mathfrak{g}_{\mathbb{R}}^{\alpha}$$

where

$$\mathfrak{m}_{\mathbb{R}} = \mathfrak{z}_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}}) \quad \text{and} \quad \mathfrak{g}_{\mathbb{R}}^{\alpha} = \big\{\xi \in \mathfrak{g}_{\mathbb{R}} \bigm| [\eta,\xi] = \alpha(\eta)\cdot\xi \text{ for all } \eta \in \mathfrak{a}_{\mathbb{R}}\big\}$$

are the *restricted root spaces* and $\Delta_{\mathfrak{a}} = \{\alpha \in \mathfrak{a}_{\mathbb{R}}^{*} \setminus 0 \mid \mathfrak{g}_{\mathbb{R}}^{\alpha} \neq 0\}$ is the set of *restricted roots*. $\Delta_{\mathfrak{a}}$ is an irreducible root system, non-reduced if B is not of tube type.

For any tripotent e, define the *Cayley element*

$$\gamma_e = \exp\big(\tfrac{\pi}{4}\cdot\xi_e^{+}\big) \in \exp i\mathfrak{p}_{\mathbb{R}} .$$

The *Cayley transform* $\mathrm{Ad}(\gamma_e) \in \mathrm{Int}(\mathfrak{g})$ has order 8. Furthermore, if e and c are orthogonal, proposition 2.1.19 shows that $\big[\xi_e^{+},\xi_c^{+}\big] = 0$, so $\gamma_e\gamma_c = \gamma_{e+c}$.

In particular, for $e^k = e_1 + \cdots + e_k$, $1 \leqslant k \leqslant r$, $\gamma_{e^k} = \gamma_{e_1}\cdots\gamma_{e_k}$ leaves $\mathfrak{t}_{\mathbb{R}}^{+}$ pointwise fixed, and $\mathfrak{a}_{\mathbb{R}} = \mathrm{Ad}(\gamma_{e^r})\big(i\mathfrak{t}_{\mathbb{R}}^{-}\big)$. It follows that

$$\Delta_{\mathfrak{a}} \cup 0 = \Delta \circ \mathrm{Ad}(\gamma_{e^r}^{-1})\big|_{\mathfrak{a}_{\mathbb{R}}} .$$

This allows for the definition of a positive system by

$$\Delta_{\mathfrak{a}}^{+} \cup 0 = \Delta^{++} \circ \mathrm{Ad}(\gamma_{e^r}^{-1})\big|_{\mathfrak{a}_{\mathbb{R}}} .$$

We define a nilpotent subalgebra by setting $\mathfrak{n}_{\mathbb{R}} = \sum\nolimits_{\alpha\in\Delta_{\mathfrak{a}}^{+}}^{\oplus} \mathfrak{g}_{\mathbb{R}}^{\alpha}$. Then [Kna02, prop. 6.43] implies that $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}_{\mathbb{R}}$ is the *Iwasawa decomposition* associated to the choices of $\mathfrak{a}_{\mathbb{R}}$ and $\Delta_{\mathfrak{a}}^{+}$. By [Kna02, th. 6.46], we also have a global decomposition $G = KAN$ as a direct product of manifolds, where $A = \exp\mathfrak{a}_{\mathbb{R}}$ and $N = \exp\mathfrak{n}_{\mathbb{R}}$.

We describe the restricted roots explicitly. Define $\alpha_k = \gamma_k \circ \gamma_{e^r}^{-1}$, i.e.

$$\alpha_k\big(\xi_{e_\ell}^{-}\big) = \delta_{k\ell} \quad \text{for all } 0 \leqslant k \leqslant r,\ 1 \leqslant \ell \leqslant r .$$

Set, for $0 \leqslant k \leqslant \ell \leqslant r$ and $\varepsilon^2 = 1$, $\alpha_{k\ell}^{\varepsilon} = \alpha_\ell - \varepsilon\cdot\alpha_k$. Then

$$\Delta_{\mathfrak{a}}^{+} \subset \{\alpha_{k\ell}^{\varepsilon} \neq 0 \mid 0 \leqslant k \leqslant \ell \leqslant r,\ \varepsilon^2 = 1\} .$$

More precisely, $\Delta_{\mathfrak{a}}^{+}$ contains exactly those $\alpha_{k\ell}^{\varepsilon}$ for which

$$\mathfrak{g}_{\mathbb{R}}^{\alpha_{k\ell}^{\varepsilon}} = \Big\{\xi_u^{-} + (2-\delta_{k\ell})\cdot\big(e_k \,\Box\, u^* - u \,\Box\, e_k^*\big) \Bigm| u \in Z_{k\ell},\ k > 0 \ \Rightarrow\ u^* = \varepsilon\cdot u\Big\}$$

is non-zero, where $u^* = \{e^r u^* e^r\}$ for $u \in Z_1(e^r)$, cf. [AU03, lem. 5.1.1].

Lemma 2.1.25. Let $1 \leqslant k \leqslant \ell \leqslant r$. If $\alpha \in \Delta_{\mathfrak{a}}^{+}$ such that $\alpha + \alpha_{k\ell}^{-} \in \Delta_{\mathfrak{a}}^{+}$, then

$$\alpha + \alpha_{k\ell}^{-} = \begin{cases} \alpha_{j\ell}^{-} & 1 \leqslant j \leqslant k\,, \\ \alpha_{kj}^{-} & k \leqslant j \leqslant \ell\,. \end{cases}$$

Proof. Obvious, since $\alpha_{ij}^{-} + \alpha_{pq}^{-} \notin \Delta_{\mathfrak{a}}$ and $\alpha_{ij}^{+} + \alpha_{k\ell}^{-} \in \Delta_{\mathfrak{a}}^{+}$ implies $j \in \{k, \ell\}$. □

Lemma 2.1.26. For $u \in Z$ and $\delta \in \mathrm{aut}(Z) = \mathfrak{k}_{\mathbb{R}}$,

$$[\delta, X_u^{\pm}] = X_{\delta u}^{\pm}\,.$$

In particular, $\mathrm{Ad}(k)(X_u^{\pm}) = X_{ku}^{\pm}$ for all $k \in K$.

Proof. We have

$$\begin{aligned} 2 \cdot [\delta, X_u^{\pm}] &= \xi_{-i\delta u}^{-} \pm [\delta, i \cdot u \,\square\, u^{*}] = \xi_{-i\delta u}^{-} \pm (i\delta u) \,\square\, u^{+} - u \,\square\, (i\delta u)^{*} \\ &= \xi_{-i\delta u}^{-} \pm \frac{1}{2} \cdot [\xi_u^{-}, \xi_{-i\delta u}^{-}] = 2 \cdot X_{\delta u}^{\pm}\,, \end{aligned}$$

whence the assertion. □

Theorem 2.1.27. For a tripotent $e \in Z$, the orbit

$$\mathcal{O}_e = \mathrm{Ad}(G)(X_e^{+}) = -\,\mathrm{Ad}(G)(X_e^{-}) \subset \Omega^{-}$$

depends only on the rank of e. Moreover,

$$\mathrm{rk}\, e \neq \mathrm{rk}\, c \;\Rightarrow\; \mathcal{O}_e \cap \mathcal{O}_c = \varnothing \quad \text{and} \quad \mathrm{rk}\, e \leqslant \mathrm{rk}\, c \;\Rightarrow\; \mathcal{O}_e \subset \overline{\mathcal{O}_c}\,.$$

Finally, for e primitive, we have

$$\mathcal{O}_e = \mathbb{R}_{>} \cdot \mathrm{Ad}(K)(X_e^{+}) = \{\alpha \cdot X_c^{+} \mid \alpha > 0\,, c \text{ primitive tripotent}\}$$

and $\Omega^{-} \setminus 0 = \mathrm{co}(\mathcal{O}_e)$.

Proof. By corollary 2.1.20, $\mathcal{O}_e \subset \Omega^{-}$, and $\pm X_e^{\pm}$ are conjugate. If e and c have equal rank, $k(e) = c$ for some $k \in K$, by [Loo75, cor. 5.12]. Hence, lemma 2.1.26 shows that $\mathrm{Ad}(k)(X_e^{+}) = X_c^{+}$, and hence the orbits are equal.

For the tripotent $e^k = e_1 + \cdots + e_k$,

$$X_{e^k}^{+} = \sum_{j=1}^{k} X_{e_j}^{+} \in \sum\nolimits_{1 \leqslant j \leqslant k}^{\oplus} \mathfrak{g}_{\mathbb{R}}^{\alpha_{jj}^{-}} \subset \sum\nolimits_{1 \leqslant i \leqslant j \leqslant k}^{\oplus} \mathfrak{g}_{\mathbb{R}}^{\alpha_{ij}^{-}}\,.$$

By lemma 2.1.25, the latter space is AN-invariant. By lemma 2.1.26 $X_{e^{k+1}}^{+} \in \mathcal{O}_{e^k}$ if and only if this orbit contains some rank $k+1$ tripotent. But for $\ell \in K$,

$$\ell(e^k) = \ell(e_1) + \cdots + \ell(e_k) \quad \text{where} \quad \ell(e_j)\,,\ j = 1, \ldots, r\,,$$

form a frame. Then the same argument as above applies to the restricted root decomposition associated to the maximal Abelian subspace $\gamma_{\ell(e^r)}(\mathfrak{t}_{\mathbb{R}}^-) \subset \mathfrak{p}_{\mathbb{R}}$. Hence, $\mathcal{O}_{e^k}$ contains a rank $k+1$ tripotent if and only if $X^+_{e^{k+1}}$ is contained in the AN-orbit of $X^+_{e^k}$. Since this is not the case, the orbits must be disjoint.

Let $k \geqslant 2$. For $t \in \mathbb{R}$,

$$\mathrm{Ad}\big(\exp\big(t\xi^-_{e_k}\big)\big)\big(X_{e^k}\big) = e^t \cdot X^+_{e_k} + X^+_{e^{k-1}}\,.$$

In particular,

$$X^+_{e^{k-1}} = \lim_{t\to-\infty} \mathrm{Ad}\big(\exp t\xi^-_{e_k}\big)\big(X^+_{e^k}\big) \in \overline{\mathcal{O}_{e^k}}\,.$$

Now, consider the case of $k = 1$. Lemma 2.1.25 implies

$$\big[\mathfrak{n}_{\mathbb{R}}, X^+_{e_1}\big] \in \big[\mathfrak{n}_{\mathbb{R}}, \mathfrak{g}_{\mathbb{R}}^{\alpha^-_{11}}\big] = 0\,.$$

Moreover, $\big[\mathfrak{a}_{\mathbb{R}}, X^+_{e_1}\big] = \mathbb{R}\cdot X^+_{e_1}$. This proves that

$$\mathrm{Ad}(G)(X^+_{e_1}) = \mathbb{R}_> \cdot \mathrm{Ad}(K)(X^+_{e_1}) = \mathbb{R}_> \cdot \big\{X^+_c \bigm| c \text{ primitive tripotent}\big\}\,,$$

where the last statement follows from lemma 2.1.26.

Clearly, $0 \notin C = \mathrm{co}\big(\mathrm{Ad}(G)(X^+_{e_1})\big)$, C is invariant, and

$$0 \cup C = \mathbb{R}_\geqslant \cdot \mathrm{co}\big(\mathrm{Ad}(K)(X^+_{e_1})\big)$$

is a closed cone, because the convex hull of a compact set is compact.

We contend that this cone is Ω^-. Since we already know by proposition 2.1.19 that $X^+_{e_1} \in \Omega^-$, it remains to be shown that $iz\,\frac{\partial}{\partial z} \in C$. We have $i \cdot e_1 \Box e_1^* = X^+_{e_1} - X^-_{e_1} \in C$ by corollary 2.1.20. Thus, $\omega^- \setminus 0 \subset C$, by lemma 2.1.13. But we have already noted $iz\,\frac{\partial}{\partial z} \in \omega^-$. □

Remark 2.1.28. Theorem 2.1.27 is contained in [Vin80, th. 2] and [HNØ94, th. III.9]. The last step of our proof follows [Pan83, lem. 4].

Hilgert, Neeb and Ørsted [HNØ94] show that all nilpotent orbits of convex type (those contained in the dual of Ω^-) are of the form $\mathcal{O}_e$, $e \in Z$ tripotent. By proposition 2.1.22, this follows in our framework from their refined Jacobson-Morosov theorem [HNØ94, th. II.10].

2.1.29. Define the dual cone

$$\Omega^+ = \Omega^{-*} = \big\{\xi \in \mathfrak{g}_{\mathbb{R}} \bigm| \big(\xi : \mathrm{Ad}(k)(X^+_{e_1})\big) \geqslant 0 \text{ for all } k \in K\big\}\,.$$

By duality, Ω^+ is a maximal pointed invariant cone.

Lemma 2.1.30. We have $\Omega^- \subset \Omega^+$ and $\omega^\pm = \Omega^\pm \cap \mathfrak{t}_\mathbb{R}$.

Proof. The vector field $iz\frac{\partial}{\partial z}$ is fixed by K. Moreover, for all tripotents $e \in Z \setminus 0$,

$$\Bigl(iz\frac{\partial}{\partial z} : X_e^+\Bigr) = -B\Bigl(iz\frac{\partial}{\partial z}, i \cdot e\square e^*\Bigr) = \mathrm{tr}_Z\bigl(e\square e^*\bigr) > 0$$

by lemma 2.1.9. In particular, this is true for $e = e_1$, so $\Omega^- \subset \Omega^+$. If $\xi \in \omega^+$ and $e \in Z$ is a tripotent, then

$$\bigl(\xi : \mathrm{Ad}(k)(X_e^+)\bigr) = \bigl(\xi : \mathrm{Ad}(k)(i \cdot e\square e^*)\bigr) \geqslant 0 \quad \text{for all } k \in K$$

since the projection of $\mathrm{Ad}(K)(i \cdot e\square e^*)$ onto $\mathfrak{t}_\mathbb{R}$ is $\mathrm{co}\bigl(W_c\,(i \cdot e\square e^*)\bigr) \subset \omega^-$ by Kostant's convexity theorem [Kos74]. Hence, $\xi \in \Omega^+$.

Now $\omega^+ \subset \Omega^+ \cap \mathfrak{t}_\mathbb{R}$. But

$$\Omega^+ \cap \mathfrak{t}_\mathbb{R} \subset (\Omega^- \cap \mathfrak{t}_\mathbb{R})^* \subset \omega^{-*} = \omega^+ ,$$

so $\omega^+ = \Omega^+ \cap \mathfrak{t}_\mathbb{R}$. Similarly

$$\Omega^- \cap \mathfrak{t}_\mathbb{R} \subset (\Omega^+ \cap \mathfrak{t}_\mathbb{R})^* = \omega^{+*} = \omega^- ,$$

so $\omega^- = \Omega^- \cap \mathfrak{t}_\mathbb{R}$. □

Remark 2.1.31. By [Pan83, th. 2], the map $\Omega \mapsto \Omega \cap \mathfrak{t}_\mathbb{R}$ is an order-preserving bijection between the set of closed pointed G-invariant cones $\Omega \subset \mathfrak{g}_\mathbb{R}$ with non-trivial interior and the closed W_c-invariant cones $\omega^- \subset \omega \subset \omega^+$. The cones Ω are given by

$$\Omega = \bigl\{\xi \in \mathfrak{g}_\mathbb{R} \bigm| p_\mathfrak{t}\bigl(\mathrm{Ad}(G)(\xi)\bigr) \subset \omega\bigr\}$$

where $p_\mathfrak{t}$ denotes the orthogonal projection onto $\mathfrak{t}_\mathbb{R}$. Moreover, [Pan83, th. 3] shows that $\Omega^* \cap \mathfrak{t}_\mathbb{R} = \bigl(\Omega \cap \mathfrak{t}_\mathbb{R}\bigr)^*$. Moreover, any orbit in Ω° has a non-trivial intersection with the relative interior of $\Omega \cap \mathfrak{t}_\mathbb{R}$.

Lemma 2.1.32. If $\xi \in \Omega^+$ is regular, it is conjugate to an element of $\omega^{+\circ}$, and hence contained in $\Omega^{+\circ}$.

Proof. The centraliser $\mathfrak{g}_\mathbb{R}^\xi$ of ξ is a Cartan subalgebra. By [War72, prop. 1.3.4.1], the subset $U = \mathrm{Ad}(G)(\mathfrak{h}_{\mathbb{R},*})$ is open. Since $\Omega^{+\circ}$ is dense in Ω^+ and $\xi \in U \cap \Omega^+$, we find an element $\eta \in \Omega^{+\circ} \cap U \neq \varnothing$.

Since $\eta \in \Omega^{+\circ}$, η is conjugate to an element ζ of $\mathfrak{t}_\mathbb{R}$, necessarily also regular. Then $\mathfrak{t}_\mathbb{R} = \mathfrak{g}_\mathbb{R}^\zeta$, so $\mathfrak{h}_\mathbb{R} = \mathfrak{g}_\mathbb{R}^\eta$ and $\mathfrak{t}_\mathbb{R}$ are conjugate. Hence, ξ is conjugate to an element of $\mathfrak{t}_\mathbb{R}$.

The remaining assertions follow from $\omega^+ \cap \mathfrak{t}_{\mathbb{R},*} = W_c \cdot c_+ \subset \omega^{+\circ}$, a consequence of proposition 2.1.17. □

2.2 Faces of invariant cones

Given the description of the orbits $\mathcal{O}_e$ in theorem 2.1.27, one might naively suspect that the faces of $\Omega^{\pm}$ have a similar description as in the case of symmetric cones, cf. proposition 1.3.6. In truth, the convex geometry of the cones $\Omega^{\pm}$ is a little more involved.

2.2.1. In this subsection, e and c shall always denote tripotents, without further mention. Associated to e, the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ has a natural $\mathbb{Z}$-grading

$$\mathfrak{g}_{\mathbb{R}}^e[k] = \ker\big(\operatorname{ad}\xi_e^- - k\big) \quad \text{for all } \ k \in \mathbb{Z}\,.$$

By [Loo75, lem. 9.14], $\mathfrak{g}_{\mathbb{R}}^e[k] = 0$ unless $|k| \leqslant 2$. Moreover,

$$\mathfrak{g}_{\mathbb{R}}^e[0] = \mathfrak{k}_{\mathbb{R}}^e \oplus \big\{\xi_u^- \bigm| u \in Z_0(e) \oplus X_1(e)\big\}$$

where $\mathfrak{k}_{\mathbb{R}}^e = \big\{\delta \in \mathfrak{k}_{\mathbb{R}} \bigm| \delta e = 0\big\}$. Furthermore,

$$\mathfrak{g}_{\mathbb{R}}^e[\pm 1] = \big\{\eta_u^{e,\pm} \bigm| u \in Z_{1/2}(e)\big\} \quad \text{and} \quad \mathfrak{g}_{\mathbb{R}}^e[\pm 2] = \big\{\eta_u^{e,\pm} \bigm| u \in iX_1(e)\big\}$$

where $\eta_u^{e,\pm} = \xi_u^- \pm \frac{1}{2}\cdot\big[\xi_e^-, \xi_u^-\big]$. In particular, $\pm 2\cdot X_e^{\pm} = \eta_{ie}^{e,\pm}$.

From the restricted root decomposition of $\mathfrak{g}_{\mathbb{R}}$ (2.1.24) it is clear that

$$\mathfrak{k}_{\mathbb{R}}^e = \big(\mathfrak{k}_{\mathbb{R},0}(e) \oplus \mathfrak{k}_{\mathbb{R},1}(e)\big) + \mathfrak{m}_{\mathbb{R}}$$

where we recall $\mathfrak{k}_{\mathbb{R},0}(e) = \operatorname{aut} Z_0(e)$ and $\mathfrak{k}_{\mathbb{R},1}(e) = \operatorname{aut} X_1(e)$, the set of algebra derivations of the formally real Jordan algebra $X_1(e)$. The sum with $\mathfrak{m}_{\mathbb{R}}$ is usually not direct.

By a *parabolic subalgebra* (or *parabolic* for short) we mean a subalgebra of $\mathfrak{g}_{\mathbb{R}}$ that is its own normaliser and contains a maximal solvable subalgebra. Then it is clear that $\mathfrak{q}_{\mathbb{R}}^e = \mathfrak{g}_{\mathbb{R}}^e[0,1,2]$ is parabolic. To wit, $\mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}_{\mathbb{R}} \subset \mathfrak{q}_{\mathbb{R}}^e$ if $e_j \leqslant e$ or $e_j \perp e$ for all $j = 1, \dots, r$. In fact, by [Loo75, prop. 9.21], $\mathfrak{q}_{\mathbb{R}}^e$ is a maximal proper parabolic.

Let $\mathfrak{n}_{\mathbb{R}}^e = \mathfrak{g}_{\mathbb{R}}^e[1,2]$ be the nilpotent part of $\mathfrak{q}_{\mathbb{R}}^e$. Recall that a *generalised Heisenberg algebra* $\mathfrak{h}_{\mathbb{R}}$ is simply a step 2 nilpotent Lie algebra, i.e. $\big[\mathfrak{h}_{\mathbb{R}}, \mathfrak{h}_{\mathbb{R}}\big] \subset \mathfrak{z}(\mathfrak{h}_{\mathbb{R}})$.

Lemma 2.2.2. The following map is an isomorphism of Lie algebras:

$$(u,v) \mapsto \eta_{u+v}^{e,+} : \mathfrak{h}_{\mathbb{R}}^e = Z_{1/2}(e) \ltimes iX_1(e) \to \mathfrak{n}_{\mathbb{R}}^e$$

where $\mathfrak{h}_{\mathbb{R}}^e$ is a *generalised Heisenberg algebra* with bracket relations

$$\big[(u,v),(u',v')\big] = \big(0, \{u'u^*e\} - \{uu'^*e\}\big)$$

for all $u, u' \in Z_{1/2}(e)$, $v, v' \in iX_1(e)$.

Proof. If $k = 1, 2$, then

$$\Big[\mathfrak{g}_{\mathbb{R}}^e[k], \mathfrak{g}_{\mathbb{R}}^e[2]\Big] \subset \mathfrak{g}_{\mathbb{R}}^e[k+2] = 0\,.$$

Thus, $\mathfrak{g}^e_{\mathbb{R}}[2]$ is central in $\mathfrak{n}^e_{\mathbb{R}}$.

For $u,v \in Z_{1/2}(e)$, $[\eta^{e,+}_u, \eta^{e,+}_v] \in \mathfrak{g}^e_{\mathbb{R}}[2]$ and hence equals $\eta^{e,+}_w$ for some $w \in iX_1(e)$. Since $\eta^{e,+}_w(0) = \xi^-_w(0) = w$,

$$\begin{aligned} w &= [\eta^{e,+}_u, \eta^{e,+}_v](0) = \frac{1}{2} \cdot [\xi^-_u, [\xi^-_e, \xi^-_v]](0) + \frac{1}{2} \cdot [[\xi^-_e, \xi^-_u], \xi^-_v](0) \\ &= \{ve^*u\} - \{ev^*u\} + \{eu^*v\} - \{ue^*v\} = \{vu^*e\} - \{uv^*e\} . \end{aligned}$$

This proves the required bracket relation. □

Remark 2.2.3. The map in lemma 2.2.2 is related to $\mathrm{Ad}(\gamma_e)$, the Cayley transform associated to e. It identifies $\mathfrak{g}^e_{\mathbb{R}}[2]$ with constant vector fields, and $\mathfrak{g}^e_{\mathbb{R}}[1]$ with certain affine vector fields, cf. [Upm85, cor. 21.17].

It is interesting to note that the product $\circ$ of the formally real Jordan algebra $X_1(e)$ can be expressed in terms of bracket relations as

$$\eta^{e,+}_{i\cdot(u\circ v)} = \frac{1}{2} \cdot [\xi^-_u, \eta^{e,+}_{iv}] \quad \text{for all } u,v \in X_1(e) .$$

2.2.4. We define

$$h_e(u,v) = \{uv^*e\} \in Z_1(e) \quad \text{for all } u,v \in Z_{1/2}(e) .$$

Then [Loo75, 10.4] shows that h_e satisfies

$$h_e(u,v)^* = h_e(v,u) \quad \text{and} \quad h_e(u,u) \in \Omega_1(e)$$

where $\Omega_1(e)$ is the closed cone of squares in the formally real Jordan algebra $X_1(e)$. Moreover, $h_e(u,u) = 0$ if and only if $u = 0$. We shall call such an h_e an *$\Omega_1(e)$-positive Hermitian map.*

Associated to h_e is a skew-symmetric map q_e, given by

$$q_e(u,v) = \operatorname{Im} h_e(u,v) = i \cdot \frac{\{vu^*e\} - \{uv^*e\}}{2} \in X_1(e) \quad \text{for all } u,v \in Z_{1/2}(e) .$$

Hence, the bracket of $\mathfrak{h}^e_{\mathbb{R}}$ can be expressed as

$$\big[(u,v),(u',v')\big] = \big(0, -2i \cdot q_e(u,u')\big) \quad \text{for all } (u,v),(u',v') \in Z_{1/2}(e) \ltimes iX_1(e) .$$

Lemma 2.2.5. For any $x \in \Omega_1(e)^\circ$, the form

$$(u,v) \mapsto (q_e(iu,v) \mid x) : Z_{1/2}(e) \times Z_{1/2}(e) \to \mathbb{R}$$

is positive symmetric. In particular, $\mathfrak{z}(\mathfrak{h}^e_{\mathbb{R}}) = iX_1(e)$.

Proof. We have

$$q_e(iu, v) = \frac{i}{2} \cdot \big(\{v(iu)^*e\} - \{(iu)v^*e\}\big) = \frac{\{vu^*e\} + \{uv^*e\}}{2} \,.$$

In particular, the form is symmetric. For $u = v$, we find

$$(q_e(iu, u) \mid x) = (h_e(u, u) \mid x) \geqslant 0 \,,$$

since $\Omega_1(e)$ is self-dual. Since $x \in \Omega_1(e)^\circ$, the face $\Omega_1(e) \cap x^\perp$ dual to the face generated by x is 0. Hence, the form is non-degenerate.

We know that $\mathfrak{z}(\mathfrak{h}^e_{\mathbb{R}}) \subset iX_1(e)$, and if $u \in Z_{1/2}(e) \setminus 0$, then

$$\big[(iu, 0), (u, 0)\big] = \big(0, -2iq_e(iu, u)\big)$$

where $(q_e(iu, u) \mid e) > 0$. So, equality follows. □

Remark 2.2.6. Recall (1.2.2) that $iX_1(e) = X_1(ie)$ is a formally real Jordan algebra with the product induced by the tripotent ie. Therefore, the centre of the generalised Heisenberg algebra $\mathfrak{h}^e_{\mathbb{R}}$ is a formally real Jordan algebra such that the product induces an $i\Omega_1(e)$-positive symmetric form (in the obvious sense). Such Lie algebras $\mathfrak{h}^e_{\mathbb{R}}$ are called *conal Heisenberg algebras* by Hilgert, Neeb and Ørsted, and are the subject of study in [HNØ96].

Since $\mathfrak{h}^e_{\mathbb{R}} = \mathfrak{n}^e_{\mathbb{R}}$ is nilpotent, the exponential map is an isomorphism onto the corresponding *generalised Heisenberg group* $H_e = Z_{1/2}(e) \ltimes iX_1(e)$ with composition law

$$(u, v) \cdot (u', v') = \left(u + u', v + v' + \frac{\{u'u^*e\} - \{uu'^*e\}}{2}\right)$$

for all $u, u' \in Z_{1/2}(e)$, $v, v' \in iX_1(e)$.

The classical Heisenberg group (with one-dimensional centre) corresponds to the special case of a *primitive* tripotent e.

The groups H_e are not the most general kind of generalised Heisenberg groups that can occur. They correspond to *symmetric* Siegel domains D (i.e. D is biholomorphically equivalent to a bounded symmetric domain B), whereas the general case corresponds to homogeneous Siegel domains (of type II) which are not necessarily symmetric.

2.2.7. Let $\Omega_1(e)$ be the closed cone of squares in the formally real Jordan algebra $X_1(e)$. $i\Omega_1(e) \subset iX_1(e) \subset \mathfrak{h}^e_{\mathbb{R}}$ can be thought of a subset of $\mathfrak{g}^e_{\mathbb{R}}[2] \subset \mathfrak{g}_{\mathbb{R}}$ under the identification from lemma 2.2.2.

Applying this consideration to $-e$, we see that $\Omega_1(e) \subset X_1(e) = X_1(-e)$ maps isomorphically into $\mathfrak{g}^e_{\mathbb{R}}[-2]$. We denote its image by $i\Omega_1(-e)$.

Proposition 2.2.8. We have $\Omega^\pm \cap \mathfrak{h}^e_{\mathbb{R}} = i\Omega_1(e)$.

Proof. Let $\varepsilon^2 = 1$. The intersection $\widetilde{\Omega} = \Omega^\varepsilon \cap \mathfrak{h}^e_{\mathbb{R}}$ is a closed pointed cone invariant under inner automorphisms. Hence, [HNØ96, lem. I.13] implies that $\widetilde{\Omega} \subset \mathfrak{g}^e_{\mathbb{R}}[2]$. On the other hand, $X_e^+ \in \widetilde{\Omega}$, and this element corresponds to the unit e of the formally real Jordan algebra $X_1(e)$ under the isomorphisms from lemma 2.2.2 and 1.2.2. Identifying $\widetilde{\Omega}$ with its image in $X_1(e)$, this implies

$$\Omega_1(e) \subset \widetilde{\Omega} \quad \text{and} \quad \widetilde{\Omega}^* \subset \Omega_1(e)^* = \Omega_1(e) \ .$$

Since $\widetilde{\Omega}$ is pointed, the interior of $\widetilde{\Omega}^*$ is non-void. Hence, $\widetilde{\Omega}^*$ contains an interior element $x \in \Omega_1(e)^\circ$. Thus,

$$\Omega_1(e)^\circ = \mathrm{GL}(\Omega_1(e)).x \subset \widetilde{\Omega}^* \subset \Omega_1(e) \ .$$

It follows that $\widetilde{\Omega}^* = \Omega_1(e)$, and by duality, $\widetilde{\Omega} = \Omega_1(e)$. □

Lemma 2.2.9. For $u \in Z_{1/2}(e) \oplus iX_1(e)$ and $c \leqslant e$,

$$\left(\eta_u^{e,+} \ : \ X_c^-\right) = 0 \quad \text{and} \quad \left(\eta_u^{e,-} \ : \ X_c^-\right) = 4i \operatorname{tr}_Z\left(u \,\Box\, c^* - c \,\Box\, u^*\right) \ .$$

Likewise, if $e \leqslant c$,

$$\left(\eta_u^{e,-} \ : \ X_c^+\right) = 0 \quad \text{and} \quad \left(\eta_u^{e,+} \ : \ X_c^+\right) = 4i \operatorname{tr}_Z\left(c \,\Box\, u^* - u \,\Box\, c^*\right) \ .$$

Proof. Note $X_{-c}^+ = -X_c^-$ and $\eta_u^{-e,+} = -\eta_u^{e,-}$. Hence, the second set of formulae follows from the first. Now, $-X_c^- = \frac{1}{2} \cdot \xi_{ic}^- + i \cdot c \,\Box\, c^*$. This implies

$$\begin{aligned} -\left(\eta_u^{e,\pm} \ : \ X_c^-\right) &= \mp\frac{1}{2} \cdot B\left([\xi_e^-, \xi_u^-], i \cdot c \,\Box\, c^*\right) + \frac{1}{2} \cdot B\left(\xi_u^-, \xi_{ic}^-\right) \\ &= \mp\frac{1}{2} \cdot B\left(\xi_u^-, \left[i \cdot c \,\Box\, c^*, \xi_e^-\right]\right) + \frac{1}{2} \cdot B\left(\xi_u^-, \xi_{ic}^-\right) \\ &= \mp\frac{1}{2} \cdot B\left(\xi_u^-, \xi_{ic}^-\right) + \frac{1}{2} \cdot B\left(\xi_u^-, \xi_{ic}^-\right) \end{aligned}$$

which is 0 or

$$B\left(\xi_u^-, \xi_{ic}^-\right) = -4i \operatorname{tr}_Z\left(u \,\Box\, c^* - c \,\Box\, u^*\right) ,$$

by lemma 2.1.9, proving the claim. □

Lemma 2.2.10. Let $\Omega \subset \mathfrak{g}_{\mathbb{R}}$ be a closed cone invariant under $\mathrm{Ad}(\exp t\xi_e^-)$ for all $t \in \mathbb{R}$. If $\xi = \sum_{j=k}^{\ell} \xi_j \in \Omega$ where $\xi_j \in \mathfrak{g}_{\mathbb{R}}^e[j]$, then $\xi_k, \xi_\ell \in \Omega$.

Proof. We have

$$\mathrm{Ad}\left(\exp t\xi_e^-\right)(\xi) = \sum_{j=k}^{\ell} e^{jt} \cdot \xi_j \in \Omega \quad \text{for all} \ \ t \in \mathbb{R} \ .$$

In particular,

$$\xi_k = \lim\nolimits_{t\to\infty} e^{kt} \cdot \mathrm{Ad}\left(\exp -t\xi_e^-\right)(\xi) \in \Omega$$

and

$$\xi_\ell = \lim\nolimits_{t\to\infty} e^{-\ell t} \cdot \mathrm{Ad}\left(\exp t\xi_e^-\right)(\xi) \in \Omega ,$$

proving the lemma. □

2.2.11. Define a convex cone $F_e^{\pm}$ contained in $\Omega^{\pm}$ by $F_e^{\pm} = \Omega^{\pm} \cap \big(X_e^-\big)^{\perp}$.

Proposition 2.2.12. We have

$$F_e^{\pm} = \Omega^{\pm} \cap \big(X_e^-\big)^{\perp} = \Omega^{\pm} \cap \mathfrak{q}_{\mathbb{R}}^e .$$

In particular, $F_e^{\pm}$ is an exposed face of $\Omega^{\pm}$.

Proof. If $e = 0$, then $X_e^- = 0$, $F_e^{\pm} = \Omega^{\pm}$, and $\mathfrak{q}_{\mathbb{R}}^e = \mathfrak{g}_{\mathbb{R}}$. W.l.o.g., we may assume $\operatorname{rk} e > 0$. From corollary 2.1.20, we know that $-X_e^- \in \Omega^- \subset \Omega^+$, so $\big(X_e^-\big)^{\perp}$ is a supporting hyperplane for $\Omega^{\pm}$, and $F_e^{\pm}$ is an exposed face.

Since they are eigenspaces of a symmetric endomorphism ($\vartheta(\xi_e^-) = -\xi_e^-$), the degrees of the grading are mutually orthogonal. In particular, $\mathfrak{q}_{\mathbb{R}}^e \perp X_e^- \in \mathfrak{g}_{\mathbb{R}}^e[-2]$.

For the converse, let $\xi \in F_e^{\pm}$, and write $\xi = \sum_{j=-2}^{2} \xi_j$ where $\xi_j \in \mathfrak{g}_{\mathbb{R}}^e[j]$. Since X_e^- is an eigenvector of $\operatorname{ad} \xi_e^-$, $F_e^{\pm}$ is invariant under $\operatorname{Ad}(\exp t\xi_e^-)$ for all $t \in \mathbb{R}$, so we can employ lemma 2.2.10.

In particular, $\xi_{-2} \in F_e^{\pm}$. Assume $\xi_{-2} \neq 0$. Then $\xi_{-2} = \eta_u^{e,-}$ for some non-zero $u = iv \in iX_1(e)$. By proposition 2.2.8, $v \in \Omega_1(e)$. Since we have $e \in \Omega_1(e)^{\circ}$, and $\Omega_1(e)$ is pointed and self-dual, $(v \mid e) > 0$. By lemma 2.2.9,

$$\big(\xi_{-2} : X_e^-\big) = 4i \cdot \operatorname{tr}_Z\big(u \,\square\, e^* - e \,\square\, u^*\big) = -\frac{8n}{k} \cdot (v \mid e) < 0$$

where $k = \operatorname{rk} e$. Contradiction! We deduce $\xi_{-2} = 0$, so $\xi_{-1} \in \Omega^{\pm}$ as above. But then $\xi_{-1} = 0$ by proposition 2.2.8. □

We have seen that the exposed face $F_e^{\pm}$ is contained in the maximal parabolic $\mathfrak{q}_{\mathbb{R}}^e$, and in particular, invariant under inner automorphisms of $\mathfrak{q}_{\mathbb{R}}^e$. However, this is not the definitive statement on $F_e^{\pm}$.

Namely, the vector subspace of $\mathfrak{q}_{\mathbb{R}}^e$ generated by $F_e^{\pm}$ is a proper ideal of $\mathfrak{q}_{\mathbb{R}}^e$. To determine it explicitly, and to gain more insight into the structure of $F_e^{\pm}$, we need to study $\mathfrak{g}_{\mathbb{R}}^e[0]$ in greater detail.

2.2.13. In the following, we assume, as we may, that the frame $e_1, \dots, e_r$ is such that $e_j \leqslant e$ for $j \leqslant k$ and $e_j \perp e$ for $j > k$, in other words,

$$e = e^k = e_1 + \cdots + e_k .$$

Recall $\mathfrak{m}_{\mathbb{R}} = \mathfrak{z}_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$ where $\mathfrak{a}_{\mathbb{R}} = \langle \xi_{e_j}^- \mid j = 1, \dots, r \rangle$. Let

$$\mathfrak{g}_{\mathbb{R},0}(e) = \mathfrak{k}_{\mathbb{R},0}(e) \oplus \mathfrak{p}_{\mathbb{R},0}(e) = \operatorname{aut} Z_0(e) \oplus \big\{\xi_u^- \;\big|\; u \in Z_0(e)\big\}$$

and

$$\mathfrak{g}_{\mathbb{R},1}(e) = \mathfrak{k}_{\mathbb{R},1}(e) \oplus \mathfrak{p}_{\mathbb{R},1}(e) = \operatorname{aut} X_1(e) \oplus \big\{\xi_u^- \;\big|\; u \in X_1(e)\big\} .$$

Then $\mathfrak{g}_{\mathbb{R},0}(e)$ is the Lie algebra of the facial subgroup G_e, i.e. the set of complete holomorphic vector fields on the circled bounded symmetric domain $B_0(e)$.

Lemma 2.2.14. We have $\mathfrak{g}_{\mathbb{R},1}(e) = \mathrm{Ad}(\gamma_e^{-1})\big(\mathfrak{gl}\,\Omega_1(e)\big)$, and a decomposition

$$\mathfrak{g}^e_{\mathbb{R}}[0] = \mathfrak{g}_{\mathbb{R},0}(e) \oplus \mathfrak{m}^e_{\mathbb{R}} \oplus \mathfrak{g}_{\mathbb{R},1}(e)$$

where $\mathfrak{m}^e_{\mathbb{R}} \subset \mathfrak{m}_{\mathbb{R}} = \mathfrak{z}_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$ is a compact subalgebra. All the factors in the decomposition are ideals of $\mathfrak{g}^e_{\mathbb{R}}[0]$.

Proof. We know from 2.2.1 that there exists a subspace

$$\mathfrak{m}^e_{\mathbb{R}} = \mathfrak{k}^e_{\mathbb{R}} \cap \big(\mathfrak{k}_{\mathbb{R},0}(e) \oplus \mathfrak{k}_{\mathbb{R},1}(e)\big)^{\perp}$$

of $\mathfrak{m}_{\mathbb{R}}$ such that

$$\mathfrak{g}^e_{\mathbb{R}}[0] = \mathfrak{g}_{\mathbb{R},0}(e) \oplus \mathfrak{m}^e_{\mathbb{R}} \oplus \mathfrak{g}_{\mathbb{R},1}(e)\ .$$

If $u \in X_1(e)$,

$$\frac{1}{4}\cdot\big[\xi_e^+,\xi_u^-\big] = \frac{i}{4}\cdot\big[\xi_{-ie}^-,\xi_u^-\big] = \frac{1}{2}\cdot\big(e\,\Box\,u^* + u\,\Box\,e^*\big) = u\,\Box\,e^* = M_u\ .$$

Hence, by [Upm85, lem. 21.16],

$$\mathrm{Ad}(\gamma_e^{-1})(2\cdot M_u) = \xi_u^- \quad\text{and}\quad \mathrm{Ad}(\gamma_e)(\delta) = \delta \quad \text{for all } \delta \in \mathrm{aut}\,X_1(e)\ .$$

This proves that

$$\mathfrak{g}_{\mathbb{R},1}(e) = \mathrm{Ad}(\gamma_e^{-1})\big(\mathfrak{gl}\,\Omega_1(e)\big)\ ,$$

in particular, this is a Lie algebra. The Peirce rules show that it commutes with $\mathfrak{g}_{\mathbb{R},0}(e)$. Since $\mathfrak{m}_{\mathbb{R}}$ leaves the restricted root spaces invariant, the orthogonal complement of $\mathfrak{m}^e_{\mathbb{R}}$ is an ideal of $\mathfrak{g}^e_{\mathbb{R}}[0]$. Hence, so is $\mathfrak{m}^e_{\mathbb{R}}$. ☐

Remark 2.2.15. Note that the factor $\mathfrak{m}^e_{\mathbb{R}}$ in the decomposition is actually independent of the choice of frame.

2.2.16. By [Sat80, prop. 4.4, cor. 4.5], each of the $\mathfrak{g}^e_{\mathbb{R}}[0]$-modules

$$\mathfrak{g}^e_{\mathbb{R}}[1] \cong Z_{1/2}(e) \quad\text{and}\quad \mathfrak{g}^e_{\mathbb{R}}[2] \cong iX_1(e)$$

is trivial or irreducible. Moreover, if non-zero, $\mathfrak{g}^e_{\mathbb{R}}[1]$ is faithful. Let $\mathfrak{l}^e_{\mathbb{R}} = \mathfrak{g}_{\mathbb{R},0}(e) \oplus \mathfrak{m}^e_{\mathbb{R}}$.

Proposition 2.2.17. The algebra $\mathfrak{l}^e_{\mathbb{R}}$ commutes with $iX_1(e)$. On $Z_{1/2}(e)$, the action is

$$\delta.v = \delta(v) \quad\text{and}\quad \xi_u^-.v = -\{uv^*e\}$$

for all $\delta \in \mathfrak{k}_{\mathbb{R},0}(e) \oplus \mathfrak{m}^e_{\mathbb{R}}$, $u \in Z_0(e)$, and $v \in Z_{1/2}(e)$. Moreover,

$$q_e(\xi.u,v) = -q_e(u,\xi.v) \quad \text{for all } \xi \in \mathfrak{l}^e_{\mathbb{R}},\ u,v \in Z_{1/2}(e)\ .$$

Proof. We have

$$\left[\delta,\eta_u^{e,+}\right]=\eta_{\delta u}^{e,+}\quad\text{for all }\ \delta\in\mathfrak{k}_{\mathbb{R}}\,,\ u\in Z_{1/2}(e)\ltimes iX_1(e)\,.$$

Moreover, $\delta(u)=0$ if $\delta\in\mathfrak{k}_{\mathbb{R},0}(e)=\operatorname{aut}Z_0(e)$ and $u\in iX_1(e)$ (Peirce rules). For all derivations $\delta\in\mathfrak{k}_{\mathbb{R}}^e$, $\delta(e)=0$. In particular, $\left[\mathfrak{m}_{\mathbb{R}}^e,X_e^+\right]=0$. By lemma 2.2.14, $\mathfrak{m}_{\mathbb{R}}^e$ is an ideal, and thence commutes with $\mathfrak{g}_{\mathbb{R},0}(e)\oplus\mathfrak{g}_{\mathbb{R},1}(e)$. By irreducibility of $\mathfrak{g}_{\mathbb{R}}^e[2]$, the vector $X_e^+\in\mathfrak{g}_{\mathbb{R}}^e[2]$ is cyclic, so we find

$$\begin{aligned}\left[\mathfrak{m}_{\mathbb{R}}^e,\mathfrak{g}_{\mathbb{R}}^e[2]\right]&=\left[\mathfrak{m}_{\mathbb{R}}^e,\left[\mathfrak{g}_{\mathbb{R}}^e[0],X_e^+\right]\right]\\&=\left[\mathfrak{m}_{\mathbb{R}}^e,\left[\mathfrak{g}_{\mathbb{R},0}(e)\oplus\mathfrak{g}_{\mathbb{R},1}(e),X_e^+\right]\right]\subset\left[\mathfrak{g}_{\mathbb{R}}^e[0],\left[\mathfrak{m}_{\mathbb{R}}^e,X_e^+\right]\right]=0\,.\end{aligned}$$

Furthermore, for all $u\in Z_0(e)$, $v\in Z_{1/2}(e)\oplus iX_1(e)$,

$$\left[\xi_u^-,\eta_v^{e,+}\right](0)=-\{ev^*u\}+\{ve^*u\}=-\{uv^*e\}\,.$$

If $v\in iX_1(e)$, this is zero. In particular, for $\xi\in\mathfrak{l}_{\mathbb{R}}^e$, $u,v\in Z_{1/2}(e)$,

$$\begin{aligned}-2i\cdot q_e(\xi.u,v)&=\left[\left[\xi,\eta_u^{e,+}\right],\eta_v^{e,+}\right](0)\\&=\left[\xi,\left[\eta_u^{e,+},\eta_v^{e,+}\right]\right](0)-\left[\eta_u^{e,+},\left[\xi,\eta_v^{e,+}\right]\right](0)=2i\cdot q_e(u,\xi.v)\end{aligned}$$

by the Jacobi identity. ☐

2.2.18. By the preceding proposition 2.2.17, the semi-direct products

$$\mathfrak{l}_{\mathbb{R}}^e\ltimes\mathfrak{h}_{\mathbb{R}}^e\quad\text{and}\quad\mathfrak{g}_{\mathbb{R},0}(e)\ltimes\mathfrak{h}_{\mathbb{R}}^e$$

make sense. Moreover, the reductive factors $\mathfrak{l}_{\mathbb{R}}^e$ and $\mathfrak{g}_{\mathbb{R},0}(e)$ leave the 'vector-valued symplectic form' q_e invariant, i.e. they act as 'generalised symplectic Lie algebras'. For this reason, $\mathfrak{l}_{\mathbb{R}}^e\ltimes\mathfrak{h}_{\mathbb{R}}^e$ and $\mathfrak{g}_{\mathbb{R},0}(e)\ltimes\mathfrak{h}_{\mathbb{R}}^e$ will be called *generalised Jacobi algebras*.

By lemma 2.2.14, the generalised Jacobi algebras $\mathfrak{l}_{\mathbb{R}}^e\ltimes\mathfrak{h}_{\mathbb{R}}^e$ and $\mathfrak{g}_{\mathbb{R},0}(e)\ltimes\mathfrak{h}_{\mathbb{R}}^e$ are ideals of the parabolic $\mathfrak{q}_{\mathbb{R}}^e$.

Lemma 2.2.19. Under the assumption from 2.2.13, $\mathfrak{t}_{\mathbb{R}}\cap\mathfrak{m}_{\mathbb{R}}=\mathfrak{t}_{\mathbb{R}}^+$, and

$$\mathfrak{t}_{\mathbb{R}}^e[0]=\mathfrak{t}_{\mathbb{R}}\cap\mathfrak{g}_{\mathbb{R}}^e[0]=\langle i\cdot e_j\Box e_j^*\mid j=k+1,\dots,r\rangle\oplus\mathfrak{t}_{\mathbb{R}}^+\subset\mathfrak{l}_{\mathbb{R}}^e\,.$$

The subalgebras $\mathfrak{g}_{\mathbb{R}}^e[0]$, $\mathfrak{g}_{\mathbb{R},0}(e)$, and $\mathfrak{m}_{\mathbb{R}}^e$ of $\mathfrak{g}_{\mathbb{R}}$ are $\mathfrak{t}_{\mathbb{R}}$-invariant. Moreover,

$$\mathfrak{t}_{\mathbb{R},0}(e)=\mathfrak{t}_{\mathbb{R}}\cap\mathfrak{g}_{\mathbb{R},0}(e)\quad\text{and}\quad\mathfrak{t}_{\mathbb{R}}^+\cap\mathfrak{m}_{\mathbb{R}}^e$$

are Cartan subalgebras of, respectively, $\mathfrak{g}_{\mathbb{R},0}(e)$ and $\mathfrak{m}_{\mathbb{R}}^e$.

Proof. Since $[\delta, \xi^-_{e_j}] = \xi^-_{\delta e_j}$ for all $j = 1, \ldots, r$ and $\delta \in \mathfrak{k}_{\mathbb{R}}$, we have

$$\mathfrak{t}^+_{\mathbb{R}} = \{\delta \in \mathfrak{t}_{\mathbb{R}} \mid \delta e_j = 0\,,\, j = 1, \ldots, r\} \subset \mathfrak{m}_{\mathbb{R}}\,.$$

Since $\{e_j e_j^* e_j\} = e_j \neq 0$, $\mathfrak{t}_{\mathbb{R}} \cap \mathfrak{m}_{\mathbb{R}} \subset \mathfrak{t}^+_{\mathbb{R}}$.

Moreover, $i \cdot e_j \Box e_j^* \in \mathfrak{k}_{\mathbb{R},0}(e)$ if $j > k$, and if $j \leqslant k$,

$$[\delta, i \cdot e_j \Box e_j^*] = i \cdot (\delta e_j) \Box e_j^* + i \cdot e_j \Box (\delta e_j)^* = 0 \quad \text{for all } \delta \in \mathfrak{k}^e_{\mathbb{R}}$$

and

$$[\xi^-_u, i \cdot e_j \Box e_j^*] = -\xi^-_{i\{e_j e_j^* u\}} = 0 \quad \text{for all } u \in Z_0(e)\,.$$

We conclude that $\mathfrak{l}^e_{\mathbb{R}}$ is $\mathfrak{t}_{\mathbb{R}}$-invariant, and that

$$\mathfrak{t}^e_{\mathbb{R}}[0] = \langle i \cdot e_j \Box e_j^* \mid j = k+1, \ldots, r\rangle \oplus \mathfrak{t}^+_{\mathbb{R}} \subset \mathfrak{l}^e_{\mathbb{R}}\,.$$

In addition, by [Bou68, ch. VIII, § 3.1, prop. 3], $\mathfrak{t}_{\mathbb{R},0}(e)$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R},0}(e)$, and $\mathfrak{m}^e_{\mathbb{R}} \cap \mathfrak{t}^+_{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{m}^e_{\mathbb{R}}$. □

2.2.20. Let $\Omega^-_0(e)$ denote the minimal cone of the Lie algebra $\mathfrak{g}_{\mathbb{R},0}(e)$, cf. definition 1.2.5. Likewise, let $\Omega^+_0(e) \subset \mathfrak{g}_{\mathbb{R},0}(e)$ be the dual cone of $\Omega^-_0(e)$.

Then, if we set $\omega^{\pm}_0(e) = \Omega^{\pm}_0(e) \cap \mathfrak{t}_{\mathbb{R},0}(e)$, lemma 2.1.30 shows that

$$\omega^+_0(e) = \omega^-_0(e)^* \quad \text{and} \quad \omega^-_0(e) = \langle iH_\alpha \mid \alpha \in \Delta^{++}_n\,,\, \mathfrak{g}^\alpha \subset \mathfrak{g}_0(e)\rangle\,.$$

Here, of course, $\mathfrak{g}_0(e)$ denotes the complexification of $\mathfrak{g}_{\mathbb{R},0}(e)$, and the set

$$\{\alpha \in \Delta^{++}_n \mid \mathfrak{g}^\alpha \subset \mathfrak{g}_0(e)\}$$

coincides with the set of positive non-compact roots for $\mathfrak{g}_{\mathbb{R},0}(e)$, since this algebra is $\mathfrak{t}_{\mathbb{R}}$-invariant and ϑ-invariant, cf. [Bou68, ch. VIII, § 3.1, prop. 3].

2.2.21. For what follows, recall that a Lie algebra is said to be *reductive* if its adjoint representation is semi-simple. A Lie algebra $\mathfrak{a}$ is *quasi-Hermitian*, if $\mathfrak{a} = \mathfrak{z}_{\mathfrak{b}}(\mathfrak{z}(\mathfrak{a} \cap \mathfrak{b}))$ for some maximal compact subalgebra $\mathfrak{b} \subset \mathfrak{a}$. If $\mathfrak{a}$ is simple and non-compact, it is called *Hermitian* if some maximal compact subalgebra has non-trivial centre. A reductive Lie algebra $\mathfrak{a}$ is quasi-Hermitian if and only if it is the direct sum of a maximal compact ideal and Hermitian simple ideals.

Lemma 2.2.22. The centre of $\mathfrak{k}_{\mathbb{R},1}(e)$ is trivial. In particular, if $\operatorname{rk} e \geqslant 2$, then the derived algebra $\mathfrak{g}_{\mathbb{R},1}(e)' = [\mathfrak{g}_{\mathbb{R},1}(e), \mathfrak{g}_{\mathbb{R},1}(e)]$ is a non-compact, non-Hermitian simple Lie algebra. If $\operatorname{rk} e \leqslant 1$, $\mathfrak{g}_{\mathbb{R},1}(e) = \mathbb{R} \cdot \xi^-_e$ is Abelian.

Proof. By lemma 2.2.14, $\operatorname{Ad}(\gamma_e)(\mathfrak{g}_{\mathbb{R},1}(e)) = \mathfrak{gl}\,\Omega_1(e)$. We know from 1.3.2 that $\mathfrak{gl}\,\Omega_1(e)$ is reductive with centre $\mathbb{R} \cdot M_e$, and the derived algebra $\mathfrak{gl}\,\Omega_1(e)'$ is simple since $X_1(e)$

is simple, whenever e is non-zero. If $\operatorname{rk} e \geqslant 2$, there exists a non-trivial idempotent $c \in X_1(e)$, $0 < c < e$. Then $M_c \subset \mathfrak{gl}\,\Omega_1(e)'$ generates an unbounded 1-parameter group, so $\mathfrak{gl}\,\Omega_1(e)'$ is non-compact.

Let $\delta \in \mathfrak{k}_{\mathbb{R},1}(e) = \operatorname{aut} X_1(e)$ be central. Since $\delta e = 0$, for all $u \in X_1(e)$,

$$0 = \left[\delta, \left[\xi_u^-, \xi_e^-\right]\right] = \left[\xi_{\delta u}^-, \xi_e^-\right] + \left[\xi_u^-, \xi_{\delta e}^-\right] = 2 \cdot (\delta u) \,\square\, e^* = 2 \cdot M_{\delta u} \,.$$

This implies $\delta u = 0$, so $\delta = 0$. ∎

Proposition 2.2.23. We have

$$\Omega^\pm \cap \mathfrak{g}_{\mathbb{R}}^e[0] = \Omega_0^\pm(e) \quad \text{and} \quad \omega^\pm \cap \mathfrak{g}_{\mathbb{R}}^e[0] = \omega_0^\pm(e) \,.$$

In particular,

$$F_e^\pm = \Omega^\pm \cap \left(\mathfrak{g}_{\mathbb{R},0}(e) \ltimes \mathfrak{h}_{\mathbb{R}}^e\right) \,.$$

Proof. If $e = 0$, there is nothing to prove. W.l.o.g. we assume $e \neq 0$. From lemma 2.2.14, we have

$$\mathfrak{g}_{\mathbb{R}}^e[0] = \mathfrak{g}_{\mathbb{R},0}(e) \oplus \mathfrak{m}_{\mathbb{R}}^e \oplus \mathfrak{g}_{\mathbb{R},1}(e)$$

where $\mathfrak{m}_{\mathbb{R}}^e \subset \mathfrak{m}_{\mathbb{R}} = \mathfrak{z}_{\mathfrak{k}_{\mathbb{R}}}(\mathfrak{a}_{\mathbb{R}})$ is a compact reductive ideal of $\mathfrak{g}_{\mathbb{R},0}(e)$.

Now, $\mathfrak{g}_{\mathbb{R},0}(e) \oplus \mathfrak{m}_{\mathbb{R}}^e$ is $\mathfrak{t}_{\mathbb{R}}$-invariant by lemma 2.2.19. Since $\mathfrak{k}_{\mathbb{R}} \perp \mathfrak{p}_{\mathbb{R}}$ and

$$\left(i \cdot e_j \,\square\, e_j^* \,:\, \delta\right) = -2i \cdot \operatorname{tr}_Z\left((\delta e_j) \,\square\, e_j^*\right) = 0 \quad \text{for all } \delta \in \mathfrak{k}_{\mathbb{R}} \cap \mathfrak{g}_{\mathbb{R}}^e[0],\, j \leqslant k \,,$$

the orthogonal projection $p_{\mathfrak{t}}$ onto $\mathfrak{t}_{\mathbb{R}}$ leaves $\mathfrak{g}_{\mathbb{R}}^e[0]$ invariant. Thus,

$$p_{\mathfrak{t}}\left(\Omega^- \cap \mathfrak{g}_{\mathbb{R}}^e[0]\right) = \omega^- \cap \mathfrak{g}_{\mathbb{R}}^e[0] \subset \mathfrak{t}_{\mathbb{R}}^e[0] \,.$$

From the lemma 2.2.19 and lemma 2.1.13, $\omega^- \cap \mathfrak{g}_{\mathbb{R},0}(e) = \omega_0^-(e) \subset \mathfrak{t}_{\mathbb{R},0}(e)$. Hence, $\Omega^- \cap \mathfrak{g}_{\mathbb{R},0}(e) = \Omega_0^-(e)$, by [Pan83, th. 2].

Let $\widetilde{\Omega}^\pm = \Omega^\pm \cap \mathfrak{g}_{\mathbb{R}}^e[0]$. Then $\widetilde{\Omega}^\pm$ is closed, pointed, and invariant under inner automorphisms. In particular, $\mathfrak{a}^\pm = \widetilde{\Omega}^\pm - \widetilde{\Omega}^\pm$ is an ideal of $\mathfrak{g}_{\mathbb{R}}^e[0]$. Since $\mathfrak{g}_{\mathbb{R}}^e[0]$ is reductive, so is $\mathfrak{a}^\pm$. By [Nee96, prop. II.2 and lem. II.4], $\mathfrak{a}^\pm$ is quasi-Hermitian. Lemma 2.2.22 implies $\mathfrak{a}^\pm \cap \mathfrak{g}_{\mathbb{R},1}(e) = 0$, since $\mathfrak{a}^\pm$ has neither proper non-compact Abelian nor non-Hermitian simple ideals. We conclude

$$\widetilde{\Omega}^\pm \subset \mathfrak{a}^\pm \subset \mathfrak{g}_{\mathbb{R},0}(e) \oplus \mathfrak{m}_{\mathbb{R}}^e \,.$$

Let $\xi \in \omega^+ \cap \mathfrak{m}_{\mathbb{R}}^e$. Seeking a contradiction, assume $\xi \neq 0$. By definition of ω^+, there exists $\alpha \in \Delta_n^{++}$ such that $\alpha(\xi) > 0$. Since $\left[\xi_e^-, \xi\right] = 0$, the one-dimensional root space $\mathfrak{g}^\alpha \subset \left[\xi, \mathfrak{p}^+\right] \subset \mathfrak{p}^+$ is $\operatorname{ad} \xi_e^-$-invariant, and hence contained $\mathfrak{g}^e[k]$, for some k. Since

$[\xi_e^-, \mathfrak{g}^\alpha] \subset \mathfrak{g}^\alpha$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, necessarily $k = 0$. But $\mathfrak{m}^e$ is an ideal of $\mathfrak{g}^e[0]$, so

$$\mathfrak{g}^\alpha \subset [\mathfrak{m}^e, \mathfrak{g}^\alpha] \subset \mathfrak{m}^e \cap \mathfrak{p}^+ = 0\,,$$

a contradiction. Therefore, $\omega^+ \cap \mathfrak{m}^e_{\mathbb{R}} = 0$. Since

$$\mathfrak{t}_{\mathbb{R}} \cap \bigl(\mathfrak{g}_{\mathbb{R},0}(e) \oplus \mathfrak{m}^e_{\mathbb{R}}\bigr) = \mathfrak{t}_{\mathbb{R},0}(e) \oplus \mathfrak{m}^e_{\mathbb{R}} \cap \mathfrak{t}^+_{\mathbb{R}}\,,$$

the projections $p_{\mathfrak{t}}$ and $p_{\mathfrak{m}^e}$ commute, and

$$p_{\mathfrak{m}^e}\bigl(\omega^\pm \cap \mathfrak{g}^e_{\mathbb{R}}[0]\bigr) = \omega^\pm \cap \mathfrak{m}^e_{\mathbb{R}} = 0\,.$$

Consequently, $\omega^- \cap \mathfrak{g}^e_{\mathbb{R}}[0] = \omega_0^-(e)$, and this entails $\widetilde{\Omega}^- = \Omega_0^-(e)$.

As for the dual cone, clearly $\Omega^+ \cap \mathfrak{g}^e_{\mathbb{R}}[0] \subset \Omega_0^-(e)^* = \Omega_0^+(e)$. In particular, we have the inclusion $\omega^+ \cap \mathfrak{g}^e_{\mathbb{R}}[0] \subset \omega_0^+(e)$. Conversely, if $\alpha \in \Delta_n^{++}$ is non-vanishing on $\omega_e^+(0)$, we have seen above that $\mathfrak{g}^\alpha \subset \mathfrak{p}^+ \cap \mathfrak{g}^e[0]$. If $\mathfrak{g}^\alpha \not\subset \mathfrak{g}_0(e)$, then, since $\mathfrak{g}^\alpha$ is one-dimensional and intersects $\mathfrak{m}^e$ trivially, $\mathfrak{g}^\alpha \subset \mathfrak{g}_1(e)$. Since $\alpha(\mathfrak{t}_0(e)) \neq 0$,

$$\mathfrak{g}^\alpha \subset [\mathfrak{t}_0(e), \mathfrak{g}_1(e)] \subset [\mathfrak{g}_0(e), \mathfrak{g}_1(e)] = 0\,,$$

which is a contradiction. So, $\mathfrak{g}^\alpha \subset \mathfrak{g}_0(e)$. By 2.2.20, this means that α is a root for $\mathfrak{g}_{\mathbb{R},0}(e)$. We have established that all $-i\alpha$, $\alpha \in \Delta_n^{++}$, are $\geqslant 0$ on $\omega_0^+(e)$. Hence,

$$\omega_0^+(e) \subset \omega^+ \cap \mathfrak{g}^e_{\mathbb{R}}[0]\,,$$

and equality follows. Now, [Pan83, th. 2] shows that $\Omega^+ \cap \mathfrak{g}^e_{\mathbb{R}}[0] = \Omega_0^+(e)$. ∎

Remark 2.2.24. It is appropriate to acknowledge that Neeb’s result [Nee00b, prop. I.11] (also available as [Nee00a, prop. VIII.3.30]) is quite similar to proposition 2.2.23. The latter part our proof owes much to the argument found there.

Now we have gained more insight into the structure of the exposed faces $F_e^\pm$, we can consider more general faces.

Lemma 2.2.25. Let $e \geqslant c$ be tripotents. Then

$$\mathfrak{h}^e_{\mathbb{R}} \cap \mathfrak{q}^c_{\mathbb{R}} = \bigl\{\eta_u^{e,+} \bigm| u \in iX_1(e-c) \oplus Z_{1/2}(e) \cap Z_0(c) \oplus Z_{1/2}(e) \cap Z_{1/2}(c) \oplus iX_1(c)\bigr\}\,.$$

In terms of the grading $\mathfrak{g}^c_{\mathbb{R}}[j]$, the first two factors correspond to $j = 0$, and the other two to $j = 1$ and $j = 2$, respectively.

Proof. Note first that $[\xi_e^-, \xi_c^-] = 2(e\,\square\,c^* - c\,\square\,e^*) = 2(c\,\square\,c^* - c\,\square\,c^*) = 0$, so the $\operatorname{ad}\xi_e^-$-eigenspaces are $\operatorname{ad}\xi_c^-$-invariant and vice versa. Thus,

$$\mathfrak{h}^e_{\mathbb{R}} \cap \mathfrak{q}^c_{\mathbb{R}} = \sum\nolimits^{\oplus}_{\lambda=1,2\,,\,\mu=0,1,2} \mathfrak{g}^e_{\mathbb{R}}[\lambda] \cap \mathfrak{g}^c_{\mathbb{R}}[\mu]\,.$$

Thus, we may consider the levels of the gradings individually and refer to the explicit form of the decomposition.

First, let us consider the case $\mu > 0$. The equation $\eta_u^{e,+} = \eta_v^{c,+}$ gives, by Cartan decomposition, $\xi_u^- = \xi_v^-$, and thus $u = v$. Consequently, the equality of the $\mathfrak{k}_{\mathbb{R}}$ components is equivalent to

$$(e-c)\,\square\, u^* + u\,\square\,(e-c)^* = 0\,.$$

If $(\lambda,\mu) = (1,2)$, then $u \in Z_{1/2}(e) \cap iX_1(c) = 0$. If we have that $(\lambda,\mu) = (2,1)$, then $u \in iX_1(e) \cap Z_{1/2}(c)$. Hence

$$0 = \{(e-c)u^*e\} - \{u(e-c)^*e\} = -u - \{cu^*c\} - \{cu^*e-c\} - \tfrac{1}{2}\cdot u = -\tfrac{3}{2}\cdot u\,,$$

because $\{cu^*c\} \in Z_{3/2}(c) = 0$ and

$$0 = \{(e-c)u^*c\} - \{u(e-c)^*c\} = \{cu^*(e-c)\}\,,$$

so $u = 0$. In case $(\lambda,\mu) = (1,1)$, we have $u \in Z_{1/2}(e) \cap Z_{1/2}(c) = Z_0(e-c) \cap Z_{1/2}(c)$, and conversely, this implies the condition explicated above for the equality of the $\mathfrak{k}_{\mathbb{R}}$ components. If $(\lambda,\mu) = (2,2)$, we have $u \in iX_1(e) \cap iX_1(c) = iX_1(c)$. Then again the Peirce rules give the converse inclusion.

Consider now the case $\mu = 0$. Then we need to determine the solutions of the equation $\eta_u^{e,+} = \xi_v^- + \delta$ where $v \in X_1(c) \oplus Z_0(c)$ and $\delta \in \mathfrak{k}_{\mathbb{R}}$ is such that $\delta(c) = 0$. This equality is equivalent to $u = v$ and

$$0 = \{eu^*c\} - \{ue^*c\} = \{eu^*c\} - \{uc^*c\}\,.$$

If $\lambda = 1$, we point out that $X_1(c) \subset Z_1(e) \perp Z_{1/2}(e)$, so $u \in Z_{1/2}(e) \cap Z_0(c)$. In this case, the $\mathfrak{k}_{\mathbb{R}}$ condition is always satisfied. If $\lambda = 2$, $u \in iX_1(e) \cap (Z_0(c) \oplus X_1(c))$ write $u = u_0 + u_1$ where $u_0 \in Z_0(c)$ and $u_1 \in X_1(c)$. Then

$$\begin{aligned}-u_1 - u_0 = -u = \{eu^*e\} &= \{eu_1{}^*e\} + \{eu_0{}^*e\} = \{cu_1{}^*c\} + \{(e-c)u_0{}^*(e-c)\}\\ &= u_1 + \{(e-c)u_0{}^*(e-c)\}\,.\end{aligned}$$

By the Peirce rules, $\{(e-c)u_0{}^*(e-c)\} \in Z_0(c)$, so we read off from the decomposition that $u_1 = -u_1 = 0$ and

$$u = u_0 = -\{(e-c)u_0{}^*(e-c)\} = \{(e-c)u^*(e-c)\}\,,$$

which means precisely that $u \in iX_1(e-c)$. By the Peirce rules, this conversely implies the $\mathfrak{k}_{\mathbb{R}}$ condition. This completes our proof. □

2.2.26. Denote $\mathfrak{g}_{\mathbb{R}}^e = \mathfrak{g}_{\mathbb{R},0}(e) \ltimes \mathfrak{h}_{\mathbb{R}}^e$. By a flag f of tripotents we mean a decreasing sequence $c_1 > \cdots > c_m$ of tripotents. If $f = (c_1 > \cdots > c_m > 0)$ is such a flag, we set

$c_{m+1} = 0$ and define

$$\mathfrak{h}_{\mathbb{R}}^{f} = \left\{ \eta_u^{c_1,+} \,\middle|\, u \in \sum_{j=1}^{m}{}^{\oplus} Z_{1/2}(c_1) \cap Z_{1/2}(c_j) \cap Z_0(c_{j+1}) \oplus \sum_{j=1}^{m}{}^{\oplus} iX_1(c_j - c_{j+1}) \right\} .$$

Proposition 2.2.27. Let $f = (c_1 > \cdots > c_m > 0)$ be a flag of non-zero tripotents. Then

$$\bigcap_{j=1}^{m} \mathfrak{g}_{\mathbb{R}}^{c_j} = \mathfrak{g}_{\mathbb{R},0}(c_1) \ltimes \mathfrak{h}_{\mathbb{R}}^{f} .$$

In particular, $\mathfrak{h}_{\mathbb{R}}^{f}$ is a $\mathfrak{g}_{\mathbb{R},0}(c_1)$-invariant subalgebra of $\mathfrak{g}_{\mathbb{R}}$.

Proof. Once the equation is proven, the subspace $\mathfrak{h}_{\mathbb{R}}^{f}$ turns out to be the nilpotent radical of the intersection.

We prove the equation by induction on the length m of the flag f . If $m = 1$, the statement is trivial, and if $m = 2$, it follows from lemma 2.2.25.

So, let $m \geqslant 2$ and the statement be true for all $1 \leqslant m' < m$. Setting $e_j = c_j - c_{j+1}$ for all $1 \leqslant j \leqslant m$, we find that $e = (e_j)$ is a system of mutually orthogonal tripotents, so the vector fields $\xi_{e_j}^{-}$ commute. The vector fields $\xi_{c_j}^{-}$ have the same span and hence also commute. Denote their joint eigenspaces by $\mathfrak{g}_{\mathbb{R}}^{c_1,\dots,c_m}[\lambda_1, \dots, \lambda_m]$.

The inductive hypothesis implies that

$$\bigcap_{j=1}^{m-1} \mathfrak{g}_{\mathbb{R}}^{c_j} = \mathfrak{g}_{\mathbb{R},0}(c_1) \ltimes \sum_{i=1}^{2}{}^{\oplus} \sum_{j=1}^{m-1}{}^{\oplus} \mathfrak{g}_{\mathbb{R}}^{c_1,\dots,c_{m-1}}[\underbrace{i, \dots, i}_{j}, 0, \dots, 0]$$

where $\mathfrak{g}_{\mathbb{R}}^{c_1,\dots,c_{m-1}}[i, \dots, i, 0, \dots, 0]$ (for j positive eigenvalues) consists of the $\eta_u^{c_1,+}$ with

$$u \in \begin{cases} Z_{1/2}(c_1) \cap Z_{1/2}(c_j) \cap Z_0(c_{j+1}) & i = 1 \,,\, 0 \leqslant j < m-1 \,, \\ Z_{1/2}(c_1) \cap Z_{1/2}(c_{m-1}) & i = 1 \,,\, j = m-1 \,, \\ iX_1(c_j - c_{j+1}) & i = 2 \,,\, 0 \leqslant j < m-1 \,, \\ iX_1(c_{m-1}) & i = 2 \,,\, j = m-1 \,. \end{cases}$$

We need to determine the intersection of these with $\mathfrak{g}_{\mathbb{R}}^{c_m}[\lambda]$ where $\lambda = 0, 1, 2$. To that end, observe

$$\mathfrak{g}_{\mathbb{R}}^{c_1,\dots,c_{m-1}}[\underbrace{i, \dots, i}_{j}, 0, \dots, 0] \cap \mathfrak{g}_{\mathbb{R}}^{c_m}[\lambda] \subset \begin{cases} \mathfrak{g}_{\mathbb{R}}^{c_{m-1},c_m}[0, \lambda] & 0 \leqslant j < m-1 \,, \\ \mathfrak{g}_{\mathbb{R}}^{c_{m-1},c_m}[i, \lambda] & j = m-1 \,, \end{cases}$$

for all $i = 1, 2$ and $\lambda = 0, 1, 2$. The $m = 2$ case shows that this is non-zero only if $\lambda = 0$

or $j = m-1$ and $i = \lambda$. The assertion follows, once we notice that

$$Z_{1/2}(c_1) \cap Z_{1/2}(c_{m-1}) \cap Z_{1/2}(c_m) = Z_{1/2}(c_1) \cap Z_{1/2}(c_m) \,,$$

which follows from lemma 2.2.28 below. $\square$

Lemma 2.2.28. Let $c_1 \geqslant c_2 \geqslant c_3$ be tripotents. Then

$$Z_{1/2}(c_1) \cap Z_{1/2}(c_2) \cap Z_{1/2}(c_3) = Z_{1/2}(c_1) \cap Z_{1/2}(c_3) \,,$$

and in particular, this is independent of c_2.

Proof. Let $e_1 = c_1 - c_2$, $e_2 = c_2 - c_3$ and $e_3 = c_3$. Then e_1, e_2, e_3 are mutually orthogonal tripotents, and we may consider the associated joint Peirce decomposition. We have $c_j = e_1 + \cdots + e_j$, $j = 1, 2, 3$, so [Loo75, th. 3.14] gives

$$Z_{1/2}(c_1) = Z_{10} \oplus Z_{20} \oplus Z_{30} \quad , \quad Z_{1/2}(c_2) = Z_{20} \oplus Z_{30} \oplus Z_{21} \oplus Z_{31} \,,$$
$$Z_{1/2}(c_3) = Z_{30} \oplus Z_{31} \oplus Z_{32} \,.$$

Therefore, both sides of the equation are equal to Z_{30}, which proves our claim. $\square$

2.2.29. Let $f = (c_1 > \cdots > c_m > 0 = c_{m+1})$ be a flag of tripotents. Define

$$\mathfrak{g}_{\mathbb{R}}^{f} = \mathfrak{g}_{\mathbb{R},0}(c_1) \ltimes \mathfrak{h}_{\mathbb{R}}^{f} = \bigcap_{j=1}^{m} \left(\mathfrak{g}_{\mathbb{R},0}(c_j) \ltimes \mathfrak{h}_{\mathbb{R}}^{c_j}\right) .$$

More generally, for $I \subset \{1, \ldots, m\}$, define

$$\mathfrak{g}_{\mathbb{R}}^{f,I} = \mathfrak{g}_{\mathbb{R},0}(c_1) \ltimes \mathfrak{h}_{\mathbb{R}}^{f,I}$$

where

$$\mathfrak{h}_{\mathbb{R}}^{f,I} = \left\{ \eta_u^{c_1,+} \;\middle|\; u \in {\sum_{j\in I}}^{\oplus} Z_{1/2}(c_1) \cap Z_{1/2}(c_j) \cap Z_0(c_{j+1}) \oplus {\sum_{j\in I}}^{\oplus} iX_1(c_j - c_{j+1}) \right\} .$$

It is easy to see that $\mathfrak{h}_{\mathbb{R}}^{f,I}$ is a Heisenberg type Lie algebra whose centre corresponds to the sum of the Peirce-1-spaces occuring in the above definition. To prove its $\mathfrak{g}_{\mathbb{R},0}(c_1)$-invariance, we give the following description of $\mathfrak{g}_{\mathbb{R}}^{f,I}$.

Proposition 2.2.30. Let $f = (c_1 > \cdots > c_m > 0 = c_{m+1})$ be a flag of tripotents and $I \subset \{1, \cdots, m\}$. Then

$$\mathfrak{g}_{\mathbb{R}}^{f,I} = \mathfrak{g}_{\mathbb{R}}^{f} \cap \bigcap_{j \notin I} \mathfrak{q}_{\mathbb{R}}^{c_{j+1}-c_j} .$$

In particular, $\mathfrak{h}_{\mathbb{R}}^{f,I}$ is the nilpotent radical of the Lie algebra $\mathfrak{g}_{\mathbb{R}}^{f,I}$.

Proof. We prove our claim by decreasing induction on $\#I$. For $\#I = m$, the statement is clear. It remains to prove

$$\mathfrak{q}_{\mathbb{R}}^{c_{k+1}-c_k}\cap\mathfrak{g}_{\mathbb{R}}^{f,I}=\mathfrak{g}_{\mathbb{R}}^{f,I\setminus k}\quad\text{whenever}\quad k\in I\,.$$

Since $\xi^-_{c_{k+1}-c_k}=\xi^-_{c_{k+1}}-\xi^-_{c_k}$ and the $\xi^-_{c_j}$ commute, we find

$$\begin{aligned}\mathfrak{q}_{\mathbb{R}}^{c_{k+1}-c_k}\cap\mathfrak{g}_{\mathbb{R}}^{f,I}&=\mathfrak{g}_{\mathbb{R},0}(c_1)\oplus\sum_{\ell\in I}^{\oplus}\sum_{i=1}^{2}{}^{\oplus}\sum_{j=0}^{2}{}^{\oplus}\mathfrak{g}_{\mathbb{R}}^{c_{k+1}-c_k}[j]\cap\mathfrak{g}_{\mathbb{R}}^{c_1,\cdots,c_m}[\underbrace{i,\dots,i}_{\ell},0,\dots,0]\\&=\mathfrak{g}_{\mathbb{R},0}(c_1)\oplus\sum_{\ell\in I}^{\oplus}\sum_{i=1}^{2}{}^{\oplus}\mathfrak{g}_{\mathbb{R}}^{c_{k+1}-c_k}[0]\cap\mathfrak{g}_{\mathbb{R}}^{c_1,\cdots,c_m}[\underbrace{i,\dots,i}_{\ell},0,\dots,0]\,,\end{aligned}$$

because the eigenvalues λ_j of $\xi^-_{c_j}$ on $\mathfrak{g}_{\mathbb{R}}^{f}$ always satisfy $\lambda_j\geqslant\lambda_{j+1}$. Now,

$$\mathfrak{g}_{\mathbb{R}}^{c_{k+1}-c_k}[0]\cap\mathfrak{g}_{\mathbb{R}}^{c_1,\cdots,c_m}[\underbrace{i,\dots,i}_{\ell},0,\dots,0]=\begin{cases}\mathfrak{g}_{\mathbb{R}}^{c_1,\dots,c_m}[\underbrace{i,\dots,i}_{\ell},0,\dots,0] & \ell<k\text{ or }k+1<\ell\,,\\ 0 & \ell=k\end{cases}$$

for all $i=1,2$, since to lie in $\mathfrak{g}_{\mathbb{R}}^{c_{k+1}-c_k}[0]$ means that $\lambda_k=\lambda_{k+1}$. But

$$\sum_{i=1}^{2}{}^{\oplus}\mathfrak{g}_{\mathbb{R}}^{c_1,\dots,c_m}[\underbrace{i,\dots,i}_{k},0,\dots,0]=\mathfrak{h}_{\mathbb{R}}^{f,k}\,,$$

which proves the assertion. □

2.2.31. Let $f=(c_1>\cdots>c_m>0=c_{m+1})$ be a flag of tripotents and $I\subset\{1,\dots,m\}$. Consider the subgroup $H_{f,I}$ of H_e associated to $\mathfrak{h}_{\mathbb{R}}^{e,c}$. Then the subgroup of G associated to $\mathfrak{g}_{\mathbb{R}}^{f,I}=\mathfrak{g}_{\mathbb{R},0}(c_1)\ltimes\mathfrak{h}_{\mathbb{R}}^{f,I}$ is the semi-direct product $G_{f,I}=G_{c_1}\ltimes H_{f,I}$, cf. [Loo75, th. 9.15].

Now, set

$$F^{\pm}_{f,I}=\Omega^{\pm}\cap\bigcap_{j=1}^{m}(X^-_{c_j})^{\perp}\cap\bigcap_{j\notin I}(X^+_{c_j-c_{j+1}})^{\perp}\,.$$

Clearly, this is an exposed face of $\Omega^{\pm}$.

Corollary 2.2.32. Let $f=(c_1>\cdots>c_m>0=c_{m+1})$ be a flag and $I\subset\{1,\dots,m\}$. We have

$$F^{\pm}_{f,I}=\Omega^{\pm}\cap\bigcap_{j=1}^{m}\mathfrak{q}_{\mathbb{R}}^{c_j}\cap\bigcap_{j\notin I}\mathfrak{q}_{\mathbb{R}}^{c_{j+1}-c_j}=\Omega^{\pm}\cap\mathfrak{g}_{\mathbb{R}}^{f,I}\,.$$

Proof. First, recall that $X^+_e=-X^-_{-e}$. Thus, the first equation follows from proposition 2.2.12. The second equation thus follows from propositions 2.2.23, 2.2.27 and 2.2.30. □

2.2.33. To see that $\mathfrak{g}_{\mathbb{R}}^{f,I}$ is spanned by $F^{\pm}_{f,I}$, it is useful to consider, for $c_1\geqslant c_2\geqslant c_3\geqslant 0$, the subalgebra $\mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3} = \mathfrak{g}_{\mathbb{R},0}(c_1) \ltimes \mathfrak{h}_{\mathbb{R}}^{c_1,c_2,c_3}$ where

$$\mathfrak{h}_{\mathbb{R}}^{c_1,c_2,c_3} = \big\{\eta_u^{e,\pm} \bigm| u \in Z_{1/2}(c_1) \cap Z_{1/2}(c_2) \cap Z_0(c_3) \oplus iX_1(c_2 - c_3)\big\} .$$

Indeed,

$$\mathfrak{h}_{\mathbb{R}}^{f,I} = \sum_{j\in I}^{\oplus} \mathfrak{h}_{\mathbb{R}}^{c_1,c_j,c_{j+1}} \quad \text{and} \quad \mathfrak{g}_{\mathbb{R}}^{f,I} = \sum_{j\in I} \mathfrak{g}_{\mathbb{R}}^{c_1,c_j,c_{j+1}}$$

whenever $f = (c_1 > \cdots c_m > 0 = c_{m+1})$ and $I \subset \{1, \ldots, m\}$.

Thus, to see that $F_{f,I}^{\pm}$ generates $\mathfrak{g}_{\mathbb{R}}^{f,I}$, it suffices to prove that $\mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3} \cap \Omega^{\pm}$ spans $\mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}$ for any tripotents $c_1 \geqslant c_2 \geqslant c_3$.

Fix tripotents $c_1 \geqslant c_2 \geqslant c_3$ and let $G^{c_1,c_2,c_3} = G_{c_1} \ltimes \exp \mathfrak{h}_{\mathbb{R}}^{c_1,c_2,c_3}$ be the analytic subgroup of G corresponding to the subalgebra $\mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}$ of $\mathfrak{g}_{\mathbb{R}}$. For all $\xi \in \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}$, consider the adjoint orbit

$$\mathcal{O}_{\xi}^{c_1,c_2,c_3} = \mathrm{Ad}\big(G^{c_1,c_2,c_3}\big)(\xi) \subset \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}$$

Moreover, define $\Omega^{\pm}(c_1, c_2, c_3)$ to be the set

$$\big\{\xi \in \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3} \bigm| \mathcal{O}_{\xi}^{c_1,c_2,c_3} \cap \big(\mathfrak{g}_{\mathbb{R},0}(c_1) \times iX_1(c_2 - c_3)\big) \subset \Omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3)\big\} .$$

The set $\mathfrak{t}_{\mathbb{R}}^{c_1,c_2,c_3} = \mathfrak{t}_{\mathbb{R},0}(c_1) \times iX_1(c_2 - c_3)$ is manifestly a compactly embedded Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}$. Denote by $p_{\mathfrak{t}}^{c_1,c_2,c_3}$ the projection onto $\mathfrak{t}_{\mathbb{R}}^{c_1,c_2,c_3}$ along $\big[\mathfrak{t}_{\mathbb{R}}^{c_1,c_2,c_3}, \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}\big]$.

Lemma 2.2.34. The set $\Omega^{\pm}(c_1, c_2, c_3)$ is a G^{c_1,c_2,c_3}-invariant convex cone, and

$$\Omega^{\pm}(c_1, c_2, c_3) = \big\{\xi \in \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3} \bigm| p_{\mathfrak{t}}^{c_1,c_2,c_3}\big(\mathcal{O}_{\xi}^{c_1,c_2,c_3}\big) \subset \omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3)\big\} .$$

In particular, $\Omega^{\pm}(c_1, c_2, c_3)$ is closed.

Proof. By linearity of the adjoint action,

$$\mathcal{O}_{\alpha\cdot\xi}^{c_1,c_2,c_3} = \alpha \cdot \mathcal{O}_{\xi}^{c_1,c_2,c_3} \quad \text{and} \quad \mathcal{O}_{\xi+\eta}^{e,c} \subset \mathcal{O}_{\xi}^{c_1,c_2,c_3} + \mathcal{O}_{\eta}^{c_1,c_2,c_3} \quad \text{for all } \alpha \geqslant 0\,,\ \xi, \eta \in \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3} .$$

Hence, $\Omega^{\pm}(c_1, c_2, c_3)$ is a convex cone, and the invariance is trivial.

Abbreviate $p_{\mathfrak{t}} = p_{\mathfrak{t}}^{c_1,c_2,c_3}$. Since $p_{\mathfrak{t}}\big(\Omega_0^{\pm}(c_1)\big) = \omega_0^{\pm}(c_1)$ by [Nee00a, prop. V.2.2] and lemma 2.1.30, $\Omega^{\pm}(c_1, c_2, c_3)$ is certainly contained in the right hand side of the equation stated above. For the converse, let $\xi \in \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}$ be such that

$$p_{\mathfrak{t}}\big(\mathcal{O}_{\xi}^{c_1,c_2,c_3}\big) \subset \omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3) .$$

Let $(\eta, \zeta) \in \mathcal{O}_{\xi}^{c_1,c_2,c_3} \cap \big(\mathfrak{g}_{\mathbb{R},0}(c_1) \times iX_1(c_2 - c_3)\big)$. Then G_{c_1} fixes ζ , and

$$p_{\mathfrak{t}}\big(\mathrm{Ad}(G_{c_1})(\eta)\big) \subset \omega_0^{\pm}(c_1) .$$

By [Pan83, th. 2], $\eta \in \mathrm{Ad}(G_{c_1})(\eta) \subset \Omega_0^{\pm}(c_1)$. Hence the lemma. $\square$

Proposition 2.2.35. The cone $\Omega^{\pm}(c_1, c_2, c_3)$ generates $\mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}$ as a vector space. We have

$$\mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3} \cap \Omega^{\pm} = \Omega^{\pm}(e, c) = \mathrm{Ad}\big(G^{c_1,c_2,c_3}\big)\big(\Omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3)\big)\,,$$

and in particular, the cone $\Omega^{\pm}(c_1, c_2, c_3)$ is closed.

Proof. Let $C = \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_2} \cap \Omega^{\pm}$. This cone is clearly G^{c_1,c_2,c_3}-invariant. Lemma 2.2.10, and propositions 2.2.23 and 2.2.8 readily entail the relation $C \subset \Omega^{\pm}(c_1, c_2, c_3)$. In particular,

$$\omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3) \subset \mathfrak{t}_{\mathbb{R}}^{c_1,c_2,c_3} \cap \Omega^{\pm} \subset \mathfrak{t}_{\mathbb{R}}^{c_1,c_2,c_3} \cap \Omega^{\pm}(c_1, c_2, c_3)$$

By lemma 2.2.34, we have

$$\mathfrak{t}_{\mathbb{R}}^{c_1,c_2,c_3} \cap \Omega^{\pm}(c_1, c_2, c_3) \subset \omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3)\,,$$

and thus, equality. Since $\omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3)$ generates $\mathfrak{t}_{\mathbb{R}}^{c_1,c_2,c_3}$, the cone

$$\big\{\xi \in \mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3} \;\big|\; p_{\mathfrak{t}}\big(\mathcal{O}_{\xi}^{c_1,c_2,c_3}\big) \subset \omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3)\big\}$$

generates $\mathfrak{g}_{\mathbb{R}}^{c_1,c_2,c_3}$, by [HHL89, prop. III.5.14 (ii)]. By lemma 2.2.34, this is $\Omega^{\pm}(c_1, c_2, c_3)$. Moreover, [HHL89, prop. III.5.14 (i)] shows that this is the smallest G^{c_1,c_2,c_3}-invariant cone containing $\omega_0^{\pm}(c_1) \times i\Omega_1(c_2 - c_3) = \mathfrak{t}_{\mathbb{R}}^{c_1,c_2,c_3} \cap C$, and is thus contained in C. This proves the proposition. □

Corollary 2.2.36. Let $f = (c_1 > \cdots > c_m > 0 = c_{m+1})$ be a flag of tripotents and $I \subset \{1, \cdots, m\}$. The face $F_{f,I}^{\pm} = \mathfrak{g}_{\mathbb{R}}^{f,I} \cap \Omega^{\pm}$ generates $\mathfrak{g}_{\mathbb{R}}^{f,I}$ as a vector space, and

$$\begin{aligned} F_{f,I}^{\pm} &= \Omega^{\pm}(f, I) = \sum_{j\in I} \Omega^{\pm}(c_1, c_j, c_{j+1}) \\ &= \left\{\xi \in \mathfrak{g}_{\mathbb{R}}^{f,I} \;\middle|\; \mathcal{O}_{\xi}^{f,I} \cap \Big[\mathfrak{g}_{\mathbb{R},0}(c_1) \oplus \sum_{j\in I} iX_1(c_j - c_{j+1})\Big] \subset \Omega_0^{\pm}(c_1) \oplus \sum_{j\in I} i\Omega_1(c_j - c_{j+1})\right\}, \end{aligned}$$

where $\mathcal{O}_{\xi}^{f,I} = \mathrm{Ad}(G_{f,I})(\xi)$.

Remark 2.2.37. This shows that $\Omega^{-}(f, I) = F_{f,I}^{-}$ is, for the Lie algebra $\mathfrak{g}_{\mathbb{R}}^{f,I}$, the minimal invariant cone W_{min} defined by Neeb in [Nee00b, def. I.3] (see also [Nee00a, ch. VIII.3, (3.1)]). Our proof in proposition 2.2.35 that $\Omega^{\pm}(f, I)$ is generating in $\mathfrak{g}_{\mathbb{R}}^{f,I}$ is similar to [Nee00b, prop. I.11] (also [Nee00a, prop. VIII.3.30]). The reader should note that Neeb's result is much more general.

Theorem 2.2.38. Let $f_i = (c_{i1} > \cdots > c_{im_i} > 0 = c_{i,m_i+1})$, $i = 1, 2$, be flags of tripotents, and $I_i \subset \{1, \ldots, m_i\}$, $i = 1, 2$. Let

$$r_i = \mathrm{rk}\, c_{i1} \quad \text{and} \quad R_i = \big\{\mathrm{rk}\big(c_{ij} - c_{i,j+1}\big) \;\big|\; j \in I_i\big\}\,.$$

The exposed faces $F^{\pm}_{f_1,I_1}$ and $F^{\pm}_{f_2,I_2}$ are K-conjugate if only if they are G-conjugate, if and only if $r_1 = r_2$ and $R_1 = R_2$.

First, note the following lemma.

Lemma 2.2.39. Let $c_1 > c_2 > c_3 \geqslant 0$ be tripotents. Then

$$Z_{1/2}(c_1) \cap Z_{1/2}(c_2) \cap Z_0(c_3) = Z_{1/2}(c_1) \cap Z_{1/2}(c_2 - c_3) \ .$$

In particular, $F^{\pm}_{f,I}$, where $f = (c_1 > \cdots > c_m > 0 = c_{m+1})$ and $I \subset \{1, \ldots, m\}$, depends only on c_1 and the $c_j - c_{j+1}$ for $j \in I$.

Proof. Consider the mutually orthogonal tripotents $e_3 = c_1 - c_2$, $e_2 = c_2 - c_3$, and $e_1 = c_3$, and their common Peirce spaces

$$Z_{ij} = Z_{ji} = \left\{ z \in Z \;\middle|\; \{e_k {e_k}^* z\} = \frac{\delta_{ik} + \delta_{jk}}{2} \cdot z \text{ for all } k = 1,2,3 \right\} \quad \text{for all } 0 \leqslant ij \leqslant 3$$

Then $c_j = e_1 + \cdots + e_j$, so by [Loo75, th. 3.14], we have

$$\begin{gathered} Z_{1/2}(c_1) = Z_{10} \oplus Z_{20} \oplus Z_{30} \ , \ Z_{1/2}(c_2) = Z_{10} \oplus Z_{20} \oplus Z_{13} \oplus Z_{23} \ , \\ Z_{1/2}(c_2 - c_3) = Z_{20} \oplus Z_{21} \oplus Z_{23} \ , \ Z_0(c_3) = Z_{00} \oplus Z_{20} \oplus Z_{30} \oplus Z_{22} \oplus Z_{23} \oplus Z_{33} \ . \end{gathered}$$

Thus, both sides of the equation equal Z_{20}. ☐

Proof of theorem 2.2.38. Assume $r_1 = r_2$ and $R_1 = R_2$. There are frames $e_{i1}, \ldots, e_{ir}$ of Z and subsets $A_{ij} \subset \{1, \ldots, r_1\}$, $j = 1, \ldots, \#R_1$, such that $A_{ik} \cap A_{i\ell} = \varnothing$ for $k \neq \ell$, and

$$c_{i1} = e_1 + \cdots + e_{r_1} \ , \ c_{ij} - c_{i,j+1} = \sum_{k \in A_{ij}} e_{ik} \quad \text{for all } i = 1,2 \, , \ j \in I_i \ .$$

There is $\ell \in K$ such that $\ell(e_{1k}) = e_{2k}$ for all $k = 1, \ldots, r$. This clearly implies the conjugacy of $\mathfrak{g}_{\mathbb{R}}^{f_1,I_1}$ and $\mathfrak{g}_{\mathbb{R}}^{f_2,I_2}$.

Conversely, if $F^{\pm}_{f_i,I_i}$, $i = 1,2$, are G-conjugate, then $\mathrm{Ad}(g)(\mathfrak{g}_{\mathbb{R}}^{f_1,I_1}) = \mathfrak{g}_{\mathbb{R}}^{f_2,I_2}$ for some $g \in G$. We have

$$\mathrm{Ad}(g)(\mathfrak{h}_{\mathbb{R}}^{f_1,I_1}) = \mathfrak{h}_{\mathbb{R}}^{f_2,I_2} \quad \text{and} \quad \mathrm{Ad}(g)\Bigl({\sum_{j \in I_1}}^{\oplus} iX_1(c_{1j} - c_{1,j+1}) \Bigr) = {\sum_{j \in I_2}}^{\oplus} iX_1(c_{2j} - c_{2,j+1}) \ ,$$

since the Lie algebra isomorphism $\mathrm{Ad}(g)$ identifies the nilpotent radicals and the centres of the two algebras.

Moreover, $\mathrm{Ad}(g)(\mathfrak{g}_{\mathbb{R},0}(c_{11}))$ is a Levi complement of $\mathfrak{h}_{\mathbb{R}}^{f_2,I_2}$ invariant under the compact Cartan subalgebra

$$\mathrm{Ad}(g)\bigl(\mathfrak{t}_{\mathbb{R},0}(c_{11}) \times \mathfrak{z}(\mathfrak{g}_{\mathbb{R}}^{f_1,I_1})\bigr) = \mathrm{Ad}(g)\bigl(\mathfrak{t}_{\mathbb{R},0}(c_{11}) \times \mathfrak{z}(\mathfrak{g}_{\mathbb{R}}^{f_2,I_2})\bigr) \ ,$$

cf. [Nee00a, th. VII.2.26]. Similarly, $\mathfrak{g}_{\mathbb{R},0}(c_{21})$ is a $\mathfrak{t}_{\mathbb{R},0}(c_{21}) \times \mathfrak{z}(\mathfrak{g}_{\mathbb{R}}^{f_2,I_2})$-invariant Levi complement. Since, for any compact Cartan subalgebra $\mathfrak{h}_{\mathbb{R}}$, the $\mathfrak{h}_{\mathbb{R}}$-invariant Levi complement is unique by [Nee00a, VII.2.5] and compact Cartan subalgebras are G_{f_2,I_2}-conjugate by [Nee00a, th. VII.1.4],

$$\mathrm{Ad}(hg)\big(\mathfrak{g}_{\mathbb{R},0}(c_{11})\big) = \mathfrak{g}_{\mathbb{R},0}(c_{21}) \quad \text{for some } h \in G_{f_2,I_2} \,.$$

W.l.o.g., we may assume $h = 1$.

