Philipps Universität Marburg

JORDAN THEORETIC G-ORBITS AND FLAG VARIETIES

Dissertation

zur Erlangung des Doktorgrades
der Naturwissenschaften (Dr. rer. nat.)

dem
Fachbereich Mathematik und Informatik
der Philipps-Universität Marburg
(Hochschulkennziffer 1180)
vorgelegt von

Benjamin Schwarz

aus Stade

Marburg, im August 2010

Erstgutachter:	Prof. Dr. H. Upmeier
Zweitgutachter:	Prof. Dr. J. Hilgert
Eingereicht am:	25. August 2010
Disputation am:	21. September 2010

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Introduction

The geometric realization of irreducible unitary representations of Lie groups via symmetric and homogeneous spaces is one of the fundamental problems in harmonic analysis. For nilpotent Lie groups, Kirillov's orbit method provides geometric realizations of the irreducible unitary representations on coadjoint orbits [**21**], and for compact Lie groups, Borel-Weil-Bott theory establishes a one-to-one correspondence between the unitary dual and certain line bundles on a corresponding flag variety [**11**]. The object of this thesis is to investigate orbit structures that are induced by *semisimple non-compact* Lie groups. It is well-known that in this general setting, the orbit method needs to be substantially extended, and the theory is still far from complete. A general construction of orbits and corresponding representations is due to J. Wolf (on partial holomorphic cohomology spaces [**45**]). However, the Lie theoretic nature of this approach still results in a rather abstract than explicit realization of the involved orbits and vector bundles. Just a few examples have been worked out explicitly in this manner.

In the hermitian case, i.e. if $G = \mathrm{Aut}(\mathcal{D})$ is the automorphism group of a bounded symmetric domain $\mathcal{D} = G/K$, the Borel imbedding of $\mathcal{D}$ into its compact dual $X = G^c/K$ gives rise to the study of G-orbits on the compact dual. Using Lie theory, J. Wolf classified the G-orbits on X and determined some of their geometric structure [**44**], but again, the result is a rather abstract than explicit description of the orbits. In this situation, the Lie theoretic treatment is complemented by the Jordan theoretic approach, which was introduced by M. Koecher and O. Loos [**24, 28**]. A fundamental difference between Lie and Jordan theory appears in the way of connecting local with global structures [**5**]: in Lie theory, charts on $X = G^c/K$ are given by the exponential map. Instead, the Jordan theoretic model of X (due to O. Loos) is algebraic geometric, since the transition maps just involve fundamental birational maps (such as determinants and quasi-inverses). By these means, Jordan theory provides a basic realization of the compact dual as an algebraic variety in the sense of Mumford [**35**], and in particular, the involved Jordan structure itself is regarded as an open and dense subset of the compact dual. Therefore, one may expect that Jordan theory yields considerably more explicit descriptions of the G-orbits on X. Indeed, for the boundary orbits of $\mathcal{D}$, this has been achieved by O. Loos [**28**], and the initial question of this thesis is how to extend this description to all G-orbits on the compact dual X. Given such explicit realizations of the G-orbits, the next step is to define additional structures (e.g. line bundles and G-invariant measures) on which the G-representations are built upon. In the case of the bounded symmetric domain $\mathcal{D}$ and the boundary orbits, this is done e.g. by J. Faraut and A. Koranyi (for tube case, [**8**]) and by H. Upmeier et al. (for non-tube case, [**1, 42**]).

More generally and extending the symmetric case, the goal of the program *"Jordan theory and geometric realizations"* is (i) to give a Jordan theoretic description of generalized flag varieties $G^{\mathbb{C}}/P$ with $P \subset G^{\mathbb{C}}$ parabolic, (ii) to determine explicitly the G-orbit structure, and (iii) to describe the corresponding representation theory. Even in the case of the compact dual X mentioned above, this task is highly non-trivial, since it also includes exceptional geometries. Therefore, based on the Jordan theoretic description of the compact dual given by O. Loos, a first main result of this thesis is the complete and explicit Jordan theoretic description of the G-orbits (and their Matsuki duals) on X, cf. Theorem 7.2.