It is clear that $\mathrm{Ad}(g)\big(\mathfrak{k}_{\mathbb{R},0}(c_{11})\big)$ is a maximal compact subalgebra of $\mathfrak{g}_{\mathbb{R},0}(c_{21})$. Denoting the Killing forms of $\mathfrak{g}_{\mathbb{R},0}(c_{i1})$ by B_i, for $i = 1, 2$, we find

$$\begin{aligned} B_2\big(\mathrm{Ad}(g)\xi, \mathrm{Ad}(g)\eta\big) &= \mathrm{tr}_{\mathfrak{g}_{\mathbb{R},0}(c_{21})}\big(\mathrm{ad}\big(\mathrm{Ad}(g)\xi\big)\,\mathrm{ad}\big(\mathrm{Ad}(g)\eta\big)\big) \\ &= \mathrm{tr}_{\mathfrak{g}_{\mathbb{R},0}(c_{21})}\big(\mathrm{Ad}(g)\,\mathrm{ad}\,\xi\,\mathrm{ad}\,\eta\,\mathrm{Ad}(g^{-1})\big) \\ &= \mathrm{tr}_{\mathfrak{g}_{\mathbb{R},0}(c_{11})}\big(\mathrm{ad}\,\xi\,\mathrm{ad}\,\eta\big) = B_1(\xi, \eta) \end{aligned}$$

for all $\xi, \eta \in \mathfrak{g}_{\mathbb{R},0}(c_{11})$, whence

$$\mathfrak{g}_{\mathbb{R},0}(c_{21}) = \mathrm{Ad}(g)\big(\mathfrak{k}_{\mathbb{R},0}(c_{11})\big) \oplus \mathrm{Ad}(g)\big(\mathfrak{p}_{\mathbb{R},0}(c_{11})\big)$$

is a Cartan decomposition, by [Hel78, prop. 7.4]. By [Hel78, th. 7.2] and [Kna02, th. 6.51],

$$r - \mathrm{rk}\, c_{11} = \text{real rk}\, \mathfrak{g}_{\mathbb{R},0}(c_{11}) = \text{real rk}\, \mathfrak{g}_{\mathbb{R},0}(c_{21}) = r - \mathrm{rk}\, c_{21} \,,$$

and therefore $r_1 = r_2$. Moreover, there exists $h \in G_{c_{12}}$, such that

$$\mathrm{Ad}(gh)\big(\mathfrak{k}_{\mathbb{R},0}(c_{11})\big) = \mathfrak{k}_{\mathbb{R},0}(c_{21}) \quad \text{and} \quad \mathrm{Ad}(gh)\big(\mathfrak{p}_{\mathbb{R},0}(c_{11})\big) = \mathfrak{p}_{\mathbb{R},0}(c_{21}) \,.$$

W.l.o.g., $h = 1$. If we let $\mathfrak{a}_{\mathbb{R}}^1 = \big\langle \xi_{e_{11}}^-, \dots, \xi_{e_{1r}}^- \big\rangle$ where e_{1j}, $j = 1, \dots, r$, is some frame such that the c_{1j} are sums of e_{1k}, then $\mathrm{Ad}(g)(\mathfrak{a}_{\mathbb{R}}^1)$ is a maximal Abelian subspace of $\mathfrak{p}_{\mathbb{R}}$ such that the span of $\xi_{e_{1j}}^-$ where $e_{1j} \leqslant c_{11}$ is a maximal Abelian subspace of $\mathfrak{p}_{\mathbb{R},0}(c_{12})$. (Because $r_1 = r_2$.) By K-conjugate of maximal Abelian subspaces, we may assume that $\mathfrak{a}_{\mathbb{R}}^2 = \mathrm{Ad}(g)(\mathfrak{a}_{\mathbb{R}}^1)$ is the span of $\xi_{e_{2j}}^-$ where e_{2j}, $j = 1, \dots, r$, is some frame such that the c_{2j} are sums of e_{2k}.

For each j, $Z_{1/2}(c_{i1}) \cap Z_{1/2}(c_{ij} - c_{i,j+1})$ is a simple $\mathfrak{g}_{\mathbb{R},0}(c_{i1}) + \mathfrak{a}_{\mathbb{R}}^i$-module, and these simple modules are mutually inequivalent because $\mathfrak{a}_{\mathbb{R}}^i$ acts differently. Hence, for each $j \in I_1$, there exists exactly one $\alpha(j) \in I_2$, such that

$$\mathrm{Ad}(g)\big(Z_{1/2}(c_{11}) \cap Z_{1/2}(c_{1j} - c_{1,j+1})\big) = Z_{1/2}(c_{21}) \cap Z_{1/2}(c_{2,\alpha(j)} - c_{2,\alpha(j)+1}) \,.$$

Since $\dim Z_{1/2}(c_{i1}) \cap Z_{1/2}(c_{ij} - c_{i,j+1}) = r_i - \mathrm{rk}\, c_{ij} + \mathrm{rk}\, c_{i,j+1} + 1$ for all i, j, we find that $R_1 = R_2$. □

2.3 Stratification of Ol'shanskiĭ semigroups

We now turn to the study of certain complex domain defined by the cones $\Omega^\pm$, the so-called *Ol'shanskiĭ domains*. Infinitesimally, they look like the tube $\mathfrak{g}_\mathbb{R} + i\Omega^\pm$. To give a rigorous introduction to these domains, we first have to recall the construction of a complexification $G^\mathbb{C}$ of the real group G.

2.3.1. For any pair $(u,v) \in Z \times Z$, define the *Bergman operator*

$$B(u,v) = 1 - 2 \cdot u \,\Box\, v^* + Q_u Q_v \in \operatorname{End} Z\,.$$

We say that (u,v) is *quasi-invertible* if there exists $w \in Z$ such that

$$B(u,v)\,w = u - Q_u(v) \quad \text{and} \quad B(u,v)\,Q_w(v) = Q_u(v)\,.$$

In this case, $B(u,v)$ is invertible, and the *quasi-inverse* w is given by

$$w = u^v = B(u,v)^{-1}\big(u - Q_u(v)\big)\,.$$

It can be seen that

$$(u,v) \sim (u',v') \;\Leftrightarrow\; (u, v - v') \text{ quasi-invertible, } u' = u^{v-v'}$$

defines an equivalence relation on $Z \times Z$, cf. [Loo75, 7.6]. We denote the class of (u,v) by $[u:v]$, and the quotient space by

$$B^* = Z \times Z/\sim = [Z:Z]\,.$$

2.3.2. We can inject Z into B^* via $u \mapsto [u:0]$. More generally, let

$$B^*_v = \big\{[u:v] \bigm| u \in Z\big\} \subset B^*\,.$$

Then $\varphi_v : [u:v] \mapsto u : B^*_v \to Z$, is a bijection. Moreover, by [Loo75, prop. 7.7], there exists a unique structure of smooth algebraic variety on B^*, such that the B^*_v are open affine subvarieties, and the φ_v are isomorphisms of algebraic varieties. (Cf. [Hum75] for the basic definitions of algebraic geometry.) In particular, the subset $Z = B^*_0 \subset B^*$ is open and dense.

In fact, B^* is a projective variety by [Loo75, th. 7.10], and in particular compact. Hence, B^* is a compactification of the vector space Z. B^* can be viewed as a 'generalised Grassmannian', since, for the triple $Z = \mathbb{C}^{p\times q}$ of rectangular matrices, $B^* = \mathrm{Gr}_p(\mathbb{C}^{p+q})$, cf. [Loo75, 7.11].

2.3.3. By [Loo75, prop. 8.2], the set $\operatorname{Aut}(B^*)$ of automorphisms of B^* has a unique structure of algebraic group, so that for any action $H \times B^* \to B^*$ of an algebraic group H

which is a morphism of algebraic varieties, the associated map $H \to \mathrm{Aut}(B^*)$ is an homomorphism of algebraic groups.

As a smooth projective variety, B^* has a complex structure, and $\mathrm{Aut}(B^*)$ coincides with the group of holomorphic automorphisms of B^*, cf. [Loo75, 8.3]. In particular, it is a complex Lie group in the compact-open topology. The connected component $G^{\mathbb{C}} = \mathrm{Aut}_0(B^*)$ (in the Zariski or compact-open topology) acts transitively on B^*, by [Loo75, cor. 8.5]. The Lie algebra of $G^{\mathbb{C}}$ is $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$, the set of all holomorphic vector fields on Z. Furthermore, $G \subset G^{\mathbb{C}}$ since every automorphism of B extends uniquely to B^*, by [Loo75, prop. 9.4]. Hence, $G^{\mathbb{C}}$ is a complexification of G. Note that $Z(G^{\mathbb{C}}) = 1$, by [Loo75, cor. 8.8].

The Cartan involution ϑ has an extension to $G^{\mathbb{C}}$ whose fixed group is the compact connected subgroup $U = K \cdot \exp i\mathfrak{p}_{\mathbb{R}}$ with Lie algebra $\mathfrak{u}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \oplus i\mathfrak{p}_{\mathbb{R}}$. Moreover, U acts transitively on B^*, and K is the fixed group at $0 \in Z \subset B^*$, by [Loo75, prop. 9.9]. Therefore, $B^* = U/K$ is the *compact symmetric space dual to* $B = G/K$.

2.3.4. A pair (k^+, k^-) with $k^{\pm} \in \mathrm{GL}(Z)$ is called a *Jordan pair automorphism* if

$$k^+ Q_u = Q_{k^+(u)}\, k^- \quad \text{for all } u \in Z\,.$$

$k^{\pm}$ determine each other uniquely. Let $\mathrm{Aut}(Z,Z)$ be the set of all Jordan pair automorphisms. Since Z is finite-dimensional, all Jordan pair automorphisms are inner, i.e.

$$\mathrm{Aut}_0(Z,Z) = \big\langle B(u,v) \;\big|\; (u,v) \text{ quasi-invertible} \big\rangle\,.$$

Note here that the set of quasi-invertible pairs $(u,v) \in Z \times Z$ is connected since it is given by a polynomial inequality, cf. [Kna02, lem. 2.14]. Hence, $B(u,v)$ is always contained in the connected component of $\mathrm{Aut}(Z,Z)$ if (u,v) is quasi-invertible.

The set $\mathrm{Aut}(Z,Z)$ is a complex Lie group whose Lie algebra $\mathrm{aut}(Z,Z)$ consists of all *Jordan pair derivations* (δ^+, δ^-), i.e. $\delta^{\pm} \in \mathrm{End}\, Z$ and

$$\big[\delta^+, u \,\square\, v^*\big] = (\delta^+ u) \,\square\, v^* + u \,\square\, (\delta^- v)^* \quad \text{for all } u, v \in Z\,.$$

All Jordan pair derivations are inner, i.e.

$$\mathrm{aut}(Z,Z) = \mathbb{C}\langle u \,\square\, v^* \mid u, v \in Z \rangle\,.$$

The set $\mathfrak{k}_{\mathbb{R}} = \mathrm{aut}(Z)$ of Jordan triple derivations sits in $\mathrm{aut}(Z,Z)$ as the diagonal. It is easy to see that $\mathrm{aut}(Z,Z) = \mathfrak{k} = \mathfrak{k}_{\mathbb{R}} \otimes \mathbb{C}$, cf. [Upm85, 22.10.9]. If we set $K^{\mathbb{C}} = \mathrm{Aut}_0(Z,Z)$, then $K \subset K^{\mathbb{C}}$ and $K^{\mathbb{C}}$ is a complexification of K.

We have $K^{\mathbb{C}} \subset G^{\mathbb{C}}$, by letting

$$(k^+, k^-)[u : v] = \big[k^+(u) : k^-(v)\big] \quad \text{for all } (k^+, k^-) \in \mathrm{Aut}(Z,Z)\,,\; u, v \in Z\,.$$

This embedding respects the embedding $K \subset G \subset G^{\mathbb{C}}$.

2.3.5. Let $P^{\pm} = \exp \mathfrak{p}^{\pm}$. Then $P^{\pm} \subset G^{\mathbb{C}}$ are closed, Abelian and contractible. Moreover, $P^{+}K^{\mathbb{C}}P^{-} \subset G^{\mathbb{C}}$ is exactly the set of all $\gamma \in G^{\mathbb{C}}$ leaving Z invariant, and is open and dense, by [Loo75, prop. 8.10]. For $g \in P^{+}K^{\mathbb{C}}P^{-}$, the decomposition $g = p^{+}kp^{-}$, where $k \in K^{\mathbb{C}}$ and $p^{\pm} \in P^{\pm}$, is unique.

We have $P^{\pm} = \{t_u^{\pm} \mid u \in Z\}$ where

$$t_w^{+}[u : v] = [u + w : v] \quad \text{and} \quad t_w^{-}[u : v] = [u : v + w]$$

are called *translations* and *quasi-translations*, respectively. On the subset $Z \subset B^*$, they are determined by the formulae

$$t_v^{+}(u) = u + v \quad \text{and} \quad t_v^{-}(u) = u^{v}$$

whenever (u, v) is quasi-invertible.

Translations and quasi-translations act according to the fundamental identity

$$t_v^{-}t_u^{+} = t_{u^v}^{+} B(u,v)^{-1} t_{v^u}^{-} \quad \text{for all } (u,v) \text{ quasi-invertible,}$$

by [Loo75, 7.3.4]. What is more, given a triple (f^{+}, f^{0}, f^{-}) of continuous homomorphisms $f^0 : K^{\mathbb{C}} \to H$, $f^{\pm} : P^{\pm} \to H$, there exists a continuous homomorphism $f : G^{\mathbb{C}} \to H$ extending f^0 and $f^{\pm}$ if and only if

$$f^{\pm}(kpk^{-1}) = f^{0}(k)f^{\pm}(p)f^{0}(k)^{-1}$$

and

$$f^{-}(t_v^{-})f^{+}(t_u^{+}) = f^{+}(t_{u^v}^{+})f^{0}\big(B(u,v)\big)^{-1}f^{-}(t_{v^u}^{-})$$

for all $k \in K^{\mathbb{C}}$, $p \in P^{\pm}$, and (u, v) quasi-invertible, by [Loo75, th. 8.11].

2.3.6. Let $e \in Z$ be a tripotent. Since all Jordan pair automorphisms are inner, we find $K_e^{\mathbb{C}} \subset K^{\mathbb{C}}$ where $K_e^{\mathbb{C}} = \mathrm{Aut}_0(Z_0(e), Z_0(e))$. Likewise, if

$$\mathfrak{p}_0(e)^{+} = \left\{ u \frac{\partial}{\partial z} \;\middle|\; u \in Z_0(e) \right\}$$

and

$$\mathfrak{p}_0(e)^{-} = \left\{ \{zu^*z\} \frac{\partial}{\partial z} \;\middle|\; u \in Z_0(e) \right\}$$

then $P_e^\pm = \exp \mathfrak{p}_0(e)^\pm$ are given by

$$P_e^\pm = \{t_u^\pm \mid u \in Z_0(e)\} .$$

Setting $G_e^\mathbb{C} = \mathrm{Aut}_0\, B_0(e)^*$, the considerations in 2.3.5 show that there exists a continuous homomorphism $G_e^\mathbb{C} \to G^\mathbb{C}$. Its kernel is discrete and normal, and hence central, by [Kna02, prop. 1.93]. Since $G_e^\mathbb{C}$ has trivial centre, the kernel is trivial, allowing us to identify $G_e^\mathbb{C}$ with its image in $G^\mathbb{C}$.

Proposition 2.3.7. Let H be a connected Lie group contained in a connected complexification $H^\mathbb{C}$, and let $\Omega \subset \mathfrak{h}_\mathbb{R}$ be a closed, pointed generating convex cone. Then

$$H \times \Omega \to \widetilde{H}^\mathbb{C} : (h, \xi) \mapsto h \cdot \exp i\xi$$

is an homeomorphism onto a closed sub-semigroup $\Gamma(\Omega) = H \cdot \exp i\Omega \subset H^\mathbb{C}$ with dense interior (in $H^\mathbb{C}$) given by $\Gamma^\circ(\Omega) = \Gamma(\Omega^\circ) = H \cdot \exp i\Omega^\circ$.

Proof. Let $\phi : \widetilde{H}^\mathbb{C} \to H^\mathbb{C}$ be the universal covering group. Then $\ker \phi \subset \widetilde{H}^\mathbb{C}$ is discrete and normal, and hence central. By [Loo69, ch. IV, th. 3.4], the fixed group $\widetilde{H} \subset \widetilde{H}^\mathbb{C}$ of the conjugation on $\widetilde{H}^\mathbb{C}$ is connected. By [Nee96, prop. II.2], $\mathrm{ad}\,\xi$ has imaginary spectrum for all $\xi \in \mathfrak{h}_\mathbb{R}$. Now, Lawson's theorem [Nee00a, th. XI.1.7 and th. XI.1.10] implies that the map

$$\widetilde{H} \times \Omega \to \widetilde{H}^\mathbb{C} : (h, \xi) \mapsto h \cdot \exp i\xi$$

is an homeomorphism onto its closed image $\widetilde{\Gamma}(\Omega) = \widetilde{H} \cdot \exp i\Omega$ which is a closed sub-semigroup of $\widetilde{H}^\mathbb{C}$.

Lawson's theorem also shows that $\widetilde{H}$ is a retraction of $\widetilde{H}^\mathbb{C}$. Hence $\widetilde{H}$ is simply connected. By [Glö00, prop. 25.9], $H^\mathbb{C}$ and H are also homotopy equivalent. Thus, we have canonical isomorphisms

$$\ker \phi \cap \widetilde{H} \to \pi_1(H, 1) \to \pi_1(H^\mathbb{C}, 1) \to \ker \phi .$$

This map associates to $h \in \ker \phi$ the unique homotopy class of $\phi \circ \gamma_h$, γ_h a path in $\widetilde{H}$ from 1 to h; to this, the homotopy class in $H^\mathbb{C}$ of $\phi \circ \gamma_h$; hereto, the end point of a lifting of $\phi \circ \gamma_h$ in $H^\mathbb{C}$. Since γ_h is such a lifting and $\gamma_h(1) = h$, the map is the identity on $\ker \phi \cap \widetilde{H}$, and $\ker \phi \subset \widetilde{H}$.

So, we get an homeomorphism

$$H \times \Omega \to \phi(\widetilde{\Gamma}(\Omega)) \subset H^\mathbb{C} : (h, \xi) \mapsto h \cdot \phi(\exp_{\widetilde{H}^\mathbb{C}} i\xi) = h \cdot \exp_{H^\mathbb{C}} i\xi$$

onto a closed semigroup $\Gamma(\Omega) = \phi(\widetilde{\Gamma}(\Omega)) = H \cdot \exp i\Omega$. Moreover, the semigroup $\Gamma^\circ(\Omega) = \Gamma(\Omega^\circ)$ is its interior in $H^\mathbb{C}$ by [Nee00a, lem. XI.I.9]. □

Remark 2.3.8. Needless to say, a real Lie group H is not always contained in a complexification $H^{\mathbb{C}}$. However, Glöckner [Glö00, prop. 25.9] actually shows that the pointedness of Ω (and even the weaker condition that the endomorphisms $\operatorname{ad}\xi$ for $\xi \in \mathfrak{h}_{\mathbb{R}}$ have imaginary spectrum) implies that H is contained in a complexification $H^{\mathbb{C}}$.

Definition 2.3.9. We call the sets from proposition 2.3.7 $\Gamma(\Omega)$ and $\Gamma^\circ(\Omega)$ the *Ol'shanskiĭ semigroup* and *Ol'shanskiĭ domain* defined by Ω, respectively. In particular, we consider the closed sub-semigroups

$$\Gamma_e = \Gamma(\Omega_0^-(e)) \subset G_e^{\mathbb{C}} \quad \text{and} \quad \Gamma_{f,I} = \Gamma(F_{f,I}^-) \subset G_{f,I}^{\mathbb{C}} = G_{c_1}^{\mathbb{C}} \ltimes H_{f,I}^{\mathbb{C}}$$

and their interiors Γ_e° and $\Gamma_{f,I}^\circ$, for tripotents e, flags $f = (c_1 > \cdots > c_m > 0 = c_{m+1})$, and subsets $I \subset \{1, \ldots, m\}$. In particular, we set $\Gamma = \Gamma_0$ and $\Gamma^\circ = \Gamma_0^\circ$.

On the semigroup $\Gamma(\Omega)$, we define an involutory anti-automorphism by

$$\gamma^* = \bar{\gamma}^{-1} = \exp i\xi \cdot g^{-1} = g^{-1} \cdot \exp i\operatorname{Ad}(g)\xi \quad \text{for all } \gamma = g \cdot \exp i\xi \in \Gamma(\Omega)\,.$$

It leaves $\Gamma^\circ(\Omega)$ invariant and is anti-holomorphic on this domain.

Remark 2.3.10. The open sub-semigroup $\Gamma^\circ \subset G^{\mathbb{C}}$ is the *minimal Ol'shanskiĭ semigroup* constructed by G. I. Ol'shanskiĭ [Ol'82], whereas $\Gamma^\circ(\Omega^+)$ is the *maximal Ol'shanskiĭ semigroup* and can be characterised as the set of compressions in $G^{\mathbb{C}}$ of the domain $B \subset B^*$, by [Sta86, th. 2.2]

In [HN93, ch. 3 and 7] and [Nee00a, ch. XI], the construction of Ol'shanskiĭ semigroups is developed in full generality.

2.3.11. In the setting of proposition 2.3.7, a natural $H \times H$-action on $H^{\mathbb{C}}$ is given by

$$H \times H \times H^{\mathbb{C}} \to H^{\mathbb{C}} : (s, t, \gamma) \mapsto s\gamma t^{-1}\,.$$

Then $\Gamma(\Omega)$ and $\Gamma^\circ(\Omega)$ are clearly $H \times H$-invariant. We consider H as the diagonal in $H \times H$, i.e. the action by conjugation.

Lemma 2.3.12. In the setting of proposition 2.3.7, the map

$$\Gamma(\Omega) \times \Gamma(\Omega) \times \Gamma^\circ(\Omega) \to \Gamma^\circ(\Omega) : (\gamma_1, \gamma_2, \gamma) \mapsto \gamma_1\, \gamma\, \gamma_2^*$$

defines a continuous semigroup action sesqui-holomorphic on the interior. Moreover, for any $\gamma \in \Gamma^\circ(\Omega)$, there exist $\gamma_1, \gamma_2 \in \Gamma^\circ(\Omega)$ such that $\gamma = \gamma_1\gamma_2$.

Proof. Continuity and sesqui-holomorphy on the interior are immediate from the corresponding properties of the operations of $H^{\mathbb{C}}$. Moreover, given any $\gamma_1, \gamma_2 \in \Gamma(\Omega)$, the map

$$\gamma \mapsto \gamma_1\, \gamma\, \gamma_2^* : H^{\mathbb{C}} \to H^{\mathbb{C}}$$

leaves $\Gamma(\Omega)$ invariant because this is a semigroup. In addition, it is injective and holomorphic, hence open. This implies the invariance of $\Gamma^\circ(\Omega)$ under the action.

Write $\gamma = g \cdot \exp i\xi$ where $g \in H$ and $\xi \in \Omega$. Then $\frac{1}{2} \cdot \xi \in \Omega$, which entails

$$\gamma = g \cdot \exp \tfrac{i}{2}\xi \cdot \exp \tfrac{i}{2}\xi \in \Gamma^\circ(\Omega) \cdot \Gamma^\circ(\Omega) \,,$$

since $[\xi, \xi] = 0$. □

Proposition 2.3.13. Let $f_i = (c_{i1} > \cdots > c_{im_i} > 0 = c_{i,m_i+1})$, $i = 1,2$, be flags of tripotents, and $I_i \subset \{1, \ldots, m_i\}$, $i = 1,2$. Let

$$r_i = \operatorname{rk} c_{i1} \quad \text{and} \quad R_i = \{\operatorname{rk}(c_{ij} - c_{i,j+1}) \mid j \in I_i\} \,.$$

The semigroups Γ_{f_1,I_1} and Γ_{f_2,I_2} are K-conjugate if and only if they are G-conjugate, if and only if $r_1 = r_2$ and $R_1 = R_2$. Likewise for the interiors.

Proof. Since the Γ_{f_i,I_i} have dense interiors, conjugacy is equivalent to conjugacy of the interiors. Note

$$h\,g \cdot \exp i\xi\, h^{-1} = hgh^{-1} \cdot \exp i \operatorname{Ad}(h)(\xi) \quad \text{for all } g, h \in G \,,\ \xi \in \mathfrak{g}_{\mathbb{R}} \,.$$

Thus the conjugacy on the level of semigroups is equivalent to conjugacy on the level of cones. The statement follows from proposition 2.3.7 and theorem 2.2.38. □

Analysis & Representation Theory

THE STUDY of the geometry of the Ol'shanskiĭ domain Γ° in part I culminated in the determination of a stratification whose strata have fibres which are smaller Ol'shanskiĭ domains $\Gamma^\circ_{f,I}$. For these domains, the groups $G_{f,I}$ played the role of Shilov boundaries. Making this fact more precise, we discuss in this part the interplay between the domain and its Shilov boundary, both from point of view of Analysis, and of Representation Theory.

In 3.1, we recapitulate the theory of Hardy spaces of a complex Ol'shanskiĭ domain. These are spaces of functions holomorphic on the Ol'shanskiĭ domain, having $\mathbf{L}^2$ boundary values on the associated group. They may be considered as closed subspaces of $\mathbf{L}^2$, and the corresponding orthogonal projection, the so-called Szegö projection, is the starting point for our study of Toeplitz operators in part III. We introduce the Szegö distribution and the Szegö kernel function, which are related to the Szegö projection.

In 3.2, we review Strichartz's symbol calculus for invariant pseudo-differential operators as a tool of micro-local analysis. We then adapt to the invariant setup a method of Melrose-Seeley-Uhlman showing that self-adjoint order zero pseudo-differential operators are invariant under single-operator Weyl functional calculus, and extend this to tuples of possibly non-commuting tuples using an idea of Álvarez-Calderón. This paragraph is somewhat of an interlude, but the result indicates that methods such as those used by Guillemin-Boutet de Mouvel [Gui79, BdMG81] in their study of Toeplitz operators should be applicable in this setup.

In 3.3, we estimate the wave front set of the Szegö distribution by explicit estimates showing the uniform tempered growth of the Szegö kernel, locally close to the boundary. In the following paragraph 3.4, we prove some results estimating the singular support of the Szegö distribution.

Besides the analytic definition of the Hardy space, there is also a representation theoretic approach. Namely, it can be described as the restriction of the Plancherel decomposition of $\mathbf{L}^2$ to the holomorphic discrete series. We shall not review the proof of this fact, but we give a mostly self-contained introduction to the holomorphic discrete series. The algebraic theory of highest weight modules is treated in 4.1, and the analytic theory, including Harish-Chandra's square integrability criterion, in 4.2.

We closely follow Neeb's approach [Nee00a, ch. IX, XII] using reproducing kernel Hilbert spaces, and do not claim any originality in this point. We sometimes include references to more classical texts such as Dixmier's [Dix77] where our less general framework asks for weaker results, and take Jordan theoretical short cuts where this seems appropriate. Although this is a revision of material which in large parts has been known for almost half a century, it perhaps better motivates some of the constructions we perform in the sequel.

In 4.3, we show that the holomorphic discrete series of the boundary subgroups G_e can be embedded into the holomorphic discrete series of the group G. Here, there is some freedom in the choice of parameters, allowing us to pass to the limit.

Using our knowledge of the kernel functions associated to holomorphic discrete series, we examine in 4.4 the asymptotic behaviour matrix coefficients these representations. Also, the micro-local information we have gathered on the Szegö projection and the calculus developed in 3.2 allow us to obtain some results on asymptotics of the Szegö distribution. To be precise, we show that it bears no information at infinity in the spectrum of G in directions which in some sense are bounded away from the holomorphic discrete series. Although the latter result is not decisive for the further development, it seemed natural to include it.

In part III, the results on singularities of the Szegö distribution and asymptotics of coefficient function are important for the development of a detailed structure theory for a C^*-algebra of Toeplitz operators, defined in terms of the Szegö projection.

To fathom the C^*-algebra of Toeplitz operators alluded to here, it is however necessary to understand other parts of the Plancherel decomposition of $\mathbf{L}^2(G)$, besides the holomorphic discrete series. Specifically, an appropriate generalisation of the embedding of holomorphic discrete series of facial subgroups constructed in 4.3 should be available for the other series of representations.

In a first step, we treat the discrete series, 5.1-5.2. Our approach is simple-minded, in that we basically try to imitate what is done in the case of holomorphic discrete series representations. Namely, we identify a copy of lowest $\bar{K}$-type of a discrete series of $\bar{G}$ within a discrete series of G, and take the generated submodule. However, the machinery of Verma modules is not available (because we treat non-adapted positive systems). Instead, we use a realisation of the discrete series due to Knapp-Wallach.

The existence of an embedding more or less comes down to finding compatible choices of fundamental sequences of strongly orthogonal non-compact roots, a matter which is trivial for adapted positive systems, and non-trivial otherwise.

The discrete series form the basic building blocks of the representations weakly contained in $\mathbf{L}^2(G)$, the other series being given by induction from discrete series. Thus, the main difficulty in 5.3-5.4, where we embed the parabolic Q-series of facial subgroups, is to control the behaviour of the parabolic subgroups. Here, our Jordan theoretic setting pays off again, and we are able to give a fairly complete description of the parabolics. Then, basically the same naive idea as for the discrete series goes through in the construction of an embedding of the parabolic Q-series.

3 Local and micro-local analysis of the Szegö distribution

3.1 The Hardy space defined by Γ

3.1.1. Let $B \subset Z$ be a circled bounded symmetric domain defined by a simple JB*-triple Z, and $G = \mathrm{Aut}_0(B)$.

Consider the closed cone $\Omega^- \subset \mathfrak{g}_\mathbb{R}$, the Ol'shanskiĭ semigroup $\Gamma = G \cdot \exp i\Omega^-$ contained in $G^\mathbb{C} = \mathrm{Aut}_0(B^*)$, and its interior, the Ol'shanskiĭ domain $\Gamma^\circ = G \cdot \exp i\Omega^{-\circ}$.

If $f = (c_1 > \cdots > c_m > 0 = c_{m+1})$ is a flag of tripotents, and $I \subset \{1, \dots, m\}$, the open sub-semigroup

$$\Gamma^\circ_{f,I} = G^\mathbb{C}_{f,I} \cdot \exp iF^{-\circ}_{f,I} \subset G^\mathbb{C}_{f,I}$$

is connected by proposition 2.3.7, justifying the term Ol'shanskiĭ *domain*. It is known that it is Stein, cf. [Nee98, th. 5.18] (also [Nee00a, th. XIII.5.15]). So, it makes sense to consider the space $\mathcal{O}(\Gamma^\circ_{f,I})$ of holomorphic functions on $\Gamma^\circ_{f,I}$. We have noted (2.3.11) that $\Gamma^\circ_{f,I}$ is $G_{f,I}$-invariant for both left and right translations. Since the cone $F^-_{f,I}$ is generating in $\mathfrak{g}^{f,I}_\mathbb{R}$ by proposition 2.2.35, it follows from [Nee00a, th. VII.1.8] that $G_{f,I}$ is *unimodular*.

Remark 3.1.2. Unimodularity is clear for each of the factors G_{c_1} and $H_{f,I}$ individually, since the former is simple, and the latter is nilpotent. For $G_{f,I} = G_{c_1} \ltimes H_{f,I}$ as a whole, it is rather surprising and certainly not true of semi-direct products of unimodular groups in general.

3.1.3. In order to treat the groups $G_{f,I}$ simultaneously without too much notation, let H be a connected Lie group, contained in a connected complexification $H^\mathbb{C}$. Assume that there exists a pointed invariant closed convex cone $\Omega \subset \mathfrak{h}_\mathbb{R}$ with non-void interior.

Then there is an Ol'shanskiĭ domain $\Gamma^\circ = H \cdot \exp i\Omega^\circ \approx H \times \Omega^\circ$, open in its closure Γ, which is a closed sub-semigroup of $H^\mathbb{C}$. Moreover, H is unimodular.

Definition 3.1.4. Denote

$$f_\gamma(\gamma') = f(\gamma^*\gamma') \quad \text{for all } f \in \mathcal{O}(\Gamma^\circ)\,,\ (\gamma, \gamma') \in \big(\Gamma^\circ \times \Gamma\big) \cup \big(\Gamma \times \Gamma^\circ\big)\,.$$

Let the *Hardy space* be given by

$$\mathbf{H}^2(\Gamma) = \left\{ f \in \mathcal{O}(\Gamma^\circ) \,\middle|\, \sup\nolimits_{\gamma \in \Gamma^\circ} \int_H^* |f_\gamma(t)|^2 \, dt < \infty \right\},$$

endowed with the norm $\|\sqcup\|_{\mathbf{H}^2}$ given by

$$\|f\|^2_{\mathbf{H}^2} = \sup\nolimits_{\gamma \in \Gamma^\circ} \int_H^* |f_\gamma(t)|^2 \, dt = \sup\nolimits_{\gamma \in \Gamma^\circ} \|f_\gamma\|_2^2 \,.$$

The Haar measure on H and the domain Γ° are H-bi-invariant. Therefore,

$$(s,t)^\# f(\gamma) = f(s^{-1}\gamma t) \quad \text{for all } \ s,t \in H\,,\ \gamma \in \Gamma\,,\ f \in \mathbf{H}^2(\Gamma)$$

defines a representation of $H \times H$ on $\mathbf{H}^2(\Gamma)$. Note

$$\left\|(s,t)^\# f\right\|_{\mathbf{H}^2} = \sup\nolimits_{\gamma\in\Gamma^\circ} \|f_{\gamma s}(\sqcup t)\|_2 = \sup\nolimits_{\gamma\in\Gamma^\circ} \|f\|_2 = \|f\|_{\mathbf{H}^2}$$

for all $s,t \in H$ and $f \in \mathbf{H}^2(\Gamma)$, i.e. the action is by isometries.

3.1.5. By [Nee00a, lem. XIV.3.3], the topology on $\mathbf{H}^2(\Gamma)$ is finer than the topology of compact convergence. Moreover, the boundary value map

$$j : \left\{f_\gamma \mid \gamma \in \Gamma^\circ\,,\ f \in \mathbf{H}^2(\Gamma)\right\} \to \mathbf{L}^2(H) : f \mapsto f\big|_H$$

extends uniquely to an isometry $\mathbf{H}^2(\Gamma) \to \mathbf{L}^2(H)$, as is proved in [Nee00a, th. XIV.3.5]. In particular, the representation of the product group $H \times H$ on $\mathbf{H}^2(\Gamma)$ is unitary.

We also deduce that the Hardy space $\mathbf{H}^2(\Gamma)$ is a Reproducing Kernel Hilbert space of functions on Γ° , with a kernel function

$$K : \Gamma^\circ \times \Gamma^\circ \to \mathbb{C}$$

defined by $K(z,w) = K_w(z)$ and the reproducing property

$$f(w) = (K_w \mid f)_{\mathbf{H}^2} \quad \text{for all } \ f \in \mathbf{H}^2(\Gamma)\,,\ w \in \Gamma^\circ\,.$$

The function K is called the *Szegö kernel*.

Proposition 3.1.6. The Szegö kernel function K is sesqui-holomorphic. In addition, it is Hermitian and H-bi-invariant in the sense that

$$K(z,w) = \overline{K(w,z)} \quad \text{and} \quad K(szt^{-1}, swt^{-1}) = K(z,w)$$

for all $z,w \in \Gamma^\circ$, $s,t \in H$.

Proof. We have

$$K(z,w) = (K_z \mid K_w) = \overline{(K_w \mid K_z)} = \overline{K(w,z)}\,.$$

Hence $K(z,w)$ is separately holomorphic in z and anti-holomorphic in w . By Hartogs's joint analyticity theorem [Hör73, th. 2.2.8], K is globally sesqui-holomorphic.

Furthermore, since the action on $\mathbf{H}^2(\Gamma)$ is isometric,

$$\left((s,t)^\# K_z \mid f\right) = \left(K_z \mid (s,t)^{-\#} f\right) = f(szt^{-1}) = \left(K_{szt^{-1}} \mid f\right)$$

for all $f \in \mathbf{H}^2(\Gamma)$. This shows $(s,t)^\# K_z = K_{szt^{-1}}$, and in particular,

$$K(szt^{-1}, swt^{-1}) = \big((s,t)^\# K_z \mid (s,t)^\# K_w\big) = (K_z \mid K_w) = K(z,w) ,$$

proving the proposition. □

Corollary 3.1.7. We have

$$K(uzv^*, w) = K(z, u^*wv) \quad \text{for all } \ z, w \in \Gamma^\circ , \ u, v \in \Gamma .$$

Proof. This follows from

$$K(szt^{-1}, w) = K(z, s^{-1}wt) \quad \text{for all } \ z, w \in \Gamma^\circ , \ s, t \in H$$

and the identity theorem [Nee00a, lem. XI.2.2]. □

Proposition 3.1.8. The boundary value map j is given by the formula

$$jf = \lim\nolimits_{\Gamma^\circ \ni \gamma \to 1} f_\gamma\big|_H \quad \text{in } \ \mathbf{L}^2(H)$$

for all $f \in \mathbf{H}^2(\Gamma)$.

Proof. Let $f \in \mathbf{H}^2(\Gamma)$. The family $(f_\gamma)_{\gamma \in \Gamma^\circ}$ is bounded in $\mathbf{H}^2(\Gamma^\circ)$, and hence also in $\mathbf{L}^2$, since j is an isometry. Therefore, the f_γ are contained in a weakly compact subset. The weak topology on the unit sphere of $\mathbf{L}^2$ is separable and therefore metrisable.

Let $\gamma_k \in \Gamma^\circ$ be a sequence converging to 1 , such that f_{γ_k} is weakly convergent. We compute, because $j\ j^*$ is the projection onto $j\big(\mathbf{H}^2(\Gamma)\big)$,

$$\begin{aligned} (j\,K_z \mid jf) = f(z) &= \lim\nolimits_{k\to\infty} f(\gamma_k^* z) \\ &= \lim\nolimits_{k\to\infty} (j\,K_z \mid f_{\gamma_k}) = (jK_z \mid \lim\nolimits_{k\to\infty} f_{\gamma_k}) \ . \end{aligned}$$

Since the vectors $j\,K_z$, $z \in \Gamma^\circ$, have dense span in $j\big(\mathbf{H}^2(\Gamma)\big)$, we deduce that $jf = \lim_k f_{\gamma_k}$ in the weak topology. By compactness, this implies

$$jf = \lim\nolimits_{\Gamma^\circ \ni \gamma \to 1} f_\gamma \quad \text{weakly in } \ \mathbf{L}^2(H) .$$

Now, consider the bounded family of real numbers $\|f_\gamma\|_2$, $\gamma \in \Gamma^\circ$. Since

$$\|jf\|_2 = \|f\|_{\mathbf{H}^2} = \sup\nolimits_{\gamma \in \Gamma^\circ} \|f_\gamma\|_2$$

and $jf_\gamma = f_\gamma\big|_H$, we find

$$\|f_{\gamma'\gamma}\|_2 = \big\|(f_\gamma)_{\gamma'}\big\|_2 \leqslant \|f_\gamma\|_{\mathbf{H}^2} = \|f_\gamma\|_2 \quad \text{for all } \ \gamma, \gamma' \in \Gamma^\circ .$$

Let $\gamma_k \to 1$ be such that $\|f_{\gamma_k}\|_2$ converges. Then, for any $k \in \mathbb{N}$, there exists $\ell \geqslant k$ so that

$\gamma_k \in \Gamma^\circ \cdot \gamma_\ell$. Consequently, there is a subsequence α satisfying

$$\gamma_{\alpha(k)} \in \Gamma^\circ \cdot \gamma_{\alpha(k+1)} \quad \text{for all } \ k \in \mathbb{N} \,.$$

Then the sequence $\|f_{\gamma_{\alpha(k)}}\|_2$ is increasing, so it converges to its supremum $\|f\|_{\mathbf{H}^2}$. Hence, $\lim_{k\to\infty} \|f_{\gamma_k}\|_2 = \|f\|_{\mathbf{H}^2}$. Since the sequence γ_k was arbitrary, by compactness,

$$\lim_{\Gamma^\circ \ni \gamma \to 1} \|f_\gamma\|_2 = \|f\|_{\mathbf{H}^2} = \|jf\|_2 \ .$$

By the Radon-Riesz theorem [Els99, prop. 5.10], $f_\gamma \to jf$ in norm. □

Remark 3.1.9. The formula for the isometry j is part of the original proof that that the Hardy space $\mathbf{H}^2(\Gamma)$ is a Hilbert space, cf. [HÓØ91, th. 2.2] and [HN93, th. 9.31]. Compare [Nee00a, th. XIV.3.5] for another proof.

Proposition 3.1.10. There exists a holomorphic function

$$K : \Gamma^\circ \to \mathbb{C} \quad \text{such that} \quad K(z,w) = K(zw^*) \quad \text{for all } \ z, w \in \Gamma^\circ \,,$$

$K(z^*) = \overline{K(z)}$, and K is H-conjugation invariant. Moreover,

$$K = \lim_{\Gamma^\circ \ni z \to 1} K_z \quad \text{compactly on } \Gamma^\circ \,,$$

and

$$\big(j\,K_w\big)(g) = K(gw^*) = K(w^*g) \quad \text{for all } \ g \in G \,,\, w \in \Gamma^\circ \,,$$

in the sense that $K(\sqcup\, w^*)$ is a representative of the $\mathbf{L}^2$ class $j\,K_z$.

Proof. Since j^*j is the identity on $\mathbf{H}^2(\Gamma)$, we deduce

$$f = j^* \lim_{\Gamma^\circ \ni \gamma \to 1} f_\gamma = \lim_{\Gamma^\circ \ni \gamma \to 1} j^* f_\gamma = \lim_{\Gamma^\circ \ni \gamma \to 1} f_\gamma \,.$$

By translation with elements of Γ , this shows that

$$\gamma \mapsto f_\gamma : \Gamma \to \mathbf{H}^2(\Gamma)$$

is continuous.

Let $z \in \Gamma^\circ$. Then there are $u, v \in \Gamma^\circ$ such that $z = v^*u$, by lemma 2.3.12. Taking a neighbourhood $v \in V \subset \Gamma^\circ$, V^*u is a neighbourhood of z , since u is invertible in $H^{\mathbb{C}}$ (or by holomorphy). Corollary 3.1.7 implies

$$\lim_{V^*u \ni \gamma \to z} K_\gamma = \lim_{V \ni \gamma \to v} K_{\gamma^* u} = \lim_{V \ni \gamma \to v} \big(K_u\big)_\gamma = \big(K_u\big)_v = K_z$$

in the topology of $\mathbf{H}^2(\Gamma)$. Therefore,

$$z \to K_z : \Gamma^\circ \to \mathbf{H}^2(\Gamma)$$

is continuous. Since the topology of compact convergence is coarser than the norm topology, we find that K_{z_λ} is a Cauchy net for any net $\Gamma^\circ \ni z_\lambda \to 1$. Hence, the limit

$$K(z) = \lim\nolimits_{\Gamma^\circ \ni w \to 1} K(z, w) \quad \text{exists in } \mathcal{O}(\Gamma^\circ) .$$

Then

$$K(zw^*) = \lim\nolimits_{\Gamma^\circ \ni \gamma \to 1} K(zw^*, \gamma) = \lim\nolimits_{\Gamma^\circ \ni \gamma \to 1} K(\gamma^* z, w) = K(z, w) .$$

If $z \in \Gamma^\circ$, then $z = uv^*$ for some $u, v \in \Gamma^\circ$. Hence

$$K(z^*) = K(v, u) = \overline{K(u, v)} = \overline{K(z)} .$$

Furthermore, $g\,\Gamma^\circ\, g^{-1} = \Gamma^\circ$ for $g \in H$, so

$$\begin{aligned} K(gzg^{-1}) &= \lim\nolimits_{\Gamma^\circ \ni w \to 1} K(gzg^{-1}, w) \\ &= \lim\nolimits_{\Gamma^\circ \ni w \to 1} K(z, gwg^{-1}) = K(z) . \end{aligned}$$

Finally, let $\gamma_k \in \Gamma^\circ$ be such that $\gamma_k \to 1$. Then

$$j\, K_w(g) = \lim\nolimits_{k \to \infty} K(\gamma_k^* g w^*) = K(g w^*)$$

where the first limit is taken in the $\mathbf{L}^2$ sense and the second convergence is uniform for g varying in compact subsets of H. By the Riesz-Fischer theorem, $K(\sqcup\, w^*)$ is a representative of the $\mathbf{L}^2$ class $j\, K_z$. □

Remark 3.1.11. Cf. [HÓØ91, th. 4.4], [HN93, th. 9.33] and [Nee00a, cor. IV.1.30] for alternative proofs. The latter is the shortest, and perhaps most in the spirit of the theory of positive definite functions on (involutive) semigroups, as an extension of the classical theory for C*-algebras.

3.1.12. Given a unimodular locally compact group H, recall that the convolution of two functions $f, g : H \to \mathbb{C}$ such that $s \mapsto g(s^{-1}t) \cdot f(s)$ is integrable for all $t \in H$ is

$$(f * g)(t) = \int_H g(s^{-1}t) f(s)\, ds \quad \text{for all } s, t \in H .$$

More generally, the convolution of two bounded measures $\mu, \nu \in \mathcal{M}^b(H)$ is given by

$$\langle f \,:\, \mu * \nu \rangle = \langle \Delta f \,:\, \mu \otimes \nu \rangle \quad \text{for all } f \in \mathbf{L}^\infty(H)$$

where $\Delta f(s, t) = f(s \cdot t)$.

If H is, moreover, a Lie group, and $\mu \in \mathcal{D}'(H)$ a distribution, then the convolution with a smooth function ψ is given by

$$\langle \varphi \,:\, \mu * \psi \rangle = \langle \varphi * \psi^\vee \,:\, \mu \rangle = \int_H \langle t \mapsto \psi(t^{-1}s)\varphi(s) : \mu \rangle\, ds ,$$

hence

$$\mu * \psi(t) = \langle t * \psi^\vee : \mu \rangle = \langle s \mapsto \psi(s^{-1}t) : \mu \rangle \quad \text{for all } t \in H\,.$$

Here, we identify $g \in H$ with the Dirac measure δ_g, and $\varphi^\vee(t) = \varphi(t^{-1})$.

Proposition 3.1.13. The adjoint j^* is given by

$$(j^* f)(z) = \int_H K(g^{-1}z) f(g)\,dg \quad \text{for all } f \in \mathbf{L}^2(H)\,.$$

The projection $j\ j^*$ onto $j(\mathbf{H}^2(\Gamma))$ is given by convolution with a uniquely determined central distribution $E \in \mathcal{D}'(H)$. In particular,

$$\langle \varphi : E \rangle = \lim_{\Gamma^\circ \ni \gamma \to 1} \int_H \varphi(g) K(g\gamma)\,dg \quad \text{for all } \varphi \in \mathcal{D}(H)\,.$$

Proof. Let $f \in \mathbf{L}^2(H)$. Then

$$(j^* f)(z) = (K_z \mid j^* f) = (j K_z \mid f) = \int_H \overline{K(z^* g)} f(g)\,dg = \int_H K(g^{-1}z) f(g)\,dg\,.$$

Since jj^* commutes with right convolutions, it is given by convolution with a unique distribution, cf. [Eym64, prop. 3.27]. In particular, it maps smooth functions to smooth functions, and the propositions 3.1.8 and 3.1.10 entail

$$\begin{aligned}\langle \varphi : E \rangle &= (E * \varphi^\vee)(1) = \lim_{\Gamma^\circ \ni \gamma \to 1} (j^* \varphi^\vee)(\gamma^*) \\ &= \lim_{\Gamma^\circ \ni \gamma \to 1} \int_H K(g^{-1}\gamma^*) \varphi(g^{-1})\,dg = \lim_{\Gamma^\circ \ni \gamma \to 1} \int_H K(g\gamma)\varphi(g)\,dg\end{aligned}$$

since H is unimodular and Γ° is invariant under $\sqcup^*$. The invariance of K now implies that E is conjugation invariant, and hence, central. □

3.2 Functional calculus of invariant pseudo-differential operators

At the beginning of this subsection, we recall some standard facts from the calculus of pseudo-differential operators, mainly to fix notation. As a general reference, we cite [Hör67], and, to a lesser extent, [Hör71a]. Textbooks would be [Kg81], [Trè80] or [Tay81].

3.2.1. Let $n, k \in \mathbb{N}$, $m \in \mathbb{R}$, $0 \leqslant 1 - \varrho \leqslant \delta < \varrho \leqslant 1$, and $U \subset \mathbb{R}^n$ an open subset. Recall the definition of Hörmander's *symbol class* $S^m_{\varrho,\delta}(U \times \mathbb{R}^k)$: It consists of smooth functions $a : U \times \mathbb{R}^m \to \mathbb{C}$, such that for all compact subsets $K \subset U$ and multi-indices α, β,

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leqslant C_{\alpha,\beta,K} \cdot \big(1 + |\xi|\big)^{m - |\alpha|\varrho + |\beta|\delta} \quad \text{for all } x \in K\,,\ \xi \in \mathbb{R}^k$$

for some $C_{\alpha,\beta,K} \geqslant 0$. The least constants $C_{\alpha,\beta,K}$ in this estimate define a system of seminorms making $S^m_{\varrho,\delta}$ into a Fréchet space. For the special case of $\varrho = 1$, $\delta = 0$, we write

$S^m = S^m_{1,0}$. Moreover,

$$S^\infty_{\varrho,\delta} = \bigcup_{m\in\mathbb{Z}} S^m_{\varrho,\delta} \quad \text{and} \quad S^{-\infty}_{\varrho,\delta} = \bigcap_{m\in\mathbb{Z}} S^m_{\varrho,\delta} \, .$$

For $a \in S^m(U \times \mathbb{R}^n)$ (i.e. $n = k$), define

$$\big[a(x,D)\varphi\big](x) = \frac{1}{(2\pi)^n} \cdot \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{i(x-y\,:\,\xi)} a(x,\xi)\varphi(y)\, dy\, d\xi \quad \text{for all } \varphi \in \mathcal{D}(U)\, ,\, x \in U\, .$$

Then $a(x,D) : \mathcal{D}(U) \to \mathcal{E}(U)$ is continuous.

3.2.2. The symbol class $S^m_{\varrho,\delta}(U \times \mathbb{R}^k)$ has an important completeness property: For symbols $a_j \in S^{m_j}_{\varrho,\delta}(U \times \mathbb{R}^k)$, $m = m_0 > m_j > m_{j+1} \to -\infty$, there exists $a \in S^m_{\varrho,\delta}(U \times \mathbb{R}^k)$, unique modulo $S^{-\infty}_{\varrho,\delta}(U \times \mathbb{R}^k)$, such that

$$a = \sum_{j=0}^{\ell-1} a_j \pmod{S^{m_\ell}_{\varrho,\delta}(U\times\mathbb{R}^k)} \quad \text{for all } \ell \in \mathbb{N}\, .$$

We write $a \sim \sum_{j=0}^\infty a_j$ and call this an *asymptotic expansion* of a .

Using this device, one shows that for $\varphi \in \mathcal{D}(U)$, $a \in S^{m_1}_{\varrho,\delta}(U \times \mathbb{R}^n)$, $b \in S^{m_2}_{\varrho,\delta}(U \times \mathbb{R}^n)$ there exists a symbol $c \in S^{m_1+m_2}_{\varrho,\delta}(U \times \mathbb{R}^n)$ such that

$$a(x,D)\varphi b(x,D) = c(x,D)\, ,$$

cf. [Hör67, th. 2.10]. Moreover, the symbol class also stable under diffeomorphisms.

3.2.3. An important subclass of S^m is the set of *classical* symbols $S^m_h(U \times \mathbb{R}^k)$. These are all $a \in S^m(U \times \mathbb{R}^k)$ which have an asymptotic expansion $a \sim \sum_{j=0}^\infty a_j$ such that $m_j - m_{j+1} \in \mathbb{N}$ and a_j is (positively) homogeneous of degree m_j for $|\xi| \to \infty$, i.e.

$$a_j(x,t\xi) = t^{m_j} \cdot a_j(x,\xi) \quad \text{for all } x \in U\, ,\, |\xi| \gg 1\, ,\, t \geqslant 1\, .$$

In fact, asymptotic expansions make sense for functions not smooth for $\xi = 0$, and one may assume a_j is homogeneous for all $\xi \neq 0$ without changing the class of a modulo $S^{-\infty}$. The set S^m_h of all classical symbols is stable under asymptotic expansions, and, for $n = k$, under composition and diffeomorphisms.

The top order term a_0 of the asymptotic expansion of $a \in S^m_h$ is uniquely determined up to terms of order $m_1 \leqslant m - 1$ and is called the *principal part* of the classical symbol a .

3.2.4. A *pseudo-differential operator* in U of order m and type (ϱ,δ) is a continuous linear map $A : \mathcal{D}(U) \to \mathcal{E}(U)$ such that for all $\varphi \in \mathcal{D}(U)$, there exists $a_\varphi \in S^m_{\varrho,\delta}(U \times \mathbb{R}^n)$ satisfying

$$A(\varphi\psi) = a_\varphi(x,D)\psi \quad \text{for all } \psi \in \mathcal{D}(U)\, .$$

The set of all such operators is denoted $\Psi^m_{\varrho,\delta}(U)$. Similarly, the set of all *classical* pseudo-differential operators $\Psi_h^m(U)$ consists of those $A \in \Psi^m(U) = \Psi_{1,0}^m(U)$ such that a_φ can be chosen to be classical for all φ.

We say that a continuous linear map $A : \mathcal{D}(U) \to \mathcal{D}'(U)$ is *properly supported*, if for the distribution kernel $\alpha \in \mathcal{D}'(U \times U)$ of A, the projections

$$U \xleftarrow{\mathrm{pr}_1} \operatorname{supp} \alpha \xrightarrow{\mathrm{pr}_2} U$$

are proper. Equivalently, for any compact $K \subset U$, there is a compact $L \subset U$ such that

$$\operatorname{supp} \varphi \subset K \;\Rightarrow\; \operatorname{supp} A\varphi \subset L \quad \text{and} \quad \varphi\big|_L = 0 \;\Rightarrow\; A\varphi\big|_K = 0\,.$$

Pseudo-differential operators are very regular and can hence be extended to continuous linear maps $\mathcal{E}'(U) \to \mathcal{D}'(U)$. If a pseudo-differential operator is properly supported, it can even be extended to a continuous linear map $\mathcal{D}'(U) \to \mathcal{D}'(U)$.

Recall that a very regular continuous linear map $A : \mathcal{D}(U) \to \mathcal{E}(U)$ is called smoothing if $A(\mathcal{E}'(U)) \subset \mathcal{E}(U)$. Any smoothing A in U is a pseudo-differential operator of order $-\infty$ (all types coincide at this order), and conversely.

Any pseudo-differential operator is the sum of a smoothing operator and a properly supported one. Any properly supported operator $A \in \Psi_{\varrho,\delta}^m(U)$ is of the form $a(x, D)$ for some $a \in S_{\varrho,\delta}^m(U \times \mathbb{R}^n)$. Similarly for classical operators.

If $A - a(x, D)$ is smoothing, a is called the *symbol* of A since it is essentially unique. If A is classical and a is the classical symbol of A, the principal part of a is called the *principal symbol* of A.

3.2.5. If M is a smooth manifold endowed with some positive density (so M is orientable), then a continuous linear map $A : \mathcal{D}(M) \to \mathcal{E}(M)$ is said to be a pseudo-differential operator of order m and type (ϱ, δ), if for any local chart (U, ϕ), the map $\phi^*(A)$ is a pseudo-differential operator in $\Psi_{\varrho,\delta}^m(U)$, where $\phi^*(A)$ is defined by

$$\phi^*(A)\varphi = A(\varphi \circ \phi^{-1})\big|_U \circ \phi \quad \text{for all } \varphi \in \mathcal{D}(\phi(U))$$

and it is understood that $\varphi \circ \phi^{-1}$ is extended to M by zero.

Similarly, we define classical pseudo-differential operators on M. One may restrict attention to a fixed covering by local charts if, in addition, it is required that the kernel of A be singular only on the diagonal of M.

An alternative definition of pseudo-differential operators which does not require expression in local coordinates is via Fourier integral operators with non-degenerate linear phase function smooth off the diagonal, cf. [Hör71a, § 2.3].

3.2.6. For a global theory for pseudo-differential operators in Euclidean space, one introduces global Sobolev spaces and proves Sobolev continuity for properly supported operators. For general non-compact manifolds, there is no established definition of global Sobolev spaces, a difficulty which can be overcome for homogeneous spaces by considering *invariant* pseudo-differential operators. We restrict ourselves to Lie groups.

Let H be a Lie group. Choose a Euclidean inner product on $\mathfrak{h}_{\mathbb{R}}$, an orthonormal basis X_j of $\mathfrak{h}_{\mathbb{R}}$, and the dual basis $\xi_j \in \mathfrak{h}_{\mathbb{R}}^*$.

This allows us to define $S^m(\mathfrak{h}_{\mathbb{R}}^*) = S^m(\{0\} \times \mathfrak{h}_{\mathbb{R}}^*)$ and $S_h^m(\mathfrak{h}_{\mathbb{R}}^*) = S_h^m(\{0\} \times \mathfrak{h}_{\mathbb{R}}^*)$. In particular, $S_h^{-\infty}(\mathfrak{h}_{\mathbb{R}}^*) = \mathcal{S}(\mathfrak{h}_{\mathbb{R}}^*)$, the Schwartz space on $\mathfrak{h}_{\mathbb{R}}^*$.

Denote by $\mathrm{I}\Psi^m(H)$ resp. $\mathrm{I}\Psi_h^m(H)$ the set of all pseudo-differential operators on H commuting with left translations. These, we call *invariant*. By [Str72, th. 1], an invariant properly supported $A : \mathcal{D}(H) \to \mathcal{E}(H)$ is in $\Psi^m(H) = \Psi_{1,0}^m(H)$ if and only if

$$A\varphi = \mathrm{op}_\chi(a) + \varphi * \psi \quad \text{for all } \varphi \text{ and some } a \in S^m(\mathfrak{h}_{\mathbb{R}}^*),\ \psi \in \mathcal{D}(H)$$

and $\chi \in \mathcal{D}(\mathfrak{h}_{\mathbb{R}})$ such that $\chi = 1$ in the neighbourhood of zero. Here,

$$\mathrm{op}_\chi(a)\varphi(t) = \frac{1}{(2\pi)^n} \cdot \iint_{\mathfrak{h}_{\mathbb{R}}^* \times \mathfrak{h}_{\mathbb{R}}} e^{-i\langle X:\xi\rangle} a(\xi)\chi(X)\varphi(t \exp X)\, dX\, d\xi .$$

The symbol a is unique modulo $S^{-\infty}(\mathfrak{h}_{\mathbb{R}}^*) = \mathcal{S}(\mathfrak{h}_{\mathbb{R}}^*)$. It is called the *Lie symbol* of A. Moreover, A is classical if and only if a be chosen to be classical. In the latter case, the principal part a_0 of the classical Lie symbol is called the *principal Lie symbol* of A.

Fix $\chi \in \mathcal{D}(\mathfrak{h}_{\mathbb{R}})$ and define the invariant pseudo-differential operator

$$\Lambda^s = \mathrm{op}_\chi\big((1 + |\sqcup|^2)^{s/2}\big) \in \mathrm{I}\Psi^s(H) .$$

Λ^s is elliptic by [Str72, th. 4]. Define $\mathrm{H}^{(s)}(H) = \mathcal{D}(\Lambda^s) \subset \mathbf{L}^2(H)$ for $s \geqslant 0$, with the graph norm. Let $\mathrm{H}^{(-s)}(H)$ be the dual of $\mathrm{H}^{(s)}(H)$, and define $\mathrm{H}^{(-\infty)}(H)$ and $\mathrm{H}^{(\infty)}(H)$ to be the union resp. the intersection of all $\mathrm{H}^{(s)}(H)$. These are the *invariant Sobolev spaces*.

The definition and the topology of the Sobolev spaces is independent of the choice of inner product and basis on $\mathfrak{h}_{\mathbb{R}}$, and of χ, by [Str72, th. 7, cor.] and [Goo80, cor. 1.1].

Any properly supported $A \in \mathrm{I}\Psi^0(H)$ is bounded on $\mathbf{L}^2(H)$. In fact, any properly supported $A \in \mathrm{I}\Psi^m(H)$ is a bounded operator $\mathrm{H}^{(s+m)}(H) \to \mathrm{H}^{(s)}(H)$ by [Str72, th. 7].

Having reviewed the basic calculus of invariant pseudo-differential operators, our next aim is to establish a functional calculus for possibly non-commuting tuples of properly supported order zero operators which are self-adjoint on $\mathbf{L}^2(H)$.

To this end, we adapt the construction of a single-operator functional calculus from [GS79] (which Guillemin-Sternberg attribute to R. Melrose, R. Seeley and G. Uhlman) to invariant pseudo-differential operators on the Lie group H. Their proof — valid for the Euclidean situation and for compact manifolds — goes through without essential changes, with the help of Strichartz's invariant pseudo-differential calculus.

In fact, using an idea of Álvarez-Calderón [ÁC83, rem. 5.7], it is straightforward to extend the proof of the single-operator calculus to non-commuting tuples.

It should be noted that in [AH96], Álvarez-Hounie construct a functional calculus for non-commuting tuples in the context of non-classical operators of Hörmander class $\Psi^m_{\varrho,\delta}(\mathbb{R}^n)$, $m \leqslant 0$, $0 \leqslant \delta < 1$, $0 < \varrho \leqslant 1$, $\delta \leqslant \varrho$. However, they rely heavily on the characterisation of pseudo-differential operators via Sobolev continuity of certain commutators due to Beals. Therefore, their proof does not immediately apply in the context of manifolds (the existence of a functional calculus is a global question).

Nonetheless, a Beals type characterisation should be valid in the invariant setup on the Lie group H. If this is the case, the methods of Álvarez-Hounie should extend to this situation (for suitable (ϱ, δ)) and give a more general functional calculus. With the applications we have in mind, we are content to restrict attention to classical operators.

Proposition 3.2.7. Let $T \in \mathrm{I}\Psi^0_h(H)$ be properly supported and self-adjoint as a bounded operator on $\mathbf{L}^2(H)$. Then, for $t \in \mathbb{R}$, the unitary operator $u(t) = e^{itT}$ is an element of $\mathrm{I}\Psi^0_h(H)$ with principal Lie symbol $e^{it\tau_0}$ where τ_0 is the principal Lie symbol of T.

In fact, if $(T^\lambda)_{\lambda\in\Lambda}$ is a family of such operators, such that the Lie symbols $(\tau^\lambda)_{\lambda\in\Lambda}$ form a bounded subset of $S^0_h(\mathfrak{h}_{\mathbb{R}})$, then a family of operators

$$u^\lambda_\infty(t) = \mathrm{op}_\chi\big(a^\lambda(\sqcup, t)\big) \in \mathrm{I}\Psi^0_h(H)$$

can be constructed such that for each $t \in \mathbb{R}$, $(u^\lambda(t) - u^\lambda_\infty(t))_{\lambda\in\Lambda}$ is a bounded family of smoothing operators in $\mathcal{L}(\mathrm{H}^{(m)}(H), \mathrm{H}^{(m+k)}(H))$ for all $m \in \mathbb{Z}$ and $k \in \mathbb{N}$.

The proof begins with a lemma which gives an algorithm for the computation of the total Lie symbol of $u(t)$.

Lemma 3.2.8. Let $T \in \mathrm{I}\Psi^0_h(H)$ be properly supported and self-adjoint with principal Lie symbol $\tau_0 : \mathfrak{h}^*_{\mathbb{R}} \to \mathbb{R}$. Fix $\chi = 1$ in a neighbourhood of unity.

There exist $b_k \in \mathcal{E}(\mathbb{R}, S^{-k}_h(\mathfrak{h}_{\mathbb{R}}))$ homogeneous of degree $-k$ in ξ such that $b_0 = 1$, $b_k(\xi, 0) = 0$ for all $\xi \in \mathfrak{h}^*_{\mathbb{R}}$, $k \geqslant 1$,

$$t \longmapsto \|(1 + |\xi|)^{|\mu|}\partial^\nu_\xi b_k(\sqcup, t)\|_\infty$$

is bounded by a polynomial in $|t|$ of degree at most $2k$ for any $|\mu| \leqslant |\nu|$, $k \in \mathbb{N}$, and

$$\dot{u}_k - iTu_k : \mathbb{R} \to \mathrm{I}\Psi^{-(k+1)}_h(H) .$$

Here, $u_k(t) = \mathrm{op}_\chi\big(e^{it\tau_0} \cdot \sum_{j=0}^k b_j(\sqcup, t)\big)$.

In fact, if $(T^\lambda)_{\lambda\in\Lambda}$ is a family such that the symbols $(\tau^\lambda)_{\lambda\in\Lambda}$ form a bounded subset of $S^0_h(\mathfrak{h}_{\mathbb{R}})$, then the polynomial bounds of $(1 + |\xi|)^{|\mu|}\partial^\nu_\xi b^\lambda_k(\xi, t)$ are independent of λ.

Proof. Proof by induction on k. The case $k = 0$ being clear, assume the statement is true for $k - 1$. Let $r_k(t) = \dot{u}_{k-1}(t) - iT \cdot u_{k-1}(t) \in \mathrm{I}\Psi^{-k}(H)$. We wish to find b_k such that

$$u_k(t) = u_{k-1}(t) + \mathrm{op}_\chi\big(e^{it\tau_0} b_k(\sqcup, t)\big)$$

has the required properties. Assuming such a b_k given for the moment, we compute

$$\begin{aligned} r_{k+1}(t) = \dot{u}_k(t) - iTu_k(t) &= r_k(t) + \partial_t \operatorname{op}_\chi\big(e^{it\tau_0} \cdot b_k(\sqcup, t)\big) - iT \cdot \operatorname{op}_\chi\big(e^{it\tau_0} b_k(\sqcup, t)\big) \\ &= r_k + \operatorname{op}_\chi\big(e^{it\tau_0} \dot{b}_k(\sqcup, t)\big) + i \operatorname{op}_\chi\big(\tau_0 e^{it\tau_0} b_k(\sqcup, t)\big) - iT \cdot \operatorname{op}_\chi\big(e^{it\tau_0} b_k(\sqcup, t)\big) \,. \end{aligned}$$

The sum of the two latter terms is in $\mathrm{I}\Psi^{-(k+1)}(H)$, since the top order term in the asymptotic expansion of the Lie symbol of $T \cdot \operatorname{op}_\chi\big(e^{it\tau_0} b_k(\sqcup, t)\big)$ is $\tau_0 e^{it\tau_0} b_k(\sqcup, t)$ by [Str72, th. 2]. Hence, the requirement is that $r_k(t) = -\operatorname{op}_\chi\big(e^{it\tau_0} \dot{b}_k(\sqcup, t)\big)$, up to terms of lower degree. So, let $a_k \in S^{-k}(\mathfrak{h}_{\mathbb{R}})$ be the principal Lie symbol of r_k, and define

$$b_k(\xi, t) = -\int_0^t e^{-is\tau_0(\xi)} a_k(\xi, s)\, ds \quad \text{for all } \xi \in \mathfrak{h}_{\mathbb{R}}^*,\, t \in \mathbb{R}\,.$$

Then $b_k \in \mathcal{E}(\mathbb{R}, S^{-k}(\mathfrak{h}_{\mathbb{R}}))$, and is homogeneous of degree $-k$ because τ_0 has degree of homogeneity 0.

By abuse of notation, we write $\deg_t(a(\xi, t))$ for the degree of the polynomial bound in $|t|$ for a function $a(\xi, t)$. In particular, let $B_{k,\ell} = \max_{|\mu| \leqslant |\nu| \leqslant \ell} \deg_t (1 + |\xi|)^{|\mu|} \partial_\xi^\nu b_k(\xi, t)$.

By definition, $r_k = \dot{u}_{k-1} - iT \cdot u_{k-1}$. The total Lie symbol of $\dot{u}_{k-1}$ is the t derivative of $e^{it\tau_0} \cdot \sum_{j=0}^{k-1} b_j(\sqcup, t)$. Since this expansion only involves only terms homogeneous of degree $> -k$, we see that a_k only depends on $iT \cdot u_{k-1}$. Then the asymptotic expansions in [Str72, lem. 3, th. 2, and rem.] show that

$$a_k(\xi, t) = \sum_{i=0}^{k-1} \sum_{j=1}^{k-i} \sum_{|\alpha+\gamma| \leqslant k-(i+j)} \sum_{|\beta| = i+j-k+|\alpha+\gamma|} c_{\alpha\beta\gamma} \cdot \xi^\beta \cdot \partial_\xi^\alpha \tau_j(\xi) \cdot \partial_\xi^\gamma \big(e^{it\tau_0(\xi)} b_i(\xi, t)\big)$$

for some $c_{\alpha\beta\gamma} \in \mathbb{C}$, where $\tau \sim \sum_{k=0}^\infty \tau_k$ is the Lie symbol of T. By Leibniz' rule,

$$\partial_\xi^\nu \big(e^{-it\tau_0(\xi)} \cdot \partial_\xi^\gamma \big(e^{it\tau_0(\xi)} \cdot b_i(\xi, t)\big)\big) = \sum_{\beta \leqslant \gamma} c_{\beta\gamma} \cdot t^{|\gamma - \beta|} \cdot \partial_\xi^{\nu+\beta} b_i(\xi, t)$$

for some $c_{\beta\gamma} \in \mathbb{C}$. Moreover,

$$|\xi^\beta \partial_\xi^\alpha \tau_j(\xi)| \lesssim (1 + |\xi|)^{|\beta| - |\alpha| - j} = (1 + |\xi|)^{k - i - |\gamma|}$$

where $k - i - |\gamma| \leqslant -j < 0$. If τ is part of a bounded family $(\tau^\lambda)_{\lambda \in \Lambda}$, $\tau^\lambda \sim \sum_{k=0}^\infty \tau_k^\lambda$, then for each $k \in \mathbb{N}$, the family $(\tau_k^\lambda)_{\lambda \in \Lambda}$ is bounded in $S^{-k}(\mathfrak{h}_{\mathbb{R}})$. Hence, the constants occurring in the bound of $\xi^\beta \partial_\xi^\alpha \tau_j^\lambda(\xi)$ are independent of $\lambda \in \Lambda$.

Since integration raises the degree by one and $B_{i,q} \leqslant 2i$, $i < k$, we have

$$\begin{aligned} &\deg_t (1 + |\xi|)^{|\mu|} \partial_\xi^\nu b_k(\xi, t) \\ &\qquad\qquad \leqslant 1 + \max_{i=0}^{k-1} \max_{|\nu| \leqslant \ell} \max_{|\gamma| \leqslant k-i} \deg_t (1 + |\xi|)^{|\mu|} \partial_\xi^\nu \big(e^{-it\tau_0(\xi)} \partial_\xi^\gamma \big(e^{it\tau_0(\xi)} b_i(\xi, t)\big)\big) \end{aligned}$$

$$\leqslant 1 + \max_{i=0}^{k-1} \max_{|\nu|\leqslant \ell} \max_{|\gamma|\leqslant k-i} \max_{\beta\leqslant\gamma} |\gamma| - |\beta| + \deg_t (1+|\xi|)^{|\mu|} \partial_\xi^{\nu+\beta} b_i(\xi,t)$$
$$\leqslant 1 + \max_{i=0}^{k-1} \big(k - i + \max_{q\leqslant k-i} B_{i,q+\ell}\big) \leqslant 2k\,,$$

for $|\mu| \leqslant |\nu| \leqslant \ell$, which is the required estimate. All the constants occurring in this polynomial bound are independent of λ, by the inductive hypothesis and the conditions on τ^λ.

Now, $r_k - \mathrm{op}_\chi(a_k) \in \mathrm{I}\Psi^{-(k+1)}(H)$, so, defining u_k as above, $r_{k+1}(t) \in \mathrm{I}\Psi^{-(k+1)}(H)$, completing the lemma's proof. □

Proof of proposition 3.2.7. Given a sequence b_k as in lemma 3.2.8, define

$$u_\infty(t) = \mathrm{op}_\chi\big(e^{it\tau_0} b(\sqcup, t)\big) \quad \text{where} \quad b(\sqcup,t) \sim \sum_{j=0}^{\infty} b_j(\sqcup,t)\,.$$

We show that $v(t) = u(t) - u_\infty(t)$ is smoothing. Observe $u_\infty(0) = 1$, and by the corresponding properties of the u_k, $r_\infty(t) = \dot u_\infty(t) - iT\cdot u_\infty(t)$ is smoothing. Moreover, lemma 3.2.11 shows that its operator norm on Sobolev space is bounded independently of $\lambda \in \Lambda$, since the polynomial bounds on $(1+|\xi|)^{|\mu|}\partial_\xi^\nu b_k(\xi,t)$, $|\mu| \leqslant |\nu|$, are.

We have

$$\dot v(t) = iT\cdot u(t) - \dot u_\infty(t) = iT\cdot v(t) + r_\infty(t) \quad \text{for all } t \in \mathbb{R}\,.$$

By integrating factors,

$$v(t) = -u(t)\int_0^t u(-s) r_\infty(s)\,ds \quad \text{for all } t\in\mathbb{R}\,.$$

The above differential equation and the integral make sense as bounded operators between suitable Sobolev spaces. In particular, $r_\infty(t) \in \mathrm{I}\Psi^{-\infty}(H)$ and is properly supported, hence bounded $\mathrm{H}^{(m)}(H) \to \mathrm{H}^{(m+k)}(H)$ for all $m\in\mathbb{Z}$ and $k\in\mathbb{N}$ by [Str72, th. 7]. $u(t)$ is bounded on $\mathrm{H}^{(s)}(H)$ for all $s\in\mathbb{R}$ since this is true of T by the same theorem. Hence $v(t) : \mathrm{H}^{(m)}(H) \to \mathrm{H}^{(m+k)}(H)$ is bounded for all $m\in\mathbb{Z}$ and $k\in\mathbb{N}$. The norm is bounded independently of $\lambda\in\Lambda$.

From lemma 3.2.10 below we conclude that $v(t)$ is smoothing, and therefore we conclude $u(t) \in \mathrm{I}\Psi^0(H)$, as required. □

Remark 3.2.9. Apart from the polynomial estimates in lemma 3.2.8, the proof of proposition 3.2.7 goes through for non-classical symbols. However, these estimates are essential in the proof of the functional calculus below.

The following lemma was employed in the proof of proposition 3.2.7.

Lemma 3.2.10. Let $A : \mathcal{D}(H) \to \mathcal{D}'(H)$ be continuous. If there are continuous extensions $A : \mathrm{H}^{(k)}(H) \to \mathrm{H}^{(k+\ell)}(H)$ for all $k\in\mathbb{Z}$ and $\ell\in\mathbb{N}$, then A is smoothing.

Proof. Choose an open exponential neighbourhood $V \subset H$ of unity. Then Haar measure, restricted to V, is equivalent to the image of Lebesgue measure on $U = \log(V)$. We get the equality $\mathbf{L}^2(V, dg) = \mathbf{L}^2(U, dX)$, with equivalence of norms. Since the topology of the local Sobolev spaces is independent of the choice of local frames of the tangent bundle, we find $\mathrm{H}_c^{(s)}(U) = \mathrm{H}_c^{(s)}(V)$ and $\mathrm{H}_{loc}^{(s)}(V) = \mathrm{H}_{loc}^{(s)}(U)$ in the category of locally convex vector spaces, for all $s \in \mathbb{R}$.

Note that by [Goo80, th. 1.1], the topology of $\mathrm{H}^{(m)}(H)$, $m \in \mathbb{N}$, is the weakest locally convex topology such that all left-invariant differential operators of order $\leqslant m$ are bounded with values in $\mathbf{L}^2(H)$. Hence, for all $\varphi \in \mathcal{D}(U)$, and $f \in \mathrm{H}^{(m)}(H)$, $\varphi \cdot f \in \mathrm{H}_{loc}^{(m)}(U)$. The Sobolev lemma [Kg81, ch.3, § 2, lem. 2.5, cor.], applied locally, shows that $\mathrm{H}^{(m+k)}(H) \subset \mathcal{C}^{(m)}(H)$ for $2k > \dim H$. By the closed graph theorem, the inclusion is continuous.

In particular, $A : \mathcal{D}(H) \to \mathcal{E}(H)$ is continuous. Therefore, A is very regular in the sense of Schwartz, and has a continuous extension $A : \mathcal{E}'(H) \to \mathcal{D}'(H)$. To see that A is smoothing, we need to see that $A : \mathcal{E}'(H) \to \mathcal{E}(H)$.

To that end, let $\mu \in \mathcal{E}'(H)$. Then μ has finite order m, in particular, defines a continuous linear form on $\mathrm{H}^{(m+k)}(H)$ where $2k > \dim H$. This implies $\mu \in \mathrm{H}^{(-(m+k))}(H)$. Hence $A\mu \in \mathrm{H}^{(\infty)}(H) \subset \mathcal{E}(H)$. This proves the lemma. □

To prove the functional calculus, we need to make the dependence of the operator norm on the Lie symbol in Strichartz's Sobolev boundedness result explicit.

Lemma 3.2.11. Let $0 \in V \subset \mathfrak{h}_{\mathbb{R}}$ be an open, relatively compact, and exponential neighbourhood, such that the Campbell-Hausdorff series converges on U. Let K be compact, so that $0 \in K^\circ \subset K \subset U$ and $1_K \leqslant \chi \leqslant 1_U$, $\chi \in \mathcal{E}(\mathfrak{h}_{\mathbb{R}})$. For all $\tau \in S^m(\mathfrak{h}_{\mathbb{R}}^*)$, $s \geqslant 0$,

$$\|T\varphi\|_{H^{(s)}} \leqslant C \cdot \sup_{|\mu|=|\nu|\leqslant n+1,\,\xi} |(1+|\xi|)^{|\mu|} \cdot \partial_\xi^\nu \tau(\xi)| \cdot \|\varphi\|_{H^{(s+m)}}$$

where $n = \dim H$, $T = \mathrm{op}_\chi(\tau)$, and the constant C is independent of τ. If H is unimodular, the estimate is valid for all $s \in \mathbb{R}$.

Proof. From the proof of [Str72, th. 1], we find

$$\chi(X)T\varphi(\exp X) = \frac{1}{(2\pi)^m} \cdot \iint_{\mathfrak{h}_{\mathbb{R}} \times \mathfrak{h}_{\mathbb{R}}^*} e^{i\langle X-Y:\xi\rangle} \cdot \tau(W(X,Y)\xi)R(X,Y) \cdot \varphi(\exp Y)\, dY\, d\xi$$

for all $X \in U$, where R is smooth with compact support in $U \times U$ and the function $W : U \times U \to \mathrm{GL}(\mathfrak{h}_{\mathbb{R}}^*)$ is smooth. Moreover, W and R are independent of τ. Let

$$\sigma(X,Y,\xi) = \tau(W(X,Y)\xi) \cdot R(X,Y) \quad \text{for all } X, Y \in U,\ \xi \in \mathfrak{h}_{\mathbb{R}}^*$$

and $V = \exp(U)$. Then, by [Trè80, th. 2.1 and proof], for all $\varphi \in \mathcal{D}(V)$,

$$\|\chi \cdot (T\varphi) \circ \exp\|_{H^{(s)}} \leqslant C \cdot \sup_{|\alpha|,|\beta|\leqslant n+1, X, Y, \xi} |\partial_X^\alpha \partial_Y^\beta \sigma(X,Y,\xi)| \cdot \|\varphi \circ \exp\|_{H^{(s+m)}}$$

where the constant C is independent of τ. Since $\mathrm{H}_c^{(s)}(U)$ and $\mathrm{H}_c^{(s)}(\exp U)$ are isomorphic with equivalent norms, we have

$$\|\psi\cdot(T\varphi)\|_{H^{(s)}} \leqslant C'\cdot \sup_{|\alpha|,|\beta|\leqslant n+1,\,X,\,Y,\,\xi} |\partial_X^\alpha\partial_Y^\beta\sigma(X,Y,\xi)|\cdot\|\varphi\|_{H^{(s+m)}} \quad \text{for all } \varphi\in\mathcal{D}(V)$$

where $\psi=\chi\circ\log$.

By induction, it is easy to prove

$$\partial_X^\alpha\partial_Y^\beta\big[\sigma(X,Y,\xi)\big] = \sum_{|\mu|=|\nu|\leqslant|\alpha+\beta|} c_{\mu\nu}(X,Y)\cdot\xi^\alpha\big(\partial_\xi^\beta\tau\big)(W(X,Y)\xi)$$

for some smooth $c_{\mu\nu}$ with compact support in $U\times U$, independent of τ. In particular,

$$\sup_{|\alpha|,|\beta|\leqslant n+1,\,X,\,Y,\,\xi} |\partial_X^\alpha\partial_Y^\beta\sigma(X,Y,\xi)| \leqslant \sum_{|\mu|=|\nu|\leqslant|\alpha+\beta|} c'_{\mu\nu}\cdot\sup\nolimits_\xi |(1+|\xi|)^{|\mu|}\cdot\partial_\xi^\nu\tau(\xi)|$$

for some positive constants $c'_{\mu\nu}$.

Combining this information, we get

$$\|\psi\cdot(T\varphi)\|_{H^{(s)}} \leqslant C''\cdot \sup_{|\mu|=|\nu|\leqslant n+1,\,\xi} |(1+|\xi|)^{|\mu|}\partial_\xi^\nu\tau(\xi)|\cdot\|\varphi\|_{H^{(s+m)}} \quad \text{for all } \varphi\in\mathcal{D}(U)$$

where C'' is independent of τ. Let $\zeta\in\mathcal{D}(H)$ such that

$$\zeta(1)=1 \quad\text{and}\quad (\operatorname{supp}\zeta)\cdot(\exp\operatorname{supp}\chi)\subset U\,,$$

and choose $\phi\in\mathcal{D}(U)$ such that $\phi=1$ on $(\operatorname{supp}\zeta)\cdot(\exp\operatorname{supp}\chi)$. Then

$$\zeta(h)\cdot\chi(Y)=\zeta(h)\cdot\chi(Y)\cdot\phi\big(h\exp Y\big) \quad\text{for all } h\in H,\ Y\in\mathfrak{h}_{\mathbb{R}}\,,$$

so we conclude

$$\zeta\cdot(T\varphi)=\zeta\cdot T(\phi\cdot\varphi)=\zeta\cdot\psi\cdot T(\phi\cdot\varphi) \quad\text{for all } \varphi\in\mathcal{D}(H)\,.$$

Introducing ζ does not change the above estimate. Plugging in $g*\varphi$ in place of φ and integrating, we see

$$\begin{aligned}&\left(\int_H \|\zeta\cdot T(h*\varphi)\|_{H^{(s)}}^2\,dh\right)^{1/2}\\ &\qquad\leqslant C''\cdot\sup_{|\mu|=|\nu|\leqslant n+1,\,\xi}|(1+|\xi|)^{|\mu|}\partial_\xi^\nu\tau(\xi)|\cdot\left(\int_H\|\phi\cdot(h*\varphi)\|_{H^{(s+m)}}\,dh\right)^{1/2}.\end{aligned}$$

By [Str72, th. 8], the integrated norms are equivalent to the usual norm on $\mathrm{H}^{(s)}(H)$ for

$s \geqslant 0$, so we have the assertion for $s \geqslant 0$.

If H is unimodular, $T^* = \mathrm{op}_\chi(\bar{\tau})$ by [Str72, th. 3], so for $s < 0$, we may apply the above to get the estimate for $T^* : \mathrm{H}^{(-s)}(H) \to \mathrm{H}^{(-s+m)}(H)$. By duality [Str72, th. 7, cor.], we get the estimate for $T : \mathrm{H}^{(s-m)}(H) \to \mathrm{H}^{(s)}(H)$. Since the $\mathrm{H}^{(s)}(H)$ form a scale [Goo80], the result follows by interpolation. □

We now assume that the group H is unimodular.

Theorem 3.2.12. Let $T = (T_1, \dots, T_m)$, $T_j \in \mathrm{I\Psi}^0(H)$ properly supported, with principal Lie symbol τ_0^j, and self-adjoint on $\mathbf{L}^2(H)$. Then, for $f \in \mathcal{E}(\mathbb{R}^m)$, the bounded operator $f(T)$ on $\mathbf{L}^2(H)$, defined by

$$f(T) = \frac{1}{(2\pi)^m} \cdot \iint_{\mathbb{R}^m \times \mathbb{R}^m} e^{-i\langle x:\xi\rangle} f(\xi) \cdot e^{i(x_1T_1+\dots+x_mT_m)}\,dx\,d\xi \quad \text{for all } f \in \mathcal{D}(\mathbb{R}^m),$$

lies in $\mathrm{I\Psi}^0(H)$, with principal Lie symbol $f(\tau_0^1, \dots, \tau_0^m)$.

Proof. Define $\hat{f}(x) = (2\pi)^{-m/2} \cdot \int_{\mathbb{R}^m} e^{-i\langle x:\xi\rangle} f(x)\,dx$ whenever this makes sense. Observe that

$$f \mapsto \frac{1}{(2\pi)^{m/2}} \cdot \int_{\mathbb{R}} \hat{f}(x) e^{i(x_1T_1+\cdots x_mT_m)}\,dx$$

is an operator-valued distribution with compact support contained in

$$K = \big[-\|T_1\|, \|T_1\|\big] \times \cdots \big[-\|T_m\|, \|T_m\|\big],$$

by the Paley-Wiener theorem, cf. [And69]. Hence, $f(T)$ makes sense for $f \in \mathcal{E}(\mathbb{R}^m)$, and only depends on $\chi \cdot f$ where $\chi \in \mathcal{D}(\mathbb{R}^m)$, $\chi = 1$ on K.

Thus, w.l.o.g., let $f \in \mathcal{D}(\mathbb{R}^m)$. For $\lambda \in \mathbb{S}^{n-1}$, let $b_k^\lambda(\sqcup, t)$ be the sequence of symbols constructed in lemma 3.2.8 for the operator $\lambda_1 T_1 + \dots + \lambda_m T_m \in \mathrm{I\Psi}^0(H)$. Since $\hat{f}$ is rapidly decreasing, the integral

$$b_k^\lambda(\xi, f) = \frac{2\pi^{n/2}}{\Gamma(m/2) \cdot (2\pi)^{m/2}} \cdot \int_0^\infty \hat{f}(r \cdot \lambda) e^{ir(\lambda_1\tau_0^1(\xi)+\dots+\lambda_m\tau_0^m(\xi))} b_k^\lambda(\xi, r)\, r^{m-1} dr$$

converges in the Fréchet space $S_h^{-k}(\mathfrak{h}_{\mathbb{R}}^*)$, for all $k \in \mathbb{N}$ and $\lambda \in \mathbb{S}^{n-1}$, because of the polynomial estimates from lemma 3.2.8. Moreover, $(b_k^\lambda(\sqcup, f))_{\lambda \in \mathbb{S}^{n-1}}$ is bounded in this space.

Let $b^\lambda(\sqcup, f) \sim \sum_{k=0}^\infty b_k^\lambda(\sqcup, f)$, and let

$$f_\infty(T) = \frac{1}{\Gamma(m/2) \cdot 2^{m/2-1}} \cdot \int_{\mathbb{S}^{m-1}} \mathrm{op}_\chi\big(b^\lambda(\sqcup, f)\big)\, d\sigma(\lambda),$$

where $d\sigma$ is surface measure on $\mathbb{S}^{m-1}$, so $f_\infty(T) \in \mathrm{I\Psi}_h^0(H)$ with principal Lie symbol

$$\xi \mapsto \frac{1}{\Gamma(m/2) \cdot 2^{m/2-1}} \cdot \int_{\mathbb{S}^{m-1}} \int_0^\infty \hat{f}(r \cdot \lambda) e^{ir(\lambda_1\tau_0^1(\xi)+\dots+\lambda_m\tau_0^m(\xi))}\, r^{m-1} dr\, d\sigma(\lambda)$$

$$= \frac{1}{(2\pi)^{m/2}} \cdot \int_{\mathbb{R}^m} \hat{f}(x) e^{i\langle x:(\tau_0^1(\xi),\dots,\tau_0^m(\xi))\rangle}\,dx = f(\tau_0^1(\xi),\dots,\tau_0^m(\xi))\,,$$

by Cavalieri's principle and Fourier inversion. We need to show that $f(T) - f_\infty(T)$ is smoothing.

Define $u^\lambda(t) = \exp(it(\lambda_1 T_1 + \cdots + \lambda_m T_m))$. Since all operators involved are properly supported, the Sobolev continuity result [Str72, th. 7] shows that the proof of [GS79, lem. A.1] goes through without changes, proving

$$\|u^\lambda(t)\psi\|_{H^{(k)}} \leqslant C_k \cdot \big(\|\psi\|_{H^{(k)}} + t^k \cdot \|\psi\|_2\big) \quad \text{for all } \psi \in \mathcal{D}(H)$$

where the constant C_k depends only on k. By interpolation, this extends to all $s \geqslant 0$.

Moreover, $\mathrm{H}^{(s)}(H) \subset \mathbf{L}^2(H)$ for all $s \geqslant 0$, and the inclusion is continuous, cf. [Str72, th. 7, cor.]. Hence, we may estimate the $\mathbf{L}^2$ norm on the right hand side by $\|\sqcup\|_{H^{(s)}}$, by increasing the constant C_s.

Recall $r_k^\lambda(t) = \dot{u}_k^\lambda(t) - iT \cdot u_k^\lambda(t) \in \mathrm{I}\Psi^{-(k+1)}(H)$ from lemma 3.2.8 where u_k^λ corresponds to the operator $\lambda_1 T_1 + \cdots + \lambda_m T_m$. By the proof of lemma 3.2.8, $r_k^\lambda = \mathrm{op}_\chi(a^\lambda)$ where $\deg_t (1+|\xi|)^\ell \partial_\xi^\beta a(\xi,t) \leqslant 2k$ for all $\ell \leqslant |\beta|$ and the polynomial bound can be chosen independent of λ. Hence, lemma 3.2.11 gives

$$\|r_k^\lambda(t)\psi\|_{H^{(s+k+1)}} \leqslant C_{sk} \cdot (1+|t|)^{2k} \cdot \|\psi\|_{H^{(s)}} \quad \text{for all } \psi \in \mathcal{D}(H)\,,\ s \geqslant 0$$

with C_{sk} independent of ψ, t and λ.

By integrating factors,

$$u^\lambda(t) - u_k^\lambda(t) = -u^\lambda(t) \cdot \int_0^t u^\lambda(s) r_k^\lambda(s)\,ds\,,$$

so we find

$$\|(u^\lambda(t) - u_k^\lambda(t))\psi\|_{H^{(s+k+1)}} \leqslant C'_{sk} \cdot (1+|t|)^{4s+1} \cdot \|\psi\|_{H^{(s)}} \quad \text{for all } \psi \in \mathcal{D}(H)\,,\ s \geqslant 0$$

with C'_{sk} independent of ψ, t and λ. This proves that

$$\begin{aligned} f(T) &- \frac{1}{\Gamma(m/2)\cdot 2^{n/2-1}} \cdot \int_{\mathrm{S}^{m-1}} \int_0^\infty \hat{f}(r\cdot\lambda) u_k^\lambda(r)\, r^{m-1} dr\, d\sigma(\lambda) \\ &= \frac{1}{\Gamma(m/2)\cdot 2^{n/2-1}} \cdot \int_{\mathrm{S}^{m-1}} \int_0^\infty \hat{f}(r\cdot\lambda)(u^\lambda(r) - u_k^\lambda(r))\, r^{m-1} dr\, d\sigma(\lambda) \end{aligned}$$

is bounded $\mathrm{H}^{(s)}(H) \to \mathrm{H}^{(s+k+1)}(H)$ for all $s \geqslant 0$. But, by definition,

$$f_\infty(T) - \frac{1}{\Gamma(n/2)\cdot 2^{n/2-1}} \cdot \int_{\mathrm{S}^{m-1}} \int_0^\infty \hat{f}(r\cdot\lambda) u_k^\lambda(r)\, r^{m-1} dr\, d\sigma(\lambda)$$

is an invariant, properly supported pseudo-differential operator of order $-(k+1)$, and consequently also bounded $\mathrm{H}^{(s)}(H) \to \mathrm{H}^{(s+k+1)}(H)$.

Therefore, the same is true of $f(T) - f_\infty(T)$. Since $f(T)^* = f^*(-T)$ where $f^*(t) = \overline{f(-t)}$, the same argument applies to $f(T)^*$, and by duality and interpolation,

$$f(T) - f_\infty(T) : \mathrm{H}^{(s)}(H) \to \mathrm{H}^{(s+k)}(H)$$

is bounded for all $s \in \mathbb{R}$ and $k \in \mathbb{N}$. So, by lemma 3.2.10, $f(T) - f_\infty(T)$ is smoothing, and $f(T) \in \mathrm{I}\Psi^0_h(H)$ with principal Lie symbol $f(\tau^1_0, \ldots, \tau^m_0)$. □

3.3 Wave front of the Szegö distribution

3.3.1. The cotangent bundle $T^*(H)$ of H is canonically trivialised, and shall therefore be identified with $H \times \mathfrak{h}^*_{\mathbb{R}}$.

So, for $\mu \in \mathcal{D}'(H)$, the wave front set $\mathrm{WF}(\mu) \subset H \times \mathfrak{h}^*_{\mathbb{R}}$. Namely,

$$\mathrm{WF}(\mu) = \bigcap \left\{ \operatorname{char} A \;\middle|\; A\mu \in \mathcal{E}(H),\ A \in \mathrm{I}\Psi^0_h(H) \text{ properly supported} \right\} .$$

Here, whenever $A \in \mathrm{I}\Psi^m_h(H)$, the *characteristic set* $\operatorname{char} A \subset H \times \mathfrak{h}^*_{\mathbb{R}}$ is

$$\operatorname{char} A = \left\{ (h, \xi) \in H \times \mathfrak{h}^*_{\mathbb{R}} \;\middle|\; \liminf_{t\to\infty} t^{-m} \left| a_m(h, \xi) \right| = 0 \right\}$$

where a_m is a principal symbol of A.

Theorem 3.3.2. Assume $H \subset H^{\mathbb{C}}$, and, moreover, the existence of the Ol'shanskiĭ domain $\Gamma^\circ = H \cdot \exp i\Omega^\circ$ defined by a closed, pointed, and generating H-invariant cone $\Omega \subset \mathfrak{h}_{\mathbb{R}}$. If E is the associated Szegö distribution, then

$$\mathrm{WF}(E) \subset H \times \Omega^*$$

where Ω^* is the closed dual cone of Ω.

Corollary 3.3.3. If $f = (c_1 > \cdots > c_m > 0 = c_{m+1})$ is a flag of tripotents of the simple JB*-triple Z, and $I \subset \{1, \ldots, m\}$, then for the Szegö distribution $E_{f,I}$ of the group $G_{f,I}$, we have

$$\mathrm{WF}(E_{f,I}) \subset G_{f,I} \times F^{-*}_{f,I} .$$

Proof of theorem 3.3.2. Fix $t \in H$. By proposition 3.3.4 below, there exists a compact neighbourhood $V \subset \Gamma$ of 1 such that $K(t(\exp_\sqcup)\gamma)$ has tempered growth at the boundary on $W \cap (\mathfrak{h}_{\mathbb{R}} + i\Omega^\circ)$ for some exponential neighbourhood $W \subset \mathfrak{h}$ of 0, uniformly in $\gamma \in V$. We assume this proposition for the moment, and postpone its proof.

By a theorem of Martineau [Mar77, ch. III,§ 1, th. 2, and proof] (see also [Iag78, app. I.1, lem. 3], or [Hör83, th. 3.1.15]), there exist distributional boundary values μ_γ,

$$\langle \varphi \,:\, \mu_\gamma \rangle = \lim_{r\to 0+} \int_{\mathfrak{h}_\mathbb{R}} \varphi(X) K(t \exp(X + irY)\gamma)\, dX \quad \text{for all} \;\; \varphi \in \mathcal{D}(W_\mathbb{R})$$

where $W_\mathbb{R} = W \cap \mathfrak{h}_\mathbb{R}$, and $Y \in \Omega^\circ$ is arbitrary, but fixed. Since the tempered growth of $K(t(\exp\sqcup)\gamma)$ is uniform in γ, the above convergence in $\mathcal{D}'(W_\mathbb{R})$ is uniform in $\gamma \in V$. In particular, $\gamma \mapsto \mu_\gamma : V \to \mathcal{D}'(W_\mathbb{R})$ is continuous.

On the other hand, because K is continuous on Γ°, by dominated convergence,

$$\begin{aligned} \int_H \varphi(\log s) K(ts\gamma)\, ds &= \int_{\mathfrak{h}_\mathbb{R}} \varphi(X) K(t(\exp X)\gamma) \varrho(X)\, dX \\ &= \lim_{r\to 0+} \int_{\mathfrak{h}_\mathbb{R}} \varphi(X) K(t \exp(X + irY)\gamma) \varrho(X)\, dX\,. \end{aligned}$$

Here, $\varrho > 0$ is the smooth function such that $ds = \exp_*(\varrho \cdot dX)$ on $\exp(W_\mathbb{R})$.

Moreover, by proposition 3.1.13,

$$\langle t * (\varphi \circ \log) \,:\, E \rangle = \lim_{\gamma\to 1} \int_H \varphi(\log s) K(ts\gamma)\, ds\,.$$

Since one of the inner limits is uniform, we can exchange limit order by Moore's theorem [DS58, I.7, lem. 5], so we find

$$E = \big(t * \exp_*(\varrho \cdot \mu_1)\big) \quad \text{on} \quad t\exp(W_\mathbb{R})\,.$$

The fibre of the analytic wave front set of μ_1 at 0 is contained in Ω^*, cf. [Sjö82, th. 6.5], [Iag78, C.1, lem. 3, p. 113]. Consequently, the fibre of $\mathrm{WF}(E)$ at t is also contained in Ω^* [Hör71b, th. 3.4], [Bon77]. □

Proposition 3.3.4. Let $K : \Gamma^\circ \to \mathbb{C}$ be the Szegö kernel function and $t \in H$. Then there exists a compact neighbourhood V of 1 in the closed Ol'shanskiĭ semigroup Γ such that the functions

$$K\big(t^{-1}(\exp\sqcup)\gamma\big) : \mathfrak{h}_\mathbb{R} + i\Omega^\circ \to \mathbb{C} : Z \mapsto K\big(t^{-1}(\exp Z)\gamma\big)$$

have tempered growth on $W \cap (\mathfrak{h}_\mathbb{R} + i\Omega^\circ)$ for some neighbourhood $W \subset \mathfrak{h}$ of 0, uniformly in $\gamma \in V$.

Remark 3.3.5. Our proof of this fact is rather technical, so we have divided it into three parts. The first part is a simple elaboration of the estimates in [HN93, proof of th. 9.31] proving that the norm topology of $\mathbf{H}^2(\Gamma^\circ)$ is finer than the topology of local uniform convergence. The next two steps, stated as lemmata below, are more technical, concerning estimates of the Baker-Campbell-Hausdorff formula.

Proof of proposition 3.3.4. Consider the smooth map

$$\phi : H^{\mathbb{C}} \times \mathfrak{h} \to H^{\mathbb{C}} : (\gamma, X + iY) \mapsto \phi_\gamma(X + iY) = t^{-1}(\exp iY)\gamma(\exp X) \,.$$

Then ϕ_0 is regular at 0, and $(\gamma, Z) \mapsto D\phi_\gamma(Z)$ is continuous, so there exists a compact neighbourhood $K \subset H^{\mathbb{C}}$ of 1 and a compact neighbourhood $L \subset \mathfrak{h}_{\mathbb{R}}$ such that

$$\det D\phi_\gamma(X + iY) \neq 0 \quad \text{for all } \gamma \in K\,,\ X, Y \in L\,.$$

In particular,

$$\phi_\gamma : L^\circ + iL^\circ \to t^{-1}(\exp iL)\gamma(\exp L)$$

is a diffeomorphism onto its open image for all $\gamma \in K$. We may also assume that

$$iL^\circ \to (\exp iY)\gamma(\exp L) : X \mapsto \phi_\gamma(X + iY)$$

is a diffeomorphism for all $\gamma \in K$ and $Y \in L$. We note that due to the regularity of left multiplication on $H^{\mathbb{C}}$, L is independent of t.

Let

$$\varepsilon = \inf_{\gamma \in K,\, X, Y \in L} |\det D_X \phi_\gamma(X + iY)| > 0\,.$$

Furthermore, for $z \in K \cap \Gamma^\circ$, let

$$\delta(z) = \sup\big\{ 0 < r \leqslant 1 \;\big|\; C_r(0) \subset L + iL\,,\ \phi_z(C_r(0)) \subset \Gamma^\circ \big\} \in]0, 1]$$

where $C_r(0) = \mathbb{B}_r \times \cdots \times \mathbb{B}_r \subset \mathfrak{h}$ is the open poly-cylinder with radius r. Because Γ° is $H \times H$-invariant, $\delta(z)$ is also independent of t.

Then, for $z \in \Gamma^\circ$ and $f \in \mathbf{H}^2(\Gamma^\circ)$,

$$\begin{aligned}
\big|f(t^{-1}z)\big|^2 = |f \circ \phi_z(0)|^2 &\leqslant \frac{1}{\pi^n \cdot \delta(z)^{2n}} \cdot \int_{C_{\delta(z)}(0)} |f \circ \phi_z(X + iY)|^2 \, dX\, dY \\
&\leqslant \frac{1}{\pi^n \cdot \delta(z)^{2n}} \cdot \int_L \int_L \Big| f\big(t^{-1}(\exp iY) z (\exp X)\big)\Big|^2 \, dX\, dY \\
&\leqslant \frac{1}{\varepsilon \cdot \pi^n \cdot \delta(z)^{2n}} \cdot \int_L \int_H \Big| f\big(t^{-1}(\exp iY) z s\big)\Big|^2 \, ds\, dY \leqslant \frac{\operatorname{vol} L}{\varepsilon \cdot \pi^n \cdot \delta(z)^n} \cdot \|f\|^2_{\mathbf{H}^2}
\end{aligned}$$

where in the first step, [Hel78, ch. VIII, § 3, prop. 3.1 and proof] was employed, and

$$\operatorname{vol} C_{\delta(z)}(0) = \pi^n \cdot \delta(z)^{2n}\,.$$

In particular, by the reproducing property

$$\|K_{t^{-1}z}\|_{\mathbf{H}^2} = \sup_{\|f\|_{\mathbf{H}^2} \leqslant 1} |(K_{t^{-1}z} \mid f)| = \sup_{\|f\|_{\mathbf{H}^2} \leqslant 1} |f(t^{-1}z)| \leqslant \sqrt{\frac{\operatorname{vol} L}{\varepsilon \cdot \pi^n \cdot \delta(z)^{2n}}}$$

for all $z \in K \cap \Gamma^\circ$.

Now, let $K_0 \subset K^\circ$ be a compact neighbourhood of 1. There exists a compact neighbourhood $U \subset K^\circ$ of 1 such that

$$K_0 \cap \Gamma^\circ \subset \bigcup_{\eta \in U} (K \cap \Gamma^\circ)\eta^* .$$

Hence, for $z \in K_0 \cap \Gamma^\circ$ and $\eta \in U$, there is $z_\eta \in K$ such that $z = z_\eta \cdot \eta^*$.

Moreover, there is a compact neighbourhood $V \subset \Gamma$ of 1 such that $V^* \cdot U \subset K$. Thus, for all $z \in K_0 \cap \Gamma^\circ$ and $\gamma \in V$,

$$\begin{aligned} |K(t^{-1}z\gamma)| &= |K(t^{-1}z_\eta\eta^*\gamma)| = |K_{\gamma^*\eta}(t^{-1}z_\eta)| \\ &\leqslant \sqrt{\frac{\operatorname{vol} L}{\varepsilon \cdot \pi^n \cdot \delta(z_\eta)^{2n}}} \cdot \|K_{\gamma^*\eta}\|_{\mathbf{H}^2} \leqslant \frac{\operatorname{vol} L}{\varepsilon \cdot \pi^n \cdot \delta(z_\eta)^n \cdot \delta(\eta)^n} \end{aligned}$$

because $\|K_{\gamma^*\eta}\|_{\mathbf{H}^2} = \|K_{\gamma^*\eta} * t\|_{\mathbf{H}^2} = \|K_{t^{-1}\gamma^*\eta}\|_{\mathbf{H}^2}$ by unimodularity of H, and

$$\delta(\gamma^*\eta) \geqslant \delta(\eta) \quad \text{for all } \ \gamma \in V\,,\ \eta \in U\,.$$

By lemma 3.3.7, for $\exp(X + iY) \in K_0 \cap \Gamma^\circ$, Y varying in a closed cone $C \setminus 0 \subset \Omega^\circ$,

$$\delta(\exp(X + iY)) \geqslant A \cdot \|Y\| \quad \text{for some } \ A > 0$$

since the norms on $\mathfrak{h}$ defining the poly-cylinder and the Euclidean ball are equivalent, and L and δ are independent of t.

By essentially the same argument as the proof of lemma 3.3.7, if $\eta \in U \cap \exp i\Omega^\circ$ is fixed, the norm of V such that

$$\exp(U + iV) \cdot \eta^* = \exp(X + iY)$$

is bounded by $B \cdot \|Y\|$ for some $B > 0$. Hence, the above estimate for $|K(z)|$ shows that, uniformly in $\gamma \in V$, $K(t^{-1}(\exp\sqcup)\gamma)$ has tempered growth on $\log(K_0 \cap \Gamma^\circ)$ if K_0 is chosen small enough to fulfill the conditions of lemma 3.3.7. □

Lemma 3.3.6. Let $X, Y \in \mathfrak{h}$ be in an exponential neighbourhood, such that the Baker-Campbell-Hausdorff series $c(X, Y) = \log\big((\exp X)(\exp Y)\big)$ is well defined. Then

$$\left\| \int_0^1 \big(f(e^{t \operatorname{ad} X} e^{\operatorname{ad} Y}) - 1\big) X \, dt \right\| \leqslant \sqrt{M \cdot (\|X\| + \|Y\|)} \cdot \|X\| \quad \text{for all } \ M \cdot (\|X\| + \|Y\|) \leqslant \frac{1}{9}$$

where $M = \|\operatorname{ad}\|$ is the operator norm of $\operatorname{ad} : \mathfrak{h} \to \operatorname{End} \mathfrak{h}$, and

$$f(z) = \frac{\log z}{z - 1} \quad \text{for all } \ |z - 1| < 1\,.$$

Proof. Note that $c(X,Y) = X + Y + \int_0^1 (f(e^{t\,\mathrm{ad}\,X} e^{\mathrm{ad}\,Y}) - 1)X\,dt$ by [KMS93, ch. I, th. 4.29].

We claim that

$$\sqrt{s} \geqslant -\log(2 - e^s) \quad \text{for all } s \in \left[0, \tfrac{1}{9}\right].$$

We establish this by a brief discussion of the function $h(s) = \sqrt{s} + \log(2 - e^s)$, defined for $0 \leqslant s < \log 2$. Observe $h'(s) = \frac{1}{2\sqrt{2}} - \frac{e^s}{2-e^s}$ for $s > 0$. Hence

$$h'(0+) = \infty \quad \text{and} \quad h'(s) \leqslant 0 \iff 2 \leqslant (2\sqrt{s} + 1) \cdot e^s \quad \text{for all } s > 0.$$

If $0 < s \leqslant \frac{1}{9}$, then

$$(2\sqrt{s} + 1)e^s \leqslant \tfrac{5}{3} \cdot e^{1/9} < 2,$$

since $e^{1/9} < \frac{6}{5}$. Hence, $h'(s) > 0$ for all $0 < s \leqslant \frac{1}{s}$. Because $h(0) = 0$, we have proved the claim. Now, for $|z - 1| < 1$,

$$|f(z) - 1| \leqslant \sum_{n=1}^{\infty} \frac{1}{n+1} \cdot |z-1|^n \leqslant \sum_{n=1}^{\infty} \frac{1}{n} \cdot |z-1|^n = -\log(1 - |z-1|).$$

Moreover, for all $0 \leqslant t \leqslant 1$,

$$\begin{aligned}\left\|e^{t\,\mathrm{ad}\,X} e^{\mathrm{ad}\,Y} - 1\right\| &= \left\|\sum_{k+\ell \geqslant 1} \frac{t^k}{k! \cdot \ell!} (\mathrm{ad}\,X)^k (\mathrm{ad}\,Y)^\ell\right\| \leqslant \sum_{k+\ell \geqslant 1} \frac{t^k}{k! \cdot \ell!} \|X\|^k \cdot \|Y\|^\ell \cdot M^{k+\ell} \\ &= e^{M \cdot (t\|X\| + \|Y\|)} - 1 \leqslant e^{M \cdot (\|X\| + \|Y\|)} - 1.\end{aligned}$$

Hence,

$$\begin{aligned}\left\|\int_0^1 \bigl(f(e^{t\,\mathrm{ad}\,X} e^{\mathrm{ad}\,Y}) - 1\bigr) X\,dt\right\| &\leqslant \sup\nolimits_{t \in [0,1]} \left\|f(e^{t\,\mathrm{ad}\,X} e^{\mathrm{ad}\,Y}) - 1\right\| \cdot \|X\| \\ &\leqslant \sup\nolimits_{t \in [0,1]} -\log\bigl(1 - \left\|e^{t\,\mathrm{ad}\,X} e^{\mathrm{ad}\,Y} - 1\right\|\bigr) \cdot \|X\| \\ &\leqslant -\log(2 - e^{M(\|X\| + \|Y\|)}) \cdot \|X\| \\ &\leqslant \sqrt{M \cdot (\|X\| + \|Y\|)} \cdot \|X\|,\end{aligned}$$

whenever $M \cdot (\|X\| + \|Y\|) \leqslant \frac{1}{9}$. □

Lemma 3.3.7. Let $X \in \mathfrak{h}_{\mathbb{R}}$ and $Y \in \Omega^\circ$, $\|X\|, \|Y\| \leqslant \frac{1}{18 \cdot M}$. Define

$$r = \min\bigl(1, \mathrm{dist}(Y, \partial\Omega)\bigr) > 0.$$

Then, for $|z| \leqslant 1$, $\mathrm{Im}\, z > 0$ we have

$$(\exp iV)(\exp(X + zY))(\exp U) \in \Gamma^\circ$$

for all $U, V \in \mathfrak{h}_{\mathbb{R}}$ such that

$$\|V\| < \min\Big[\frac{1}{18 \cdot M}, \Big(1 + \frac{1}{\sqrt{6 \cdot M}}\Big)^{-1} \cdot r \cdot \operatorname{Im} z\Big] .$$

Proof. Since Γ° is $H \times H$-invariant,

$$(\exp iV)(\exp(X + zY))(\exp U) \in \Gamma^\circ \iff (\exp iV)(\exp(X + zY)) \in \Gamma^\circ .$$

By the Baker-Campbell-Hausdorff formula [KMS93, ch. I, th. 4.29]

$$\exp c(iV, X + zY) = (\exp iV)(\exp(X + zY))$$

where

$$c(iV, X + zY) = X + zY + iV + \int_0^1 \big(f(e^{t \operatorname{ad} iV} e^{\operatorname{ad}(X+zY)}) - 1\big) iV \, dt .$$

Hence,

$$\begin{aligned} \|\operatorname{Im} c(iV, X + zY) - i \cdot \operatorname{Im} z \cdot Y\| &= \Big\| iV + \int_0^1 \big(f(e^{t \operatorname{ad} iV} e^{\operatorname{ad}(X+zY)}) - 1\big) iV \, dt \Big\| \\ &\leqslant \big(1 + \sqrt{\|V\| + \|X + zY\|}\big) \cdot \|V\| < r \cdot \operatorname{Im} z \end{aligned}$$

for all $V \in \mathfrak{h}_{\mathbb{R}}$ satisfying the condition stated above. This implies

$$\operatorname{Im} c(iV, X + zY) \in B_{\operatorname{Im} z \cdot r}(i \cdot \operatorname{Im} z \cdot Y) = \operatorname{Im} z \cdot B_r(iY) \subset \operatorname{Im} z \cdot i\Omega^\circ = i\Omega^\circ .$$

Thus, $c(iV, X + zY) \in \mathfrak{h}_{\mathbb{R}} + i\Omega^\circ$, and $(\exp iV)(\exp(X + zY)) \in \Gamma^\circ$. □

3.3.8. Recall that for $X \subset \mathfrak{h}_{\mathbb{R}}^*$,

$$\operatorname{AC}(X) = \big\{\xi \in \mathfrak{h}_{\mathbb{R}}^* \bigm| t \cdot \xi \in X \text{ for all } t \gg 1\big\} ,$$

the *asymptotic cone* of X.

Proposition 3.3.9. Let $A \in \mathrm{I}\Psi_h^m(H)$ with principal Lie symbol $a_m \in S_h^m(H)$. The characteristic set $\operatorname{char} A$ of A is

$$\operatorname{char} A = H \times \operatorname{AC}(a_m^{-1}(0)) .$$

Proof. The point is to express the principal symbol in terms of the principal Lie symbol. By [Str72, (3.8)], we see that

$$a(X, \xi) = \big[a_m\big(W(X, Y)\xi\big) \cdot R(X, Y)\big]\big|_{Y=X}$$

defines a principal symbol of A in a coordinate neighbourhood of the identity. Here,

$$W : U \times U \to \mathrm{GL}(\mathfrak{h}_{\mathbb{R}}^*) \quad \text{and} \quad R : U \times U \to \mathbb{R}_+$$

are smooth, and R has compact support. By the proof of [Str72, th. 1], it is clear that $W(X,X) = \mathrm{id}_{\mathfrak{h}_{\mathbb{R}}^*}$ for all $X \in U$. Moreover, R is non-zero at $(0,0)$.

Since a_m is homogeneous of degree m for $|\xi| \to \infty$, we conclude that $(1,\xi) \in \operatorname{char} A$ if and only if $a_m(t\cdot\xi) = 0$ for all $t > 1$. Hence, the fibre of $\operatorname{char} A$ at the identity is exactly $\mathrm{AC}\big(a_m^{-1}(0)\big)$. By left invariance of A, the contention follows. □

3.3.10. Let H be a connected Lie group and $\xi_1,\dots,\xi_n \in \mathfrak{h}_{\mathbb{R}}$ a basis of its Lie algebra. Denote by $\xi_1^*,\dots,\xi_n^*$ the dual basis. We identify $\mathfrak{h}_{\mathbb{R}}^*$ with $\mathbb{R}^n$ via this basis. Then the following corollary is obvious.

Corollary 3.3.11. Assume H is contained a complexification $H^{\mathbb{C}}$, and that there exists an Ol'shanskiĭ domain $\Gamma^\circ = H\cdot\exp i\Omega^\circ$ defined by a closed pointed H-invariant cone $\Omega \subset \mathfrak{h}_{\mathbb{R}}$. Let E be the Szegö distribution. If $A = (A_1,\dots,A_r)$ is defined by

$$A_k = -i\xi_k\big(1 - \textstyle\sum_{j=0}^n \xi_j^2\big)^{1/2} \in \mathrm{I}\Psi_h^0(H)\,,$$

then

$$f(A)(\alpha\cdot E) \in \mathcal{E}(H) \quad \text{for all } \alpha \in \mathcal{D}(H) \text{ and } f \in \mathcal{E}(\mathfrak{h}_{\mathbb{R}}^*, \mathbb{R})$$

homogeneous of degree 0 for $|\xi| \to \infty$, such that $\mathrm{AC}(f^{-1}(0)) \cap \Omega^* = 0$.

Proof. The operators A_j are well-defined, since $-\sum_{j=0}^n \xi_j^2$ has Lie symbol $|\sqcup|^2$ and is therefore elliptic. They are properly supported, because ξ_j are, as differential operators, and self-adjoint. The Lie symbol of $i\xi_j$ is ξ_j^*. Hence, theorem 3.2.12 shows that the principal Lie symbol of $f(A) \in \mathrm{I}\Psi_h^0(H)$ is $f(\xi_1^*,\dots,\xi_n^*)$, where we have used the homogeneity of the function f.

Proposition 3.3.9 implies $\operatorname{char} f(A) = \mathrm{AC}(f^{-1}(0))$. Since this intersects Ω^* trivially, and $\mathrm{WF}(E) \subset H\times\Omega^*$ by theorem 3.3.2, we find that $f(A)(\alpha\cdot E)$ is smooth. □

Besides the wave front estimates coming from the fact that the Szegö distribution is locally the boundary value of a holomorphic function, there are also estimates coming from invariance. We have the following global version of [DV90, lemme 1 (a)].

Proposition 3.3.12. Let H be a Lie group and $e \in \mathcal{D}'(H)$ an invariant distribution, i.e. $(c_g)_*(e) = e$ for all $g \in H$ where $c_g(h) = ghg^{-1}$. Then

$$\mathrm{WF}(e)_g \subset \mathrm{Ad}^*(h)\ker(1 - \mathrm{Ad}^*(h^{-1}gh)) \quad \text{for all } \; g,h \in H\,.$$

Proof. For $X \in \mathfrak{h}_{\mathbb{R}}$, define a vector field $\widetilde{X}$ on H by

$$\widetilde{X}f(g) = \frac{d}{dt} f\big(\exp(-tX)g\exp(tX)\big)\big|_{t=0} \quad \text{for all } \; g \in H\,,\ f \in \mathcal{E}(H)\,.$$

Then, for all $\psi \in \mathcal{D}(H)$,

$$\langle \psi \,:\, \widetilde{X}p\rangle = -\frac{d}{dt}\langle \varphi \circ c_{\exp(tX)} \,:\, p\rangle\big|_{t=0} = 0\,.$$

Hence, by [Hör83, th. 8.3.1], $\mathrm{WF}(e) \subset \operatorname{char} X$. So, we need to determine the principal symbol of the differential operator $\widetilde{X}$. For $g \in H$,

$$\begin{aligned}\widetilde{X}_g &= \frac{\mathrm{d}}{\mathrm{d}t}\exp(-tX)g\exp(tX)\big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}g\exp(-t\operatorname{Ad}(g^{-1})(X))\exp(tX)\big|_{t=0} = d\ell_g(1)(1-\operatorname{Ad}(g^{-1}))(X)\,,\end{aligned}$$

where $\ell_g(h) = gh$. Since these are the trivialising maps $T(H) \to H \times \mathfrak{h}_{\mathbb{R}}$,

$$\sigma(g,\xi) = \langle X \,:\, (1-\operatorname{Ad}^*(g))(\xi)\rangle \quad \text{for all } g \in H\,,\ \xi \in \mathfrak{h}_{\mathbb{R}}$$

is the principal symbol of $\widetilde{X}$. We find

$$\bigcap_{X\in\mathfrak{h}_{\mathbb{R}}} \big(\operatorname{char}\widetilde{X}\big) = \ker\big(1-\operatorname{Ad}^*(g)\big)\,.$$

This proves the equation for $h = 1$. Now let $h \in H$ be arbitrary. $c_h : H \to H$ is a diffeomorphism, so by [Hör83, th. 8.2.4],

$$\begin{aligned}d\ell_g(1)^t\,\mathrm{WF}(e)_g = d\ell_g(1)\,\mathrm{WF}\big((c_h)_*(e)\big)_g &= (dc_h(g))^t d\ell_g(1)^t\,\mathrm{WF}(e)_{hgh^{-1}} \\ &= \operatorname{Ad}^*(h^{-1})d\ell_{hgh^{-1}}(1)^t\,\mathrm{WF}(e)_{hgh^{-1}}\,,\end{aligned}$$

because $dc_h(1)^t = \operatorname{Ad}^*(h^{-1})$ and $c_h \circ \ell_g = \ell_{hgh^{-1}} \circ c_h$. Applying the trivialising maps, this proves the assertion. □

Although the preceding proposition main seem innocuous, combined with the previous wave front estimate for the Szegö distribution, it surprisingly gives some information on its singular support.

Proposition 3.3.13. Let $G = \operatorname{Aut}_0 B$ and E be the Szegö distribution associated to the cone $\Omega^- \subset \mathfrak{g}_{\mathbb{R}}$. Let $T = Z_G(\mathfrak{t}_{\mathbb{R}})$ be the torus for the compact Cartan subalgebra $\mathfrak{t}_{\mathbb{R}}$. If $g \in G_*$ is regular, and $\mathrm{WF}(E)_g$ contains regular elements, then $g \in T^G_*$. In other words, if $g \in G_* \setminus T^G_*$, then $\mathrm{WF}(E)_g$ consists of singular semi-simple elements.

Proof. Let $g \in G_*$, i.e. g is regular. Then $\mathfrak{h}_{\mathbb{R}} = \ker(1-\operatorname{Ad}(g))$ is a Cartan subalgebra. We have

$$\mathrm{WF}(E)_g \subset \Omega^{-*} \cap \ker(1-\operatorname{Ad}^*(g))\,.$$

Identifying $\mathfrak{g}_{\mathbb{R}}^* = \mathfrak{g}_{\mathbb{R}}$ w.r.t. the invariant inner product $(\sqcup : \sqcup)$, the right hand side identifies with $\Omega^+ \cap \mathfrak{h}_{\mathbb{R}}$.

In particular, all elements of $\mathrm{WF}(E)_g$ are semi-simple, since $\mathfrak{h}_{\mathbb{R}}$ is a CSA. If $X \in \Omega^+ \cap \mathfrak{h}_{\mathbb{R}}$ is regular, lemma 2.1.32 shows that X is conjugate to an element of $\omega^{+\circ} \subset \mathfrak{t}_{\mathbb{R}}$. Then $\mathfrak{h}_{\mathbb{R}}$ is conjugate to $\mathfrak{t}_{\mathbb{R}}$, and by [Kna02, th. 7.108], this implies $g \in T^G$. □

Remark 3.3.14. If something could be said about the relation of $\mathrm{WF}(E)$ to the cone of nilpotents in $\mathfrak{g}_{\mathbb{R}}$, then more conclusive information on the singular support $\operatorname{sing\,supp} E$

would be available. To wit, whenever $g \in G_*$ is regular and $\mathrm{WF}(E)_g$ consists only of nilpotents, it would follow that $\mathrm{WF}(E)_g = \varnothing$, so that $g \notin \operatorname{sing\,supp} E$.

For $\mathcal{Z}(\mathfrak{g})$-finite invariant distributions, the fibre of the wave front set is contained in the cone of nilpotents, cf. [DV90, lem. 1 (b)]. This follows from the fact that such distributions are annihilated by the differential operators built from the ideal of invariant polynomial without constant term. The common set of zeros of this ideal is the cone of nilpotent elements.

3.4 Singular support of the Szegö distribution

3.4.1. The considerations of the previous subsection suggest a closer study of the Szegö distribution on T_*^G. Consider the map

$$\phi : G/T \times T_* \to T_*^G : (\dot{g}, t) \mapsto gtg^{-1} .$$

It is a real analytic submersion, and by [OM80, th. 2], it is a $W_c : 1$ covering map. In particular, it is proper, and we can pull back arbitrary distributions along ϕ.

Proposition 3.4.2. There exists a unique $E_T \in \mathcal{D}'(T_*)$ such that

$$\langle \phi_*(\varphi) \,:\, E \rangle = \left\langle t \mapsto \int_{G/T} \varphi(\dot{g}, t)\, d\dot{g} \,:\, E_T \right\rangle \quad \text{for all } \varphi \in \mathcal{D}(G/T \times T_*) .$$

On T_*^G, E is determined uniquely by E_T. Moreover, E_T is W_c-invariant and given by

$$\langle \psi \,:\, E_T \rangle = \lim_{T_+^{\mathbb{C}} \ni \gamma \to 1} \int_T \psi(t) K(t\gamma)\, dt \quad \text{for all } \psi \in \mathcal{D}(T_*)$$

where $K : \Gamma^* \to \mathbb{C}$ is the one-variable Szegö kernel function and $T_+^{\mathbb{C}} = \Gamma^\circ \cap T^{\mathbb{C}}$.

Proof. By [HC64, th. 1], the linear map $\phi_* : \mathcal{D}(G/T \times T_*) \to \mathcal{D}(T_*^G)$, well-defined by

$$\int_{G/T} \int_T \psi(gtg^{-1}) \varphi(\dot{g}, t)\, dt\, d\dot{g} = \int_G \psi(g) \phi_*(\varphi)(g)\, dg$$

for all $\varphi \in \mathcal{D}(G/T \times T)$, $\psi \in \mathcal{D}(T_*^G)$, is weakly continuous and surjective, so that $\phi^*(E) = E \circ \phi_*$ uniquely determines E on T_*^G. Moreover, ϕ intertwines the G-action on $G/T \times T_*$ induced by left multiplication and the action by conjugation on T_*^G.

Hence, $\phi^*(E)$ is left-invariant, thus $\phi^*(E) = d\dot{g} \otimes E_T$ for some uniquely determined and W_c-invariant $E_T \in \mathcal{D}'(T_*^G)$, by [Kna86, X, § 6, lem. 10.28]. Let $\varphi \in \mathcal{D}(G/T \times T)$. Since $K(g\gamma)$ is, locally in g, of uniform tempered growth for $\gamma \to 1$, by proposition 3.3.4,

$$\lim_{\gamma \to 1} \int_T \varphi(\dot{g}, t) K(tg^{-1}\gamma g)\, dt = \lim_{\gamma \to 1} \int_T \varphi(\dot{g}, t) K(t\gamma)\, dt$$

exists and is continuous in $\dot{g} \in G/T$. The equality follows from the invariance of Γ° and the independence of the limit on the sequence $\gamma \to 1$. The projection of $\operatorname{supp} \varphi$ onto

G/T is compact, so we may apply Lebesgue's theorem to the outer integral, and

$$\begin{aligned}\langle \varphi \,:\, d\dot{g} \otimes E_T \rangle = \langle \phi_*(\varphi) \,:\, E \rangle &= \lim\nolimits_{\gamma\to 1} \int_G \phi_*(\varphi)(g) K(g\gamma)\, dg \\ &= \lim\nolimits_{\gamma\to 1} \int_{G/T} \int_T \varphi(\dot{g}, t) K(gtg^{-1}\gamma)\, dt\, d\dot{g} \\ &= \int_{G/T} \lim\nolimits_{\gamma\to 1} \int_T \varphi(\dot{g}, t) K(t\gamma)\, dt\, d\dot{g}\,,\end{aligned}$$

by propositions 3.1.13 and 3.1.10. By uniqueness of E_T, the assertion follows. $\square$

3.4.3. Ol'shanskiĭ [Ol'95, proofs of ths. 4.2 and 4.3] has computed the kernel K on $T^{\mathbb{C}}_+$. Namely, let $\Pi = B(\Delta^{++})$ be the simple system, and $\varrho = \frac{1}{2} \cdot \sum_{\alpha\in\Delta^{++}} \alpha$. Then H_α, $\alpha \in \Pi$, is a basis of $i\mathfrak{t}_{\mathbb{R}}$. Let ω_α, $\alpha \in \Pi$, denote the dual basis of $i\mathfrak{t}_{\mathbb{R}}^*$. These are the so-called *fundamental weights*, and $\varrho = \sum_{\alpha\in\Pi} \omega_\alpha$, by [Bou68, ch. VI, § 1.11, prop. 29].

The *Weyl denominator* is

$$d(s) = s^\varrho \cdot \prod\nolimits_{\alpha\in\Delta^{++}} (1 - s^{-\alpha}) \quad \text{for all } \; s \in T^{\mathbb{C}}$$

Then $d(s) \neq 0$ whenever $s \in T^{\mathbb{C}}_*$. Define

$$k(s,x) = \frac{1}{d(sx)} \cdot \sum_{\sigma\in W} \sum_{\tau\in W_c} \varepsilon(\sigma)\varepsilon(\tau) \cdot \frac{(\sigma(x)\cdot\tau(\tau_0(s)^{-1}))^\varrho}{\prod_{\beta\in\Pi}(1 - (\sigma(x)\cdot\tau(\tau_0(s)^{-1}))^{\omega_\beta})} \quad \text{for all } \; s, x \in T^{\mathbb{C}}$$

where this makes sense. Here, $\tau_0 \in W_c$ is the longest Weyl group element, and

$$\varepsilon(\sigma) = (-1)^{\ell(\sigma)} = \det\sigma \quad \text{for all } \; \sigma \in W$$

is the sign function. If $t \in T_*$, then there exist compact connected neighbourhoods

$$t \in U^\circ \subset U \subset T_*\,,\; 0 \in V^\circ \subset \mathfrak{t}_{\mathbb{R}} \text{ and } 1 \in W^\circ \subset W \subset T^{\mathbb{C}}\,,$$

such that k is a holomorphic function on $O \times W_*$ where $O = U \cdot \exp i(V \cap \omega^{-\circ})$.

Then Ol'shanskiĭ's formula is

$$K(s) = \lim\nolimits_{W_*\ni x\to 1} k(s,x) \quad \text{for all } \; s \in O\,.$$

This convergence can be sharpened, and indeed, the limit can be computed explicitly, by the same device as is used in the proof of Weyl's dimension formula from Weyl's character formula.

This is nothing but L'Hospital's rule, applied to the differential operator

$$D = \prod\nolimits_{\alpha\in\Delta^{++}} H_\alpha \quad \text{on} \quad T^{\mathbb{C}}\,.$$

We first state and prove an appropriate version of L'Hospital's rule.

Namely, for $r > 0$ let $\mathbb{B}_r = \{z \in \mathbb{C} \mid |z| < r\}$, and the punctured polydisc

$$\dot{D}_r^n = (\mathbb{B}_r \setminus 0) \times \cdots \times (\mathbb{B}_r \setminus 0) \subset \mathbb{C}^n .$$

Moreover, for any positive function $\varrho : A \to]0,\infty[$, define a norm by

$$\|f\|_\varrho = \sup_{a\in A} |\varrho(x) \cdot f(x)| \quad \text{for all } \ f : A \to \mathbb{C}$$

and a Banach space

$$\ell_\varrho^\infty(A) = \left\{ f : A \to \mathbb{C} \;\middle|\; \|f\|_\varrho < \infty \right\} .$$

Lemma 3.4.4. Let $n \in \mathbb{N}$, $r > 0$ and A be a set

(i). Let $\varrho : A \to]0,\infty[$. If

$$f : \dot{D}_r^n \times A \to \mathbb{C} \quad \text{is holomorphic in the first variable,}$$

and $(f(\sqcup, x))_{x\in L}$ is bounded in $\ell_\varrho^\infty(A)$, then

$$f^{(\alpha)}(0,a) = \lim_{z\to 0} f^{(\alpha)}(z,a) \quad \text{exists in} \quad \ell_\varrho^\infty(A) ,$$

for all multi-indices $\alpha \in \mathbb{N}^n$.

(ii). Assume f and g satisfy the assumptions of (i) for ϱ_1 and ϱ_2 . Let $\beta \in \mathbb{N}^n$ such that

$$f^{(\alpha)}(0,a) = g^{(\alpha)}(0,a) = 0 \quad \text{for all} \quad |\alpha| \leqslant |\beta| \ , \alpha \neq \beta , \, a \in A$$

and $\inf_{a\in A} |\varrho_2(a) \cdot g^{(\beta)}(0,a)| > 0$. Then, for any $\varrho_3 \geqslant \varrho_2$ and any $U \subset \mathbb{B}_r^n$ star-shaped around 0 , such that

$$\sup_{a\in A, z\in U} \left| \varrho_3(a) f^{(\alpha)}(z,a) \right| < \infty \quad \text{for} \quad \alpha = \beta \text{ and } |\alpha| = |\beta| + 1 ,$$

we have

$$\lim_{U\setminus 0 \ni z \to 0} \frac{f(z,x)}{g(z,x)} = \frac{f^{(\alpha)}(0,x)}{g^{(\alpha)}(0,x)} \quad \text{uniformly in} \quad a \in A .$$

Proof of (i). For any $a \in A$, $f(\sqcup, a)$ is a holomorphic function locally bounded near the analytic set $\mathbb{B}^n \setminus D$. Hence, by the Riemann removable singularity theorem, it is uniquely holomorphically extendible to $\mathbb{B}_r^n$. By assumption, the resulting function f is, locally on $\mathbb{B}^n$, uniformly bounded with values in $\ell_\varrho^\infty(A)$. Since $A \subset \ell_\varrho^\infty(A)'$ is norm-determining,

$$f : \mathbb{B}^n \to \ell_\varrho^\infty(A) : z \mapsto f(z, \sqcup)$$

is a holomorphic vector-valued function, cf. [Nee00a, cor. A.III.1]. In particular,

$$f^{(\alpha)}(0,\sqcup) = \lim_{z\to 0} f^{(\alpha)}(z,\sqcup) \quad \text{in} \quad \ell^\infty_\varrho(A)\,.$$

Proof of (ii). Assume $\beta \in \mathbb{N}^n$ given as stated. By Taylor's theorem,

$$\beta! \cdot \frac{f(z,a)}{z^\beta} = f^{(\beta)}(0,a) + \varphi(z,a) \quad \text{and} \quad \beta! \cdot \frac{g(z,a)}{z^\beta} = g^{(\beta)}(0,a) + \psi(z,a)$$

where

$$\lim_{z\to 0} \varphi(z,x) = 0 = \lim_{z\to 0} \psi(z,x) \quad \text{in the respective spaces} \quad \ell^\infty_{\varrho_1}(A) \text{ and } \ell^\infty_{\varrho_2}(A)\,.$$

Let $\varepsilon = \inf_{a\in A} |\varrho_2(a)\cdot g^{(\beta)}(0,a)| > 0$. Then

$$\left\| \frac{f(z,\sqcup)}{g(z,\sqcup)} - \frac{f^{(\beta)}(z,\sqcup)}{g^{(\beta)}(z,\sqcup)} \right\|_\infty \leqslant \frac{1}{\varepsilon}\cdot \|\varphi(z,\sqcup)\|_{\varrho_3} + \frac{1}{\varepsilon^2}\cdot \|f^{(\beta)}(0,\sqcup)\|_{\varrho_3}\cdot\|\psi(z,\sqcup)\|_{\varrho_2} \to 0\,,$$

since for $z \in U$ and $s\in[0,1]$, by Lagrange's formula,

$$|\varrho_3(a)\varphi(sz,a)| \leqslant \sum_{|\alpha|=|\beta|+1} \left\|\varrho_3(a) f^{(\alpha)}(\sqcup,a)\right\|_{\infty,[0,sz]} \cdot \frac{1}{s^{|\beta|}} \int_0^s t^{|\beta|}\,dt \leqslant C\cdot s$$

for some positive constant C. □

Proposition 3.4.5. Let $0<\varepsilon<1$ and denote

$$V_\varepsilon = \left\{ H \in iV \cap \omega^{-\circ} \;\middle|\; \min_{\beta\in\Pi} \omega_\beta(H) \geqslant -\log(1-\varepsilon) \right\}.$$

Then, uniformly for $s \in O_\varepsilon = U\cdot \exp V_\varepsilon$,

$$\begin{aligned} K(s) &= \lim_{T\cap W_*\ni x\to e} k(s,x) \\ &= \frac{1}{d(s)\cdot\prod_{\alpha>0}(\varrho : \alpha)} \cdot \sum\nolimits_{\tau\in W_c} \varepsilon(\tau)\cdot D_u \left.\frac{u^\varrho}{\prod_{\beta\in\Pi}(1-u^{\omega_\beta})}\right|_{u=\tau(\tau_0(s)^{-1})}. \end{aligned}$$

Proof. The Weyl denominator $d \in \mathbb{Z}[P]$ where $P = \mathbb{Z}\langle\omega_\beta \,|\, \beta\in\Pi\rangle$ is the group of weights. In fact, [Bou68, ch. VI, § 3.3, prop. 2], d is a universal divisor for any W-odd $p\in\mathbb{Z}[P]$. Consequently, the limit

$$\lim_{x\to e} \frac{1}{d(x)} \cdot \sum_{\sigma\in W_\mathfrak{t}} \varepsilon(\sigma)\cdot\sigma(x)^{\varrho+\mu} = \prod_{\alpha>0} \frac{(\varrho_\mathfrak{t}+\mu : \alpha)}{(\varrho_\mathfrak{t} : \alpha)}$$

exists and can be evaluated via L'Hospital's rule, applied the operator D as in [Kna86, IV, § 10, proof of th. 4.48]. To see that the convergence is uniform in $\mu\in P^+$, the set of dominant weights, we wish to apply lemma 3.4.4. To that end, consider

$$\varrho_1(\mu)^{-1} = \max_{\sigma\in W\,,x\in T\cap W_*} \left|\sigma(x)^\mu\right| \ ,\ \mu\in P^+$$

and $\varrho_2 = \varrho_3 = 1$. Since μ is imaginary, $|\sigma(x)^\mu| = 1$ if $x \in T\cap W_*$. So the boundedness assumptions are verified if we take $U = \log(T\cap W_*)$ in the lemma.

For $s^{-1}\in O$, the geometric series

$$\sum\nolimits_{\mu\in P^+} s^{\varrho+\mu} = \frac{s^\varrho}{\prod_{\beta\in\Pi}(1-s^{\omega_\beta})}$$

is locally uniformly convergent. Viz, if $s^{-1} = u\cdot\exp X\in O\,,\,u\in T\,,\,-X\in i\omega^{-\circ}$, then

$$|s^{\omega_\beta}| = e^{\omega_\beta(X)} < 1$$

and the convergence is uniform of subsets where $\max_\beta e^{\omega_\beta(X)} \leqslant 1-\varepsilon < 1$. Now,

$$D_s\, s^{\varrho+\mu} = \prod\nolimits_{\alpha>0}(\varrho+\mu\ :\ \alpha)\cdot s^{\varrho+\mu}\ .$$

Since the derivatives of power series have the same radius of convergence as their primitives, the series

$$\sum\nolimits_{\mu\in P^+}\prod\nolimits_{\alpha>0}(\varrho+\mu\ :\ \alpha)\cdot s^{\varrho+\mu} = D_s\sum\nolimits_{\mu\in P^+} s^{\varrho+\mu} = D_s\frac{s^\varrho}{\prod_{\beta\in\Pi}(1-s^{\omega_\beta})}$$

converges uniformly for $s^{-1} = u\cdot\exp X$, $\max_\beta\omega_\beta(X)\leqslant\log(1-\varepsilon)$. By the above considerations, the convergence for $T\cap W_*\ni x\to 1$ of the coefficients is uniform in μ, so we may exchange limit order to achieve, for $W_*\cap T\in x\to 1$,

$$\begin{aligned} d(s)\cdot k(s,x) &= \sum\nolimits_{\tau\in W_c}\varepsilon(\tau)\cdot\sum\nolimits_{\mu\in P^+}\frac{1}{d(x)}\cdot\sum\nolimits_{\sigma\in W}\varepsilon(\sigma)\cdot\sigma(x)^{\varrho+\mu}\cdot\tau(\tau_0(s)^{-1})^{\varrho+\mu}\\ &\to\frac{1}{\prod_{\alpha>0}(\varrho\ :\ \alpha)}\cdot\sum\nolimits_{\tau\in W_c}\varepsilon(\tau)\cdot D_u\left.\frac{u^\varrho}{\prod_{\beta\in\Pi}(1-u^{\omega_\beta})}\right|_{u=\tau(\tau_0(s)^{-1})}\ ,\end{aligned}$$

uniformly in $s = u\cdot\exp(-X)$, $\max_\beta\omega_\beta(X)\leqslant\log(1-\varepsilon)$. Since $d(s)$ is locally uniformly bounded away from 0, the assertion follows. □

3.4.6. For $\operatorname{rk}\mathfrak{g}_{\mathbb{R}}\geqslant 2$, there may exist fundamental ω_β not proportional to any root, so to evaluate K on a subset of T, we need to consider the set

$$T_{**} = \left\{t\in T_*\ \middle|\ t^{\omega_\beta}\neq 1 \text{ for all }\ \beta\in\Pi\right\}\ .$$

whose elements are more-than-regular.

Proposition 3.4.7. Let $U\subset T_{**}$ be a small relatively compact neighbourhood of $t\in T_{**}$.

Then the limit

$$K(u) = \lim_{i\omega^{-\circ} \ni X \to 0} K(u \cdot \exp X) \quad \text{exists uniformly in} \quad u \in U\,,$$

together with all derivatives. In particular,

$$\operatorname{sing\,supp} E_T \subset T \setminus T_{**} \quad \text{and} \quad \bigl(\operatorname{sing\,supp} E\bigr) \cap T^G \subset T^G \setminus T^G_{**}\,.$$

Proof. The function f defined by

$$f(s) = \frac{s^{\varrho}}{\prod_{\beta\in\Pi}(1 - s^{\omega_\beta})}$$

is holomorphic and bounded on a neighbourhood of U^{-1} in $T^{\mathbb{C}}_{**}$. Hence, so are its derivatives. But then

$$D_s\, f(s)\big|_{s=\tau(\tau_0(u)^{-1})} = \lim_{i\omega^{-\circ} \ni X \to 0} D_s\, f(s)\big|_{s=\tau(\tau_0(u\cdot \exp X)^{-1})}$$

uniformly in $u \in U$, proving the assertion by propositions 3.4.5 and 3.4.2. □

Remark 3.4.8. One should note that Hecht [Hec76] has computed the characters of the holomorphic discrete series on a set of conjugacy classes of Cartan subgroups. (In fact, the formula is due to Martens in the case of a regular parameter λ, and Hecht extends this to singular cases.) The formula is the same as on the torus. Basically, the same proof as Ol'shanskiĭ has given on the compact Cartan subgroup T should go through in general. This would allow for the computation of E on the other Cartan subgroups.

4 Holomorphic discrete series, and the Hardy space

4.1 Algebraic theory of highest weight modules

4.1.1. As above, consider the group $G = \operatorname{Aut}_0 B$ where $B \subset Z$ is a circled bounded symmetric domain defined by a JB*-triple Z. Fix a frame $e_1, \dots, e_r$ of Z, and choose a torus $\mathfrak{t}_{\mathbb{R}} \subset \mathfrak{k}_{\mathbb{R}}$, as in 2.1.3. For the corresponding root system Δ, choose any positive system Δ_c^+ of Δ_c, and define the corresponding adapted positive system Δ^{++} of Δ, according to lemma 2.1.6. Then

$$\mathfrak{b} = \mathfrak{t} \oplus \sum\nolimits^{\oplus}_{\alpha\in\Delta^{++}} \mathfrak{g}^{\alpha} = \mathfrak{t} \oplus \mathfrak{k}^+ \oplus \mathfrak{p}^+$$

is a maximal solvable subalgebra of $\mathfrak{g} = \mathfrak{g}_{\mathbb{R}} \otimes \mathbb{C}$, containing $\mathfrak{p}^+$ as an Abelian ideal. Denote its nilpotent radical by $\mathfrak{n}^+ = \mathfrak{k}^+ \oplus \mathfrak{p}^+$.

To any $\Lambda \in \mathfrak{t}^*$, we associate the one-dimensional $\mathfrak{b}$-module $\mathbb{C}_\Lambda = \mathbb{C}$,

$$H\,z = \Lambda(H)\cdot z \quad \text{and} \quad X\,z = 0 \quad \text{for all } \; z \in \mathbb{C}_\Lambda\,,\, H \in \mathfrak{t}\,,\, X \in \mathfrak{n}^+\,.$$

The universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ is naturally a $(\mathfrak{g}, \mathfrak{b})$-bimodule. The *Verma module*

$$V_\Lambda^{\mathfrak{g}} = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{b})} \mathbb{C}_\Lambda$$

is therefore naturally a $\mathfrak{g}$-module. Denote by 1_Λ the image of $1 \otimes 1$ in this tensor product.

4.1.2. Recall that for a $\mathfrak{t}$-module V, a linear form $\mu \in \mathfrak{t}^*$ is called a $\mathfrak{t}$-*weight of* V if

$$H\,v = \mu(H) \cdot v \quad \text{for some non-zero} \quad v \in V\,.$$

In this case, v is called a *μ-weight vector*. The subspace of all μ-weight vectors in V is denoted $V[\mu]$.

The Verma module $V_\Lambda^{\mathfrak{g}}$ is the algebraic direct sum of its $\mathfrak{t}$-weight spaces. The weights are of the form $\Lambda - \sum_{\alpha\in\Pi} n_\alpha \cdot \alpha$ where $n_\alpha \in \mathbb{N}$ and $\Pi = B(\Delta^{++})$ is the set of simple roots, cf. [Bou68, ch. VI, § 1.5]. The weight spaces are finite-dimensional, and the highest weight space $V_\Lambda^{\mathfrak{g}}[\Lambda] = \mathbb{C} \cdot 1_\Lambda$. Moreover, the vector 1_Λ is annihilated by $\mathfrak{U}(\mathfrak{n}^+)$. For these statements, cf. [Dix77, prop. 7.1.6].

A cyclic $\mathfrak{g}$-module $V = \mathfrak{U}(\mathfrak{g})\,v$ such that $v \in V[\mu]$ and $\mathfrak{U}(\mathfrak{n}^+)\,v = 0$ is called a *highest weight module*. Then $V[\mu] = \mathbb{C} \cdot v$ and v is called a *highest weight vector*. Moreover, one has $V = \mathfrak{U}(\mathfrak{n}^-)\,v$ where $\mathfrak{n}^- = \sum_{\alpha\in\Delta^{++}} \mathfrak{g}^{-\alpha}$ is opposite to $\mathfrak{n}^+$, and there is a unique equivariant surjection $V_\mu^{\mathfrak{g}} \to V$ mapping $1_\mu \mapsto v$, cf. [Dix77, prop. 7.1.7].

There is a largest proper submodule $U \subset V_\Lambda^{\mathfrak{g}}$, by [Dix77, prop. 7.1.11]. Consequently, $L_\Lambda^{\mathfrak{g}} = V_\Lambda^{\mathfrak{g}}/U$ is simple, and by what was said above, the unique simple quotient of $V_\Lambda^{\mathfrak{g}}$.

4.1.3. Define an conjugate linear involution $\sqcup^* : \mathfrak{g} \to \mathfrak{g}$ by

$$(X + iY)^* = -X + iY \quad \text{for all} \;\; X, Y \in \mathfrak{g}_{\mathbb{R}}\,.$$

Then $\sqcup^*$ is an anti-automorphisms of $\mathfrak{g}$, i.e.

$$[Z, W]^* = [W^*, Z^*] = -[Z^*, W^*] \quad \text{for all} \;\; Z, W \in \mathfrak{g}\,.$$

With respect to this involution, $\mathfrak{g}_{\mathbb{R}} = \{Z \in \mathfrak{g} \mid Z^* = -Z\}$, and it is clear that this sets up a bijection between real forms and conjugate linear, involutive anti-automorphism of $\mathfrak{g}$.

If $\alpha \in \Delta \subset i\mathfrak{t}_{\mathbb{R}}^*$ and $Z \in \mathfrak{g}^\alpha$, then

$$[H, Z^*] = -[H^*, Z]^* = [H, Z]^* = \big(\alpha(H) \cdot Z\big)^* = -\alpha(H) \cdot Z^* \quad \text{for all} \;\; H \in \mathfrak{t}_{\mathbb{R}}\,,$$

and this shows that $(\mathfrak{g}^\alpha)^* = \mathfrak{g}^{-\alpha}$. We deduce $(\mathfrak{n}^\pm)^* = \mathfrak{n}^\mp$.

The involution $\sqcup^*$ can be naturally carried over to the dual space $\mathfrak{g}^*$ by

$$\langle Z : \mu^* \rangle = \overline{\langle Z^* : \mu \rangle} \quad \text{for all} \;\; Z \in \mathfrak{g}\,,\; \mu \in \mathfrak{g}^*\,.$$

Then $\sqcup^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is a conjugate linear involution.

For this involution on $\mathfrak{g}^*$, $\mathfrak{g}^*_{\mathbb{R}} = \{\mu \in \mathfrak{g}^* \mid \mu^* = -\mu^*\}$. In particular, the weight $\Lambda \in \mathfrak{t}$ satisfies $\Lambda^* = \Lambda$ if and only if $\Lambda \in i\mathfrak{t}^*_{\mathbb{R}}$.

4.1.4. By the universal property [Dix77, lem. 2.1.3] of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, the conjugate linear involutive Lie algebra anti-automorphism $\sqcup^* : \mathfrak{g} \to \mathfrak{g}$ extends to a conjugate linear involutive algebra anti-automorphism $\sqcup^* : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$.

If V is a $\mathfrak{g}$-module, an $\mathbb{R}$-bilinear form $(\sqcup : \sqcup) : V \times V \to \mathbb{C}$ is *invariant* if

$$(X^* u : v) = (u : X v) \quad \text{for all } X \in \mathfrak{U}(\mathfrak{g}),\ u, v \in V.$$

If $(\sqcup : \sqcup)$ is invariant, then the *left and right annihilators* of submodules $U \subset V$

$$U^{\perp} = \{u \in V \mid (U : u) = 0\} \quad \text{and} \quad {}^{\perp}U = \{u \in V \mid (u : U) = 0\}$$

are submodules of V. The form $(\sqcup : \sqcup)$ is *non-degenerate* if $V^{\perp} = {}^{\perp}V = 0$. Of course, there are one-sided notions of non-degeneracy.

A sesqui-linear form $(\sqcup \mid \sqcup)$, conjugate linear in the first variable, is *Hermitian* if

$$\overline{(u \mid v)} = (v \mid u) \quad \text{for all } u, v \in V.$$

If it is invariant and Hermitian, then left and right annihilators coincide.

Define an involution $h \mapsto h^*$ on the space of invariant sesqui-linear forms on V by

$$h^*(u, v) = \overline{h(v, u)} \quad \text{for all } u, v \in V.$$

Then h is Hermitian if and only if $h^* = h$. Moreover, any invariant sesqui-linear form h decomposes as

$$h = \frac{1}{2} \cdot (h + h^*) + \frac{1}{2} \cdot (h - h^*)$$

where $h \pm h^*$ are invariant sesqui-linear forms which are, respectively, Hermitian and skew-Hermitian.

Let $(\sqcup \mid \sqcup)$ be sesqui-linear and invariant, and λ, μ weights of V such that $\lambda^* \neq \mu$. Then, for all $X \in \mathfrak{U}(\mathfrak{g})$, $u \in V[\lambda]$, $v \in V[\mu]$,

$$\begin{aligned}(\lambda^*(X) - \mu(X)) \cdot (u \mid v) &= (\lambda(X^*) \cdot u \mid v) - (u \mid \mu(X) \cdot v) \\ &= (X^* u \mid v) - (u \mid X v) = 0.\end{aligned}$$

There exists $X \in \mathfrak{g}$ such that $\lambda^*(X) \neq \mu(X)$. Hence $V[\lambda] \perp V[\mu]$.

The set of invariant sesqui-linear forms is exactly $\mathrm{Hom}_{\mathfrak{g}}(\bar{V} \otimes V, \mathbb{C})$ where the conjugate space $\bar{V}$ is a a $\mathfrak{g}$-module via

$$X \bar{v} = -\overline{X^* v} \quad \text{for all } v \in V$$

where $\bar{\sqcup} : V \to \bar{V} : v \mapsto \bar{v}$ is the (conjugate linear) identity. We denote the set of Hermitian elements in $\mathrm{Hom}_{\mathfrak{g}}$ by $\mathrm{Hom}_{\mathfrak{g}}^{+}$.

Proposition 4.1.5. Let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$. For the modules $V = V_\Lambda^{\mathfrak{g}}$ and $L = L_\Lambda^{\mathfrak{g}}$,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(\bar{V} \otimes V, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(\bar{L} \otimes L, \mathbb{C}) = 1 .$$

and

$$\dim_{\mathbb{R}} \mathrm{Hom}_{\mathfrak{g}}^{+}(\bar{V} \otimes V, \mathbb{C}) = \dim_{\mathbb{R}} \mathrm{Hom}_{\mathfrak{g}}^{+}(\bar{L} \otimes L, \mathbb{C}) = 1 ,$$

Any non-zero $h \in \mathrm{Hom}_{\mathfrak{g}}^{+}(\bar{L} \otimes L, \mathbb{C})$ is non-degenerate.

Proof. Since all the weights of $V = V_\Lambda^{\mathfrak{g}}$ are of the form $\Lambda - \sum_{\alpha\in\Pi} n_\alpha \cdot \alpha \in i\mathfrak{t}_{\mathbb{R}}^*$ and therefore fixed by $\sqcup^*$, the weight spaces are mutually orthogonal. Hence, if $(\sqcup \mid \sqcup)$ is an invariant form on V such that $(1_\Lambda \mid 1_\Lambda) = 0$, we see that $1_\Lambda \in {}^{\perp}V \cap V^{\perp}$.

Since 1_Λ is a cyclic vector, we find ${}^{\perp}V = V^{\perp} = V$, which means that $(\sqcup \mid \sqcup) = 0$. In particular, $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(\bar{V} \otimes V, \mathbb{C}) \leqslant 1$.

Let $h = (\sqcup \mid \sqcup) \in \mathrm{Hom}_{\mathfrak{g}}(\bar{V} \otimes V, \mathbb{C})$, and set $z = (1_\Lambda \mid 1_\Lambda)$. If $z \in \mathbb{R}$, then the skew-Hermitian part of h vanishes on 1_Λ and is hence zero. We conclude that h is Hermitian in this case. Clearly, the converse is also true. Consequently, $\dim_{\mathbb{R}} \mathrm{Hom}_{\mathfrak{g}}^{+}(\bar{V} \otimes V, \mathbb{C}) \leqslant 1$.

Since any form on $L = L_\Lambda^{\mathfrak{g}}$ lifts to V, the dimensions of the corresponding spaces are at most one, too.

As a quotient of V, $L = L_\Lambda^{\mathfrak{g}}$ is the algebraic direct sum of its weight spaces. So, there is a unique $h = (\sqcup \mid \sqcup) \in \mathrm{Hom}_{\mathfrak{g}}^{+}(\bar{L} \otimes L, \mathbb{C})$ such that $(1_\Lambda \mid 1_\Lambda) = 1$. Namely,

$$(1_\Lambda \mid V[\mu]) = 0 \text{ and } (X 1_\Lambda \mid u) = (1_\Lambda \mid X^* u) \quad \text{for all } \mu \neq \Lambda,\, X \in \mathfrak{U}(\mathfrak{g}),\, u \in L .$$

Hence,

$$1 \leqslant \dim_{\mathbb{R}} \mathrm{Hom}_{\mathfrak{g}}^{+}(\bar{L} \otimes L, \mathbb{C}) \leqslant \dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(\bar{L} \otimes L, \mathbb{C}) \leqslant 1 .$$

The same is true of V, since the form h just constructed pulls back.

The remaining claim is that any non-zero invariant sesqui-linear form on L is non-degenerate. But $L^{\perp}$ is a submodule of L, and it is non-trivial since $1_\Lambda \notin L^{\perp}$. Since L is simple, $L^{\perp} = 0$, so h is non-degenerate (since it is Hermitian). Because any other invariant sesqui-linear form is proportional to h, we are done. □

Remark 4.1.6. Our proof of proposition 4.1.5 closely follows [Enr79, prop. 6.8]. The main difference being that Enright deals with bilinear instead of sesqui-linear forms (symmetric instead of Hermitian). This does not place any conditions on the parameter Λ and is therefore a crucial difference. Moreover, he treats slightly different involutions (they are linear instead of conjugate linear). The construction of the unique Hermitian invariant sesqui-linear form satisfying $(1_\Lambda \mid 1_\Lambda) = 1$ goes back to Shapovalov which is why it is often called by this name.

Neeb generalises this theory in [Nee00a, IX.1] to Lie algebras including, among others, the Lie algebras $\mathfrak{g}_{\mathbb{R}}^{f,I}$ generated by the faces $F_{f,I}^{-}$ of the cone Ω^{-}, and even infinite-dimensional cases.

Definition 4.1.7. If V is a $\mathfrak{g}$-highest weight module of weight Λ, and we have fixed a highest weight vector 1_Λ, we call the unique invariant Hermitian form such that $(1_\Lambda \mid 1_\Lambda) = 1$ the *Shapovalov form of* V.

Proposition 4.1.5 allows for the classification of highest weight modules admitting a non-degenerate invariant Hermitian form.

Proposition 4.1.8. Let V be a highest weight module of weight $\Lambda \in \mathfrak{t}^*$. Then V permits a non-zero invariant Hermitian form if and only if $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$. In this case,

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(\bar{V} \otimes V, \mathbb{C}) = \dim_{\mathbb{R}} \operatorname{Hom}_{\mathfrak{g}}^{+}(\bar{V} \otimes V, \mathbb{C}) = 1\,.$$

Moreover, there exists a non-degenerate invariant Hermitian form on the $\mathfrak{g}$-module V if and only if it is simple, i.e. $V = L_\Lambda^{\mathfrak{g}}$.

Proof. We know that V is a quotient of $V_\Lambda^{\mathfrak{g}}$. If $(\sqcup \mid \sqcup) \in \operatorname{Hom}_{\mathfrak{g}}^{+}(\bar{V} \otimes V, \mathbb{C})$ is non-zero, then so is its pull-back to $V_\Lambda^{\mathfrak{g}}$. This shows that $(v \mid v) \neq 0$ for any non-zero $v \in V[\Lambda]$, compare the proof of proposition 4.1.5. Hence

$$\Lambda^*(H) \cdot (v \mid v) = (\Lambda(H^*)\, v \mid v) = (H^*\, v \mid v) = (v \mid H\, v) = \Lambda(H) \cdot (v \mid v)$$

for all $H \in \mathfrak{t}$, $v \in V[\Lambda]$, shows that $\Lambda = \Lambda^* \in i\mathfrak{t}_{\mathbb{R}}^*$.

Conversely, let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$. As a highest weight module, $V \neq 0$. The kernel of the epimorphism $V_\Lambda^{\mathfrak{g}} \to V$ is hence contained in the largest proper submodule. In other words, there is a canonical $\mathfrak{g}$-linear epimorphism $V \to L_\Lambda^{\mathfrak{g}}$. Hence

$$1 = \dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(\bar{L}_\Lambda^{\mathfrak{g}} \otimes L_\Lambda^{\mathfrak{g}}, \mathbb{C}) \leqslant \dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(\bar{V} \otimes V, \mathbb{C}) \leqslant \dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(\bar{V}_\Lambda^{\mathfrak{g}} \otimes V_\Lambda^{\mathfrak{g}}, \mathbb{C}) = 1\,,$$

by proposition 4.1.5, and analogously for Hermitian forms.

If V is simple, then clearly $V = L_\Lambda^{\mathfrak{g}}$. Hence proposition 4.1.5 shows that V admits a non-degenerate Hermitian invariant form. Assume, conversely, that V admits such a form. Let $U \subset V$ be a proper submodule. So is its preimage in $V_\Lambda^{\mathfrak{g}}$, and we infer that $V[\Lambda] \cap U = 0$. Hence U is contained in the sum of weight spaces $V[\mu]$ with $\mu \neq \Lambda$. All of these are orthogonal to $V[\Lambda]$, so $V[\Lambda] \subset U^\perp$. But $U^\perp$ is invariant, and $V[\Lambda]$ contains a cyclic vector. Consequently, $U^\perp = V$. By non-degeneracy, this implies $U = 0$. So we have proved that V is simple. ∎

4.1.9. If e_j, $j = 1, \dots, n$, is a basis of $\mathfrak{g}$, and f_j, $j = 1, \dots, n$ a B-dual basis, i.e.

$$B(e_i, f_j) = \delta_{ij} \quad \text{for all } 1 \leqslant i, j \leqslant n\,,$$

then $\Omega = \sum_{j=1}^{n} e_j f_j \in \mathfrak{U}(\mathfrak{g})$ is called the *Casimir operator*.

A preimage of Ω under the canonical epimorphism $\bigotimes \mathfrak{g} \to \mathfrak{U}(\mathfrak{g})$ is $\sum_{j=0}^{n} e_j \otimes f_j$. Under the representation of the tensor algebra $\bigotimes \mathfrak{g}$ derived from $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}\,\mathfrak{g}$, this tensor is the identity of $\mathfrak{g}$. Consequently, Ω is independent of the choice the basis (e_j), and invariant under the adjoint action of $\mathfrak{g}$ on $\mathfrak{U}(\mathfrak{g})$, cf. [Bou68, ch. I, § 3.7, prop. 11]. So, it is a canonical non-zero element on $\mathcal{Z}(\mathfrak{g})$, the centre of $\mathfrak{U}(\mathfrak{g})$.

It is desirable to express Ω in terms of the root space decomposition of $\mathfrak{g}$.

Lemma 4.1.10. Consider for $\alpha \in \Delta$ elements $H_\alpha \in i\mathfrak{t}_{\mathbb{R}}$ such that $\alpha(H_\alpha) = 2$. There exist, for all $\alpha \in \Delta$, $X_\alpha \in \mathfrak{g}^\alpha$ such that

$$X_\alpha^* = \pm X_{-\alpha} \quad \text{and} \quad [X_\alpha, X_{-\alpha}] = \frac{|\alpha|^2}{2} \cdot H_\alpha$$

where the sign is $+$ or $-$ according to whether $\alpha \in \Delta_c$ or $\alpha \in \Delta_n$, and we consider the inner product dual to B on $i\mathfrak{t}_{\mathbb{R}}^*$.

In particular, the Casimir Ω of $\mathfrak{g}$ is given by

$$\Omega = \sum_{j=1}^{R} H_j^2 + \frac{1}{2} \cdot \sum_{\alpha\in\Delta^{++}} |\alpha|^2 \cdot H_\alpha + 2 \cdot \sum_{\alpha\in\Delta_c^{++}} X_\alpha^* X_\alpha - 2 \cdot \sum_{\alpha\in\Delta_n^{++}} X_\alpha^* X_\alpha$$

where H_j, $j = 1, \ldots, R = \mathrm{rk}\,\mathfrak{g}$ is any B-orthonormal basis of $i\mathfrak{t}_{\mathbb{R}}$.

Proof. Recall from proposition 2.1.11 that B is positive definite on $\mathfrak{p}_{\mathbb{R}}$, and negative definite on $\mathfrak{k}_{\mathbb{R}}$. In particular, B is an inner product on $i\mathfrak{t}_{\mathbb{R}}$. Hence H_α is determined by $B(H_\alpha, H) = 2\,|\alpha|^{-2} \cdot \alpha(H)$ for all $H \in \mathfrak{t}$.

By [Kna02, II.4, lem. 2.18], $[X, Y] = \frac{|\alpha|^2}{2} \cdot B(X, Y) \cdot H_\alpha$ for all $X \in \mathfrak{g}^\alpha$, $Y \in \mathfrak{g}^{-\alpha}$. Now,

$$h : (Z, W) \mapsto B(Z^*, W) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

is a sesqui-linear form which, on $\mathfrak{g}_{\mathbb{R}}$, coincides with $-B$. Hence, h is negative on $\mathfrak{p}_{\mathbb{R}}$, and by sesqui-linearity, this remains true on $\mathfrak{p}$. Likewise, h is positive on $\mathfrak{k}$. Hence, for $\alpha \in \Delta^{++}$, we may choose $X_\alpha \in \mathfrak{g}^\alpha$ in such a way that $h(X_\alpha, X_\alpha) = \pm 1$ with the sign positive or negative according to whether α is compact or non-compact.

Now, for $\alpha \in \Delta^{++}$, let $X_{-\alpha} = h(X_\alpha, X_\alpha) \cdot X_\alpha^* = B(X_\alpha^*, X_\alpha) \cdot X_\alpha^*$. Then $X_\alpha^* = \pm X_{-\alpha}$ and $X_{-\alpha}^* = \pm X_\alpha$. Moreover, $B(X_\alpha, X_{-\alpha}) = \pm h(X_\alpha, X_\alpha) = 1$, so $[X_\alpha, X_{-\alpha}] = \frac{|\alpha|^2}{2} \cdot H_\alpha$.

Since B is positive on $i\mathfrak{t}_{\mathbb{R}}$, there is an orthonormal basis H_j, $j = 1, \ldots, R = \mathrm{rk}\,\mathfrak{g}$ of $i\mathfrak{t}_{\mathbb{R}}$. By definition of Ω,

$$\begin{aligned}
\Omega &= \sum_{j=1}^{R} H_j^2 + \sum_{\alpha\in\Delta_c} X_\alpha^* X_\alpha - \sum_{\alpha\in\Delta_n} X_\alpha^* X_\alpha \\
&= \sum_{j=1}^{R} H_j^2 + \frac{1}{2} \cdot \sum_{\alpha\in\Delta^{++}} |\alpha|^2 \cdot H_\alpha + 2 \cdot \sum_{\alpha\in\Delta_c^{++}} X_\alpha^* X_\alpha - 2 \cdot \sum_{\alpha\in\Delta_n^{++}} X_\alpha^* X_\alpha ,
\end{aligned}$$

because, for $\alpha \in \Delta^{++}$,

$$\left[X_{-\alpha}^*, X_{-\alpha}\right] = \pm\left[X_{-\alpha}, X_\alpha\right] = \mp\frac{|\alpha|^2}{2}\cdot H_\alpha\,,$$

with the usual sign convention. □

We can now compute the Casimir on highest weight modules.

Proposition 4.1.11. Let V be a highest weight module of highest weight $\Lambda \in \mathfrak{t}^*$. Then

$$\Omega\big|_V = |\Lambda + \varrho|^2 - |\varrho|^2$$

where $\varrho = \frac{1}{2}\cdot\sum_{\alpha\in\Delta^{++}}\alpha$.

Proof. All $\mathfrak{g}$-linear endomorphisms of V are scalar by [Dix77, prop. 7.1.8]. Hence, it suffices to evaluate Ω on some non-zero $v \in V[\Lambda]$. Since v is annihilated by elements of $\mathfrak{U}(\mathfrak{b})$ without constant term, we find, by lemma 4.1.10,

$$\begin{aligned}\Omega\, v &= \sum_{j=1}^R \Lambda(H_j)^2\cdot v + \frac{1}{2}\cdot\sum_{\alpha\in\Delta^{++}}|\alpha|^2\cdot\Lambda(H_\alpha)\cdot v\\ &= |\Lambda|^2\cdot v + \sum_{\alpha\in\Delta^{+}+}(\alpha : \Lambda)\cdot v = \left(|\Lambda|^2 + 2\,(\Lambda : \varrho)\right)\cdot v\end{aligned}$$

which is the desired expression since $|\Lambda|^2 + 2\,(\Lambda : \varrho) = |\Lambda+\varrho|^2 - |\varrho|^2$. □

Remark 4.1.12. It should be noted that none of the above propositions depend on the choice the Lie algebra $\mathfrak{g}_\mathbb{R}$, the Cartan subalgebra $\mathfrak{t}_\mathbb{R}$, or the positive system Δ^{++}. The only fact that was needed was the semi-simplicity of $\mathfrak{g}_\mathbb{R}$. In fact, if the Killing form had been replaced by another non-degenerate invariant form negative on $\mathfrak{k}_\mathbb{R}$ and positive on $\mathfrak{p}_\mathbb{R}$, the definition and expression the Casimir could been extended to this situation. This applies to reductive Lie algebras.

In particular, the operator

$$\Omega_c = \sum_{j=1}^R H_j^2 + \frac{1}{2}\cdot\sum_{\alpha\in\Delta_c^{++}}|\alpha|^2\cdot H_\alpha + 2\cdot\sum_{\alpha\in\Delta_c^{++}}X_\alpha^* X_\alpha \in \mathcal{Z}(\mathfrak{k})\ .$$

If we consider the subalgebra of $\mathfrak{k}$ given by

$$\mathfrak{b}_\mathfrak{k} = \mathfrak{b}\cap\mathfrak{k} = \mathfrak{t}\oplus\mathfrak{k}^+ = \mathfrak{t}\oplus\textstyle\sum^{\oplus}_{\alpha\in\Delta_c^{++}}\mathfrak{g}^\alpha\,,$$

then we can naturally define, for $\Lambda \in \mathfrak{t}^*$, the $\mathfrak{k}$-*Verma module* $V_\Lambda^\mathfrak{k} = \mathfrak{U}(\mathfrak{k})\otimes_{\mathfrak{U}(\mathfrak{b}_\mathfrak{k})}\mathbb{C}_\Lambda$. Moreover, a cyclic $\mathfrak{k}$-module $V = \mathfrak{U}(\mathfrak{k})\,v$ where $v \in V[\Lambda]\setminus 0$ is annihilated by $\mathfrak{b}_\mathfrak{k}$, is called a $\mathfrak{k}$-*highest weight module* and v a $\mathfrak{k}$-*highest weight vector*.

The same theory as before applies. In particular, on a $\mathfrak{k}$-highest weight module V of weight Λ, the Casimir Ω_c acts by

$$\Omega_c\big|_V = |\Lambda + \varrho_c|^2 - |\varrho_c|^2$$

where $\varrho_c = \frac{1}{2} \cdot \sum_{\alpha \in \Delta_c^{++}} \alpha$.

If V is a highest weight module, we say that V is *unitary* if it admits a positive definite invariant Hermitian form.

Proposition 4.1.13. Let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$ and denote $F_\Lambda = L_\Lambda^{\mathfrak{k}}$, the unique irreducible quotient of the $\mathfrak{k}$-Verma module $V_\Lambda^{\mathfrak{k}}$. Then F_Λ is finite-dimensional if and only if it is unitary, and this is the case precisely if Λ is Δ_c^{++}-dominant and integral, i.e.

$$\Lambda(H_\alpha) \in \mathbb{N} \quad \text{for all } \alpha \in \Delta_c^{++} .$$

Proof. If F_Λ is finite-dimensional, the representation of $\mathfrak{k}_{\mathbb{R}}$ integrates to one of the universal covering group $\widetilde{K}$ of K. The kernel of the covering map is discrete and normal, and hence central. Any $\mathfrak{k}$-linear endomorphism of F_Λ is scalar by [Dix77, prop. 7.1.8]. Fix a highest weight vector 1_Λ and consider the Shapovalov form $(\sqcup \mid \sqcup)$ on F_Λ. It is positive on $V[\Lambda]$, and hence, $Z(\widetilde{K})$ acts by elements of $\mathrm{U}(1)$. We conclude that the image of $\widetilde{K}$ in End F_Λ is compact. By integrating any inner product F_Λ over this compact group, we conclude that F_Λ is unitary.

Assume that F_Λ is unitary. Then the Shapovalov form is positive definite. For the root vectors $X_\alpha \in \mathfrak{g}^\alpha$ chosen in lemma 4.1.10, we define

$$E_\alpha = \frac{|\alpha|}{\sqrt{2}} \cdot X_\alpha \quad \text{for all } \alpha \in \Delta .$$

Then

$$\big[H_\alpha, E_{\pm\alpha}\big] = \pm 2 \cdot E_{\pm\alpha} \quad \text{and} \quad \big[E_\alpha, E_{-\alpha}\big] = H_\alpha .$$

Hence [Dix77, lem. 7.1.14] applies to show

$$\big[E_\alpha, E_{-\alpha}^m\big] = m \cdot E_{-\alpha}^{m-1}(H_\alpha - m + 1) \quad \text{for all } m \in \mathbb{N} ,$$

in the universal enveloping algebra $\mathfrak{U}(\mathfrak{k})$.

Fix $\alpha \in \Delta_c^{++}$. Because $X_\alpha 1_\Lambda = 0$, we find

$$E_\alpha^m E_{-\alpha}^m 1_\Lambda = E_\alpha^{m-1}\big[E_\alpha, E_{-\alpha}^m\big] 1_\Lambda = m\big(\Lambda(H_\alpha) - m + 1\big) \cdot E_\alpha^{m-1} E_{-\alpha}^{m-1} 1_\Lambda = m!(\Lambda(H_\alpha))_m \cdot 1_\Lambda$$

where $(z)_m = z(z-1)\cdots(z-m+1)$ is the falling factorial. We infer

$$0 \leqslant (E_{-\alpha}^m 1_\Lambda \mid E_{-\alpha}^m 1_\Lambda) = (1_\Lambda \mid E_\alpha^m E_{-\alpha}^m 1_\Lambda) = m! \cdot (\Lambda(H_\alpha))_m \quad \text{for all } m \in \mathbb{N} .$$

In particular $\Lambda(H_\alpha) \geqslant 0$, and is an integer since otherwise, $(\Lambda(H_\alpha))_m$ would be negative for some $m > \Lambda(H_\alpha) + 1$.

Finally, assume Λ is dominant and integral. The canonical generator $1_\Lambda \in V_\Lambda^{\mathfrak{k}}$ lies above $1_\Lambda \in F_\Lambda$. If $\alpha \in \Pi_c = B(\Delta_c^{++})$, then $m = (\Lambda + \varrho_c)(H_\alpha) = \Lambda(H_\alpha) + 1 \in \mathbb{N}$ by [Bou68, ch. VI, § 1.10, prop. 29]. Moreover, the vector $v = E_{-\alpha}^m 1_\Lambda \in V_\Lambda^{\mathfrak{k}}$ is non-zero, and is annihilated by $\mathfrak{b}_{\mathfrak{k}}$, cf. [Dix77, prop. 7.1.15]. The weight of v is $\Lambda - m \cdot \alpha \neq \Lambda$, so v lies in a proper submodule of $V_\Lambda^{\mathfrak{k}}$. This means that $E_{-\alpha}^m 1_\Lambda = 0$ in F_Λ. Now, [Dix77, lem. 7.2.4] implies that F_Λ is finite-dimensional. □

4.1.14. If V is a $\mathfrak{g}$-module, $U \subset V$ a subspace and $\mathfrak{h} \subset \mathfrak{g}$ some subalgebra, we denote by $U^{\mathfrak{h}}$ the subspace of U consisting of those vectors annihilated by $\mathfrak{h}$. In particular, we are interested in the subalgebra $\mathfrak{b}_{\mathfrak{k}}$.

Proposition 4.1.15. Let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$ be Δ_c^{++}-dominant and integral. Then $L = L_\Lambda^{\mathfrak{g}}$ is unitary if and only if

$$|\Lambda + \varrho| < |\mu + \varrho| \quad \text{for all } \mu \in \mathfrak{t}^*,\ L[\mu]^{\mathfrak{b}_{\mathfrak{k}}} \neq 0\,.$$

Proof. Let $v \in L$ be a $\mathfrak{k}$-highest weight vector of weight μ. Then $\mathfrak{U}(\mathfrak{k})\,v$ is a highest weight module, and hence proposition 4.1.11 shows that Ω_c acts on v by $|\mu + \varrho_c|^2 - |\varrho_c|^2$. Since Ω acts by $|\Lambda + \varrho|^2 - |\varrho|^2$, we find

$$\begin{aligned}
\left(|\Lambda + \varrho|^2 - |\mu + \varrho|^2\right) \cdot (v \mid v) &= \left([\Omega - \Omega_c - 2 \cdot (\mu : \varrho_n)]\, v \mid v\right) \\
&= -2\,(\mu : \varrho_n) \cdot (v \mid v) + \sum_{\alpha \in \Delta_n^{++}} \left([(\mu : \alpha) - 2X_\alpha^* X_\alpha]\, v \mid v\right) \\
&= -2 \cdot \sum_{\alpha \in \Delta_n^{++}} (X_\alpha v : X_\alpha v)
\end{aligned}$$

where $\varrho_n = \varrho - \varrho_c = \frac{1}{2} \cdot \sum_{\alpha \in \Delta_n^{++}} \alpha$.

Hence, if $(\sqcup \mid \sqcup)$ is an inner product, then $|\Lambda + \varrho| \leqslant |\mu + \varrho|$ with equality if and only if v is annihilated by $\mathfrak{p}^+$. If this is the case, then $U = \mathfrak{U}(\mathfrak{g})\,v$ is a non-zero submodule of L with highest weight μ. Since L is simple, we conclude $U = L$, so $\mu = \Lambda$. We have proved one implication.

As for the converse, we argue by contraposition and assume that the Shapovalov form be non-positive. It is obvious that the $\mathfrak{k}$-submodule $\mathfrak{U}(\mathfrak{k})\,1_\Lambda$ of the Verma module $V_\Lambda^{\mathfrak{g}}$ is isomorphic to $V_\Lambda^{\mathfrak{k}}$. The Shapovalov form on $V_1 = V_\Lambda^{\mathfrak{g}}$ induces the one on $V_2 = V_\Lambda^{\mathfrak{k}}$, so $V_2 \cap V_1^\perp \subset V_2^\perp$. Since the $\mathfrak{g}$-Shapovalov form is non-degenerate on L, $V_1^\perp$ is the maximal proper submodule of V_1. Hence $V_2^\perp = V_2 \cap V_1^\perp$. This implies that the $\mathfrak{k}$-submodule $\mathfrak{U}(\mathfrak{k})\,1_\Lambda \subset L$ equals F_Λ.

The PBW theorem [Dix77, th. 2.1.11] shows that $\mathfrak{U}(\mathfrak{p}^-)\,F_\Lambda = L$, so

$$L = \sum\nolimits_{j \in \mathbb{N}}^{\oplus} L_j \quad \text{with} \quad L_j = \mathfrak{U}_j(\mathfrak{p}^-)\,F_\Lambda$$

is a decomposition into $\mathfrak{k}$-submodules, where we consider the grading of $\mathfrak{U}(\mathfrak{p}^-)$ into homogeneous parts. The Shapovalov form is not positive on some L_j where j is minimal.

Since L_j is finite-dimensional, it is fully reducible, and the decomposition into isotypic components is orthogonal with respect to the Shapovalov form. Hence there is an isotypic component $L_j \supset V = F_\mu \otimes \mathbb{C}^{m(\mu)}$ such that the Shapovalov form is not positive when restricted to V. Since

$$V = \sum\nolimits_{0\neq\mathbb{C}\cdot v\subset V^{\mathfrak{b}_\mathfrak{k}}}^{\oplus} \mathfrak{U}(\mathfrak{k})\, v \quad \text{where} \quad \mathfrak{U}(\mathfrak{k})\, v = F_\mu \text{ for all } v \in V^{\mathfrak{b}_\mathfrak{k}} \setminus 0\,,$$

there exists $v \in V^{\mathfrak{b}_\mathfrak{k}}$ such that $(v \mid v) < 0$. Write $v = \sum_\ell X_{1\ell} \cdots X_{j\ell} Y 1_\Lambda$ where $Y \in \mathfrak{U}(\mathfrak{k})$ and $X_{i\ell} \in \mathfrak{p}^+$. Then

$$X_\alpha v = \sum\nolimits_\ell \Big[\sum_{i=1}^{j} X_{1\ell} \cdots [X_\alpha, X_{i\ell}] \cdots X_{j\ell} Y 1_\Lambda + X_{1\ell} \cdots X_{j\ell}([X_\alpha, Y] + Y X_\alpha) 1_\Lambda \Big] \in L_{j-1}$$

since $[X_\alpha, X_{i\ell}], [X_\alpha, Y] \in \mathfrak{k}$ and $X_\alpha 1_\Lambda = 0$. Hence $(X_\alpha v \mid X_\alpha v) \leqslant 0$. The above calculation shows $|\Lambda + \varrho| \geqslant |\mu + \varrho|$, so we have completed the proof. ☐

4.1.16. Since $\mathfrak{p}^+$ is $\mathfrak{k}$-invariant by lemma 2.1.2, the subspace $\mathfrak{k} + \mathfrak{p}^+$ is a subalgebra of $\mathfrak{k}$. In an abstract Lie algebraic setting, this is related to the fact that the positive system Δ^{++} is adapted. In fact, if $\mathfrak{p}^+$ is the sum of the positive non-compact root spaces for some positive system Δ^+, then $\mathfrak{k} + \mathfrak{p}^+$ is a subalgebra if and only if Δ^+ is adapted.

Since $\mathfrak{k} + \mathfrak{p}^+$ is an algebra, we can define, for $\Lambda \in i\mathfrak{t}_\mathbb{R}^*$ Δ_c^{++}-dominant and integral

$$N(\Lambda) = \mathfrak{U}(\mathfrak{g}) \otimes_{\mathfrak{U}(\mathfrak{k}+\mathfrak{p}^+)} F_\Lambda$$

where $\mathfrak{p}^+$ acts trivially on F_Λ. Then the canonical generator $1_\Lambda = 1 \otimes 1_\Lambda$ is annihilated by $\mathfrak{b} = \mathfrak{b}_\mathfrak{k} + \mathfrak{p}^+$. Moreover, it is cyclic and has weight Λ. Hence, $N(\Lambda)$ is a $\mathfrak{g}$-highest weight module.

From PBW [Dix77, th. 2.1.11], it follows that $N(\Lambda) = \mathfrak{U}(\mathfrak{p}^-) \otimes F_\Lambda$ as $\mathfrak{k}$-modules, where consider the adjoint action on the first factor, i.e.

$$X(u \otimes v) = [X, u] \otimes v + u \otimes X v \quad \text{for all } u \in \mathfrak{U}\left(\mathfrak{p}^-\right),\, v \in F_\Lambda,\, X \in \mathfrak{k}\,.$$

4.1.17. We can give a convenient Jordan theoretic description of $\mathfrak{U}(\mathfrak{p}^\pm)$. Since $\mathfrak{p}^\pm$ is Abelian, the universal enveloping algebra is just $\mathfrak{U}(\mathfrak{p}^\pm) = S(\mathfrak{p}^\pm)$, the symmetric algebra of $\mathfrak{p}^\pm$. Recall from lemma 2.1.2 the isomorphisms

$$\partial : Z \longrightarrow \mathfrak{p}^+ : u \longmapsto \partial_u = u\frac{\partial}{\partial z} \quad \text{and} \quad \partial^* : Z \longrightarrow \mathfrak{p}^- : u \longmapsto \vartheta(\partial_u) = \partial_u^* = \{zu^*z\}\frac{\partial}{\partial z}$$

respectively linear and conjugate linear. Here, the involution $\sqcup^*$ is the one defined in 4.1.3. On $\mathfrak{p}$, it coincides with the Cartan involution on $\mathfrak{p}$, because the latter is the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{u}_\mathbb{R} = \mathfrak{k}_\mathbb{R} \oplus i\mathfrak{p}_\mathbb{R}$, cf. 2.1.1.

On the symmetric algebras, the above isomorphisms extend to isomorphisms

$$\partial : \mathrm{S}(Z) \to \mathfrak{U}\left(\mathfrak{p}^+\right) = \mathrm{S}(\mathfrak{p}^+) \quad \text{and} \quad \partial^* : \mathrm{S}(Z) \to \mathfrak{U}\left(\mathfrak{p}^-\right) = \mathrm{S}(\mathfrak{p}^-)$$

of graded commutative algebras, the first linear, the second conjugate linear. Observe that ∂ is equivariant for $K^{\mathbb{C}} = \mathrm{Aut}_0(Z,Z)$ (2.3.4) where the action on $\mathfrak{U}\left(\mathfrak{p}^+\right)$ is induced by the adjoint action.

Define an involutive anti-automorphism of $K^{\mathbb{C}}$ by

$$(k^+,k^-)^* = (k^-,k^+)^{-1} = (k^{-\,-1},k^{+\,-1}) \quad \text{for all } \ k \in K^{\mathbb{C}}\,.$$

On $\mathfrak{k}$, this involution of $K^{\mathbb{C}}$ induces the involution $\sqcup^*$ defined in 4.1.3. Indeed,

$$(\delta^+,\delta^-)^* = -(\delta^-,\delta^+) \quad \text{for all } \ (\delta^+,\delta^-) \in \mathfrak{k} = \mathrm{aut}(Z,Z)\,.$$

Moreover, the dependence of δ^- on δ^+ is conjugate linear by 2.3.4. Since $\mathfrak{k}_{\mathbb{R}}$ is the diagonal in $\mathfrak{k}$ and $i\mathfrak{k}_{\mathbb{R}}$ the anti-diagonal, the identity of the two involutions follows.

Hence, the involution of $K^{\mathbb{C}}$ satisfies

$$\mathrm{Ad}(k)(\xi^*) = \mathrm{Ad}\big(k^{-1*}\big)(\xi)^* = \mathrm{Ad}(k^-,k^+)(\xi)^* \quad \text{for all } \ k = (k^+,k^-) \in K^{\mathbb{C}}\,.$$

In particular,

$$\partial^*_{k(p)} = \mathrm{Ad}(k^{-1*})(\partial^*_p) \quad \text{for all } \ p \in \mathrm{S}(Z)\,,\, k \in K^{\mathbb{C}}\,.$$

Of course, the algebras $\mathfrak{U}\left(\mathfrak{p}^{\pm}\right)$ act naturally, by holomorphic differential operators, on the algebra $\mathcal{P}(Z)$ of polynomials on Z. If we endow $\mathcal{P}(Z)$ with the gradation by homogeneous terms,

$$\partial_p : \mathcal{P}^m(Z) \to \mathcal{P}^{m-k}(Z) \text{ and } \partial^*_p : \mathcal{P}^m(Z) \to \mathcal{P}^{m+k}(Z) \text{ for all } p \in \mathrm{S}^k(Z)\,,\, m \in \mathbb{Z}\,,\, k \in \mathbb{N}$$

where, as is customary, $\mathcal{P}^m(Z) = 0$ for $m < 0$. There is a natural bilinear pairing

$$\langle p \,:\, q\rangle = (\partial_p q)(0) \quad \text{for all } \ p \in \mathrm{S}(Z)\,,\, q \in \mathcal{P}(Z)\,,$$

If we define a $K^{\mathbb{C}}$-action on $\mathcal{P}(Z)$ by

$$(k^+,k^-)^{-1}\,p = p \circ k^+ \quad \text{for all } \ (k^+,k^-) \in K^{\mathbb{C}}\,,$$

then the pairing satisfies

$$\langle k\,p \,:\, q\rangle = \langle p \,:\, k^{-1} q\rangle \quad \text{for all } \ k \in K^{\mathbb{C}}\,,\, p \in \mathrm{S}(Z)\,,\, q \in \mathcal{P}(Z)\,.$$

The different degrees of the gradings are mutually orthogonal. By [Upm82, lem. 3.10],

$$\langle u^m \,:\, p\rangle = m! \cdot p(u) \quad \text{for all } \; u \in Z \; p \in \mathcal{P}^m(Z)\,,\, m \in \mathbb{N}\,.$$

Consequently, the pairing is non-degenerate.

The canonical inner product $(\sqcup \mid \sqcup)$ on Z gives rise to an isomorphism

$$Z^* \to Z : \alpha \to \alpha^*\,,\, \alpha(u) = (u \mid \alpha^*) \quad \text{for all } \; u \in Z$$

which is conjugate linear because the canonical inner product is conjugate linear in the second variable. Since $\mathcal{P}(Z) = \mathrm{S}(Z^*)$ as graded algebras, this induces a conjugate linear graded algebra isomorphism

$$\mathcal{P}(Z) \to \mathrm{S}(Z) : p \mapsto p^*\,.$$

We can now define a sesqui-linear form on $\mathcal{P}(Z)$ by

$$(p \mid q)_{\mathcal{P}(Z)} = \langle p^* \,:\, q\rangle = \big(\partial_{p^*} q\big)(0) \quad \text{for all } \; p, q \in \mathcal{P}(Z)\,.$$

This in an inner product by [Upm82, lem. 3.11], and extends the conjugate of the canonical inner product. Since

$$(k\,u \mid v) = (u \mid k^*\, v) \quad \text{for all } \; u, v \in Z\,,\, k \in K^{\mathbb{C}}\,,$$

where we have

$$(k\,p \mid q)_{\mathcal{P}(Z)} = (p \mid k^*\, q)_{\mathcal{P}(Z)} \quad \text{for all } \; k \in K^{\mathbb{C}}\,,\, p, q \in \mathcal{P}(Z)$$

and

$$k\,p^* = \big(k^{-1*}\, p\big)^* \quad \text{for all } \; p \in \mathcal{P}(Z)\,.$$

Hence, the inner product $(\sqcup \mid \sqcup)_{\mathcal{P}(Z)}$ is $\mathfrak{k}$-invariant, and the map

$$p \mapsto \partial^*_{p^*} : \mathcal{P}(Z) \to \mathrm{S}(\mathfrak{p}^-) = \mathfrak{U}\left(\mathfrak{p}^-\right)$$

is $K^{\mathbb{C}}$-equivariant. This enables us to define $\mathfrak{k}$-invariant inner products on $\mathfrak{U}\left(\mathfrak{p}^{\pm}\right)$,

$$\big(\partial_{p^*} \mid \partial_{q^*}\big)_{\mathfrak{U}(\mathfrak{p}^+)} = (q \mid p)_{\mathcal{P}(Z)} = \big(\partial_{q^*} p\big)(0) \quad \text{for all } \; p, q \in \mathcal{P}(Z)$$

and

$$(\partial^*_{p^*} \mid \partial^*_{q^*})_{\mathfrak{U}(\mathfrak{p}^-)} = (p \mid q)_{\mathcal{P}(Z)} = \big(\partial_{p^*} q\big)(0) \quad \text{for all } \; p, q \in \mathcal{P}(Z)\,.$$

Theorem 4.1.18. Let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$ be Δ_c^{++}-dominant and integral. If

$$(\Lambda + \varrho)\big(H_\alpha\big) < 0 \quad \text{for all } \alpha \in \Delta_n^{++} ,$$

then $N(\Lambda)$ is unitary, and, in particular, simple.

Proof. We have noted $N(\Lambda) = \mathfrak{U}(\mathfrak{p}^-) \otimes F_\Lambda$, as $\mathfrak{k}$-modules. Hence, we can define a $\mathfrak{k}$-invariant inner product on $N(\Lambda)$ by the requirement

$$\big(u \otimes v \mid u' \otimes v'\big) = \big(u \mid u'\big)_{\mathfrak{U}(\mathfrak{p}^-)} \big(v \mid v'\big) \quad \text{for all } u, u' \in \mathfrak{U}\big(\mathfrak{p}^-\big) , v, v' \in F_\Lambda .$$

Here, we consider the Shapovalov form on F_Λ, an inner product by proposition 4.1.13.

We claim that the $\mathfrak{k}$-highest weights of $N(\Lambda)$ are all contained in $\Lambda - \mathbb{N}\langle\Delta_n^{++}\rangle$. To that end, let

$$U = \mathfrak{U}\big(\mathfrak{p}^-\big) \otimes \mathfrak{n}_{\mathfrak{k}}^- F_\Lambda \quad \text{where} \quad \mathfrak{n}_{\mathfrak{k}}^- = \textstyle\sum_{\alpha \in \Delta_c^{++}} \mathfrak{g}^{-\alpha} .$$

Then $\mathfrak{n}_{\mathfrak{k}}^- F_\Lambda$ is the sum of weight spaces of weight $\neq \Lambda$. In particular,

$$N(\Lambda) = U \oplus \big(\mathfrak{U}\big(\mathfrak{p}^-\big) \otimes 1_\Lambda\big)$$

is an orthogonal direct sum. Since $\mathfrak{U}(\mathfrak{p}^-) \otimes 1_\Lambda$ is $\mathfrak{b}_{\mathfrak{k}}$-invariant and $\mathfrak{n}_{\mathfrak{k}}^- \subset \mathfrak{b}_{\mathfrak{k}}^*$, the image under the involution $\sqcup^*$, U is $\mathfrak{n}_{\mathfrak{k}}^-$-invariant. If $v \in U^{\mathfrak{b}_{\mathfrak{k}}}$, then

$$\mathfrak{U}(\mathfrak{k})\ v = \mathfrak{U}\big(\mathfrak{n}_{\mathfrak{k}}^-\big)\ v \subset U$$

by PBW, so $\big(\mathfrak{U}(\mathfrak{k})\ v\big)^\perp \supset \mathfrak{U}(\mathfrak{p}^-) \otimes 1_\Lambda$. But the latter is a generating set for the $\mathfrak{k}$-module $N(\Lambda)$, so we conclude $v = 0$. This proves that all $\mathfrak{k}$-highest weight vectors of $N(\Lambda)$ are contained in $\mathfrak{U}(\mathfrak{p}^-) \otimes 1_\Lambda$, in particular, their weights are in $\Lambda - \mathbb{N}\langle\Delta_n^{++}\rangle$.