Turning to generalized flag varieties, we note that the existence of a Jordan theoretic description of these is far from obvious: The Jordan structures corresponding to bounded symmetric domains $\mathcal{D} = G/K$ inherit the characteristics of the real Lie group G, but these characteristics are not present in the generalized flag variety $G^{\mathbb{C}}/P$. In fact, this indicates that it is too much to expect a Jordan theoretic model for *all* flag varieties. Instead, the second main result of this thesis is the Jordan theoretic description of flag varieties $G^{\mathbb{C}}/P$, where $P = Q^{\mathbb{C}}$ is the complexification of some real parabolic subgroup Q of G, cf. Theorem 8.11 and Theorem 8.20. Concerning the representation theory of G, we note that the restriction to real parabolic subgroups is sufficient, e.g. the principal series is realized on $G^{\mathbb{C}}/Q^{\mathbb{C}}$ for minimal parabolic $Q \subset G$, cf. [**22, 4**].

Regarding the proposed program on Jordan theory and geometric realizations, this thesis provides a large part of the geometric background for the representation theoretic questions of this program. We also give a Jordan theoretic description of some fundamental line bundles on the generalized flag varieties, and as a first application (and a third main result), we generalize the *determinant functions* introduced by L. Barchini, S.G. Gindikin and H.W. Wong on ordinary flag manifolds. For further reading on the particular importance of these functions to both geometric and representation theoretic questions we refer to [**3, 4**].

Parallel to the G-orbit structure we determine the $K^{\mathbb{C}}$-orbits on the compact dual. In the 80s of the 20th century, T. Matsuki worked out a one-to-one correspondence between these two orbit structures, now called *Matsuki duality.* Using Jordan theoretic arguments, we are able to verify this duality by explicit computations. There are close connections of the Matsuki duality to the theory of cycle spaces of flag domains, which in turn provides substantial contributions to the geometric realization of representations of semisimple Lie groups, see [**9**].

Methods and results. We now review the main methods used in this thesis and the results thus obtained. In addition, we will give first brief descriptions of important concepts.

In Chapters 1 and 2 we introduce the basic algebraic structures, Jordan algebras and Jordan triple systems. The review on Jordan algebras is purely classical and can be found in any standard text on Jordan algebras [**6, 8**]. We recall these results for convenience and to fix some notation. The same holds for Sections 2.1 through 2.4, where we introduce the basic notions of positive hermitian Jordan triple systems (phJTS), from now on denoted by Z. The (dis-)advantages of using phJTS instead of Jordan pairs with positive hermitian involution are discussed in Remark 2.1. Starting in Section 2.5, we deviate from the classical treatment of phJTS by introducing *pseudo-inverses* and a *generalized Peirce decomposition.* We adopt both concepts from the work of W. Kaup in [**18**]. The systematic application of the concept of pseudo-inverses and generalized Peirce decompositions is new. In particular, we emphasize the non-trivial interaction of the structure group with Peirce decompositions and pseudo-inverse elements.

> *On pseudo-inverses.* The pseudo-inverse $a^\dagger \in Z$ of an element $a \in Z$ is uniquely defined by the relations
>
> $$Q_a a^\dagger = a\ , \quad Q_{a^\dagger} a = a^\dagger\ , \quad Q_a Q_{a^\dagger} = Q_{a^\dagger} Q_a\ ,$$
>
> where Q_x denotes the quadratic operator of the Jordan triple system Z. This generalizes the Moore-Penrose inverse of rectangular

matrices [**37**]. The eigenspaces of the box-operator $a \,\square\, a^\dagger$ define the *generalized Peirce decomposition*