If $\mu = \Lambda - \nu$ is the weight of a $\mathfrak{k}$-highest weight vector, then

$$(\Lambda + \varrho : \nu) = \frac{|\nu|^2}{2} \cdot \big(\Lambda + \varrho\big)\big(H_\nu\big) \leqslant 0 \quad \text{since} \quad \nu \in \mathbb{N}\langle\Delta_n^{++}\rangle .$$

Hence

$$|\mu + \varrho|^2 - |\Lambda + \varrho|^2 = |\nu|^2 - 2 \cdot (\Lambda + \varrho : \nu) > 0 .$$

Since any $\mathfrak{k}$-highest weight vector in $L_\Lambda^{\mathfrak{g}}$ lies below one in $N(\Lambda)$, this implies, by proposition 4.1.15, that $L_\Lambda^{\mathfrak{g}}$ is unitary. In particular, the $\mathfrak{g}$-invariant Shapovalov form on $N(\Lambda)$ is positive semi-definite.

By [Dix77, prop. 7.6.1], the Verma module $V_\Lambda^{\mathfrak{g}}$ has a Jordan-Hölder series

$$0 = V_m \subset V_{m-1} \subset \cdots \subset V_0 = V_\Lambda^{\mathfrak{g}} .$$

Assume that $N(\Lambda)$ is not simple. Then there exists a non-trivial proper submodule of

$N(\Lambda)$, and in particular, some V_j, $0 < j < m$, is not mapped to 0 in $N(\Lambda)$. Let j be maximal with this property.

Again by [Dix77, prop. 7.6.1],

$$V_j/V_{j+1} = L^{\mathfrak{g}}_{\mu} \quad \text{for some} \quad \mu \in \Lambda - \mathbb{N}\langle\Delta^{++}\rangle \ .$$

Hence, there is a non-zero vector $v \in V_j[\mu]$ such that $\mathfrak{b}\,v \in V_{j+1}$. Since V_{j+1} is mapped to 0 in $N(\Lambda)$, we conclude that the image $u \in N(\Lambda)$ of v is a highest weight vector, and $W = \mathfrak{U}(\mathfrak{k})\ v$ is a highest weight module. By proposition 4.1.11,

$$\Omega\big|_W = |\mu + \varrho|^2 - |\varrho|^2 \ .$$

But we already know that $\Omega\big|_{N(\Lambda)} = |\Lambda + \varrho|^2 - |\varrho|^2$. Hence $|\Lambda + \varrho| = |\mu + \varrho|$, a contradiction. We conclude that $N(\Lambda)$ is simple, so the Shapovalov form is non-degenerate, and consequently, an inner product. □

Remark 4.1.19. The condition from theorem 4.1.18,

$$\lambda(H_\alpha) < 0 \quad \text{for all} \ \ \alpha \in \Delta^{++}_n$$

where $\lambda = \Lambda + \varrho$, is called *Harish-Chandra's square integrability condition*. First introduced by Harish-Chandra in [HC56, th. 4], this condition guarantees that the module $N(\Lambda)$ is the Harish-Chandra $(\mathfrak{g}, K)$-module of K-finite vectors for a square-integrable irreducible unitary representation of G. This will become evident in the sequel.

4.2 Analytic theory of the holomorphic discrete series

Our next aim is to globalise the representation $N(\Lambda)$ under additional conditions on the parameter Λ. To this end, we need a different realisation of this $\mathfrak{g}$-module.

Lemma 4.2.1. For $g \in P^+K^{\mathbb{C}}P^-$ and $z \in Z$, the holomorphic derivative $g'(z) : Z \to Z$ uniquely defines an element of $K^{\mathbb{C}}$. Specifically, we have the formulae

$$t^{+\prime}_v(u) = 1\ ,\ k'(u) = k \quad \text{and} \quad t^{-\prime}_v(u) = B(u,v)^{-1} \quad \text{for all} \ \ u, v \in Z\,,\ k \in K^{\mathbb{C}}\ ,$$

where in the latter equation, (u, v) is quasi-invertible.

Proof. By the chain rule, it suffices to check $g'(z) \in K^{\mathbb{C}}$ for generators. Moreover, since $1'(0) = 1 \in K^{\mathbb{C}}$ and $P^+K^{\mathbb{C}}P^- \times Z$ is connected, $g'(z)$ varies in a connected set. Hence, we need only see that $g'(z) \in \mathrm{Aut}(Z, Z)$.

The action of $K^{\mathbb{C}}$ on Z is linear, so the statement is trivial for $k \in K^{\mathbb{C}}$. If $v \in Z$, then

$$t^+_v(u) = u + v \quad \text{for all} \ \ u \in Z\ ,$$

so $t^{+\prime}_v(u) = 1$ which defines an element of $K^{\mathbb{C}} = \mathrm{Aut}_0(Z, Z)$.

The computation of $t_v^{-\prime}(u)$ is contained in [Loo75, 7.8], we repeat it the for sake of completeness. Namely, since (u,v) is quasi-invertible,

$$(u+w)^v = u^v + B(u,v)^{-1} \cdot w^{v^u} \quad \text{for all } \; w \in Z$$

such that (w, v^u) is quasi-invertible, by [Loo75, 7.3.4]. Now, by [Loo75, 7.3.1],

$$(tw)^{v^u} = t \cdot w^{t \cdot v^u} \quad \text{for all } \; t \in \mathbb{C} \; .$$

Differentiating, we get

$$\frac{\mathrm{d}}{\mathrm{d}t} t \cdot w^{t \cdot v^u} = w^{t \cdot v^u} + t \cdot \frac{\mathrm{d}}{\mathrm{d}t} w^{t \cdot v^u}$$

which is $w^0 = w$ for $t = 0$. We compute

$$t_v^{-\prime}(u)w = \frac{\mathrm{d}}{\mathrm{d}t}(u+tw)^v\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\big[t \cdot B(u,v)^{-1} \cdot w^{t \cdot v^u}\big]\big|_{t=0} = B(u,v)^{-1} w \; ,$$

proving the assertion, since $B(u,v) \in K^{\mathbb{C}} = \mathrm{Aut}_0(Z,Z)$. □

4.2.2. Note that the map

$$J : G \times B \to K^{\mathbb{C}} : (g,z) \mapsto g'(z)$$

satisfies the cocycle relation

$$J(gh,z) = J(g,h(z)) \cdot J(h,z) \quad \text{for all } \; g,h \in G \, , \, z \in B$$

by the chain rule. For this reason, it is sometimes called the *canonical automorphy factor*, e.g. [Sat80, § 5]. We shall check in a moment that our definition coincides with the one given by Satake.

Lemma 4.2.3. Let $g \in G$ and $u \in Z$. Then, in the $P^+K^{\mathbb{C}}P^-$ decomposition of

$$g \exp\Big(u \frac{\partial}{\partial z}\Big) \in P^+K^{\mathbb{C}}P^- \; ,$$

the $K^{\mathbb{C}}$ component is $g'(u)$. Similarly,

$$\exp\Big(v \frac{\partial}{\partial z}\Big)^* \exp\Big(u \frac{\partial}{\partial z}\Big) \in P^+K^{\mathbb{C}}P^-$$

if and only if (u,v) is quasi-invertible. In this case, its $K^{\mathbb{C}}$ component is the inverse of the Bergman operator, $B(u,v)^{-1} \in K^{\mathbb{C}}$.

Proof. Let $k \in K^{\mathbb{C}}$ and $u,v,w \in Z$. By [Loo75, 8.6],

$$t_u^+ = \exp\Big(u \frac{\partial}{\partial z}\Big) \quad \text{and} \quad t_v^- = \exp\Big(\{zv^*z\} \frac{\partial}{\partial z}\Big) = \Big(\exp v \frac{\partial}{\partial z}\Big)^* \, .$$

In particular,

$$k\,t_u^+ = \exp\mathrm{Ad}(k)\Big(u\frac{\partial}{\partial z}\Big)\,k = \exp\Big(k^{-1\prime}(z)^{-1}u\frac{\partial}{\partial z}\Big)\,k = t_{k(u)}^+\,k\,,$$

by the formula for the adjoint action from 1.1.1. Then [Loo75, th. 8.11] shows that for $g = t_w^+ k t_v^- \in G$,

$$g\,\exp\Big(u\frac{\partial}{\partial z}\Big) = t_w^+\,k\,t_v^-\,t_u^+ = t_w^+\,k\,t_{u^v}^+\,B(u,v)^{-1}\,t_{v^u}^- = t_w^+\,t_{k(u^v)}^+\,k\,B(u,v)^{-1}\,t_{v^u}^-\,,$$

if (u,v) is quasi-invertible. But the first equality shows that $g\,\exp u\,\frac{\partial}{\partial z} \in P^+K^{\mathbb{C}}P^-$ if and only if this is the case for $t_v^- t_u^+$. By [Loo75, th. 8.11], this is equivalent to that requirement that (u,v) be quasi-invertible.

The $K^{\mathbb{C}}$ component in the above formula is

$$k\,B(u,v)^{-1} = t_w^{-\prime}(kt_v^-(u))\,k'(t_v^-(u))\,t_v^{-\prime}(u) = g'(u)\,,$$

by lemma 4.2.1 and the chain rule. Since $G \subset P^+K^{\mathbb{C}}P^-$ by [Loo75, prop. 8.10], we have proved the first assertion.

As for the second, the statement about quasi-invertibility is [Loo75, th. 8.11]. For (u,v) quasi-invertible,

$$\Big(\exp v\frac{\partial}{\partial z}\Big)^*\Big(\exp u\frac{\partial}{\partial z}\Big) = t_v^-\,t_u^+ = t_{u^v}^+\,B(u,v)^{-1}\,t_{v^u}^-\,,$$

completing the proof. □

4.2.4. The sesqui-holomorphic Bergman operator

$$B : Z\times Z \longrightarrow \mathrm{Aut}_0(Z,Z) = K^{\mathbb{C}} : (u,v) \longmapsto B(u,v) = 1 - 2\,u\,\square\,v^* + Q_uQ_v$$

is sometimes also called the *universal kernel function* of G. Observe that it behaves covariantly under the group action of G,

$$B\big(g(z),g(w)\big) = g'(z)\,B(z,w)\,g'(w)^* = J(g,z)\,B(z,w)\,J(g,w)^* \quad \text{for all } g\in G,\ z,w\in B\,,$$

by [Loo75, lem. 2.11]. Recall that we have noted in 4.1.17 that the adjoint with respect to the trace form on Z coincides on $K^{\mathbb{C}}$ with the involution defined in 4.1.3.

The Bergman operator encodes the Hermitian Bergman metric of B. Indeed, at any point $z\in B$,

$$h_z(u,v) = \big(B(z,z)^{-1}\,u \mid v\big) \quad \text{for all } u,v\in Z$$

by [Loo75, th. 2.10], where on the right hand side, we take the trace form on Z. Moreover, the Bergman kernel is simply $\det B(u,v)^{-1}$, for a suitable normalisation of measures.

Under certain conditions on the parameter Λ, a global realisation of the $\mathfrak{g}$-module $N(\Lambda)$ is obtained by generalising these considerations to certain homogeneous Hermitian vector bundles on B.

Lemma 4.2.5. Let λ be Lebesgue measure on Z, normalised such that $\lambda(B) = 1$. Then $d\mu_0(z) = \det B(z,z)^{-1}\,d\lambda(z)$ defines a G-invariant measure on B.

Proof. We only need to check invariance. Let $g \in G$. Then

$$|\det dg(z)| = \det g'(z) \cdot \det g'(z)^* \,,$$

as follows from the Cauchy-Riemann equations. Consequently,

$$g_*(d\mu_0(z)) = |\det dg(z)| \cdot \det B(g(z),g(z))^{-1}\,d\lambda(z) = \det B(z,z)^{-1}\,d\lambda(z) = d\mu_0(z)$$

by the change of variables formula. This proves the assertion. □

4.2.6. Let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$ be Δ_c^{++}-dominant and integral, and let $\lambda = \Lambda + \varrho$. In particular, F_Λ is a finite-dimensional unitary $\mathfrak{k}$-module with Shapovalov form $(\sqcup \mid \sqcup)_\Lambda$.

If $T = N_K(\mathfrak{t})$ is the torus, then a necessary and sufficient condition that F_Λ integrates to K is that $(\exp H)^\Lambda = e^{\Lambda(H)}$ for all $H \in \mathfrak{t}_{\mathbb{R}}$ defines a character of T, by [Kna02, th. 5.110]. Equivalently, Λ is *analytically integral*, i.e.

$$\Lambda\big(\exp^{-1}(1)\big) \subset 2\pi i\mathbb{Z} \,,$$

cf. [Kna02, prop. 4.58]. If this is the case, the representation of K on F_Λ is unitary, and it extends to an involutive holomorphic representation of $K^{\mathbb{C}}$. Slightly abusing notation, we denote the action of $k \in K^{\mathbb{C}}$ on F_Λ by k^Λ, and the action of k^{-1} by $k^{-\Lambda}$.

Consider the space $\mathcal{O}(B,F_\Lambda)$ of holomorphic functions $f : B \to F_\Lambda$. Define operators $g^{-\pi_\lambda} = \pi_\lambda(g^{-1})$ on $\mathcal{O}(B,F_\Lambda)$ by

$$\big(g^{-\pi_\lambda} f\big)(z) = g'(z)^{-\Lambda} f(g(z)) \quad \text{for all } f \in \mathcal{O}(B,F_\Lambda)\,,\, g \in G\,,\, z \in B\,.$$

Then it is obvious that π_λ is a representation of G on $\mathcal{O}(B,F_\Lambda)$. Since $z \mapsto g'(z)^{-1}$ is a polynomial (of degree at most 2) by [Loo75, prop. 8.13], the subspace $\mathcal{P}(Z,F_\Lambda)$ of F_Λ-valued polynomials on Z is π_λ-invariant.

Moreover, consider the *weighted Bergman space* $\mathcal{O}^2_\Lambda = \mathcal{O}^2(B,F_\Lambda)$,

$$\mathcal{O}^2(B,F_\Lambda) = \left\{ f \in \mathcal{O}(B,F_\Lambda) \;\middle|\; \|f\|^2_{\mathcal{O}^2_\Lambda} = \int_B^* \big(f(z) \mid B(z,z)^{-\Lambda} f(z)\big)_\Lambda \,d\mu_0(z) < \infty \right\} .$$

It is easy to see that $\mathcal{O}^2_\Lambda$, endowed with the norm $\|\sqcup\|_{\mathcal{O}^2_\Lambda}$, is a reproducing kernel Hilbert space, cf. [Nee00a, lem. XII.5.2]. Moreover, the invariance of μ_0 implies that it is invariant under π_λ, and this defines an action by unitary operators, cf. [Nee00a, lem. XII.5.3].

The representation is unitary, since the strong continuity of π_λ follows from dominated convergence.

Remark 4.2.7. It should be noted that the action of G on $\mathcal{O}(B,F_\Lambda)$ is in some sense natural. Since this is not immediately obvious from the definitions, we briefly comment on this fact.

Indeed, the domain B is homogeneous and thus, canonically identified with G/K. It is therefore the base of G, considered as principal fibre bundle for the group K. Hence, the vector bundle

$$\mathbb{F}_\Lambda = G\times_K F_\Lambda \ni [g,v]=[gk,k^{-\Lambda}v]$$

associated to the representation F_Λ of K, has base B. Consider the map

$$\phi:\mathbb{F}_\Lambda\to B\times F_\Lambda:[g,v]\mapsto\big(g(0),g'(0)^\Lambda v\big)\ .$$

It is well-defined, since

$$[gk,k^{-\Lambda}v]\mapsto\big(g(k(0)),g'(k(0))^\Lambda k'(0)^\Lambda k^{-\Lambda}v\big)=\big(g(0),g'(0)^\Lambda v\big)$$

for all $k\in K$. Also, ϕ is a bundle map, and it can be shown that for the natural vector bundle structure of $\mathbb{F}_\Lambda$, ϕ is a diffeomorphism. (Local trivialisations of $\mathbb{F}_\Lambda$ are constructed from local sections of the K-principal bundle $G\to B:g\mapsto g(0)$.)

Hence, ϕ is a trivialisation. Indeed, for the canonical Hermitian structure on $\mathbb{F}_\Lambda$, it is a holomorphic trivialisation.

The vector bundle $\mathbb{F}_\Lambda$ is naturally G-homogeneous, for the action defined by

$$h\,[g,v]=[hg,v]\quad\text{for all }\ g,h\in G\,,\ v\in F_\Lambda\ .$$

Then ϕ is G-equivariant if we endow the trivial bundle $B\times F_\Lambda$ with the G-action

$$h\,(z,v)=\big(h(z),h'(z)^\Lambda v\big)\quad\text{for all }\ h\in G\,,\ z\in B\,,\ v\in F_\Lambda\ .$$

If $\sigma:B\to\mathbb{F}_\Lambda$ is a holomorphic section, then

$$F=\phi\circ\sigma:B\to B\times F_\Lambda\quad\text{is of the form}\quad F(z)=(z,f(z))\quad\text{for some}\quad f\in\mathcal{O}(B,F_\Lambda)\ .$$

The natural action of G on σ, given by $g\,\sigma=g\circ\sigma\circ g^{-1}$, then corresponds to $g^{\pi_\lambda}f$.

4.2.8. In order to characterise when the weighted Bergman space $\mathcal{O}^2_\Lambda$ is non-zero, we very briefly recall some basic facts about reproducing kernel Hilbert spaces.

Given a set X and a Hilbert space V, a Hilbert space $\mathcal{H}$ is called a *reproducing kernel Hilbert space (RKH) of V-valued functions on X* if $\mathcal{H}\subset V^X$ and this inclusion mapping is continuous for the product topology on V^X. An equivalent condition is that the point evaluations on $\mathcal{H}$ be continuous.

The space V^X of V-valued functions on X can be naturally identified with the space of continuous conjugate linear functionals on $V^{\oplus X}$, the algebraic direct sum of X copies of V (V-valued functions with finite support). Here, $V^{\oplus X}$ is endowed with the final locally convex topology with respect to the inclusions $V^K \subset V^{\oplus X}$ for $K \subset X$ finite. Then the product topology of V^X equals the $\sigma(V^X, V^{\oplus X})$-topology, and the weak topology on $V^{\oplus X}$ equals the $\sigma(V^{\oplus X}, V^X)$-topology. The pairing of $V^{\oplus X}$ and V^X which induces this identification is given by

$$\langle \varphi \,|\, f \rangle = \sum_{x \in X} (\varphi(x) \mid f(x))_V \quad \text{for all } \varphi \in V^{\oplus X}, f \in V^X .$$

Since this pairing is non-degenerate, $V_\sigma^{\oplus X}$ can be identified with the space of continuous conjugate linear functionals on V^X.

These considerations show that, by duality, $\mathcal{H}$ is isometrically isomorphic to an RKH of V-valued functions on X if and only if there is continuous linear map $V^{\oplus X} \to \mathcal{H}$ with dense image, i.e., $\mathcal{H}$ is the completion of a quotient of $V^{\oplus X}$.

Given an RKH $\mathcal{H}$, the concatenation of the maps $V^{\oplus X} \to H \to V^X$ is denoted by h and called the *kernel operator* of $\mathcal{H}$. Then h is continuous, and

$$\langle \varphi \,|\, h\psi \rangle = \overline{\langle \psi \,|\, h\varphi \rangle} \quad \text{and} \quad \langle \varphi \,|\, h\varphi \rangle \geqslant 0 \quad \text{for all } \varphi, \psi \in V^{\oplus X} ,$$

i.e., h is Hermitian positive.

Since the functions of point support span $V^{\oplus X}$, there is a bijective correspondence between the set of Hermitian positive continuous linear operators $h : V^{\oplus X} \to V^X$ and functions $h : X \times X \to \mathcal{L}(V)$ which are of *positive type*, meaning that, for any choice of elements $x_1, \ldots, x_m \in X$, the matrix $\big(h(x_i, x_j)\big)$ is a positive self-adjoint element of the C*-algebra $\mathcal{L}(V)^{m \times m}$. By a standard argument, this is equivalent to $h(x, y) = h(y, x)^*$ for all $x, y \in X$, and

$$\sum_{i,j=1}^{m} \big(v_i \mid h(x_i, x_j)\, v_j\big) \geqslant 0 \quad \text{for all } x_1, \ldots, x_m \in X,\, v_1, \ldots, v_m \in V .$$

The correspondence between Hermitian positive operators and functions of positive type given by

$$\langle \varphi \,|\, h\psi \rangle = \sum_{x,y \in X} (\varphi(x) \mid h(x, y)\, \psi(y))_V \quad \text{for all } \varphi, \psi \in V^{\oplus X} ,$$

and is a conoid isomorphism. The function $h(\sqcup, \sqcup)$ associated to the RKH $\mathcal{H}$ is called the *kernel function of* $\mathcal{H}$.

Given a function $h(\sqcup, \sqcup)$ of positive type, and the associated Hermitian positive operator h, we can associate by a generalised GNS construction to h an RKH $\mathcal{H}$ such that its kernel is h. Indeed, we endow $h(V^{\oplus X})$ with an inner product by

$$(h\varphi \mid h\psi) = \langle \varphi \mid h\psi \rangle \quad \text{for all } \varphi, \psi \in V^{\oplus X} ,$$

well-defined by the properties of h. Completing, we obtain a Hilbert space $\mathcal{H}$. As the completion of a quotient of $V^{\oplus X}$, we can identify $\mathcal{H}$ with an RKH. Specifically, for all $f \in V^X$, we have $f \in \mathcal{H}$ if and only

$$\sup_{\langle \varphi \mid h\varphi \rangle \leqslant 1} |\langle \varphi \mid f \rangle| < \infty ,$$

and in this case, $\|f\|_{\mathcal{H}}$ is given by this quantity. Clearly, the kernel of $\mathcal{H}$ is h.

It is now an easy matter to show that the correspondence between Hermitian positive kernels (functions or operators) and RKH is bijective. Note here that equality of RKH is understood as the equality as subspaces of V^X and identity of Hilbert space norms. Two RKH in V^X are equal as vector subspaces if and only they have equivalent norms. This is equivalent to the fact that their kernels be proportional by a strictly positive constant. The main ingredients in the proof of these facts are Riesz's theorem and the closed graph theorem, or a replacement thereof (such as the barreledness of the space $V^{\oplus X}$).

There is a property analogous to the reproducing property for RKH of scalar functions (corresponding to $V = \mathbb{C}$). Namely, denote for $x \in X$ by $\varepsilon_x : \mathcal{H} \to V$ the continuous evaluation at x. If $v_x \in V^{\oplus X}$ denotes the function such that

$$\varepsilon_y(v_x) = \delta_{xy} \cdot v \quad \text{for all } y \in X ,$$

then the image of v_x in $\mathcal{H}$ is given by

$$(hv_x)(y) = \sum_{z \in X} h(y,z)\, v_x(z) = h(y,x)\, v \quad \text{for all } y \in X .$$

Moreover, for all $x \in X$, $v \in V$ and $f \in \mathcal{H}$,

$$(v \mid f(x))_V = \sum_{y \in X} \delta_{xy} \cdot (v \mid f(y)) = \langle v_x \mid f \rangle = (hv_x \mid f)_{\mathcal{H}} ,$$

so the adjoint of $\varepsilon_x : \mathcal{H} \to V$ is given by $hv_x = \varepsilon_x^* v$. We conclude

$$h(x,y)\, v = \varepsilon_x h v_y = \varepsilon_x \varepsilon_y^* v \quad \text{for all } x, y \in X ,\, v \in V .$$

This is the required generalisation of the reproducing property.

Properties of the RKH $\mathcal{H}$ can be expressed in terms of the kernel function h. E.g., if X is a topological space, $\mathcal{H}$ consists of continuous functions if and only if h is separately continuous. The topology on $\mathcal{H}$ is always weaker than the topology of uniform convergence on subsets of $Y \subset X$ such that the operator norm of $h(x,x)$ remains bounded when

x varies in Y. By Hartogs's joint analyticity theorem [Hör73, th. 2.2.8], if X is a complex manifold, then $\mathcal{H}$ consists of holomorphic functions if and only h is sesqui-holomorphic on $X \times X$.

For a more thorough treatment of RKH of V-valued functions, in particular, their connection to involutive semigroups. we refer to [Nee00a, ch. I-IV].

Using these facts on RKH, we can characterise precisely when $\mathcal{O}^2_\Lambda$ is non-zero.

Theorem 4.2.9. If $\Lambda \in i\mathfrak{t}^*_{\mathbb{R}}$ is Δ_c^{++}-dominant and analytically integral, then the following conditions are equivalent.

(i). The weighted Bergman space $\mathcal{O}^2_\Lambda = \mathcal{O}^2(B, F_\Lambda)$ is non-zero.

(ii). The function

$$B^\Lambda : B \times B \to \operatorname{End} F_\Lambda : (z,w) \mapsto B(z,w)^\Lambda$$

is of positive type, and proportional to the kernel function of $\mathcal{O}^2_\Lambda$.

(iii). The integral

$$\int_B \operatorname{tr} B(z,z)^{-\Lambda}\, d\mu_0(z) < \infty .$$

(iv). We have $\mathcal{P}(Z, F_\Lambda) \subset \mathcal{O}^2(B, F_\Lambda)$.

In this case, $\mathcal{O}^2_\Lambda$ is an irreducible square-integrable representation of G whose space of K-finite vectors is $\mathcal{P}(Z, F_\Lambda)$.

Proof. Let K be the kernel function of $\mathcal{O}^2_\Lambda$. Then, for $z = g(0) \in B$ and all $u, v \in F_\Lambda$,

$$\begin{aligned}(u \mid K(z,z)\,v)_\Lambda &= (u \mid (\varepsilon_z^* v)(z))_\Lambda = \big(u \mid g'(0)^\Lambda (g^{-\pi_\lambda}\varepsilon_z^* v)(0)\big)_\Lambda \\ &= \big(g^{\pi_\lambda}\varepsilon_0^*(g'(0)^{*\Lambda}u) \mid \varepsilon_z^* v\big)_{\mathcal{O}^2_\Lambda} = \big(g^{-1\prime}(g(0))^{-\Lambda}(\varepsilon_0^*(g'(0)^{*\Lambda}u))(0) \mid v\big)_\Lambda \\ &= \big(u \mid g'(0)^\Lambda\, \varepsilon_0\varepsilon_0^*\, g'(0)^{*\Lambda}\, v\big)_\Lambda = \big(u \mid g'(0)^\Lambda\, K(0,0)\, g'(0)^{*\Lambda}\, v\big)_\Lambda .\end{aligned}$$

In particular, $K(0,0)$ commutes with the action of K. Since F_Λ is simple, $C = K(0,0) \geqslant 0$ is a scalar constant, by Schur's lemma. If $\mathcal{O}^2_\Lambda$ is non-zero, so is ε_z for some $z \in B$. Then $K(z,z) \neq 0$, and consequently, so is $K(0,0)$. Hence, $C > 0$.

Moreover, $B(0,0)^\Lambda = 1$, and B^Λ is also covariant, so $K = C \cdot B^\Lambda$ on the diagonal. Since both functions are sesqui-holomorphic, the identity is valid everywhere. In particular, B^Λ is of positive type, and the RKH it defines equals $\mathcal{O}^2_\Lambda$ as a vector space, and has an equivalent norm.

If the kernel functions K and B^Λ are proportional, then

$$\mathcal{O}^2(B, F_\Lambda) \ni B(\sqcup, 0)^\Lambda v \quad \text{for all } v \in F_\Lambda .$$

But $B(z,0) = 1$, so these are the constant F_Λ-valued functions on B. This means that for any $v \in F_\Lambda$,

$$\infty > \|v\|^2_{\mathcal{O}^2_\Lambda} = \int_B (v \mid B(z,z)^{-\Lambda}\, v)\ d\mu_0(z)\ .$$

Applying this to an orthonormal basis of F_Λ, the integrability of $\operatorname{tr} B(z,z)^{-\Lambda}$ follows.

Assuming the finiteness of the integral, the Cauchy-Schwarz and Hölder inequalities show that

$$\begin{aligned}\int_B^* (p(z) \mid B(z,z)^{-\Lambda}\, p(z))_\Lambda\ d\mu_0(z) &\leqslant \int_B^* \|p(z)\|^2_\Lambda \cdot \|B(z,z)^{-\Lambda}\|\ d\mu_0(z) \\ &\leqslant \|p\|^2_\infty \cdot \int_B \|B(z,z)^{-\Lambda}\|\ d\mu_0(z) \\ &\leqslant \|p\|^2_\infty \cdot \int_B \operatorname{tr} B(z,z)^{-\Lambda}\, d\mu_0(z)\end{aligned}$$

for all $p \in \mathcal{P}(Z, F_\Lambda)$. The right hand side is finite, because p is continuous on Z and hence bounded on the relatively compact subset B. Hence $\mathcal{P}(Z, F_\Lambda) \subset \mathcal{O}^2(B, F_\Lambda)$.

Finally, if $\mathcal{O}^2_\Lambda$ contains the polynomials, then it is clearly non-zero.

Assume that one of the equivalent conditions is fulfilled. The representation of G on the RKH $\mathcal{O}^2_\Lambda$ is irreducible by a theorem of Kobayashi [Nee00a, th. IV.1.10], since B is homogeneous, and F_Λ is simple. In order to check the square-integrability, it is necessary and sufficient that the matrix coefficient $(v \mid \pi_\lambda\, v)_{\mathcal{O}^2_\Lambda}$ be square integrable for some constant $v \in F_\Lambda$, by [Dix69, def. 14.1.3, cor. 14.3.2].

To this end, note that the set of constants in $\mathcal{O}^2_\Lambda$ is K-equivalent to F_Λ. Moreover, by G-invariance of μ_0, the integral

$$I = \int_B B(z,z)^{-\Lambda}\, d\mu_0(z) \in \operatorname{End} F_\Lambda$$

defines a K-equivariant operator. By Schur's lemma, $c = I$ is a scalar constant, namely

$$c = \frac{\operatorname{tr} I}{\dim F_\Lambda} = \big(\dim F_\Lambda\big)^{-1} \int_B \operatorname{tr} B(z,z)^{-\Lambda}\, d\mu_0(z)\ .$$

But this implies that

$$\|v\|^2_\Lambda = c^{-1} \cdot \int_B (v \mid B(z,z)^{-\Lambda}\, v)_\Lambda\ d\mu_0(z) = c^{-1} \cdot \|v\|^2_{\mathcal{O}^2_\Lambda}\ .$$

By polarisation, the inner products are also proportional.

For $v \in F_\Lambda$ and $f \in \mathcal{O}^2_\Lambda$, we compute

$$(v \mid f)_{\mathcal{O}^2_\Lambda} = (B(\sqcup,0)^\Lambda\, v \mid f)_{\mathcal{O}^2_\Lambda} = \frac{1}{C} \cdot (\varepsilon_0^*\, v \mid f)_{\mathcal{O}^2_\Lambda} = \frac{1}{C} \cdot (v \mid f(0))_\Lambda = \frac{c}{C} \cdot (v \mid f(0))_\Lambda\ .$$

Hence $\frac{c}{C} \cdot \varepsilon_0$ is the orthogonal projection onto the constants. But $\varepsilon_0^2 = \varepsilon_0$, so $C = c$, and the projection is just ε_0 . This implies

$$(v \mid (gk)^{\pi_\lambda} v)_{\mathcal{O}^2_\Lambda} = (g^{-\pi_\lambda} v \mid k^\Lambda v)_{\mathcal{O}^2_\Lambda} = ((g^{-\pi_\lambda} v)(0) \mid k^\Lambda v)_\Lambda = (g'(0)^{-\Lambda} v \mid k^\Lambda v)_\Lambda \ .$$

For a suitable normalisation of Haar measure dg on G , $dg = d\mu_0(g(0))\, dk$. Hence, the orthogonality relations on K [Dix69, th. 14.3.3, prop. 15.2.3] imply for $\|v\|_\Lambda = 1$

$$\begin{aligned}\int_G \left|(v \mid g^{\pi_\lambda} v)_{\mathcal{O}^2_\Lambda}\right|^2 dg &= \int_B \int_K |(g'(0)^{-\Lambda} v \mid k^\Lambda v)_\Lambda|^2 \, dk\, d\mu_0(g(0)) \\ &= \frac{1}{\dim F_\Lambda} \cdot \int_B \|g'(0)^{-\Lambda} v\|^2_\Lambda \, d\mu_0(g(0)) \\ &= \frac{1}{\dim F_\Lambda} \int_B (v \mid (g'(0)^\Lambda g'(0)^{*\Lambda})^{-1} v) \, d\mu_0(g(0)) \\ &= \frac{1}{\dim F_\Lambda} \int_B (v \mid B(z,z)^{-\Lambda} v) \, d\mu_0(z) = \frac{1}{\dim F_\Lambda} \cdot \|v\|^2_{\mathcal{O}^2_\Lambda} < \infty \ .\end{aligned}$$

Hence, π_λ is square-integrable.

To complete the theorem's proof, we have to show that the module of K-finite vectors coincides with $\mathcal{P}(Z, F_\Lambda)$. For any polynomial p , and for $k \in K$, we have

$$\big(k^{\pi_\lambda} p\big)(z) = k^\Lambda\, p(k(z)) \quad \text{for all } \ z \in B \ .$$

Since the action of K on B is linear, $\deg(k^{\pi_\lambda} p) = \deg p$. Since $\mathcal{P}^m(Z)$ is of finite dimension for any $m \in \mathbb{N}$, this proves that all elements of $\mathcal{P}(Z, F_\Lambda)$ are K-finite in the representation π_λ . On the other hand, $\mathcal{P}(Z, F_\Lambda)$ is G-invariant, and since π_λ is a topologically irreducible unitary representation, the space of K-finite vectors is algebraically irreducible, by [War72, th. 4.5.2.11, th. 4.5.5.4]. Hence, we have equality. □

We have reduced the non-triviality of the weighted Bergman space $\mathcal{O}^2_\Lambda$ to the finiteness of an integral.

Our next goal is to characterise this finiteness in terms of an algebraic condition on the parameter $\lambda = \Lambda + \varrho$, namely, Harish-Chandra's square-integrability condition. To this end, we digress a little on the meaning of this condition.

4.2.10. Recall from 4.1.1 that we have fixed a frame $e_1, \ldots, e_r$, an associated torus and positive system. The Killing form is an inner product on $i\mathfrak{t}_\mathbb{R}$, and $i\mathfrak{t}^*_\mathbb{R}$ is endowed with the dual product. Via this duality, to $\mu \in i\mathfrak{t}^*_\mathbb{R}$, there corresponds $\mu^* \in i\mathfrak{t}_\mathbb{R}$. Then

$$-\mu(H_\alpha) = -B(\mu^*, H_\alpha) = (\mu^* \,:\, H_\alpha) = (-i\mu^* \,:\, iH_\alpha) \quad \text{for all } \ \alpha \in \Delta \ .$$

Hence, $\mu(H_\alpha) < 0$ for all $\alpha \in \Delta_n^{++}$ simply means that $\mu \in i(\omega^-)^*$, if we identify the Euclidean vector spaces $i\mathfrak{t}^*_\mathbb{R} = i\mathfrak{t}_\mathbb{R}$ by $\mu \mapsto \mu^*$. Moreover, since $\sigma(i \cdot e_j \,\square\, e_j^*)$, $\sigma \in W_c$, $j = 1, \ldots, r$, are the generators of extremal rays in ω^- by lemma 2.1.13,

$$\mu \in i(\omega^-)^* \iff \mu(H_\alpha) < 0 \quad \text{for all } \ \alpha \in \Delta_n^{++}$$

$$\iff \quad \langle \sigma(e_j \,\Box\, e_j^*) \,:\, \mu \rangle < 0 \quad \text{for all} \quad \sigma \in W_c\,,\; j = 1, \dots, r\,.$$

If $\mu \in i\mathfrak{t}_{\mathbb{R}}^*$ is assumed to be Δ_c^{++}-dominant and integral, then this can be sharpened. In fact, $2 \cdot e_r \,\Box\, e_r^* = H_{\gamma_r}$, and γ_r is the largest root. Since $\Delta_n^{++} \subset \gamma_r - \mathbb{N}\langle \Delta_c^{++} \rangle$ by [Nee00a, proof of lem. IX.5.8], the dominance of μ implies that

$$\mu(H_\alpha) \leqslant 2 \cdot \langle e_r \,\Box\, e_r^* \,:\, \mu \rangle \quad \text{for all} \;\; \alpha \in \Delta_n^{++}\,.$$

This shows that in the above equivalence, it is already sufficient that $\langle e_r \,\Box\, e_r^* \,:\, \mu \rangle < 0$.

The following proposition gives some indication why Harish-Chandra's condition is related to the non-triviality of $\mathcal{O}^2_\Lambda$. The precise theorem will be stated and proved below.

Proposition 4.2.11. Let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$ be Δ_c^{++}-dominant and analytically integral. On the orbit of the Iwasawa A component $A = \exp\langle \zeta_{e_1}^-, \dots, \zeta_{e_r}^- \rangle$, the kernel B^Λ is given by

$$B\big(a(0), a(0)\big)^\Lambda v = \prod_{j=1}^r (\cosh t_j)^{-4\mu_j} \cdot v \quad \text{for all} \;\; a = \exp\Big(\sum_{j=1}^r t_j \cdot \zeta_{e_j}^-\Big)\,,\; v \in F_\Lambda[\mu]$$

where $\mu_j = \langle e_j \,\Box\, e_j^* \,:\, \mu \rangle$, $j = 1, \dots, r$.

Proof. For $\log a = \sum_{j=1}^r t_j \cdot \zeta_{e_j}^-$, we have

$$a = \prod_{j=1}^r \exp \zeta_{t_j e_j}^- = \prod_{j=1}^r t^+_{\tanh t_j e_j} \, B(\tanh t_j e_j, \tanh t_j e_j)^{1/2} \, t^-_{-\tanh t_j e_j}\,,$$

by [Loo75, prop. 9.8]. By [Loo75, prop. 8.10], $K^{\mathbb{C}} P^-$ fixes the origin. Since the above factors commute, we have, by the chain rule and lemma 4.2.1,

$$\begin{aligned} a'(0) &= \prod_{j=1}^r \Big(t^+_{\tanh t_j e_j} \, B(\tanh t_j e_j, \tanh t_j e_j)^{1/2} \, t^-_{-\tanh t_j e_j} \Big)'(0) \\ &= \prod_{j=1}^r B((\tanh t_j) \cdot e_j, (\tanh t_j) \cdot e_j)^{1/2}\,, \end{aligned}$$

because $\tanh(te) = (\tanh t) \cdot e$ for any tripotent $e \in Z$. Hence,

$$B\big(a(0), a(0)\big) = a'(0) a'(0)^* = \prod_{j=1}^r B((\tanh t_j) \cdot e_j, (\tanh t_j) \cdot e_j) = \prod_{j=1}^r B(e_j, (\tanh^2 t_j) \cdot e_j)\,.$$

If $e \in Z$ is a tripotent, then $g_t = B(e, (1 - e^t) \cdot e)$ is a one-parameter group in $K^{\mathbb{C}}$, by [Upm85, proof of 21.9]. Moreover,

$$\dot{g}_0 = 4 \cdot e \,\Box\, e^* (1 - e \,\Box\, e^*) + 2 \cdot e \,\Box\, e^* (2\, e \,\Box\, e^* - 1) = 2 \cdot e \,\Box\, e^*\,.$$

Therefore,

$$B\big(a(0),a(0)\big) = \exp\Big(2\sum_{j=1}^{r}\log(1-\tanh^2 t_j)\cdot e_j\,\square\, e_j^*\Big) = \exp\Big(-4\sum_{j=1}^{r}\log\cosh t_j\cdot e_j\,\square\, e_j^*\Big)\;.$$

The vector v has weight μ . In particular,

$$B\big(a(0),a(0)\big)^{\Lambda}\,v = \prod_{j=1}^{r}\big(\cosh t_j\big)^{-4\mu_j}\cdot v\;,$$

since $e_j\,\square\, e_j^*\in i\mathfrak{t}_{\mathbb{R}}$. $\square$

4.2.12. The normalised Lebesgue measure dz on B is given by

$$\int_B f(z)\,dz = C\cdot\int\cdots\int_{1>t_r>\cdots>t_1>0}\int_K f\big(k\sum_{j=1}^{r}t_je_j\big)\,dk\prod_{1\leqslant i<j\leqslant r}(t_j^2-t_i^2)^a\prod_{j=1}^{r}t_j^{2b+1}\,dt_1\cdots dt_r$$

for all Lebesgue-integrable f , and some constant $C>0$, cf. [Upm96, prop. 1.5.84]. Here, $a=\dim Z_{ij}$ ($1\leqslant i<j\leqslant r$), and $b=\dim Z_{0j}$ ($0<j\leqslant r$). For an evaluation of the constant C , we refer to [FK94, ex. VI.3] and the references given there.

Lemma 4.2.13. The invariant measure μ_0 of B is given by

$$\int_B f(z)\,d\mu_0(z) = C\cdot\int\cdots\int_{1>t_r>\cdots>t_1>0}\int_K f\big(k\sum_{j=1}^{r}t_je_j\big)\,dk\,\frac{\prod_{1\leqslant i<j\leqslant r}(t_j^2-t_i^2)^a\,\prod_{j=1}^{r}t_j^{2b+1}}{\prod_{j=1}^{r}(1-t_j^2)^{2+(r-1)a+b}}\,dt_1\cdots dt_r$$

for all μ_0-integrable f . Here, $C>0$ is the constant from 4.2.12.

Proof. The calculation in proposition 4.2.11 shows that for $\log a=\sum_{j=1}^{r}s_j\cdot\xi_{e_j}^-$,

$$a(0) = \Big[\prod_{j=1}^{r}t^-_{\tanh s_je_j}\Big](0) = \sum_{j=1}^{r}(\tanh s_j)\cdot e_j\;,$$

since $K^{\mathbb{C}}P^-$ fixes 0 . So, for $z=\sum_{j=1}^{r}t_j\cdot e_j=a(0)$, we have

$$B(z,z) = \exp\Big(2\sum_{j=1}^{r}\log(1-\tanh^2 s_j)\cdot e_j\,\square\, e_j^*\Big) = \exp\Big(2\sum_{j=1}^{r}\log(1-t_j^2)\cdot e_j\,\square\, e_j^*\Big)\;.$$

The action of k on Z is unitary, so $\det B(kz,kz)=\det k\,B(z,z)\,k^*=\det B(z,z)$. Moreover,

$$\det\exp\big(e_j\,\square\, e_j^*\big) = \exp\operatorname{tr}\big(e_j\,\square\, e_j^*\big)\;.$$

Here, $e_j\,\square\, e_j^*$ acts by λ on $Z_\lambda(e_j)$. By [Loo75, th. 3.14],

$$Z_1(e_j) = Z_{jj}\quad\text{and}\quad Z_{1/2}(e_j) = \textstyle\sum^{\oplus}_{0\leqslant i<j}Z_{ij}\oplus\sum^{\oplus}_{j<i\leqslant r}Z_{ji}\;,$$

so $\dim Z_1(e_j) = 1$ and $\dim Z_{1/2}(e_j) = \dim Z_{0j} + (r-1)\cdot\dim Z_{ij} = (r-1)\cdot a + b$. Hence,

$$\operatorname{tr}\big(e_j \,\square\, e_j^*\big) = 1 + \frac{1}{2}\cdot\big((r-1)\cdot a + b\big)\,.$$

We conclude

$$\det B\Big(k\sum_{j=1}^r t_j e_j, k\sum_{j=1}^r t_j e_j\Big) = \prod_{j=1}^r (1-t_j^2)^{2+(r-1)\cdot a+b}\,.$$

Hence, the assertion follows from lemma 4.2.5 and the formula from 4.2.12. ——— □

Theorem 4.2.14. Let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$ be Δ_c^{++}-dominant and analytically integral. Then the weighted Bergman space $\mathcal{O}^2_\Lambda = \mathcal{O}^2(B, F_\Lambda)$ is non-zero if and only if Harish-Chandra's square-integrability condition

$$\lambda(H_\alpha) < 0 \quad \text{for all } \alpha \in \Delta_n^{++}$$

is satisfied for $\lambda = \Lambda + \varrho$, $\varrho = \frac{1}{2}\cdot\sum_{\alpha\in\Delta^{++}}\alpha$.

Proof. First, we need to evaluate ϱ on $e_j \,\square\, e_j^*$. To this end, recall the facts from 2.1.24. In particular, $\mathrm{Ad}^*(\gamma_e)(\varrho) = \varrho_{\mathfrak{a}}$, the half sum of the positive restricted roots (with multiplicity). Moreover,

$$\mathrm{Ad}(\gamma_e)(e_j \,\square\, e_j^*) = \frac{1}{2i}\cdot\big[\mathrm{Ad}(\gamma_e)(\xi^-_{ie_j}), \mathrm{Ad}(\gamma_e)(\xi^-_{e_j})\big] = \frac{1}{4i}\big[\xi^-_{ie_j}, [\xi^+_e, \xi^-_{e_j}]\big] = \frac{1}{2}\cdot\xi^-_{e_j}$$

by [Upm85, lem. 21.16]. We deduce

$$\langle e_j \,\square\, e_j^* : \varrho\rangle = \langle \xi^-_{e_j} : \varrho_{\mathfrak{a}}\rangle = \frac{1}{2}\cdot\sum_{\alpha\in\Delta^+_{\mathfrak{a}}} \dim_{\mathbb{R}} \mathfrak{g}^\alpha_{\mathbb{R}}\cdot\langle \xi^-_{e_j} : \alpha\rangle$$

Recall that $\alpha^\varepsilon_{k\ell} = \alpha_\ell - \varepsilon\cdot\alpha_k$ where $\alpha_0 = 0$ and $\alpha_1,\dots,\alpha_r$ is dual to $e_1 \,\square\, e_1^*, \dots, e_r \,\square\, e_r^*$, and that $\Delta^+_{\mathfrak{a}}$ consists of non-zero $\alpha^\varepsilon_{k\ell}$ for $0 \leqslant k \leqslant \ell \leqslant r$ and $\varepsilon^2 = 1$. Now,

$$\langle \xi^-_{e_j} : \alpha^\varepsilon_{k\ell}\rangle = \delta_{\ell j} - \varepsilon\cdot\delta_{kj} = \begin{cases} 0 & j \notin \{k,\ell\} \text{ or } j = k = \ell,\ \varepsilon = +1\,,\\ 2 & j = k = \ell,\ \varepsilon = -1\,,\\ -\varepsilon & j = k < \ell\,,\\ 1 & k < \ell = j\,. \end{cases}$$

Hence,

$$\langle \xi^-_{e_j} : \varrho_{\mathfrak{a}}\rangle = \frac{1}{2}\cdot\big(2\cdot\dim_{\mathbb{R}} iX_1(e_j) + \dim_{\mathbb{R}} Z_{0j} + (r-1)\cdot\dim_{\mathbb{R}} Z_{ij}\big) = 1 + (j-1)\cdot a + b\,.$$

We conclude that $\varrho_j = \frac{1}{2}\cdot(1 + (j+1)\cdot a + b)$.

By proposition 4.2.11,

$$\operatorname{tr} B\Big(k\sum_{j=1}^r t_j e_j, k\sum_{j=1}^r t_j e_j\Big)^{-\Lambda} = \sum_{F_\Lambda[\mu]\neq 0} \dim F_\Lambda[\mu]\cdot\prod_{j=1}^r (1-t_j^2)^{-2\mu_j}$$

If we denote $m_\mu = \dim F_\Lambda[\mu]$, then by lemma 4.2.13,

$$\int_B^* \operatorname{tr} B(z,z)^{-\Lambda}\,d\mu_0(z) = C\cdot \int\cdots\int_{1>t_r>\cdots>t_1>0} \sum\nolimits_\mu m_\mu \cdot \frac{\prod_{i<j}(t_j^2-t_i^2)^a\cdot\prod_{j=1}^r t_j^{2b+1}}{\prod_{j=1}^r(1-t_j^2)^{2\mu_j+2+(r-1)a+b}}\,dt_1\cdots t_r\,.$$

By theorem 4.2.9, this integral is finite if and only if $\mathcal{O}^2_\Lambda \neq 0$. The numerator of the fraction is bounded and non-zero for t_j close to 1 . Moreover, μ_j is an half-integer because $2\cdot e_j \,\square\, e_j^* = H_{\gamma_j}$ is a coroot and all weights of F_Λ are integral. Since $\int_0^1 \frac{dr}{1-t^2} = \infty$, the above integral is finite if and only if

$$2(\mu_j+\varrho_r)+1 = 2\mu_j+2+(r-1)a+b \leqslant 0 \quad \text{for all } \mu\in\mathfrak{t}^*\,,\ m_\mu>0\,,\ j=1,\ldots,r\,.$$

By integrality, this is equivalent to

$$2(\mu_j+\varrho_r) = 2\mu_j+1+(r-1)a+b < 0 \quad \text{for all } \mu\in\mathfrak{t}^*\,,\ m_\mu>0\,,\ j=1,\ldots,r\,.$$

Since $\mu \in \Lambda - \mathbb{N}\langle\Delta_c^{++}\rangle$ for any weight μ of F_Λ , this is the case of and only if it is true for Λ. Since $\lambda_j = \Lambda_j + \varrho_j$, this requirement implies Harish-Chandra's condition. On the other hand, in the presence of Harish-Chandra's condition, we have $\lambda_r < 0$. But $\Lambda_j \leqslant \Lambda_r$ for all $j = 1,\ldots,r$ by 4.2.10, and the above condition follows. □

We have characterised the non-triviality of the weighted Bergman space in terms of Harish-Chandra's condition. To complete this discussion, we need to see that $\mathcal{O}^2_\Lambda$ is indeed a globalisation of $N(\Lambda)$.

Lemma 4.2.15. The action of G on $\mathcal{O}(Z,F_\Lambda)$ is analytic. Infinitesimally, it is given by

$$\Big[\pi_\lambda\Big(u\frac{\partial}{\partial z}\Big)f\Big](z) = -f'(z)u$$
$$\big[\pi_\lambda(\delta^+,\delta^-)f\big](z) = (\delta^+,\delta^-)^\Lambda f(z) - f'(z)\delta^+(z)$$
$$\Big[\pi_\lambda\Big(\{zv^*z\}\frac{\partial}{\partial z}\Big)f\Big](z) = 2\cdot(z\,\square\, v^*)^\Lambda f(z) - f'(z)\{zv^*z\}$$

for all $z\in B$, $u,v\in Z$, $(\delta^+,\delta^-)\in \operatorname{aut}(Z,Z)$, and $f\in\mathcal{O}(Z,F_\Lambda)$.

Proof. Let $\xi \in \mathfrak{g}$ be an holomorphic vector field on B and let $g_t = \exp t\xi$ be its local holomorphic flow. We compute

$$(\xi^{\pi_\lambda} f)(z) = \frac{\mathrm{d}}{\mathrm{d}t} g'_{-t}(z)^{-\Lambda} f(g_{-t}(z))\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} g'_{-t}(z)^{-\Lambda}\Big|_{t=0} f(z) - f'(z)\xi(z)\,.$$

If $\xi = u\,\frac{\partial}{\partial z}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} g'_{-t}(z)\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} t^{+\prime}_{-tu}(z)\big|_{t=0} = 0$$

since $t^{+\prime}_{-tu}(z) = 1$. If, $\xi = (\delta^+, \delta^-)$, then the local flow is linear.

Finally, let $\xi = v\,\frac{\partial}{\partial z}$. Then

$$g'_{-t}(z)^{-1} = t^{-\prime}_{-tv}(z)^{-1} = B(z, -tv)\ .$$

by lemma 4.2.1. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t} g'_{-t}(z)^{-1}\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} 2t \cdot z \,\square\, v^* + t^2 Q_u Q_v\big|_{t=0} = 2 \cdot z \,\square\, v^*\ ,$$

proving the assertion. $\square$

Proposition 4.2.16. Let $\Lambda \in i\mathfrak{t}^*_{\mathbb{R}}$ be Δ^{++}_c-dominant and analytically integral. Whenever $\mathcal{O}^2_\Lambda = \mathcal{O}^2(B, F_\Lambda)$ is non-zero, its module of K-finite vectors $\mathcal{P}(Z, F_\Lambda)$ is a $\mathfrak{g}$-highest weight module of highest weight Λ. In particular, it is $\mathfrak{g}$-isomorphic to $N(\Lambda)$.

Proof. If $\mathcal{O}^2(B, F_\Lambda)$ is non-zero, then $\mathcal{P}(Z, F_\Lambda)$ is simple by theorem 4.2.9. Hence, the constant 1_Λ is cyclic. By lemma 4.2.15, it is annihilated by $\mathfrak{p}^+$. The constants in $\mathcal{O}^2_\Lambda$ are K-equivalent to F_Λ, so 1_Λ has weight Λ. Therefore, $\mathcal{P}(Z, F_\Lambda) = L^{\mathfrak{g}}_\Lambda$ as $\mathfrak{g}$-modules. But theorem 4.2.14 shows that Harish-Chandra's condition is fulfilled, so theorem 4.1.18 shows that $N(\Lambda) = L^{\mathfrak{g}}_\Lambda$, completing the proposition's proof. $\square$

Remark 4.2.17. In principle, the $\mathfrak{g}$-isomorphism $\mathcal{P}(Z, F_\Lambda) = N(\Lambda)$ is computable with the help of lemma 4.2.15, by considering the action on $\mathfrak{U}(\mathfrak{p}^-)$ on the highest weight vector 1_Λ. However, the action of $\mathfrak{p}^-$ on $\mathcal{P}(Z, F_\Lambda)$ is quite complicated. In fact, the main technical difference between these two realisations of the module of K-finite vectors of $\mathcal{O}^2(B, F_\Lambda)$ is the following: The action of $\mathfrak{p}^+$ on $\mathcal{P}(Z, F_\Lambda)$ is straightforward, whereas it is rather involved if computed in terms of the basis of $N(\Lambda)$ given by the $\mathfrak{k}$-isomorphic identification with $\mathfrak{U}(\mathfrak{p}^-) \otimes F_\Lambda$, and conversely for the action of $\mathfrak{p}^-$.

4.2.18. We have seen that the representations $\mathcal{O}^2_\Lambda$ of G, where Λ is Δ^{++}_c-dominant and analytically integral, and $\lambda = \Lambda + \varrho$ satisfies Harish-Chandra's condition, are square-integrable irreducible unitary representations, i.e., belong to the discrete series of G. In fact, their totality is called the *holomorphic discrete series* of G.

Which part of $\mathbf{L}^2(G)$ is spanned by the coefficient functions of the holomorphic discrete series can be characterised quite precisely. Indeed, any holomorphic discrete series representation extends to a holomorphic representation of the involutive semigroup Γ°, by [Ol'82, cor. 5.8] or [Sta86, th. 3.5], and its coefficient functions therefore belong to $\mathbf{H}^2(\Gamma)$. Conversely, the Hardy space is spanned by these matrix coefficients (loc.cit.). In fact, this was the original motivation for the construction of the domain Γ°, cf. [GG77].

The characterisation of the unitary highest modules which are square-integrable (module centre) as holomorphic semigroup representations, respectively as representations contained in some Hardy space, has been considerably extended, encompassing the groups $G_{f,I}$ associated to the faces $F^-_{f,I}$ of the cone Ω^-. As a blanket reference we give [Nee00a, ch. XIV]. The theory we have reviewed in the last two sections applies, among the Lie groups associated to the faces of Ω^-, only to the G_e, where $c = 0$. In order to complete the programme set forth in this work for all faces of Ω^-, a treatment of the case $c > 0$ would, however, be necessary.

4.3 Embedding of the holomorphic discrete series

4.3.1. Let $e \in Z$ be a tripotent, and consider the facial subgroup $G_e = \mathrm{Aut}_0\, B_0(e) \sqsubset G$ where $B_0(e) = Z_0(e) \cap B$. We may assume $e = e_{k+1} + \cdots + e_r$ where $0 \leqslant k \leqslant r$. We abbreviate $\bar{G} = G_e$ and $\bar{B} = B_0(e)$. Then the extreme cases $k = r$ and $k = 0$ correspond to $\bar{G} = G$, $\bar{B} = B$ and $\bar{G} = 1$, $\bar{B} = 0$, respectively.

The definition 1.2.5 of the embedding $\bar{G} \sqsubset G$ implies the inclusions $\bar{K} = K_e \subset K$, $\bar{\mathfrak{k}}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R},0}(e) \subset \mathfrak{k}_{\mathbb{R}}$, and $\bar{\mathfrak{p}}_{\mathbb{R}} = \mathfrak{p}_{\mathbb{R},0}(e) \subset \mathfrak{p}_{\mathbb{R}}$.

Moreover, $\bar{Z} = Z_0(e) \subset Z$ is a subtriple, which implies that $\bar{\mathfrak{p}}^{\pm} = \mathfrak{p}_0(e)^{\pm} \subset \mathfrak{p}^{\pm}$. Since $e_1, \ldots, e_k$ is a frame of $\bar{Z}$, lemma 2.2.19 shows that $\bar{\mathfrak{t}}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R},0}(e) = \mathfrak{t}_{\mathbb{R}} \cap \bar{\mathfrak{g}}_{\mathbb{R}}$ is a compactly embedded Cartan subalgebra of $\bar{\mathfrak{g}}_{\mathbb{R}} = \mathfrak{g}_{0,\mathbb{R}}(e)$, such that $\bar{\mathfrak{t}}^{\pm}_{\mathbb{R}} \subset \mathfrak{t}^{\pm}_{\mathbb{R}}$.

By 2.2.20 and the arguments therein,

$$\bar{\Delta} = \left\{\alpha \in \Delta \;\middle|\; \mathfrak{g}^{\alpha} \subset \bar{\mathfrak{g}}\right\}$$

is the set of roots for $\bar{\mathfrak{g}}_{\mathbb{R}}$. We have already seen that the Cartan decompositions of $\bar{\mathfrak{g}}$ and $\mathfrak{g}$ are compatible, so $\bar{\Delta}_c = \bar{\Delta} \cap \Delta_c$ and $\bar{\Delta}_n = \bar{\Delta} \cap \Delta_n$. Moreover, $\bar{\Delta}^{++} = \bar{\Delta} \cap \Delta^{++}$ is an adapted positive system constructed as in lemma 2.1.6. By proposition 2.2.23,

$$\bar{\omega}^{\pm} = \omega_0^{\pm}(e) = \bar{\mathfrak{g}}_{\mathbb{R}} \cap \omega^{\pm} \quad \text{and} \quad \bar{\Omega}^{\pm} = \Omega_0^{\pm}(e) = \bar{\mathfrak{g}}_{\mathbb{R}} \cap \Omega^{\pm}\,.$$

The Killing forms of $\bar{\mathfrak{g}}_{\mathbb{R}}$ and $\mathfrak{g}_{\mathbb{R}}$ are proportional on $\bar{\mathfrak{g}}_{\mathbb{R}}$, and the canonical inner products of $\bar{Z}$ and Z coincide on $\bar{Z}$.

We may assume that Δ_c^{++} was chosen in such a way that the compact simple roots $\bar{\Pi}_c = B(\bar{\Delta}_c^{++})$ are also Δ_c^{++}-simple, cf. lemma 2.1.6 and [Bou68, ch. VI, § 1.7, prop. 24]. In addition, γ_1 is the only simple non-compact root, and it is contained in $i\bar{\mathfrak{t}}^*_{\mathbb{R}}$. Hence, the set $\bar{\Pi} = B(\bar{\Delta}^{++})$ of $\bar{\Delta}^{++}$-simple roots is contained in $\Pi = B(\Delta^{++})$. Since the half sum of positive roots ϱ is also the sum of the fundamental weights [Bou68, ch. VI, § 1.10, prop. 29], this implies that $\varrho\big|_{\bar{\mathfrak{t}}} = \bar{\varrho}$ where $\bar{\varrho} = \frac{1}{2} \cdot \sum_{\alpha \in \bar{\Delta}^{++}} \alpha$.

4.3.2. Given a Δ_c^{++}-dominant and analytically integral parameter $\Lambda \in i\mathfrak{t}^*_{\mathbb{R}}$, its restriction $\bar{\Lambda} = \Lambda\big|_{\bar{\mathfrak{t}}}$ is manifestly $\bar{\Delta}_c^{++}$-dominant and analytically integral, since the exponential map restricts as $\exp_G\big|_{\bar{G}} = \exp_{\bar{G}}$. Besides, the analytically integral functionals form a spanning lattice in $i\mathfrak{t}_{\mathbb{R}}$, so any such parameter $\bar{\Lambda}$ can be extended to a parameter Λ, with $\mathrm{rk}\, \mathfrak{g}_{\mathbb{R}} - \mathrm{rk}\, \bar{\mathfrak{g}}_{\mathbb{R}}$ degrees of freedom.

Since $\bar{\omega}^- = \bar{\mathfrak{t}}_{\mathbb{R}} \cap \omega^-$ and $\varrho\big|_{\bar{\mathfrak{t}}} = \bar{\varrho}$, 4.2.10 shows that $\bar{\lambda} = \bar{\Lambda} + \bar{\varrho}$ fulfils Harish-Chandra's condition if this is the case for $\lambda = \Lambda + \varrho$. Conversely, $\bar{\Lambda}$ can be extended to such a Λ with sufficiently many degrees of freedom.

In order to state matters more succinctly, we say that $(\bar{\Lambda}, \Lambda)$ is an *adapted pair* if $\Lambda\big|_{\bar{\mathfrak{t}}} = \Lambda$. We say $(\bar{\Lambda}, \Lambda)$ is compactly dominant and integral if both parameters are, and similarly for analytic integrality. We that $(\bar{\Lambda}, \Lambda)$ is an adapted pair of *holomorphic discrete series parameters* if it is compactly dominant and analytically integral, and the shifted parameters $(\bar{\lambda}, \lambda)$ satisfy Harish-Chandra's condition.

Lemma 4.3.3. Let $(\bar{\Lambda}, \Lambda)$ be a compactly dominant and integral adapted pair. Then the $\bar{\mathfrak{k}}$-submodule $\mathfrak{U}(\bar{\mathfrak{k}})\, 1_\Lambda$ of the simple highest weight module F_Λ is $\bar{\mathfrak{k}}$-equivalent to the simple highest weight module $F_{\bar{\Lambda}}$ of $\bar{\mathfrak{k}}$.

Proof. Clearly, $\mathfrak{U}(\bar{\mathfrak{k}})\, 1_\Lambda$ is a $\bar{\mathfrak{k}}$-highest weight module of weight $\bar{\Lambda}$. Moreover, the subalgebra $\mathfrak{n}_{\bar{\mathfrak{k}}}^- = \sum_{\alpha \in \bar{\Delta}_c^{++}}^{\oplus} \mathfrak{k}^{-\alpha}$ acts locally nilpotently, since $\dim F_\Lambda < \infty$. By [Dix77, lem. 7.2.4], $\mathfrak{U}(\bar{\mathfrak{k}})\, 1_\Lambda$ is simple, so the assertion follows. □

Proposition 4.3.4. Let $(\bar{\Lambda}, \Lambda)$ be an adapted pair of holomorphic discrete series parameters. Then there is a unique $\bar{G}$-equivariant isometry

$$\mathcal{O}^2(\bar{B}, F_{\bar{\Lambda}}) \to \mathcal{O}^2(B, F_\Lambda)$$

determined by $1_{\bar{\Lambda}} \mapsto 1_\Lambda$, where $1_{\bar{\Lambda}}$ and 1_Λ are normalised highest weight vectors.

Proof. Clearly, $\mathfrak{U}(\bar{\mathfrak{g}})\, 1_\Lambda \subset V_\Lambda^{\mathfrak{g}}$ is a $\bar{\mathfrak{g}}$-highest weight module, and since any $X \in \mathfrak{g}^{-\alpha} \setminus 0$, $\alpha \in \bar{\Delta}^{++}$, acts injectively, [Dix77, prop. 7.1.8] implies that it is $\bar{\mathfrak{g}}$-equivalent to $V_{\bar{\Lambda}}^{\bar{\mathfrak{g}}}$. The identities

$$N(\bar{\Lambda}) = \mathfrak{U}\left(\bar{\mathfrak{p}}^-\right) \otimes F_{\bar{\Lambda}} \quad \text{and} \quad N(\Lambda) = \mathfrak{U}\left(\mathfrak{p}^-\right) \otimes F_\Lambda$$

which are $\bar{\mathfrak{k}}$-, respectively $\mathfrak{k}$-equivariant, show, together with lemma 4.3.3, that the map $V_{\bar{\Lambda}}^{\bar{\mathfrak{g}}} \to V_\Lambda^{\mathfrak{g}}$ descends to a well-defined and $\bar{\mathfrak{g}}$-equivariant injection $N(\bar{\Lambda}) \to N(\Lambda)$.

In particular, there is an injection $j : \mathcal{P}(\bar{Z}, F_{\bar{\Lambda}}) \to \mathcal{P}(Z, F_\Lambda)$ which is $\bar{\mathfrak{g}}$-equivariant, and, by connectedness, also $\bar{G}$-equivariant. Since $\mathcal{P}(\bar{Z}, F_{\bar{\Lambda}})$ is algebraically irreducible, the requirement $1_{\bar{\Lambda}} \mapsto 1_\Lambda$ uniquely determines this map.

Denote by V the image of j. It consists of $\bar{K}$-finite vectors, and is hence the algebraic sum of its isotypic components, each of which has finite multiplicity. If V_μ is some isotypic component of type F_μ, then $j^{-1}(V_\mu) = U_\mu$ is the corresponding isotype of $U = \mathcal{P}(\bar{Z}, F_{\bar{\Lambda}})$. The adjoint $j^* : V_\mu \to U_\mu$ with respect to the inner products induced by $\mathcal{O}^2_\Lambda$ and $\mathcal{O}^2_{\bar{\Lambda}}$ is well-defined, and by summing over all μ, we get a $\bar{G}$-equivariant map $j^* : V \to U$. Then $j^* j$ is a $\bar{G}$-equivariant endomorphism of U, and hence a scalar constant α, by Schur's lemma. We compute

$$\alpha = (1_{\bar{\Lambda}} \mid j^* j\, 1_{\bar{\Lambda}})_{\mathcal{O}^2_{\bar{\Lambda}}} = (j\, 1_{\bar{\Lambda}} \mid j\, 1_{\bar{\Lambda}})_{\mathcal{O}^2_{\Lambda}} = \|1_\Lambda\|^2_{\mathcal{O}^2_{\Lambda}} = 1\,,$$

and this proves the proposition. □

Remark 4.3.5. In principle, the embedding $\mathcal{O}^2(\bar{B}, F_{\bar{\Lambda}}) \to \mathcal{O}^2(B, F_\Lambda)$ can be computed, at least on the level of $\bar{K}$-finite vectors, by applying lemma 4.2.15. Here, it however quickly becomes evident that the extension of a polynomial $p \in \mathcal{P}(\bar{Z}, F_{\bar{\Lambda}})$ to Z, with values in F_Λ, is not constant in all directions orthogonal to $\bar{Z}$, viz, its behaviour in the direction of $Z_{1/2}(e)$ could be quite complicated. So the embedding, which is somewhat innocuous on the level of the highest weight modules $N(\bar{\Lambda})$ and $N(\Lambda)$, is not what one might naively expect in the natural analytic realisation.

4.4 Asymptotic behaviour of matrix coefficients

4.4.1. For $g \in G$, there is, by [Hel78, ch. IX, § 1, th. 1.1], a decomposition $g = kal$ where $k, l \in K$ and $a \in A$, such that

$$\log a = \sum_{j=1}^{r} t_j \cdot \xi_{e_j}^- \quad \text{for some} \quad t_1, \ldots, t_r \geqslant 0 \,.$$

In this decomposition, a is unique. Define a open cover of the set $G \setminus (K \cdot \bar{G} \cdot K)$, indexed by $\varepsilon > 0$,

$$U_\varepsilon(\bar{G}) = \left\{ g \;\middle|\; g^{-1} = kal \,,\, k, l \in K \,,\, \log a = \sum_{j=1}^{r} t_j \cdot \xi_{e_j}^- \,,\, t_1, \ldots, t_k \geqslant 0 \,,\, t_{k+1}, \ldots, t_r \geqslant \varepsilon \right\} .$$

Proposition 4.4.2. Consider adapted pairs $(\bar{\Lambda}, \Lambda)$ of holomorphic discrete series parameters. For any constant $v \in F_\Lambda$, define the real analytic functions Δ_v^Λ by

$$\Delta_v^\Lambda(g) = (v \mid g^{\pi_\Lambda} v)_{\mathcal{O}^2_\Lambda} \quad \text{for all} \;\; g \in G \,.$$

Then

$$\lim\nolimits_{\Lambda_{k+1}, \ldots, \Lambda_r \to -\infty} \Delta_v^\Lambda = 0 \quad \text{on} \quad G \setminus (K \cdot \bar{G} \cdot K) \,,$$

uniformly with all derivatives on each of the subsets $U_\varepsilon(\bar{G})$ with $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ and take $g \in U_\varepsilon(\bar{G})$, and set $w = g(0)$. For any $p \in \mathcal{P}(Z, F_\Lambda)$,

$$\begin{aligned}
\left|(v \mid g^{-\pi_\Lambda} p)\right|^2 &= |(v \mid g'(0)^{-\Lambda} p(w))_\Lambda|^2 \leqslant \|v\|_\Lambda^2 \cdot \|g'(0)^{-\Lambda} p(w)\|_\Lambda^2 \\
&= \|v\|_{\bar{\Lambda}} \cdot (p(w) \mid B(w,w)^{-\Lambda} p(w))_\Lambda \leqslant \|v\|_{\bar{\Lambda}}^2 \cdot \|p(w)\|_\Lambda^2 \cdot \operatorname{tr} B(w,w)^{-\Lambda} \\
&= \|v\|_{\bar{\Lambda}}^2 \cdot \|p(w)\|_\Lambda^2 \cdot \sum_{F_\Lambda[\mu] \neq 0} \dim F_\Lambda[\mu] \cdot \prod_{j=1}^{r} \big(\cosh t_j\big)^{4\mu_j} \,,
\end{aligned}$$

by proposition 4.2.11. We have $\sum_{F_\Lambda[\mu] \neq 0} \dim F_\Lambda[\mu] = \dim F_\Lambda = d_\Lambda$. Moreover, $\mu_j \leqslant \Lambda_j$, so the above sum is majorised by $d_\Lambda \cdot \prod_{j=1}^r \big(\cosh t_j\big)^{4\Lambda_j}$. The number d_Λ is a polynomial in Λ by Weyl's dimension formula [Kna02, th. 5.84]. So, since $\cosh t_j \geqslant \cosh \varepsilon > 1$, this quantity falls exponentially for $\Lambda_{k+1}, \ldots, \Lambda_r \to -\infty$, uniformly in $g \in U_\varepsilon(\bar{G})$.

It remains to study the polynomial p. We need only consider the case $p = \pi_\lambda(u)\, v$ where $u \in \mathfrak{U}(\mathfrak{g})$ is fixed, independently of Λ. Indeed,

$$(\xi\, \Delta_v^\Lambda)(g) = \frac{\mathrm{d}}{\mathrm{d}t}(v \mid \pi_\lambda(g \exp(-t\xi))\, v)\big|_{t=0} = -\,(v \mid g^{\pi_\lambda}\pi_\lambda(\xi)v) \quad \text{for all } \xi \in \mathfrak{g}\,.$$

Lemma 4.2.15 shows that $\pi_\lambda(u)\, v \in \mathcal{P}(Z, F_\Lambda)$ for any $u \in \mathfrak{U}(\mathfrak{g})$.

Since $\mathfrak{k} \oplus \mathfrak{p}^+$ leaves the constants $F_{\bar\Lambda}$ invariant, PBW [Dix77, th. 2.1.11] allows us to restrict attention to $u \in \mathfrak{U}(\mathfrak{p}^-)$. We contend the following: For all holomorphic discrete series parameters Λ, all $m \in \mathbb{N}$, and all $a_1, \ldots, a_m \in Z$,

$$\pi_\lambda\Big(\{za_1^* z\}\frac{\partial}{\partial z} \cdots \{za_m^* z\}\frac{\partial}{\partial z}\Big) u = \textstyle\sum_j a_{mj} \cdot \displaystyle\prod_{i=1}^{b_{mj}} p_{mji}(a_1, \ldots, a_m)(z)^\Lambda u \quad \text{for all } u \in F_\Lambda\,,$$

where $a_{mj} \in \mathbb{Z}$, $b_{mj} \in \mathbb{N}$, and $p_{mji}(a_1, \ldots, a_m)$ are homogeneous polynomials of degree n_{mji}, with values in $\mathfrak{k}$, such that

$$\sum_{i=1}^{b_{mj}} n_{mji} = m \quad \text{and} \quad \prod_{i=1}^{b_{mj}} p_{mji}(a_1, \ldots, a_m)(z)^\Lambda$$

depends m-conjugate linearly on $a_1, \ldots, a_m$. Moreover, a_{mj}, b_{mj} and p_{mji} are independent of the parameter Λ.

The proof hereof is a straightforward induction on m. Indeed, for $q \in \mathcal{P}^{m-1}(Z, \mathfrak{k})$,

$$\pi_\lambda\Big(\{za^*z\}\frac{\partial}{\partial z}\Big)(q^\Lambda v)(z) = 2 \cdot (z \,\square\, a^*)^\Lambda q(z)^\Lambda - \big(q'(z)\{za^*z\}\big)^\Lambda\,,$$

as follows from lemma 4.2.15. Now, $q'(z)\{za^*z\}$ is an m-homogeneous polynomial with values in $\mathfrak{k}$, so the contention follows.

Now we apply this fact to our estimate. Namely, let $p(z) = \big(\partial^*_{a_1 \cdots a_m}\big)^{\pi_\lambda} v$. Then, by lemma 4.4.3 below,

$$\|p(w)\|_\Lambda \leqslant \big(|\Lambda + \varrho_c|^2 - |\varrho_c|^2 + 1\big)^{m/2} \cdot \textstyle\sum_j |a_{mj}| \cdot \displaystyle\prod_{i=1}^{b_{mj}} c_{mji}$$

where the constants c_{mji} depend only on $p_{mji}(a_1, \ldots, a_m)$. Since the sum on the right hand side is constant, the left hand side is polynomially bounded in Λ, and we have proved the proposition. □

The following lemma was used in the proof of proposition 4.4.2.

Lemma 4.4.3. Let $\delta \in \mathfrak{k}$ be fixed. Then, for the operator norm $\|\delta^\Lambda\|$,

$$\|\delta^\Lambda\| \leqslant c \cdot \sqrt{|\Lambda + \varrho_c|^2 - |\varrho_c|^2 + 1}\,.$$

where the constant c is independent of the Δ_c^{++}-dominant and analytically integral parameter $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$.

Proof. The Casimir Ω_c is an elliptic invariant differential operator of degree 2. Hence $A = \delta(1+\Omega)^{-1/2} \in \mathrm{I}\Psi_h^0(K)$, and this operator is properly supported because K is compact. Hence, it is bounded on $\mathbf{L}^2(K)$, by [Str72, th. 6]. Writing $d_\Lambda = \dim F_\Lambda$, proposition 4.1.11 and the orthogonality relations on K [Dix69, th. 14.3.3, prop. 15.2.3] imply

$$\big(|\Lambda+\varrho_c|^2 - |\varrho_c|^2 + 1\big)^{-1} \cdot \|\delta^\Lambda v\|_\Lambda^2 = \|A^\Lambda v\|_\Lambda^2$$
$$= d_\Lambda \cdot \|A\Delta_v^\Lambda\|_2^2 \leqslant d_\Lambda \cdot \|A\|^2 \cdot \|\Delta_v^\Lambda\|_2^2 = \|A\|^2$$

for all $v \in F_\Lambda$, $\|v\|_\Lambda = 1$. This proves the lemma. □

4.4.4. A fundamental theorem on the Hardy space $\mathbf{H}^2(\Gamma)$ associated to the Lie group $G = \mathrm{Aut}_0\, B$ states that it is the largest invariant subspace of $\mathbf{L}^2(G)$ such that for all $\xi \in \Omega^-$, the action of $-i\xi$ on $\mathbf{H}^2(\Gamma)$ is given by a positive semi-definite self-adjoint (unbounded) operator, cf. [HÓØ91, th. 3.4].

This can interpreted in the following manner: Let π be any irreducible unitary representation of G, weakly contained in $\mathbf{L}^2(G)$. Then we may define the *moment map* μ

$$\mu : \langle G\rangle_\pi \setminus 0 \to \mathfrak{g}_{\mathbb{R}}^* \quad \text{where} \quad \langle \xi : \mu(\varphi)\rangle = -\frac{i}{\|\varphi\|^2} \cdot (\varphi \mid \pi(\xi)\,\varphi)$$

where $\langle G\rangle_\pi$ denotes the representation space of π. Then the above theorem states that if π is not contained in the holomorphic discrete series, then the image of the moment map contains an orbit which is disjoint from the dual $(\Omega^-)^*$ of Ω^-.

In other words, the 'orbit picture' given by the moment map separates the holomorphic discrete series from the remainder of the reduced dual.

Such considerations entail, with the micro-local information on the Szegö distribution, the following statement on the behaviour of E on sequences of matrix coefficients.

Proposition 4.4.5. Let π_k be a sequence of irreducible unitary representations of G, and $\varphi_j \in \langle G\rangle_{\pi_j}$ be smooth unit vectors, of weight $\mu_j \in i\mathfrak{t}_{\mathbb{R}}^*$. If π_j tends to infinity in the reduced dual of G, and $-i\mu_j$ is eventually contained in the complement of a neighbourhood of the dual $(\omega^-)^*$ of ω^-, then

$$\Delta_k \cdot E \to 0 \quad \text{weakly, where} \quad \Delta_k(g) = \big(\varphi_j \mid \pi_j(g)\,\varphi_j\big) \quad \text{for all } g \in G\,.$$

Here, $\pi_j \to \infty$ means that it is eventually contained in the complement of every quasi-compact subset of the reduced dual of G, in the Jacobson topology.

Proof. By a parameter shift, we may assume that $-i\mu_j$ is contained in $\mathfrak{t}_{\mathbb{R}}^* \setminus U$ for all $j \in \mathbb{N}$, for some neighbourhood U of $(\omega^-)^*$. Then there exists a smooth function $f : \mathfrak{t}_{\mathbb{R}}^* \to \mathbb{R}$, homogeneous of degree 0 as $|\xi| \to \infty$, such that $f(-i\mu_j) = 1$ for all $j \in \mathbb{N}$, and $f = 0$ on $(\omega^-)^*$.

If we chose an orthonormal basis ξ_j, $j = 1, \ldots, R$, of $\mathfrak{t}_{\mathbb{R}}$, then the Lie symbol of the associated order zero operators A_j is ξ_j^*, cf. corollary 3.3.11. Similarly as in the proof of this corollary, the symbol of $f(A) = f(A_1, \ldots, A_R)$ is $f \circ (\xi_1^*, \ldots, \xi_R^*) = f \circ p_{\mathfrak{t}^*}$ where $p_{\mathfrak{t}^*}$ is the orthogonal projection $\mathfrak{g}_{\mathbb{R}}^* \to \mathfrak{t}_{\mathbb{R}}^*$. Its characteristic set is hence contained in $p_{\mathfrak{t}^*}(\mathfrak{t}_{\mathbb{R}}^* \setminus U)$.

By [Pan83, th. 2 (12)], this set is disjoint from $(\Omega^-)^*$, and hence non-characteristic for E, by theorem 3.3.2. It follows that $f(A)(\alpha \cdot E)$ is smooth for any $\alpha \in \mathcal{D}(G)$. On the other hand, since φ_j is a simultaneous eigenvector of A_j, we find

$$f(A)\Delta_k = f\big(-i\mu_j \cdot \pi_j(\Omega^{-1/2})\big) \cdot \Delta_k = f(-i\mu_j) \cdot \Delta_k = \Delta_k$$

where $\Omega = \sum_{j=1}^n \xi_j^2$ is the Casimir operator of $\mathfrak{g}_{\mathbb{R}}$ ($\xi_1, \ldots, \xi_n$ an extension to a basis of $\mathfrak{g}_{\mathbb{R}}$), and the homogeneity of f was used.

Consequently, we find that

$$\langle \alpha : \Delta_k \cdot E \rangle = \langle f(A)\Delta_k : \alpha \cdot E \rangle = \langle \Delta_k : f(A)(\alpha \cdot E) \rangle \to 0 \quad \text{for all } \alpha \in \mathcal{D}(G)\,,$$

by dominated convergence, because $f(A)(\alpha \cdot E)$ is a smooth function of compact support, and $\Delta_k \to 0$ point-wise a.e., by the Riemann-Lebesgue lemma [Dix69, prop. 3.3.8, 18.2.4], because π_j tends to infinity. □

Remark 4.4.6. The device used in this proof is similar to the first step in the proof of [Upm96, lem. 3.6.33]. Here, Upmeier used a theorem of Guillemin stating that (in his setting) the Szegö projection is a Hermite operator (a special class of Fourier integral operator) whose distribution kernel has a certain prescribed singular set. So, whereas Upmeier's use of the symbol calculus was local, our use is micro-local.

However, in the applications we have in mind, the above result is not as conclusive as Upmeier's is in his setting. Indeed, the projection onto the torus $\mathfrak{t}_{\mathbb{R}}$ does not separate adjoint orbits. This can already be seen for the case of the unit disc, where $\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(1,1)$. Here, the complement of the double light cone consists of adjoint orbits (time-like hyperboloids) which all have non-trivial projection onto the cone $\omega^+ = (\omega^-)^*$.

Hence, if π_j is a sequence of representation not belonging to the holomorphic discrete series, the above proposition can only be applied to vectors φ_j in the representation π_j whose image under the moment map lies in a part of these orbits which does not project onto ω^+. Since such vector can always be moved above ω^+ by the action of some inner automorphism, we will not be able to apply the proposition to a spanning subset of $\langle G \rangle_{\pi_j}$, which is a severe restriction.

5 Embedding of representations of facial subgroups

In order to study the asymptotic behaviour of the Szegö distribution on sequences of matrix coefficients associated to representations of G belonging to the support of the Plancherel measure, but not to the holomorphic discrete series, it is necessary to embed such representations of the facial subgroups $\bar{G} = G_e$ into those of G. I.e., we have to prove an appropriate generalisation of proposition 4.3.4 for the other parts of the reduced spectrum.

However, what is nearly trivial for the holomorphic discrete series, is comparatively difficult for the discrete series, and even more so, for the parabolic Q-series which are induced from discrete series. The main complication is that the positive systems corresponding to non-holomorphic discrete series are not related to the complex structure of the Jordan triple system Z, as is the adapted positive system Δ^{++}. Hence, a convenient Jordan theoretic description is not available.

We have only been able to overcome these difficulties by case-by-case arguments, using the classification of Hermitian symmetric spaces. Moreover, we omit the exceptional cases. In would of course be desirable to have a uniform argument, independent of classification.

5.1 An embeddability theorem for fundamental sequences

5.1.1. Let $G = \mathrm{Aut}_0 B$ where $B \subset Z$ is an irreducible circled bounded symmetric domain. Two roots $\alpha, \beta \in \Delta$ are said to be *strongly orthogonal* if

$$\alpha \notin \mathbb{Q}\beta \quad \text{and} \quad \alpha \pm \beta \notin \Delta\,.$$

Given a positive system $\Delta^+ \subset \Delta$, a *fundamental sequence* of positive non-compact roots $\alpha_1, \ldots, \alpha_r$ is defined by the following properties

(FS1). The $\alpha_i \in \Delta_n^+$ are strongly orthogonal and r is maximal with this property.

(FS2). Every α_i is simple in the set of roots strongly orthogonal to $\alpha_1, \ldots, \alpha_{i-1}$. I.e. if $\alpha, \beta \in \Delta^+$ are strongly orthogonal to $\alpha_1, \ldots, \alpha_{i-1}$, then $\alpha + \beta \neq \alpha_i$.

(FS3). Whenever $\alpha \in \Delta_n$ is strongly orthogonal to $\alpha_1, \ldots, \alpha_{i-1}$ but not strongly orthogonal to α_i, we have $|\alpha_i| \geqslant |\alpha|$.

It is important to insist that the length r, and not only the sequence $\alpha_1, \ldots, \alpha_r$ be maximal. Indeed, $\varepsilon_1 + \varepsilon_2$ is a maximal strongly orthogonal sequence of positive non-compact roots for $\mathfrak{sp}(4, \mathbb{R})$, but its length is not maximal, since $2\varepsilon_2, 2\varepsilon_1$ are also strongly orthogonal. The number r is independent of Δ^+ and coincides with the *real rank* of $\mathfrak{g}_{\mathbb{R}}$. Moreover, r is also the rank of B, and of Z.

Knapp-Wallach [KW76, prop. 4.5] show that fundamental sequences always exist, even for non-Hermitian groups, except in the case that the Lie algebra contains a split G_2 factor. In this case, (FS3) has to be replaced with another condition. We have avoided this difficulty, since in the Hermitian symmetric case, only the root systems A–E occur.

We have already seen in 2.1.12 that Harish-Chandra's roots $\gamma_1, \ldots, \gamma_r$ form a fundamental sequence for the positive system Δ^{++}. Moreover, if we consider, for the tripotent $e = e_1 + \cdots + e_k$, the subgroup $\bar{G} = G_e$ and the associated substructures as in 4.3.1, it is a trivial matter that the corresponding Harish-Chandra sequence is simply the terminal segment $\gamma_{k+1}, \ldots, \gamma_r$ where $\bar{r} = r - k$ is the rank of $\bar{Z}$.

We wish to generalise this statement to arbitrary positive systems. More precisely, given a positive system $\Delta^+ \subset \Delta$ and a fundamental sequence $\alpha_1, \ldots, \alpha_r$, we define

$$m_j = \#\{\alpha \in \Delta_n^+ \mid \alpha \text{ strongly orthogonal to } \alpha_1, \ldots, \alpha_{j-1} \text{ , and } \alpha - \alpha_j \in \Delta\} \, .$$

The sequence $m = (m_j)_{j=1,\ldots,r}$ is called *lower signature* of $\alpha_1, \ldots, \alpha_r$.

Theorem 5.1.2. Let Z be classical and $\bar{G} \sqsubset G$ be a facial subgroup. Then, for any positive system $\bar{\Delta}^+ \subset \bar{\Delta}$, there exists a positive system $\Delta^+ \subset \Delta$ and a fundamental sequence $\alpha_1, \ldots, \alpha_r \in \Delta_n^+$ such that

(i). Δ^+ extends $\bar{\Delta}^+$, i.e. $\bar{\Delta}^+ = \bar{\Delta} \cap \Delta^+$,

(ii). for $\bar{\alpha}_j = \alpha_{r-\bar{r}+j}$, $1 \leqslant j \leqslant \bar{r}$, the sequence

$$\bar{\alpha}_1, \ldots, \bar{\alpha}_{\bar{r}} \in \bar{\Delta}_n^+$$

is fundamental for $\bar{\Delta}^+$, and

(iii). for the lower signature $\bar{m} = (\bar{m}_j)$ of $\bar{\alpha}_1, \ldots, \bar{\alpha}_{\bar{r}}$, we have

$$\bar{m}_j = m_{r-\bar{r}+j} \quad \text{for all } \ 1 \leqslant j \leqslant \bar{r} \, .$$

Remark 5.1.3. Since the theorem is a purely infinitesimal statement, it is also valid for any finite cover of G.

Proof of theorem 5.1.2. Since Z is simple, all subtriples $\bar{Z} = Z_0(e)$ with e non-zero tripotent of fixed rank $r - \bar{r}$ are conjugate, it is also sufficient to consider one tripotent for each rank $\bar{r} < r$. Finally, since the conditions (i)–(iii) are manifestly invariant under $\bar{W}_c$-conjugacy, we need only consider a set of representatives of the $\bar{W}_c$-conjugacy classes of positive systems in $\bar{\Delta}$.

So, the proof reduces to a case-by-case study, using the classification of irreducible bounded symmetric spaces of non-compact type. Hence propositions 5.1.11, 5.1.17, 5.1.23, 5.1.29 and 5.1.35 below complete the proof. □

Definition 5.1.4. Let $\bar{G} \sqsubset G$ be a closed connected subgroup. Whenever $\bar{G}$ fulfils the conclusion of theorem 5.1.2 for any positive system, we shall say that it is an *embeddable subgroup*. The etymology being that the discrete series of an embeddable $\bar{G}$ can be embedded into that of G, see corollary 5.2.25 below. We have seen that $\bar{G}$ is embeddable if it is a facial subgroup, and if Z is classical.

In the following case-by-case considerations, we refer to the table in 1.1.7 for a summary of the classification. Moreover, refer to [Bou68, ch. IV-VI, planches I-IV] and [Tit67], or to [Kna02, app. C], for tables of root systems and Weyl groups.

5.1.1 Proof for type $\mathrm{I}_{p,q}$

5.1.5. As is customary, we identify $i\mathfrak{t}_{\mathbb{R}}^*$ with $\mathbb{R}^R$, $R = \operatorname{rk} \mathfrak{g}_{\mathbb{R}}$ (not to be confused with the real rank r). Denote the standard orthonormal basis by $\varepsilon_1, \dots, \varepsilon_R$.

If $G = \mathrm{Aut}_0 B$ where B is of type $\mathrm{I}_{p,q}$, $R = p + q$. Furthermore,

$$\Delta = \left\{ \pm(\varepsilon_i - \varepsilon_j) \,\middle|\, 1 \leqslant i < j \leqslant p + q \right\}$$

and

$$\Delta_c = \left\{ \pm(\varepsilon_i - \varepsilon_j) \,\middle|\, 1 \leqslant i < j \leqslant p \text{ or } p < i < j \leqslant p + q \right\} .$$

Moreover, the respective Weyl groups W and W_c are

$$W = \mathfrak{S}_{p+q} \quad \text{and} \quad W_c = \mathfrak{S}_p \times \mathfrak{S}_q ,$$

acting by permutation of labels. The adapted positive system Δ^{++} is given by

$$\Delta^{++} = \left\{ \varepsilon_i - \varepsilon_j \,\middle|\, 1 \leqslant i < j \leqslant p + q \right\} .$$

For an appropriate ordering of the simple roots, one calculates

$$\gamma_j = \varepsilon_{p-j+1} - \varepsilon_{p+j} \quad \text{for all } 1 \leqslant j \leqslant r = p$$

for the Harish-Chandra sequence.

5.1.6. Since $\#(W/W_c) = \binom{p+q}{p}$, a system of representatives of W_c-conjugacy classes of positive systems is indexed by subsets

$$A = \{a_1 < \cdots < a_p\} \subset \{1, \dots, p + q\} .$$

Write

$$\{1, \dots, p + q\} \setminus A = \{b_1 < \cdots < b_q\} .$$

Then the permutations

$$\sigma_A = \begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & p+q \\ a_1 & \cdots & a_p & b_1 & \cdots & b_q \end{pmatrix} \in \mathfrak{S}_{p+q} = W$$

are clearly pairwise non-W_c-conjugate. Hence the sets $\Delta^{+,A} = \sigma_A.\Delta^{++}$ constitute a set of representatives of positive systems.

Lemma 5.1.7. Let $A \subset \{1,\ldots,p+q\}$ and retain the above notation. Set

$$p_A = \#\bigl(A \cap \{1,\ldots,p\}\bigr) .$$

Then the subsets of $\Delta^{+,A}$ of compact resp. non-compact roots are

$$\begin{aligned} \Delta_c^{+,A} =& \Bigl\{ \varepsilon_{a_i} - \varepsilon_{a_j} \Bigm| 1 \leqslant i < j \leqslant p_A \text{ or } p_A < i < j \leqslant p \Bigr\} \\ &\cup \Bigl\{ \varepsilon_{b_i} - \varepsilon_{b_j} \Bigm| 1 \leqslant i < j \leqslant p - p_A \text{ or } p - p_A < i < j \leqslant q \Bigr\} \\ &\cup \Bigl\{ \varepsilon_{a_i} - \varepsilon_{b_j} \Bigm| 1 \leqslant i \leqslant p_A\,,\, 1 \leqslant j \leqslant p - p_A \text{ or } p_A < i \leqslant p\,,\, p - p_A < j \leqslant q \Bigr\} \end{aligned}$$

and

$$\begin{aligned} \Delta_n^{+,A} =& \Bigl\{ \varepsilon_{a_i} - \varepsilon_{a_j} \Bigm| 1 \leqslant i \leqslant p_A < j \leqslant p \Bigr\} \\ &\cup \Bigl\{ \varepsilon_{b_i} - \varepsilon_{b_j} \Bigm| 1 \leqslant i \leqslant p - p_A < j \leqslant q \Bigr\} \\ &\cup \Bigl\{ \varepsilon_{a_i} - \varepsilon_{b_j} \Bigm| 1 \leqslant i \leqslant p_A\,,\, p - p_A < j \leqslant q \text{ or } p_A < i \leqslant p\,,\, 1 \leqslant j \leqslant p - p_A \Bigr\} . \end{aligned}$$

Proof. From the definition of σ_A, it is clear that $\alpha \in \Delta^{+,A}$ if and only if α is of the form

$$\alpha = \begin{cases} \varepsilon_{a_i} - \varepsilon_{a_j} & 1 \leqslant i < j \leqslant p\,, \\ \varepsilon_{b_i} - \varepsilon_{b_j} & 1 \leqslant i < j \leqslant q\,, \\ \varepsilon_{a_i} - \varepsilon_{b_j} & 1 \leqslant i \leqslant p\,, 1 \leqslant j \leqslant q\,. \end{cases}$$

The compact roots among these are precisely those where either both indices are left of p or both on the right hand side. Since $a_j \leqslant p$ if and only if $j \leqslant p_A$ and $b_j \leqslant p$ if and only if $j \leqslant p - p_A$, the assertion follows immediately. ☐

Proposition 5.1.8. A fundamental sequence for $\Delta^{+,A}$ is given by

$$\gamma_j^A = \varepsilon_{a_{p-j+1}} - \varepsilon_{b_j} \quad \text{for all } 1 \leqslant j \leqslant p\,.$$

Proof. The sequence manifestly consists of strongly orthogonal roots. Moreover, in case $1 \leqslant j \leqslant p - p_A$, then $b_j \leqslant p$, and $p - j + 1 > p_A$, so $p < a_j$. Hence, $\gamma_j^A \in \Delta_n^{+,A}$ in this case. If $p - p_A < j \leqslant p$, then $b_j > p$ and $p - j + 1 \leqslant p_A$, so $a_j \leqslant p$. Hence, $\gamma_j^A \in \Delta_n^{+,A}$ in this case, too. This verifies (FS1).