$$Z = Z_1^a \oplus Z_{1/2}^a \oplus Z_0^a \;.$$

For tripotent elements $e \in Z$, this coincides with the usual Peirce decomposition since $e^\dagger = e$. Besides the Peirce decomposition, we systematically generalize further concepts which usually are defined just for tripotents to concepts for arbitrary elements of the triple system, e.g. particular Jordan algebra structures on Peirce 1-spaces (Proposition 2.14), Peirce equivalence (Section 2.6), Frobenius transformations (Lemma 3.14) and partial Cayley mappings (Section 4.3). Moreover, in Lemma 2.26 we obtain for elements $a, z \in Z$ the relation

$$a^{a^\dagger - z} = {z_1}^\dagger = Q_a z_1^{-1} \quad \text{with} \quad z_1 = Q_a Q_{a^\dagger} z \;,$$

which relates certain quasi-inverses, pseudo-inverses and inverses of the unital Jordan algebra Z_1^a. This relation is also included in the following formula, relating a denominator δ of the quasi-inverse on Z with a denominator of the inverse on Z_1^a,

$$\delta(a^\dagger - z, a) = \delta^a(z) \quad \text{for all } z \in Z_1^a.$$

We apply these formulas, e.g. to prove that certain maps involving pseudo-inverses are complex analytic (Theorem 3.18), and to define line bundles on various manifolds (Section 6.3).

The crucial advantage of the use of pseudo-inverses becomes apparent when we study the action of the structure group $\mathrm{Str}(Z)$ on various objects involving these generalized concepts. Since the set of tripotents is not invariant under the action of the structure group, there are no analogues of these group actions in the usual treatment. For example, we show that for $a \in Z$ and $h \in \mathrm{Str}(Z)$, the Peirce decomposition with respect to a and ha satisfies (Lemma 2.32)

$$Z_1^{ha} = h Z_1^a \;, \quad Z_0^{ha} = h^{-*} Z_0^a \;.$$

We note that the relation between structure automorphisms and pseudo-inverses is highly non-trivial: For a counterexample to the formula $(ha)^\dagger = h^{-*} a^\dagger$ given in [**18**], see Section 2.8. Instead one just obtains the formula (Lemma 2.32)

$$a^\dagger = Q_a Q_{a^\dagger} h^* (ha)^\dagger \;.$$

Therefore, the stated results on the action of the structure group on various objects such as sets of Peirce spaces (see above), Peirce varieties (Theorem 6.5) and the compact dual (Section 7.1) involve some non-trivial algebraic calculations.

Besides the algebraic benefit of the use of pseudo-inverses, we also gain analytic advantages: The set of tripotents is a real analytic manifold, whereas the set of rank-j elements is a complex analytic manifold. The generalization of concepts involving tripotents to concepts involving arbitrary elements now implies that certain real analytic maps from the set of tripotents become complex analytic sets from the set of rank-j elements, e.g. the canonical projection map onto Peirce Grassmannians (Theorem 6.1).

In Section 2.6, we use the generalized Peirce decomposition to extend E. Neher's equivalence relation on the set of tripotents[1] to an equivalence relation on all of Z, which we call *Peirce equivalence relation*. Sections 2.7 through 2.10 pick up the threads of the usual presentation. However, we have to rephrase and prove again some of the results to fit them into the concepts of generalized Peirce decompositions and pseudo-inverses. In particular, the identities mentioned above are proved in these sections.

In Chapter 3, we prepare the study of analytic aspects in Jordan theory. Section 3.1 is a brief collection of well-known results on imbedded and immersed submanifolds. In this thesis, the term 'submanifold' without further qualification means an imbedded submanifold. Section 3.2 deals with equivalence relations and their connection to analytic structures. Given a manifold M and an equivalence relation $R \subset M \times M$, we recall a well-known criterion for the quotient space M/R also to be a manifold (Godement's Theorem). The main result of this section is a global description of vector bundles on such *quotient manifolds* based on *cocycles* on the equivalence relation R (Theorem 3.8). We call this the *Godement approach* to analytic structures on the quotient M/R. Since most of the manifolds we describe in this thesis are based on Jordan theoretically defined equivalence relations, the Godement approach is of particular importance to us. Besides the global viewpoint via equivalence relations, we indicate how to obtain local descriptions of quotient manifolds and their vector bundles.

> *On the Godement approach.* In text books such as [**38, 40**], Godement's Theorem is used to prove the classical result that the quotient of a Lie group by a closed subgroup is a manifold (cf. homogeneous spaces). We apply Godement's Theorem in the context of equivalence relations which are *not* induced by group actions. There are basically two fundamental equivalence relations on Jordan triple systems, on which all the manifolds discussed in this thesis are built upon. The first one is the *Peirce equivalence relation*, defined on rank-j elements Z_j by
>
> $$u \approx \tilde{u} \quad \text{if and only if} \quad Z_1^u = Z_1^{\tilde{u}} \ ,$$
>
> and the second one is O. Loos' equivalence relation on $Z \times \overline{Z}$, given by
>
> $$(z,a) \sim (\tilde{z},\tilde{a}) \iff \begin{cases} (z, a-\tilde{a}) \text{ is quasi-invertible} \\ \text{and } \tilde{z} = z^{a-\tilde{a}} \ . \end{cases}$$
>
> Both equivalence relations are regular ones, i.e. the corresponding quotient admits a manifold structure. On the one hand we obtain the Peirce Grassmannian $\mathbb{P}_j = Z_j/\approx$ of type j (cf. Chapter 6) and on the other hand the Grassmannian $\mathbb{G}(Z) = (Z \times \overline{Z})/\sim$ of the phJTS Z (cf. Chapter 4).
>
> Weakening the Peirce equivalence relation to inclusions, we define a partial order on Z by
>
> $$u \subset \tilde{u} \quad \text{if and only if} \quad Z_1^u \subset Z_1^{\tilde{u}} \ .$$
>
> Using this, we generalize the manifold Z_j of rank-j elements to the manifold of *pre-Peirce flags* Z_J defined by
>
> $$Z_J := \{(u_1,\ldots,u_k) \,|\, u_1 \subset \ldots \subset u_k,\ \mathrm{rk}\, u_i = j_i\} \ ,$$

[1] Two tripotents are equivalent if their Peirce decompositions coincide, see [**36**].

where $J = (j_1, \dots, j_k)$ is called the *type* of the pre-Peirce flag manifold. In Section 3.3, we prove that this indeed defines a complex analytic submanifold of Z^k. By extending the Peirce equivalence relation to the tuples in Z_J in an obvious way, we then obtain the *Peirce flag varieties* $\mathbb{P}_J = Z_J/\approx$, to which the Godement approach applies (Theorem 6.15).

Finally, we note that a highly non-trivial interconnection of the Peirce equivalence relation and O. Loos' equivalence relation yields the basis for the definition of Jordan flag varieties, see below. In this case, the Godement approach applies also in full. On each of these manifolds, line bundles are defined globally via cocycles, which are given by concise formulas using a denominator of the quasi-inverse (see Sections 4.1, 6.3 and 8.6).

In Section 3.3 we apply the methods introduced so far (1) to show that the subset of rank-j elements Z_j is a complex analytic submanifold (Theorem 3.15), and (2) to generalize this manifold to the so-called pre-Peirce flag manifolds Z_J (Theorem 3.19), which form the basis of the definition of Peirce flag varieties discussed in Chapter 6. We note that (1) also follows by abstract arguments, since the structure group $\mathrm{Str}(Z)$ is a complex algebraic group and acts transitively on each connected component of Z_j. Instead, our proof is by explicit calculations, which also provide deeper insight into the structure of Z_j, cf. Corollary 3.16. This explicit approach seems to be new. For (2), the abstract argument fails, since the structure group does not act transitively on the components of Z_J. Section 3.4 starts with a review of the functional calculus defined on the phJTS Z, cf. [**28**]. By a modification of a well-known result on real analytic functions around 0 to the corresponding result for real analytic functions on $\mathbb{R}\setminus\{0\}$, see Proposition 3.25, we are able to prove that the pseudo-inverse map $z \mapsto z^\dagger$ is real analytic on each submanifold Z_j of rank-j elements, and to determine its derivative. We note that even in the case $Z = \mathbb{C}^{r\times s}$, where the pseudo-inverse corresponds to the Moore-Penrose inverse, this is a substantial result (Theorem 3.27). In the same way, we show that the projection of Z_j onto the set of rank-j tripotents is real analytic, and we determine its derivative. In the last section of Chapter 3, we briefly recall the connection between phJTS and bounded symmetric domains. This material is standard. Throughout this thesis, we set