If $\alpha, \beta \in \Delta^{+,A}$ such that $\alpha + \beta = \gamma_j^A$, then either

$$\alpha = \varepsilon_{a_{p-j+1}} - \varepsilon_{a_k} \quad \text{and} \quad \beta = \varepsilon_{a_k} - \varepsilon_{b_j} ,$$

or

$$\alpha = \varepsilon_{a_{p-j+1}} - \varepsilon_{b_k} \quad \text{and} \quad \beta = \varepsilon_{b_k} - \varepsilon_{b_j} .$$

In the first case, $p - j + 1 < k \leqslant p$, so $j > p - k + 1$ and $\alpha \not\perp \gamma_{p-k+1}^A$, in particular, not strongly orthogonal. In the second case, $1 \leqslant k < j$, so $\beta \not\perp \gamma_k^A$, in particular, not strongly orthogonal. This proves (FS2).

Finally, condition (FS3) is trivial in this case since A_{p+q} is simply laced, and therefore only one root length occurs. □

Proposition 5.1.9. For lower signature $m^A = (m_j^A)$ of $\gamma_1^A, \ldots, \gamma_p^A$ is

$$m_j^A = \begin{cases} 2\big(p - p_A - j\big) & j \leqslant p - p_A , \\ p + q - 2j & j > p - p_A . \end{cases}$$

Proof. Let $\alpha \in \Delta_n^{+,A}$. We need to determine when $\alpha - \gamma_k^A \in \Delta$. As a consequence of (FS2), this the case if and only if $\alpha - \gamma_k^{+,A} \in \Delta_c^{+,A}$. It is clear that $\alpha - \gamma_k^{+,A}$ cannot be of the form $\varepsilon_{a_i} - \varepsilon_{b_j}$. Hence, α must be of this form.

This entails that either

$$j = k \quad \text{and} \quad \alpha - \gamma_k^A = \varepsilon_{a_i} - \varepsilon_{a_{p-k+1}} \in \Delta_c^{+,A} ,$$

or

$$i = p - k + 1 \quad \text{and} \quad \alpha - \gamma_k^A = \varepsilon_{b_k} - \varepsilon_{b_i} \in \Delta_c^{+,A} .$$

If $k \leqslant p - p_A$, then $p - k + 1 > p_A$, so the first case occurs exactly for $p_A < i \leqslant p - k$, and the second, if and only if $k < j \leqslant p - p_A$. If $k > p - p_A$, then since $p - k + 1 \leqslant p_A$, the first case occurs if and only if $1 \leqslant i \leqslant p - k$, and the second if and only if $k < j \leqslant q$. In summary,

$$\alpha = \begin{cases} \left.\begin{matrix} \varepsilon_{a_i} - \varepsilon_{b_k} & p_A < i \leqslant p - k \\ \varepsilon_{a_{p-k+1}} - \varepsilon_{b_j} & k < j \leqslant p - p_A \end{matrix}\right\} & k \leqslant p - p_A \\ \\ \left.\begin{matrix} \varepsilon_{a_i} - \varepsilon_{b_k} & 1 \leqslant i \leqslant p - k \\ \varepsilon_{a_{p-k+1}} - \varepsilon_{b_j} & k < j \leqslant q \end{matrix}\right\} & k > p - p_A \end{cases}$$

Since, in all cases, α is strongly orthogonal to $\gamma_1^A, \ldots, \gamma_{k-1}^A$, the value of m_k^A is easily determined. □

5.1.10. For the subtriples $\bar{Z} = Z_0(e) \sqsubset Z$, it suffices to consider the tripotents

$$e^k = \begin{array}{c c} & \begin{array}{cc} k & q-k \end{array} \\ \begin{array}{c} k \\ \\ p-k \end{array} & \begin{pmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} \end{array} \quad \text{for all } 1 \leqslant k \leqslant p \, .$$

Hence, we find $\bar{Z} \cong \mathbb{C}^{\bar{p} \times \bar{q}}$, where

$$1 \leqslant \bar{p} < p \, , \, 1 \leqslant \bar{q} < q \, , \, q - p = \bar{q} - \bar{p} \, .$$

So,

$$\bar{\Delta} = \left\{ \pm(\varepsilon_i - \varepsilon_j) \;\middle|\; p - \bar{p} < i < j \leqslant p + \bar{q} \right\} ,$$

and we need to consider

$$\bar{A} = \{\bar{a}_1 < \cdots < \bar{a}_{\bar{p}}\} \subset \{p - \bar{p} + 1, \ldots, p + \bar{q}\}$$

with

$$\{p - \bar{p} + 1, \ldots, p + \bar{q}\} \setminus \bar{A} = \{\bar{b}_1 < \cdots < \bar{b}_{\bar{q}}\} \, .$$

Set

$$a_j = \bar{a}_j \text{ for all } 1 \leqslant j \leqslant \bar{p} \text{ and } b_{q - \bar{q} + j} = \bar{b}_j \text{ for all } 1 \leqslant j \leqslant \bar{q} \, .$$

This allows us to define

$$A = \bar{A} \cup \{p + \bar{q} + 1, \ldots, p + q\} = \{a_1 < \cdots < a_p\}$$

where $a_{p + \bar{q} + j} = p + \bar{q} + j$ for $1 \leqslant j \leqslant q - \bar{q} = p - \bar{p}$, and

$$\{1, \ldots, p + q\} \setminus A = \{b_1 < \cdots < b_q\} \quad \text{where} \quad b_j = j \quad \text{for all } 1 \leqslant j \leqslant q - \bar{q} \, .$$

Proposition 5.1.11. We have $\bar{\Delta}^{+,\bar{A}} = \bar{\Delta} \cap \Delta^{+,A}$. The fundamental sequence

$$\bar{\gamma}_j^{\bar{A}} = \varepsilon_{\bar{a}_{\bar{p}-j+1}} - \varepsilon_{\bar{b}_j} \quad \text{for all } 1 \leqslant j \leqslant \bar{p}$$

for $\bar{\Delta}^{+,\bar{A}}$, satisfies

$$\bar{\gamma}_j^{\bar{A}} = \gamma_{p - \bar{p} + j}^{A} \quad \text{for all } 1 \leqslant j \leqslant \bar{p} \, ,$$

and for its lower signature $\bar{m}^{\bar{A}} = \left(\bar{m}_j^{\bar{A}}\right)$, we have

$$\bar{m}_j^{\bar{A}} = m_{p - \bar{p} + j}^{A} \quad \text{for all } 1 \leqslant j \leqslant \bar{p} \, .$$

Proof. Clearly,

$$\varepsilon_{\bar{a}_i} - \varepsilon_{\bar{b}_j} = \varepsilon_{a_i} - \varepsilon_{b_{p-\bar{p}+i}} \in \Delta^{+,A} ,$$

and similarly for $\varepsilon_{\bar{a}_i} - \varepsilon_{\bar{a}_j}$ and $\varepsilon_{\bar{b}_i} - \varepsilon_{\bar{b}_j}$, so $\bar{\Delta}^{+,\bar{A}} \subset \Delta^{+,A}$. The equality $\bar{\Delta}^{+,\bar{A}} = \bar{\Delta} \cap \Delta^{+,A}$ follows, since $-\Phi \cap \Phi = \varnothing$ for any positive system Φ.

Furthermore, since $p - \bar{p} = q - \bar{q}$,

$$\gamma^A_{p-\bar{p}+j} = \varepsilon_{a_{\bar{p}-j+1}} - \varepsilon_{b_{q-\bar{q}+j}} = \varepsilon_{\bar{a}_{\bar{p}-j+1}} - \varepsilon_{\bar{b}_j} = \bar{\gamma}^{\bar{A}}_j .$$

Finally, we have

$$\bar{p}_{\bar{A}} = \#\bigl(\bar{A} \cap \{p - \bar{p} + 1, \dots, p\}\bigr) = \#\bigl(A \cap \{1, \dots, p\}\bigr) = p_A ,$$

so $j \leqslant \bar{p} - \bar{p}_{\bar{A}}$ if and only if $p - \bar{p} + j \leqslant p - p_A$. Hence,

$$\begin{aligned} m^A_{p-\bar{p}+j} &= \begin{Bmatrix} 2\bigl(p - p_A - (p - \bar{p} + j)\bigr) & p - \bar{p} + j \leqslant p - p_A \\ p + q - 2(p - \bar{p} + j) & p - \bar{p} + j > p - p_A \end{Bmatrix} \\ &= \begin{Bmatrix} 2\bigl(\bar{p} - \bar{p}_{\bar{A}} - j\bigr) & j \leqslant \bar{p} - \bar{p}_{\bar{A}} \\ \bar{p} + \bar{q} - 2j & j > \bar{p} - \bar{p}_{\bar{A}} \end{Bmatrix} = \bar{m}^{\bar{A}}_j , \end{aligned}$$

where, once again, the equation $p - \bar{p} = q - \bar{q}$ was used. □

This completes the proof of theorem 5.1.2 in case B is of type $\mathrm{I}_{p,q}$. The other types are less demanding combinatorially.

5.1.2 Proof for type II_n

5.1.12. If $G = \mathrm{Aut}_0\, B$ where B is of type II_n, then $R = n$, and

$$\Delta = \Bigl\{\pm(\varepsilon_i \pm \varepsilon_j) \Bigm| 1 \leqslant i < j \leqslant n\Bigr\} \quad \text{and} \quad \Delta_c = \Bigl\{\pm(\varepsilon_i - \varepsilon_j) \Bigm| 1 \leqslant i < j \leqslant n\Bigr\} ,$$

where $n \geqslant 3$. (Note that $D_2 = A_1 + A_1$ is not simple.) The Weyl groups are

$$W = \mathbb{A}_n \ltimes \mathfrak{S}_n \quad \text{and} \quad W_c = \mathfrak{S}_n ,$$

where $\mathbb{A}_n = \bigl\{\varkappa \in \mathbb{Z}_2^n \bigm| \prod_i \varkappa_i = 1\bigr\}$. Here, our convention is $\mathbb{Z}_2 = \{\pm 1\}$. The adapted positive system is

$$\Delta^{++} = \Bigl\{\varepsilon_i \pm \varepsilon_j \Bigm| 1 \leqslant i < j \leqslant n\Bigr\} .$$

The Harish-Chandra sequence is then (for an appropriate ordering of the simple roots)

$$\gamma_j = \varepsilon_{n-2j+1} + \varepsilon_{n-2j+2} \quad \text{for all } 1 \leqslant j \leqslant r ,$$

where $r = \lfloor \frac{n}{2} \rfloor$. A system of representatives of W_c-conjugacy classes is indexed by elements $\varkappa \in \mathbb{A}_n = W/W_c$. Namely, we set $\Delta^{+,\varkappa} = \varkappa.\Delta^{++}$. These positive systems are manifestly pairwise non-W_c-conjugate.

Lemma 5.1.13. Let $\varkappa \in \mathbb{A}_n$. Then

$$\Delta_c^{+,\varkappa} = \Big\{ \varkappa_i(\varepsilon_i - \varepsilon_j) \,\Big|\, 1 \leqslant i < j \leqslant n \Big\} \quad \text{and} \quad \Delta_n^{+,\varkappa} = \Big\{ \varkappa_i(\varepsilon_i + \varepsilon_j) \,\Big|\, 1 \leqslant i < j \leqslant n \Big\} .$$

Proof. Clearly, $\alpha \in \Delta^{+,\varkappa}$ if and only if $\varkappa = \varkappa_i(\varepsilon_i \pm \varepsilon_j)$, for some $1 \leqslant i < j \leqslant n$. The compact roots among these are precisely those for which the sign in $\varepsilon_i \pm \varepsilon_j$ is negative. This proves the lemma. □

Proposition 5.1.14. Let $\varkappa \in \mathbb{A}_n$. A fundamental sequence for $\Delta^{+,\varkappa}$ is given by

$$\gamma_j^\varkappa = \varkappa_{n-2j+1} \cdot \gamma_j = \varkappa_{n-2j+1} \cdot (\varepsilon_{n-2j+1} + \varepsilon_{n-2j+2}) \quad \text{for all } 1 \leqslant j \leqslant r .$$

Proof. Clearly, $\gamma_k^\varkappa \in \Delta_n^{+,\varkappa}$. Moreover, these roots are strongly orthogonal since this is true of γ_k, proving (FS1). Since D_n is simply laced and hence only one root length occurs, the statement (FS3) is trivial.

As to (FS2), assume $\alpha, \beta \in \Delta^{+,\varkappa}$ such that $\alpha + \beta = \gamma_k^\varkappa$. We may assume α is non-compact and β is compact, moreover, that α, β are strongly orthogonal, in particular, orthogonal, to $\gamma_1^\varkappa, \dots, \gamma_{k-1}^\varkappa$.

This means that

$$\alpha = \varkappa_i(\varepsilon_i + \varepsilon_j) \quad \text{and} \quad \beta = \varkappa_a(\varepsilon_a - \varepsilon_b)$$

where $1 \leqslant i < j \leqslant n - 2k + 2$ and $1 \leqslant a < b \leqslant n - 2k + 2$. Hence, if $\varkappa_i = \varkappa_a$, then necessarily $i = b$, so

$$a = n - 2k + 1 \quad \text{and} \quad j = n - 2k + 2 .$$

But this implies $n - 2k + 2 = b = i < n - 2k + 2$, contradiction. If $\varkappa_i = -\varkappa_a$, necessarily $i \neq a$, so $a = j$. Thus

$$i = n - 2k + 1 \quad \text{and} \quad b = n - 2k + 2 ,$$

contradicting $i < j = a < b$. This proves (FS2) and hence, the lemma. □

Proposition 5.1.15. Let $\varkappa \in \mathbb{A}_n$. For the lower signature $m^\varkappa = (m_j^\varkappa)$ of the fundamental sequence $\gamma_1^\varkappa, \dots, \gamma_r^\varkappa$, we have

$$m_j^\varkappa = 2 \cdot \#\Big\{ 1 \leqslant i \leqslant n - 2j \,\Big|\, \varkappa_i = \varkappa_{n-2j+1} \Big\} \quad \text{for all } 1 \leq j \leq r .$$

Proof. Let $\alpha = \varkappa_i(\varepsilon_i + \varepsilon_j) \in \Delta_n^{+,\varkappa}$ be strongly orthogonal to $\gamma_1^\varkappa, \dots, \gamma_{k-1}^\varkappa$, which is equivalent to $1 \leqslant i < j \leqslant n - 2k + 2$. We wish to determine when $\alpha - \gamma_k^\varkappa \in \Delta$, which is the case if and only if $\alpha - \gamma_k^\varkappa \in \Delta_c^{+,\varkappa}$.

I.e., we are searching for solutions i , j of the equation

$$\varkappa_i(\varepsilon_i+\varepsilon_j)-\varkappa_{n-2k+1}(\varepsilon_{n-2k+1}+\varepsilon_{n-2k+2})=\varkappa_u(\varepsilon_u-\varepsilon_v)$$

where $1\leqslant i<j\leqslant n-2k+2$ and $1\leqslant u<v\leqslant n-2k+2$. A solution exists only if $\varkappa_i=\varkappa_{n-2k+1}$.

Since $i<j\leqslant n-2k+2$, we see that $i<n-2k+1$. Then a solution exists if and only if $j=n-2k+1,n-2k+2$, for arbitrary i . We may summarise this as

$$\alpha=\left\{\begin{matrix}\varkappa_i(\varepsilon_i+\varepsilon_{n-2k+1})\\ \varkappa_i(\varepsilon_i+\varepsilon_{n-2k+2})\end{matrix}\right\}\ 1\leqslant i\leqslant n-2k\,.$$

This entails the formula. □

5.1.16. A rank j tripotent, $1\leqslant 2j\leqslant r$, of $Z=\mathbb{C}_-^{n\times n}$ is given by

$$e^j=\begin{array}{cc} & \begin{array}{cc} n-2j & \qquad 2j \end{array} \\ \begin{array}{c} n-2j \\ \\ \\ 2j \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} & & \\ \vdots & & \ddots & \\ 0 & & & \begin{smallmatrix}0&-1\\1&0\end{smallmatrix} \end{pmatrix} \end{array}.$$

The corresponding subtriples $\bar{Z}\cong\mathbb{C}_-^{\bar{n}\times\bar{n}}$ for some $1\leqslant\bar{n}<n$, where we have $2\mid(n-\bar{n})$. Hence, the rank $\bar{r}=\lfloor\frac{\bar{n}}{2}\rfloor$ of $\bar{Z}$ satisfies $2\cdot(r-\bar{r})=n-\bar{n}$. Furthermore,

$$\bar{\Delta}=\left\{\pm(\varepsilon_i\pm\varepsilon_j)\,\middle|\,1\leqslant i<j\leqslant\bar{n}\right\},$$

unless $\bar{n}=2$. Indeed, defining $\bar{\Delta}$ as above in this case, $\bar{\Delta}=D_2=A_1+A_1$ is reducible. Moreover,

$$\bar{\Delta}_c=\{\pm(\varepsilon_1-\varepsilon_2)\}\quad\text{and}\quad\bar{\Delta}_n=\{\pm(\varepsilon_1+\varepsilon_2)\}$$

are precisely the two summands of type A_1 . This corresponds to the isomorphism $\mathfrak{so}^*(4)=\mathfrak{su}(2)\oplus\mathfrak{sl}(2,\mathbb{R})$. We have $\operatorname{aut}\bar{B}=\mathfrak{sl}(2,\mathbb{R})$, but the set of non-compact roots is the same for $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}^*(4)$, and the compact roots are strongly orthogonal to the non-compact ones. Hence, we may as well check the embeddability for $\mathfrak{so}^*(4)$.

So, we need to consider $\bar{\varkappa}\in\mathbb{A}_{\bar{n}}$. Define $\varkappa=(\bar{\varkappa},1,\dots,1)\in\mathbb{A}_n$.

Proposition 5.1.17. Let $\bar{\varkappa}\in\mathbb{A}_{\bar{n}}$. Then $\bar{\Delta}^{+,\bar{\varkappa}}=\bar{\Delta}\cap\Delta^{+,\varkappa}$. The fundamental sequence

$$\bar{\gamma}_j^{\bar{\varkappa}}=\bar{\varkappa}_{\bar{n}-2j+1}(\varepsilon_{\bar{n}-2j+1}+\varepsilon_{\bar{n}-2j+2})\quad\text{for all }\ 1\leqslant j\leqslant\bar{r}$$

satisfies

$$\bar{\gamma}_j^{\bar{\varkappa}} = \gamma_{r-\bar{r}+j}^{\varkappa} \quad \text{for all } 1 \leqslant j \leqslant \bar{r}\,.$$

For the lower signature $\bar{m}^{\bar{\varkappa}} = \left(\bar{m}_j^{\bar{\varkappa}}\right)$ of $\bar{\gamma}_1^{\bar{\varkappa}}, \ldots, \bar{\gamma}_{\bar{r}}^{\bar{\varkappa}}$, we have

$$\bar{m}_j^{\bar{\varkappa}} = m_{r-\bar{r}+j}^{\varkappa} \quad \text{for all } 1 \leqslant j \leqslant \bar{r}\,.$$

Proof. The first statement is trivial, and the second follows from

$$\begin{aligned}\gamma_{r-\bar{r}+j}^{\varkappa} &= \varkappa_{n-2(r-\bar{r}+j)+1}\left(\varepsilon_{n-2(r-\bar{r}+j)+1} + \varepsilon_{n-2(r-\bar{r}+j)+2}\right) \\ &= \bar{\varkappa}_{\bar{n}-j+1}\left(\varepsilon_{\bar{n}-2j+1} + \varepsilon_{\bar{n}-2j+2}\right) = \bar{\gamma}_j^{\bar{\varkappa}}\,,\end{aligned}$$

since $n - \bar{n} = 2(r - \bar{r})$. The last assertion now follows from the fact that $m_j^{\varkappa}$ only depends on $\varkappa_1, \ldots, \varkappa_{j-1}$. ☐

This completes the proof of theorem 5.1.2 in case B is of type II_n.

5.1.3 Proof for type III_n

5.1.18. For B of type III_n, we have $R = n$,

$$\Delta = \left\{\pm\left(\varepsilon_i \pm \varepsilon_j\right) \neq 0 \,\middle|\, 1 \leqslant i \leqslant j \leqslant n\right\} \quad \text{and} \quad \Delta_c = \left\{\pm\left(\varepsilon_i - \varepsilon_j\right) \,\middle|\, 1 \leqslant i < j \leqslant n\right\},$$

so $\pm 2\varepsilon_i$ are long, whereas $\pm\left(\varepsilon_i \pm \varepsilon_j\right)$, $1 \leqslant i < j \leqslant n$, are short. The Weyl groups are

$$W = \mathbb{Z}_2^n \ltimes \mathfrak{S}_n \quad \text{and} \quad W_c = \mathfrak{S}_n\,.$$

The adapted positive system Δ^{++} is

$$\Delta^{++} = \left\{\varepsilon_i \pm \varepsilon_j \neq 0 \,\middle|\, 1 \leqslant i \leqslant j \leqslant n\right\}.$$

For an appropriate ordering of the simple roots, Harish-Chandra's roots $\gamma_1, \ldots, \gamma_n$ are

$$\gamma_j = 2\varepsilon_{n-j+1} \quad \text{for all } 1 \leqslant j \leqslant n = r\,.$$

Since $W/W_c = \mathbb{Z}_2^n$, a system of representatives of positive systems is indexed by elements $\varkappa \in \mathbb{Z}_2^n$. Set $\Delta^{+,\varkappa} = \varkappa.\Delta^{++}$. Clearly, this is a system of representatives of W_c-conjugacy classes of positive systems.

Lemma 5.1.19. Let $\varkappa \in \mathbb{Z}_2^n$. Then

$$\Delta_c^{+,\varkappa} = \left\{\varkappa_i\left(\varepsilon_i - \varepsilon_j\right) \,\middle|\, 1 \leqslant i < j \leqslant n\right\} \quad \text{and} \quad \Delta_n^{+,\varkappa} = \left\{\varkappa_i\left(\varepsilon_i + \varepsilon_j\right) \,\middle|\, 1 \leqslant i \leqslant j \leqslant n\right\}.$$

Proof. The proof is essentially the same as in the case II_n. ☐

Proposition 5.1.20. Let $\varkappa \in \mathbb{Z}_2^n$. A fundamental sequence for $\Delta^{+,\varkappa}$ is given by

$$\gamma_j^{\varkappa} = 2\varkappa_{n-j+1}\varepsilon_{n-j+1} \quad \text{for all } 1 \leqslant j \leqslant n \,.$$

Proof. It is clear that for this particular root system, a root is strongly orthogonal to a given long root if and only if it is orthogonal. Now, $\gamma_j^{\varkappa} = \varkappa.\gamma_j$ for all $1 \leqslant j \leqslant n$, and are non-compact. Moreover, $\Delta^{+,\varkappa} = \varkappa.\Delta^{++}$, and $\varkappa$ leaves the orthogonal complement $\langle\gamma_1, \ldots, \gamma_j\rangle^{\perp}$ invariant for any j. Hence the assertions (FS1) and (FS2) for $\gamma_j^{\varkappa}$ follow from those for γ_j. (FS3) is clear, since the $\gamma_j^{\varkappa}$ are long. □

Proposition 5.1.21. Let $\varkappa \in \mathbb{Z}_2^n$. For the lower signature $m^{\varkappa} = \left(m_j^{\varkappa}\right)$ of the fundamental sequence $\gamma_1^{\varkappa}, \ldots, \gamma_n^{\varkappa}$, we have

$$m_j^{\varkappa} = \#\left\{1 \leqslant i \leqslant n-j \,\middle|\, \varkappa_i = \varkappa_{n-j+1}\right\} \quad \text{for all } 1 \leqslant j \leqslant n \,.$$

Proof. Let $\alpha \perp \gamma_1^{\varkappa}, \ldots, \gamma_{k-1}^{\varkappa}$, $\alpha \in \Delta_n^{+,\varkappa}$ such that $\alpha - \gamma_k^{\varkappa} \in \Delta_c^{+,\varkappa}$. This means

$$\alpha = \varkappa_i\left(\varepsilon_i + \varepsilon_j\right) \quad \text{for some } 1 \leqslant i < j \leqslant n-k+1 \,.$$

Hence

$$j = n - 2k + 1 \,,\; i \leqslant n-k \,,\; \varkappa_i = \varkappa_{n-k+1} \,,$$

i.e.

$$\alpha = \varkappa_i\left(\varepsilon_i + \varepsilon_{n-k+1}\right) \,,\; 1 \leqslant i \leqslant n-k \,,\; \varkappa_i = \varkappa_{n-k+1} \,,$$

proving the formula. □

5.1.22. A rank k tripotent, $1 \leqslant k \leqslant n = r$, is given by

$$e^k = \begin{array}{c c} & \begin{array}{cc} n-k & \quad k \end{array} \\ \begin{array}{c} n-k \\ \\ k \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{pmatrix} \end{array} .$$

Hence, the subtriples $\bar{Z} \cong \mathbb{C}_+^{\bar{n}\times\bar{n}}$, where $1 \leqslant \bar{n} < n$. We have

$$\bar{\Delta} = \left\{\pm\left(\varepsilon_i \pm \varepsilon_j\right) \neq 0 \,\middle|\, 1 \leqslant i \leqslant j \leqslant \bar{n}\right\} .$$

Consider $\bar{\varkappa} \in \mathbb{Z}_2^{\bar{n}}$, and set $\varkappa = \left(\bar{\varkappa}, 1, \ldots, 1\right) \in \mathbb{Z}_2^n$. The following is immediate.

Proposition 5.1.23. Let $\bar{\varkappa} \in \mathbb{Z}_2^{\bar{n}}$. Then $\bar{\Delta}^{+,\bar{\varkappa}} = \bar{\Delta} \cap \Delta^{+,\varkappa}$. The fundamental sequence

$$\bar{\gamma}_j^{\bar{\varkappa}} = 2\bar{\varkappa}_{\bar{n}-j+1}\varepsilon_{\bar{n}-j+1} \quad \text{for all } 1 \leqslant j \leqslant \bar{n} \,,$$

satisfies

$$\bar{\gamma}_j^{\bar{\varkappa}} = \gamma_{n-\bar{n}+j}^{\varkappa} \quad \text{for all } 1 \leqslant j \leqslant \bar{n} \, .$$

For its lower signature $\bar{m}^{\bar{\varkappa}} = \left(\bar{m}_j^{\bar{\varkappa}}\right)$, we have

$$\bar{m}_j^{\bar{\varkappa}} = m_{n-\bar{n}+j}^{\varkappa} \quad \text{for all } 1 \leqslant j \leqslant \bar{n} \, .$$

This completes the proof of theorem 5.1.2 in case B is of type III_n. In the remaining cases, B has rank 2. Therefore, for the proof of the theorem, they could be omitted, since all non-trivial facial subtriples have rank 1. We include them for the sake of completeness.

5.1.4 Proof for type IV_n, $q = 2k$

5.1.24. For B of type IV_n, $B = \mathrm{SO}(2,q)/(\mathrm{SO}(2) \times \mathrm{SO}(q))$, $q = 2k$,

$$\Delta = \left\{ \pm(\varepsilon_i \pm \varepsilon_j) \;\middle|\; 1 \leqslant i < j \leqslant k+1 \right\} \quad \text{and} \quad \Delta_c = \left\{ \pm(\varepsilon_i \pm \varepsilon_j) \;\middle|\; 2 \leqslant i < j \leqslant k+1 \right\} .$$

The respective Weyl groups are

$$W = \mathbb{A}_{k+1} \ltimes \mathfrak{S}_{k+1} \quad \text{and} \quad W_c = \mathbb{A}_k \ltimes \mathfrak{S}_k$$

where W_c acts only on $\varepsilon_2, \dots, \varepsilon_{k+1}$. The adapted positive system is

$$\Delta^{++} = \left\{ \varepsilon_i - \varepsilon_j \;\middle|\; 1 \leqslant i < j \leqslant k+1 \right\} ,$$

and the associated Harish-Chandra sequence is

$$\gamma_1 = \varepsilon_1 - \varepsilon_2 \, , \; \gamma_2 = \varepsilon_2 + \varepsilon_2 \, .$$

Since $\#W/W_c = 2(k+1)$, a set of representatives of W_c-conjugacy classes of positive systems is indexed by $\varkappa \in \mathbb{Z}_2 = \{\pm 1\}$ and $1 \leqslant n \leqslant k+1$. We define $\Delta^{+,\varkappa,n}$ by setting

$$\Delta_c^{+,\varkappa,n} = \left\{ \varkappa(\varepsilon_j \pm \varepsilon_i) \;\middle|\; 2 \leqslant i < j \leqslant n \right\} \cup \left\{ \varkappa(\varepsilon_i \pm \varepsilon_j) \;\middle|\; 2 \leqslant i < j \leqslant k+1 \, , \, n < j \right\} ,$$

and

$$\Delta_n^{+,\varkappa,n} = \left\{ \varkappa(\varepsilon_j \pm \varepsilon_1) \;\middle|\; 1 < j \leqslant n \right\} \cup \left\{ \varkappa(\varepsilon_1 \pm \varepsilon_j) \;\middle|\; 1 \leqslant n < j \leqslant k+1 \right\} .$$

Lemma 5.1.25. The sets $\Delta^{+,\varkappa,n} = \Delta_c^{+,\varkappa,n} \cup \Delta_n^{+,\varkappa,n}$ for $\varkappa \in \mathbb{Z}_2$, $1 \leqslant n \leqslant k+1$, are pairwise non-$W_c$-conjugate positive systems for Δ.

Proof. Clearly,

$$\Delta = -\Delta^{+,\varkappa,n} \,\dot{\cup}\, \Delta^{+,\varkappa,n}$$

where the union is disjoint. To see that $\Delta^{+,\varkappa,n}$ is a positive system, we need to prove that $\Delta^{+,\varkappa,n}$ is closed, i.e. $\Delta \cap (\Phi + \Phi) \subset \Phi$ for $\Phi = \Delta^{+,\varkappa,n}$, by [Bou68, ch. VI, § 1.7, cor. 1].

If $\alpha = \varkappa(\varepsilon_i + \varepsilon_j)$, $1 \leqslant i \neq j \leqslant k+1$ and $\beta \in \Delta^{+,\varkappa,n}$ are such that $\alpha + \beta \in \Delta$, then $\beta = \varkappa(\varepsilon_p - \varepsilon_q)$ where $q = i, j$. In any case

$$\alpha + \beta = \varkappa(\varepsilon_p + \varepsilon_{i+j-q}) \in \Delta^{+,\varkappa,n} .$$

The only other possible situation is that $\alpha = \varkappa(\varepsilon_i - \varepsilon_j)$ and $\beta = \varkappa(\varepsilon_j - \varepsilon_p)$. Here, the interesting cases are

$$1 \leqslant j < i \leqslant n \,,\; 1 \leqslant j < p \leqslant k+1 \,,\; n < p \,,$$

and

$$1 \leqslant i < j \leqslant k+1 \,,\; n < j \,,\; 1 \leqslant p < j \leqslant n \,.$$

The second case is impossible, and in the first case,

$$\alpha + \beta = \varkappa(\varepsilon_i - \varepsilon_p) \,,\; i \leqslant n < p \,,$$

so $\alpha + \beta \in \Delta^{+,\varkappa,n}$. Hence $\Delta^{+,\varkappa,n}$ is closed and therefore a positive system.

If two of these positive systems are W_c-conjugate, then the signs $\varkappa$ are identical, and we may assume $\varkappa = 1$. Let $\sigma \in \mathfrak{S}_k$ be such that $\sigma.\Delta^{+,1,n_1} = \Delta^{+,1,n_2}$. Then $n_1 = n_2$, and σ fixes a positive system of the root system

$$A_{k-1} = \left\{ \pm(\varepsilon_i - \varepsilon_j) \;\middle|\; 2 \leqslant i < j \leqslant k+1 \right\} ,$$

so $\sigma = 1$. Hence the positive systems $\Delta^{+,\varkappa,n}$ are pairwise non-conjugate. ☐

Proposition 5.1.26. Let $\varkappa \in \mathbb{Z}_2$ and $1 \leqslant n \leqslant k+1$. The sequence

$$\gamma_1^{\varkappa,n} = \begin{cases} \varkappa(\varepsilon_1 - \varepsilon_2) & n = 1 \\ \varkappa(\varepsilon_2 - \varepsilon_1) & n > 1 \end{cases} ,\; \gamma_2^{\varkappa,n} = \varkappa(\varepsilon_1 + \varepsilon_2)$$

is fundamental for the positive system $\Delta^{+,\varkappa,n}$.

Proof. Condition (FS1) is obvious, and so is (FS3), since D_{k+1} is simply laced.

As to (FS2), first note that for $n = 1$, $\varkappa(\varepsilon_1 - \varepsilon_2)$ is simple. Indeed, this is clear since $\Delta^{+,\varkappa,1} = \varkappa.\Delta^{++}$ and $\varepsilon_1 - \varepsilon_2$ is simple in Δ^{++}.

For $n > 1$, $\gamma_1^{\varkappa,n} = \varkappa(\varepsilon_2 - \varepsilon_1) \in \Delta_n^{+,\varkappa,n}$ is simple, too. Indeed, if $\alpha \in \Delta_n^{+,\varkappa,n}$ and $\beta \in \Delta_c$ such that $\alpha + \beta = \gamma_1^{\varkappa,n}$, then

$$\alpha = \varkappa(\varepsilon_j - \varepsilon_1) \quad \text{and} \quad \beta = \varkappa(\varepsilon_2 - \varepsilon_j) \,,$$

so β is negative. Therefore, $\gamma_1^{\varkappa,n}$ is simple. The only positive non-compact root strongly orthogonal to $\gamma_1^{\varkappa,n}$ is $\gamma_2^{\varkappa,n}$. This proves (FS2). □

Proposition 5.1.27. Let $\varkappa \in \mathbb{Z}_2$ and $1 \leqslant n \leqslant k+1$. Then for the fundamental sequence $\gamma_1^{\varkappa,n}, \gamma_2^{\varkappa,n}$, the lower signature $m^{\varkappa,n} = \left(m_1^{\varkappa,n}, m_2^{\varkappa,n}\right)$ is given by

$$m_1^{\varkappa,n} = \begin{cases} 2(k-1) & n=1 \\ n-1 & n>1 \end{cases}, \quad m_2^{\varkappa,n} = 0\,.$$

Proof. As we have already seen, the only non-compact positive root strongly orthogonal to $\gamma_1^{\varkappa,n}$ is $\gamma_2^{\varkappa,n}$, so $m_2^{\varkappa,n} = 0$.

Now let $\alpha \in \Delta_n^{+,\varkappa,n}$ such that $\alpha - \gamma_1^{\varkappa,n} \in \Delta_c^{+,\varkappa,n}$. If $n = 1$, then

$$\alpha = \varkappa\left(\varepsilon_1 \pm \varepsilon_j\right) \quad \text{for some } 2 \leqslant j \leqslant k+1\,.$$

Clearly, there are $2(k-1)$ possibilities for α.

If $n > 1$, then

$$\alpha = \varkappa\left(\varepsilon_j - \varepsilon_1\right) \quad \text{for some } 2 \leqslant j \leqslant n\,.$$

Then there are $n-1$ possibilities for α. □

5.1.28. The JB*-triple $Z = V_{k+1}$ has rank 2, and the only non-trivial subtriple $\bar{Z} = Z_0(e)$ is $\bar{Z} \cong \mathbb{C}$. Since $\bar{\gamma}_1 = \gamma_2$ (recall that we already know that the Harish-Chandra sequence for $\bar{Z}$ occurs as the tail of that for Z), the rank-one root system $\bar{\Delta}$ is

$$\bar{\Delta} = \{\pm\gamma_2\} = \left\{\pm\left(\varepsilon_1 + \varepsilon_2\right)\right\}.$$

We have $\bar{W}_c = 1$, and the two positive systems of $\bar{\Delta}$ are indexed by $\varkappa \in \mathbb{Z}_2$:

$$\bar{\Delta}^{+,\varkappa} = \left\{\varkappa\left(\varepsilon_1 + \varepsilon_2\right)\right\}.$$

Since $\bar{\gamma}_1^{\varkappa} = \gamma_2^{\varkappa,n} = \varkappa\left(\varepsilon_1 + \varepsilon_2\right)$ is the only non-compact positive root for $\bar{\Delta}^{+,\varkappa}$, the following proposition is clear.

Proposition 5.1.29. Let $\varkappa \in \mathbb{Z}_2$. For any $1 \leqslant n \leqslant k+1$, $\bar{\Delta}^{\varkappa} = \bar{\Delta} \cap \Delta^{+,\varkappa,n}$, the sequence $\bar{\gamma}_1^{\varkappa} = \gamma_2^{\varkappa,n}$ is fundamental, and its lower signature is $\bar{m}_1^{\varkappa} = m_2^{\varkappa,n} = 0$.

5.1.5 Proof for type IV$_n$, $q = 2k+1$

5.1.30. If $B = \mathrm{SO}(2,q)/(\mathrm{SO}(2) \times \mathrm{SO}(q))$ where $q = 2k+1$, then

$$\Delta = \left\{\pm\varepsilon_i \,\middle|\, 1 \leqslant i \leqslant k+1\right\} \cup \left\{\pm\left(\varepsilon_i \pm \varepsilon_j\right) \,\middle|\, 1 \leqslant i < j \leqslant k+1\right\},$$

and

$$\Delta_c = \left\{\pm\varepsilon_i \,\middle|\, 2 \leqslant i \leqslant k+1\right\} \cup \left\{\pm\left(\varepsilon_i \pm \varepsilon_j\right) \,\middle|\, 2 \leqslant i < j \leqslant k+1\right\},$$

so the roots $\pm\varepsilon_i$ are short, whereas $\pm(\varepsilon_i \pm \varepsilon_j)$ are long. The Weyl groups are

$$W = \mathbb{Z}_2^{k+1} \ltimes \mathfrak{S}_{k+1} \quad \text{and} \quad W_c = \mathbb{Z}_2^k \ltimes \mathfrak{S}_k \,,$$

where W_c acts only on $\varepsilon_2, \ldots, \varepsilon_{k+1}$. The adapted positive system Δ^{++} is

$$\Delta^{++} = \left\{\varepsilon_i \,\middle|\, 1 \leqslant i \leqslant k+1\right\} \cup \left\{\varepsilon_i \pm \varepsilon_j \,\middle|\, 1 \leqslant i < j \leqslant k+1\right\}.$$

As in the case $q = 2k$, $\#(W/W_c) = 2(k+1)$, so a set of representatives of W_c-conjugacy classes of positive systems is indexed by $\varkappa \in \mathbb{Z}_2$, $1 \leqslant n \leqslant k+1$. Define $\Delta^{+,\varkappa,n}$ by

$$\begin{aligned}\Delta_c^{+,\varkappa,n} =& \left\{\varkappa\varepsilon_i \,\middle|\, 2 \leqslant i \leqslant k+1\right\} \cup \left\{\varkappa(\varepsilon_j \pm \varepsilon_i) \,\middle|\, 2 \leqslant i < j \leqslant n\right\} \\ & \cup \left\{\varkappa(\varepsilon_i \pm \varepsilon_j) \,\middle|\, 2 \leqslant i < j \leqslant k+1\,,\ j > n\right\},\end{aligned}$$

and

$$\Delta_n^{+,\varkappa,n} = \left\{\varkappa\varepsilon_1\right\} \cup \left\{\varkappa(\varepsilon_j \pm \varepsilon_1) \,\middle|\, 2 \leqslant j \leqslant n\right\} \cup \left\{\varkappa(\varepsilon_1 \pm \varepsilon_j) \,\middle|\, n < j \leqslant k+1\right\}.$$

Lemma 5.1.31. The sets $\Delta^{+,\varkappa,n} = \Delta_c^{+,\varkappa,n} \cup \Delta_n^{+,\varkappa,n}$ are pairwise non-W_c-conjugate positive systems for Δ.

Proof. Clearly, we have a disjoint union

$$\Delta = -\Delta^{+,\varkappa,n} \,\dot\cup\, \Delta^{+,\varkappa,n}.$$

Hence, we need only establish the closedness of $\Delta^{+,\varkappa,n}$. The subset

$$\Phi = \Delta \setminus \left\{\pm\varepsilon_i \,\middle|\, 1 \leqslant i \leqslant k+1\right\}$$

is a subsystem of type D_{k+1}. $\Phi \cap \Delta^{+,\varkappa,n}$ is the positive system $\Delta^{+,\varkappa,n}$ for Φ from lemma 5.1.25 above, in particular, closed. So it remains to consider sums where at least one root is short.

Moreover, w.l.o.g., we may assume $\varkappa = 1$. $\varepsilon_1 + \varepsilon_i$ and $\varepsilon_i + \varepsilon_j$, $2 \leqslant i < j \leqslant k+1$, are always positive roots. $\varepsilon_1 + (\varepsilon_j - \varepsilon_1) = \varepsilon_j$ is positive. Similarly for $\varepsilon_i + (\varepsilon_j - \varepsilon_i)$ and $\varepsilon_j + (\varepsilon_1 - \varepsilon_j)$. Since these are all possible cases, the lemma follows. □

Proposition 5.1.32. et $\varkappa \in \mathbb{Z}_2$ and $1 \leqslant n \leqslant k+1$. The sequence

$$\gamma_1^{\varkappa,n} = \begin{cases} \varkappa(\varepsilon_1 - \varepsilon_2) & n = 1 \\ \varkappa(\varepsilon_2 - \varepsilon_1) & n > 1 \end{cases}, \ \gamma_2^{\varkappa,n} = \varkappa(\varepsilon_1 + \varepsilon_2)$$

is fundamental for $\Delta^{+,\varkappa,n}$.

Proof. $\gamma_1^{\varkappa,n}$ and $\gamma_2^{\varkappa,n}$ are strongly orthogonal positive non-compact roots, whence (FS1), and they are also both long, whence (FS3).

Since the only positive non-compact root strongly orthogonal to $\gamma_1^{\varkappa,n}$ is $\gamma_2^{\varkappa,n}$, for the latter, (FS2) is true. As for $\gamma_1^{\varkappa,n}$, it is certainly never the sum of two short roots (since these sums are $\varkappa(\varepsilon_i+\varepsilon_j)$), or of a short and a long root (since these sums are short). Moreover, $\gamma_1^{\varkappa,n}$ is simple in $\Phi\cap\Delta^{+,\varkappa,n}$ by proposition 5.1.26 (notation from the proof of lemma 5.1.31), so $\gamma_1^{\varkappa,n}$ is simple in $\Delta^{+,\varkappa,n}$ and (FS2) follows. □

Proposition 5.1.33. Let $\varkappa\in\mathbb{Z}_2$ and $1\leqslant n\leqslant k+1$. For the fundamental sequence $\gamma_1^{\varkappa,n},\gamma_2^{\varkappa,n}$, the lower signature $m^{\varkappa,n}=\big(m_1^{\varkappa,n},m_2^{\varkappa,n}\big)$ is given by

$$m_1^{\varkappa,n}=\begin{cases}2(k-1) & n=1\\ n-1 & n>1\end{cases}\;,\; m_2^{\varkappa,n}=0\,.$$

Proof. Let $\alpha\in\Delta_n^{+,\varkappa,n}$ such that $\alpha-\gamma_1^{\varkappa,n}\in\Delta$. Apart from those stated in the proof of proposition 5.1.27, there are no further possibilities for α, since clearly $\alpha\neq\varkappa\varepsilon_1$. This proves the proposition. □

5.1.34. The JB*-triple Z is the same as in case BD I, $q=2k$. Hence the same considerations as in that case show that we need only consider

$$\bar\Delta=\big\{\pm(\varepsilon_1+\varepsilon_2)\big\}\,,$$

and that $\bar\gamma_1^{\varkappa}=\varkappa(\varepsilon_1+\varepsilon_2)$ with signature $\bar m_1^{\varkappa}=0$. The following proposition ensues.

Proposition 5.1.35. Let $\varkappa\in\mathbb{Z}_2$. For any $1\leqslant n\leqslant k+1$, $\bar\Delta^{\varkappa}=\bar\Delta\cap\Delta^{+,\varkappa,n}$, the sequence $\bar\gamma_1^{\varkappa}=\gamma_2^{\varkappa,n}$ is fundamental, and its lower signature is $\bar m_1^{\varkappa}=m_2^{\varkappa,n}=0$.

5.2 Embedding of the discrete series of embeddable subgroups

5.2.1. Let $G=\mathrm{Aut}_0 B$, and $\bar G\sqsubset G$ an embeddable facial subgroup. By theorem 5.1.2, this is the case if Z is classical. Both G and $\bar G$ have equal rank, i.e. $\operatorname{rk}G=\operatorname{rk}\bar K$ and $\operatorname{rk}\bar G=\operatorname{rk}\bar K$. This condition is equivalent to the existence of a discrete series.

If $\mathfrak{t}_{\mathbb{R}}$ is Cartan subalgebra (CSA) of $\mathfrak{g}_{\mathbb{R}}$ contained in $\mathfrak{k}_{\mathbb{R}}$, then $\bar{\mathfrak{t}}_{\mathbb{R}}=\bar{\mathfrak{g}}_{\mathbb{R}}\cap\mathfrak{t}_{\mathbb{R}}$ is a CSA of $\bar{\mathfrak{g}}_{\mathbb{R}}$ contained in $\bar{\mathfrak{k}}_{\mathbb{R}}$. Fix a positive system $\bar\Delta^+\subset\bar\Delta$, and let $\Delta^+\subset\Delta$ and a fundamental sequence $\alpha_1,\dots,\alpha_r$ be given as in theorem 5.1.2.

We now construct compatible Iwasawa decompositions of $\mathfrak{g}_{\mathbb{R}}$ and $\bar{\mathfrak{g}}_{\mathbb{R}}$, related to the fundamental sequence $\alpha_1,\dots,\alpha_r$ in the same way as the Iwasawa decomposition constructed in 2.1.24 was related to Harish-Chandra's fundamental sequence $\gamma_1,\dots,\gamma_r$.

For $\alpha\in\Delta$, there are unique $H_\alpha\in\mathfrak{t}$, such that $\alpha(H_\alpha)=2$. Moreover, for $\alpha\in\Delta^+$, we may choose $E_{\pm\alpha}\in\mathfrak{g}_{\pm\alpha}$ such that

$$H_\alpha=\big[E_\alpha,E_{-\alpha}\big]\,,\;\vartheta\overline{E_\alpha}=-E_{-\alpha}\,,\;B\big(E_\alpha,E_{-\alpha}\big)=\frac{2}{|\alpha|^2}\,,$$

cf. [Hel78, ch. IV, § 3, lemma 3.1]. Here, B is the Killing form and ϑ the Cartan involution w.r.t. the Cartan decomposition chosen above. Then

$$E_\alpha + E_{-\alpha}\ ,\ i\big(E_\alpha - E_{-\alpha}\big)$$

are in $\mathfrak{g}_{\mathbb{R}}$ resp. in $i\mathfrak{g}_{\mathbb{R}}$, depending on whether α is non-compact resp. compact.

We define the *Cayley transforms* (following Korányi-Wolf),

$$c = \prod_{j=1}^{r} \operatorname{Ad}\exp\Big[\frac{\pi}{4}\big(E_{\alpha_j} - E_{-\alpha_j}\big]\Big] \quad \text{and} \quad \bar{c} = \prod_{j=1}^{\bar{r}} \operatorname{Ad}\exp\Big[\frac{\pi}{4}\big(E_{\bar{\alpha}_j} - E_{-\bar{\alpha}_j}\big)\Big]\ ,$$

cf. [KW76, Sch75]. Here, recall that $\bar{\alpha}_j = \alpha_{r-\bar{r}+j}$ for all $1 \leqslant j \leqslant \bar{r}$.

Due to the strong orthogonality of the root α_j , c and $\bar{c}$ are products of commuting automorphisms of $\mathfrak{g}$ resp. $\bar{\mathfrak{g}}$. Moreover,

$$c\big(iH_{\alpha_j}\big) = E_{\alpha_j} + E_{-\alpha_j} \quad \text{for all}\ \ 1 \leqslant j \leqslant r$$

and similarly for $\bar{c}$. Define real Abelian subspaces

$$\mathfrak{a}_{\mathbb{R}} = \sum\nolimits^{\oplus}_{1\leqslant j\leqslant r} \mathbb{R}\big\langle E_{\alpha_j} + E_{-\alpha_j}\big\rangle \quad \text{and} \quad \bar{\mathfrak{a}}_{\mathbb{R}} = \sum\nolimits^{\oplus}_{1\leqslant j\leqslant \bar{r}} \mathbb{R}\big\langle E_{\bar{\alpha}_j} + E_{-\bar{\alpha}_j}\big\rangle\ ,$$

and denote their complexifications by $\mathfrak{a}$ and $\bar{\mathfrak{a}}$. Let $\varrho_{\mathfrak{a}}$ and $\bar{\varrho}_{\bar{\mathfrak{a}}}$ be the respective weighted half sums of positive restricted roots, i.e.

$$\varrho_{\mathfrak{a}} = \frac{1}{2} \cdot \sum\nolimits_{\alpha\in\Delta^+_{\mathfrak{a}}} \dim \mathfrak{g}^{\alpha} \cdot \alpha\ ,$$

and similarly for $\bar{\varrho}_{\bar{\mathfrak{a}}}$. Here, $\Delta^+_{\mathfrak{a}}$ denotes the set of positive restricted roots w.r.t. a compatible ordering.

Denote by (m_j) and $(\bar{m}_j)$ the lower signatures of (α_j) and $(\bar{\alpha}_j)$. Define *upper signatures*

$$n_j = \#\Big\{\alpha \in \Delta^+_n\ \Big|\ \alpha \text{ strongly orthogonal to } \alpha_1, \ldots, \alpha_{j-1}\text{ , and } \alpha + \alpha_j \in \Delta\Big\}\ ,$$

and

$$\bar{n}_j = \#\Big\{\alpha \in \bar{\Delta}^+_n\ \Big|\ \alpha \text{ strongly orthogonal to } \bar{\alpha}_1, \ldots, \bar{\alpha}_{j-1}\text{ , and } \alpha + \bar{\alpha}_j \in \bar{\Delta}\Big\}\ .$$

Proposition 5.2.2. The sets $\mathfrak{a}_{\mathbb{R}}$ resp. $\bar{\mathfrak{a}}_{\mathbb{R}}$ are maximal Abelian subspaces of $\mathfrak{p}_{\mathbb{R}}$ resp. $\bar{\mathfrak{p}}_{\mathbb{R}}$. Moreover,

$$\bar{\mathfrak{a}}_{\mathbb{R}} = \bar{\mathfrak{g}} \cap \mathfrak{a}_{\mathbb{R}} \quad \text{and} \quad \bar{c}^{-1}\big|_{\bar{\mathfrak{a}}} = c^{-1}\big|_{\mathfrak{a}}$$

Let $\Lambda \in i\mathfrak{t}_{\mathbb{R}}^*$ and $\bar{\Lambda} = \Lambda\big|_{\bar{\mathfrak{t}}}$, and define $\nu \in \mathfrak{a}_{\mathbb{R}}^*$ resp. $\bar{\nu} \in \bar{\mathfrak{a}}_{\mathbb{R}}^*$ by

$$(\nu - \varrho_{\mathfrak{a}})\big(E_{\alpha_j} + E_{-\alpha_j}\big) = -\frac{2}{|\alpha_j|^2} \cdot \big(\Lambda + n_j \cdot \alpha_j \,:\, \alpha_j\big) \quad \text{for all } \ 1 \leqslant j \leqslant r\,,$$

and

$$(\bar{\nu} - \bar{\varrho}_{\bar{\mathfrak{a}}})\big(E_{\bar{\alpha}_j} + E_{-\bar{\alpha}_j}\big) = -\frac{2}{|\bar{\alpha}_j|^2} \cdot \big(\bar{\Lambda} + \bar{n}_j \cdot \bar{\alpha}_j \,:\, \bar{\alpha}_j\big) \quad \text{for all } \ 1 \leqslant j \leqslant \bar{r}\,.$$

Then

$$(\nu + \varrho_{\mathfrak{a}})\big|_{\bar{\mathfrak{a}}} = \bar{\nu} + \bar{\varrho}_{\bar{\mathfrak{a}}}\,.$$

Proof. The first statement follows from [Sug71, th. 7]. The next two are immediate consequences of the above considerations. As for the last statement,

$$\varrho_{\mathfrak{a}}\big(E_{\alpha_j} + E_{-\alpha_j}\big) = 1 + n_j + m_j$$

by [KW76, lemma 8.5], and similarly for $\bar{\varrho}_{\bar{\mathfrak{a}}}$. Moreover,

$$\nu - \varrho_{\mathfrak{a}} = -\Big(\Lambda + \sum_{j=1}^{r} n_j \cdot \alpha_j\Big) \circ c^{-1}\,\Big|_{\mathfrak{a}}\,,$$

since

$$\alpha_i(H_{\alpha_j}) = \frac{2(\alpha_i \,:\, \alpha_j)}{|\alpha_j|^2} = 2\delta_{ij}\,.$$

Thus, by theorem 5.1.2 (iii),

$$\begin{aligned}
(\nu + \varrho_{\mathfrak{a}})\big(E_{\bar{\alpha}_j} + E_{-\bar{\alpha}_j}\big) &= -\Lambda\big(H_{\alpha_{r-\bar{r}+j}}\big) - 2n_{r-\bar{r}+j} + 2\big(1 + n_{r-\bar{r}+j} + m_{r-\bar{r}+j}\big) \\
&= -\bar{\Lambda}\big(H_{\bar{\alpha}_j}\big) - 2\bar{n}_j + 2\big(1 + \bar{n}_j + \bar{m}_j\big) \\
&= (\bar{\nu} + \bar{\varrho}_{\bar{\mathfrak{a}}})\big(E_{\bar{\alpha}_j} + E_{-\bar{\alpha}_j}\big)
\end{aligned}$$

for all $1 \leqslant j \leqslant \bar{r}$. □

5.2.3. Let us recall the Iwasawa decomposition associated to the choices of K and $\mathfrak{a}_{\mathbb{R}}$. The CSA

$$\mathfrak{h}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}}^{+} + \mathfrak{a}_{\mathbb{R}} \quad \text{where} \quad \mathfrak{t}_{\mathbb{R}}^{+} = \big\{\delta \in \mathfrak{t}_{\mathbb{R}} \;\big|\; (\delta \,:\, \gamma_j) = 0 \text{ for all } \ j = 1, \dots, r\big\}$$

equals $\mathfrak{g}_{\mathbb{R}} \cap c(\mathfrak{t})$, cf. [Sch75, § 2, lemma 2.15]. (Note that this is the appropriate generalisation of $\mathfrak{t}_{\mathbb{R}}^{+}$, as defined for the adapted positive system Δ^{++} in 2.1.3.)

Therefore, the root system $\Delta(\mathfrak{h} : \mathfrak{g})$ is just $\Delta \circ c^{-1}$, and the root spaces are the same as for Δ. Hence, with the induced compatible ordering, the positive restricted roots are

$$\Delta_{\mathfrak{a}}^{+}=\left\{\left(\alpha\circ c^{-1}\right)\big|_{\mathfrak{a}}\;\middle|\;\alpha\circ\left(c^{-1}\vartheta c\right)\neq\alpha\,,\,\alpha\in\Delta^{+}\right\},$$

and the associated nilpotent subalgebra $\mathfrak{n}_{\mathbb{R}}$ is

$$\mathfrak{n}_{\mathbb{R}}=\mathfrak{g}_{\mathbb{R}}\cap\sum\nolimits^{\oplus}_{\substack{\alpha\in\Delta^{+},\\ \alpha\circ(c^{-1}\vartheta c)\neq\alpha}}\mathfrak{g}^{\alpha}\,,$$

cf. [Hel78, thm. 3.4]. On the Lie algebra level, the Iwasawa decomposition is

$$\mathfrak{g}_{\mathbb{R}}=\mathfrak{k}_{\mathbb{R}}\oplus\mathfrak{a}_{\mathbb{R}}\oplus\mathfrak{n}_{\mathbb{R}}\,.$$

On the group level, let A resp. N be the connected closed subgroups of G with Lie algebras $\mathfrak{a}_{\mathbb{R}}$ resp. $\mathfrak{n}_{\mathbb{R}}$. Then $G=KAN$ by [Hel78, thm. 5.1].

Consider further the Iwasawa decomposition of $\bar{\mathfrak{g}}_{\mathbb{R}}$,

$$\bar{\mathfrak{g}}_{\mathbb{R}}=\bar{\mathfrak{k}}_{\mathbb{R}}\oplus\bar{\mathfrak{a}}_{\mathbb{R}}\oplus\bar{\mathfrak{n}}_{\mathbb{R}}$$

and $\bar{G}=\bar{K}\bar{A}\bar{N}$, defined similarly w.r.t. $\bar{\mathfrak{a}}_{\mathbb{R}}$, $\bar{K}$, $\bar{c}$ and $\bar{\Delta}^{+}$.

Proposition 5.2.4. We have $\bar{\mathfrak{n}}_{\mathbb{R}}=\bar{\mathfrak{g}}_{\mathbb{R}}\cap\mathfrak{n}_{\mathbb{R}}$ and $\bar{N}=\bar{G}\cap N$. Moreover, for the centralisers $M=Z_K(\mathfrak{a})$ and $\bar{M}=Z_{\bar{K}}(\bar{\mathfrak{a}})$, we have $\bar{M}=\bar{K}\cap M=\bar{G}\cap M$.

Proof. The intersection $\bar{\mathfrak{g}}_{\mathbb{R}}\cap\mathfrak{n}_{\mathbb{R}}$ is the sum of restricted root spaces $\bar{\mathfrak{g}}_{\mathbb{R}}^{\alpha}$ where $\alpha\in\bar{\Delta}^{+}$ does not vanish on $\mathfrak{a}_{\mathbb{R}}$. In particular, $\bar{\mathfrak{n}}_{\mathbb{R}}\subset\bar{\mathfrak{g}}_{\mathbb{R}}\cap\mathfrak{n}_{\mathbb{R}}$. Since $\bar{\mathfrak{n}}_{\mathbb{R}}$ is maximally nilpotent, equality follows. In particular, we have $\bar{N}=\bar{G}\cap N$.

It is clear that $\bar{G}\cap M=\bar{K}\cap M\subset M$. To prove the converse inclusion, since $\bar{K}$ is connected, it suffices to see that $\bar{\mathfrak{k}}_{\mathbb{R}}$ centralises $\mathfrak{a}_{\mathbb{R}}\ominus\bar{\mathfrak{a}}_{\mathbb{R}}$. This is true of

$$\bar{\mathfrak{t}}_{\mathbb{R}}=\bar{\mathfrak{t}}_{\mathbb{R}}^{+}\oplus\langle iH_{\bar{\alpha}_j}\mid 1\leqslant j\leqslant\bar{r}\rangle\,.$$

In fact, $\bar{\mathfrak{t}}_{\mathbb{R}}^{+}\subset\mathfrak{t}_{\mathbb{R}}^{+}$ commutes with $\mathfrak{a}$, and for $1\leqslant i\leqslant\bar{r}$ and $1\leqslant j\leqslant r-\bar{r}$,

$$\left[H_{\bar{\alpha}_i},E_{\alpha_j}+E_{-\alpha_j}\right]=\alpha_j\left(H_{\alpha_{r-\bar{r}+i}}\right)\cdot E_{\alpha_j}-\alpha_j\left(H_{\alpha_{r-\bar{r}+i}}\right)\cdot E_{-\alpha_j}=0\,.$$

So, let $\alpha\in\bar{\Delta}_c$. Then $\alpha\perp\alpha_1,\dots,\alpha_{r-\bar{r}}$. By [KW76, lemma 4.2], α is strongly orthogonal to these roots. Thus

$$\left[E_{\alpha},E_{\alpha_j}+E_{-\alpha_j}\right]=0\quad\text{for all }\;1\leqslant j\leqslant r-\bar{r}\,,$$

so $E_{\alpha}\in\mathfrak{g}^{\alpha}\subset\mathfrak{k}$ centralises $\mathfrak{a}\ominus\bar{\mathfrak{a}}$, proving the assertion, i.e. $\bar{\mathfrak{k}}_{\mathbb{R}}$ centralises $\mathfrak{a}_{\mathbb{R}}\ominus\bar{\mathfrak{a}}_{\mathbb{R}}$. Hence $\bar{M}\subset\bar{K}$ centralises $\mathfrak{a}_{\mathbb{R}}\ominus\bar{\mathfrak{a}}_{\mathbb{R}}$, and is therefore contained in $\bar{G}\cap M$. ☐

5.2.5. Let the tori $T=Z_K(\mathfrak{t})$ and $\bar{T}=Z_{\bar{K}}(\bar{\mathfrak{t}})$. Let a parameter $\bar{\lambda}\in i\bar{\mathfrak{t}}_{\mathbb{R}}^{*}$ be given that is

$\bar{\Delta}^+$-dominant regular, i.e.

$$\bar{\lambda}(H_\alpha) > 0 \quad \text{for all} \;\; \alpha \in \bar{\Delta}^+ \,,$$

and such that $e^{\bar{\lambda}-\bar{\varrho}}$ is a character of $\bar{T}$, i.e. $\bar{\lambda} - \bar{\varrho}$ is analytically integral.

Set $\bar{\Lambda} = \bar{\lambda} + \bar{\varrho} - 2\bar{\varrho}_c$ and take $\Lambda \in i\mathfrak{t}^*_{\mathbb{R}}$ such that $\Lambda\big|_{\bar{\mathfrak{t}}} = \bar{\Lambda}$, $\lambda = \Lambda + 2\varrho_c - \varrho$ is Δ^+-dominant regular, and $e^{\lambda-\varrho}$ is a character of T. Then $(\bar{\Lambda}, \Lambda)$ will be called an *adapted pair of discrete series parameters*.

For the root system $\Delta^+ = -\tau_0(\Delta^{++})$ where $\tau_0 \in W_c$ is the longest element of W_c with respect to the Bruhat order induced by Δ_c^{++} (cf. [Bou68, ch. VI, § 1.6, cor. 3]), this reduces to the adapted pairs of *holomorphic* discrete series parameters, cf. 4.3.2.

Fix an adapted pair $(\bar{\Lambda}, \Lambda)$ of discrete series parameters. Recalling that the facts on Verma modules up to proposition 4.1.13 were independent of the adaptedness of the positive system, we conclude as in lemma 4.3.3 that $F_{\bar{\Lambda}} = \langle \bar{K}.1_\Lambda \rangle \sqsubset F_\Lambda$, and this is an isometric $\bar{K}$-invariant embedding if $1_{\bar{\Lambda}}$ and 1_Λ are normalised.

To stress the dependence on the group, we use the uniform notation $\langle H \rangle_\pi$ for the space of the representation π of the group H (unless we have to distinguish between different realisations). In particular, we have $\langle K \rangle_\Lambda = F_\Lambda$ and $\langle \bar{K} \rangle_{\bar{\Lambda}} = F_{\bar{\Lambda}}$.

5.2.6. Since G/K and $\bar{G}/\bar{K}$ are Hermitian symmetric spaces, [KW76, prop. 5.5] and [KW80, remark 2], show that

$$\langle M \rangle_\Lambda = \langle M.1_\Lambda \rangle \quad \text{and} \quad \langle \bar{M} \rangle_{\bar{\Lambda}} = \langle \bar{M}.1_{\bar{\Lambda}} \rangle = \langle \bar{M}.1_\Lambda \rangle \,,$$

i.e., the finite-dimensional irreducible representations of M resp. $\bar{M}$ of highest weight $\Lambda\big|_{\mathfrak{t}\cap\mathfrak{m}}$ and $\bar{\Lambda}\big|_{\bar{\mathfrak{t}}\cap\bar{\mathfrak{m}}}$ are the respective cyclic subrepresentations generated by 1_Λ in $\langle K \rangle_\Lambda$.

Hence, by proposition 5.2.4, there are M- resp. $\bar{M}$-equivariant projections

$$p_{\langle M \rangle_\Lambda} : \langle K \rangle_\Lambda \to \langle M \rangle_\Lambda \quad \text{and} \quad p_{\langle \bar{M} \rangle_{\bar{\Lambda}}} : \langle \bar{K} \rangle_{\bar{\Lambda}} \to \langle \bar{M} \rangle_{\bar{\Lambda}}$$

which give rise to a commutative square

$$\begin{tikzcd}
\langle K \rangle_\Lambda \arrow[r, "p_{\langle \bar{K} \rangle_{\bar{\Lambda}}}"] \arrow[d, "p_{\langle M \rangle_\Lambda}"'] & \langle \bar{K} \rangle_{\bar{\Lambda}} \arrow[d, "p_{\langle \bar{M} \rangle_{\bar{\Lambda}}}"] \\
\langle M \rangle_\Lambda \arrow[r, "p_{\langle \bar{K} \rangle_{\bar{\Lambda}}}\big|_{\langle M \rangle_\Lambda}"'] & \langle \bar{M} \rangle_{\bar{\Lambda}}
\end{tikzcd}$$

5.2.7. Define $\nu \in \mathfrak{a}^*_{\mathbb{R}}$ resp. $\bar{\nu} \in \bar{\mathfrak{a}}^*_{\mathbb{R}}$ as in proposition 5.2.2, i.e.

$$(\nu - \varrho_{\mathfrak{a}})\big(E_{\alpha_j} + E_{-\alpha_j}\big) = -\frac{2}{|\alpha_j|^2} \cdot \big(\Lambda + n_j \cdot \alpha_j \;:\; \alpha_j\big) \quad \text{for all} \;\; 1 \leqslant j \leqslant r \,,$$

and

$$(\bar{\nu} - \bar{\varrho}_{\bar{\mathfrak{a}}})\big(E_{\bar{\alpha}_j} + E_{-\bar{\alpha}_j}\big) = -\frac{2}{|\bar{\alpha}_j|^2} \cdot \big(\bar{\Lambda} + \bar{n}_j \cdot \bar{\alpha}_j \,:\, \bar{\alpha}_j\big) \quad \text{for all } 1 \leqslant j \leqslant \bar{r}\,,$$

where $\varrho_{\mathfrak{a}}$, n_j, $\bar{\varrho}_{\bar{\mathfrak{a}}}$ and $\bar{n}_j$ are defined in 5.2.1 and proposition 5.2.2. Let

$$H_\nu^\infty = \Big\{ f : K \to \langle M \rangle_\Lambda \;\Big|\; f \in \mathcal{C}^\infty\,,\, f(km) = m^{-\Lambda} f(k) \text{ for all } \; m \in M\,,\, k \in K \Big\}\,.$$

Then H_ν^∞ is a pre-Hilbert space with respect to the norm

$$\|f\|^2_{H_\nu} = \int_K^* \|f(k)\|^2_\Lambda \, dk$$

and G acts (non-unitarily) on H_ν^∞ by

$$(g^\nu f)(k) = a(g^{-1}k)^{-(\nu+\varrho_{\mathfrak{a}})} \cdot f\big(k(g^{-1}k)\big)\,.$$

Here, $g = k(g)a(g)n(g)$ denotes the Iwasawa decomposition of G associated to the choices of $\mathfrak{a}_{\mathbb{R}}$ and K. Let H_ν be the completion of H_ν^∞. The Hilbert space H_ν is a continuous representation of G. The action of K is unitary.

5.2.8. The space H_ν^∞ is naturally identified with the space of smooth sections of the vector bundle $K \times_M \langle M \rangle_\Lambda$ associated to the principal M-bundle $K \to K/M$ and the representation $\langle M \rangle_\Lambda$. The identification $K/M = G/Q$ given by the Iwasawa decomposition, where $Q = MAN$ is a minimal parabolic subgroup, allows for the identification

$$K \times_M \langle M \rangle_\Lambda = G \times_Q \big(\langle M \rangle_\Lambda \otimes \langle A \rangle_{\nu+\varrho_{\mathfrak{a}}} \otimes \mathbb{C}\big)\,.$$

Here, $\mathbb{C}$ is the trivial N-module. Under this identification, the G-action on H_ν becomes natural. Namely, extend $f \in H_\nu^\infty$ to a section on G/Q by

$$f(g) = a(g)^{-(\nu+\varrho_{\mathfrak{a}})} \cdot f\big(k(g)\big) \quad \text{for all } \; g \in G\,.$$

Then the action is by left translations.

The space H_ν^∞ is a dense subspace of smooth vectors of the non-unitary principal series representation $H_\nu = \operatorname{ind}_Q^G\big(\langle M \rangle_\Lambda \otimes \langle A \rangle_\nu \otimes \mathbb{C}\big)$, by [Kna86, lem. 3.13]. Moreover, H_ν is admissible (has finite-dimensional K-types) by the Frobenius reciprocity theorem. Hence [Kna86, th. 8.7] implies that the set $H_{\nu,K}$ of K-finite vectors in H_ν coincides with the set of K-finite vectors in H_ν^∞.

5.2.9. Following Knapp-Wallach [KW76], we define

$$S_\Lambda f(g) = \int_K k^\Lambda f(gk)\, dk \quad \text{for all } \; f \in H_\nu\,,\, g \in G\,.$$

We have the transformation rule

$$\int_K f(k)\,dk = \int_K f\big(k(gk)\big)a(gk)^{-2\varrho_\mathfrak{a}}\,dk \quad \text{for all } f\in \mathbf{L}^1(K)\,,\, g\in G\,.$$

Applying this to the map

$$k\mapsto k^\Lambda f(gk) = a(gk)^{-(\nu+\varrho_\mathfrak{a})}k^\Lambda f\big(k(gk)\big)\,,$$

we see that

$$\begin{aligned} S_\Lambda f(g) &= \int_K a(g^{-1}k)^{-2\varrho_\mathfrak{a}} a\big(gk(g^{-1}k)\big)^{-(\nu+\varrho_\mathfrak{a})} k(g^{-1}k)^\Lambda f\big(k(gk(g^{-1}k))\big)\,dk \\ &= \int_K a(g^{-1}k)^{\nu-\varrho_\mathfrak{a}} k(g^{-1}k)^\Lambda f(k)\,dk \end{aligned}$$

since $k\big(gk(g^{-1}k)\big) = k$ and $a\big(gk(g^{-1}k)\big) = a^{-1}$, as follows easily from Iwasawa decomposition, because A normalises N.

Note the parameter shift, compared with [KW76]. Knapp-Wallach realise the induced representation H_ν by right translations. We prefer to use left translations, and a more conventional parametrisation.

Since $P_\nu(g,k) = a(g^{-1}k)^{\nu-\varrho_\mathfrak{a}}$ is the Poisson kernel of the symmetric space G/K, S_Λ may be thought of as a matrix-valued Poisson transformation. On the other hand, in limiting cases on the unit disc, the kernel

$$S_\Lambda(g,k) = P_\nu(g,k)k(g^{-1}k)^\Lambda = a(g^{-1}k)^{\nu-\varrho_\mathfrak{a}}k(g^{-1}k)^\Lambda$$

coincides with the Szegö kernel. For this reason, Knapp-Wallach refer to S_Λ as a Szegö map. In fact, as we shall elaborate below for the case of the unit disc, for the holomorphic discrete series, S_Λ is the (weighted) Bergman kernel. For a certain limit of holomorphic discrete series, this gives the Szegö kernel of the unit disc.

5.2.10. The map S_Λ has values in

$$\Gamma\big(G/K, G\times_K \langle K\rangle_\Lambda\big) = \Big\{ f : G\to\langle K\rangle_\Lambda \;\Big|\; f\in\mathcal{C}^\infty\,,\, f(gk) = k^{-\Lambda}f(g) \text{ for all } k,g\Big\}$$

and is G-equivariant with respect to the G-action by left translations on this space of smooth sections.

Let $H_{\nu,K}$ denote the space of K-finite vectors in H_ν. Knapp-Wallach [KW76, th. 1.1, th. 10.8] show that the image of $H_{\nu,K}$ under S_Λ equals the Harish-Chandra $(\mathfrak{g},K)$-module $\langle G\rangle_{\pi_\lambda,K}$ of K-finite vectors of the discrete series representation $\langle G\rangle_{\pi_\lambda}$.

More precisely, the $S_\Lambda\big(H_{\nu,K}\big)$ is non-zero and contained in the K-finite part of the kernel of Schmid's differential operator $\mathcal{D}$, by [KW76, th. 6.1]. If λ is strongly dominant, then the K-finite part of $\ker\mathcal{D}$ equals $\langle G\rangle_{\pi_\lambda,K}$ [KW76, cor. 9.6], in particular, it is irreducible and hence equals the image of S_Λ. In any case, the image of S_Λ is irreducible and equivalent to $\langle G\rangle_{\pi_\lambda,K}$, by [KW76, th. 10.8].

Remark 5.2.11. We have already noted that Knapp-Wallach use a somewhat different parametrisation of the induced representation H_ν. To check that our choice of parameters indeed guarantees that their results remain valid, it suffices to ascertain that the image of S_Λ is contained in $\ker\mathcal{D}$. In order to do this, we recapitulate part of [KW76, proof of th. 6.1] for our realisation of H_ν.

Namely, the integral kernel of S_Λ is

$$S_\Lambda(g,k) = a\big(g^{-1}k\big)^{\nu-\varrho_{\mathfrak{a}}} k\big(g^{-1}k\big)^\Lambda ,$$

cf. 5.2.9, and if $E_\beta = X_\beta + iY_\beta$, then

$$\begin{aligned} E_\beta S_\Lambda(\sqcup,1)(1) &= \frac{d}{dt}\Big[S_\Lambda\big(\exp(-tX_\beta),1\big) + iS_\Lambda\big(\exp(-tY_\beta),1\big)\Big]_{t=0} \\ &= (\nu-\varrho_{\mathfrak{a}})\big(\mathrm{pr}_{\mathfrak{a}} E_\beta\big) + \big(\mathrm{pr}_{\mathfrak{k}} E_\beta\big)^\Lambda \end{aligned}$$

where it is understood that the left invariant action of $\mathfrak{U}(\mathfrak{g})$ is given by

$$Xf(g) = \frac{d}{dt} f\big(g\exp(-tX)\big)\Big|_{t=0} \qquad \text{for all } X \in \mathfrak{g}_{\mathbb{R}},\ f \in \mathcal{C}^\infty(G),\ g \in G .$$

Now, Schmid's differential operator $\mathcal{D}$ is defined as

$$\mathcal{D}f(g) = \sum_{\beta\in\Delta_n} \frac{|\beta|^2}{2}\cdot P\big(E_\beta f(g)\otimes E_{-\beta}\big) ,$$

where P is the orthogonal projection of $\langle K\rangle_\Lambda \otimes \mathfrak{p}$ onto the K-submodule given by the sum of the simple submodules with highest weights $\Lambda-\beta$, $\beta\in\Delta_n^+$. Hence,

$$\begin{aligned} \mathcal{D}S_\Lambda(\sqcup,1)1_\Lambda(1) &= \frac{1}{2}\sum_{\beta\in\Delta_n} |\beta|^2\cdot P\Big[\big[(\nu-\varrho_{\mathfrak{a}})\big(\mathrm{pr}_{\mathfrak{a}} E_\beta\big) + \big(\mathrm{pr}_{\mathfrak{k}} E_\beta\big)^\Lambda\big]1_\Lambda \otimes E_{-\beta}\Big] \\ &= \frac{1}{4}\sum_{j=1}^m c_j(\nu,\Lambda)\cdot\big|\alpha_j\big|^2\cdot P\big(1_\Lambda\otimes E_{-\alpha_j}\big) \end{aligned}$$

where the constants

$$c_j(\nu,\Lambda) = (\nu-\varrho_{\mathfrak{a}})\big(E_{\alpha_j}+E_{-\alpha_j}\big) + \frac{2(\Lambda+n_j\alpha_j:\alpha_j)}{\big|\alpha_j\big|^2} = 0$$

by our assumptions. Compare [KW76] for the last step of the above calculation. As there, it follows that $\ker\mathcal{D}$ contains the image of S_Λ. This proves that for our parametrisation of the induced representation, ν should be defined as indicated.

5.2.12. Since $H_{\Lambda,K} = S_\Lambda\big(H_{\nu,K}\big)$ is equivalent to $\langle G\rangle_{\pi_\lambda,K}$, it carries a pre-Hilbert space norm $\|\cdot\|_{H_\Lambda}$ which makes the action of G unitary. By Schur's lemma, the norm is unique up to constant multiples.

Lemma 5.2.13. The Szegö map S_Λ extends to a continuous map

$$S_\Lambda : H_\nu \longrightarrow \langle G\rangle_{\pi_\lambda} = H_\Lambda = \overline{H_{\Lambda,K}}\,.$$

In fact, there is $C > 0$ such that $S_\Lambda S_\Lambda^* = C$. In particular, $C^{-1/2}\cdot S_\Lambda$ is a partial isometry, and $C^{-1}\cdot S_\Lambda^* S_\Lambda$ is the orthogonal projection onto $\operatorname{im} S_\Lambda^* = \big(\ker S_\Lambda\big)^\perp \cong \langle G\rangle_{\pi_\lambda}$.

Remark 5.2.14. It is not natural to normalise $\|\cdot\|_{H_\Lambda}$ in such a way that $S_\Lambda S_\Lambda^* = 1$. The reason is that S_Λ depends on the choice of a fundamental sequence of roots, and different choices of fundamental sequences may give rise to different realisations of the discrete series π_λ as quotients of non-unitary principal series, cf. [KW76].

Proof of lemma 5.2.13. By the remark in 5.2.8, H_ν is admissible, and so is H_Λ. I.e. for any $\delta\in\hat K$, the K-isotypic components $H_{\nu,\delta}$ and $H_{\Lambda,\delta}$ are finite dimensional. So, we may define $S_\Lambda^* : H_{\Lambda,K}\to H_{\nu,K}$ by $S_\Lambda^*\big|_{H_{\Lambda,\delta}} = \big(S_\Lambda\big|_{H_{\nu,\delta}}\big)^*$, cf. the proof of proposition 4.3.4. Then $S_\Lambda S_\Lambda^*$ is $\mathfrak{g}$-equivariant and hence, by Schur's lemma, equals a positive constant $C > 0$. Hence the assertion. □

Proposition 5.2.15. The evaluation map

$$\varepsilon_1 : H_\Lambda \longrightarrow \langle K\rangle_\Lambda : f \longmapsto f(1)$$

is well-defined, continuous and K-equivariant. In particular, $H_\Lambda = \langle G\rangle_{\pi_\lambda}$ is a reproducing kernel Hilbert space of sections of $G\times_K \langle K\rangle_\Lambda$. Moreover, the lowest K-type Λ is precisely $\big(\ker\varepsilon_1\big)^\perp$.

Proof. Clearly, ε_1 is well-defined on $H_{\Lambda,K}$ and K-equivariant.

To see the continuity, we proceed as in [ÓØ88, § 2]. Namely, let

$$pf(g) = \dim\langle K\rangle_\Lambda \cdot \int_K \operatorname{tr} k^\Lambda\cdot f(k^{-1}g)\,dk \quad \text{for all } f\in S_\Lambda(H_\nu^\infty)\,,\ g\in G\,.$$

By the equivariance, linearity and continuity of S_Λ,

$$\dim\langle K\rangle_\Lambda^{-1}\cdot\big(pS_\Lambda f\big)(g) = \int_K S_\Lambda\big(\operatorname{tr} k^\Lambda\cdot k^\nu f\big)\,dk = S_\Lambda\Big[\int_K \operatorname{tr} k^\Lambda\cdot k^\nu f\,dk\Big]\,.$$

In particular, p extends by continuity to H_Λ. Since, for $f\in H_{\nu,K}$, the integral

$$\int_K \operatorname{tr} k^\Lambda\cdot k^\nu f\,dk \in H_\nu$$

and moreover, is K-finite, we see that $p(H_{\Lambda,K})\subset H_{\Lambda,K}$.

By the Schur orthogonality relations and the Peter-Weyl theorem, p is the projection onto the K-type Λ, which is isomorphic to $\langle K\rangle_\Lambda$ since the latter occurs without multiplicity. Hence,

$$(pf)(1) = \dim \langle K\rangle_\Lambda \cdot \int_K \operatorname{tr} k^\Lambda \cdot k^\nu \, dk \cdot f(1) = f(1)\,,$$

so $\varepsilon_1 = \varepsilon_1 p$ on $H_{\Lambda,K}$, and this formula defines an extension of ε_1 to H_Λ. Because the minimal K-type is finite-dimensional, ε_1 is continuous on pH_Λ. Hence, it is continuous on H_Λ. Moreover, ε_1 vanishes on $\ker p$. Since

$$\varepsilon_g = \varepsilon_1 g^{-\pi_\lambda} \quad \text{for all } g \in G\,,$$

we have $\varepsilon_1 \neq 0$, for otherwise, H_Λ would be zero. Hence, ε_1 restricts to an isomorphism on the K-type Λ, by irreducibility.

The maps ε_g, $g \in G$, extend to $\langle G\rangle_{\pi_\lambda}$ by continuity. Because π_λ is irreducible, the union of the subspaces

$$\varepsilon_g^* \langle K\rangle_\Lambda = g^{\pi_\lambda} pH_\Lambda\,,\ g \in G\,,$$

has dense span in $\langle G\rangle_{\pi_\lambda}$. Hence, for any $f \in \langle G\rangle_{\pi_\lambda}$, $\varepsilon_g f = 0$ for all $g \in G$ implies $f = 0$. So, the $f \in \langle G\rangle_{\pi_\lambda}$ are $\langle K\rangle_\Lambda$-valued functions on G.

By the formula for ε_g, all the evaluations at points of G are continuous. Hence, $\langle G\rangle_{\pi_\lambda}$ is a reproducing kernel Hilbert space of $\langle K\rangle_\Lambda$-valued functions on G. The transformation rule $f(gk) = k^{-\Lambda} f(g)$ remains valid on the completion, so these functions may be viewed as sections of $G \times_K \langle K\rangle_\Lambda$. □

Remark 5.2.16. We do not say anything about continuity or differentiability properties of the functions or sections the space H_Λ consists of.

Corollary 5.2.17. For a suitable normalisation of $\|\cdot\|_{H_\Lambda}$,

$$\|\varepsilon_1^* 1_\Lambda\|_{H_\Lambda} = 1 \quad \text{where } 1_\Lambda \in \langle K\rangle_\Lambda$$

is a normalised highest weight vector. Moreover, $\varepsilon_1\varepsilon_1^* = 1$ for this norm, and $\varepsilon_1^*\varepsilon_1$ is the orthogonal projection onto the lowest K-type Λ.

Remark 5.2.18. The above normalisation is natural in the sense that is turns the — up to the choice of $1_\Lambda \in \langle K\rangle_\Lambda$ — canonical realisation $\varepsilon_1^* 1_\Lambda$ of the highest weight vector in the lowest K-type into a unit vector.

Proof of corollary 5.2.17. Since ε_1 is continuous, we may consider $\varepsilon_1^* : \langle K\rangle_\Lambda \to H_\Lambda$. Then $C = \varepsilon_1\varepsilon_1^* > 0$ is a constant by Schur's lemma. If $\|\cdot\|'_{H_\Lambda}$ is the given norm on H_Λ, let $\|\cdot\|_{H_\Lambda} = \sqrt{C} \cdot \|\cdot\|'_{H_\Lambda}$. Since this does not affect the operator norm on endomorphisms of H_Λ, this space is a unitary G-representation with the new norm.

However, the adjoint of ε_1 with respect to this norm is $C^{-1} \cdot \varepsilon_1^*$. Hence, $\varepsilon_1\varepsilon_1^* = 1$ where the adjoint is now computed for the new norm. In particular, ε_1^* is an isometry, $\|\varepsilon_1\| = 1$, and $\varepsilon_1^*\varepsilon_1$ is a projection, so the assertion follows. □

Fixing the above normalisation of $\|\cdot\|_{H_\Lambda}$, we can compute the norm of S_Λ.

Proposition 5.2.19. For $\xi \in \langle K\rangle_\Lambda$, let $f_\xi \in H_{\nu,K}$ be defined by

$$f_\xi(k) = p_{\langle M\rangle_\Lambda} k^{-\Lambda}\xi \quad \text{for all } k \in K\,.$$

Then

$$f_\xi = S_\Lambda^* \varepsilon_1^* \xi \quad \text{and} \quad S_\Lambda f_\xi(1) = c_\Lambda \cdot \xi \quad \text{where} \quad c_\Lambda = \frac{\dim \langle M\rangle_\Lambda}{\dim \langle K\rangle_\Lambda}\,.$$

In particular, $\|S_\Lambda\| = \sqrt{c_\Lambda}$.

Proof. It is obvious that $f_\xi \in H_\nu^\infty$. Moreover,

$$K^\nu f_\xi \subset \langle f_\eta \mid \eta \in \langle K\rangle_\Lambda\rangle\,,$$

so f_ξ is K-finite. By irreducibility of $\langle K\rangle_\Lambda$ and Schur's lemma,

$$\begin{aligned} S_\Lambda f_\xi(1) &= \int_K k^\Lambda p_{\langle M\rangle_\Lambda} k^{-\Lambda}\xi\,dk \\ &= \frac{1}{\dim \langle K\rangle_\Lambda} \cdot \int_K \operatorname{tr}\bigl(k^\Lambda p_{\langle M\rangle_\Lambda} k^{-\Lambda}\bigr)\,dk \cdot \xi = \frac{\dim \langle M\rangle_\Lambda}{\dim \langle K\rangle_\Lambda} \cdot \xi\,. \end{aligned}$$

Finally, for $f \in H_\nu$, we compute

$$\begin{aligned} (f \mid f_\xi)_{H_\nu} &= \int_K \bigl(f(k) \mid p_{\langle M\rangle_\Lambda} k^{-\Lambda}\xi\bigr)_\Lambda\,dk = \int_K \bigl(k^\Lambda f(k) \mid \xi\bigr)_\Lambda\,dk \\ &= (S_\Lambda f(1) \mid \xi)_\Lambda = (\varepsilon_1 S_\Lambda f \mid \xi)_\Lambda \end{aligned}$$

by the projection theorem. Hence, $f_\xi = S_\Lambda^* \varepsilon_1^* \xi$. In particular, letting $C = S_\Lambda S_\Lambda^* > 0$,

$$C \cdot \xi = C \cdot \varepsilon_1 \varepsilon_1^*\,\xi = \varepsilon_1 S_\Lambda S_\Lambda^* \varepsilon_1^*\,\xi = S_\Lambda f_\xi(1) = c_\Lambda \cdot \xi\,.$$

This proves the proposition. □

Remark 5.2.20. Proposition 5.2.19 gives rise to an integral formula for the End $\langle K\rangle_\Lambda$-valued reproducing kernel of H_Λ,

$$K(g,h) = \varepsilon_1 (h^{-1}g)^{\pi_\Lambda} \varepsilon_1^* \quad \text{for all } g, h \in G\,,$$

cf. [ÓØ88]. Indeed,

$$\begin{aligned} c_\Lambda \cdot (\xi \mid K(g,h)\,\eta)_\Lambda &= c_\Lambda \cdot \bigl(\xi \mid \varepsilon_1 (h^{-1}g)^{\pi_\Lambda} \varepsilon_1^* \eta\bigr)_\Lambda \\ &= \bigl(\xi \mid \varepsilon_1 (h^{-1}g)^{\pi_\Lambda} S_\Lambda S_\Lambda^* \varepsilon_1^* \eta\bigr)_\Lambda = \bigl(\xi \mid S_\Lambda f_\eta(g^{-1}h)\bigr)_\Lambda \\ &= \int_K a(h^{-1}gk)^{\nu - \varrho_{\mathfrak{a}}} \bigl(\xi \mid k(h^{-1}gk)^\Lambda p_{\langle M\rangle_\Lambda} k^{-\Lambda}\eta\bigr)_\Lambda\,dk\,. \end{aligned}$$

However, this integral is difficult to compute in general, so it appears that this formula may be more useful in evaluating the right hand side once the left hand side is known. At least, the analyticity of this integral implies that H_Λ consists of real analytic functions.

5.2.21. For the subgroup $\bar{G} \sqsubset G$, we may define, analogously to S_Λ, a map

$$\bar{S}_{\bar{\Lambda}} : \bar{H}^\infty_{\bar{\nu}} = \Gamma\big(\bar{K}/\bar{M}, \bar{K} \times_{\bar{M}} \langle \bar{M} \rangle_{\bar{\Lambda}}\big) \to \Gamma\big(\bar{G}/\bar{K}, \bar{G} \times_{\bar{K}} \langle \bar{K} \rangle_{\bar{\Lambda}}\big)$$

such that $\bar{H}_{\bar{\Lambda},\bar{K}} = \bar{S}_{\bar{\Lambda}}(\bar{H}_{\bar{\nu},\bar{K}}) \cong \langle \bar{G} \rangle_{\pi_{\bar{\lambda}},\bar{K}}$. Moreover, recall that by 5.2.6,

$$\langle \bar{M} \rangle_{\bar{\Lambda}} = \bar{M}.1_\Lambda \sqsubset M.1_\Lambda = \langle M \rangle_\Lambda \ .$$

This allows us to define a map $H^\infty_\nu \to \bar{H}^\infty_{\bar{\nu}}$, as follows.

Proposition 5.2.22. The formula

$$R : H^\infty_\nu \to \bar{H}^\infty_{\bar{\nu}} : f \mapsto p_{\langle \bar{M} \rangle_{\bar{\Lambda}}} f\big|_{\bar{K}} \ .$$

defines a $\bar{G}$-equivariant map.

Proof. R is well-defined, since

$$Rf(km) = p_{\langle \bar{M} \rangle_{\bar{\Lambda}}} m^{-\bar{\Lambda}} f(k) = m^{-\Lambda} p_{\langle \bar{M} \rangle_{\bar{\Lambda}}} f(k) = m^{-\Lambda} Rf(k)$$

for all $k \in \bar{K}$ and $m \in \bar{M}$. Moreover, for $k \in \bar{K}$, $g \in \bar{G}$ and $f \in H_\nu$

$$\begin{aligned} \big(g^{\bar{\nu}} Rf\big)(k) &= \bar{a}(g^{-1}k)^{-(\bar{\nu}+\bar{\varrho}_{\bar{\mathfrak{a}}})} \cdot p_{\langle \bar{M} \rangle_{\bar{\Lambda}}} f\big(\bar{k}(g^{-1}k)\big) \\ &= p_{\langle \bar{M} \rangle_{\bar{\Lambda}}} \Big[a(g^{-1}k)^{-(\nu+\varrho_{\mathfrak{a}})} \cdot f\big(k(g^{-1}k)\big)\Big] = \big(Rg^\nu f\big)(k) \ , \end{aligned}$$

since the Iwasawa decompositions of G and $\bar{G}$ are compatible,

$$\big(\nu + \varrho_{\mathfrak{a}}\big)\big|_{\bar{\mathfrak{a}}} = \bar{\nu} + \bar{\varrho}_{\bar{\mathfrak{a}}}$$

by proposition 5.2.2, and by 5.2.6. So, R is indeed $\bar{G}$-equivariant. □

Remark 5.2.23. If H_ν were a $\bar{G}$-admissible representation, R would be automatically continuous. However, generically, this will be false. Consider, for instance, the extreme case of the trivial subgroup $\bar{G} = \bar{K} = 1$: The multiplicity of its unique representation equals the dimension of H_ν !

Theorem 5.2.24. The map

$$\bar{S}_{\bar{\Lambda}} R S^*_\Lambda : H_{\Lambda,K} = \langle G \rangle_{\pi_\lambda,K} \to \bar{H}_{\bar{\Lambda},\bar{K}} = \langle \bar{G} \rangle_{\pi_{\bar{\lambda}},\bar{K}}$$

is a $\bar{G}$-equivariant surjection. Moreover, $c_{\bar{\Lambda}}^{-1} \cdot S_{\bar{\Lambda}} R S^*_\Lambda$, where $c_{\bar{\Lambda}} = \frac{\dim \langle \bar{K} \rangle_{\bar{\Lambda}}}{\dim \langle \bar{M} \rangle_{\bar{\Lambda}}}$, restricts to an orthogonal projection on the minimal K-type.

Proof. In order to prove surjectivity, we need only ascertain that $\bar{S}_{\bar{\Lambda}} R S_{\Lambda}^* \neq 0$ (since $\bar{H}_{\bar{\Lambda},\bar{K}}$ is irreducible).

To this end, consider $f_\xi \in H_{\nu,K}$ from proposition 5.2.19 and consider, by analogy,

$$\bar{f}_\xi(k) = p_{\langle \bar{M}\rangle_{\bar{\Lambda}}} k^{-\Lambda}\xi \quad \text{for all } k \in \bar{K}\,,\, \xi \in \langle \bar{K}\rangle_{\bar{\Lambda}}\ .$$

Let $\xi \in \langle \bar{K}\rangle_{\bar{\Lambda}} \setminus 0$. Then $S_{\bar{\Lambda}} \bar{f}_\xi \neq 0$ by proposition 5.2.19. Now, for $k \in \bar{K}$,

$$R f_\xi(k) = p_{\langle \bar{M}\rangle_{\bar{\Lambda}}} p_{\langle M\rangle_\Lambda} k^{-\Lambda}\xi = p_{\langle \bar{M}\rangle_{\bar{\Lambda}}} k^{-\Lambda}\xi = \bar{f}_\xi(k)\ .$$

Hence, again by proposition 5.2.19,

$$S_{\bar{\Lambda}} R S_{\Lambda}^* \varepsilon_1^* \xi = S_{\bar{\Lambda}} R f_\xi = S_{\bar{\Lambda}} \bar{f}_\xi = S_{\bar{\Lambda}} S_{\bar{\Lambda}}^* \bar{\varepsilon}_1^* \xi = c_{\bar{\Lambda}} \cdot \bar{\varepsilon}_1^* \xi \neq 0$$

where $\bar{\varepsilon}_1$ denotes the evaluation map on $\bar{H}_{\bar{\Lambda}}$. So, $p = S_{\bar{\Lambda}} R S_{\Lambda}^*$ is a surjection. Moreover, it restricts to a map

$$p : \varepsilon_1^* \langle K\rangle_\Lambda \to \bar{\varepsilon}_1^* \langle \bar{K}\rangle_{\bar{\Lambda}}$$

which is surjective and of norm $c_{\bar{\Lambda}}$. Since, by Schur's lemma, p differs only by a constant from a projection, the assertion follows. □

Corollary 5.2.25. The $(\bar{\mathfrak{g}}, \bar{K})$-module $H_0 = \mathfrak{U}(\bar{\mathfrak{g}})\, f_\Lambda$ where $f_\Lambda = f_{1_\Lambda}$, is $\bar{K}$-finite and admissible. In particular, if q is the projection onto $H = \overline{H_0}$, and $p = c_{\bar{\Lambda}}^{-1} \cdot S_{\bar{\Lambda}} R S_{\Lambda}^*$ is the map constructed in theorem 5.2.24, then

$$j = (pq)^* : \langle \bar{G}\rangle_{\pi_{\bar{\lambda}}} \to \langle G\rangle_{\pi_\lambda}$$

is a $\bar{G}$-equivariant isometry.

Proof. Since $f_\Lambda = \varepsilon_1^*\, 1_\Lambda$ is K-finite, it is $\bar{K}$-finite, and since the $\bar{K}$-finite vectors form a $\bar{\mathfrak{g}}$-module by [War72, prop. 4.4.5.18], H_0 consists of $\bar{K}$-finite vectors. As a $\mathfrak{U}(\bar{\mathfrak{g}})$-module, H_0 is finitely generated. Moreover, its set of weights is bounded from below, since this is the case for $\langle G\rangle_{\pi_\lambda,K}$. Hence, by essentially the same proof of this fact as for Verma modules (cf. [Dix69, 7.1.6]), the $\bar{\mathfrak{t}}$-weight spaces of H_0 are finite dimensional. In particular, its $\bar{K}$-types have finite multiplicity, viz, H_0 is admissible.