$$\mathcal{D} = \{z \in Z \,|\, |z| < 1\}\ , \quad \mathcal{D} = G/K \quad \text{with} \quad G = \mathrm{Aut}(\mathcal{D})^0\ , \quad K = \mathrm{Aut}(Z)^0\ ,$$

where the index 0 labels the identity component of the corresponding group. Furthermore, $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$ denote the complexifications of G and K, respectively, and we use G^c for a compact real form of $G^{\mathbb{C}}$ containing K. This completes the first part.

The aim of Part 2 is the description of the G- and the $K^{\mathbb{C}}$-orbit structure on the compact dual of a bounded symmetric symmetric domain $\mathcal{D} = G/K$. Chapter 4 introduces the Jordan theoretic model of the compact dual, which is due to O. Loos [**28**]. Initiated by the matrix case $Z = \mathbb{C}^{r\times s}$, one defines

$$\mathbb{G}(Z) = (Z \times \overline{Z})/\sim \quad \text{with} \quad (z,a) \sim (\tilde{z},\tilde{a}) \iff \begin{cases} (z, a-\tilde{a}) \text{ is quasi-invertible} \\ \text{and } \tilde{z} = z^{a-\tilde{a}}\,. \end{cases}$$

We call $\mathbb{G}(Z)$ the *Grassmannian variety* of Z, and note that the identification of the Grassmannian with the compact dual X is verified not until the group action of $G^{\mathbb{C}}$ is defined, and the stabilizer of a fixed point of $\mathbb{G}(Z)$ is determined (cf. Theorem 4.7). All this material is standard and worked out in detail in [**28**]. In Section 4.1, we recall Loos' construction and reconsider it from the point of view of the Godement approach, see above. We describe the standard vector and line bundles of the Grassmannian via cocycles on the equivalence relation. Section 4.2 is a review of the standard facts on the automorphism group of the Grassmannian and its identity component, which coincides with the complexification of G, i.e. $\mathrm{Aut}(\mathbb{G}(Z))^0 = G^{\mathbb{C}}$. Here we fix the notation for (quasi-)translations. In Section 4.3, we generalize the notion of partial Cayley mappings and partial inverse mappings to concepts which admit arbitrary elements of Z instead of only tripotents (cf. the account on pseudo-inverses above). For further reading on the importance of partial inverse mappings within this thesis, we refer to the outline on Peirce varieties below. The main result of this chapter is the description of two distinct systems of representatives for the elements of the Grassmannian (Theorem 4.12). In addition, the result is demonstrated in the matrix case $Z = \mathbb{C}^{r\times s}$. In Chapter 7, we show that these systems of representatives are well-suited for the description of the G- and the $K^{\mathbb{C}}$-orbit structure on the Grassmannian.