Now the argument from the proof of lemma 5.2.13 applies to show that the restriction of p is bounded with $pp^* = c$ on $H = \overline{H_0}$, some positive constant $c > 0$. But $pp^* = 1$ on $\varepsilon_1^* \langle K\rangle_\Lambda$ by theorem 5.2.24, so $c = 1$. An application of q entails our claim. □

5.2.26. We conclude this section with a discussion of the case of the unit disc $B = \mathbb{B}$. As we shall see, the Knapp-Wallach Szegö kernel S_Λ turns out to be the weighted Bergman kernel. This is a general fact for the holomorphic discrete series that can be established by Jordan theoretic methods, where K/M is interpreted as the manifold of frames.

As this will be irrelevant for our further programme concerning Toeplitz operators, we shall not dwell upon this point, and content ourselves with the nonetheless revealing unit disc case.

We may consider $G = \mathrm{SU}(1,1)$ and $K = \mathrm{U}(1)$, acting by Möbius transformations. Then G is the connected double cover of $\mathrm{Aut}_0 \mathbb{B}$. The torus

$$\mathfrak{t}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} = \mathbb{R} \cdot H \,, \; H = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} .$$

The root system $\Delta = \Delta_n = \{\pm\alpha\}$ where $\alpha(H) = 2i$. Hence $H_\alpha = -iH$, and $\lambda \in i\mathfrak{t}_{\mathbb{R}}^*$ is determined by its value on H_α, also denoted by $\lambda = \lambda(H_\alpha) \in \mathbb{R}$.

$\lambda - \varrho$ is a weight if $\lambda \in \mathbb{Z}$. If, moreover, λ is dominant regular, then $\lambda \in \mathbb{N}$, $\lambda \geqslant 1$. Then $\Lambda = \lambda + \varrho - 2\varrho_c = \lambda + 1 \geqslant 2$.

Up to normalising constants,

$$E_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E_{-\alpha} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

Of course, $\alpha = \gamma_1$ is the Harish-Chandra fundamental sequence. Then

$$Y = c(H_\alpha) = E_\alpha + E_{-\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Consequently,

$$a_t = \exp(tY) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \in A \,.$$

Any $\nu \in \mathfrak{a}_{\mathbb{R}}^*$ is determined by its value on Y, also denoted by $\nu = \nu(Y) \in \mathbb{R}$. Since $\mathfrak{g}_{\mathbb{R}}$ is three-dimensional, there is exactly one positive reduced root, and it has multiplicity one. So, $\varrho_{\mathfrak{a}} = \varrho_{\mathfrak{a}}(Y) = 1$. Hence, $\nu = 1 - \Lambda \in \mathbb{Z}$, $-1 \geqslant \nu$.

We also compute the N component of the associated Iwasawa decomposition. Recall that the Cayley transform c is Ad of the element

$$\exp \frac{\pi}{4} \cdot (E_\alpha - E_{-\alpha}) = \exp \begin{pmatrix} 0 & -\frac{\pi}{4} \\ \frac{\pi}{4} & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} .$$

Hence

$$\exp c(2x \cdot E_\alpha) = \begin{pmatrix} 1 - ix & -ix \\ ix & 1 + ix \end{pmatrix} \in N \,.$$

5.2.27. The holomorphic discrete series π_λ is the Bergman space of weight Λ,

$$H_\Lambda = \mathcal{O}^2_\Lambda(\mathbb{B}) = \left\{ f \in \mathcal{O}(\mathbb{B}) \;\middle|\; \|f\|^2_{H_\Lambda} = \int_{\mathbb{B}} |f(z)|^2 \, d\mu_\Lambda(z) < \infty \right\} .$$

Here,

$$d\mu_\Lambda(z) = \frac{\Lambda-1}{\pi} \cdot (1-|z|^2)^{\Lambda-2}\, dz$$

where dz is Lebesgue measure. The space $\mathcal{O}^2_\Lambda(\mathbb{B})$ has the reproducing kernel function $B^\Lambda(z,w) = (1-\bar{z}w)^{-\Lambda}$.

The action of G on $\mathcal{O}^2_\Lambda(\mathbb{B})$ is given by

$$g^{-\pi_\Lambda} f(z) = g'(z)^{\Lambda/2} \cdot f\bigl(g(z)\bigr) \quad \text{for all } \ g\in G\,,\, z\in\mathbb{B}\,,$$

where the denominator 2 in the exponent of the cocycle is the order of the covering map $\mathrm{SU}(1,1) \to \mathrm{Aut}_0\,\mathbb{B}$.

The highest weight representation of weight Λ (in this case, simply a character of T) is given by $k^\Lambda = u^\Lambda$ for all $k = \exp(\vartheta\cdot H) = \bigl(\begin{smallmatrix}\bar{u} & 0\\ 0 & u\end{smallmatrix}\bigr)$, $u = e^{i\vartheta}$. Hence, $f\in\mathcal{O}^2_\Lambda(\mathbb{B})$ is identified with a section s of $G\times_K \langle K\rangle_\Lambda$ via

$$s(g) = g'(0)^{\Lambda/2}\cdot f\bigl(g(0)\bigr) \quad \text{for all } \ g\in G\,.$$

(Note $k'(0)^{\Lambda/2} = u^{-\Lambda}$ if $k = \bigl(\begin{smallmatrix}\bar{u} & 0\\ 0 & u\end{smallmatrix}\bigr)$.)

We demonstrated in theorem 4.2.9 that the space $H_{\Lambda,K} = \mathcal{O}^2_{\Lambda,K}$ of K-finite vectors is $\mathbb{C}[z]$. In fact, the K-type of weight $\Lambda+2k$ is spanned by the polynomial z^k.

5.2.28. The map

$$K/M \to \mathbb{T} : \bigl(\begin{smallmatrix}\bar{u} & 0\\ 0 & u\end{smallmatrix}\bigr)M \mapsto \bar{u}^2$$

is a K-equivariant isomorphism (because $M=\{\pm 1\}$). A section $s\in H_\nu$ corresponds to a function $f\in\mathbf{L}^2(\mathbb{T})$ by

$$f(\vartheta) = k^\Lambda \cdot h(k)\,,\ k = \bigl(\begin{smallmatrix}\bar{u} & 0\\ 0 & u\end{smallmatrix}\bigr)\,,\ \bar{u}^2 = \vartheta\,.$$

Let $A(z,\vartheta)\in\mathbb{R}$ be the (signed) hyperbolic distance from 0 to the horocycle through $z\in\mathbb{B}$ tangential to $\vartheta\in\mathbb{T}$. By [Hel70, ch. I, § 1], this horocycle is defined as

$$\xi(g(0),kM) = ka(g^{-1}k)^{-1}N.0 \quad \text{for all } \ g\in G\,,\, k\in K\,.$$

We find

$$\exp\bigl(-A(g(0),k)\cdot Y\bigr) = a(g^{-1}k) \quad \text{for all } \ g\in G\,,\, k\in K\,.$$

If the centre of $\xi(z,\vartheta)$ is $r\cdot\vartheta$, then the point closest to 0 is $w = (2r-1)\cdot\vartheta$. The modulus r of the centre is characterised by $|z-r\vartheta| = 1-r$, so

$$2r-1 = \frac{\mathrm{Re}(z\vartheta^{-1}) - |z|^2}{1-\mathrm{Re}(z\vartheta^{-1})} \quad \text{for all } \ z\in\mathbb{B}\,,\, \vartheta\in\mathbb{T}\,.$$

By [Hel84, Intro., § 4, (3)], the (unsigned) distance is $d(0,w) = \frac{1}{2}\cdot\log\frac{1+|w|}{1-|w|}$. Since the sign is precisely that of $\operatorname{Re}(z\vartheta^{-1}) - |z|^2$, we find

$$A(z,\vartheta) = \frac{1}{2} \cdot \log\left[\frac{1-|z|^2}{|z-\vartheta|^2}\right] .$$

In particular, if $k = \left(\begin{smallmatrix} \bar{u} & 0 \\ 0 & u \end{smallmatrix}\right)$, then

$$a(g^{-1}k)^{\nu-\varrho_\mathfrak{a}} = \left(\frac{1-|g(0)|^2}{|g(0)-\bar{u}^2|^2}\right)^{\Lambda/2} \quad \text{for all } g \in G .$$

Since

$$\begin{pmatrix} \bar{v} & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} c & s \\ s & c \end{pmatrix} \begin{pmatrix} 1-ix & -ix \\ ix & 1+ix \end{pmatrix} = \begin{pmatrix} * & * \\ v(s+i(c-s)x) & v(c+i(c-s)x) \end{pmatrix}$$

it follows that for $g = \left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{smallmatrix}\right)$, $k = \left(\begin{smallmatrix} \bar{u} & 0 \\ 0 & u \end{smallmatrix}\right)$, we have

$$k(g^{-1}k) = \begin{pmatrix} \bar{v} & 0 \\ 0 & v \end{pmatrix} \quad \text{where } v = \frac{\alpha u - \bar{\beta}\bar{u}}{|\alpha u - \bar{\beta}\bar{u}|} .$$

Hence,

$$S_\Lambda f(z) = \int_{\mathbb{T}} s_\Lambda(z,\vartheta) f(\vartheta)\, d\vartheta \quad \text{for all } f \in H_\nu,\, z \in \mathbb{B} ,$$

where (taking the trivialising factors $g'(0)^{-\Lambda/2}$ and $\vartheta^{\Lambda/2}$ into account)

$$\begin{aligned} s_\Lambda(g(0),\vartheta) &= g'(0)^{-\Lambda/2} \cdot \left(\frac{1-|g(0)|^2}{|g(0)-\vartheta|^2}\right)^{\Lambda/2} \cdot \left(\frac{\alpha\vartheta^{-1/2} - \bar{\beta}\vartheta^{1/2}}{|\alpha\vartheta^{-1/2} - \bar{\beta}\vartheta^{1/2}|}\right)^{\Lambda} \cdot \vartheta^{\Lambda/2} \\ &= \bar{\alpha}^\Lambda \cdot \left(\frac{\alpha - \bar{\beta}\vartheta}{|\bar{\alpha} - \beta\bar{\vartheta}|^2}\right)^{\Lambda} = \left(1 - \tfrac{\bar{\beta}}{\alpha} \cdot \bar{\vartheta}\right)^{-\Lambda} , \end{aligned}$$

so that $s_\Lambda(z,\vartheta) = (1 - z\bar{\vartheta})^{-\Lambda} = K^\Lambda(z,w)$. So S_Λ is just the weighted Bergman projection. Note that we could have established this result by using the formula from remark 5.2.20.

5.2.29. The only subgroup $\bar{G}$ we can consider is the trivial one, associated to the unique non-zero tripotent in $Z = \mathbb{C}$. In this case $\bar{\Lambda} = 0$, $\bar{H}_{\bar{\Lambda}} = \bar{H}_{\bar{\nu}} = \mathbb{C}$, and $S_{\bar{\Lambda}} = 1$. So R is simply evaluation at $1 \in \mathbb{T}$, and we are considering the map

$$RS_\Lambda^* : H_{\Lambda,K} = \mathbb{C}[z] \to \mathbb{C} ,$$

where

$$RS_\Lambda^* p = \int_{\mathbb{B}} (1-\bar{z})^{-\Lambda} p(z)\, d\mu_\Lambda(z) = \left(K_1^\Lambda \mid p\right) = p(1) .$$

Here, $B_w^\Lambda(z) = B^\Lambda(z,w)$. Clearly, RS_Λ^* is not continuous, but its restriction to the constants $\mathbb{C} = \mathfrak{U}(\bar{\mathfrak{g}})^{\pi_\lambda} \varepsilon_1^* \langle \bar{K} \rangle_{\bar{\Lambda}}$ is, trivially, an isometry onto $\bar{H}_{\bar{\Lambda}} = \mathbb{C}$.

5.3 An embeddability theorem for parabolic subgroups

So far, we have given an embedding of the discrete series of an embeddable subgroup $\bar{G} \sqsubset G$ into the discrete series of G. It is natural to ask whether the entire reduced unitary dual of $\bar{G}$ can be embedded in this way.

The support of the Plancherel measure for $\bar{G}$ decomposes into finitely many series of representations, each of which is associated to the conjugacy class of a cuspidal parabolic subgroup $\bar{Q} \sqsubset \bar{G}$. Here, *cuspidal* means that $\bar{M}$ has a discrete series, where $\bar{Q} = \bar{M}\bar{A}\bar{N}$. The $\bar{Q}$-series then consists of the representations $\operatorname{ind}_{\bar{Q}}^{\bar{G}}(\pi \otimes e^{\nu} \otimes 1)$ where π is a discrete series representation of $\bar{M}$. Therefore, an embedding of the reduced dual of $\bar{G}$ should be induced by an embedding of $\bar{M}$-discrete series once a sensible relationship between cuspidal parabolics $\bar{Q}$ of $\bar{G}$ and Q of G has been established.

5.3.1. We return to our setting with $G = \operatorname{Aut}_0 B$ and $\bar{G} = \operatorname{Aut}_0 \bar{B}$ where Z is a simple JB*-triple, $\bar{B} = B_0(e)$, and $e \in Z$ is a tripotent.

A subalgebra $\mathfrak{q}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ is called *parabolic*, if it is its own normaliser and contains a maximal solvable subalgebra. The normaliser $Q = N_G(\mathfrak{q}_{\mathbb{R}})$ of a parabolic subalgebra is said to be a *parabolic subgroup*. Its Lie algebra is $\mathfrak{q}_{\mathbb{R}}$.

A subalgebra $\mathfrak{s}_{\mathbb{R}}$ is *ϑ-stable* if $\mathfrak{s}_{\mathbb{R}} = \mathfrak{k}_{\mathbb{R}} \cap \mathfrak{s}_{\mathbb{R}} \oplus \mathfrak{p}_{\mathbb{R}} \cap \mathfrak{s}_{\mathbb{R}}$. A parabolic $\mathfrak{q}_{\mathbb{R}}$ containing a fixed minimal parabolic, compatible the Cartan decomposition, has a unique ϑ-stable maximal reductive subalgebra, namely $\mathfrak{s}_{\mathbb{R}} = \mathfrak{q}_{\mathbb{R}} \cap \vartheta\mathfrak{q}_{\mathbb{R}}$, called the *Levi component*. Then $\mathfrak{q}_{\mathbb{R}} = \mathfrak{s}_{\mathbb{R}} \ltimes \mathfrak{n}_{\mathbb{R}}$, as the semi-direct product of Lie algebras, where $\mathfrak{n}_{\mathbb{R}} \subset \mathfrak{q}_{\mathbb{R}}$ is the nilpotent radical.

The Levi component $\mathfrak{s}_{\mathbb{R}}$ and its centraliser $S = Z_G(\mathfrak{s}_{\mathbb{R}})$ have a canonical decomposition. Namely, let $\mathfrak{a}_{\mathbb{R}}$ be the vector part of the centre, i.e. $\mathfrak{a}_{\mathbb{R}} = \mathfrak{z}(\mathfrak{s}_{\mathbb{R}}) \cap \mathfrak{p}_{\mathbb{R}}$, and let $\mathfrak{m}_{\mathbb{R}}$ be the orthogonal complement with respect to an $\mathfrak{s}_{\mathbb{R}}$-invariant inner product.

Let $N = \exp \mathfrak{n}_{\mathbb{R}}$, $A = \exp \mathfrak{a}_{\mathbb{R}}$, and let M be the largest subgroup of S with Lie algebra $\mathfrak{m}_{\mathbb{R}}$. These are closed subgroups of G, and $S = MA$, as the direct product of Lie groups. Similarly, $Q = SN = MAN$, as the semi-direct product of $S = MA$ and N. This is called the *Langlands decomposition* of Q.

A parabolic $\mathfrak{q}_{\mathbb{R}}$ or Q is called *cuspidal* if the component $\mathfrak{m}_{\mathbb{R}}$ is an *equal-rank algebra*, i.e. $\operatorname{rk} \mathfrak{m}_{\mathbb{R}} = \operatorname{rk} \mathfrak{m}_{\mathbb{R}} \cap \mathfrak{k}_{\mathbb{R}}$.

5.3.2. Let $c \in Z$ be a tripotent. In 2.2.1, we introduced the parabolic

$$\mathfrak{q}_{\mathbb{R}}^c = \mathfrak{g}_{\mathbb{R}}^c[0,1,2] \quad \text{where} \quad \mathfrak{g}_{\mathbb{R}}^c[k] = \ker\bigl(\operatorname{ad} \zeta_c^- - k\bigr) .$$

The parabolic is maximal for c non-zero, and equals G otherwise. Let $Q_c \subset G$ denote corresponding the parabolic subgroup. Then Q_c coincides with the normaliser of the face $c + B_0(c)$, by [Loo75, thm. 9.15]. By [Loo75, prop. 9.21], the parabolics Q_c associated to non-zero tripotents $c \in E_Z \setminus 0$ exhaust the set of *maximal* proper parabolic subgroups.

In order to describe all proper parabolics, we introduce the following concept. By a *flag of tripotents* we mean a sequence

$$f = (0 < f_1 < \cdots < f_k) \quad \text{of non-zero tripotents.}$$

For such a flag $f = (f_1, \ldots, f_k)$, define

$$Q_f = Q_{f_1} \cap \cdots Q_{f_k} .$$

This is a proper parabolic subgroup, and by [Loo75, 9.22], $f \mapsto Q_f$ is a bijection from the set of non-trivial flags of tripotents onto the set of parabolic subgroups of G, manifestly K-equivariant.

5.3.3. Consider the Lie algebra of Q_f,

$$\mathfrak{q}_{\mathbb{R}}^f = \mathfrak{q}_{\mathbb{R}}^{f_1} \cap \cdots \cap \mathfrak{q}_{\mathbb{R}}^{f_k} ,$$

and its Levi component $\mathfrak{s}_{\mathbb{R}}^f = \mathfrak{q}_{\mathbb{R}}^f \cap \vartheta \mathfrak{q}_{\mathbb{R}}^f$. For a single tripotent, this is simply $\mathfrak{g}_{\mathbb{R}}^c[0]$. According to lemma 2.2.14, it decomposes as

$$\mathfrak{s}_{\mathbb{R}}^c = \mathfrak{g}_{\mathbb{R}}^c[0] = \mathfrak{g}_{0,\mathbb{R}}(c) \oplus \mathfrak{m}_{\mathbb{R}}^c \oplus \mathfrak{g}_{1,\mathbb{R}}(c) ,$$

the sum of ideals. As for the Levi component $\mathfrak{s}_{\mathbb{R}}^f$ of the parabolic $\mathfrak{q}_{\mathbb{R}}^f$ associated to a flag $0 < f_1 < \cdots < f_k$, it is clearly given by

$$\mathfrak{s}_{\mathbb{R}}^f = \mathfrak{s}_{\mathbb{R}}^{f_1} \cap \cdots \cap \mathfrak{s}_{\mathbb{R}}^{f_k} = \mathfrak{g}_{\mathbb{R}}^{f_1}[0] \cap \cdots \cap \mathfrak{g}_{\mathbb{R}}^{f_k}[0] .$$

Evidently, this intersection contains the direct sum of Lie algebras,

$$\mathfrak{g}_{0,\mathbb{R}}(f_k) \oplus \textstyle\sum_{1 \leqslant j \leqslant k}^{\oplus} \mathfrak{g}_{1,\mathbb{R}}(f_j - f_{j-1})$$

where we set $f_0 = 0$. However, the orthogonal complement of this subalgebra in $\mathfrak{s}_{\mathbb{R}}^f$ is difficult to determine.

5.3.4. Given a flag $f = \big(0 < f_1 < \cdots < f_k\big)$, there exists a frame $e_1, \ldots, e_r$ of Z and a sequence $1 \leqslant m_1 < \cdots < m_k \leqslant r$ of integers, such that

$$f_j = e_1 + \cdots + e_{m_j} \quad \text{for all } 1 \leqslant j \leqslant k .$$

Consider the maximal Abelian subspace

$$\mathfrak{a}_{\mathbb{R}} = \langle \zeta_{e_1}^-, \ldots, \zeta_{e_r}^- \rangle \subset \mathfrak{p}_{\mathbb{R}} .$$

introduced in 2.1.24. The basis $\alpha_1, \dots, \alpha_r \in \mathfrak{a}_{\mathbb{R}}^*$ dual to $\xi_{e_1}^-, \dots, \xi_{e_r}^-$, and $\alpha_0 = 0$, give rise to the linear forms $\alpha_{k\ell}^{\varepsilon} = \alpha_\ell - \varepsilon\alpha_k$, $0 \leqslant k \leqslant \ell \leqslant r$, $\varepsilon^2 = 1$. Then the positive restricted roots are exactly

$$\alpha_{k\ell}^+ \ (1 \leqslant k < \ell \leqslant r)\,,\ \alpha_{k\ell}^- \ (1 \leqslant k \leqslant \ell \leqslant r) \text{ and } \alpha_k = \alpha_{0k}^+ = \alpha_{0k}^- \ (1 \leqslant k \leqslant r\,,\ b = Z_{0k} \neq 0)\,.$$

Hence, the α_k are roots if and only if B is not of tube type. Consequently, $2\alpha_r$ is simple if and only if B is of tube type. In summary, the simple system for $\Delta_{\mathfrak{a}}^+$ is given by

$$\Pi_{\mathfrak{a}} = \begin{cases} \alpha_2 - \alpha_1, \dots, \alpha_r - \alpha_{r-1}, 2\alpha_r & B \text{ of tube type}\,, \\ \alpha_2 - \alpha_1, \dots, \alpha_r - \alpha_{r-1}, \alpha_r & B \text{ not of tube type}\,. \end{cases}$$

In the first case, the root system $\Delta_{\mathfrak{a}}$ is of type C_r, in the second, of type BC_r. We enumerate the simple roots as $\omega_1, \dots, \omega_r$.

By [Loo75, 9.20] (cf. [Kna86, prop. 5.27]), Q_f is the parabolic subgroup associated to the subset $\Pi_{\mathfrak{a}} \setminus \{\omega_{m_1}, \dots, \omega_{m_k}\}$ of $\Pi_{\mathfrak{a}}$. In particular, the Levi component decomposes as $S_f = M_f A_f$ where $A_f = \exp \mathfrak{p}_{\mathbb{R}} \cap Z(S_f)$ is given by

$$A_f = \exp \mathfrak{a}_{\mathbb{R}}^f \quad \text{with} \quad \mathfrak{a}_{\mathbb{R}}^f = \langle \xi_{e_{m_1}}^-, \dots, \xi_{e_{m_k}}^- \rangle\,,$$

cf. [Loo75, proof of prop. 9.19].

5.3.5. The structure of the Lie algebra of the component M_f of the parabolic Q_f is already quite complicated, but to gain control of Q_f, we need to understand also the degree of its disconnectedness. Fortunately, the disconnectedness of M_f is already completely determined by that of $M = Z_K(\mathfrak{a}_{\mathbb{R}})$.

For $G = \mathrm{Aut}_0 B$, it turns out that M can be described quite well. This is the content of the next proposition. In its proof, we need to know the value of the characteristic multiplicities $a = \dim Z_{ij}$ and $b = \dim Z_{0j}$ for all simple JB*-triples. We list these in the following table, cf. [Upm96, p. 60].

Invariants of the simple JB*-triples

type	$\dim Z$	r	a	b
$\mathrm{I}_{p,q}$	pq	$\min(p,q)$	2	$\max(p,q) - \min(p,q)$
II_n	$\frac{n(n-1)}{2}$	$\lfloor \frac{n}{2} \rfloor$	4	ε, $n = 2k + \varepsilon$
III_n	$\frac{n(n+1)}{2}$	n	1	0
IV_n	n	2	$n-2$	0
V	16	2	6	4
VI	27	3	8	0

Proposition 5.3.6. Consider $M = Z_K(\mathfrak{a}_\mathbb{R})$ where $\mathfrak{a}_\mathbb{R} = \langle \xi^-_{e_1}, \dots, \xi^-_{e_r} \rangle$ and $e_1, \dots, e_r$ is a frame. Let $F \subset G$ be the finite Abelian group generated by the *Peirce reflections*

$$s_{e_j} = B(e_j, 2e_j) = \exp i\pi(e_j \,\Box\, e_j^*) = \gamma_{e_j}^4 = \exp \pi\xi^+_{ie_j} \quad \text{for all } j = 1, \dots, r$$

each of which has order 2. Then

$$F \subset Z(M) \quad \text{and} \quad M = F \cdot M_0$$

where M_0 is the connected component of M.

Proof. Let $H = K' \cap \exp i\mathfrak{a}_\mathbb{R}$, where K' is the split component of K, given as the analytic subgroup of K with Lie algebra $\mathfrak{k}_\mathbb{R} \cap \left[\mathfrak{z}_{\mathfrak{g}_\mathbb{R}}(\mathfrak{t}^+_\mathbb{R}), \mathfrak{z}_{\mathfrak{g}_\mathbb{R}}(\mathfrak{t}^+_\mathbb{R})\right]$. Here, $\mathfrak{t}^+_\mathbb{R}$ is part of the $\mathbb{Z}_2$-grading of $\mathfrak{t}_\mathbb{R}$ defined in 2.1.3. Then, because G is semi-simple, connected, and contained in $G^\mathbb{C}$, by [Kna02, ch. VII, th. 4.53], $H \subset Z(M)$ is finite, Abelian, consists of elements of order two (apart from 1), and $M = HM_0$.

Moreover, consider the CSA $\mathfrak{h}_\mathbb{R} = \mathfrak{t}^+_\mathbb{R} \oplus \mathfrak{a}_\mathbb{R}$ obtained by Cayley transformation from $\mathfrak{t}_\mathbb{R}$, cf. 2.1.24. Whenever $\alpha \in \Delta(\mathfrak{h} : \mathfrak{g})$ is a real root, i.e. vanishes on $\mathfrak{t}^+_\mathbb{R}$, we can consider the elements $H_\alpha \in \mathfrak{a}_\mathbb{R}$ determined by

$$H_\alpha \in \left[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}\right] \quad \text{and} \quad \alpha\big(H_\alpha\big) = 2 \,.$$

Then we may define $\gamma_\alpha = \gamma_{-\alpha} = \exp i\pi H_\alpha$. These elements generate H, by [Kna02, ch. VII, th. 7.55]. More precisely, consider the real root α as a restricted root. Letting $m_\beta = \dim \mathfrak{g}^\beta_\mathbb{R}$ for all $\beta \in \mathfrak{a}^*_\mathbb{R}$, we have $\gamma_\alpha \in M_0$ whenever $\sum_{c \in \mathbb{Q}\,,\, c>0} m_{c\alpha} > 1$, by [Kna02, ch. VII, cor. 7.69]. Here, recall that M_0 denotes the connected component of M. In other words, M is generated by M_0 and the γ_α where $\alpha \in \Delta^+_\mathfrak{a}$ is reduced and has multiplicity one. (Recall that a root β is called *reduced* if the only roots in $\mathbb{Q} \cdot \beta$ are $\pm\beta$.)

The multiplicity of the roots $\alpha^\varepsilon_{k\ell}$ defined in 2.1.24 is

$$m_{\alpha^-_{kk}} = 1 \;(1 \leqslant k \leqslant r)\,,\; m_{\alpha^\varepsilon_{k\ell}} = a \;(1 \leqslant k < \ell \leqslant r) \quad \text{and} \quad m_{\alpha^\varepsilon_{0k}} = b \;(1 \leqslant k \leqslant r)$$

where we recall $a = \dim Z_{k\ell}$ for $1 \leqslant k < \ell \leqslant r$ and $b = \dim Z_{0k}$ for $1 \leqslant k \leqslant r$. The roots $\alpha^-_{kk} = 2\alpha_k$ are reduced if and only if $b = 0$ (i.e. B of tube type). If $b > 0$, the linear forms $\alpha_k = \alpha^\pm_{0k}$ are roots, but they are always non-reduced. The roots $\alpha^\varepsilon_{k\ell}$ for $1 \leqslant k < \ell \leqslant r$ are always reduced. As for their multiplicity, it equals a.

Seeing that in the spin factor case IV_n, we may assume $n \geqslant 5$, we have $a > 1$ unless $Z = \mathbb{C}^{n\times n}_+$, by 5.3.5. Except in this case, we have $\gamma_{\alpha^\varepsilon_{k\ell}} \in M_0$ for all $1 \leqslant k < \ell \leqslant r$, $\varepsilon^2 = 1$. Postponing the treatment of type III_n for the moment, we determine γ_α for $\alpha = 2\alpha_k$, $k = 1, \dots, r$.

Since $\alpha_k(\xi^-_{e_\ell}) = \delta_{k\ell}$, it is clear that $H_{2\alpha_k} = \xi^-_{e_k}$. But $i\xi^-_{e_k} = \xi^+_{ie_k}$, and hence

$$\gamma_{\alpha_k} = \exp \pi\xi^+_{ie_k} = \gamma^4_{ie_k} = B(ie_k, 2ie_k) = B(e_k, 2e_k) ,$$

by [Loo75, 10.1.(5)]. Moreover, [Loo75, th. 5.6] implies $B(e_k, 2e_k) = \exp i\pi(e_k \,\square\, e_k^*) \in K$. Except for type III_n, we have proved the proposition.

As for type III_n, a finite cover of G is the matrix group $\mathrm{Sp}(2n, \mathbb{R})$. A cover of K in $\mathrm{Sp}(2n, \mathbb{R})$ is given by $\mathrm{U}(n)$. We may identify $\mathfrak{a}_{\mathbb{R}}$ with the set of matrices

$$\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} \quad \text{for all } t_1, \dots, t_r \in \mathbb{R} ,$$

on which the preimage $\widetilde{M} \subset \mathrm{U}(n)$ of M acts by conjugation. The set of $u \in \mathrm{U}(n)$ leaving these diagonal matrices invariant consists of diagonal matrices with entries ± 1, and is therefore a group isomorphic to $\mathbb{Z}_2^r$. Its image in K is isomorphic to $\mathbb{Z}_2^r$ modulo diagonal, so it is generated by the Peirce reflections. This completes the proof in case Z is of type III_n, and hence, in general. □

Corollary 5.3.7. Let $f = (0 < f_1 < \cdots < f_k)$ be a flag of tripotents in Z. Let $e_1, \dots, e_r$ be a frame such that $f_j = e_1 + \cdots + e_{m_j}$ for all $j = 1, \dots, k$. Then the component M_f of the parabolic Q_f is given by

$$M_f = F_f \cdot M_{f,0} \quad \text{where} \quad F_f = \langle s_{e_j} = B(e_j, 2e_j) \mid j = 1, \dots, m_k \rangle$$

is a finite Abelian subgroup, and $M_{f,0}$ is the connected component of M_f.

Proof. If $M = Z_K(\mathfrak{a}_{\mathbb{R}})$, then $M \subset M_f$. Moreover, $M_f = M \cdot M_{f,0}$ by [Kna02, ch. VII, prop. 7.82], and $M = F \cdot M_0$ by proposition 5.3.6. But $\mathfrak{g}_{\mathbb{R},0}(f_k) \subset \mathfrak{m}^f_{\mathbb{R}}$, the Lie algebra of M_f, so $G_{f_k} \subset M_{f,0}$. Since $f_k = e_1 + \cdots + e_{m_k}$ implies $e_j \in Z_0(f_k)$ for $j > m_k$,

$$s_{e_j} = \exp i\pi(e_j \,\square\, e_j^*) \in \exp \mathfrak{k}_{\mathbb{R},0}(f_k) = K_{f_k} \subset G_{f_k} \subset M_{f,0} \quad \text{for all } j > m_k .$$

This proves the claim. □

5.3.8. Consider now a facial subgroup $\bar{G} = G_e \sqsubset G$ and the corresponding subtriple $\bar{Z} = Z_0(e) \sqsubset Z$. Take a flag $f = (0 = f_0 < f_1 < \cdots < f_k)$ of tripotents in $\bar{Z}$. f is also a flag in Z, and there exists a frame $e_1, \dots, e_r$ of Z, such that $e_1, \dots, e_{\bar{r}}$ is a frame of $\bar{Z}$ and

$$f_j = e_1 + \cdots + e_{m_j} \quad \text{for some } 1 \leqslant m_1 < \cdots < m_k \leqslant \bar{r} .$$

Let $\bar{Q}_f = \bar{M}_f \bar{A}_f \bar{N}_f$ denote the parabolic of $\bar{G}$ associated to f.

Theorem 5.3.9. Under the assumptions of 5.3.8, $\bar{Q}_f = \bar{G} \cap Q_f$. Moreover,

$$\bar{A}_f = A_f \quad \text{and} \quad \bar{N}_f \subset N_f .$$

Assume further that Z is classical. Then there are closed reductive subgroups $M' \sqsubset M_f$, $\bar{M}' \sqsubset M_f$ and a closed connected reductive subgroup $L \sqsubset K$ such that

$$M' = L \times \bar{M}' \,,\ \bar{M}_f = \bar{G}_{f_k} \times \bar{M}' \quad \text{and} \quad M_f = G_{f_k} \times M' \,.$$

In particular, $\bar{Q}_f$ is cuspidal if and only if Q_f is.

Proof. Clearly,

$$\bar{\mathfrak{a}}^f_{\mathbb{R}} = \langle \xi^-_{e_{m_1}}, \dots, \xi^-_{e_{m_k}} \rangle = \mathfrak{a}^f_{\mathbb{R}} \,,$$

so $\bar{A}_f = A_f$. The positive systems $\bar{\Delta}^+_{\bar{\mathfrak{a}}}$ and $\Delta^+_{\mathfrak{a}}$ are compatible, so $\bar{N}_f \subset N_f$. Since $\bar{\mathfrak{g}}^c_{\mathbb{R}}[0] = \bar{\mathfrak{g}}_{\mathbb{R}} \cap \mathfrak{g}^c_{\mathbb{R}}[0]$ for any tripotent $c \in \bar{Z}$, we have $\bar{\mathfrak{s}}^f_{\mathbb{R}} \subset \mathfrak{s}^f_{\mathbb{R}}$.

The Levi component S_f has the Cartan decomposition $S_f = K^f \cdot \exp\big(\mathfrak{s}^f_{\mathbb{R}} \cap \mathfrak{p}_{\mathbb{R}}\big)$ where

$$K^f = K^{f_1} \cap \dots \cap K^{f_k} = \big\{ k \in K \bigm| k(f_j) = f_j \text{ for all } j = 1, \dots, k \big\} \,,$$

and an analogous formula is valid for $\bar{S}_f$. Since $\bar{K}^f = \bar{K} \cap K^f$, we conclude $\bar{S}_f = \bar{G} \cap S_f$ and $\bar{Q}_f = \bar{G} \cap Q_f$.

Let $\mathfrak{r}^f_{\mathbb{R}}$ denote the orthogonal complement of

$$\mathfrak{k}_{\mathbb{R},0}(f_k) \oplus \textstyle\sum_{1 \leqslant j \leqslant k} \mathfrak{k}_{\mathbb{R},1}(f_j - f_{j-1})$$

in $\mathfrak{k}^f_{\mathbb{R}} = \big\{ \delta \in \mathfrak{k}_{\mathbb{R}} \bigm| \delta(f_j) = 0 \,,\ j = 1, \dots, k \big\}$. Then $\mathfrak{r}^f_{\mathbb{R}}$ is an ideal of $\mathfrak{m}^f_{\mathbb{R}}$, and since

$$\mathfrak{m}^f_{\mathbb{R}} = \mathfrak{r}^f_{\mathbb{R}} \oplus \mathfrak{g}_{\mathbb{R},0}(f_k) \oplus \textstyle\sum_{1 \leqslant j \leqslant k} \mathfrak{g}_{\mathbb{R},1}(f_j - f_{j-1})$$

is the sum of ideals, all of these factors commute. Define

$$\mathfrak{m}'_{\mathbb{R}} = \mathfrak{r}^f_{\mathbb{R}} \oplus \textstyle\sum_{1 \leqslant j \leqslant k} \mathfrak{g}_{\mathbb{R},1}(f_j - f_{j-1}) \,,$$

and let M'_0 be the analytic subgroup of G with this Lie algebra. Then M'_0 commutes with G_{f_k} and $G_{f_k} \cap M'_0 = 1$. By its mere definition, F_f commutes with G_{f_k}, so $M' = F_f \cdot M'_0$ is a subgroup of G, commuting with G_{f_k}.

Now, since $f_j - f_{j-1} \perp e$ for $1 \leqslant j \leqslant k$, $\mathfrak{g}_{\mathbb{R},1}(f_j - f_{j-1}) \subset \bar{\mathfrak{g}}_{\mathbb{R}}$. Moreover, if Z is classical, by proposition 5.3.11 below, either $\mathfrak{r}^f_{\mathbb{R}} \subset \bar{\mathfrak{g}}_{\mathbb{R}}$, or it commutes with $\bar{\mathfrak{g}}_{\mathbb{R}}$. Hence, $M'_0 = L \times \bar{M}'_0$ where $\bar{M}'_0 \subset \bar{G}$ is defined for $\bar{G}$ as M'_0 is for G, and L is a compact factor. Since $F_f \subset \bar{G}$, we may define $M' = F_f \cdot M'_0 \sqsubset G$ and $\bar{M}' = F_f \cdot \bar{M}'_0 \subset \bar{G}$. Then $M' = L \times \bar{M}'$. Since L is compact,

$$\operatorname{rk} \mathfrak{m}'_{\mathbb{R}} - \operatorname{rk}(\mathfrak{k}_{\mathbb{R}} \cap \mathfrak{m}'_{\mathbb{R}}) = \operatorname{rk} \bar{\mathfrak{m}}'_{\mathbb{R}} - \operatorname{rk}(\bar{\mathfrak{k}}_{\mathbb{R}} \cap \bar{\mathfrak{m}}'_{\mathbb{R}}) \,,$$

so M' has equal rank if and only $\bar{M}'$ does.

In particular, $M_f = G_{f_k} \times M'$ and $\bar{M}_f = \bar{G}_{f_k} \times \bar{M}'$. Finally, because G_{f_k} and $\bar{G}_{f_k}$ are

Hermitian, and therefore have equal rank, these are equal rank groups if and only if M' has equal rank, and therefore always simultaneously so. □

Remark 5.3.10. In principle, theorem 5.3.9 should be true for all JB*-triples. This is a matter of generalising the following proposition, which was essential in the theorem's proof.

Proposition 5.3.11. Assume Z is classical. If $e \perp c$ are tripotents in Z, then either

$$\mathfrak{k}_{\mathbb{R}}^{c} \cap \big(\mathfrak{k}_{\mathbb{R},0}(c)\big)^{\perp} \subset \mathfrak{k}_{\mathbb{R},0}(e)\,,$$

or $\mathfrak{k}_{\mathbb{R}}^{c} \cap \big(\mathfrak{k}_{\mathbb{R},0}(c)\big)^{\perp}$ has trivial intersection with $\mathfrak{k}_{\mathbb{R},0}(e)$ and commutes with it.

Proof. There exists a frame $e_1,\dots,e_r$ of Z such that $c = e_1+\dots+e_k$, $e = e_\ell+\dots+e_r$. Because the correspondence $e \mapsto \mathfrak{k}_{\mathbb{R},0}(e)$ is decreasing, we may w.l.o.g. restrict attention to the case $\ell = k+1$. By K-conjugacy of the assertion, it suffices to prove it for a fixed frame. Hence, it follows from propositions 5.3.13, 5.3.15, 5.3.17 and 5.3.19. □

5.3.1 Proof for type $\mathrm{I}_{p,q}$

5.3.12. The JB*-triple $Z = \mathbb{C}^{p\times q}$, $p \leqslant q$, of rank p, has triple product given by

$$\{uv^*w\} = \tfrac{1}{2}\big(uv^*w + wv^*u\big) \quad \text{for all } u,v,w \in \mathbb{C}^{p\times q}\,,$$

where v^* denotes the conjugate transpose matrix of v. A tripotent e^k of rank k is given by the formula in 5.1.10. We consider the tripotents e^k defined in 5.1.10. The Lie algebra

$$\mathfrak{k}_{\mathbb{R}} = \mathfrak{s}\big(\mathfrak{u}(p)\times\mathfrak{u}(q)\big) \cong \Big\{\big(\begin{smallmatrix}A&0\\0&D\end{smallmatrix}\big)\;\Big|\; A\in\mathfrak{u}(p)\,,\,D\in\mathfrak{u}(q)\,,\,\operatorname{tr}A+\operatorname{tr}D=0\Big\}\,,$$

every $\big(\begin{smallmatrix}A&0\\0&D\end{smallmatrix}\big) \in \mathfrak{k}_{\mathbb{R}}$ acting on Z by $z \mapsto Az + zD^t$.

Proposition 5.3.13. For $c = e^k$ and $e = e^r - e^k$, we have $\mathfrak{k}_{\mathbb{R}}^{c} \cap \big(\mathfrak{k}_{\mathbb{R},0}(c)\big)^{\perp} \subset \mathfrak{k}_{\mathbb{R},0}(e)$.

Proof. The Peirce space decomposition of Z w.r.t. c is given by

$$Z = \begin{array}{c c} & \begin{array}{cc} k & q-k \end{array} \\ \begin{array}{c} k \\ p-k \end{array} & \begin{pmatrix} Z_1(c) & Z_{1/2}(c) \\ Z_{1/2}(c) & Z_0(c) \end{pmatrix} \end{array}.$$

According to this decomposition, we write

$$A = \begin{array}{c c} & \begin{array}{cc} k & p-k \end{array} \\ \begin{array}{c} k \\ p-k \end{array} & \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array} \quad\text{and}\quad D = \begin{array}{c c} & \begin{array}{cc} k & q-k \end{array} \\ \begin{array}{c} k \\ q-k \end{array} & \begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \end{array}.$$

Then $\big(\begin{smallmatrix}A&0\\0&D\end{smallmatrix}\big) \in \mathfrak{k}_{\mathbb{R}}^{c}$ implies $\big(\begin{smallmatrix}\alpha+\alpha'^t & \gamma'^t\\ \gamma & 0\end{smallmatrix}\big) = \big(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\big)$. Hence, $\gamma = 0 = \gamma'$. Since A and D are skew-Hermitian, this implies $\beta = 0 = \beta'$. Therefore, α, α', δ and δ' are skew-Hermitian. Moreover, from

$$\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \perp \mathfrak{k}_{\mathbb{R},0}(c) = \mathfrak{s}(\mathfrak{u}(p-k) \times \mathfrak{u}(q-k)) ,$$

one deduces that $\delta = 0 = \delta'$. Thus,

$$\left(\begin{smallmatrix} \alpha & 0 \\ 0 & -\alpha^t \end{smallmatrix}\right) \in \mathfrak{k}_{\mathbb{R},1}(c) \subset \mathfrak{k}_{\mathbb{R},0}(e) = \mathfrak{s}(\mathfrak{u}(k) \times \mathfrak{u}(q-p+k)) ,$$

proving the assertion. □

5.3.2 Proof for type II_n

5.3.14. In the JB*-triple $Z = \mathbb{C}^{n\times n}_-$ of rank $r = \lfloor \frac{n}{2} \rfloor$, the Jordan triple product is given by

$$\{uv^*w\} = \tfrac{1}{2} \cdot \big(uv^*w + wv^*u\big) \quad \text{for all } u, v, w \in \mathbb{C}^{n\times n}_- = Z ,$$

as in the case of the full matrix algebra. Since, for $n = 2$, $Z \cong \mathbb{C} = \mathbb{C}^{1\times 1}$, we may assume $n \geqslant 3$. We consider a different set of tripotents than in 5.1.16. Namely, we take

$$e^j = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathbb{C}^{2j\times 2j} .$$

The Lie algebra $\mathfrak{k}_{\mathbb{R}} \cong \mathfrak{u}(n)$, where $u \in \mathfrak{u}(n)$ acts by $z \mapsto uzu^t$.

Proposition 5.3.15. For $c = e^k$ and $e = e^r - e^k$, we have $\mathfrak{k}^c_{\mathbb{R}} \cap \big(\mathfrak{k}_{\mathbb{R},0}(c)\big)^\perp \subset \mathfrak{k}_{\mathbb{R},0}(e)$.

Proof. The Peirce space decomposition is

$$Z = \begin{array}{c} \\ 2j \\ n-2j \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2j & n-2j \end{array} \\ \begin{pmatrix} Z_1(e) & Z_{1/2}(e) \\ * & Z_0(e) \end{pmatrix} \end{array} .$$

We choose a corresponding decomposition of u,

$$u = \begin{array}{c} \\ 2j \\ n-2j \end{array} \!\!\begin{array}{c} \begin{array}{cc} 2j & n-2j \end{array} \\ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \end{array} .$$

Then

$$\begin{pmatrix} \alpha J + J\alpha^t & \gamma J \\ J\gamma^t & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} ,$$

so we find $\gamma = 0 = \beta$. Thus, α and δ are skew-Hermitian, and $u \perp \mathfrak{k}_{\mathbb{R},0}(c) = \mathfrak{u}(n-2j)$ implies $\delta = 0$. Hence the assertion, since $\mathfrak{k}_{\mathbb{R},0}(e) = \mathfrak{u}(2j)$. □

5.3.3 Proof for type III_n

5.3.16. In this case, $Z = \mathbb{C}_+^{r\times r}$ has rank r, and the triple product is

$$\{uv^*w\} = \tfrac{1}{2}\big(uv^*w + wv^*u\big) \quad \text{for all } u,v,w \in \mathbb{C}_+^{r\times r} = Z\,,$$

as in the other matrix cases. A rank $1 \leqslant k \leqslant r$ tripotent e^k is given as in 5.1.22. The compact Lie algebra $\mathfrak{k}_{\mathbb{R}} \cong \mathfrak{u}(r)$, $u \in \mathfrak{u}(r)$ acting by $z \mapsto uzu^t$. By a similar proof as for the other matrix cases, we have the following proposition.

Proposition 5.3.17. For $c = e^k$ and $e = e^r - e^k$, we have $\mathfrak{k}^c_{\mathbb{R}} \cap \big(\mathfrak{k}_{\mathbb{R},0}(c)\big)^{\perp} \subset \mathfrak{k}_{\mathbb{R},0}(e)$.

5.3.4 Proof for type IV_n

5.3.18. In this case, the JB*-triple is $Z = V_n$, the complex spin factor of dimension n. There is no loss of generality in assuming $n \geqslant 5$, since in the other cases $n = 1,3,4$ in which Z is simple, Z is isomorphic to $\mathbb{C}$, $\mathbb{C}_+^{2\times 2}$, and $\mathbb{C}^{2\times 2}$, respectively.

The rank 2 triple Z is the vector space $\mathbb{C}^n$ of column vectors, with triple product

$$2\cdot\{uv^*w\} = u^t\bar{v}\cdot w + w^t\bar{v}\cdot u - u^t w\cdot\bar{v} \quad \text{for all } u,v,w \in \mathbb{C}^n = Z\,.$$

Here v^t denotes transposition and $\bar{v}$ complex conjugation.

A standard frame of tripotents is given by

$$e_1 = \big(\tfrac{1}{2},\tfrac{i}{2},0,\dots,0\big)^t \quad \text{and} \quad e_2 = \big(\tfrac{1}{2},-\tfrac{i}{2},0,\dots,0\big)^t\,.$$

The compact Lie algebra is $\mathfrak{k}_{\mathbb{R}} \cong \mathbb{R}\oplus\mathfrak{so}(n)$, where (t,u) acts by $z \mapsto (t+u)z$.

Proposition 5.3.19. For $c = e_1$ and $e = e_2$, the subalgebra $\mathfrak{k}^c_{\mathbb{R}} \cap \big(\mathfrak{k}_{\mathbb{R},0}(c)\big)^{\perp}$ commutes with $\mathfrak{k}_{\mathbb{R},0}(e)$, and has a trivial intersection with it.

Proof. The Peirce decomposition is given by

$$Z_{11} = \mathbb{C}\cdot e_1\,,\ Z_{22} = \mathbb{C} = e_2 \quad \text{and} \quad Z_{12} = 0_2 \oplus \mathbb{C}^{n-2}\,.$$

Write $u \in \mathfrak{so}(n)$ as

$$u = \begin{array}{c c} & \begin{array}{cc} 2 & n-2 \end{array} \\ \begin{array}{c} 2 \\ n-2 \end{array} & \begin{pmatrix} \alpha & \beta \\ -\beta^t & \delta \end{pmatrix} \end{array}.$$

Then $\alpha = \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right)$ for some $a \in \mathbb{R}$. Hence,

$$0 = 2\big(t\cdot e_1 + ue_1\big) = 2t\cdot(t - ia, it + a, 0,\dots,0)^t\,,$$

so $t = 0$ and $a = 0$. But then it is clear that $\mathfrak{k}_{\mathbb{R},0}(e_2)$ commutes with (t,u), since the latter is generated by $4i\,e_1 \,\square\, e_1^* = (1,v)$ where $v = \left(\begin{smallmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. □

5.4 Embedding of Q-series of facial subgroups

5.4.1. By Harish-Chandra's Plancherel theorem for the space $\mathbf{L}^2(G)$ [HC70, th. 19, cor.], [HC76, th. 27.3], the reduced unitary dual of G is exhausted by the set of representations

$$\pi^Q_{\lambda,\nu} = \operatorname{ind}^G_Q(\pi_\lambda \otimes e^{\nu+\varrho_{\mathfrak{a}}} \otimes 1)$$

induced from cuspidal parabolic subgroups $Q = M_Q A_Q N_Q$, where π_λ is a discrete series representation of M_Q (M_Q has a discrete series because Q is cuspidal) and e^ν is a unitary character of A_Q. For a fixed cuspidal parabolic Q, the set of the $\pi^Q_{\lambda,\nu}$ is called the *Q-series*. The discrete series corresponds to the case $Q = G$, if G is an equal-rank group.

More precisely, let $Q = Q_f = M_f A_f N_f$ be the Langlands decomposition of the cuspidal parabolic Q_f with $S_f = M_f A_f$ ϑ-stable. We may assume that $A_f = \exp \mathfrak{a}^f_{\mathbb{R}}$ where $\mathfrak{a}^f_{\mathbb{R}} \subset \mathfrak{a}_{\mathbb{R}} = \langle \xi^-_{e_1}, \ldots, \xi^-_{e_r} \rangle$ for some fixed frame $e_1, \ldots, e_r$. In fact, we may assume that $f_j = e_1 + \cdots + e_{m_j}$ for some $0 < m_1 < \ldots < m_k$.

The intersection $\mathfrak{t}^f_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} \cap \mathfrak{m}^f_{\mathbb{R}}$ is a compact CSA of M_f, let T_f be the corresponding Cartan subgroup (CSG). Then $H_f = T_f \times A_f$ is a ϑ-stable CSG of G. If π_λ is a discrete series representation of M (associated to a representation of $Z_M(M_0)$ and a character of T_0 subject to a certain consistency condition, cf. [Wol80, ch. III, § 22]), and $\nu \in i\mathfrak{a}^{f*}_{\mathbb{R}}$ is arbitrary, then $\pi_{\lambda,\nu}$ acts unitarily on the completion $H_{\lambda,\nu}$ of the pre-Hilbert space

$$H^\infty_{\lambda,\nu} = \left\{ h : K \to \langle M \rangle_{\pi_\lambda} \;\middle|\; h \in \mathcal{C}^\infty,\, h(km) = m^{-\pi_\lambda} h(k),\, m \in M_f \cap K,\, k \in K \right\}$$

with respect to the norm

$$\|h\|^2_{H_{\lambda,\nu}} = \int_K^* \|h(k)\|^2_{\pi_\lambda}\, dk\,.$$

The action of G on $H_{\lambda,\nu}$ is defined by

$$g^{\lambda,\nu} h(k) = a^{-(\nu+\varrho_{\mathfrak{a}})} m^{-\pi_\lambda} f(l) \quad \text{for all } h \in H^\infty_{\lambda,\nu},\, g \in G,\, k \in K\,,$$

whenever $g^{-1}k = lman$ for some $l \in K$, $m \in M_f$, $a \in A$ and $n \in N$. (The decomposition exists on a dense subset of G.) Note that although such a decomposition is not unique ($M_f \cap K$ being non-trivial), the covariance condition of h guarantees that the left hand side is well-defined.

$H_{\lambda,\nu}$ is an irreducible unitary representation, and if f runs through all flags constructed from $e_1, \ldots, e_r$, π_λ runs through the discrete series of M_f, and ν runs through $\mathfrak{a}^{f*}_{\mathbb{R}}$, then the $H_{\lambda,\nu}$ exhaust a thick subset of the support of the Plancherel measure for $\mathbf{L}^2(G)$. (However, they are not necessarily all mutually inequivalent.)

5.4.2. We have to assume that Z be classical. Then, so is $\bar{Z} = Z_0(e)$, since no exceptional Jordan triple occurs within a classical one. Take a facial subgroup $\bar{G} = G_e \sqsubset G$ and a cuspidal parabolic $\bar{Q}_f \sqsubset \bar{G}$. At least if Z is classical, theorem 5.3.9 implies that the parabolic $Q_f \sqsubset G$ is cuspidal, and that for the respective Langlands decompositions $\bar{Q}_f = \bar{M}_f \bar{A}_f \bar{N}_f$, $Q_f = M_f A_f N_f$, we have

$$\bar{A}_f = A_f \;,\; \bar{N}_f \subset N_f \;,\; \bar{M}_f = \bar{M}^0 \times M' \;,\; M = M^0 \times L \times M'$$

where $\bar{M}^0$ is a facial subgroup of M^0 and L is compact reductive. Moreover, since M^0 and $\bar{M}^0$ are facial subgroups of G and $\bar{G}$, they are also classical. Theorem 5.1.2 applies, and $\bar{M}^0$ is an embeddable subgroup of M^0 .

Choose $\bar{\nu} \in i\bar{\mathfrak{a}}^*_{\mathbb{R}} = i\mathfrak{a}^*_{\mathbb{R}}$. Define ν by

$$\nu = \bar{\nu} + \bar{\varrho}_{\bar{\mathfrak{a}}} - \varrho_{\mathfrak{a}} \;.$$

A discrete series representation of $\bar{M}$ is of the form $\pi_{\bar{\lambda}} \otimes \eta$ where η is a discrete series representation of M' . As in 5.2.5, $\bar{\lambda} = \bar{\Lambda} + 2\bar{\varrho}_c - \bar{\varrho}$ where $\bar{\Lambda} \in i\bar{\mathfrak{t}}^*$ is subject to the conditions explicated there. By theorem 5.2.24, for any extension Λ of $\bar{\Lambda}$, $\langle \bar{M}^0 \rangle_{\pi_{\bar{\lambda}}}$ embeds as a subrepresentation of $\langle M^0 \rangle_{\pi_\lambda}$. Hence so does

$$\langle \bar{M}_f \rangle_{\bar{\lambda}} = \langle \bar{M}^0 \rangle_{\pi_{\bar{\lambda}}} \otimes \langle M' \rangle_\eta \sqsubset \langle M^0 \rangle_{\pi_\lambda} \otimes \langle L \rangle_1 \otimes \langle M' \rangle_\eta = \langle M_f \rangle_\lambda$$

where 1 is the trivial representation of L and we use the shorthand $\bar{\lambda} = \pi_{\bar{\lambda}} \otimes \eta$ and $\lambda = \pi_\lambda \otimes 1 \otimes \eta$, slightly abusing notation.

Denote the $\bar{M}_f$-equivariant projection onto this subspace by

$$p_{\langle \bar{M}_f \rangle_{\bar{\lambda}}} : \langle M_f \rangle_\lambda \to \langle \bar{M}_f \rangle_{\bar{\lambda}} \;.$$

As above, let

$$\pi_{\bar{\lambda},\bar{\nu}} = \mathrm{ind}_{\bar{Q}_f}^{\bar{G}} \big((\pi_{\bar{\lambda}} \otimes \eta) \otimes e^{\bar{\nu}} \otimes 1 \big) \quad \text{and} \quad \pi_{\lambda,\nu} = \mathrm{ind}_{Q_f}^{G} \big((\pi_\lambda \otimes 1 \otimes \eta) \otimes e^{\nu} \otimes 1 \big) \,,$$

so that $\pi_{\bar{\lambda},\bar{\nu}}$ acts on $\bar{H}_{\lambda,\nu}$ and $\pi_{\lambda,\nu}$ on $H_{\lambda,\nu}$. Then, in analogy to the discrete series, we may define a map

$$R_{\lambda,\nu} : H^\infty_{\lambda,\nu} \to \bar{H}^\infty_{\bar{\lambda},\bar{\nu}} : h \mapsto p_{\langle \bar{M}_f \rangle_{\bar{\lambda}}} h\big|_{\bar{K}} \;.$$

Proposition 5.4.3. The map $R_{\lambda,\nu}$ from 5.4.2 is well-defined and $\bar{G}$-equivariant.

Proof. Indeed, for $h \in H^\infty_{\lambda,\nu}$, $R_{\lambda,\nu} h : \bar{K} \to \langle \bar{M}_f \rangle_{\bar{\lambda}}$, and is smooth. Moreover, for any $m \in \bar{M}_f \cap \bar{K} \subset M_f \cap K$, $k \in \bar{K} \subset K$,

$$\big(R_{\lambda,\nu} h\big)(km) = p_{\langle \bar{M} \rangle_{\bar{\lambda}}} m^{-\bar{\lambda}} h(k) = m^{-\lambda} p_{\langle \bar{M} \rangle_{\bar{\lambda}}} h(k) = m^{-\lambda} \big(R_{\lambda,\nu} h\big)(k) \,,$$

so $R_{\lambda,\nu}$ is well-defined. Moreover, for $g \in \bar{G}$ and $k \in \bar{K}$, write $g^{-1}k = lman$ with

$$l \in \bar{K} \subset K \;,\; m \in \bar{M} \subset M \;,\; a \in \bar{A} = A \;,\; n \in \bar{N} \subset N \;.$$

Then, by the definition of ν and the equivariance of $p_{\langle \bar{M}_f \rangle_{\bar{\lambda}}}$,

$$\begin{aligned}(g^{\bar{\lambda},\bar{\nu}} R_{\lambda,\nu} h)(k) &= a^{-(\bar{\nu}+\bar{\varrho}_{\bar{\mathfrak{a}}})} m^{\bar{\lambda}} p_{\langle \bar{M} \rangle_{\bar{\lambda}}} h(l) \\ &= p_{\langle \bar{M}_f \rangle_{\bar{\lambda}}} a^{-(\nu+\varrho_{\mathfrak{a}})} m^{\lambda} h(l) = (R_{\lambda,\nu} g^{\lambda,\nu} h)(k)\,,\end{aligned}$$

proving the equivariance and hence the proposition. □

Theorem 5.4.4. Under the assumptions of 5.4.2, $R_{\lambda,\nu}$ is non-zero when restricted to the K-finite part of $H_{\lambda,\nu}$. Hence,

$$R_{\lambda,\nu} : \langle G \rangle_{\pi_{\lambda,\nu},K} \to \langle \bar{G} \rangle_{\pi_{\bar{\lambda},\bar{\nu}},\bar{K}}$$

is a $\bar{G}$-equivariant surjection.

Proof. As in the proof of theorem 5.2.24, all we need to see is that $R_{\lambda,\nu} F \neq 0$ for some K-finite $F \in H_{\lambda,\nu}$.

The lowest K-type $\langle M^0 \cap K \rangle_\Lambda$ of $\langle M^0 \rangle_{\pi_\lambda}$ occurs multiplicity freely, and up to normalising constants, the uniquely determined embedding is given by

$$\varepsilon_1^* : \langle M^0 \cap K \rangle_\Lambda \to \langle M^0 \rangle_{\pi_\lambda}\,,$$

by corollary 5.2.17. Moreover, by 5.2.5, $\langle M^0 \cap K \rangle_\Lambda$ coincides with the cyclic space generated by the highest weight vector $1_\Lambda \in \langle K \rangle_\Lambda$ (here we extend Λ arbitrarily to a K-weight). In particular, we have an $M^0 \cap K$-equivariant projection

$$p_{\langle M^0 \cap K \rangle_\Lambda} : \langle K \rangle_\Lambda \to \langle M^0 \cap K \rangle_\Lambda\,.$$

Similarly, $\langle \bar{M}^0 \cap \bar{K} \rangle_{\bar{\Lambda}}$ can be realised in $\langle M^0 \cap K \rangle_\Lambda$.

Moreover, since any $M' \cap K$-type of $\langle M' \rangle_\eta$ is contained (not necessarily multiplicity freely) in some finite-dimensional simple K-module, there exists a K-weight δ, a non-zero irreducible $M' \cap K$-submodule $\langle M' \cap K \rangle_\delta$ of $\langle M' \rangle_\eta$, and a non-zero $M' \cap K$-equivariant projection

$$p_\delta : \langle K \rangle_\delta \to \langle M' \cap K \rangle_\delta \sqsubset \langle M' \rangle_\eta\,.$$

We fix an arbitrary non-zero vector $\xi_\delta \in \langle M' \cap K \rangle_\delta \setminus 0$.

This allows us to define, for $\xi \in \langle \bar{M}^0 \cap \bar{K} \rangle_{\bar{\Lambda}}$,

$$F_\xi(k) = (\varepsilon_1^* p_{\langle M^0 \cap K \rangle_\Lambda} k^{-\Lambda} \xi) \otimes (p_\delta k^{-\delta} \xi_\delta) \quad \text{for all } k \in K\,.$$

Then

$$F_\xi : K \to \langle M_f \rangle_\lambda = \langle M^0 \rangle_{\pi_\lambda} \otimes \langle L \rangle_1 \otimes \langle M' \rangle_\eta$$

is a smooth map. By the definition of M^0,

$$M_f \cap K = \left(M^0 \cap K\right) \times L \times \left(M' \cap K\right),$$

and we find that for $m \in M_f \cap K$,

$$\begin{aligned} F_\xi(km) &= \left(\varepsilon_1^* p_{\langle M^0 \cap K\rangle_\Lambda} m^{-\Lambda} k^{-\Lambda} \xi\right) \otimes \left(p_\delta m^{-\delta} k^{-\delta} \xi_\delta\right) \\ &= \left(m^{-\pi_\lambda} \varepsilon_1^* p_{\langle M^0 \cap K\rangle_\Lambda} k^{-\Lambda} \xi\right) \otimes \left(m^{-\eta} p_\delta k^{-\delta} \xi_\delta\right) \\ &= m^{-\pi_\lambda \otimes \eta} F_\xi(k) = m^{-\lambda} F_\xi(k), \end{aligned}$$

by the equivariance of the various maps involved. Hence $F_\xi \in H^\infty_{\lambda,\nu}$, and is K-finite by definition.

Recall from theorem 5.2.24 that there exists a constant $C > 0$ such that

$$C \cdot p_{\langle \bar{M}_f \rangle_{\bar{\lambda}}} = \bar{S}_{\bar{\Lambda}} R S_\Lambda^* \otimes 1 \otimes 1 \quad \text{on} \quad H \otimes \langle L \rangle_1 \otimes \langle M' \rangle_\eta$$

where H is the closed linear span of $\mathfrak{U}\left(\bar{\mathfrak{m}}^0\right) 1_\Lambda$. Hence, for non-zero $\xi \in \langle \bar{M}^0 \cap \bar{K}\rangle_{\bar{\Lambda}} \setminus 0$, we compute

$$\begin{aligned} C \cdot \left(R_{\lambda,\nu} F_\xi\right)(1) = C \cdot p_{\langle \bar{M}\rangle_{\pi_{\bar{\lambda}}}} F_\xi(1) &= \left(\bar{S}_{\bar{\Lambda}} R S_\Lambda^* \varepsilon_1^* \xi\right) \otimes \xi_\delta \\ &= \left(\bar{S}_{\bar{\Lambda}} R f_\xi\right) \otimes \xi_\delta = \left(\bar{S}_{\bar{\Lambda}} \bar{f}_\xi\right) \otimes \xi_\delta \neq 0, \end{aligned}$$

by the proof of theorem 5.2.24. □

In order to prove the boundedness of the map constructed in theorem 5.4.4 on some subspace of $H_{\lambda,\nu}$, we cannot use the same argument as in the proof of corollary 5.2.25, since there is no lowest K-type. Rather, in order to achieve the admissibility of the appropriately chosen subspace, we need to apply a result of Harish-Chandra on quasi-simple representations.

5.4.5. Let $R_{\lambda,\nu}$ be the map constructed in theorem 5.4.4. Chose a highest weight vector $1_{\bar{\Lambda}} \in \langle \bar{M}_f \cap \bar{K}\rangle_{\bar{\Lambda}}$, and let

$$F_{\bar{\Lambda}}(k) = p_{\langle \bar{M}_f \cap \bar{K}\rangle_{\bar{\Lambda}}} k^{-\bar{\Lambda}} 1_{\bar{\Lambda}} \quad \text{for all } k \in \bar{K}$$

be the corresponding $\bar{K}$-finite element of $H_{\bar{\lambda},\bar{\nu}}$. Let $F_\Lambda \in H_{\lambda,\nu}$ be K-finite, such that $R_{\lambda,\nu} F_\Lambda = F_{\bar{\Lambda}}$.

If F_Λ is $\mathcal{Z}(\bar{\mathfrak{g}})$-finite where $\mathfrak{U}(\bar{\mathfrak{g}})$ is the centre of $\mathfrak{U}(\bar{\mathfrak{g}})$, then consider $V = \mathcal{Z}(\bar{\mathfrak{g}})\, F_\Lambda$. Since $H_{\bar{\lambda},\bar{\nu}}$ is irreducible, $\mathcal{Z}(\bar{\mathfrak{g}})$ acts by a non-trivial character, and hence V has trivial intersection with $\ker R_{\lambda,\nu}$.

Moreover, since V is finite-dimensional, it decomposes as the direct sum of one-dimensional $\mathcal{Z}(\bar{\mathfrak{g}})$-invariant subspaces. (The algebra $\mathcal{Z}(\bar{\mathfrak{g}})$ is Abelian and involutive, so every element is normal.) Hence, we may chose a simultaneous eigenvector $v \in V$ of $\mathcal{Z}(\bar{\mathfrak{g}})$ such that $R_{\lambda,\nu}v = F_{\bar{\Lambda}}$. Denote $H_0 = \mathfrak{U}(\bar{\mathfrak{g}})\ v$, and let H be its closure in $H_{\lambda,\nu}$.

Corollary 5.4.6. Under the assumptions of 5.4.5, if $p : H_{\lambda,\nu} \to H$ denotes the orthogonal projection onto H , then $R_{\lambda,\nu}p$ is bounded, and there exists a constant $C > 0$, such that

$$C \cdot (R_{\lambda,\nu}p)^* : H_{\bar{\lambda},\bar{\nu}} \to H_{\lambda,\nu}$$

is a $\bar{G}$-equivariant isometry.

Proof. The vector v is K-finite, and a simultaneous eigenvector of $\mathcal{Z}(\bar{\mathfrak{g}})$. Hence, the latter acts by scalars on H_0 , i.e., this is a quasi-simple $(\bar{\mathfrak{g}}, \bar{K})$-module.

Thus, [HC54, lem. 33] implies that H is admissible, and the same proof as corollary 5.2.25 gives the assertion. □

Remark 5.4.7. It is plausible that the $\mathcal{Z}(\bar{\mathfrak{g}})$-finiteness assumption in 5.4.5 follows automatically from the $\mathfrak{g}$-quasi-simplicity of the irreducible G-representation $H_{\lambda,\nu}$. Indeed, this were the case if one could prove that $\mathcal{Z}(\bar{\mathfrak{g}})$ is contained in a finitely generated $\mathcal{Z}(\mathfrak{g})$-submodule of the centraliser of $\mathfrak{U}(\bar{\mathfrak{g}})$ in $\mathfrak{U}(\mathfrak{g})$.

We assume that this can be proved with the help of [War72, th. 2.1.3.6] which states that the set of $\mathrm{S}(\mathfrak{t})^{\bar{W}}$ of $\bar{W}$-invariants in the symmetric algebra $\mathrm{S}(\mathfrak{t})$ is a finitely generated $\mathrm{S}(\mathfrak{t})^{W}$-module (which contains $\mathrm{S}(\bar{\mathfrak{t}})^{\bar{W}}$). However, the incompatibility of the respective Harish-Chandra homomorphisms makes it difficult to pull this fact back to the centres $\mathcal{Z}(\bar{\mathfrak{g}})$ and $\mathcal{Z}(\mathfrak{g})$.

C*-algebras of Toeplitz operators

THE HARDY SPACE, introduced in section III, naturally gives rise to E, the Szegö projection, or, equivalently, the associated distribution. Hence, it is reasonable to define, for continuous functions f vanishing at infinity, Toeplitz operators EM_fE of symbol f. These are bounded operators on $\mathbf{L}^2(G)$ (or $\mathbf{H}^2(\Gamma)$), and one can consider the C*-algebra generated by these operators.

It is a natural and non-trivial question to ask how much of the geometry of the domain Γ, and of its boundary, is captured in the structure of this C*-algebra (compare the introduction). Extrapolating from the known case of the Hardy space on the Shilov boundary of a bounded symmetric domain, what one would expect is a lattice of ideals corresponding to the lattice of faces of the cone Ω^-. Each of the ideals should be given as the intersection of kernels of certain irreducible representations of the Toeplitz C*-algebra associated in an essential way to the subgroups $G_{f,I}$ of inner automorphisms of the faces $F^-_{f,I}$.

The simplest case of these is that f consists only of 0, so $G_{f,I} = G$. Associated to this subgroup is the identical representation on $\mathbf{H}^2(\Gamma)$.

Section 6 treats the irreducibility of this identity representation. Moreover, it develops a natural framework for Toeplitz C*-algebras which will also be useful in constructing other irreducible representations.

In 6.1, the formalism of Hopf von Neumann and Hopf C*-algebras, and their coactions, is introduced. We have attempted to be as gentle as possible here, proving some facts in detail which could be found in the literature, while at the same time being sensibly terse.

In 6.2, we introduce a notion of support affiliated with module structures over the Fourier algebra. The basic properties are well-known or minor modifications of known results. Nonetheless, we found a complete development more appropriate than an informal discussion laden with references. We apply these considerations to define coactions whose co-crossed product serve as models for Toeplitz C*-algebras. The basic idea behind this is due to Wassermann [Was84], and was developed in the generality necessary for the treatment of compact groups by Upmeier [Upm96].

A new feature of our presentation is the treatment of locally compact groups, and, more importantly, arbitrary coactions (instead of merely the 'identical' coaction on the reduced group von Neumann algebra). We feel that this makes proofs more transparent than e.g. in [AU02]. Moreover, it appears to allow for the description Toeplitz operators living on Hardy spaces for more general affine symmetric spaces than G itself. Admittedly, the operator theoretic details are quite technical, but they provide a pleasing algebraic framework for the more concrete geometry and representation theory.

In subsection 6.3, the abstract theory developed beforehand is applied to establish the irreducibility of the identical representation.

In section 7, we consider the problem of constructing other representations of the Toeplitz C*-algebra. The basic philosophy is that the compact operators correspond, upon Fourier transformation, to continuous functions vanishing at infinity. Other representations of the Toeplitz C*-algebra (with the compact operators contained in their kernel) should correspond to limits to infinity. Specifically, thinking of Pontryagin duality of locally compact Abelian groups, the representations associated to some subgroup of G should be given as limits in the 'orthogonal' directions.

This philosophy is given a precise formulation in 7.1, in terms of a 'spectral boundary condition'. We show how this gives rise to representations of the Toeplitz C*-algebra. The basic idea how use the Fourier transform to construct representations is of course contained in [Upm96]. There, it is applied in the context of compact groups. The Functional Analysis problems which occur in the general setup for locally compact unimodular type I groups are, however, not hard to overcome.

Of course, the difficulty is to verify the spectral boundary condition for our particular geometric setup. Using some ideas on supports at infinity discussed in 7.2, we reduce this problem to a support condition. This states, in general terms, that the support of a distribution associated to a limiting process in directions orthogonal to some subgroup of G should have support contained in that subgroup. In this reduction, we use Grothendieck's double limit criterion which characterises relative compactness in the topology of simple convergence.

6 Irreducibility of Toeplitz C*-algebras

In the following, denote by H a unimodular, locally compact group. We shall develop certain aspects of the theory of Toeplitz C*-algebras in an abstract framework, without specific reference to an underlying geometry. Our aim is then to apply these results to the groups $G = \mathrm{Aut}_0(B)$ and $G_{f,I}$, taking our knowledge of the harmonic analysis on their associated Ol'shanskiĭ semigroups into account.

In the beginning, the restriction to unimodular groups is inessential and mainly for convenience. In later sections, where we further assume that H be of type I, we apply the Plancherel theorem. Here the assumption of unimodularity becomes crucial. Of course, since there is a generalised Plancherel theorem due to Duflo-Moore [DM76] which also encompasses the non-unimodular case, an extension, at least in parts, to non-unimodular groups is conceivable and possibly interesting, albeit daunting.

6.1 Hopf von Neumann and Hopf C*-algebras

6.1.1. Consider the Hilbert space $\mathbf{L}^2(H)$ and the set $\mathcal{M}^b(H)$ of finite complex-valued measures on H. Any $\mu \in \mathcal{M}^b(H)$ gives rise to a bounded (left) convolution operator

$$\mu^\# : \mathbf{L}^2(H) \to \mathbf{L}^2(H) : f \mapsto \mu * f = \Big(g \mapsto \int_H f(h^{-1}g)\, d\mu(h)\Big) .$$

This defines an injection of the set $\mathcal{M}^b(H)$ into the bounded operators $\mathcal{L}(\mathbf{L}^2(H))$ on $\mathbf{L}^2(H)$. The measure μ is reconstructed from $\mu^\#$ by

$$\langle \varphi \,:\, \mu\rangle = \int_H \varphi(h)\, d\mu(h) = (\mu * \varphi^\vee)(1) \quad \text{for all } \varphi \in \mathcal{K}(H)$$

where $\mathcal{K}(H)$ denotes the set of continuous functions on H with compact support, and $\varphi^\vee(t) = \varphi(t^{-1})$ for all $t \in H$. Note that $\mu^\#$ leaves $\mathbf{L}^2(H) \cap \mathcal{C}(H)$ invariant, so the value at identity is well-defined.

The injection $\mathcal{M}^b(H) \to \mathcal{L}(\mathbf{L}^2(H))$ is an involutive algebra morphism. The content $\|\mu\| = |\mu|\,(H)$ does not define a C*-norm (unless H is trivial), so it is not an isometry.

Any element $h \in H$ gives rise to a Dirac measure $\delta_h \in \mathcal{M}^b(H)$, so we may think of H as a subset of $\mathcal{M}^b(H)$, and hence of $\mathcal{L}(\mathbf{L}^2(H))$. The linear span of H in $\mathcal{M}^b(H)$ is the set of finitely supported measures.

Let $\mathrm{W}^*(H) = H''$ be the von Neumann algebra generated by H. It is called the *reduced group von Neumann algebra*. Since $\mathcal{M}^b(H)$ commutes with H', as follows from the formula

$$\mu^\# = \int_H h^\#\, d\mu(h) \quad \text{for all } \mu \in \mathcal{M}^b(H)$$

and standard facts of vector-valued integration, $\mathcal{M}^b(H) \subset \mathrm{W}^*(H)$. Therefore, $\mathrm{W}^*(H)$ is the weak, strong, ultraweak, and ultrastrong closure of $\mathcal{M}^b(H)$ in $\mathcal{L}(\mathbf{L}^2(H))$, by the von Neumann density theorem. In particular, the set of finitely supported measures is weakly dense in $\mathcal{M}^b(H)$.

Similarly, $\mathrm{W}^*(H)$ is the weak, strong, ultraweak, and ultrastrong closure of the set $\mathbf{L}^1(H) \subset \mathcal{M}^b(H)$ of integrable complex-valued functions on H, since Dirac nets define (bounded) approximate units for $\mathbf{L}^1(H)$.

Denote

$$\mathrm{C}^*_\#(H) = \mathrm{C}^*\langle f^\# \mid f \in \mathbf{L}^1(H)\rangle \subset \mathcal{L}(\mathbf{L}^2(H))$$

where we recall $f^\# g = f * g$. The C*-algebra $\mathrm{C}^*_\#(H)$ is called the *(reduced) group C*-algebra of H*. The above considerations show that $\mathrm{C}^*_\#(H) \subset \mathrm{W}^*(H)$, and that $\mathrm{W}^*(H)$ is the von Neumann algebra generated by $\mathrm{C}^*_\#(H)$.

Remark 6.1.2. It is easy to see that $\mathrm{W}^*(H)$ is the commutant of the set of right convolutions $h_\#$, $h \in H$. If H is a Lie group, then $x \in \mathrm{W}^*(H)$ defines a continuous linear map $x : \mathcal{D}(H) \to \mathcal{D}'(H)$ from the smooth compactly supported functions on H to the set of distributions on H. Hence, by the Schwartz kernels theorem, it is given on $\mathcal{D}(H)$ by a kernel operator with distribution kernel $\xi \in \mathcal{D}'(H \times H)$. Since x commutes with right translations, $\xi(s,t) = \mu(t^{-1}s)$ for some $\mu \in \mathcal{D}'(H)$. Hence, x is given on $\mathcal{D}(H)$ by left convolution with a distribution.

This construction can be extended to general H by applying Bruhat's generalised distributions which are defined via projective limits of Lie groups, cf. [Eym64, prop. 27]. However, it seems to be more natural to introduce a new set of test functions for $\mathrm{W}^*(H)$ (the Fourier algebra), which we do below.

Remark 6.1.3. If H is Abelian, then $\mathrm{W}^*(H) \cong \mathbf{L}^\infty(\hat{H})$ as von Neumann algebras, where $\hat{H}$ is the dual group, and the isomorphism is given by conjugation with the Fourier transform. Under this isomorphism, $\mathrm{C}^*_\#(H) \cong \mathcal{C}_0(\hat{H})$. (This is essentially the Riemann-Lebesgue lemma.)

In the Abelian case, the algebra product of $\mathrm{W}^*(H)$ and $\mathrm{C}^*_\#(H)$, namely, convolution, is related via Fourier transform to the point-wise product on $\mathbf{L}^\infty(H)$ and $\mathcal{C}_0(H)$. The starting point of non-commutative group duality is to ask whether there exists on $\mathrm{W}^*(H)$ and $\mathrm{C}^*_\#(H)$ some algebraic structure, a 'coproduct', which induces on a suitable 'dual' of these algebras a (commutative) algebra product, without reference to the Fourier transform. After some preliminaries, we describe the relevant structures for $\mathrm{W}^*(H)$ and $\mathrm{C}^*_\#(H)$.

6.1.4. Recall that for a C^*-algebra A, the *multiplier algebra* $\mathrm{M}(A)$ is given by

$$\mathrm{M}(A) = \{a \in \mathcal{L}(\mathcal{H}) \mid aA \cup Aa \subset A\}$$

where A acts faithfully and non-degenerately on the Hilbert space $\mathcal{H}$. Recall that a normed A-module E is called *non-degenerate*, if the linear span of $A \cdot E$ is dense in E.

The definition of $\mathrm{M}(A)$ is independent of $\mathcal{H}$. Abstractly, $\mathrm{M}(A)$ can be introduced via double centralisers, cf. [Bus68].

$\mathrm{M}(A)$ is a unital C^*-algebra, and acts non-degenerately on $\mathcal{H}$. It is the largest closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$ containing A as an essential ideal. Here, the requirement that an ideal $J \lhd \mathrm{M}(A)$ be *essential* means that $I \cap J \neq 0$ whenever $I \lhd \mathrm{M}(A)$ is non-zero. Finally, we have $\mathrm{M}(A) \subset A'' \subset \mathcal{L}(\mathcal{H})$.

The natural topology to consider on $\mathrm{M}(A)$ is not the norm topology, but the *strict* topology (which is weaker). It is the weakest locally convex topology such that the multiplication maps

$$\mathrm{M}(A) \times A \to A : (a,b) \mapsto ab \quad \text{and} \quad A \times \mathrm{M}(A) \to A : (a,b) \mapsto ab$$

are *separately* continuous in the norm topologies. Any element of $\mathrm{M}(A)$ is the strict limit of a bounded net in A, and $\mathrm{M}(A)$ is strictly complete. Approximate units of A are just the nets in A converging strictly to $1 \in \mathrm{M}(A)$, cf. [Bus68, prop. 3.5-6].

6.1.5. Since A is strictly dense in $\mathrm{M}(A)$, it is natural to ask for classes of linear maps on A which extend to $\mathrm{M}(A)$. If B is another C*-algebra and $\alpha : A \to \mathrm{M}(B)$ is an involutive algebra morphism, then α extends to a unital involutive algebra morphism which is strictly continuous *on bounded subsets* of $\mathrm{M}(A)$ if and only if

$$\alpha(u_\lambda) \to 1 \quad \text{strictly in } \mathrm{M}(B) \text{ for all approximate units } u_\lambda \in A\,,$$

by [Val85, lem. 0.2.5]. Such morphisms are called *strict*, and their extensions to $\mathrm{M}(A)$ are denoted by the same letter. It follows from the non-degeneracy of the action of A on $\mathcal{H}$ that if α is injective on A, then so is its extension to $\mathrm{M}(A)$.

Clearly, any continuous linear functional $\mu \in A^*$ has a strictly continuous extension μ to $\mathrm{M}(A)$. Moreover, by [Tay70, cor. 2.2], the set of strictly continuous linear functionals on $\mathrm{M}(A)$ is $A \cdot \mathrm{M}(A)^* = \mathrm{M}(A)^* \cdot A$. Here, $\mathrm{M}(A)^*$ is the set of norm continuous functionals, which is an (A, A)-bimodule via

$$\langle c : a \cdot \mu \cdot b \rangle = \langle bca : \mu \rangle \quad \text{for all } a, b, c \in A\,,\ \mu \in \mathrm{M}(A)^*\,.$$

6.1.6. Let us return to our setting with the group H. The C*-algebra $\mathrm{C}^*_\#(H)$ acts non-degenerately on $\mathbf{L}^2(H)$, so $\mathrm{M}(\mathrm{C}^*_\#(H)) \subset \mathrm{C}^*_\#(H)'' = \mathrm{W}^*(H)$.

For $s \in H$, the left convolution (translation) operator $s^\# \in \mathrm{W}^*(H)$, and $\mathrm{C}^*_\#(H)$ is bi-invariant under multiplication with $s^\#$. Hence $s^\# \in \mathrm{M}(\mathrm{C}^*_\#(H))$, leading us to consider the map

$$W_H : H \to \mathrm{M}(\mathrm{C}^*_\#(H)) : s \mapsto s^\#\,.$$

It is bounded and strictly continuous, so

$$W_H \in \mathcal{C}^b(H, \mathrm{M}(\mathrm{C}^*_\#(H))) = \mathrm{M}(\mathcal{C}_0(H) \otimes \mathrm{C}^*_\#(H))$$

where $\otimes$ is the spatial tensor product, by [APT73, cor. 3.4].

The latter identity is induced by the usual isomorphism

$$\mathbf{L}^2(H, \mathbf{L}^2(H)) \to \mathbf{L}^2(H \times H) : f \mapsto \big((s,t) \mapsto f(s)(t)\big)\,.$$

Consequently, the multiplier W_H acts on $\mathbf{L}^2(H \times H)$ by

$$(W_H f)(s,t) = f(s, s^{-1}t) \quad \text{for all } f \in \mathbf{L}^2(H \times H)\,,\ s,t \in H\,.$$

Hence, it is a unitary operator, sometimes referred to as the *Kac-Takesaki fundamental unitary*. Observe that $W = W_H$ obeys the *pentagonal equation*

$$W_{12} W_{13} W_{23} = W_{23} W_{12}$$

where we use the *leg notation* $W_{12} = W \otimes 1$, $W_{23} = 1 \otimes W$, and

$$W_{13}(f \otimes g \otimes h) = \sum_i f_i \otimes g \otimes h_i \quad \text{if} \quad W(f \otimes h) = \sum_i f_i \otimes h_i \,,$$

cf. [BS93].

If we set

$$\delta_H(x) = \mathrm{Ad}(W)(x \otimes 1) = W(x \otimes 1)W^* \quad \text{for all} \quad x \in \mathrm{W}^*(H)\,,$$

then $\delta = \delta_H$ is *coassociative*, i.e.

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta\,,$$

as follows from the pentagonal equation. Note that

$$\delta = \delta_H : \mathrm{W}^*(H) \to \mathrm{W}^*(H) \bar{\otimes} \mathrm{W}^*(H) = \mathrm{W}^*(H \times H)$$

where $\bar{\otimes}$ denotes the tensor product of von Neumann algebras. Moreover, δ is a normal morphism. It is called the *coproduct* of $\mathrm{W}^*(H)$. A von Neumann algebra M with an injective coassociative normal involutive algebra morphism $\delta : M \to M \bar{\otimes} M$ is called a *Hopf von Neumann algebra*.

Define a unitary operator $V = V_H$ by

$$(Vf)(s,t) = f(ts,t) \quad \text{for all} \ \ f \in \mathbf{L}^2(H \times H)\,,\ s,t \in H\,.$$

Then V satisfies the pentagonal equation, and

$$d_H f = \mathrm{Ad}(V)(f \otimes 1) = V(f \otimes 1)V^* \quad \text{for all} \ \ f \in \mathbf{L}^\infty(H)$$

defines a normal involutive morphism

$$d = d_H : \mathbf{L}^\infty(H) \to \mathbf{L}^\infty(H) \bar{\otimes} \mathbf{L}^\infty(H) = \mathbf{L}^\infty(H \times H)$$

making $\mathbf{L}^\infty(H)$ into a Hopf von Neumann algebra. It is interesting to note the formula

$$df(s,t) = f(st) \quad \text{for all} \ \ f \in \mathbf{L}^\infty(H)\,.$$

6.1.7. Given C*-algebras A and B, an involutive morphism $\alpha : A \to \mathrm{M}(B)$ is called *non-degenerate*, if $\alpha(A)$ acts non-degenerately on B. If α is non-degenerate, it follows from [LPRS87, lemma 1.1] that α is strict.

Consider the following subalgebra of $\mathrm{M}(A \otimes B)$,

$$\overleftarrow{\mathrm{M}}(A,B) = \big\{ x \in \mathrm{M}(A \otimes B) \ \big|\ x\,(\mathbb{C} \otimes B) \cup (\mathbb{C} \otimes B)\,x \subset A \otimes B \big\}\,.$$

Then, given an injective non-degenerate $*$-morphism $\delta : A \to \overleftarrow{\mathrm{M}}(A, A)$ such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta \,,$$

(A, δ) is called a *Hopf C*-algebra*.

Remark 6.1.8. From an algebraic perspective, to call a von Neumann algebra M with a coproduct δ, or the corresponding C*-structure, a *Hopf algebra*, is not completely appropriate. Classically, the existence of a counit ε and an antipode S is what distinguishes Hopf algebras from mere bialgebras, cf. [Swe69, MM65]. For this, one needs additional structure.

Most often, one assumes the existence of (the non-commutative analogue of) Haar measure, cf. [ES92]. Such bialgebras are called *Kac algebras*, since they were first studied independently by Kac-Vainerman and Enock-Schwartz. Under weaker invariance conditions, the term *locally compact quantum group* has been coined, [KV00, KV03].

Another strategy is to start with a unitary implementation of the coproduct, this was initiated in [BS93].

Finally, let us note that Vaes-Van Daele [VD01] have, using the so-called Haagerup tensor product, given topological conditions on C*-coproducts that guarantee the existence of a (densely defined) counit and antipode. It should be noted that Effros-Ruan [ER03] place the theory of Hopf von Neumann algebras in a similar framework, the category of operator spaces (also using the Haagerup tensor product).

We refer to the introductions of [VD01, KV00] for more thorough discussions of the different approaches to Hopf operator algebras.

The following proposition is well-known, and will in fact follow from our more general proposition 6.2.10 below. We give a proof, because the (standard) technique used therein will reappear presently, in a more abstract guise.

Proposition 6.1.9. With the coproduct $\delta = \delta_H$ induced from $\mathrm{W}^*(H)$, the reduced group C*-algebra $\mathrm{C}^*_\#(H)$ is a Hopf C*-algebra.

Proof. Note that the spatial tensor product $\mathrm{C}^*_\#(H) \otimes \mathrm{C}^*_\#(H) = \mathrm{C}^*_\#(H \times H)$, as follows immediately from the equality $\mathbf{L}^2(H) \otimes \mathbf{L}^2(H) = \mathbf{L}^2(H \times H)$.

For $r \in H$, $\delta(r^\#) = r^\# \otimes r^\#$. Hence,

$$\delta(\varphi^\#) = \int_H \varphi(r) \cdot r^\# \otimes r^\# \, dr \quad \text{for all} \;\; \varphi \in \mathcal{K}(H) \,.$$

For all $\psi, \chi \in \mathcal{K}(H \times H)$, we get

$$\delta(\varphi^\#)\psi^\# \chi(u, v) = \int_H \int_H \int_H \chi(s^{-1}u, t^{-1}v)\psi(rs, rt)\varphi(r) \, dr \, ds \, dt = \zeta^\# \chi(u, v)$$

where $\zeta \in \mathcal{K}(H \times H)$ is given by

$$\zeta(s,t) = \int_H \psi(rs,rt)\varphi(r)\,dr \quad \text{for all } s,t \in H\,.$$

Hence, $\delta(\varphi^{\#})\psi^{\#} \in \mathrm{C}^*_{\#}(H)$. Similarly for $\psi^{\#}\delta(\varphi^{\#})$, so $\delta(\varphi^{\#}) \in \mathrm{M}(\mathrm{C}^*_{\#}(H \times H))$.

Using Dirac sequences, we can replace $\psi \in \mathcal{K}(H \times H)$ in the above integral by $\delta_1 \otimes \psi$ where $\psi \in \mathcal{K}(H)$ and δ_1 denotes the Dirac measure at unity. Then we have

$$\delta(\varphi^{\#})(1 \otimes \psi^{\#})\chi(u,v) = \int_H \int_H \chi(s^{-1}u, t^{-1}v)\psi(s^{-1}t)\varphi(s)\,ds\,dt = \zeta^{\#}\chi(u,v)$$

where $\zeta \in \mathcal{K}(H \times H)$ is given by

$$\zeta(s,t) = \psi(s^{-1}t)\varphi(s) \quad \text{for all } s,t \in H\,.$$

This shows that $\delta(\varphi^{\#}) \in \overleftarrow{\mathrm{M}}(\mathrm{C}^*_{\#}(H), \mathrm{C}^*_{\#}(H))$.

If we take a Dirac net $\varphi_\lambda \in \mathcal{K}(H)$, then

$$\int_H \psi(rs,rt)\varphi_\lambda(r)\,dr \to \psi(s,t) \quad \text{for all } \psi \in \mathcal{K}(H \times H)\,,$$

uniformly on $H \times H$. Hence, $\delta(\varphi_\lambda^{\#})\psi^{\#} \to \psi^{\#}$ in the norm topology. This shows that δ is non-degenerate. Moreover, δ is injective since W is unitary. Since

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \delta) \circ \delta\,,$$

our contention is proven. □

Remark 6.1.10. One can prove easily that, as for $(\mathrm{C}^*_{\#}(H), \delta)$, the restriction of $d = d_H$ to $\mathcal{C}_0(H)$ turns this C*-algebra into a Hopf C*-algebra.

Note that for $\mu, \nu \in \mathcal{M}^b(H)$, we have

$$\langle df \,:\, \mu \otimes \nu \rangle = \int_H \int_H f(st)\,d\mu(s)\,d\nu(t) = \langle f \,:\, \mu * \nu \rangle \quad \text{for all } f \in \mathcal{C}_0(H)\,.$$

So, convolution, which defines the algebra products of $\mathrm{W}^*(H)$ and $\mathrm{C}^*_{\#}(H)$, is in some sense dual to the coproduct $d = d_H$. It is therefore natural to ask which product is dual to the coproduct $\delta = \delta_H$.

6.1.11. Define $\mathrm{B}_{\#}(H) = \mathrm{C}^*_{\#}(H)^*$, the set of norm continuous functionals, endowed with the dual norm. $\mathrm{B}_{\#}(H)$ is called the *(reduced) Fourier-Stieltjes algebra of H*. (Enock and Schwartz [ES92] call it the *Eymard algebra*.)

We have a contractive map $\mathbf{L}^1(H) \to \mathrm{C}^*_{\#}(H)$ with dense image. By duality, we get a contractive injection $\mathrm{B}_{\#}(H) \to \mathbf{L}^\infty(H)$.

Hence, for any $\varphi \in \mathrm{B}_\#(H)$, there exists a (unique) $\psi \in \mathbf{L}^\infty(H)$ such that

$$\|\varphi\|_{\mathrm{B}_\#(H)} = \sup_{f\in\mathbf{L}^1(H)\,,\,\|f^\#\|\leqslant 1}\left|\int_H f(s)\psi(s)\,ds\right| < \infty$$

and

$$\langle f^\# : \varphi\rangle = \int_H f(s)\psi(s)\,ds \text{ for all } f \in \mathbf{L}^1(H)\,.$$

In fact, $\psi \in \mathcal{C}^b(H)$, cf. [Eym64, (1.19), prop. 2.1].

Hence, $\mathrm{B}_\#(H)$ is identified with a Banach space of bounded continuous functions. It consists of coefficient functions of the unitary representations π of H weakly contained in the regular representation $\sqcup^\#$ of H on $\mathbf{L}^2(H)$. Here, π is *weakly contained* in ϱ if

$$\|f^\pi\| \leqslant \|f^\varrho\| \quad \text{for all } f \in \mathbf{L}^1(H)$$

where as usual $f^\pi = \int_H f(s)s^\pi\,ds$.

Proposition 6.1.12. For all $\varphi, \psi \in \mathrm{B}_\#(H)$, define $\varphi\cdot\psi \in \mathrm{B}_\#(H)$ by

$$\langle a : \varphi\cdot\psi\rangle = \langle\delta_H(a) : \varphi\otimes\psi\rangle \quad \text{for all } a \in \mathrm{C}^*_\#(H)\,.$$

This defines a commutative, associative product, identical to point-wise multiplication of functions. Hence, $\mathrm{B}_\#(H)$ is a commutative Banach algebra.

Proof. Since $\varphi\otimes\psi$ extends to the multiplier algebra $\mathrm{M}(\mathrm{C}^*_\#(H\times H))$, the definition makes sense. Moreover, whenever $a_\lambda \to a$ in $\mathrm{C}^*_\#(H)$, for the norm topology, a_λ is a bounded strictly convergent net. Hence $\delta(a_\lambda) \to a$ strictly, and this net is bounded. Therefore, $\varphi\cdot\psi$ is norm continuous on $\mathrm{C}^*_\#(H)$, and defines an element of $\mathrm{B}_\#(H)$. The coassociativity of $\delta = \delta_H$ implies that $\cdot$ is associative.

For the commutativity, consider the transposition $\sigma = (12) \in \mathfrak{S}_2$ and its natural action on $\mathbf{L}^2(H)\otimes\mathbf{L}^2(H)$. For any operator T on $\mathbf{L}^2(H\times H)$, define $T^\sigma = \sigma T\sigma$.

Let $\mu \in \mathcal{M}^b(H)$, and consider the image measure $\Delta_*(\mu) \in \mathcal{M}^b(H\times H)$ under the diagonal map $\Delta(s) = (s,s)$. By [Val85, lem. 4.1], $\delta(\mu^\#) = \Delta_*(\mu)^\#$. We find

$$\left.\begin{aligned}\delta(\mu^\#)^\sigma\varphi(s,t) &= \int_H(\sigma\varphi)(r^{-1}t, r^{-1}s)\,d\mu(r)\\ &= \int_H\varphi(r^{-1}s, r^{-1}t)\,d\mu(r) = \delta(\mu^\#)\varphi(s,t)\end{aligned}\right.\quad\text{for all } \varphi\in\mathcal{K}(H\times H)\,.$$

In particular, this is true for $\mu = f \in \mathbf{L}^1(H)$. By density of $\mathbf{L}^1(H) \subset \mathrm{C}^*_\#(H)$, $\delta(\sqcup)^\sigma = \delta$. This implies that $\cdot$ is commutative.

For $f \in \mathbf{L}^1(H)$, we have

$$\begin{aligned}\int_H f(s)(\varphi \cdot \psi)(s)\,ds &= \langle f^\# : \varphi \cdot \psi \rangle = \langle \delta(f^\#) : \varphi \otimes \psi \rangle \\ &= \int_H (\varphi \otimes \psi)(s,s) \cdot f(s)\,ds = \int_H \varphi(s) \cdot \psi(s) \cdot f(s)\,ds\end{aligned}$$

because $\mathrm{C}^*_\#(H \times H) = \mathrm{C}^*_\#(H) \otimes \mathrm{C}^*_\#(H)$. Hence, the product $\cdot$ is just point-wise multiplication.

The product on $\mathrm{B}_\#(H)$ is norm-contractive for $\|\sqcup\|_{\mathrm{B}_\#(H)}$ by definition. This implies that $\mathrm{B}_\#(H)$ is a Banach algebra. □

6.1.13. Recall that for a von Neumann algebra M, the *predual* M_* is the set of ultraweakly continuous linear functionals on M. This is a closed subspace of the Banach dual M^*, and M is its Banach space dual. The $\sigma(M, M_*)$-topology coincides with the ultraweak topology on M.

Let $\mathrm{A}(H) = \mathrm{W}^*(H)_*$, called the *Fourier algebra*. Since $\mathrm{C}^*_\#(H)$ is $\sigma(\mathrm{W}^*(H), \mathrm{A}(H))$ dense in $\mathrm{W}^*(H)$, the natural map $\mathrm{A}(H) \to \mathrm{B}_\#(H)$ is an isometry. Hence, we may think of $\mathrm{A}(H)$ as a Banach space of bounded continuous functions.

Since $\delta = \delta_H : \mathrm{W}^*(H) \to \mathrm{W}^*(H \times H)$ is normal, it is ultraweakly continuous, and we see that $\mathrm{A}(H)$ is invariant for the product of $\mathrm{B}_\#(H)$. The Fourier algebra is a commutative Banach algebra of functions on H.

Both the reduced Fourier-Stieltjes algebra and the Fourier algebra are invariant for the usual involution

$$\varphi^*(t) = \overline{\varphi(t^{-1})} \quad \text{for all } \varphi : H \to \mathbb{C}\,,\ t \in H\,.$$

Namely, for $\varphi \in \mathcal{C}^b(H)$ and $f \in \mathbf{L}^1(H)$, we compute

$$\langle \varphi^* : f \rangle = \int_H \overline{\varphi(t^{-1})} \cdot f(t)\,dt = \int_H \overline{\varphi(t)} \cdot f(t^{-1})\,dt = \overline{\langle \varphi : f^* \rangle}$$

by unimodularity. (For non-unimodular groups, the involution has to be defined differently.) Since $(f^*)^\# = (f^\#)^*$, the involution leaves $\mathrm{B}_\#(H)$ and $\mathrm{A}(H)$ invariant, and is an isometry. Hence, both are commutative Banach $*$-algebras.

The Fourier algebra can be described as

$$\mathrm{A}(H) = \left\{ \bar{\xi} * \eta^\vee \mid \xi, \eta \in \mathbf{L}^2(H) \right\}\,,$$

by [Eym64, th. 3.25]. I.e., $\mathrm{A}(H)$ is the space of coefficient functions of the regular representation: Observe that

$$\bar{\xi} * \eta^\vee(t) = \int_H \eta(t^{-1}s)\overline{\xi(s)}\,ds = (\xi \mid t^\# \eta) \quad \text{for all } \xi, \eta \in \mathbf{L}^2(H)\,,\ t \in H\,.$$

By [Eym64, lem. 3.1],

$$\|\bar{\xi} * \eta^{\vee}\|_{\mathrm{A}(H)} \leqslant \|\xi\|_2 \cdot \|\eta\|_2 \quad \text{for all } \xi, \eta \in \mathbf{L}^2(H)\,.$$

Hence the span of all

$$\varphi * \varphi^*\,, \quad \varphi \in \mathcal{K}(H)$$

and, in particular, $\mathcal{K}\mathrm{A}(H) = \mathrm{A}(H) \cap \mathcal{K}(H)$ is dense in $\mathrm{A}(H)$. Since we have the norm inequality $\|\sqcup\|_\infty \leqslant \|\sqcup\|_{\mathrm{A}(H)}$, we conclude $\mathrm{A}(H) \subset \mathcal{C}_0(H)$.

6.1.14. An $\mathrm{A}(H)$-module structure on $\mathrm{W}^*(H)$ can be defined by

$$\langle \alpha' : \alpha \cdot x \rangle = \langle \alpha \cdot \alpha' : x \rangle \quad \text{for all } \alpha, \alpha' \in \mathrm{A}(H)\,,\ x \in \mathrm{W}^*(H)\,.$$

Since $\mathrm{A}(H) \triangleleft \mathrm{B}_\#(H)$, this even makes sense for $\alpha \in \mathrm{B}_\#(H)$.

Since $\alpha \cdot \alpha' = (\alpha \otimes \alpha') \circ \delta_H$, the formula for the $\mathrm{A}(H)$-action amounts to

$$\alpha \cdot x = (\mathrm{id} \otimes \alpha)(\delta(x)) \quad \text{for all } \alpha \in \mathrm{A}(H)\,,\ x \in \mathrm{W}^*(H)\,.$$

This formula shows that

$$\|\alpha \cdot x\| \leqslant \|\alpha\|_{\mathrm{A}(H)} \cdot \|x\| \quad \text{for all } \alpha \in \mathrm{A}(H)\,,\ x \in \mathrm{W}^*(H)\,.$$

If $f \in \mathbf{L}^1(H)$, we have for all $\alpha, \alpha' \in \mathrm{A}(H)$

$$\langle \alpha' : \alpha \cdot f^\# \rangle = \langle \alpha \cdot \alpha' : f \rangle = \int_H \alpha(s) \cdot \alpha'(s) \cdot f(s)\, ds = \langle \alpha' : \alpha \cdot f \rangle\,.$$

Hence, the module structure on $\mathrm{W}^*(H)$ extends the usual multiplication of functions.

Moreover, since $\alpha \cdot f \in \mathbf{L}^1(H)$, this shows that $\mathrm{C}^*_\#(H)$ is $\mathrm{A}(H)$-invariant. In fact, the formula

$$\beta \cdot a = (\mathrm{id} \otimes \beta)(\delta(a)) \quad \text{for all } \beta \in \mathrm{B}_\#(H)\,,\ a \in \mathrm{C}^*_\#(H)$$

defines a compatible $\mathrm{B}_\#(H)$-module structure on $\mathrm{C}^*_\#(H)$.

Remark 6.1.15. Let $\mathrm{C}^*(H)$ be the universal enveloping C*-algebra of $\mathbf{L}^1(H)$. It is called the *universal group C*-algebra of H*. It has a coproduct dual to the multiplication on $\mathbf{L}^\infty(H)$, defined in terms of the minimal tensor product $\mathrm{C}^*(H) \otimes_\iota \mathrm{C}^*(H) = \mathrm{C}^*(H \times H)$, rather than the spatial tensor product. Its dual $\mathrm{B}(H) = \mathrm{C}^*(H)^*$ is commutative Banach $*$-algebra of bounded continuous functions containing $\mathrm{A}(H)$ and $\mathrm{B}_\#(H)$ as closed ideals. In particular, $\mathrm{W}^*(H)$ is a $\mathrm{B}(H)$-module in a compatible way.

6.1.16. If (B, d) is a Hopf C*-algebra, an *action* of B on the C*-algebra is a non-degenerate $*$-morphism $\delta : A \to \overleftarrow{\mathrm{M}}(A, B)$ such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes d) \circ \delta\,.$$

If $B = \mathcal{C}_0(H)$, one says that δ is an *action of H on A*. If $B = \mathrm{C}^*_{\#}(H)$, δ is called a *(reduced) coaction of H on A*.

Remark 6.1.17. There is a notion of *full* coaction where the universal group C*-algebra $\mathrm{C}^*(H)$ of H comes in. As we have noted above,

$$\mathrm{C}^*(H \times H) = \mathrm{C}^*(H) \otimes_\iota \mathrm{C}^*(H) ,$$

so in this setup one uses the minimal tensor product, rather than the spatial one. However, we are mainly interested in type I groups H, and for these, $\mathrm{C}^*(H)$, and, as a quotient of $\mathrm{C}^*(H)$, the reduced group C*-algebra $\mathrm{C}^*_{\#}(H)$, are of type I. In particular, they are nuclear, so all C*-cross norms on the algebraic tensor product with these algebras coincide, by [Tak03, prop. 1.6].

An action of H in the sense defined above coincides with the usual definition, i.e., H acts by automorphisms on A. If H is Abelian, full coactions correspond exactly to actions of the dual group $\hat{H}$.

6.1.18. Similarly to the C*-case, if (N, d) is a Hopf von Neumann algebra, an *action* of N on M is a normal $*$-morphism $\delta : M \to M \bar{\otimes} N$, such that

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \circ d) \circ \delta .$$

If $N = \mathbf{L}^\infty(H)$, one says that δ is an *action of H on M*, and if $N = \mathrm{W}^*(H)$, δ is said to be a *(reduced) coaction of H on M*.

6.2 Localisation, co-crossed products, and Toeplitz operators

The object of this subsection is to define a general notion of support for $\mathrm{A}(H)$-module structures which are naturally induced by coactions of H. We then define a large class of 'localised' C*-coactions, and, using the notion of (co-) crossed products by coactions, show how these relate to 'Toeplitz-type' C*-algebras.

6.2.1. Given a coaction δ of H on a von Neumann algebra M we can, as for $\mathrm{W}^*(H)$, declare an $\mathrm{A}(H)$-action by

$$\alpha \cdot x = (\mathrm{id} \otimes \alpha)(\delta(x)) \quad \text{for all } \alpha \in \mathrm{A}(H), \, x \in M .$$

Analogously, for a coaction δ of H on a C*-algebra A, we may define a $\mathrm{B}_{\#}(H)$-action by

$$\beta \cdot a = (\mathrm{id} \otimes \beta)(\delta(a)) \quad \text{for all } \beta \in \mathrm{B}_{\#}(H), \, a \in A .$$

This is well-defined by [LPRS87, lem. 1.5].

If M is a von Neumann algebra and M_* its predual, we can define a right Banach space action of M on M_* by

$$\langle \omega \rtimes x : y \rangle = \langle \omega : x \cdot y \rangle \quad \text{for all } x, y \in M, \omega \in M_* .$$

This is well defined, because multiplication in M is separately ultraweakly continuous.

6.2.2. Given an $\mathrm{A}(H)$-module E, let

$$e^\perp = \{\alpha \in \mathrm{A}(H) \mid \alpha \cdot e = 0\} \quad \text{for all } e \in E$$

be the annihilator. It is an ideal $e^\perp \lhd \mathrm{A}(H)$, so we may consider the hull

$$\operatorname{supp} e = \operatorname{supp}_E e = \operatorname{hull}(e^\perp) = \{\mathfrak{m} \in \operatorname{Sp}\mathrm{A}(H) \mid \mathfrak{m} \supset e^\perp\} = \operatorname{Sp}(\mathrm{A}(H)/e^\perp) ,$$

and we call this the *support of* e. Here, $\operatorname{Sp}\mathrm{A}(H)$ is the maximal ideal space of $\mathrm{A}(H)$. By [Eym64, th. 3.34], $\operatorname{Sp}\mathrm{A}(H) = H$, so

$$\operatorname{supp} e = \{h \in H \mid \alpha \cdot e = 0 \Rightarrow \alpha(h) = 0 \text{ for all } \alpha \in \mathrm{A}(H)\} .$$

It is easy to see that for $E = \mathrm{B}_\#(H)$, one gets the usual support of functions, and by [Eym64, rem. 4.7], if H is a Lie group, the support in $\mathrm{W}^*(H)$ is just the usual support of distributions. Similarly for $\mathcal{M}^b(H) \subset \mathrm{W}^*(H)$.

6.2.3. Let E be an $\mathrm{A}(H)$-module. Define

$$\mathcal{K}E = \{e \in E \mid \operatorname{supp} e \text{ is compact}\} .$$

If E is normed, let $\bar{\mathcal{K}}E$ denote the norm closure of $\mathcal{K}E$.

A C*-coaction δ is said to be *non-degenerate* if for each non-zero $\beta \in \mathrm{B}_\#(H)$, there exists $\beta' \in \mathrm{B}_\#(H)$ such that

$$(\beta \otimes \beta') \circ \delta \neq 0 .$$

Equivalently, $A \otimes \mathrm{C}^*_\#(H)$ is the closed span of $\delta(A)(\mathbb{C} \otimes \mathrm{C}^*_\#(H))$; or even, $A = \bar{\mathcal{K}}A$, by [Kat84, th. 5] and [Qui92] (see proposition 6.2.4 (v) below for an alternative description of $\bar{\mathcal{K}}A$). Hence this notion of non-degeneracy is the usual one for normed modules if we consider the action of $\mathrm{A}(H)$ induced by δ.

Admittedly, this use of the term 'non-degenerate' is rather trite. Nonetheless, it seems inappropriate to alter the prevalent terminology at this point, since its meaning will always be evident from the context.

A related concept is the following one. We call E a *regular* $\mathrm{A}(H)$-module, if

$$\mathrm{A}(H)^\perp = \{e \in E \mid \alpha \cdot e = 0 \text{ for all } \alpha \in \mathrm{A}(H)\} = 0 .$$

We collect some basic facts on supports. The proofs are standard, compare [Eym64, prop. 4.8] or [NT79, lem. 1.2]. We give them for the reader's and our own convenience.

Proposition 6.2.4. Let $\mathrm{A}(H) \subset J \subset \mathrm{B}_{\#}(H)$ be a closed subalgebra, and E a normed J-module.

(i). Let $e \in E$ such that $\operatorname{supp} e = \varnothing$. Then $\alpha \cdot e = 0$ for all $\alpha \in \mathrm{A}(H)$. If E is regular, then $e = 0$. Conversely, if $\operatorname{supp} e = \varnothing$ implies $e = 0$ for all $e \in E$, then E is regular.

(ii). We have

$$\operatorname{supp}(\alpha \cdot e) \subset \operatorname{supp} \alpha \cap \operatorname{supp} e \quad \text{for all } \ \alpha \in J \,, \, e \in E \,.$$

(iii). Let $e \in E$. Any $\alpha \in \mathcal{K}\mathrm{A}(H)$ vanishing on a neighbourhood of $\operatorname{supp} e$ annihilates e, i.e. $\alpha \cdot e = 0$. Moreover, $\operatorname{supp} e$ is the smallest closed subset of H with this property.

(iv). We have

$$\operatorname{supp}(\lambda \cdot e) = \operatorname{supp} e \quad \text{and} \quad \operatorname{supp}(e + e') \subset \operatorname{supp} e \cup \operatorname{supp} e'$$

for all $\lambda \in \mathbb{C} \setminus 0$, $e, e' \in E$. Equality holds in the second formula if $\operatorname{supp} e \cap \operatorname{supp} e' = \varnothing$.

(v). The sets $\mathcal{K}E$ and $\bar{\mathcal{K}}E$ are J-submodules. If E is regular, then

$$\mathcal{K}E = \left\{ \alpha \cdot e \;\middle|\; \alpha \in \mathcal{K}\mathrm{A}(H) \,, \, e \in E \right\} ,$$

and $\bar{\mathcal{K}}E$ is the closure of $\mathrm{A}(H) \cdot E$.

Proof of (i). For all $h \in H$, there is $\alpha \in e^{\perp}$ such that $\alpha(h) \neq 0$. Moreover, $e^{\perp}$ is a closed ideal. The Tauberian theorem [Eym64, cor. 3.38] implies that $e^{\perp} = \mathrm{A}(H)$, as required.

For the converse, let $\alpha \cdot e = 0$ for all $\alpha \in \mathrm{A}(H)$. Since $\mathrm{A}(H)$ separates the points of H, we have $\operatorname{supp} e = \varnothing$. By assumption, this gives $e = 0$, so E is regular.

Proof of (ii). Let $h \notin \operatorname{supp} \alpha$. Take a neighbourhood $h \in U \subset H$ such that $\alpha\big|_U = 0$. There is $\beta \in \mathrm{A}(H)$ such that $\operatorname{supp} \beta \subset U$ and $\beta(h) \neq 0$. Then

$$\beta \cdot (\alpha \cdot e) = (\beta \cdot \alpha) \cdot e = 0 \,.$$

Hence, $h \notin \operatorname{supp}(\alpha \cdot e)$.

Let $h \in \operatorname{supp}(\alpha \cdot e)$. For $\beta \in e^{\perp}$, we have $\beta \cdot (\alpha \cdot e) = \alpha \cdot (\beta \cdot e) = 0$, so $\beta(h) = 0$. This implies $h \in \operatorname{supp} e$.

Proof of (iii). Let $\alpha \in \mathcal{K}\mathrm{A}(H)$, $\operatorname{supp} \alpha \subset U$, where $U \subset H \setminus \operatorname{supp} e$ is open. There exists $\chi \in \mathcal{K}\mathrm{A}(H)$ such that $\operatorname{supp} \chi \subset U$ and $\chi = 1$ on $\operatorname{supp} \alpha$. Since $\alpha = \alpha \cdot \chi$, we find $\alpha \cdot e = \alpha \cdot (\chi \cdot e)$. Now,

$$\operatorname{supp}(\chi \cdot e) \subset \operatorname{supp} \chi \cap \operatorname{supp} e = \varnothing$$

by part (ii). Part (i) implies that $\alpha\cdot e=\alpha\cdot(\chi\cdot e)=0$. Hence, $\operatorname{supp}e$ fulfils the stated condition, and it is, moreover, manifestly closed.

Let the closed subset $C\subset H$ fulfill the conditions, and take $h\notin C$. Then there is $\alpha\in\mathcal{K}\mathrm{A}(H)$, $\operatorname{supp}\alpha\subset H\setminus C$, such that $\alpha(h)\neq 0$. But $\alpha\cdot e=0$ by the first part, so $h\notin\operatorname{supp}e$.

Proof of (iv). The first formula is obvious. As for the second, let $\alpha\in\mathcal{K}\mathrm{A}(H)$ vanish in the neighbourhood of $\operatorname{supp}e\cup\operatorname{supp}e'$. Then α vanishes in the neighbourhood of $\operatorname{supp}e$ and $\operatorname{supp}e'$. By (iii), this implies $\alpha\cdot(e+e')=\alpha\cdot w+\alpha\cdot e'=0$. Since we may assume that $\alpha(h)\neq 0$ for any fixed $h\notin\operatorname{supp}e\cup\operatorname{supp}e'$, the conclusion follows.

If $\operatorname{supp}e\cap\operatorname{supp}e'=\varnothing$, let $\alpha\in\mathcal{K}\mathrm{A}(H)$ be such that $\alpha\cdot(e+e')=0$. Then

$$\operatorname{supp}(\alpha\cdot e)=\operatorname{supp}(\alpha\cdot e')\subset\operatorname{supp}e\cap\operatorname{supp}e'=\varnothing\,,$$

by part (ii). Let $\chi\in\mathcal{K}\mathrm{A}(H)$, $\chi\cdot\alpha=\alpha$. Then part (i) implies

$$\alpha\cdot e=\chi\cdot(\alpha\cdot e)=0\,,$$

and equally $\alpha\cdot e'=0$.

This implies $\operatorname{supp}e\cup\operatorname{supp}e'\subset\operatorname{supp}(e+e')$, by the definition of supp.

Proof of (v). By (iv), $\mathcal{K}E$ is a linear subspace of E, and by (ii), a J-submodule. By continuity, so is $\bar{\mathcal{K}}E$.

By (ii), we have $\mathcal{K}\mathrm{A}(H)\cdot E\subset\mathcal{K}E$. On the other hand, let $e\in E$, and choose a compact neighbourhood K of $\operatorname{supp}e$. There exists $\chi\in\mathcal{K}\mathrm{A}(H)$ such that $\chi|_K=1$. Then, for all $\alpha\in\mathcal{K}\mathrm{A}(H)$, $\alpha-\alpha\cdot\chi=0$ on K. By (iii), $\alpha\cdot(e-\chi\cdot e)=(\alpha-\alpha\cdot\chi)\cdot e=0$.

Since $\mathcal{K}\mathrm{A}(H)$ is dense in $\mathrm{A}(H)$, we find that

$$\alpha\cdot(e-\chi\cdot e)=0\quad\text{for all }\ \alpha\in\mathrm{A}(H)\,.$$

If E is regular, this implies $e=\chi\cdot e\in\mathrm{A}(H)\cdot E$. ∎

Proposition 6.2.5. Let (M,δ) be a von Neumann algebra coaction.

(i). The $\mathrm{A}(H)$-module structure on M is regular.

(ii). For any $x\in M$, $\operatorname{supp}x^*=(\operatorname{supp}x)^{-1}$.

(iii). We have the equation

$$\operatorname{supp}x=\overline{\bigcup_{\omega\in M_*}\operatorname{supp}(\omega\otimes\mathrm{id})(\delta(x))}\quad\text{for all }\ x\in M$$

where we note that $(\omega\otimes\mathrm{id})(\delta(x))\in\mathrm{W}^*(H)$ for all $\omega\in M_*$.

(iv). Let $x, y \in M$. If either x or y has compact support, then

$$\operatorname{supp}(xy) \subset \operatorname{supp} x \cdot \operatorname{supp} y \ .$$

Proof of (i). Let $x \in M$ such that $\alpha \cdot x = 0$ for all $\alpha \in \mathrm{A}(H)$. Then we have

$$\langle \omega \otimes \alpha \,:\, \delta(x) \rangle = \langle \omega \,:\, \alpha \cdot x \rangle = 0 \quad \text{for all } \ \omega \in M_* \ .$$

Since the algebraic tensor product $M_* \odot \mathrm{A}(H)$ is dense in $(M \bar{\otimes} \mathrm{W}^*(H))_*$, we conclude that $\delta(x) = 0$, and hence $x = 0$, δ being injective.

Proof of (ii). Let $\alpha \in \mathrm{A}(H)$. Note that since δ is involutive,

$$\langle \omega \,:\, (\alpha \cdot x)^* \rangle = \overline{\langle \omega^* \otimes \alpha \,:\, \delta(x) \rangle} = \langle \omega \otimes \alpha^* \,:\, \delta(x^*) \rangle = \langle \omega \,:\, \alpha^* \cdot x^* \rangle \quad \text{for all } \ \omega \in M_*$$

where $\langle \omega^* \,:\, y \rangle = \overline{\langle \omega \,:\, y^* \rangle}$. Hence, $(\alpha \cdot x)^* = \alpha^* \cdot x^*$. Because $\sqcup^*$ is an isometry on M and and on $\mathrm{A}(H)$, and $\alpha^*(s) = \overline{\alpha(s^{-1})}$ for all $s \in H$, we conclude

$$\big(\alpha \cdot x = 0 \ \Rightarrow \ \alpha(s) = 0\big) \quad \Longleftrightarrow \quad \big(\alpha \cdot x^* = 0 \ \Rightarrow \ \alpha(s^{-1}) = 0\big) \quad \text{for all } \ s \in H \ .$$

Hence our contention.

Proof of (iii). We note the formula

$$\langle \beta \,:\, \alpha \cdot (\omega \otimes \mathrm{id})(\delta(x)) \rangle = \langle \omega \otimes (\alpha \cdot \beta) \,:\, \delta(x) \rangle = \langle \omega \,:\, \alpha \cdot \beta \cdot x \rangle$$

for all $\alpha, \beta \in \mathrm{A}(H)$, $\omega \in M_*$.

Let $s \in \operatorname{supp}(\omega \otimes \mathrm{id})(\delta(x))$. Then for $\alpha \in \mathrm{A}(H)$ such that $\alpha \cdot x = 0$, we find $\alpha \cdot (\omega \otimes \mathrm{id})(\delta(x)) = 0$. This implies $\alpha(t) = 0$, so $t \in \operatorname{supp} x$. Inasmuch $\operatorname{supp} x$ is closed, we infer that it contains the right hand side.

Conversely, let $s \notin \overline{\bigcup_{\omega \in M_*} \operatorname{supp}(\omega \otimes \mathrm{id})(\delta(x))}$. Then, by proposition 6.2.4 (iii), there is a neighbourhood $U \subset H$ of s such that

$$\alpha \cdot (\omega \otimes \mathrm{id})(x) = 0 \quad \text{for all } \ \omega \in M_* \, , \, \alpha \in \mathcal{K}\mathrm{A}(H) \, , \, \operatorname{supp} \alpha \subset U \ .$$

Given any such α, there is $\beta \in \mathrm{A}(H)$ such that $\alpha \cdot \beta = \alpha$. The equation we initially stated shows that $\alpha \cdot x = \alpha \cdot \beta \cdot x = 0$. Another application of proposition 6.2.4 (iii) shows that $s \notin \operatorname{supp} x$, proving our claim.

Proof of (iv). Because of (ii), we may assume that $\operatorname{supp} y$ is compact. Let $\alpha_\lambda \in \mathcal{K}\mathrm{A}(H)$ define an approximate unit in $\mathrm{C}^*_\#(H)$. Then, by lemma 6.2.6 (i) below,

$$\delta(y) = \lim\nolimits_\lambda \delta(y)(1 \otimes \alpha_\lambda^{\vee \#}) = \lim\nolimits_\lambda \int_H (s * \alpha_\lambda) \cdot y \otimes s^\# \, ds \quad \text{ultraweakly in} \quad M \bar{\otimes} \mathrm{W}^*(H) \, ,$$

where the integral converges since $s \mapsto (s * \alpha_\lambda) \cdot y$ has compact support by proposition 6.2.4 (iii). We deduce, for all $\omega \in M_*$ and $\alpha \in \mathrm{A}(H)$,

$$\begin{aligned}\langle \omega \otimes \alpha \,:\, \delta(xy)\rangle &= \lim_\lambda \int_H \langle \omega \otimes \alpha \,:\, \delta(x)\big((s * \alpha_\lambda) \cdot y \otimes s^\#\big)\rangle \, ds \\ &= \lim_\lambda \int_H \Big\langle \omega \rtimes \big((s * \alpha_\lambda) \cdot y\big) \otimes \alpha * s^{-1} \,:\, \delta(x)\Big\rangle \, ds \\ &= \lim_\lambda \int_H \langle \alpha \,:\, \big(\omega \rtimes \big((s * \alpha_\lambda) \cdot y\big) \otimes \mathrm{id}\big)(\delta(x))s^\#\rangle \, ds \,.\end{aligned}$$

Let $U \subset H$ be a neighbourhood of the identity. Then there exists λ_0 such that

$$\operatorname{supp} \alpha_\lambda \subset U \quad \text{for all } \lambda \geqslant \lambda_0 \,.$$

In particular,

$$(s * \alpha_\lambda) \cdot y = 0 \quad \text{for all } \lambda \geqslant \lambda_0 \,,\, s \notin U^{-1}U(\operatorname{supp} y) \,,$$

by proposition 6.2.4 (iii). Moreover, (iii) shows that

$$\operatorname{supp}\big(\omega \rtimes \big((s * \alpha_\lambda) \cdot y\big) \otimes \mathrm{id}\big)(\delta(x))s^\# \subset (\operatorname{supp} x)s \,.$$

Hence,

$$\operatorname{supp}(\omega \otimes \mathrm{id})(\delta(xy)) \subset (\operatorname{supp} x)U^{-1}U(\operatorname{supp} y) \,,$$

and since ω and U were arbitrary, the claim follows from (iii). □

The following lemma was used in the proof of proposition 6.2.5 (iv), and will also be useful in the sequel.

Lemma 6.2.6. Let δ be a coaction of H on the von Neumann algebra M.

(i). For $x \in M$ and $\alpha \in \mathcal{K}\mathrm{A}(H)$,

$$\delta(x)(1 \otimes \alpha^{\vee\#}) = \int_H (s * \alpha) \cdot x \otimes s^\# \, ds \quad \text{in} \quad M \bar{\otimes} \mathrm{W}^*(H)$$

provided the integral on the right hand side is ultraweakly convergent.

(ii). For $\alpha \in \mathcal{K}\mathrm{A}(H)$ and $\beta \in \mathrm{B}(H)$, we have the norm convergent integral

$$\int_H (\alpha \cdot \beta) * s^{-1} \, ds = \int_H \alpha(s)\beta(s) \, ds \quad \text{in} \quad \mathrm{B}(H) \,.$$

(iii). For $x \in M$ and $\alpha \in \mathcal{K}\mathrm{A}(H)$, we have

$$x \otimes \alpha^\# = \int_H \delta(\alpha * s^{-1} \cdot x)(1 \otimes s^\#) \, ds \quad \text{in} \quad M \bar{\otimes} \mathrm{W}^*(H) \,.$$

provided the integral on the right hand side is ultraweakly convergent.

Proof of (i). For $\omega \in M_*$ and $\beta \in \mathrm{A}(H)$, we compute

$$\begin{aligned}\int_H \langle \omega \otimes \beta : (s * \alpha) \cdot x \otimes s^{\#} \rangle \, ds &= \int_H \beta(s)(\omega \otimes s * \alpha)(\delta(x)) \, ds \\ &= \int_H \beta(s) \, \langle s * \alpha : (\omega \otimes \mathrm{id})(\delta(x)) \rangle \, ds \\ &= \int_H \beta(s) \big((\omega \otimes \mathrm{id})(\delta(x)) \alpha^{\vee} \big)(s) \, ds \\ &= \big\langle \omega \otimes \beta : \delta(x)(1 \otimes \alpha^{\vee \#}) \big\rangle \ ,\end{aligned}$$

since $(\omega \otimes \mathrm{id})(\delta(x)) \in \mathrm{W}^*(H)$, and

$$\langle s * \gamma : y \rangle = (y\gamma^{\vee})(s) \quad \text{for all } \ \gamma \in \mathcal{K}\mathrm{A}(H)\,,\, y \in \mathrm{W}^*(H)\,,\, s \in H\,,$$

by [Eym64, prop. 3.7].

Proof of (ii). Because the integrand is norm continuous and has compact support, the integral is normally convergent in $\mathrm{B}(H) = \mathrm{C}^*(H)^*$. For all $t \in H$

$$\int_H (\alpha \cdot \beta) * s^{-1}(t) \, ds = \int_H \alpha(ts)\beta(ts) \, ds = \int_H \alpha(s)\beta(s) \, ds \,,$$

hence the assertion.

Proof of (iii). By (ii), we have for all $\omega \in M_*$ and $\beta \in \mathrm{A}(H)$,

$$\begin{aligned}\int_H \Big\langle \omega \otimes \beta : \delta\big((\alpha * s^{-1}) \cdot x\big)(1 \otimes s^{\#}) \Big\rangle \, ds &= \int_H \Big\langle \omega \otimes (\beta * s^{-1}) : \delta\big((\alpha * s^{-1}) \cdot x\big) \Big\rangle \, ds \\ &= \int_H \Big\langle \omega : \big((\alpha \cdot \beta) * s^{-1}\big) \cdot x \Big\rangle \, ds \\ &= \big\langle \omega \otimes \beta : x \otimes \alpha^{\#} \big\rangle \ ,\end{aligned}$$

proving the claim. □

Remark 6.2.7. The formulae in lemma 6.2.6 are cited in [Qui92, lem. 2.3], without proof (although with references).

Corollary 6.2.8. Let (M, δ) be a von Neumann algebra coaction. The $\mathrm{A}(H)$-submodule $\bar{\mathcal{K}}M \sqsubset M$ is a C^*-algebra. In particular, this is true of $\bar{\mathcal{K}}\mathrm{W}^*(H) \subset \mathrm{W}^*(H)$.

Proof. Let $a, b \in \mathcal{K}M$. Then, by proposition 6.2.5 (iv), $\operatorname{supp}(a \cdot b) \subset \operatorname{supp} a \cdot \operatorname{supp} b$, and is hence compact, i.e. $a \cdot b \in \mathcal{K}M$. By continuity, this implies that $\bar{\mathcal{K}}M$ is a subalgebra of M. Since $\operatorname{supp} x^* = (\operatorname{supp} x)^{-1}$ for all $x \in M$, by proposition 6.2.5 (ii), $\bar{\mathcal{K}}M$ is also involutive. Since it is norm closed, it is a C^*-subalgebra of M. □

Now we define a large class of C^*-coactions which, as we shall see below, are intimately related to C^*-algebras of Toeplitz operators.

6.2.9. Let δ be a coaction of H on a von Neumann algebra M and $\mathcal{E} \subset M$ such that $\mathcal{E}^* = \mathcal{E}$. Define

$$\mathrm{C}^*_{\mathcal{E}}(M,\delta) = \mathrm{C}^*\langle \alpha \cdot e \bigm| \alpha \in \mathrm{A}(H)\,,\, e \in \mathcal{E} \rangle \subset M\,.$$

In particular, if $(M,\delta) = (\bar{\mathcal{K}}\mathrm{W}^*(H), \delta_H)$, we write $\mathrm{C}^*_{\mathcal{E}}(H) = \mathrm{C}^*_{\mathcal{E}}(M,\delta)$. Then

$$\mathrm{C}^*_{\mathrm{L}^1(H)}(H) = \mathrm{C}^*_{\#}(H) \quad \text{and} \quad \mathrm{C}^*_{\mathrm{W}^*(H)}(H) = \bar{\mathcal{K}}\mathrm{W}^*(H)\,,$$

where the second statement follows from corollary 6.2.8.

Proposition 6.2.10. Assume the conditions of 6.2.9 satisfied. Then δ defines a non-degenerate C*-coaction on $\mathrm{C}^*_{\mathcal{E}}(M,\delta)$.

Corollary 6.2.11. For any $\mathcal{E} = \mathcal{E}^* \subset \mathrm{W}^*(H)$, δ_H defines a non-degenerate C*-coaction on $\mathrm{C}^*_{\mathcal{E}}(H)$. In particular, this is the case for $\mathrm{C}^*_{\#}(H)$ and $\bar{\mathcal{K}}\mathrm{W}^*(H)$.

Proof of proposition 6.2.10. For the sake of brevity, we denote $A = \mathrm{C}^*_{\mathcal{E}}(M,\delta)$. Let $e \in \mathcal{E}$ and $\alpha \in \mathcal{K}\mathrm{A}(H)$. Then, for all $\beta \in \mathcal{K}\mathrm{A}(H)$ and $b \in \mathrm{C}^*_{\#}(H)$,

$$\delta(\alpha \cdot e)(1 \otimes \alpha^{\vee\#} b) = \int_H \big((s * \alpha) \cdot \beta \cdot e\big) \otimes s^{\#} b \, ds \in A \otimes \mathrm{C}^*_{\#}(H)\,.$$

Namely, the equation is true by lemma 6.2.6 (i). Moreover, α and $\beta \cdot e$ have compact support, so by proposition 6.2.5 (i) and proposition 6.2.4 (i)-(ii),

$$s \longmapsto (s * \alpha) \cdot \beta \cdot e \in \mathcal{K}(H, A)\,.$$

So, the integrand has compact support and is norm continuous. Hence the integral converges in norm, and the statement is true.

Since $\alpha \cdot e$, $\alpha \in \mathcal{K}\mathrm{A}(H)$, $e \in \mathcal{E}$, generate A , and $\mathcal{E}$ is $*$-invariant,

$$\delta(A) \subset \overleftarrow{\mathrm{M}}(A, \mathrm{C}^*_{\#}(H))\,,$$

by [Bus68, th. 3.9], as required.

In order to prove that δ defines a C*-coaction, by self-adjointness of A , $\mathrm{C}^*_{\#}(H)$ and their tensor product, it remains to show that the span of $\delta(A)(\mathbb{C} \otimes \mathrm{C}^*_{\#}(H))$ is dense in $A \otimes \mathrm{C}^*_{\#}(H)$. Let $e \in E$, $\alpha, \beta \in \mathcal{K}\mathrm{A}(H)$, and $a \in \mathrm{C}^*_{\#}(H)$. We have

$$(\alpha \cdot e)(\beta \cdot f) \otimes \gamma^{\#} b = \int_H \delta\big((\gamma * s^{-1}) \cdot \alpha \cdot e\big)\big((\beta \cdot f) \otimes s^{\#} b\big)\, ds \in \overline{\delta(A)(A \otimes \mathrm{C}^*_{\#}(H))}\,.$$

The set of all $(\alpha \cdot e)(\beta \cdot f)$, $e, f \in \mathcal{E}$, $\alpha, \beta \in \mathcal{K}\mathrm{A}(H)$ generates A as a Banach algebra, because $A = A \cdot A$, as follows, e.g., from Cohen's theorem [HR79, thm. 32.50]. Similarly, the span of $\gamma^{\#} b$, $\gamma \in \mathcal{K}\mathrm{A}(H)$, $b \in \mathrm{C}^*_{\#}(H)$, is dense in $\mathrm{C}^*_{\#}(H)$. This implies the required density result since the right hand side of the above equation is closed under finite products by the first part of this proof. Hence, δ is a coaction.

The same argument, taking 1 in place of $\beta \cdot f$ in the integral, shows that the subset $\delta(A)(\mathbb{C} \otimes \mathrm{C}^*_\#(H))$ has dense span in $A \otimes \mathrm{C}^*_\#(H)$. Therefore, δ is non-degenerate as a coaction, and we have established our claim. $\square$

Remark 6.2.12. Our technique of proof is essentially borrowed from [Qui92, lem. 2.2, lem 2.3, cor. 2.4].

6.2.13. Recall that

$$W_H \in \mathrm{M}(\mathcal{C}_0(H) \otimes \mathrm{C}^*_\#(H)) \subset \mathbf{L}^\infty(H) \bar{\otimes} \mathrm{W}^*(H) \,.$$

If $\alpha \in \mathrm{A}(H)$, then $(\mathrm{id} \otimes \alpha)(W_H) \in \mathbf{L}^\infty(H)$, if fact, one computes $(\mathrm{id} \otimes \alpha)(W_H) = M_\alpha$, the multiplication operator with symbol α.

This motivates the following definition: A unitary operator

$$W \in \mathcal{L}(\mathcal{H}) \bar{\otimes} \mathrm{W}^*(H) \quad \text{such that} \quad W_{12} W_{13} (W_H)_{23} = (W_H)_{23} W_{12} \,,$$

or, equivalently, $W_{12} W_{13} = (\mathrm{id} \otimes \delta_H)(W)$, is called a *corepresentation* of $\mathrm{A}(H)$.

The associated map

$$\mu : \mathrm{A}(H) \to \mathcal{L}(\mathcal{H}) : \alpha \mapsto (\mathrm{id} \otimes \alpha)(W)$$

is a non-degenerate $*$-representation of $\mathrm{A}(H)$ on the Hilbert space $\mathcal{H}$.

Conversely, any such representation μ gives rise to a corepresentation via

$$W = (\mu \otimes \mathrm{id})(W_H) \in \mathcal{L}(\mathcal{H}) \bar{\otimes} \mathrm{W}^*(H) \,,$$

cf. [NT79, th. A.1 (b)].

Since $H = \mathrm{Sp}\,\mathrm{A}(H)$, the universal enveloping C^*-algebra of $\mathrm{A}(H)$ is just $\mathcal{C}_0(H)$, cf. [Dix69, 2.7.2]. A non-degenerate $*$-representation of $\mathrm{A}(H)$ can be uniquely extended to one of $\mathcal{C}_0(H)$, by [Dix69, prop. 2.7.4].

6.2.14. Let δ be a coaction of H on the C^*-algebra A. Assume given a non-degenerate $*$-representation of A on $\mathcal{H}$, and a corepresentation $W \in \mathcal{L}(\mathcal{H}) \bar{\otimes} \mathrm{W}^*(H)$ with associated representation μ. Then (π, μ) is called a *covariant pair of representations* if

$$(\pi \otimes \mathrm{id})(\delta(a)) = W(\pi(a) \otimes 1) W^* \quad \text{for all } a \in A \,.$$

If $B \subset \mathcal{L}(\mathcal{H})$ is a C^*-algebra acting non-degenerately, we say that (π, μ) is a covariant pair *in* B if π and μ define non-degenerate $*$-morphisms into $\mathrm{M}(B)$.

The von Neumann algebra $M = \pi(A)''$ is then the ultraweak closure of $\pi(A)$, by non-degeneracy. If we define

$$\delta_W(x) = W(x \otimes 1) W^* \quad \text{for all } x \in M \,,$$

then the covariance condition shows that $\delta_W(\pi(A)) \subset M\bar{\otimes}\mathrm{W}^*(H)$, and by ultraweak continuity of δ_W, we see $\delta_W : M \to M\bar{\otimes}\mathrm{W}^*(H)$. Then δ_W is clearly normal and injective, and it satisfies the coaction identity because this is true for its restriction to $\pi(A)$.

This shows that δ_W is a von Neumann algebra coaction of H on M. We say that W *implements* this coaction.

Proposition 6.2.15. Let (A,δ) be a C*-algebra coaction and (π,μ) a covariant pair of representations on $\mathcal{H}$. Then

$$\mathrm{C}^*(A,\pi,\mu) = \overline{\langle \pi(A)\cdot\mu(\mathrm{A}(H))\rangle} \subset \mathcal{L}(\mathcal{H})$$

is the C*-algebra generated by $\pi(A)\mu(\mathrm{A}(H))$.

The proposition's proof is preceded by a lemma.

Lemma 6.2.16. Let $W \in \mathcal{L}(\mathcal{H})\bar{\otimes}\mathrm{W}^*(H)$ be a corepresentation implementing the von Neumann algebra coaction δ on $M \subset \mathcal{L}(\mathcal{H})$. If μ is the associated representation,

$$\mu(\alpha)x \in (x) \text{ and } \mu(\alpha)y\mu(\beta) \in (y) \quad \text{for all } x \in \bar{\mathcal{K}}M\,,\ y \in M\,,\ \alpha,\beta \in \mathrm{A}(H)\,,$$

where

$$(x) = \overline{\langle \mathrm{A}(H)\cdot x\cdot\mu(\mathrm{A}(H))\rangle}$$

is the norm closed linear span of all $(\varphi\cdot x)\mu(\psi)$, $\varphi,\psi \in \mathrm{A}(H)$.

Proof. Let $\alpha,\beta \in \mathrm{A}(H)$, $x \in M$. We compute

$$\mu(\alpha)x = (\mathrm{id}\otimes\alpha)(W(x\otimes 1)) = (\mathrm{id}\otimes\alpha)(\delta(x)W)\,.$$

There exist nets $\alpha_{j\lambda} \in \mathrm{A}(H)$, $j = 1,2$, such that $\alpha = \lim_\lambda \alpha_{1\lambda} \rtimes \alpha_{2\lambda}^\#$ in $\mathrm{A}(H)$. Then

$$\mu(\alpha)x = \lim\nolimits_\lambda(\mathrm{id}\otimes\alpha_{1\lambda})\big((1\otimes\alpha_{2\lambda}^\#)\delta(x)W\big) \quad \text{normally in} \quad \mathcal{L}(\mathcal{H})\,.$$

Provided the integral is ultraweakly convergent, by lemma 6.2.6 (i), we have

$$\begin{aligned}
(\mathrm{id}\otimes\alpha_{1\lambda})\big((1\otimes\alpha_{2\lambda}^\#)\delta(x)W\big) &= \Big\langle \mathrm{id}\otimes\alpha_{1\lambda} \,:\, \int_H \big((\alpha_{2\lambda}^\vee * s^{-1})\cdot x\big)\otimes s^\#\,ds\cdot W\Big\rangle \\
&= \int_H \big(\mathrm{id}\otimes(\alpha_{1\lambda}\rtimes s^\#)\big)\big(((\alpha_{2\lambda}^\vee * s^{-1})\cdot x\otimes 1)W\big)\,ds \\
&= \int_H \big((\alpha_{2\lambda}^\vee * s^{-1})\cdot x\big)\mu(\alpha_{1\lambda} * s^{-1})\,ds\,,
\end{aligned}$$

because $\alpha_{1\lambda}\rtimes s^\# = \alpha_{1\lambda} * s^{-1}$. If x has compact support, the integral is normally convergent, proving the first assertion.

For general x, to show that $\mu(\alpha)x\mu(\beta) \in \overline{\langle \mathrm{A}(H)\cdot x\cdot\mu(\mathrm{A}(H))\rangle}$, we may assume w.l.o.g. that $\beta \in \mathcal{K}\mathrm{A}(H)$. We conclude

$$\mu(\alpha)x\mu(\beta) = \lim\nolimits_\lambda \int_H \big((\alpha_{2\lambda}^\vee * s^{-1})\cdot x\big)\mu\big((\alpha_{1\lambda} * s^{-1})\cdot\beta\big)\,ds \in \overline{\langle \mathrm{A}(H)\cdot x\cdot\mu(\mathrm{A}(H))\rangle}\,,$$

because the function

$$\big(s \mapsto (\alpha_{1\lambda} * s^{-1})\cdot\beta\big) \in \mathcal{K}(H,\mathrm{A}(H))\,,$$

and the integral inside the limit is hence normally convergent. This proves the second assertion. □

Proof of proposition 6.2.15. Observe that

$$\alpha\cdot\pi(a) = (\mathrm{id}\otimes\alpha)(\pi\otimes\mathrm{id})(\delta(a)) = \pi(\alpha\cdot a) \quad \text{for all } a\in A\,,\ \alpha\in\mathrm{A}(H)\,.$$

Hence, $\pi(A)\subset M$ is an $\mathrm{A}(H)$-submodule. By lemma 6.2.16, we find

$$\mu(\alpha)x\mu(\beta)y\mu(\gamma) \in \mu(\alpha)x\cdot(y) \subset \overline{\big\langle(\mathrm{A}(H)\cdot x)(\mathrm{A}(H)\cdot y)\mu(\mathrm{A}(H))\big\rangle} \subset \mathrm{C}^*(A,\pi,\mu)$$

for all $x,y\in\pi(A)$, $\alpha,\beta,\gamma\in\mathrm{A}(H)$. Hence, $\mathrm{C}^*(A,\pi,\mu)$ is closed under products. Moreover, lemma 6.2.16 also proves it is invariant for the involution $\sqcup^*$. Since $\mathrm{C}^*(A,\pi,\mu)$ is norm closed, it is a C^*-algebra. □

6.2.17. We have seen that W_H is the corepresentation associated to

$$M : \mathrm{A}(H) \to \mathcal{L}(\mathbf{L}^2(H)) : \alpha \mapsto M_\alpha\,.$$

Hence, the coaction identity shows that $(\delta, 1\otimes M)$ is a covariant pair of representations in $A\otimes\mathbb{K}(\mathbf{L}^2(H))$ for any C^*-algebra coaction (A,δ) .

By proposition 6.2.15,

$$A\otimes_\delta \mathcal{C}_0(H) = \mathrm{C}^*(A,\delta,1\otimes M) \subset \mathrm{M}(A\otimes\mathbb{K}(\mathbf{L}^2(H)))$$

is a C^*-algebra, called the *co-crossed product of A with H*. From the non-degeneracy one concludes easily that $\pi(A)\cup\mu(\mathrm{A}(H)) \subset M(A\otimes_\delta\mathcal{C}_0(H))$.

By [LPRS87, th. 3.7], the co-crossed product $A\otimes_\delta\mathcal{C}_0(H)$ can be characterised by the following universal property: $(\delta,1\otimes M)$ is (up to isomorphism) the unique covariant pair (π,μ) such that for any covariant pair (π',μ') of non-degenerate $*$-representations, there is a $*$-morphism $\vartheta : \mathrm{C}^*(A,\pi,\mu) \to \mathrm{C}^*(A,\pi',\mu')$ satisfying

$$\vartheta(\pi(a)\mu(\alpha)) = \pi'(a)\mu'(\alpha) \quad \text{for all } a\in A\,,\ \alpha\in\mathrm{A}(H)\,.$$

We define $\pi'\otimes_\delta\mu' = \vartheta$.

Similarly, if (M,δ) is a von Neumann algebra coaction, the von Neumann algebra

$$M\bar{\otimes}_\delta \mathbf{L}^\infty(H) = \big[\delta(M)(1\otimes \mathbf{L}^\infty(H))\big]'' \subset M\bar{\otimes}\mathcal{L}(\mathbf{L}^2(H))$$

is the ultraweakly closed span of $\delta(M)(1\otimes \mathbf{L}^\infty(H))$, and is called the *co-crossed product of M with H*.

6.2.18. Let $M \subset \mathcal{L}(\mathcal{H})$ be a von Neumann algebra and δ a coaction thereon, implemented by a unitary $W \in \mathcal{L}(\mathcal{H})\bar{\otimes}\mathrm{W}^*(H)$. Take $\mathcal{E} \subset M$ such that $\mathcal{E}^* = \mathcal{E} = \mathcal{E}\cdot\mathcal{E}$ (we do not assume $\mathcal{E}$ is a linear subspace). Define the *Toeplitz C*-algebra*

$$\mathcal{T}_\mathcal{E}(M,\delta) = \mathrm{C}^*\langle e\mu(\alpha)f \mid e,f\in\mathcal{E},\, \alpha\in \mathrm{A}(H)\rangle \subset \mathcal{L}(\mathcal{H})\,.$$

Typically, $\mathcal{E} = \{e\}$ where e is a projection. If $(M,\delta) = (\mathrm{W}^*(H),\delta_H)$ and e is the projection onto Hardy space, the Toeplitz C*-algebra is the C*-algebra generated by the Toeplitz operators eM_fe, as it should be.

Proposition 6.2.19. Given the conditions of 6.2.18, we have

$$\mathcal{T}_\mathcal{E}(M,\delta) = \overline{\langle \mathcal{E}\cdot \mathrm{C}^*\big(\mathrm{C}^*_\mathcal{E}(M,\delta),\mathrm{id},\mu\big)\cdot\mathcal{E}\rangle} = \overline{\langle \mathcal{E}\cdot(\mathrm{id}\otimes_\delta\mu)\big(\mathrm{C}^*_\mathcal{E}(M,\delta)\otimes_\delta \mathcal{C}_0(H)\big)\cdot\mathcal{E}\rangle}\,,$$

the closed linear span of $ea\mu(\alpha)f$, for $e,f\in\mathcal{E}$, $a\in \mathrm{C}^*_\mathcal{E}(M,\delta)$, and $\alpha\in\mathrm{A}(H)$.

In particular, if $e\in M$ is a projection,

$$\mathcal{T}_e(M,\delta) = e\big[(\mathrm{id}\otimes_\delta\mu)\big(\mathrm{C}^*_e(M,\delta)\otimes_\delta\mathcal{C}_0(H)\big)\big]e\,.$$

Proof. Since $\mathcal{T}_\mathcal{E}(M,\delta)$ is a C*-algebra, Cohen's theorem applies to it. Hence, it is generated as a Banach algebra by products

$$e\mu(\alpha)f\mu(\beta)g \quad \text{with} \quad e,f,g\in\mathcal{E}\,,\ \alpha,\beta\in\mathrm{A}(H)\,.$$

By lemma 6.2.16,

$$e\mu(\alpha)f\mu(\beta)g \in e(f)g \subset \mathcal{E}\cdot \mathrm{C}^*\big(\mathrm{C}^*_\mathcal{E}(M,\delta),\mathrm{id},\mu\big)\cdot\mathcal{E}\,.$$

Inductively, we find that the Toeplitz C*-algebra is contained in the right hand side.

On the other hand, let A be the C*-algebra generated by $\mu(\alpha)e\mu(\beta)$, $\alpha,\beta\in\mathrm{A}(H)$, $e\in\mathcal{E}$. Then $\mathcal{E}\cdot A\cdot\mathcal{E}\subset\mathcal{T}_\mathcal{E}(M,\delta)$.

Let $\alpha,\beta,\gamma\in\mathcal{K}\mathrm{A}(H)$. We have, by [NT79, th. A.4],

$$\big((\alpha*\beta^\vee)\cdot e\big)\mu(\gamma) = \int_H \mu(\alpha*t^{-1})\cdot e\cdot\mu\big((\beta*t^{-1})\cdot\gamma\big)\,dt\,,$$

the integral being ultraweakly convergent. Since

$$\chi : H \times H \to \mathbb{C} : (s,t) \mapsto \beta(st)\gamma(s) = \big((\beta * t^{-1}) \cdot \gamma\big)(s)$$

is a function in $\mathcal{K}\mathrm{A}(H \times H)$ by lemma 6.2.21 below, we may write $\chi = \sum_{j=0}^{\infty} \varphi_j \otimes \psi_j$ in $\mathrm{A}(H \times H)$, for some $\varphi_j, \psi_j \in \mathcal{K}\mathrm{A}(H)$. We find that $\beta * t^{-1} \cdot \gamma = \sum_{j=0}^{\infty} \psi_j(t) \cdot \varphi_j$, so

$$\begin{aligned}
\big((\alpha * \beta^\vee) \cdot e\big)\mu(\gamma) &= \sum_{j=0}^{\infty} \int_H \mu(\alpha * t^{-1}) \cdot e \cdot \psi_j(t)\, dt \cdot \mu(\varphi_j) \\
&= \sum_{j=0}^{\infty} \mu(\alpha * \psi_j^\vee) \cdot e \cdot \mu(\varphi_j) \,.
\end{aligned}$$

Since, for all $\varphi, \psi \in \mathrm{A}(H)$,

$$\begin{aligned}
\big\|\mu(\alpha * \psi^\vee) \cdot e \cdot \mu(\varphi)\big\| &\leqslant \big\|\alpha * \psi^\vee\big\|_\infty \cdot \|e\| \cdot \|\varphi\|_\infty \\
&\leqslant \|\alpha\|_1 \cdot \|\psi\|_\infty \cdot \|e\| \cdot \|\varphi\|_\infty = \|\alpha\|_1 \cdot \|e\| \cdot \|\varphi \otimes \psi\|_\infty \,,
\end{aligned}$$

the mapping $\varphi \otimes \psi \mapsto \mu(\alpha * \psi^\vee)e\mu(\varphi)$ is continuous for the topology of $\mathcal{C}_0(H \times H)$. Since the series for χ converges in this topology (it being weaker than the usual one on $\mathrm{A}(H \times H)$), we conclude $\big((\alpha * \beta^\vee) \cdot e\big)\mu(\gamma) \in A$.

If $\alpha_\lambda \in \mathcal{C}_0(H)$ is an approximate unit, $\mu(\alpha_\lambda)(\beta \cdot e)\mu(\gamma) \to (\beta \cdot e)\mu(\gamma)$ in norm for all $\beta, \gamma \in \mathrm{A}(H)$. This shows that as a C*-algebra, $\mathrm{C}^*\big(\mathrm{C}^*_{\mathcal{E}}(M,\delta), \mathrm{id}, \mu\big)$ is generated by $(\alpha \cdot e)\mu(\beta)$, $\alpha, \beta \in \mathrm{A}(H)$. Therefore,

$$\mathrm{C}^*\big(\mathrm{C}^*_{\mathcal{E}}(M,\delta), \mathrm{id}, \mu\big) \subset A \,,$$

and hence the conclusion. □

Remark 6.2.20. The use of the universal enveloping C*-algebra $\mathcal{C}_0(H \times H)$ of the commutative Banach $*$-algebra $\mathrm{A}(H \times H)$ in the proof of proposition 6.2.19 is essential.

The appropriate framework to treat completions of the tensor product $\mathrm{A}(H) \odot \mathrm{A}(H)$ directly would be that of operator spaces.

For, if M and N are von Neumann algebras, then, in the category of operator spaces and completely bounded maps, $(M \bar{\otimes} N)_* = M_* \hat{\otimes} N_*$, the projective tensor product of operator spaces, by [ER00, th. 7.2.4].

Note that in [ER03], Effros-Ruan develop the theory of Hopf von Neumann algebras in this framework.

The following lemma was used in the proof of proposition 6.2.19.

Lemma 6.2.21. Let $\alpha, \beta \in \mathcal{K}\mathrm{A}(H)$. Then $\gamma \in \mathcal{K}\mathrm{A}(H \times H)$ where

$$\gamma : H \times H \to \mathbb{C} : (s,t) \mapsto \alpha(st)\beta(s)$$

Moreover, $\|\gamma\|_{A(H\times H)} \leqslant \|\alpha\|_{A(H)} \cdot \|\beta\|_{A(H)}$.

Proof. For all $f, g \in \mathbf{L}^1(H)$, we have

$$\begin{aligned} \left| \int_H \int_H f(s)g(t)\gamma(s,t)\,ds\,dt \right| &= \left| \int_H \big(f * (\beta \cdot g)\big)(t)\alpha(t)\,dt \right| \\ &\leqslant \|f^\#\| \cdot \|(\beta \cdot g)^\#\| \cdot \|\alpha\|_{A(H)} \\ &\leqslant \|f^\#\| \cdot \|g^\#\| \cdot \|\alpha\|_{A(H)} \cdot \|\beta\|_{A(H)} \\ &= \|(f \otimes g)^\#\| \cdot \|\alpha\|_{A(H)} \cdot \|\beta\|_{A(H)} \ . \end{aligned}$$

Since $\mathbf{L}^1(H) \odot \mathbf{L}^1(H)$ is dense in $\mathbf{L}^1(H \times H)$, this implies that

$$\gamma \in B_\#(H \times H) \quad \text{and} \quad \|\gamma\|_{B_\#(H\times H)} \leqslant \|\alpha\|_{A(H)} \cdot \|\beta\|_{A(H)} \ .$$

Since γ has compact support, [Eym64, prop. 3.4] entails the lemma. □

6.3 Local regularity, compact operators, and irreducibility

6.3.1. We shall now restrict attention to the von Neumann algebra $M = W^*(H)$, acting on the Hilbert space $\mathbf{L}^2(H)$. Also, we assume $\mathcal{E}$ consists of a single projection $e \in W^*(H)$. Then the coproduct $\delta = \delta_H$ is a von Neumann algebra coaction, and together with the representation $M : \mathcal{C}_0(H) \to \mathcal{L}(\mathbf{L}^2(H)) : f \mapsto M_f$ by multiplication operators, forms a covariant pair (δ, M) whose associated corepresentation is the fundamental unitary $W = W_H$.

Since δ is the dual coaction of the trivial action of H on $\mathbb{C}$, Takesaki's duality theorem [NT79, th. 2.5] states that

$$\mathrm{id} \otimes_\delta M : W^*(H) \bar{\otimes}_\delta \mathbf{L}^\infty(H) \to \mathcal{L}(\mathbf{L}^2(H))$$

is an isomorphism which, by [Val85, lem. 5.2.8], restricts to an isomorphism

$$\mathrm{id} \otimes_\delta M : C^*_\#(H) \otimes_\delta \mathcal{C}_0(H) \to \mathbb{K}(\mathbf{L}^2(H)) \ .$$

By proposition 6.2.19, we get the equality

$$\mathcal{T}_e(H) = e\big[(\mathrm{id} \otimes_\delta M)(C^*_e(H) \otimes_\delta \mathcal{C}_0(H))\big]e = e\,\overline{\langle aM_\alpha \mid a \in C^*_e(H)\,,\, \alpha \in A(H)\rangle}\,e$$

for the Toeplitz C*-algebra. Hence, there is an obvious way to proving when $\mathcal{T}_e(H)$ contains the ideal of compact operators on the range of e .

Lemma 6.3.2. The C*–algebra $C^*_\#(H)$ is an ideal of $\overline{\mathcal{K}}W^*(H)$. In particular, we may consider $\overline{\mathcal{K}}W^*(H) \subset M(C^*_\#(H))$.

Proof. Clearly $\mathcal{K}\mathrm{W}^*(H)$ is dense in $\overline{\mathcal{K}}\mathrm{W}^*(H)$ and $\mathcal{K}\mathrm{A}(H)$ is dense in $\mathrm{A}(H)$. So, consider $x \in \mathrm{W}^*(H)$ with $\operatorname{supp} x$ compact and $\alpha \in \mathcal{K}\mathrm{A}(H)$. Let $x\alpha \in \mathrm{A}(H)$ be the element defined by the natural $\mathrm{W}^*(H)$-module structure on its predual $\mathrm{A}(H)$.

By [Eym64, prop. 3.17],

$$x\alpha^\# = (x\alpha)^\# \quad \text{is convolution by} \quad x\alpha \in \mathrm{A}(H) \,.$$

Since $x\alpha$ has compact support, it is contained in $\mathcal{K}\mathrm{A}(H) \subset \mathbf{L}^1(H)$, so $x\alpha^\# \in \mathrm{C}^*_\#(H)$. Since $\mathrm{C}^*_\#(H)$ is involutive, it is a two-sided ideal.

As to the second statement, $\mathrm{C}^*_\#(H)$ acts faithfully and non–degenerately on $\mathbf{L}^2(H)$, so $\mathrm{M}(\mathrm{C}^*_\#(G))$ is faithfully represented as the idealiser of $\mathrm{C}^*_\#(H)$ in its ultraweak closure $\mathrm{W}^*(H)$, by the von Neumann density theorem and [Bus68, 3.9 theorem]. Since $\overline{\mathcal{K}}\mathrm{W}^*(G) \subset \mathrm{W}^*(G)$, this entails the assertion. □

Proposition 6.3.3. Let the projection $e \in \mathrm{W}^*(H)$ be of full support $\operatorname{supp} e = H$. Moreover, assume that e is a.e. locally in $\mathrm{A}(H)$, i.e.

$$A = \bigl\{ g \in H \bigm| \alpha \cdot e \in \mathrm{A}(H) \text{ for some } \alpha \in \mathcal{K}\mathrm{A}(H)\,,\, \alpha(g) \neq 0 \bigr\}$$

is a thick subset of H. Then $\mathrm{C}^*_\#(H) \triangleleft \mathrm{C}^*_e(H)$.

Proof. Let $\alpha \in \mathcal{K}\mathrm{A}(H)$, $\operatorname{supp}\alpha \subset A$. Let $K \subset A$ be a compact neighbourhood of $\operatorname{supp}\alpha$, and $\chi \in \mathcal{K}\mathrm{A}(H)$, $1_K \leqslant \chi \leqslant 1_A$. Then

$$\psi = \chi \cdot e \in \mathrm{A}(H)\,, \quad \text{and} \quad \inf\nolimits_K |\psi| > 0 \,.$$

Since $\mathrm{A}(G)$ is Shilov-regular, there is $\varphi \in \mathcal{K}\mathrm{A}(H)$ so that $(\varphi \cdot \psi)\big|_K = 1$, cf. [Eym64, proof of prop. 4.4]. Then

$$\alpha \cdot \varphi \cdot e = \alpha \cdot \varphi \cdot \chi \cdot e = \alpha \cdot \varphi \cdot \psi = \alpha \,,$$

whence $\alpha \in \mathrm{C}^*_e(H)$. Since A is thick, $\{\alpha \in \mathcal{K}\mathrm{A}(H) \mid \operatorname{supp}\alpha \subset A\}$ is dense in $\mathbf{L}^1(H)$ and hence in $\mathrm{C}^*_\#(H)$. Thus, $\mathrm{C}^*_\#(H) \subset \mathrm{C}^*_e(H)$, and by lemma 6.3.2, it is an ideal. □

Proposition 6.3.4. Let the projection $e \in \mathrm{W}^*(H)$ have full support and be a.e. locally contained in $\mathrm{A}(H)$. Then $\mathbb{K}(e\mathbf{L}^2(H)) \triangleleft \mathcal{T}_e(H)$, in particular, the latter is faithfully and irreducibly represented on $e\,\mathbf{L}^2(H)$.

Proof. By proposition 6.3.3, $\mathrm{C}^*_\#(H) \subset \mathrm{C}^*_e(H)$, so by 6.3.1

$$\mathbb{K}(\mathbf{L}^2(H)) = (\mathrm{id} \otimes_\delta M)(\mathrm{C}^*_\#(H) \otimes_\delta \mathcal{C}_0(H)) \subset (\mathrm{id} \otimes_\delta M)(\mathrm{C}^*_e(H) \otimes_\delta \mathcal{C}_0(H)) \,.$$

Since $\mathbb{K}(e\mathbf{L}^2(H)) = e\,\mathbb{K}(\mathbf{L}^2(H))\,e$, we have

$$\mathbb{K}(e\mathbf{L}^2(H)) \subset e\bigl[(\mathrm{id} \otimes_\delta M)(\mathrm{C}^*_e(H) \otimes_\delta \mathcal{C}_0(H))\bigr]e = \mathcal{T}_e(H)\,,$$

by proposition 6.2.19. Now the irreducibility follows from Schur's lemma, because

$$\mathbb{K}(e\mathbf{L}^2(H))' = \mathcal{L}(e\mathbf{L}^2(H))' = \mathbb{C}$$

by the von Neumann density theorem. The representation is obviously faithful, since we have realised $\mathcal{T}_e(H)$ by operators on $e\,\mathbf{L}^2(H)$. □

Corollary 6.3.5. The conclusion of proposition 6.3.4 is true if H is a Lie group, e has full support, and $\operatorname{sing\,supp} e$ is thin. In particular, it applies to the Szegö distributions $e = E^{f,I}$ associated the subgroups $G_{f,I}$ of $G = \operatorname{Aut}_0 B$ where B is an irreducible bounded symmetric domain.

Proof. Suffices to remark that $\mathcal{D}(H) \subset \mathcal{K}\mathrm{A}(H)$, by [Eym64, prop. 3.26]. □

7 Asymptotics, singularities, and representations

7.1 Fourier coefficients and covariant pairs

7.1.1. In proposition 6.2.19, we have represented the Toeplitz C*-algebra $\mathcal{T}_e(H)$ associated to a projection $e \in \mathrm{W}^*(H)$ as a 'corner' of a co-crossed product. This suggests that its representations should be constructed from covariant pairs (π, μ) of representations π of $\mathrm{C}^*_e(H)$ and μ of $\mathrm{A}(H)$.

Having the application to Toeplitz operators defined by the Szegö distribution E on G and its subgroups $G_e \ltimes H_{e,c}$ in mind, we shall be interested particularly in the case where μ is restriction to a closed unimodular subgroup $\bar{H}$. To see how this fits into our framework, we cite the following theorem.

Theorem 7.1.2. Let $\bar{H} \sqsubset H$ be a closed unimodular subgroup.

(i). The restriction map

$$\operatorname{res}_{\bar{H}} : \mathrm{A}(H) \to \mathrm{A}(\bar{H}) : \alpha \mapsto \alpha\big|_{\bar{H}}$$

is well-defined, and an extremal epimorphism of Banach spaces. I.e., the induced map

$$\mathrm{A}(H)/\ker \operatorname{res}_{\bar{H}} \to \mathrm{A}(\bar{H})$$

is an isometry.

(ii). The adjoint $\operatorname{ext}_H = \operatorname{res}'_{\bar{H}}$ coincides on $\mathcal{M}^b(\bar{H})$ with the natural injection into $\mathcal{M}^b(H)$. It is an ultraweakly continuous isometry $\mathrm{W}^*(\bar{H}) \to \mathrm{W}^*(H)$, whose image is the set

$$\mathrm{W}^*_{\bar{H}}(H) = \left\{x \in \mathrm{W}^*(H) \;\middle|\; \operatorname{supp} x \subset \bar{H}\right\}.$$

In particular, for all $x \in W^*_{\bar{H}}(H)$, there exists a unique $x_{\bar{H}} \in W^*(\bar{H})$ such that

$$\langle \alpha\big|_{\bar{H}} : x_{\bar{H}} \rangle = \langle \alpha : x \rangle \quad \text{for all } \alpha \in A(\bar{H}) . \tag{7.1}$$

Proof. This is the content of [Her73, th. A and th. 1] and [TT72, th. 3]. □

7.1.3. The equation (7.1) and [Eym64, (3.16) and prop. 3.17] show that $\mathrm{res}_{\bar{H}}$ and its adjoint are algebra morphisms and $A(H)$-module maps.

In particular, $\mu = \mathrm{res}_{\bar{H}}$ is a non-degenerate representation of $A(H)$ on the Hilbert space $\mathbf{L}^2(\bar{H})$. Moreover,

$$Wf(s,t) = \big(W_{H,\bar{H}}\big)f(s,t) = f(s,s^{-1}t) \quad \text{for all } s \in \bar{H}\,,\ t \in H\,,\ f \in \mathbf{L}^2(\bar{H} \times H)$$

clearly defines the associated corepresentation. Note $(\mathrm{id} \otimes \mathrm{ext}_H)(W_{\bar{H}}) = W_{H,\bar{H}}$.

7.1.4. For the remainder of this subsection, we fix a closed subgroup $\bar{H} \sqsubset H$, and we assume H and $\bar{H}$ are both unimodular and of type I. Denote their reduced duals by $H^\wedge_\#$ and $\bar{H}^\wedge_\#$. These are precisely the sets of irreducible $*$-representations of the reduced group C*-algebras $C^*_\#(H)$ and $C^*_\#(\bar{H})$, and they extend to normal morphisms on the von Neumann algebras $W^*(H)$ and $W^*(\bar{H})$.

Fix projections $e \in W^*(H)$ and $\bar{e} \in W^*(\bar{H})$, such that $\bar{e} \leqslant e$. For almost every $\bar{\pi} \in \bar{H}^\wedge_\#$, assume given a sequence $\pi = (\pi_k) \subset H^\wedge_\#$ converging to ∞. This means that π is eventually contained in the complement of every quasi-compact subset of $H^\wedge_\#$ in the Jacobson topology. Assume given a sequence of $\bar{G}$-equivariant isometries $j_\pi = (j_{\pi,k})$

$$j_k = j_{\pi,k} : \langle \bar{H} \rangle_{\bar{\pi}} \longrightarrow \langle H \rangle_{\pi_k} \ ,$$

such that

$$j_k^*\, \pi_j(\alpha \cdot e)\, j_k \longrightarrow \bar{\pi}\big(\bar{\alpha}\big|_{\bar{H}} \cdot \bar{e}\big) \quad \text{for all } \alpha \in A(H) \tag{7.2}$$

in the weak topology on $\mathcal{L}(\langle \bar{H} \rangle_{\bar{\pi}})$.

The image $\pi(\alpha \cdot e)$ can be thought of as a generalised Fourier coefficient of $\alpha \cdot e$. Then the above relation could be viewed as a boundary condition on the Fourier transform of $\alpha \cdot e$. We shall now see how such a *spectral boundary condition* gives rise to a representation π of $C^*_e(H)$ on $C^*_{\bar{e}}(\bar{H})$ which is covariant w.r.t. the restriction representation $\mu = \mathrm{res}_{\bar{H}}$. Moreover, the associated representation $\pi \otimes_\delta \mu$ of the co-crossed product restricts to a representation of $\mathcal{T}_e(H)$.

Proposition 7.1.5. Assume the spectral boundary condition 7.1.4 given. Then, for every $a \in C^*_e(H)$, there exists $\bar{a} = \pi_{\bar{H}}(a) \in C^*_{\bar{e}}(\bar{H})$ such that

$$\bar{\pi}(\bar{a}) = \lim_k j_k^*\, \pi(a)\, j_k \quad \text{strongly in} \quad \mathcal{L}(\langle \bar{H} \rangle_{\bar{\pi}})$$

for a.e. $\bar{\pi} \in \bar{H}^{\wedge}_{\#}$. This defines a surjective $*$-morphism

$$\pi_{\bar{H}} : \mathrm{C}^*_e(H) \to \mathrm{C}^*_{\bar{e}}(\bar{H}) \quad \text{so that} \quad \pi_{\bar{H}}(\alpha \cdot e) = \alpha\big|_{\bar{H}} \cdot \bar{e} \quad \text{for all } \alpha \in \mathrm{A}(H) .$$

The proof of this proposition requires a series of lemmata.

Lemma 7.1.6. Assume that the spectral boundary condition 7.1.4 is satisfied. Then the convergence in (7.2) is strong, and

$$\lim_k \big\| (1 - j_k j_k^*) \pi_k(\alpha \cdot e) j_k \psi \big\| = 0 \quad \text{for all } \psi \in \langle \bar{H} \rangle_{\bar{\pi}} \, , \, \alpha \in \mathrm{A}(H) .$$

Proof. Recall that # denotes the left regular representation of H on $\mathbf{L}^2(H)$. Define orthogonal projections e_k and p_k on $\langle H \rangle_{\pi_k} \otimes \mathbf{L}^2(H)$ and $\langle H \rangle_{\pi_k}$ by

$$e_k = (\pi_k \otimes \#)(e) \quad \text{and} \quad p_k = j_k j_k^* .$$

Set

$$A_k = (j_k^* \otimes 1)\, e_k \,(j_k \otimes 1) \quad \text{and} \quad C_k = ((1 - p_k) \otimes 1)\, e_k \,(j_k \otimes 1) .$$

Then

$$A_k^2 + C_k^* C_k = (j_k^* \otimes 1) e_k \big((p_k + 1 - p_k) \otimes 1 \big)\, e_k (j_k \otimes 1) = A_k .$$

If $\varphi, \psi \in \mathbf{L}^2(H)$, then for $\alpha = \overline{\varphi} * \psi^{\vee}$, we have

$$(u \otimes \varphi \mid A_k \, v \otimes \psi) = (j_k \, u \otimes \varphi \mid e_k \, (j_k \, v) \otimes \psi) = (u \mid j_k^* \pi_k(\alpha \cdot e) j_k \, v)$$

for all $u, v \in \langle \bar{H} \rangle_{\bar{\pi}}$. Hence,

$$(u \otimes \varphi \mid A_k \, v \otimes \psi) \to \big(u \mid \bar{\pi}\big(\alpha\big|_{\bar{H}} \cdot \bar{e}\big)\, v \big) = \big(u \otimes \varphi \mid (\bar{\pi} \otimes \#)(\bar{e})\, v \otimes \psi \big) ,$$

by (7.1). So, A_k converges weakly to a projection $p = (\bar{\pi} \otimes \#)(\bar{e})$.

By $\bar{G}$-equivariance of j_k ,

$$p = (j_k^* \otimes 1)(\pi_k \otimes \#)(\bar{e})(j_k \otimes 1) \leqslant (j_k^* \otimes 1)\, e_k \,(j_k \otimes 1) = A_k ,$$

and $[p, A_k] = 0$. Since the product of commuting positive operators is positive, and $A_k^2 - p = (A_k - p)(A_k + p)$, we find $p \leqslant A_k^2 \leqslant A_k \to p$, so $C_k^* C_k = A_k - A_k^2 \to 0$ in the weak topology. But this means that $C_k \to 0$ in the strong topology. Analogously, the weak convergence of A_k^2 implies the strong convergence of A_k , so the convergence in (7.2) is strong. □

Lemma 7.1.7. Assume the spectral boundary condition 7.1.4 fulfilled. For all finite sequences $(\alpha_j) \subset \mathrm{A}(H)$,

$$\lim_k j_k^* \Big[\textstyle\prod_j \pi_k(\alpha_j \cdot e) \Big] j_k = \textstyle\prod_j \bar{\pi}\big(\alpha_j\big|_{\bar{H}} \cdot \bar{e}\big) \quad \text{strongly in} \quad \mathcal{L}(\langle \bar{H} \rangle_{\bar{\pi}}) .$$

Proof. By lemma 7.1.6, for all j, the sequences $(j_k^*\pi_k(\alpha_j\cdot p)j_k)$ are bounded and strongly convergent. In particular, their product converges strongly. Moreover,

$$\lim_k j_k^*\pi_k(\alpha_1\cdot e)(1-j_kj_k^*)\pi_k(\alpha_2\cdot e)j_k = 0\,,$$

so we deduce

$$\begin{aligned}\lim_k \prod_j j_k^*\pi_k(\alpha_k\cdot e)j_k &= \lim_k j_k^*\pi_k(\alpha_1\cdot e)\big[j_kj_k^* + (1-j_kj_k^*)\big]\pi_k(\alpha_2\cdot e)j_k\prod_{j>2} j_k^*\pi_k(\alpha_j\cdot e)j_k\\ &= \bar\pi\big(\alpha_1\big|_{\bar H}\cdot\bar e\big)\lim_k j_k^*\pi_k(\alpha_2\cdot e)j_k\prod_{j>2} j_k^*\pi_k(\alpha_j\cdot e)j_k\,.\end{aligned}$$

Inductively, the claim follows. □

Proof of proposition 7.1.5. Let $A\subset C_e^*(H)$ be the dense $*$-algebra generated by the set $\{\alpha\cdot e \mid \alpha\in \mathrm{A}(H)\}$. Since $e^*=e$, A is the linear span of

$$(\alpha_1\cdot e)\cdots(\alpha_n\cdot e)\quad\text{where}\quad n\in\mathbb{N}\,,\ (\alpha_j)\subset \mathrm{A}(H)\,.$$

Define $\pi_{\bar H}((\alpha_1\cdot e)\cdots(\alpha_n\cdot e)) = (\alpha_1\big|_{\bar H}\cdot\bar e)\cdots(\alpha_n\big|_{\bar H}\cdot\bar e)$ and extend linearly. We need to see that $\pi_{\bar H}$ is well-defined. To that end, let

$$\textstyle\sum_j(\alpha_{1,j}\cdot e)\cdots(\alpha_{n_j,j}\cdot e) = \sum_j(\beta_{1,j}\cdot e)\cdots(\beta_{m_j,j}\cdot e)\,.$$

By lemma 7.1.7, for a.e. $\bar\pi\in\bar H_\#^\wedge$

$$\begin{aligned}\textstyle\sum_j\prod_{i=1}^{n_j}\bar\pi\big(\alpha_{ij}\big|_{\bar H}\cdot\bar e\big) &= \textstyle\lim_k j_k^*\sum_j\prod_{i=1}^{n_j}\pi_k(\alpha_{ij}\cdot e)\,j_k\\ &= \textstyle\lim_k j_k^*\sum_j\prod_{i=1}^{m_j}\pi_k(\beta_{ij}\cdot e)\,j_k = \sum_j\prod_{i=1}^{m_j}\bar\pi\big(\beta_{ij}\big|_{\bar H}\cdot\bar e\big)\,.\end{aligned}$$

So $\pi_{\bar H}$ is well-defined, in fact, a $*$-morphism. Moreover, for all $a\in A$ and a.e. $\bar\pi\in\bar H_\#^\wedge$, we have for $\bar a = \pi_{\bar H}(a)$,

$$\|\bar\pi(\bar a)\|\leqslant \sup_k\|\pi_k(a)\|\leqslant\|a\|\ ,$$

from the uniform boundedness principle. So,

$$\|\bar a\| = \operatorname{ess\,sup}_{\bar\pi}\|\bar\pi(\bar a)\|\leqslant\|a\|\ ,$$

cf. [Tak76]. By continuity, $\pi_{\bar H}$ extends to a $*$–morphism on $C_e^*(H)$ whose image is manifestly $C_{\bar e}^*(\bar G)$. Moreover, $\pi_{\bar H}$ is given on A by strong limits as stated above. Since $\pi_{\bar H}$ is a contraction, this formula extends to all of $C_e^*(H)$. □

7.1.8. We introduce the following uniform notation for matrix coefficients,

$$\Delta^{\pi}_{u,v}(g) = (u \mid \pi(g)\,v) \quad \text{for all } g \in H\,,\, u,v \in \langle H\rangle_{\pi}\ .$$

Moreover, we abbreviate $\Delta^{\pi}_{u} = \Delta^{\pi}_{u,u}$, $\bar{\Delta}_{u,v} = \Delta^{\bar{\pi}}_{u,v}$ and $\Delta^{k}_{u,v} = \Delta^{\pi_k}_{u,v}$.

Proposition 7.1.9. Assume given the spectral boundary condition 7.1.4. If $C^*_{\bar{e}}(\bar{H})$ acts non-degenerately on $\mathbf{L}^2(\bar{H})$, then $(\pi_{\bar{H}}, \mathrm{res}_{\bar{H}})$ is a covariant pair of representations on $\mathbf{L}^2(\bar{H})$ for the coaction $\delta = \delta_H$ on $C^*_e(H)$. The non-degeneracy condition is fulfilled if $\bar{e}$ has full support and is a.e. locally contained in $A(\bar{H})$.

Proof. If $C^*_{\bar{e}}(\bar{H})$ acts non-degenerately on $\mathbf{L}^2(\bar{H})$, $\pi_{\bar{H}}$ is non-degenerate. If $\bar{e}$ has full support and is a.e. locally contained in $A(\bar{H})$, proposition 6.3.3 shows $C^*_{\#}(\bar{H}) \subset C^*_{\bar{e}}(\bar{H})$, and the former acts non-degenerately.

Let $\alpha \in A(H)$, $a \in C^*_e(H)$ and $b \in C^*_{\#}(H)$. Write $\pi = \pi_{\bar{H}}$. Then, for a.e. $\bar{\pi} \in \bar{H}^{\wedge}_{\#}$,

$$(\bar{\pi}\otimes\#)\big[(\pi\otimes\mathrm{id})(\delta(\alpha\cdot e)(a\otimes b))\big] = \lim_k (j_k^*\otimes 1)[(\pi_k\otimes\#)\delta(\alpha\cdot e)(\pi_k(a)\otimes b)](j_k\otimes 1)\ .$$

Let $\varphi,\psi \in \mathbf{L}^2(H)$ and set $\beta = \overline{\varphi} * \psi^{\vee}$. For all $u,v \in \langle H\rangle_{\pi_k}$,

$$(u\otimes\varphi \mid (\pi_k\otimes\#)(\delta(a))\,v\otimes\psi) = \langle \Delta^k_{u,v}\otimes\beta \,:\, \delta(a)\rangle = \langle \Delta^k_{u,v} \,:\, \beta\cdot a\rangle = (u \mid \pi_k(\beta\cdot a)\,v)\ .$$

Consequently, if we take $\beta = \overline{b\varphi} * \psi^{\vee}$, for all $u,v \in \langle \bar{H}\rangle_{\bar{\pi}}$,

$$((j_k\,u)\otimes\varphi \mid (\pi_k\otimes\#)(\delta(\alpha\cdot e))(\pi_k(a)\otimes b)\,(j_k\,v)\otimes\psi) = (j_k\,u \mid \pi_k(\alpha\beta\cdot e)\pi_k(a)\,j_k\,v)$$

which converges to

$$\begin{aligned}
\Big(u \;\Big|\; \bar{\pi}\big[(\alpha\beta|_{\bar{H}}\cdot\bar{e})\cdot\bar{a}\big]\,v\Big) &= \Big\langle \bar{\Delta}_{u,\bar{\pi}(\bar{a})v}\otimes\beta|_{\bar{H}} \,:\, \delta(\alpha|_{\bar{H}}\cdot\bar{e})\Big\rangle \\
&= \Big\langle \bar{\Delta}_{u,\bar{\pi}(\bar{a})v}\otimes\beta \,:\, \mathrm{Ad}(W)\big(\alpha|_{\bar{H}}\cdot\bar{e}\otimes 1\big)\Big\rangle \\
&= \Big(u\otimes\varphi \;\Big|\; (\bar{\pi}\otimes\#)\big[\mathrm{Ad}(W)\big(\alpha|_{\bar{H}}\cdot\bar{e}\otimes 1\big)(\pi(a)\otimes b)\big]\,v\otimes\psi\Big)
\end{aligned}$$

where we denote $W = W_{\bar{H},H}$ and $\bar{a} = \pi(a)$. Since π and $C^*_{\#}(\bar{H})$ are non-degenerate,

$$(\pi\otimes\mathrm{id})(\delta(\alpha\cdot e)) = \mathrm{Ad}(W)\big[\alpha|_{\bar{H}}\cdot\bar{e}\otimes 1\big]\,,$$

which is the required covariance condition. □

Remark 7.1.10. Observe the following consequence of proposition 7.1.9.

Assume that $C^*_{\bar{e}}(\bar{H})$ acts non-degenerately. We noted in 6.2.14 that the covariant pair $(\pi,\mu) = (\pi_{\bar{H}}, \mathrm{res}_{\bar{H}})$ defines a von Neumann algebra coaction of H on the bicommutant $\pi(C^*_e(H))'' = W^*(\bar{H})$. The relation $W_{\bar{H},H} = (\mathrm{id}\otimes\mathrm{ext}_{\bar{H}})(W_{\bar{H}})$ shows that

$$(\mathrm{id}\otimes\alpha)\big(\mathrm{Ad}(W_{H,\bar{H}})(a\otimes 1)\big) = \alpha|_{\bar{H}}\cdot a \quad \text{for all } \alpha \in A(H)\,,\, a \in C^*_{\bar{e}}(\bar{H})\,,$$

in particular, $\pi = \pi_{\bar{H}}$ is $\mathrm{A}(H)$-linear (even $\mathrm{B}_{\#}(H)$-linear).

Theorem 7.1.11. Assume that the spectral boundary condition 7.1.4 is fulfilled. If the C^*-algebra $\mathrm{C}^*_{\bar{e}}(\bar{H})$ acts non-degenerately on $\mathbf{L}^2(\bar{H})$, then

$$\varrho_{\bar{H}}(eae) = \bar{e}\left(\pi_{\bar{H}} \otimes_\delta \mathrm{res}_{\bar{H}}\right)(a)\,\bar{e} \quad \text{for all } a \in \mathrm{C}^*_e(H) \otimes_\delta \mathrm{C}^*_{\#}(H)$$

defines a non-degenerate $*$-representation of $\mathcal{T}_e(H)$ on $\bar{e}\,\mathbf{L}^2(\bar{H})$. Moreover,

$$\varrho_{\bar{H}}(T_f) = \bar{T}_{f|\bar{H}} \quad \text{for all } f \in \mathcal{C}_0(\bar{H})$$

where $T_f = eM_f e$, $\bar{T}_f = \bar{e}M_f\bar{e}$ are the Toeplitz operators of symbol f.

The non-degeneracy condition is satisfied if $\bar{e}$ has full support and is a.e. locally contained in $\mathrm{A}(H)$. In this case, $\varrho_{\bar{H}}$ is irreducible.

Proof. We need to ascertain the well-definedness of $\varrho = \varrho_{\bar{H}}$. To that end, we abbreviate $\nu = \pi_{\bar{H}} \otimes_\delta \mathrm{res}_{\bar{H}}$. Let $a \in \mathrm{C}^*_e(H) \otimes_\delta \mathcal{C}_0(G)$, so that $eae = 0$. By lemma 6.2.16 and its proof, for all $b, c \in \mathrm{C}^*_e(H) \otimes \mathcal{C}_0(H)$, $bec \in \mathrm{C}^*_e(H) \otimes \mathcal{C}_0(H)$ and $\nu(bec) = \nu(b)\bar{e}\nu(c)$.

Write $a = a_1 \cdot a_2$ for some $a_1, a_2 \in \mathrm{C}^*_e(H) \otimes \mathcal{C}_0(H)$ (Cohen's theorem). Then

$$\nu(b)\bar{e}\nu(a)\bar{e}\nu(c) = \nu(b)\bar{e}\nu(a_1)\nu(a_2)\bar{e}\nu(c) = \nu(bea_1)\nu(a_2ec) = \nu(beaec) = 0$$

for all $b, c \in \mathrm{C}^*_e(H) \otimes_\delta \mathcal{C}_0(H)$. Since ν is non-degenerate by assumption, $\bar{e}\nu(a)\bar{e} = 0$, and ϱ is well-defined.

Clearly, $\varrho_{\bar{H}}$ is linear and involutive. Furthermore,

$$\varrho(eaebe) = \bar{e}\nu(aeb)\bar{e} = \bar{e}\nu(a)\bar{e}\nu(b)\bar{e} = \varrho(eae)\varrho(ebe) \quad \text{for all } a, b \in \mathrm{C}^*_e(H) \otimes_\delta \mathcal{C}_0(H)\,,$$

so ϱ is a $*$-morphism. Since $(\pi_{\bar{H}}, \mathrm{res}_{\bar{H}})$ is a covariant pair,

$$\nu(aM_f) = \pi_{\bar{H}}(a)M_{f|\bar{H}} \quad \text{for all } a \in \mathrm{C}^*_e(H)\,,\ f \in \mathrm{A}(H)\,.$$

This implies that $\varrho(T_f) = \bar{T}_{f|\bar{H}}$ for all $f \in \mathrm{A}(H)$, and by density, for all $f \in \mathcal{C}_0(H)$. Hence, ϱ is surjective onto $\mathcal{T}_{\bar{e}}(\bar{H})$, and in particular, non-degenerate on $\bar{e}\,\mathbf{L}^2(\bar{H})$. If $\bar{e}$ has full support and is a.e. locally contained in $\mathrm{A}(\bar{H})$, then proposition 6.3.4 shows that ϱ is irreducible. □

7.1.12. Let e be central, i.e.

$$\mathrm{Ad}(g)(e) = c_{g*}(e) = g^{\#}eg^{-1\#} = e \quad \text{for all } g \in H\,.$$

Then $\mathrm{C}^*_e(H)$ is $\mathrm{Ad}(H)$-invariant. What is more, if we set

$$d_{g,h} := \mathrm{Ad}\big((g,g)^{\#}(1,h)_{\#}\big) \quad \text{for all } g, h \in H\,,$$

then $d : H \times H \to \operatorname{Aut}\big(\mathrm{C}^*_e(H) \otimes_\delta \mathcal{C}_0(H)\big)$ is an action of $H \times H$.

If (π, μ) is a covariant pair of representations of $\mathrm{C}^*_e(H)$ for the coaction $\delta = \delta_H$, then

$$\big((\pi \otimes_\delta \mu) \circ d_{g,h}^{-1}\big)(\delta(a)(1 \otimes f)) = \pi\big(\operatorname{Ad}(g^{-1\#})(a)\big)\mu(g * f * h)$$

for all $a \in \mathrm{C}^*_e(H)$, $f \in \mathcal{C}_0(H)$, since $\delta \circ \operatorname{Ad}(g^\#) = (g,g)^\# \circ \delta$. Here, recall that for all $t \in H$, $(g * f * h)(t) = f(g^{-1}th)$. Moreover,

$$(\pi', \mu') = \big(\pi \circ \operatorname{Ad}(g^{-1\#}), \mu \circ g^\# \circ h_\#\big)$$

is a covariant pair of representations such that

$$\pi' \otimes_\delta \mu' = (\pi \otimes_\delta \mu) \circ d_{g,h}^{-1},$$

cf. [LPRS87, lem. 5.4]. This establishes a natural action of $H \times H$ on the set of non-degenerate representations of $\mathrm{C}^*_e(H) \otimes_\delta \mathcal{C}_0(H)$. With this in mind, the proof of the following corollary it straightforward.

Corollary 7.1.13. Let the spectral boundary condition 7.1.4 be satisfied, and assume that e is central and $\bar{e}$ has full support and is a.e. locally contained in $\mathrm{A}(\bar{H})$. Let $g, h \in H$,

$$\pi_{g\bar{H}h^{-1}} = \pi_{\bar{H}} \circ \operatorname{Ad}\big(g^{-1\#}\big) \quad \text{and} \quad \operatorname{res}_{g\bar{H}h^{-1}} = \operatorname{res}_{\bar{H}} \circ g^\# \circ h_\#\,.$$

Then

$$\varrho_{g\bar{H}h^{-1}}(eae) = \bar{e}\,\big(\pi_{g\bar{H}h^{-1}} \otimes_\delta \mu_{g\bar{H}h^{-1}}\big)(a)\,\bar{e} \quad \text{for all } a \in \mathrm{C}^*_e(H) \otimes_\delta \mathcal{C}_0(H)$$

defines an irreducible $*$-representation of $\mathcal{T}_e(H)$ on $\bar{e}\,\mathbf{L}^2(\bar{H})$ such that

$$\varrho_{g\bar{H}h^{-1}}(T_f) = \bar{T}_{g*f*h|\bar{H}} \quad \text{for all } f \in \mathcal{C}_0(H)\,.$$

7.1.14. A feature of the construction of representations of the Toeplitz C*-algebra is that it behaves naturally under restriction to subgroups. To emphasise the dependence of $\varrho_{\bar{H}}$ on H, we also write $\varrho^H_{\bar{H}}$.

Corollary 7.1.15. Let $H_1 \sqsubset H_2 \sqsubset H$ be closed subgroups such that H_2 satisfies the spectral boundary condition 7.1.4 relative H, and H_1 satisfies it relative H_2. Then H_1 satisfies the spectral boundary condition relative H.

If the projections $e_1 \in \mathrm{W}^*(H_1)$ and $e_2 \in \mathrm{W}^*(H_2)$ have full support and are a.e. locally contained in the Fourier algebra, then, for the representations from theorem 7.1.11,

$$\varrho^H_{H_1} = \varrho^{H_2}_{H_1} \circ \varrho^H_{H_2} \quad \text{and} \quad \ker \varrho^H_{H_2} \lhd \ker \varrho^H_{H_1}$$

holds, and similarly for their shifted versions from corollary 7.1.13.

Proof. Clearly, we can concatenate the respectively H_2- and H_1-equivariant embeddings j^2_k and j^1_ℓ and choose an appropriate enumeration of $\mathbb{N}^2$ to get H_1-equivariant embeddings of $\pi^1 \in H^\wedge_{1\#}$ into $\pi_m \in H^\wedge_\#$. The convergence statement is then also trivial.

As to the formula for the representation of the Toeplitz C*-algebras,

$$\varrho^H_{H_1}(T_f) = e_1 M_{f|H_1} e_1 = \varrho^{H_2}_{H_1}(e_2 M_{f|H_2} e_2) = \varrho^{H_2}_{H_1}\big(\varrho^H_{H_2}(T_f)\big)$$

for all $f \in \mathcal{C}_0(H)$, by theorem 7.1.11. Hence, the inclusion of kernels. □

7.2 Supports at infinity

In the construction of representations of $\mathcal{T}_e(H)$ from the spectral boundary condition 7.1.4, we considered limits of sequences $\beta_k \cdot e$ where $\beta_k \in \mathrm{B}_\#(H)$ were $\beta_k = \Delta^k_{j_k u, j_k v}$ for fixed $u, v \in \langle \bar{H} \rangle_{\bar{\pi}}$. In order to verify the spectral boundary condition in the concrete situation of the Szegö distributions, we need to compute the limits of $\beta_k \cdot e$. A first step in the computation of these limits is to bound their support. Therefore, it is useful to have general principles at hand which allow for such estimates.

7.2.1. Since $\mathrm{C}^*_\#(H)$ is an $\mathrm{A}(H)$-submodule, $E = \mathrm{W}^*(H)/\mathrm{C}^*_\#(H)$ is an $\mathrm{A}(H)$-module. Hence, the notion of support makes sense, and we define the *singular set*

$$\operatorname{sing} x = \operatorname{supp}_E[x] \quad \text{for all } \ x \in \mathrm{W}^*(H) \ .$$

Clearly, if x is locally integrable at $g \in H$, then $g \notin \operatorname{sing} x$. In particular, if H is a Lie group, $\operatorname{sing} x \subset \operatorname{sing\,supp} x$.

Moreover, if $\beta = (\beta_j) \subset \mathrm{B}(H)$ is a norm bounded sequence, let $\operatorname{supp}^\infty \beta$ be

$$\operatorname{supp}^\infty \beta = \bigcap \big\{ E \subset G \text{ closed} \bigm| \forall\, \alpha \in \mathcal{K}\mathrm{A}(G) \ : \ \operatorname{supp} \alpha \subset G \setminus E \ \Rightarrow \ \lim\nolimits_j \lVert \alpha \cdot \beta_j \rVert = 0 \big\} \ .$$

i.e. the smallest closed subset of H such that for all $\alpha \in \mathcal{K}\mathrm{A}(H)$ vanishing on a neighbourhood of its complement, $\alpha \cdot \beta_j \to 0$ in norm. This is the support for the $\mathrm{A}(H)$-module

$$\ell^\infty(\mathbb{N}, \mathrm{B}(H)) / c_0(\mathbb{N}, \mathrm{B}(H)) \ ,$$

by proposition 6.2.4 (iii).

Proposition 7.2.2. Let $\beta = (\beta_j) \subset \mathrm{B}_\#(G)$ be bounded, such that $\lim_j \beta_j = 0$ a.e. on H. For any $x \in \mathrm{W}^*(H)$ such that $\bar{x} = \lim_j \beta_j \cdot x$ exists in $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$,

$$\operatorname{supp} \bar{x} \subset \operatorname{sing} x \ .$$

Proof. Boundedness and pointwise a.e. convergence $\beta_j \to 0$ implies convergence in $\sigma(\mathrm{B}_\#(H), \mathbf{L}^1(H))$ by the dominated convergence theorem. The σ-topologies induced by $\mathbf{L}^1(H)$ and $\mathrm{C}^*_\#(H)$ coincide on bounded subsets, because $\mathbf{L}^1(H) \subset \mathrm{C}^*_\#(H)$ is norm dense. Hence, $\beta_j \to 0$ in $\sigma(\mathrm{B}_\#(H), \mathrm{C}^*_\#(H))$.

If $\alpha \in \mathcal{K}\mathrm{A}(H)$ is such that $\operatorname{supp} \alpha \cap \operatorname{sing} x = \varnothing$, then $\alpha \cdot x \in \mathrm{C}^*_{\#}(H)$ by proposition 6.2.4 (iii). This implies

$$\langle \gamma : \alpha \cdot \bar{x} \rangle = \lim_j \big\langle \beta_j : \gamma \cdot (\alpha \cdot x) \big\rangle = 0 \quad \text{for all } \gamma \in \mathrm{A}(H)\,,$$

so $\alpha \cdot \bar{x} = 0$, and the assertion follows by proposition 6.2.4 (iii). □

Proposition 7.2.3. Let $\beta = (\beta_j) \subset \mathrm{B}(H)$ be bounded. For any $x \in \mathrm{W}^*(H)$ such that $\bar{x} = \lim_j \beta_j \cdot x$ exists in the $\sigma(\mathrm{W}^*(H), \mathrm{A}(H))$-topology,

$$\operatorname{supp} \bar{x} \subset \operatorname{supp}^\infty \beta\,.$$

Proof. Let $\alpha \in \mathrm{A}(H)$ such that $\lim_j \|\alpha \cdot \beta_j\| = 0$. Then

$$\|\alpha \cdot \bar{x}\| = \lim_j \big\|\alpha \cdot \beta_j \cdot x\big\| \leqslant \|x\| \cdot \lim_j \big\|\alpha \cdot \beta_j\big\| = 0\,,$$

so $\alpha \cdot \bar{x} = 0$. The assertion follows from proposition 6.2.4 (iii). □

Proposition 7.2.4. Let H be a Lie group and $\beta = (\beta_j) \subset \mathrm{B}(H)$ be smooth. If $g \in H$ is such that $\beta_j \to 0$ uniformly with all derivatives, locally on a neighbourhood U of g, then $g \notin \operatorname{supp}^\infty \beta$.

Proof. Let $\varphi \in \mathcal{D}(H)$ be such that $\varphi(g) \neq 0$ and $\operatorname{supp} \varphi \subset U$. We have $\varphi \cdot \beta_j \to 0$ in the usual (LF) topology on $\mathcal{D}(H)$. By [Eym64, prop. 3.26], this topology is finer than the norm topology of $\mathrm{A}(H)$. Hence, $g \notin \operatorname{supp}^\infty \beta$, by 6.2.2. □

7.2.5. Besides being useful in the computation of the limits of $\beta_k \cdot e$, as we shall see below, the concept of singular set also has consequences for the kernels of the representations $\varrho_{\bar{H}}$ constructed in the previous subsection.

Proposition 7.2.6. Let the spectral boundary condition 7.1.4 be verified, and assume $\bar{e}$ has full support and is a.e. locally contained in $\mathrm{A}(\bar{H})$. Then $\mathrm{C}^*_{\#}(H) \triangleleft \ker \pi_{\bar{H}}$, and consequently $\mathbb{K}(\bar{e}\mathbf{L}^2(\bar{H})) \triangleleft \ker \varrho_{\bar{H}}$.

Proof. The proposition is an immediate consequence of proposition 6.3.3 and the following lemma. □

Lemma 7.2.7. Assume the spectral boundary condition 7.1.4. If $a \in \mathrm{C}^*_e(H)$ is such that

$$\operatorname{sing} a \cap \operatorname{supp}^\infty(\Delta^k_{j_k u}) = \varnothing \quad \text{for all } u \in \langle \bar{H} \rangle_{\bar{\pi}}$$

a.e. $\bar{\pi} \in \bar{H}^\wedge_{\#}$ and (π_k) associated to $\bar{\pi}$, then $\pi_{\bar{H}}(a) = 0$.

Proof. Fix $\bar{\pi}$ and $u \in \langle \bar{H} \rangle_{\bar{\pi}}$, and write $\bar{a} = \pi_{\bar{H}}(a)$. For all $\alpha \in \mathrm{A}(H)$,

$$\begin{aligned}\big\langle \alpha|_{\bar{H}} : \bar{\Delta}_u \cdot \bar{a} \big\rangle &= \big(u \mid \bar{\pi}\big(\alpha|_{\bar{H}} \cdot \bar{a}\big)\, u\big) \\ &= \lim_k \left(j_k\, u \,:\, \pi_k(\alpha \cdot a)\, j_k\, v\right) = \lim_k \left\langle \alpha \,:\, \Delta^k_{j_k\, u} \cdot a \right\rangle\,,\end{aligned}$$

by proposition 7.1.5. Hence,

$$\bar{\Delta}_u \cdot \bar{a} = \lim_k \Delta^k_{j_k u} \cdot a \quad \text{in} \quad \sigma(\mathrm{W}^*(H), \mathrm{A}(H))\,,$$

where we consider $\mathrm{W}^*(\bar{H}) \subset \mathrm{W}^*(H)$, by theorem 7.1.2. Since $\pi_k \to \infty$ in $H^\wedge_\#$, $\Delta^k_{j_k u} \to 0$ a.e. on H, by the Riemann-Lebesgue lemma [Dix69, prop. 3.3.8, 18.2.4].

By propositions 7.2.2 and 7.2.3,

$$\operatorname{supp}(\bar{\Delta}_u \cdot \bar{a}) \subset \operatorname{sing} a \cap \operatorname{supp}^\infty(\Delta^k_{j_k u}) = \varnothing\,.$$

As a submodule of its bicommutant, $\mathrm{C}^*_{\bar{e}}(\bar{H})$ is a non-degenerate $\mathrm{A}(\bar{H})$-module, by proposition 6.2.5 (i). Proposition 6.2.4 (i) implies that $\bar{\Delta}_u \cdot \bar{a} = 0$. Thus, $\bar{\pi}(\alpha \cdot \bar{a}) = 0$ for all $\alpha \in \mathrm{A}(\bar{H})$. Since this condition is verified for a.e. $\bar{\pi} \in \bar{H}^\wedge_\#$, we have $\alpha \cdot \bar{a} = 0$ for all $\alpha \in \mathrm{A}(\bar{H})$. Again applying non-degeneracy, $\bar{a} = 0$. ☐

7.3 Computation of the limits

7.3.1. We now wish to apply the theory developed above to construct representations of the Toeplitz C*-algebra $\mathcal{T}_E(G)$ associated to the Szegö distribution E of the connected automorphism group $G = \mathrm{Aut}_0\, B$. The subgroups $G_e \ltimes H_{e,c}$ associated to the faces of Ω^- are unimodular and of type I. While the construction of the representations for all of the faces seems a rather ambitious project, the facial subgroups $\bar{G} = G_e$ (where $c = 0$) appear to be more tractable. We only treat this case.

We need to make some additional assumptions. First, we assume Z is classical. Then we have constructed embeddings of the discrete series of $\bar{G} = G_e$, in corollary 5.2.25. (For the holomorphic discrete series, we have done this general, cf. proposition 4.3.4.) Assuming that the $\mathcal{Z}(\bar{\mathfrak{g}})$-finiteness condition in corollary 5.4.6 can always be satisfied, we have embeddings of all parabolic Q-series, and therefore, of a thick subset of the reduced spectrum of $\bar{G}^\wedge_\#$.

We need to see that these embeddings give rise to an appropriate spectral boundary condition. To do so, we use the following idea: If the sequences $\beta_k \cdot E$ we shall be considering lie in a *sequentially compact* set, to show that they converge to some prescribed limit, it would sufficient to see that all convergent subsequences have the same limit. To apply this idea, we establish the following compactness result.

Proposition 7.3.2. If H is countable at infinity, then the unit ball $\mathbb{B}(\mathrm{W}^*(H))$ is a compact metrisable space in the $\sigma(\mathrm{W}^*(H), \mathrm{A}(H))$-topology, in particular, sequentially compact.

Proof. The unit ball $\mathbb{B}(\mathrm{W}^*(H))$ is a $\sigma(\mathrm{W}^*(G), \mathrm{A}(G))$-compact subset of $\mathrm{W}^*(H)$ by the Alaoğlu theorem. We need to prove metrisability of $\mathbb{B}(\mathrm{W}^*(H))$, and this follows from the separability of $\mathrm{A}(H)$. Suffices to validate the latter.

Since H is countable at infinity, $\mathbf{L}^2(H)$ contains a dense countable-dimensional subspace and is hence $\|\sqcup\|_2$-separable. By [Eym64, th. 3.25], $\mathrm{A}(H)$ consists of the elements

$\overline{\varphi} * \psi^\vee$ with $\varphi, \psi \in \mathbf{L}^2(H)$, and

$$\|\overline{\varphi} * \psi^\vee\|_{\mathrm{A}(H)} \leqslant \|\varphi\|_2 \cdot \|\psi\|_2$$

by [Eym64, lem. 3.1]. Hence $\mathrm{A}(H)$ is $\|\sqcup\|_{\mathrm{A}(H)}$-separable. □

Remark 7.3.3. The proposition is contained in [Mia99, proof of th. 3.7].

7.3.4. Fix a facial subgroup $\bar{G} = G_e$ and for $\bar{\pi}$ in a thick subset of $\bar{G}^\wedge_\#$, let $j_k^{\bar{\pi}}$ denote the embeddings of $\bar{\pi}$ which exist under the assumptions of 7.3.1. For $u \in \langle \bar{G} \rangle_{\bar{\pi}}$, let $\Delta_u^{\bar{\pi},k}$ be the associated sequence of coefficient functions.

Lemma 7.3.5. Up to a set of set of measure 0, for the representations (π_k) associated to $\bar{\pi}$ and $j_k = j_k^{\bar{\pi}}$, we have

$$\lim_k j_k^* \pi_k(E) j_k = \varkappa \in \{0,1\}$$

where $\varkappa = 1$ or 0 depending on whether $\bar{\pi}$ is contained in $\mathbf{H}^2(\bar{\Gamma})$ or not.

Proof. Any weak accumulation point of $j_k^* \pi_k(E) j_k$ is $\bar{G}$-equivariant and hence scalar, by Schur's lemma. In particular, it suffices to check equality on a single vector. By construction of the embeddings, π_k is a holomorphic discrete series representation depending on whether this is the case of $\bar{\pi}$ or not. Hence the assertion. □

7.3.6. Let $\alpha \in \mathrm{A}(G)$ and μ be a sublimit of $\Delta_u^{\bar{\Lambda},k} \cdot E$ where $\Delta_u^{\bar{\Lambda},k}$ is associated to the holomorphic discrete series $\pi_{\bar{\lambda}}$ of $\bar{G}$.

If $u \in F_{\bar{\Lambda}}$, the submodule of constant functions in $\langle \bar{G} \rangle_{\pi_{\bar{\lambda}}} = \mathcal{O}^2(\bar{B}, F_{\bar{\Lambda}})$, proposition 4.4.2 and proposition 7.2.4 imply that $\operatorname{supp}^\infty(\Delta_u^{\bar{\Lambda},k}) \subset K\bar{G}K$. If we had estimates on the singular support of E, improving on proposition 3.4.7 in such a way that $\operatorname{sing\,supp} E \cap (K \cdot \bar{G} \cdot K) = \bar{G}$, then proposition 7.2.2 would imply that $\operatorname{supp} \mu \subset \bar{G}$, because of the Riemann-Lebesgue lemma [Dix69, prop. 3.3.8, 18.2.4].

Then by the following device, this could be extended to all vectors $v \in \langle \bar{G} \rangle_{\pi_{\bar{\lambda}}}$. For any element $g \in \bar{G}$,

$$c_{g*}(E) = E \quad \text{and} \quad \Delta_u^{\bar{\Lambda},k} \circ c_{g^{-1}} = \Delta_{\pi_{\bar{\lambda}}(g)\,u}^{\bar{\Lambda},k} \quad \text{for all } u \in \langle G \rangle_{\pi_k} ,$$

so

$$\operatorname{supp} \mu \subset \operatorname{supp}^\infty(\Delta_v^{\bar{\Lambda},k}) \cap \operatorname{sing} E \subset \bigcup_{g \in \bar{G}} gK\bar{G}Kg^{-1} \cap \operatorname{sing} E \subset \bar{G} \quad \text{for all } v \in \langle \bar{G} \rangle_{\pi_{\bar{\lambda}}} ,$$

by proposition 6.2.4 (iv).

At this point, we do not have complete information on the singular set of E, so that we can, as yet, not complete this argument (compare, however, the discussion of the rank one case in part IV). Moreover, we need to have similar information for all series of representations.

To be precise, let, for a.e. $\bar{\pi} \in \bar{G}^{\wedge}_{\#}$, a sequence $(\pi_k) \subset G^{\wedge}_{\#}$ be given, and assume that for any $u \in \langle \bar{G} \rangle_{\bar{\pi}}$, and any weak sublimit $\mu \in \mathrm{W}^*(G)$ of the sequence $\Delta^k_u \cdot E$, we have $\operatorname{supp} \mu \subset \bar{G}$. We shall call this the *support condition* for $\bar{G}$.

Theorem 7.3.7. Let the additional assumptions 7.3.1 and the support condition 7.3.6 be given. Denote by $\bar{E} = E^e$ the Szegö distribution of $\bar{G} = G_e$. The spectral boundary condition,

$$\bar{\pi}\big(\alpha\big|_{\bar{G}} \cdot \bar{E}\big) = \lim_k j^*_k \pi_k(\alpha \cdot E) j_k \quad \text{weakly for all} \quad \alpha \in \mathrm{A}(H)$$

and a.e. $\bar{\pi} \in \bar{G}^{\wedge}_{\#}$, is verified.

Proof. We note that for $u \in \langle \bar{G} \rangle_{\bar{\pi}}$,

$$\big(u \mid \bar{\pi}\big(\alpha|_{\bar{G}} \cdot \bar{E}\big)\, u\big) = \big\langle \alpha \,:\, \bar{\Delta}^{\bar{\pi}}_u \cdot \bar{E} \big\rangle \ ,$$

so it suffices to prove

$$\bar{\Delta}^{\bar{\pi}}_u \cdot \bar{E} = \lim_k \Delta^{\bar{\pi},k}_u \cdot E \quad \text{in} \quad \sigma(\mathrm{W}^*(G), \mathrm{A}(G)) \ .$$

For a.e. $\bar{\pi} \in \bar{G}^{\wedge}_{\#}$, and all $u \in \langle \bar{G} \rangle_{\bar{\pi}}$, choose sublimits $\bar{\mu}_{\bar{\pi},u}$ of $\Delta^{\bar{\pi},k}_u \cdot E$. Since $\Delta^{\bar{\pi},k}_u$ restricts to $\bar{\Delta}^{\bar{\pi}}_u$ on $\bar{G}$, we have $\operatorname{supp} \bar{\mu}_{\bar{\pi},u} \subset \operatorname{supp} \bar{\Delta}^{\bar{\pi}}_u$, and there exist distributions $\nu_{\bar{\pi},u}$ such that $\mu_{\bar{\pi},u} = \Delta^{\bar{\pi}}_u \cdot \nu_{\bar{\pi},u}$.

We have

$$\bar{\Delta}^{\bar{\pi}}_u \cdot \bar{\Delta}^{\bar{\pi}}_v \cdot \nu_{\bar{\pi},u} = \lim_j \lim_i \Delta^{\bar{\pi},i}_u \cdot \Delta^{\bar{\pi},j}_v \cdot E \ ,$$

and, symmetrically,

$$\bar{\Delta}^{\bar{\pi}}_u \cdot \bar{\Delta}^{\bar{\pi}}_v \cdot \nu_{\bar{\pi},v} = \lim_j \lim_i \Delta^{\bar{\pi},i}_u \cdot \Delta^{\bar{\pi},j}_v \cdot E \ ,$$

where we omit the choice of subsequences from the notation. By corollary 7.3.11 below, the double limits are equal, so

$$\bar{\Delta}^{\bar{\pi}}_u \cdot \bar{\Delta}^{\bar{\pi}}_v \cdot \nu_{\bar{\pi},u} = \bar{\Delta}^{\bar{\pi}}_u \cdot \bar{\Delta}^{\bar{\pi}}_v \cdot \nu_{\bar{\pi},v} \ ,$$

and $\nu_{\bar{\pi},u} = \nu_{\bar{\pi},v}$ on $\operatorname{supp} \bar{\Delta}^{\bar{\pi}}_u \cap \operatorname{supp} \bar{\Delta}^{\bar{\pi}}_v$. Letting u and v vary, we find that $\nu_{\bar{\pi}} = \nu_{\bar{\pi},u}$, independent of u. Since the embeddings $j^{\bar{\pi}}_k$ are $\bar{G}$-equivariant, we find that $\nu_{\bar{\pi}}$ is a $\bar{G}$-central distribution.

By the same device, $\nu = \nu_{\bar{\pi}}$ is independent of $\bar{\pi}$. But then, for u in the Gårding space of $\bar{\pi}$, the Fourier coefficient

$$(u \mid \bar{\pi}(\nu)\, u) = \big\langle \bar{\Delta}^{\bar{\pi}}_u \,:\, \nu \big\rangle$$

makes sense. By lemma 7.3.5, it equals $\|u\|^2$ or 0 according to whether $\bar{\pi}$ is a holomorphic discrete series representation or not. But then ν coincides with $\bar{E}$, since their Fourier transforms coincide. Hence, the limits of all convergent subsequences are equal, so the limits exists and are of the required form. □

Corollary 7.3.8. For any facial subgroup $G_e \sqsubset G$ and $g,h \in G$, there exists an irreducible $*$-representation

$$\varrho_{gG_eh^{-1}} : \mathcal{T}_E(G) \to \mathcal{T}_{E^e}(G_e) \text{ so that } \varrho_{gG_eh^{-1}}(T_f) = E^e M_{g*f*h|G_e} E^e \text{ for all } f \in \mathcal{C}_0(G)\,.$$

Moreover, if $I_j = \bigcap_{g,h\in G} \ker \varrho_{gG_{e^j}h^{-1}}$ for some rank j tripotent $e^j \in Z$, then

$$\mathbb{K}(\mathbf{H}^2(\Gamma)) \triangleleft I_0 \triangleleft I_1 \triangleleft \cdots \triangleleft I_r \triangleleft \mathcal{T}_E(G)$$

is an ascending chain of ideals.

Proof. Apply corollaries 7.1.13 and 7.1.15 and proposition 7.2.6 to theorem 7.3.7. □

7.3.9. In the proof of theorem 7.3.7, we used an exchange of limit theorem. To complete the proof, we need to establish this technical device. To that end, we fix some notation. In a normed vector space, the ball is denoted $\mathbb{B}$, and the sphere by $\mathbb{S}$. The set of elements in a subset $A \subset \mathrm{A}(H)$ which are of positive type will be abbreviated A_+. Moreover, for a compact subset $L \subset H$, we let

$$\mathrm{A}_L(H) = \left\{\alpha \in \mathrm{A}(H) \,\middle|\, \operatorname{supp}\alpha \subset L\right\}.$$

Proposition 7.3.10. Let $L \subset H$ be compact, and $(x_j) \subset \mathbb{B}(\mathrm{W}^*(H))$, $(\alpha_i) \subset \mathbb{S}(\mathrm{A}_L(H))_+$ be sequences. Then

$$\lim_i \lim_j \langle \alpha_i : x_j \rangle = \lim_j \lim_i \langle \alpha_i : x_j \rangle$$

whenever the iterated limits exist.

Proof. The set $E = \mathbb{B}(\mathrm{W}^*(H))$ is a compact Hausdorff space in the $\sigma(\mathrm{W}^*(H), \mathrm{A}(H))$-topology. To prove the proposition, by [Gro52, th. 2, cor. 2], it suffices to prove that the set $A = \mathbb{S}(\mathrm{A}_L(H))_+$ is relatively compact in $\mathcal{C}(E)$, endowed with the topology of simple convergence.

To that end, observe that since the elements of A are linear, simple convergence on points of E is equivalent to point-wise convergence on $\mathrm{W}^*(H)$. Furthermore, the set $L(\mathrm{W}^*(G), \mathbb{C})$ of linear forms is closed in $\mathbb{C}^{\mathrm{W}^*(H)}$, so limits of nets in A are linear. Also, Tychonov's theorem implies that $\mathbb{B}(\mathbb{C})^E$ is compact, so A is relatively compact in $\mathbb{C}^E$.

Remains to prove that the simple closure of A lies in $\mathcal{C}(E)$. Let $(u_\alpha) \subset A$ be a net converging point-wise on E to $u \in \mathbb{B}(\mathbb{C})^E \cap L(\mathrm{W}^*(H), \mathbb{C})$. The net (u_α) is norm bounded in $\mathrm{A}(H)$, and $\|\sqcup\|_{\mathrm{A}(H)}$ coincides on $\mathrm{A}(H)$ with the dual norm of $\mathrm{W}^*(H)$ because $\|\sqcup\|_{\mathrm{W}^*(H)}$ is the dual norm of $\mathrm{A}(H)$ by [Eym64, th. 3.10]. Thus, we may apply the Banach-Steinhaus theorem [Trè67, th 33.1, cor.], and u is norm-continuous on $\mathrm{W}^*(H)$. In particular, it is norm–continuous on $\mathrm{C}^*_\#(H)$, so $u \in \mathrm{B}_\#(H) \subset \mathrm{B}(H)$.

The point evaluation have norm 1, so $H \subset E$. For any choice of finite sequences $(z_i) \subset \mathbb{C}$ and $(g_i) \subset H$,

$$\sum_{i,j=0}^{n} \bar{z}_i z_j u(g_i^{-1} g_j) = \lim_\alpha \sum_{i,j=0}^{n} \bar{z}_i z_j u_\alpha(g_i^{-1} g_j) \geqslant 0 \,.$$

Thus u is of positive type, and $\|u\|_{\mathrm{B}(H)} = u(1) = \lim_\alpha u_\alpha(1) = 1$. Hence, $u \in \mathbb{S}(\mathrm{B}(H))_+$. On bounded subsets, the topology $\sigma(\mathrm{B}(H), \mathrm{C}^*(H))$ is weaker than the topology of pointwise convergence, and $\mathrm{A}_L(H)$ is $\sigma(\mathrm{B}(H), \mathrm{C}^*(H))$-closed by [GL81, proof of th. B_1]. We conclude $u \in A = \mathbb{S}(\mathrm{A}_L(H))_+$. This proves the assertion. □

Corollary 7.3.11. Let $a \in \mathrm{W}^*(H)$, $x, y \in \mathrm{W}^*(\bar{H})$, $\bar{\beta}, \bar{\gamma} \in \mathbb{S}(\mathrm{B}(\bar{H}))_+$ and sequences $(\beta_i), (\gamma_j) \subset \mathbb{S}(\mathrm{B}(H))_+$ be given. Assume that

$$(\alpha \cdot \beta_i)\big|_{\bar{H}} \to \alpha\big|_{\bar{H}} \cdot \bar{\beta} \quad \text{and} \quad (\alpha \cdot \gamma_j)\big|_{\bar{H}} \to \alpha\big|_{\bar{H}} \cdot \bar{\gamma}$$

in $\mathrm{A}(\bar{H})$ for all $\alpha \in \mathrm{A}(H)$, and

$$x = \lim_i \beta_i \cdot a \quad \text{and} \quad y = \lim_j \gamma_j \cdot a$$

in $\sigma(\mathrm{W}^*(H), \mathrm{A}(H))$. Then

$$\bar{\gamma} \cdot x = \lim_j \lim_i \beta_i \cdot \gamma_j \cdot a = \lim_i \lim_j \beta_i \cdot \gamma_j \cdot a = \bar{\beta} \cdot y \,.$$

Proof. Note that $\mathrm{A}(H) \triangleleft \mathrm{B}(H)$. For $\alpha \in \mathrm{A}(H)$, by (7.1),

$$\left\langle \alpha\big|_{\bar{H}} : \bar{\gamma} \cdot x \right\rangle = \lim_j \left\langle \alpha \cdot \gamma_j \,:\, x \right\rangle = \lim_j \lim_i \left\langle \alpha \,:\, \beta_i \cdot \gamma_j \cdot a \right\rangle \,.$$

Similarly, the other double limit exists. If $\alpha \in \mathbb{S}(\mathrm{A}(H))_+$ has compact support L, then $\operatorname{supp}(\alpha \cdot \gamma_j) \cup \operatorname{supp}(\alpha \cdot \beta_i) \subset L$. Moreover, $\alpha \cdot \beta_i$ and $\alpha \cdot \gamma_j$ are of positive type. Since the norm of a function of positive type is given by evaluation at 1, both products are contained in $\mathbb{S}(\mathrm{A}_L(H))_+$. By proposition 7.3.10, the double limits are equal on α. By polarisation and [Eym64, prop. 3.4], the set $\mathbb{S}(\mathcal{K}\mathrm{A}(H))_+$ spans a dense subspace of $\mathrm{A}(H)$, so we have equality on $\mathrm{A}(H)$. Since $\mathrm{res}_{\bar{H}}$ is surjective by theorem 7.1.2, the assertion follows. □

A special case, a general strategy

In this final part, we develop, in section 8, the theory established in general above, for the important special case of the unit disc. Here, slightly improved results on the local and harmonic analysis of the group allow us to complete the construction of the composition series of $\mathcal{T}_E(G)$ with the help of the methods we have introduced beforehand. The upshot is that even in this case, the composition series is longer than for the Toeplitz C*-algebra defined on the Hardy space of underlying symmetric domain, the unit disc, itself.

In section 9, we elaborate on the points where our results for the general case are incomplete or preliminary, and give detailed indications on the further steps that have to be undertaken to complete our programme, and on the required methods.

8 The case of the unit disc

We give a complete account of the theory of the Toeplitz C*-algebra $\mathcal{T}_E(G)$ for the case of the unit disc, where $G = \mathbb{P}\,\mathrm{SU}(1,1)$. The results were partially obtained in the master's thesis [All99]. They were completed and published in [AU02], in an ad hoc form, tailored to the group $\mathrm{SL}(2,\mathbb{R})$, which is isomorphic to the connected double cover $\mathrm{SU}(1,1)$ of G . Our present development of the rank one case is perhaps better adapted to the general theory we have presented in the previous chapters.

8.1 Geometry of the minimal cone

8.1.1. The unit disc $\mathbb{B} = \{z \in \mathbb{C} \mid |z| < 1\}$ is the bounded symmetric domain corresponding to the simple JB*-triple $Z = \mathbb{C}$, with triple product given by

$$\{uv^*w\} = u\bar{v}w \quad \text{for all } u,v,w \in Z = \mathbb{C} .$$

In particular, the set of all non-zero tripotents (which are all primitive) is just the unit circle, as it should be, because the Shilov boundary $\partial_1\mathbb{B}$ equals the entire boundary for rank 1 .

The automorphism group of $\mathbb{B}$ is $\mathbb{P}\,\mathrm{SU}(1,1) = \mathrm{SU}(1,1)/\mathbb{Z}_2$, where

$$\mathrm{SU}(1,1) = \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{smallmatrix}\right) \in \mathbb{C}^{2\times 2} \;\middle|\; |\alpha|^2 - |\beta|^2 = 1 \right\} ,$$

and the action is given by Möbius transformations. Its Lie algebra is

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{su}(1,1) = \left\{ \left(\begin{smallmatrix} \alpha & \beta \\ \bar{\beta} & -\alpha \end{smallmatrix}\right) \;\middle|\; \alpha \in i\mathbb{R}\,,\, \beta \in \mathbb{C} \right\}.$$

We relate the Jordan theoretic viewpoint to this matrix realisation. Consider the one-parameter group $k_t = \left(\begin{smallmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{smallmatrix}\right) \in \mathrm{SU}(1,1)$. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t} k_t(z)\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} e^{it} \cdot z\big|_{t=0} = iz\,,$$

so the generator is

$$iz\frac{\partial}{\partial z} = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}.$$

Similarly, consider the one-parameter group $a_t = \left(\begin{smallmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{smallmatrix}\right)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} a_t(z)\big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\cosh t \cdot z + \sinh t}{\sinh t \cdot z + \cosh t}\Big|_{t=0} = 1 - \left(\frac{\cosh t \cdot z + \sinh t}{\sinh t \cdot z + \cosh t}\right)^2\Big|_{t=0} = 1 - z^2\,,$$

so its generator is

$$\xi_e^- = (1 - z^2)\frac{\partial}{\partial z} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for the tripotent $e = 1$. Hence, $\mathfrak{a}_{\mathbb{R}} = \mathbb{R} \cdot \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, so one calculates easily that the restricted root spaces have the respective generators $X^+ = \frac{1}{2} \cdot \left(\begin{smallmatrix} i & -i \\ i & -i \end{smallmatrix}\right)$ and $X^- = \frac{1}{2} \cdot \left(\begin{smallmatrix} -i & -i \\ i & i \end{smallmatrix}\right)$.

These matrices generate the one-parameter subgroups

$$n_t^+ = \begin{pmatrix} 1 + \frac{it}{2} & -\frac{it}{2} \\ \frac{it}{2} & 1 - \frac{it}{2} \end{pmatrix} \quad \text{and} \quad n_t^- = \begin{pmatrix} 1 - \frac{it}{2} & -\frac{it}{2} \\ \frac{it}{2} & 1 + \frac{it}{2} \end{pmatrix}.$$

One then computes

$$\frac{\mathrm{d}}{\mathrm{d}t} n_t^+(z)\big|_{t=0} = -\tfrac{i}{2} \cdot (z-1)^2 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t} n_t^-(z)\big|_{t=0} = -\tfrac{i}{2} \cdot (z+1)^2\,,$$

so $X_e^\pm = X^\pm$ for $e = 1$. Now, we can determine Ω^- explicitly.

Lemma 8.1.2. For $Z = \mathbb{C}$, the cone Ω^- is

$$\Omega^- = \left\{ \xi = \left(\begin{smallmatrix} ir & z \\ \bar{z} & -ir \end{smallmatrix}\right) \;\middle|\; r \geqslant 0\,,\, \det \xi = r^2 - |z|^2 \geqslant 0 \right\},$$

the three-dimensional *forward light cone*.

Proof. Recall from theorem 2.1.27 that $\Omega^- \setminus 0 = \mathbb{R}_> \cdot \mathrm{co}\big(\mathrm{Ad}(K)(X_e^+)\big)$ where e is an arbitrary primitive tripotent. So, we may take $e = 1$. We compute

$$\mathrm{Ad}(k_t)(X_e^+) = \frac{1}{2}\cdot\begin{pmatrix} i & -ie^{it} \\ ie^{-it} & -i \end{pmatrix} \quad \text{for all } t \in \mathbb{R}\,.$$

As t varies, $-ie^{it}$ runs through all points of the circle. Since $\overline{\mathbb{B}}$ is the convex hull of its extremal points, we find

$$\mathrm{co}\big(\mathrm{Ad}(K)(2\cdot X_e^+)\big) = \big\{\xi = \big(\begin{smallmatrix} i & z \\ \bar z & -i\end{smallmatrix}\big) \bigm| z \in \overline{\mathbb{B}}\big\}\,.$$

Note that $\det\xi = 1 - |z| > 0$ if $z \in \mathbb{B}$. Hence Ω^- is contained in the right hand side, and the converse is also true, by dividing with $r > 0$. ☐

8.1.3. We note $\mathfrak{k}_{\mathbb{R}} = \mathfrak{t}_{\mathbb{R}} = \mathfrak{z}(\mathfrak{k}_{\mathbb{R}})$ for the unit disc. Hence, it is obvious that

$$\omega^- = \mathbb{R}_{\geqslant}\cdot iz\frac{\mathrm{d}}{\mathrm{d}z} = \mathbb{R}_{\geqslant}\cdot\begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{1}{2}\end{pmatrix} = \omega^+\,,$$

since this cone is manifestly self-dual. Hence, $\Omega^+ = \Omega^-$.

The faces of these cones are easy to determine. For the zero flag 0 one gets, as usual, the equality $F^{\pm}_{0,\varnothing} = \Omega^{\pm}$. For $f = (e = 1 > 0)$, we have $Z_0(e) = 0$, and $Z_1(e) = Z = \mathbb{C}$. Hence, $F^{\pm}_{(1,0),\{1\}} = i\Omega_1(e) = \mathbb{R}_{\geqslant}\cdot X_e^+$. This is an extremal ray that generates a maximal nilpotent subalgebra of (the Iwasawa $\mathfrak{n}_{\mathbb{R}}$ component) of $\mathfrak{g}_{\mathbb{R}}$. Finally, if $f = (1,0)$, we find $F^{\pm}_{(1,0),\varnothing} = 0$. At any rate, Ω^- has non-trivial faces, unlike $\mathbb{B}$ itself.

8.1.4. It is obvious that any orbit $\mathcal{O} \subset \Omega^{-\circ}$ intersecting $\mathbb{R}_{>}\cdot iz\,\frac{\partial}{\partial z}$ is G-isomorphic to the underlying domain $B = G/K$, in this case, to $\mathbb{B}$. From the above considerations, we can describe this isomorphism explicitly.

Lemma 8.1.5. Define a map

$$\Phi : \Omega^{-\circ} \to \mathbb{B} : \xi = \tfrac{1}{2}\cdot\big(\begin{smallmatrix} ir & z \\ \bar z & -ir\end{smallmatrix}\big) \mapsto \frac{i\sqrt{\det\xi}-r}{z}\,.$$

Then Φ is G-equivariant, and its restriction to any orbit is a bijection.

Proof. We observe that $G = KNK$, and that K fixes $\mathfrak{z}(\mathfrak{k}_{\mathbb{R}}) = \mathbb{R}\cdot iz\,\frac{\partial}{\partial z}$ and $0 \in \mathbb{B}$. Hence it suffices to apply elements $k_t n_x^+$. We have $k_t n_x^+(0) = e^{it}\cdot\frac{ix}{ix-2}$. On the other hand,

$$\mathrm{Ad}(k_t n_x^+)\Big(iz\frac{\partial}{\partial z}\Big) = \frac{ir}{2}\cdot\begin{pmatrix} 1+\frac{x^2}{2} & e^{it}\cdot\big(ix-\frac{x^2}{2}\big) \\ e^{-it}\cdot\big(ix+\frac{x^2}{2}\big) & -1-\frac{x^2}{2}\end{pmatrix}\,.$$

Under Φ, this element maps to

$$\frac{\frac{ix^2}{2}}{e^{-it}\cdot\frac{x(ix-2)}{2}} = e^{it}\cdot\frac{ix}{ix-2}\,,$$

proving the assertion. □

8.2 Singularities and asymptotics

8.2.1. For the case of the unit disc, the Szegö kernel has been known explicitly for a while. In fact, a formula is given by Gel'fand-Gindikin [GG77]. Ol'shanskiĭ [Ol'95] has determined it explicitly for the classical domains of type $\mathrm{I}_{p,q}$ and III_n. The unit disc is, of course, a special case.

The formula for the unit disc is easy to describe. For $\gamma \in \Gamma^\circ = G \cdot \exp(i\Omega^{-\circ})$, any representative of γ has a unique eigenvalue λ of modulus $|\lambda| < 1$. This eigenvalue is uniquely determined by γ, up to multiplication by ± 1. By abuse of notation, we write $q(\gamma) = \lambda$. Then

$$K(\gamma) = \frac{q(\gamma)^2}{(1-q(\gamma))^3(1+q(\gamma))} = \frac{q(\gamma)^2}{(1-q(\gamma)^2)(1+q(\gamma)^2)} \quad \text{for all } \gamma \in \Gamma^\circ .$$

Note that the right hand side of this formula is well-defined, since it is independent of factors ± 1 in $q(\gamma)$.

Denote by $\mathcal{N} \subset G = \mathbb{P}\,\mathrm{SU}(1,1)$ the unipotent cone, consisting of all elements of G whose eigenvalues (as elements of the adjoint group) are all equal to 1. Since all $\mathrm{Ad}(g)$, $g \in G$, are special endomorphisms, this means that if $\mathrm{Ad}(g)$ has only one eigenvalue λ, it satisfies $\lambda^2 = 1$. Because $\lambda = -1$ is excluded, this implies $\lambda = 1$. Hence, $\mathcal{N}$ coincides with the set of $g \in G$ such that $\mathrm{Ad}(g)$ has only one eigenvalue, put differently, the characteristic polynomial is not irreducible, and therefore distinct from its minimal polynomial.

Since G contains no non-trivial singular semi-simple elements, $\mathcal{N}$ coincides with the set of all singular elements in G.

Lemma 8.2.2. The function q locally has smooth extensions to $G_* = G \setminus \mathcal{N}$. Hence, the Szegö distribution E is regular on G_*, and $\operatorname{sing\,supp} E \subset \mathcal{N} = G \setminus G_*$ is thin.

Proof. The function Δ that associates to g the discriminant of the characteristic polynomial $\mathrm{Ad}(g)$ is polynomial, and G_* is the complement of its zero set. For any $g \in G_*$, there exists a connected neighbourhood U of g in Γ such that $\Delta(U)$ is contained in a simply connected region of $\mathbb{C} \setminus 0$.

The characteristic polynomial of $\mathrm{Ad}(g)$ is quadratic, so wherever a smooth square root $\sqrt{\Delta}$ can be chosen, eigenvalues of $\mathrm{Ad}(g)$ can be chosen in a fashion depending smoothly on g. Thus, a smooth extension of q exists on $G \cap U$, and by the above formula for the Szegö kernel K, the formula for the Szegö distribution in proposition 3.1.13, and dominated convergence, E is smooth on $G \cap U$. □

8.2.3. We consider now the embedding of reduced spectra for the subgroups of G associated to the faces of the minimal cone Ω^-. Associated to the trivial face $F^-_{(1,0),\varnothing}$ is the trivial subgroup $\bar{G} = G_{(1,0),\varnothing} = 1$. Its sole irreducible unitary representation is the trivial representation on $\mathbb{C}$, also equal to $\mathbf{L}^2(\bar{G}) = \mathbf{H}^2(\bar{\Gamma})$. The relative Szegö distribution is in this case simply δ, the Dirac distribution supported by $\bar{G} = 1$.

The embedding of the 'holomorphic discrete series' of $\bar{G} = 1$, consisting of $\mathbb{C}$, corresponds to the choice of the constant $1 = 1_\Lambda$ in the Bergman space $\mathcal{O}^2_\Lambda = \mathcal{O}^2(\mathbb{B}, \mathbb{C}_\Lambda)$. Then propositions 4.4.2 and 7.2.4 show that $\operatorname{supp}^\infty(\Delta^\Lambda) \subset K$, where

$$\Delta^\Lambda(g) = (1_\Lambda \mid \pi_\lambda(g)\, 1_\Lambda) \quad \text{for all } g \in G\,.$$

The non-trivial face $F^-_{(1,0),\{1\}} = \mathbb{R}_> \cdot X^+$, is associated to the subgroup N, the Iwasawa N component of G with its canonical positive system. This is the image of the one-parameter group n_t^+ in G. The unitary representations of N are the characters $e^{i\nu}$ where $\nu \in \mathfrak{n}^*_\mathbb{R}$ is determined by its value $\nu(X^+)$, which we also denote by the letter ν. All of these are weakly contained in $\mathbf{L}^2(N) = \mathbf{L}^2(\mathbb{R})$, as follows from the Plancherel theorem for the Euclidean Fourier transform. The Plancherel measure is absolutely continuous w.r.t. Lebesgue measure on $N^\wedge \cong \mathbb{R}$, so we may omit finitely many representations when constructing embeddings.

An asymptotically equivariant embedding of the space $\mathbb{C}_\nu$ corresponding to the character e^ν is constructed in the following manner. We identify the dual $\mathfrak{g}^*_\mathbb{R}$ with $\mathfrak{g}_\mathbb{R}$. Any representation of G gives rise to co-adjoint orbit, which identifies with an adjoint orbit.

To the holomorphic discrete series representation π_λ on the Bergman space $\mathcal{O}^2_\Lambda$, there corresponds via the moment map

$$\mu : \pi_\lambda(G)\, 1_\Lambda \to \mathfrak{g}^*_\mathbb{R} \quad \text{defined by} \quad \langle X : \mu(\varphi)\rangle = -i \cdot (\varphi \mid \pi_\lambda(X)\, \varphi)$$

the adjoint orbit $\mathcal{O}_\Lambda$ through $\Lambda \cdot z\frac{\partial}{\partial z}$, where we identify $\Lambda \in i\mathfrak{t}^*_\mathbb{R}$ with its value on the element $2 \cdot e \,\square\, e^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \in i\mathfrak{t}_\mathbb{R}$, $e = 1$ (the positive coroot). Then $\mathcal{O}_\Lambda$ is precisely the set of all $\xi \in \Omega^-$ such that $\det \xi = \frac{\Lambda^2}{4}$.

An embedding of $\mathbb{C}_\nu$ into $\mathcal{O}^2_\Lambda$ corresponds hence to the choice of a point on the orbit $\mathcal{O}_\Lambda$. We construct such a choice by considering ν as an element of $\mathbb{R} \cdot X^+$, and considering the intersection of $\nu + \mathbb{R} \cdot X^-$ with Ω^-.

The natural generator of $\mathbb{R} \cdot X^-$ is in fact $-X^- \in \Omega^-$. Hence we are searching for $t > 0$ such that $\nu X^+ - tX^- \in \mathcal{O}_\Lambda$, i.e.

$$\frac{\Lambda^2}{4} = \det\bigl(\nu X^+ - tX^-\bigr) = t\nu\,.$$

We get $t = \frac{\Lambda^2}{4\nu}$ for $\nu > 0$. For this value of t,

$$z_{\nu,\Lambda} = \Phi\bigl(\nu X^+ - tX^-\bigr) = \frac{\Lambda - 2\nu}{\Lambda + 2\nu}\,.$$

Hence, the embedding $\mathbb{C}_\nu \to \mathcal{O}^2_\Lambda$ is given by associating to $1 \in \mathbb{C}_\nu$ the element $f_{\nu,\Lambda} \in \mathcal{O}^2_\Lambda$ determined by

$$f_{\nu,\Lambda}(z) = B(z_{\nu,\Lambda}, z_{\nu,\Lambda})^{-\Lambda} \cdot B(z, z_{\nu,\Lambda})^\Lambda 1_\Lambda \quad \text{for all } z \in \mathbb{B}\,.$$

Here, the first factor normalises this vector in $\mathcal{O}^2_\Lambda$. We need to compute this quantity.

Lemma 8.2.4. We have

$$f_{\nu,\Lambda}(z) = \left(\frac{(\Lambda+2\nu)^2}{8\nu\Lambda}\right)^\Lambda \cdot \left(1 - z\cdot\frac{\Lambda-2\nu}{\Lambda+2\nu}\right)^\Lambda \quad \text{for all } z \in \mathbb{B}\,.$$

Moreover, the matrix coefficient $\Delta^\Lambda_\nu(g) = (f_{\nu,\Lambda} \mid \pi_\lambda(g)\, f_{\nu,\Lambda})$ is given by

$$\Delta^\Lambda_\nu(g^{-1}) = (8\nu\Lambda)^{-\Lambda}\cdot\big((\bar\beta-\beta)(\Lambda^2-4\nu^2) + \bar\alpha(\Lambda+2\nu)^2 - \alpha(\Lambda-2\nu)^2\big)^\Lambda$$

whenever g is represented by $\left(\begin{smallmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{smallmatrix}\right) \in \mathrm{SU}(1,1)$.

Proof. As a linear map, $B(u,v)z = z - 2u\bar v z + u^2\bar v^2 z = (1-u\bar v)^2 z$. Hence, as an element of $K^\mathbb{C}$, $B(u,v)$ is represented in the double cover $\mathrm{SL}(2,\mathbb{C}) \twoheadrightarrow G^\mathbb{C}$, by the matrix

$$B(u,v) = \begin{pmatrix} 1-u\bar v & 0 \\ 0 & (1-u\bar v)^{-1}\end{pmatrix} \quad \text{for all } u,v \in \mathbb{B}\,.$$

Recalling that we have identified Λ with its value on $2\cdot e\,\Box\, e^* = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, we find that $B(u,v)^\Lambda = (1-u\bar v)^\Lambda$. Hence, the first formula.

Up to a normalising factor, B^Λ equals the kernel function K^Λ of $\mathcal{O}^2_\Lambda$. Thus,

$$\Delta^\Lambda_\nu(g^{-1}) = c\cdot g'(z_{\nu,\Lambda})^{-\Lambda} B(g(z_{\nu,\Lambda}), z_{\nu,\Lambda})^\Lambda = c\cdot(\bar\beta z_{\nu,\Lambda}+\bar\alpha)^\Lambda \cdot \left(1 - \frac{\alpha z_{\nu,\Lambda}+\beta}{\bar\beta z_{\nu,\Lambda}+\bar\alpha}\cdot z_{\nu,\Lambda}\right)^\Lambda,$$

for some constant c. Since the value of Δ^Λ_ν at the identity is one, this constant is

$$c = (1-z^2_{\nu,\Lambda})^{-\Lambda} = \left(\tfrac{(\Lambda+2\nu)^2}{8\nu\Lambda}\right)^\Lambda.$$

The assertion follows. ∎

8.2.5. We note that since we have identified Λ with its value on the positive (non-compact) co-root, Harish-Chandra's square integrability criterion (theorem 4.2.14) states that Λ should be a negative integer < -1.

The normaliser of N in $G = \mathbb{P}\,\mathrm{SU}(1,1)$ is AN (since $M = Z_K(\mathfrak{a}_\mathbb{R}) = 1$). We define a covering of $G \setminus AN$ by

$$U_\varepsilon(N) = \left\{ g^{-1} \equiv \left(\begin{smallmatrix}\alpha & \beta\\ \bar\beta & \bar\alpha\end{smallmatrix}\right) \pmod{\mathbb{Z}_2} \;\middle|\; |\alpha - \bar\alpha + \beta - \bar\beta| \geqslant \varepsilon \right\}.$$

We can estimate the rate of decay of Δ^Λ_ν as $\Lambda \to -\infty$ on these sets.

Lemma 8.2.6. Let $\varepsilon > 0$. On compact subsets of $U_\varepsilon(N)$, $\Delta_\nu^\Lambda \to 0$ ($\Lambda \to -\infty$), uniformly with all derivatives. In particular, $\mathrm{supp}^\infty(\Delta_\nu^\Lambda) \subset AN$.

Proof. We have

$$\left(\Delta_\nu^\Lambda(g^{-1})\right)^{1/\Lambda} = \frac{\Lambda}{8\nu} \cdot (\bar\beta - \beta + \bar\alpha - \alpha) + \frac{\nu}{2\Lambda} \cdot (\beta - \bar\beta + \bar\alpha - \alpha) + \frac{1}{2} \cdot (\alpha + \bar\alpha) \,.$$

The modulus of the last two summands is bounded by a constant C, independently of Λ, and g in a compact subset of $U_\varepsilon(N)$. Then

$$|\Delta_\nu^\Lambda(g^{-1})|^{1/\Lambda} \geqslant \frac{|\Lambda|}{8\nu} \cdot \varepsilon - C > 1$$

for sufficiently large $|\Lambda|$. Since $\Lambda \to -\infty$, Δ_ν^Λ vanishes exponentially in the limit. Along the lines of the proof of proposition 4.4.2, it can be seen that derivatives introduce only factors of polynomial decay. Hence, the result. □

8.2.7. So far, we have only treated the case $\nu > 0$.

For $\nu < 0$, one gets an analogous embedding of $\mathbb{C}_\nu$ into the *anti-holomorphic discrete series*, associated to the positive system $-\Delta^{++}$. It is realised on the conjugate space of $\mathcal{O}^2_\Lambda$, so it quickly turns out that the corresponding sequence of matrix coefficients Δ_ν^Λ ($\nu < 0$) is simply the conjugate of $\Delta_{-\nu}^\Lambda$.

The next issue is that the constructed embedding is in fact asymptotically equivariant. Point-wise, for g represented by n_x^+, we find

$$\Delta_\nu^\Lambda(g^{-1}) = \left(1 - \frac{i\nu x}{\Lambda}\right)^\Lambda \to e^{-i\nu x} = e^{i\nu}(g^{-1}) \quad (\Lambda \to -\infty) \,,$$

as it should be. We wish to strengthen this convergence. To that end, in the following lemma, we tacitly identify n_x^+ with its image in G.

Lemma 8.2.8. The inclusion $j_{\nu,\Lambda} : \mathbb{C}_\nu \to \mathcal{O}^2_\Lambda : 1 \mapsto f_{\nu,\Lambda}$ is asymptotically equivariant in the sense that

$$j_{\nu,\Lambda}^* \pi_\lambda(n) j_{\nu,\Lambda} - e^{i\nu}(n) \to 0 \quad \text{strongly on} \quad \mathbb{C}_\nu$$

for all $n \in N$. Moreover,

$$\alpha \cdot \Delta_\nu^\Lambda|_N \to \alpha \cdot e^{i\nu} \quad \text{in} \quad \mathrm{A}(N) \quad \text{for all } \alpha \in \mathrm{A}(N) \,.$$

Proof. We calculate

$$\left\| (\pi_\lambda(n_x^+) - e^{i\nu x}) f_{\nu,\Lambda} \right\|^2_{\mathcal{O}^2_\Lambda} = 2\left(1 - \mathrm{Re}(e^{i\nu x} \Delta_\nu^\Lambda(n_{-x}^+))\right) \to 0 \quad (\Lambda \to \infty) \,.$$

This proves the first assertion.

As for the second, the restriction mapping $\mathrm{res}_N : \mathrm{A}(G) \to \mathrm{A}(N)$ is surjective by theorem 7.1.2. Hence, for $\alpha \in \mathrm{A}(N)_+$, we may assume that α is the restriction of some element of $\mathrm{A}(G)$. Hence, $\alpha \cdot \Delta_\nu^\Lambda\big|_N \in \mathrm{A}(N)$. We may assume $\alpha(1) = 1$. Then the sequence $\alpha \cdot \Delta_\nu^\Lambda\big|_N \in \mathbb{S}(\mathrm{A}(N))_+$.

The convergence statement is true point-wise. By [GL81, th. B_2], it remains valid in norm. □

8.3 Construction of a series of ideals

8.3.1. First, we consider the problem of irreducibility of the Toeplitz C*-algebras. We have already noted that for the trivial subgroup $G_{(1,0),\varnothing} = 1$, the Szegö distribution is simply $E^{(1,0),\varnothing} = \delta$. On $N = G_{(1,0),\{1\}}$, the Hardy space the classical Hardy space of the upper half plane, isomorphic to $\mathbf{L}^2(\mathbb{R}_\geqslant)$. Hence, the Szegö distribution $E^{(1,0),\{1\}}$ is the Fourier transform of the characteristic function of $\mathbb{R}_\geqslant$. Hence the following result is immediate.

Proposition 8.3.2. The Toeplitz C*-algebras $\mathcal{T}_E(G)$, $\mathcal{T}_N(E^{(1,0),\{1\}})$, and $\mathcal{T}_1(E^{(1,0),\varnothing}) = \mathbb{C}$ contain the ideal of compact operators, and hence, act irreducibly on their respective Hardy spaces.

Proof. This is proposition 6.3.4, applied to lemma 8.2.2. □

8.3.3. The next step is to construct the representations associated to the faces. We first treat the case of the trivial subgroup, associated to the face $F_{1,0}^- = 0$.

Proposition 8.3.4. For every $g \in G$, there is a character

$$\varrho_g : \mathcal{T}_E(G) \to \mathbb{C} \quad \text{determined by} \quad \varrho_g(T_f) = f(g) \quad \text{for all } f \in \mathcal{C}_0(G)\ .$$

Proof. We have seen in 8.2.3 that $\operatorname{supp}^\infty(\Delta^\Lambda) \subset K$, and $\operatorname{sing} E \subset \mathcal{N}$, by lemma 8.2.2. Since K contains no non-trivial unipotent elements, all sublimits of $\Delta^\Lambda \cdot E$ have support contained in the trivial subgroup 1. Hence, the assertion follows from theorem 7.3.7 and corollary 7.1.13. □

8.3.5. Now to the case of the subgroup N, associated to the face $F_{(1,0),\{1\}}^-$.

We note that the only point where the equivariance of the embeddings in the spectral boundary condition was used in the construction of the representations in theorem 7.1.11 was the passage from weak to strong convergence in the proof of lemma 7.1.6. For the group N, this is however trivial, since $\mathbb{C}_\nu$ is finite-dimensional. Hence, the same technique is applicable.

Moreover, for $\nu > 0$, $\Delta_\nu^\Lambda \in \mathbf{H}^2(\Gamma)$, whereas for $\nu < 0$, $\Delta_\nu^\Lambda \perp \mathbf{H}^2(\Gamma)$. Hence, we have

$$\lim_{\Lambda \to -\infty} j_{\nu,\Lambda}^* \pi_\lambda(E) j_{\nu,\Lambda} = \varkappa \in \{0, 1\}$$

corresponding to whether $\nu > 0$ or $\nu < 0$. This is analogous to lemma 7.3.5.

Finally, we note $\operatorname{supp}^\infty(\Delta_\nu^\Lambda) \subset AN$ by lemma 8.2.6. Since $\operatorname{sing} E \subset \mathcal{N}$, and the set of unipotents in AN is N, all sublimits of $\Delta_\nu^\Lambda \cdot E$ have support contained in N. Lemma 8.2.8

shows that we can apply corollary 7.3.11. Hence, by the same proof as for theorem 7.3.7, we obtain the following proposition.

Proposition 8.3.6. For any $g, h \in G$, there exists an irreducible representation

$$\varrho_{gNh^{-1}} : \mathcal{T}_E(G) \to \mathcal{T}_{E^{1,1}}(N) \quad \text{determined by} \quad \varrho_{gNh^{-1}}(T_f) = E^{(1,0),\{1\}} M_{g*f*h|N} E^{(1,0),\{1\}}$$

for all $f \in \mathcal{C}_0(G)$.

8.3.7. Define ideals

$$I_{(1,0),\{1\}} = \bigcap_{g,h\in G} \ker \varrho_{gNh^{-1}} \quad \text{and} \quad I_{(1,0),\varnothing} = \bigcap_{g\in G} \ker \varrho_g \,.$$

Then we have the following theorem.

Theorem 8.3.8. We have the following chain of ideals

$$\mathbb{K}(\mathbf{H}^2(\Gamma)) \triangleleft I_{(1,0),\{1\}} \triangleleft I_{(1,0),\varnothing} \triangleleft \mathcal{T}_E(G)$$

where $I_{1,1} \neq I_{1,0}$ and $I_{1,0} \neq \mathcal{T}_E(G)$.

Proof. The existence of the ideals follows from the above. The inclusions follow from corollary 7.1.15. That $I_{(1,0),\varnothing} \neq \mathcal{T}_E(G)$ is clear, because $T_f \in I_{(1,0),\varnothing}$ means that $f = 0$. As for the other inequality, $\mathcal{T} / I_{(1,0),\varnothing}$ is manifestly isomorphic to the sum of the representations ϱ_g, $g \in G$, and similarly for $I_{(1,0),\{1\}}$ (where the sum extends over a system of mutually inequivalent representations). Since the representations ϱ_g and $\varrho_{gNh^{-1}}$ are clearly inequivalent, the assertion follows. □

9 A strategy for the general case

In some parts, the results we have obtained fall short of the completion of the programme set forth in the introduction, namely, to establish a Principle of Restriction-Induction for the Toeplitz C*-algebra $\mathcal{T}_E(G)$. In this final section, we present a strategy for the general case, by presenting the methods that should allow for the definitive achievement of this bold objective.

9.1 Singularities and wave front of the Szegö distribution

The results on the geometry of the Ol'shanskiĭ domain we have obtained are complete and valid in full generality, for any bounded symmetric domain B, and the associated automorphism group $G = \mathrm{Aut}_0 B$. Likewise, the estimate of the fibre of the wave front set $\mathrm{WF}(E)_g$ of the Szegö distribution (theorem 3.3.2) is also valid in complete generality, but so far, we have not been able to fully exploit this information.

More specifically, as we have seen in theorem 7.3.7, the construction of the representation of $\mathcal{T}_E(G)$ associated to a facial subgroup $\bar{G} = G_e$ reduces to the proof of the statement that the support of a sublimit distribution is contained in $\bar{G}$. The tools we have at hand for the proof of such a statement (and which are, in fact, successful in the rank one case) were established in section 7.2, and are purely local (not micro-local).

To verify the support condition with the help of these methods, we need to have more information available on the singular support of E, that is, the base, and not the fibre, of the wave front set. Of course, one can infer that a distribution is smooth at some point in the presence of sufficiently detailed information on the wave front set. Using such micro-local methods, Duflo-Vergne [DV90] reproved Harish-Chandra's famous regularity theorem on invariant $\mathcal{Z}(\mathfrak{g})$-finite distributions. Needless to say, their proof is much shorter than Harish-Chandra's original argument.

However, the result (proposition 3.3.13) we have obtained by attempting to imitate this train of thought is, as yet, of a rather preliminary nature. Perhaps the existent information on the boundary of the cone $\Omega^+ = (\Omega^-)^*$ and its relation to the (non-convex) cone of nilpotents (theorem 2.1.27, which we recall is due to Hilgert, Neeb, and Ørsted) can be used to improve on the result lemma 2.1.32, which was important in the proof of proposition 3.3.13.

Another approach to the singular support would be to determine a formula for the Szegö kernel function K, at least on a system of representatives for the conjugacy classes of Cartan subgroups. This idea was pursued in 3.4, were the location of the singularities on the regular part T_*^G of the conjugacy classes of the maximal torus T was determined by extending Ol'shanskiĭ's explicit formula for the Szegö kernel on $T^{\mathbb{C}} \cap \Gamma^\circ$ to the boundary. This was in fact also the approach we take in the case of the unit disc.

Here, however, the Szegö kernel is known explicitly on all of Γ°. Likewise, this is true for all other the classical domains of type $\mathrm{I}_{p,q}$ or III_n. For these domains, the explicit formulae could be used to determine the singular support of E. This is rather unsatisfactory, since they are quite different from each other and depend on the specific structure of the algebra $\mathrm{S}(\mathfrak{t})^W$ of invariant polynomials in each of these cases.

As we have already remarked, at the end of section 3.4, it might well be possible to get similar formulae for the Szegö kernel on $H^{\mathbb{C}} \cap \Gamma^\circ$, where H is any one of a system of representatives of ϑ-stable CSG. The idea would be to imitate Ol'shanskiĭ's proof for the torus, which consisted in computing the sum over all characters of the holomorphic discrete series. On the torus T, these are given by the well-known rational function which resembles Weyl's character formula. The series summation is a sophisticated application of the geometric series.

Martens and Hecht have proved that this character formula essentially remains valid on (an 'octant' of) any ϑ-stable CSG constructed by Cayley transformation from T. Hence, a similar summation scheme should go through. The details of this approach will certainly be part of further work we intend to undertake in this direction.

Finally, we note that in order to complete the proof of irreducibility of the Toeplitz C*-algebras $\mathcal{T}_{E^{f,Ic}}(G_{f,I})$, the statement that the Szegö distributions are a.e. regular is needed. To that end, the required information on the singular set need not be very detailed. All that is required is the statement that it be thin.

9.2 Embedding of representations

Our results on the embedding of representations of facial subgroups are reasonably complete. Of course, a treatment of the exceptional domains, or even better, a unified treatment of all domains, would be desirable.

The construction of embeddings of the discrete series (corollary 5.2.25) relied on theorem 5.1.2 on the existence of compatible fundamental sequences, which was proved by case-by-case considerations. This is really a theorem on $\mathbb{Z}_2$-graded root systems, and perhaps this is the framework in which it should be proved in general. In any case, what is missing for a general picture is a good understanding of strong orthogonality of non-compact roots. This seems to be delicate matter, and even Knapp-Wallach [KW76] prove the existence of fundamental sequences case-by-case.

As for the result on parabolic subgroups of facial subgroups (theorem 5.3.9) where the last step was proved by a case-by-case argument, this can probably be made into a unified argument quite easily, since we allow for the appearance of a connected compact factor whose detailed structure is not needed. Up to this minor point, the embedding of the parabolic Q-series follows automatically from the existence of an embedding of the discrete series.

We note, however, that there is a boundedness issue here which is has not been completely solved. We commented on this fact in remark 5.4.7. Our approach to the boundedness of the embedding of parabolic Q-series was via Harish-Chandra's admissibility theorem for quasi-simple representations. This reduced the question to the $\mathcal{Z}(\bar{\mathfrak{g}})$-finiteness of a certain $\bar{\mathfrak{g}}$-module. If $\mathcal{Z}(\bar{\mathfrak{g}})$ is contained in a finitely generated $\mathcal{Z}(\mathfrak{g})$-submodule of the centraliser of $\mathfrak{U}(\bar{\mathfrak{g}})$ in $\mathfrak{U}(\mathfrak{g})$, then this $\mathcal{Z}(\bar{\mathfrak{g}})$-finiteness is automatic. We think that this is plausible, and we discussed the relation of this statement to a problem in invariant theory (which is probably not hard for the experts) in the aforementioned remark.

Besides the case of the exceptional domains, a more important task is the construction of the reduced spectra of the generalised Jacobi groups $G_{f,I}$ (which are also unimodular and of type I) associated to the other faces of the cone Ω^- . Of course, Mackey's machinery is in principle applicable to these groups, and general considerations of such semi-direct products have been carried out, e.g., by Lipsman.

However, this would ignore the special structure of these groups, where the semi-simple factor G_{c_1} acts on the generalised Heisenberg group $H_{f,I}$ as a vector-valued symplectic group, compare section 2.2. Hence, one expects that the representations of the group $G_{f,I} = G_{c_1} \ltimes H_{f,I}$ should have the general form of the classical metaplectic representation. This fact has been established by Neeb [Nee00a, th. X.3.7], in his metaplectic factorisation theorem. It represents every unitary representation of $G_{f,I}$ as the tensor product of a unitary representation of G_{c_1} with a highest weight representation of $H_{f,I}$.

In particular, the already constructed embeddings of the representations of G_{c_1} have only to be made to accommodate the nilpotent factor. Moreover, the highest weight representations of $H_{f,I}$ can be described explicitly, either by Neeb's generalised theory of highest weight representations, or by the classical construction due to Ogden-Vági [OV79]. This is closely related to the decomposition of the set of invertible elements in a Euclidean Jordan algebra into connected components (cones, most of them non-convex), and can hence be well related to the structure theory of the cone Ω^-.

A difficulty which we encountered already for the case of rank one will probably reappear for the nilpotent groups $H_{f,I}$ occurring in the semi-direct products: The embedding of their representations into representations of G will turn out to be only asymptotically equivariant. Here, by a sequence $j_k : \langle \bar{G} \rangle_{\bar{\pi}} \to \langle G \rangle_{\pi_k}$ to be *asymptotically equivariant*, we mean

$$j_k^* \pi(g) j_k - \bar{\pi}(g) \to 0 \quad \text{strongly for all} \quad g \in \bar{G} .$$

Along the lines of proof for lemma 8.2.8, the necessary convergence statements in the Fourier algebra follow, so that corollary 7.3.11 shall be applicable to prove the spectral boundary condition from the support condition on the sublimit distributions, as in theorem 7.3.7. The asymptotic equivariance condition should also be sufficient to extend lemma 7.1.6 to these subgroups. Then the entire construction of representations, as presented in section 7.2, goes through, as we have seen for the rank one case.

So the essential part is again the asymptotic behaviour of the matrix coefficients.

9.3 Asymptotic behaviour of matrix coefficients

Having only the asymptotic behaviour of the matrix coefficients of the holomorphic discrete series at hand, we can only estimate the supports of limit distributions for these representations. Moreover, since our information on the singular set of the Szegö distribution is incomplete, the estimates are preliminary at this stage.

To complete the construction of representations for the facial subgroups, the appropriate estimates have to be proved for the other series of representations, as well. This situation does not occur for the unit disc, since there are no other series of representations for the unique proper facial subgroup (the trivial one).

They main step should be to get an understanding of the discrete series. We point out that in our construction of the embeddings of the parabolic Q-series, we arranged matters in such a way to make the A component of the parabolic subgroup of G as small as possible, so that the parameter of the Q-series representation is fixed on A by the representation of the corresponding subgroup of $\bar{G}$, i.e. the ν in $\pi_{\lambda,\nu}$ is fixed.

The asymptotics we can consider are therefore only in the parameter λ of the inducing discrete series representation, which has some degrees of freedom.

We note that Duistermaat-Kolk-Varadarajan [DKV83] have obtained precise statements on the asymptotic behaviour of matrix coefficients of unitary principal series representations π_ν, as $\nu \to \infty$, by interpreting these as oscillatory integrals, and applying the method of stationary phase. An approach for the estimation of the asymptotics of matrix coefficients of the discrete series might have been to consider Knapp-Wallach's embedding into a non-unitary principal series, and applying similar ideas. However, this leads to oscillatory integrals with complex phase, and the method of stationary phase is, hence, not applicable.

Our approach for the holomorphic discrete series was to use the kernel function and its expression in terms of the Bergman operator. To use such an approach for the discrete series in general, an explicit realisation of the discrete series in a reproducing kernel Hilbert space has to used. Of course there is a host of such realisations, such as those due to Narasimhan-Okamoto, Hotta, Schmid, and Parthasarathy. All of these have the drawback that one has to treat non-holomorphic kernels.

Holomorphic realisations can be given by appropriate generalisations of the Borel-Weil theorem. These all have the drawback that they are given in some sense as sections of line bundles on 'flag manifolds', or 'flag domains' which are huge compared to the symmetric $B = G/K$, on which the holomorphic discrete series can be realised to neatly.

An interesting recent development in this direction is the application of the 'Penrose transform', related to a double fibration associated to to the flag domains, by considering the so-called space of linear cycles. This leads to a realisation of the discrete series in spaces of holomorphic sections of a vector bundle on the product of B with its conjugate. The drawback is here that the representation space identifies with the kernel of a differential operator which is defined in highly abstract terms, namely, as the image of a relative $\bar{\partial}$ operator in some degree of a Leray spectral sequence. This idea has been developed by Barchini, Gindikin, Matsuki, Huckleberry, Wolf, and Zierau, among others. We mention the survey [WZ00].

It is conceivable that a more or less explicit formula for the kernel be found for such a realisation of the discrete series, but, needless to say, the techniques necessary for such a venture will have to be rather more sophisticated than for the case of the holomorphic discrete series.

We note in passing that of course a lot of substantial work has been done on the asymptotics of matrix coefficients. However, with few exceptions, this treated the behaviour for fixed representation parameter, as the group variable tends to infinity in prescribed directions.

A different approach to bounding would be to combine micro-local information on the matrix coefficients and on the Szegö distribution. A first attempt to do this is given by proposition 4.4.5. The applicability of this result to the construction of representation of the Toeplitz C*-algebra $\mathcal{T}_E(G)$ is somewhat hampered by the fact that we were only able to apply information on the torus, whereas it should by now be obvious that the orbit picture in the entire (dual) Lie algebra has to be taken into consideration. We discussed the limitations of the cited proposition in remark 4.4.6.

Nonetheless, micro-local methods will certainly ultimately come to bear upon the problem, and it seems promising to further investigate how the information on the wave-front of the Szegö distribution can be brought to effect.

9.4 Inequivalence and exhaustion

Besides the open problems in the construction of irreducible representations of $\mathcal{T}_E(G)$, there are of course further steps that have to be undertaken in order to fully establish the Principle of Restriction-Induction.

One point is the proof of inequivalence for the representations $\varrho_{g\bar{G}h^{-1}}$ associated to a fixed subgroup $\bar{G} = G_{f,I}$. The conjectural result is that $\varrho_{g_j\bar{G}h_j^{-1}}$, $j = 1, 2$, are inequivalent whenever $(g_1g_2^{-1}, h_1h_2^{-1})$ is not contained in the normaliser of $\bar{G}$ in $G \times G$. This should be quite easy to see from the defining equation of these representations on generators.

Moreover, the normaliser of $G_{f,I}$ with respect to the action of the diagonal subgroup $G \sqsubset G \times G$ has already been more or less determined, since we have related the associated face $F^-_{f,I}$ to the intersection of Ω^- with two opposed parabolics. Since parabolics are self-normalising, this looks rather promising. To extend this to the action of $G \times G$ is only a matter of invoking the Langlands decomposition.

In analogy to the case of bounded symmetric domains, one may conjecture that the successive quotients of the constructed ideals is Morita equivalent to the commutative algebra $\mathcal{C}_0((G \times G)/N_{G\times G}(\bar{G}))$, where the normaliser occurring in the denominator is computable along the lines sketched above.

In proving such a formula for the successive quotients, the first step is to identify the ideals associated to the maximal faces, with the ideal of compact operators. By the machinery of coactions, this reduces to showing equality in the inclusion $\mathrm{C}^*_\#(G) \subset \bigcap_{g,h\in G} \ker \pi_{g\bar{G}h^{-1}}$ where $\bar{G}$ belongs to a maximal face.

Our approach is again to tackle this problem by the local analysis of singularities. First, we would prove that $\operatorname{sing} a = 0$ for any a contained in the intersection of kernels. An indication as to which methods should allow for the proof of such a statement is given by remark 7.1.10 and lemma 7.2.7, in fact, we are searching for a converse of the latter.

The next point would be to prove that $\operatorname{sing} a = \varnothing$ implies that $a \in \mathrm{C}^*_\#(G)$, at least if $a \in \overline{\mathcal{K}}\mathrm{W}^*(G)$. This is a *passage from local to global* statement. Alternatively, one could try to prove that the above intersection of kernels is generated by compactly supported elements (this is not obvious).

Once the ideals at the lowest level of the composition series have been identified, the computation of the quotient should amount to a relative version of this argument, compare the proof of [Upm96, th. 4.11.133]. One should note that Upmeier makes great use of the (commutative) subalgebra of K-invariants in $\mathrm{C}^*_E(K)$, which in this case was easy to define, because of the compactness of K. In our setting, the analogous object is the algebra of $(G \times G)$-invariants. Such an object does exist, for a dense subalgebra of $\mathrm{C}^*_E(G)$: Quigg [Qui92] has constructed an algebra of invariants, by using methods of non-commutative integration. To extend this to our framework might turn out be a crucial step in the computation of the quotients.

Anther useful approach could be to apply the same idea which lead to the definition of the *singular set* sing to the successive quotients of the other ideals. Namely, recall that sing was defined in terms of the $\mathrm{A}(G)$-action on a quotient module. The counterparts of the ideals $I_{k,\ell}$ in the algebra $\mathrm{C}^*_E(G)$ are also submodules, so the local deviation of an element from the ideal in the denominator of a successive quotient can be measured. Proving the formula for the quotients would then again be a question of passing from local to global.

Once the successive quotients have thus been computed, it follows that any maximal chain in the aforementioned lattice of ideals is a composition series. In particular, all of the representations of the full Toeplitz C^*-algebra are induced by one of the ideals, i.e. are supported by one of the faces. This solves the problem of exhaustion in the Principle of Restriction-Induction.

References

[ÁC83] J. D. Álvarez Alonso and A. P. Calderón. Functional Calculi for Pseudodifferential Operators. II. In *Studies in Applied Mathematics*, Adv. Math. Suppl. Stud. **8**, pages 27–72. Academic Press, New York, 1983.

[AH96] J. D. Alvarez and J. Hounie. Functions of Pseudo-Differential Operators of Non-Positive Order. *J. Funct. Anal.*, 141(1):45–59, 1996.

[All99] A. Alldridge. Toeplitz-Operatoren auf dem Hardyraum von $\mathrm{SL}(2, \mathbb{R})$. Diplomarbeit, Philipps-Universiät Marburg, 1999.

[And69] R. F. V. Anderson. The Weyl Functional Calculus. *J. Func. An.*, 4:240–267, 1969.

[APT73] C. A. Akemann, G. K. Pedersen, and J. Tomiyama. Multipliers of C^*-Algebras. *J. Func. An.*, 13:277–301, 1973.

[AU02] A. Alldridge and H. Upmeier. Toeplitz Operators on Hardy Spaces over $\mathrm{SL}(2, \mathbf{R})$: Irreducibility and Representations. In *Geometry and Analysis on Finite- and Infinite-Dimensional Lie Groups (Będlewo, 2000)*, Banach Center Publ. **55**, pages 173–209. Polish Acad. Sci., Warsaw, 2002.

[AU03] J. Arazy and H. Upmeier. Boundary Measures for Symmetric Domains and Integral Formulas for the Discrete Wallach Points. *Integral Equations Operator Theory*, 47(4):375–434, 2003.

[BC79] C. A. Berger and L. A. Coburn. Wiener-Hopf Operators on U_2. *Integral Equations Operator Theory*, 2(2):139–173, 1979.

[BCK80] C. A. Berger, L. A. Coburn, and A. Korányi. Opérateurs de Wiener-Hopf sur les sphères de Lie. *C. R. Acad. Sci. Paris Sér. A-B*, 290(21):A989–A991, 1980.

[BdMG81] L. Boutet de Monvel and V. Guillemin. *The Spectral Theory of Toeplitz Operators*. Ann. of Math. Stud. **99**. Princeton Univ. Press, 1981.

[Bon77] J. M. Bony. Équivalence des diverses notions de spectre singulier analytique. In *Séminaire Goulaouic-Schwartz (1976/1977), Exp. No. 3*. Centre Math., École Polytech., Palaiseau, 1977.

[Bou68] N. Bourbaki. *Éléments de mathematiques: Groupes et algèbres de Lie*. Act. Sci. Ind. **1337**. Hermann, Paris, 1968.

[BR99] J. Bernstein and A. Reznikov. Analytic Continuation of Representations and Estimates of Automorphic Forms. *Ann. of Math. (2)*, 150(1):329–352, 1999.

[BS93] S. Baaj and G. Skandalis. Unitaires multiplicatifs et dualité pour les produits croisés de C^*-algèbres. *Ann. Sci. École Norm. Sup. (4)*, 26(4):425–488, 1993.

[Bus68] R. C. Busby. Double Centralizers and Extensions of C^*-Algebras. *Trans. Amer. Math. Soc*, 132:79–99, 1968.

[Car35] E. Cartan. Sur les domaines bornés de l'éspace de n variables complexes. *Abh. Math. Sem. Univ. Hamburg*, 11:116–162, 1935.

[Cob74] L. A. Coburn. Singular Integral Operators and Toeplitz Operators on Odd Spheres. *Indiana Univ. Math. J.*, 23:433–439, 1973/74.

[DH71] R. G. Douglas and Roger Howe. On the C^*-algebra of Toeplitz operators on the quarterplane. *Trans. Amer. Math. Soc.*, 158:203–217, 1971.

[Dix69] J. Dixmier. *Les C^*-algèbres et leurs representations*. Cahiers Scientifiques, Fasc. **XXIX**. Gauthiers-Villars Éditeur, Paris, 2 edition, 1969.

[Dix77] J. Dixmier. *Enveloping Algebras*. Math. Library **14**. North-Holland, Amsterdam, 1977.

[DKV83] J. J. Duistermaat, J. A. C. Kolk, and V. S. Varadarajan. Functions, Flows and Oscillatory Integrals on Flag Manifolds and Conjugacy Classes in Real Semisimple Lie Groups. *Compositio Math.*, 49(3):309–398, 1983.

[DM76] M. Duflo and C. C. Moore. On the Regular Representation of a Nonunimodular Locally Compact Group. *J. Func. An.*, 21(2):209–243, 1976.

[DS58] N. Dunford and J.T. Schwartz. *Linear Operators. I. General Theory*. Pure Appl. Math. **7**. John Wiley & Sons, New York, 1958.

[DV90] M. Duflo and M. Vergne. Orbites Coadjointes et Cohomologie Équivariante. In M. Duflo, N. V. Pedersen, and M. Vergne, editors, *The Orbit Method in Representation Theory, Proceedings of a Conference (Copenhagen, Aug–Sep 1988)*, Progress in Math. **82**, Boston, MA, 1990. Birkhäuser.

[Els99] J. Elstrodt. *Maß- und Integrationstheorie*. Springer, Berlin, 1999.

[Enr79] T. J. Enright. On the Fundamental Series of a Real Semisimple Lie Algebra: Their Irreducibility, Resolutions and Multiplicity Formulae. *Ann. Math.*, 110(1):1–82, 1979.

[ER00] E. G. Effros and Z.-J. Ruan. *Operator Spaces*. London Mathematical Society Monographs. New Series **23**. Clarendon Press, Oxford, 2000.

[ER03] E. G. Effros and Z.-J. Ruan. Operator Space Tensor Products and Hopf Convolution Algebras. *J. Operator Theory*, 50(1):131–156, 2003.

[ES92] M. Enock and J.-M. Schwartz. *Kac Algebras and Duality of Locally Compact Groups*. Springer-Verlag, Berlin, 1992.

[Eym64] P. Eymard. L'algèbre de Fourier d'un groupe localement compact. *Bull. Soc. Math. France*, 92:181–236, 1964.

[FK94] J. Faraut and A. Korányi. *Analysis on Symmetric Cones*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1994.

[GG77] I. M. Gel'fand and S. G. Gindikin. Complex Manifolds whose Skeletons are Semi-Simple Lie Groups, and Analytic Discrete Series of Representations. *Funct. Anal. Appl.*, 11(4), 1977.

[GK58] I. C. Gohberg and M. G. Kreĭn. Systems of Integral Equations on the Half-Line with Kernels depending on the Difference of the Arguments. *Uspehi Mat. Nauk (N.S.)*, 13(2 (80)):3–72, 1958.

[GL81] E. E. Granirer and M. Leinert. On Some Topologies which Coincide on the Unit Sphere of the Fourier-Stieltjes Algebra $B(G)$ and of the Measure Algebra $M(G)$. *Rocky Mountain J. Math.*, 11(3):459–472, 1981.

[Glö00] H. Glöckner. Infinite-Dimensional Complex Groups and Semigroups: Representations of Cones, Tubes, and Conelike Semigroups. Dissertation, TU Darmstadt, 2000.

[Goo80] R. Goodman. Elliptic and Subelliptic Estimates for Operators in an Enveloping Algebra. *Duke Math. J.*, 47(4):819–833, 1980.

[Gro52] A. Grothendieck. Critères de compacité dans les espaces fonctionnels généraux. *Amer. J. Math.*, 74:168–186, 1952.

[GS79] V. Guillemin and S. Sternberg. On the Spectra of Commuting Pseudodifferential Operators: Recent Work of Kac-Spencer, Weinstein, and others. In *Partial Differential Equations and Geometry (Proc. Conf., Park City, Utah, 1977)*, Lect. Notes in Pure and Appl. Math. **48**, pages 149–165. Dekker, New York, 1979.

[Gui79] V. Guillemin. Some Microlocal Aspects of Analysis on Compact Symmetric Spaces. In *Seminar on Microlocal Analysis*, Ann. of Math. Stud. **93**, pages 79–111. Princeton Univ. Press, 1979.

[Hag99] U. Hagenbach. Hardy-Toeplitz C*-Algebren über nicht-konvexen Kegeln. Dissertation, Philipps-Universität Marburg, 1999.

[HC54] Harish-Chandra. Representations of Semisimple Lie Groups. III. *Trans. Amer. Math. Soc.*, 76:234–253, 1954.

[HC56] Harish-Chandra. Representations of Semisimple Lie Groups. VI. Integrable and Square-Integrable Representations. *Amer. J. Math.*, 78:564–628, 1956.

[HC64] Harish-Chandra. Invariant Distributions on Lie Algebras. *Amer. J. Math.*, 86:271–309, 1964.

[HC70] Harish-Chandra. Harmonic Analysis on Semisimple Lie Groups. *Bull. Amer. Math. Soc.*, 76:529–551, 1970.

[HC76] Harish-Chandra. Harmonic Analysis on Real Reductive Groups III. The Maass-Selberg Relations and The Plancherel Formula. *Ann. Math.*, 104(1):117–201, 1976.

[Hec76] H. Hecht. The Characters of some Representations of Harish-Chandra. *Math. Ann.*, 219(3):213–226, 1976.

[Hel70] S. Helgason. A Duality for Symmetric Spaces with Applications to Group Representations. *Adv. Math.*, 5:1–154, 1970.

[Hel78] S. Helgason. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Pure and Appl. Math. **80**. Academic Press, 1978.

[Hel84] S. Helgason. *Groups and Geometric Analysis*. Pure Appl. Math. **113**. Academic Press, San Diego, 1984.

[Her73] C. Herz. Harmonic Synthesis for Subgroups. *Ann. Inst. Fourier*, 23(3):91–123, 1973.

[HH88] J. Hilgert and K. H. Hofmann. Invariant Cones in Lie Algebras. *Semigroup Forum*, 37(2):241–252, 1988.

[HHL89] J. Hilgert, K. H. Hofmann, and J. D. Lawson. *Lie Groups, Convex Cones, and Semigroups*. Oxford Mathematical Monographs. Clarendon Press, Oxford, 1989.

[HN93] J. Hilgert and K.-H. Neeb. *Lie Semigroups and their Applications*. Lect. Notes Math. **1552**. Springer-Verlag, Berlin, 1993.

[HN95] J. Hilgert and K.-H. Neeb. Wiener-Hopf Operators on Ordered Homogeneous Spaces. I. *J. Funct. Anal.*, 132(1):86–118, 1995.

[HNØ94] J. Hilgert, K.-H. Neeb, and B. Ørsted. The Geometry of Nilpotent Coadjoint Orbits of Convex Type in Hermitian Lie Algebras. *J. Lie Theory*, 4(2):185–235, 1994.

[HNØ96] J. Hilgert, K.-H. Neeb, and B. Ørsted. Conal Heisenberg Algebras and Associated Hilbert Spaces. *J. Reine Angew. Math.*, 474:67–112, 1996.

[HÓØ91] J. Hilgert, G. Ólafsson, and B. Ørsted. Hardy Spaces on Affine Symmetric Spaces. *J. Reine Angew. Math.*, 415:189–218, 1991.

[Hör67] L. Hörmander. Pseudo-Differential Operators and Hypoelliptic Equations. In *Singular Integrals (Proc. Sympos. Pure Math., Vol. X, Chicago, Ill., 1966)*, pages 138–183. Amer. Math. Soc., Providence, RI, 1967.

[Hör71a] L. Hörmander. Fourier Integral Operators I. *Acta Math.*, 127:79–183, 1971.

[Hör71b] L. Hörmander. Uniqueness Theorems and Wave Front Sets for Solutions of Linear Differential Equations with Analytic Coefficients. *Comm. Pure Appl. Math.*, 24:671–704, 1971.

[Hör73] L. Hörmander. *An Introduction to Complex Analysis in Several Variables*. North-Holland, Amsterdam, 1973.

[Hör83] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*. Grundlehren **256**. Springer-Verlag, Berlin, 1983.

[HR79] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis. Vol. I*. Grundlehren **115**. Springer-Verlag, Berlin, 1979.

[HU98] U. Hagenbach and H. Upmeier. Toeplitz C^*-Algebras over Non-Convex Cones and Pseudo-Symmetric Spaces. In *Operator Theory for Complex and Hypercomplex Analysis (Mexico City, 1994)*, Contemp. Math. **212**, pages 109–131. Amer. Math. Soc., Providence, RI, 1998.

[Hum75] J. E. Humphreys. *Linear Algebraic Groups*. Springer, Berlin, 1975.

[Iag78] D. Iagolnitzer. *The S Matrix*. North-Holland, Amsterdam, 1978.

[Kat84] Y. Katayama. Takesaki's Duality for a Nondegenerate Co-Action. *Math. Scand.*, 55(1):141–151, 1984.

[Kg81] H. Kumano-go. *Pseudodifferential Operators*. MIT Press, Cambridge, MA, 1981.

[KMS93] I. Kolář, P.W. Michor, and J. Slovák. *Natural Operations in Differential Geometry*. Springer-Verlag, Berlin, 1993.

[Kna86] A. W. Knapp. *Representation Theory of Semisimple Groups — An Overview Based on Examples*. Princeton Mathematical Series **36**. Princeton Univ. Press, Princeton, NJ, 1986.

[Kna02] A. W. Knapp. *Lie Groups Beyond an Introduction*. Progress in Math. **140**. Birkhäuser, Boston, MA, 2002.

[Koe69] M. Koecher. *An Elementary Approach to Bounded Symmetric Domains*. Rice University, Houston, TX, 1969.

[Kos74] B. Kostant. On Convexity, the Weyl group and the Iwasawa Decomposition. *Ann. Sci. École Norm. Sup. (4)*, 6:413–455, 1974.

[KS] B. Krötz and R. J. Stanton. Holomorphic Extension of Representations. II. Geometry and Harmonic Analysis. 46 p., submitted.

[KS04] B. Krötz and R. J. Stanton. Holomorphic Extension of Representations. I. Automorphic Functions. To appear in *Ann. Math.*, 159(2), 2004. 84 p.

[KV00] J. Kustermans and S. Vaes. Locally Compact Quantum Groups. *Ann. Sci. École Norm. Sup. (4)*, 33(6):837–934, 2000.

[KV03] J. Kustermans and S. Vaes. Locally Compact Quantum Groups in the Von Neumann Algebraic Aetting. *Math. Scand.*, 92(1):68–92, 2003.

[KW76] A. W. Knapp and N. R. Wallach. Szegö Kernels Associated with Discrete Series. *Inventiones math.*, 34(3):163–200, 1976.

[KW80] A. W. Knapp and N. R. Wallach. Correction and Addition to: Szegö Kernels Associated with Discrete Series. *Inventiones math.*, 62(2):341–346, 1980.

[LN02] O. Loos and E. Neher. Locally Finite Root Systems. Preprint, 2002.

[Loo69] O. Loos. *Symmetric Spaces. I: General Theory*. W. A. Benjamin, New York, 1969.

[Loo75] O. Loos. *Bounded Symmetric Domains and Jordan Pairs.* Lecture Notes. Univ. of California, Irvine, 1975.

[LPRS87] M. B. Landstad, J. Phillips, I. Raeburn, and C. E. Sutherland. Representations of Crossed Products by Coactions and Principal Bundles. *Trans. Amer. Math. Soc.*, 299(2):747–784, 1987.

[Mar77] A. Martineau. Distributions et valeurs au bord des fonctions holomorphes. In *Oeuvres de André Martineau*, pages 439–582. Éditions du CNRS, Paris, 1977.

[Mia99] T. Miao. Decomposition of $B(G)$. *Trans. Amer. Math. Soc.*, 351(11):4675–4692, 1999.

[MM65] J. W. Milnor and J. C. Moore. On the Structure of Hopf Algebras. *Ann. of Math.*, 81(2):211–264, 1965.

[Moo64] C. C. Moore. Compactifications of Symmetric Spaces II: The Cartan Domains. *Amer. J. Math.*, 86:358–378, 1964.

[Mos55] G. D. Mostow. Some New Decomposition Theorems for Semi-Simple Groups. *Mem. Amer. Math. Soc.*, 1955(14):31–54, 1955.

[MR82] P. S. Muhly and J. N. Renault. C^*-Algebras of Multivariable Wiener-Hopf Operators. *Trans. Amer. Math. Soc.*, 274(1):1–44, 1982.

[Nee94] K.-H. Neeb. The Classification of Lie algebras with Invariant Cones. *J. Lie Theory*, 4(2):139–183, 1994.

[Nee96] K.-H. Neeb. Invariant Convex Sets and Functions in Lie Algebras. *Semigroup Forum*, 53(2):230–261, 1996.

[Nee98] K.-H. Neeb. On the Complex and Convex Geometry of Ol'shanskiĭ Semigroups. *Ann. Inst. Fourier*, 48(1):149–203, 1998.

[Nee00a] K.-H. Neeb. *Holomorphy and Convexity in Lie Theory*. Expositions in Mathematics **28**. De Gruyter, 2000.

[Nee00b] K.-H. Neeb. Representation Theory and Convexity. *Transform. Groups*, 5(4):325–350, 2000.

[NT79] Y. Nakagami and M. Takesaki. *Duality for Crossed Products of von Neumann Algebras*. Lect. Notes in Math. **731**. Springer-Verlag, Berlin, 1979.

[Ol'82] G. I. Ol'shanskiĭ. Invariant Cones in Lie Algebras, Lie Semigroups, and the Holomorphic Discrete Series. *Funct. Anal. Appl.*, 15:275–285, 1982.

[Ol'91] G. I. Ol'shanskiĭ. Complex Lie Semigroups, Hardy Spaces, and the Gelfand-Gindikin Program. *Diff. Geom. Appl.*, 1:297–308, 1991.

[Ol'95] G. I. Ol'shanskiĭ. Cauchy-Szegö Kernels for Hardy Spaces on Simple Lie Groups. *J. Lie Theory*, 5(2):241–273, 1995.

[OM80] T. Oshima and T. Matsuki. Orbits on Affine Symmetric Spaces under the Action of the Isotropy Subgroups. *J. Math. Soc. Japan*, 32(2):399–114, 1980.

[ÓØ88] G. Ólafsson and B. Ørsted. The Holomorphic Discrete Series of An Affine Symmetric Space and Representations with Reproducing Kernels. Institut for Matematik og Datalogi, Odense Universitet, Preprint No. 13, 1988.

[OV79] R. D. Ogden and S. Vági. Harmonic Analysis of a Nilpotent Group and Function Theory of Siegel Domains of Type II. *Adv. in Math.*, 33(1):31–92, 1979.

[Pan83] S. M. Paneitz. Determination of Invariant Convex Cones in Simple Lie Algebras. *Ark. Mat.*, 21:217–228, 1983.

[Qui92] J. C. Quigg. Landstad Duality for C^*-Algebras. *Math. Scand.*, 71:277–294, 1992.

[Rup88] W. A. F. Ruppert. A Geometric Approach to the Bohr Compactification of Cones. *Math. Z.*, 199:209–232, 1988.

[Sat80] I. Satake. *Algebraic Structures of Symmetric Domains*. Kanō Memorial Lectures **4**. Iwanami Shōten, Tōkyō, 1980.

[Sch75] W. Schmid. On the Characters of The Discrete Series. The Hermitian Symmetric Case. *Inventiones math.*, 30:47–144, 1975.

[Sjö82] J. Sjöstrand. Singularités Analytiques Microlocales. Astérisque **95**, pages 1–166. Soc. Math. France, Paris, 1982.

[Sta86] R. J. Stanton. Analytic Extension of the Holomorphic Discrete Series. *Amer. J. Math.*, 108:1411–1424, 1986.

[Str72] R. S. Strichartz. Invariant Pseudo-Differential Operators on a Lie Group. *Ann. Scuola Norm. Sup. Pisa (3)*, 26:587–611, 1972.

[Sug71] M. Sugiura. Correction to My Paper: Conjugate Classes of Cartan Subalgebras in Real Semisimple Lie Algebras. *J. Math. Soc. Japan*, 23(2):379–383, 1971.

[Swe69] M. E. Sweedler. *Hopf Algebras*. Mathematics Lecture Note Series. W. A. Benjamin, New York, 1969.

[Tak64] M. Takeuchi. On the Fundamental Group and the Group of Isometries of a Symmetric Space. *J. Fac. Sci. Univ. Tokyo*, 10:88–123, 1964.

[Tak76] M. Takesaki. *Theory of Operator Algebras. I.* Springer-Verlag, Berlin, 1976.

[Tak03] M. Takesaki. *Theory of operator algebras. III*. Encyclopaedia of Mathematical Sciences **127**. Springer-Verlag, Berlin, 2003.

[Tay70] D. C. Taylor. The Strict Topology for Double Centralizer Algebras. *Trans. Amer. Math. Soc.*, 150:633–643, 1970.

[Tay81] M.E. Taylor. *Pseudodifferential Operators*. Princeton Mathematical Series **34**. Princeton Univ. Press, Princeton, N.J., 1981.

[Tit67] J. Tits. *Tabellen zu den einfachen Lie-Gruppen und ihren Darstellungen*. Lect. Notes in Math. **40**. Springer-Verlag, Berlin, 1967.

[Trè67] F. Trèves. *Topological Vector Spaces, Tensor Products, Distributions and Kernels*. Pure Appl. Math. **25**. Academic Press, New York, 1967.

[Trè80] F. Trèves. *Introduction to Pseudodifferential and Fourier Integral Operators. Vol. 1*. Plenum Press, New York, 1980.

[TT72] M. Takesaki and N. Tatsūma. Duality and Subgroups, II. *J. Func. An.*, 11:184–190, 1972.

[Upm82] H. Upmeier. Harmonische Analysis und Toeplitz-Operatoren auf beschränkten symmetrischen Gebieten. Habilitationsschrift, Eberhard-Karls-Universität Tübingen, 1982.

[Upm84] H. Upmeier. Toeplitz C^*-Algebras on Bounded Symmetric Domains. *Ann. of Math. (2)*, 119(3):549–576, 1984.

[Upm85] H. Upmeier. *Symmetric Banach Manifolds and Jordan C^*-Algebras*. Math. Studies **104**. North-Holland, 1985.

[Upm86] H. Upmeier. Jordan Algebras and Harmonic Analysis on Symmetric Spaces. *Amer. J. Math.*, 108:1–25, 1986.

[Upm91] H. Upmeier. Toeplitz C^*-Algebras and Noncommutative Duality. *J. Operator Theory*, 26(2):407–432, 1991.

[Upm96] H. Upmeier. *Toeplitz Operators and Index Theory in Several Complex Variables*. Op. Theory Adv. Appl. **81**. Birkhäuser, Basel, 1996.

[Val85] J.-M. Vallin. C*–algèbres de Hopf et C*–algèbres de Kac. *Proc. London Math. Soc.*, 50:131–174, 1985.

[VD01] S. Vaes and A. Van Daele. Hopf C*-Algebras. *Proc. London Math. Soc. (3)*, 82(2):337–384, 2001.

[Vin80] E. B. Vinberg. Invariant Convex Cones and Orderings in Lie Groups. *Funct. Anal. Appl.*, 14:1–10, 1980.

[War72] G. Warner. *Harmonic Analysis on Semi-Simple Lie Groups I, II*. Grundlehren **188**/**189**. Springer-Verlag, Berlin, 1972.

[Was84] A. Wassermann. Algèbres d'opérateurs de Toeplitz sur les groupes unitaires. *C. R. Acad. Sci. Paris Ser. I Math.*, 299:871–874, 1984.

[Wol80] J. A. Wolf. Foundations of Representation Theory for Semisimple Lie Groups. In J. A. Wolf, M. Cahen, and M. De Wilde, editors, *NATO Advanced Study Institute on Representations of Lie Groups and Harmonic Analysis, Liège*

1977. Harmonic Analysis and Representation Theory of Semisimple Lie Groups, Math. Phys. Appl. Math. **5**, pages 69–130, Dordrecht, 1980. D. Reidel.

[WZ00] J. A. Wolf and R. Zierau. Holomorphic Double Fibration Transforms. In *The mathematical legacy of Harish-Chandra (Baltimore, MD, 1998)*, volume 68 of *Proc. Sympos. Pure Math.*, pages 527–551. Amer. Math. Soc., Providence, RI, 2000.

Lebenslauf

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Erklärung

Ich versichere, dass ich meine Dissertation

'Toeplitz Operators on Semi-Simple Lie Groups'

selbstständig und ohne unerlaubte Hilfe angefertigt und mich dabei keiner anderen als der von mir ausdrücklich bezeichneten Quellen und Hilfen bedient habe.

Die Dissertation wurde in der jetzigen oder einer ähnlichen Form noch bei keiner anderen Hochschule eingereicht und hat noch keinen sonstigen Prüfungszwecken gedient.

Paderborn, 11. August 2004

(Alexander Alldridge)