> *On representatives of elements on the Grassmannian.* Since the Grassmannian $\mathbb{G}(Z)$ is defined by an equivalence relation on $Z\times\overline{Z}$ (see above) elements of $\mathbb{G}(Z)$ refer to equivalence classes, denoted by $[z : a]$. Since this equivalence relation is regular, Godement's Theorem implies that the canonical projection of $Z\times\overline{Z}$ onto $\mathbb{G}(Z)$ is a submersion. We note that for fixed $a\in\overline{Z}$, the restriction of this projection to $Z\times\{a\}$ exactly describes the (Jordan theoretic) charts of the Grassmannian. In this way, the factor $\overline{Z}$ may be considered as some parameter space of the chart maps on $\mathbb{G}(Z)$. Regarded differently, we might say that for fixed $a\in\overline{Z}$, the subset $Z\times\{a\}$ forms a partial system of representatives for the elements of the Grassmannian.
>
> Obviously it is possible to choose quite different systems of representatives, so we are led to the question, whether one might take further advantage of the Godement approach by choosing systems of representatives which are well-suited to solve given problems. Theorem 4.12 affirms this question for the problem of the G- and $K^{\mathbb{C}}$-orbit structure. Together with Remark 4.13, it states that any element $\chi\in\mathbb{G}(Z)$ is representable as[2]
>
> (i) $\chi = [e + d_e : c + d_c]$ with $e, c\in S,\ c\le e,\ d_e\in\mathcal{D}^e_0,\ d_c\in\mathcal{D}^c_1$,
>
> (ii) $\chi = [u + z : u^\dagger]$ with $u, z\in Z,\ u \perp\!\!\!\perp z$.
>
> These representatives are unique up to Peirce equivalence in c and in u. In the outline of Chapter 7 we describe the application of this system of representatives to the orbit structure on the Grassmannian.

Chapters 5 and 6 are intermediate chapters on the way of determining the G- and $K^{\mathbb{C}}$-orbit structures on the Grassmannian $\mathbb{G}(Z)$. In this context, Chapter 5 provides a G-invariant on $\mathbb{G}(Z)$, which classifies the G-orbits (Corollary 5.10), and Chapter 6 gives a description of particularly important closed $K^{\mathbb{C}}$-orbits, which can be identified with Peirce Grassmannians (Theorem 6.5).

The derivation of the G-invariant on the Grassmannian $\mathbb{G}(Z)$ in Chapter 5 happens in an indirect way. We imbed the Grassmannian diagonally as a real submanifold into the product manifold $\mathbb{G}(Z)\times\overline{\mathbb{G}}(Z)$, where $\overline{\mathbb{G}}(Z)$ denotes the conjugate Grassmannian (cf. Section 4.1) and investigate a $G^{\mathbb{C}}$-action on this product which

[2]Here, S is the set of tripotents, $c\le e$ denotes the usual order of tripotents, $\mathcal{D}^e_\nu$ represents the bounded symmetric domain within the phJTS Z^c_ν, i.e. $\mathcal{D}^e_\nu = \mathcal{D}\cap Z^c_\nu$, and $u \perp\!\!\!\perp z$ denotes strong orthogonality, i.e. $u\,\square\,z = 0$.

coincides in the restriction to $G \subset G^{\mathbb{C}}$ on the diagonal with the usual G-action on $\mathbb{G}(Z)$. The advantage of this procedure is that $G^{\mathbb{C}}$-actions are considerably easier to handle (via generators and relations) than G-actions. This idea is adopted from the theory of cycle spaces, where the diagonal imbedding of a bounded symmetric domain $\mathcal{D} = G/K$ into the complex manifold $G^{\mathbb{C}}/K^{\mathbb{C}}$ is studied, cf. [**9**]. We prove that $G^{\mathbb{C}}/K^{\mathbb{C}}$ is an open and dense subset of the product manifold $\mathbb{G}(Z) \times \overline{\mathbb{G}}(Z)$, see Theorem 5.12. In Section 5.1, we motivate the results of this chapter by examining the matrix case using usual geometric arguments. In Section 5.2, we define the $G^{\mathbb{C}}$-action on the product manifold $\mathbb{G}(Z) \times \overline{\mathbb{G}}(Z)$ and describe certain vector bundles on it. The central section of this chapter is the third one. Here, we introduce $G^{\mathbb{C}}$-equivariant sections on $\mathbb{G}(Z) \times \overline{\mathbb{G}}(Z)$ and corresponding invariants (Propositions 5.4 and 5.9). In the case of the restriction to G and the diagonal $\mathbb{G}(Z) \hookrightarrow \mathbb{G}(Z) \times \overline{\mathbb{G}}(Z)$, we prove a further refinement of these invariants (Corollaries 5.6 and 5.10). We emphasize that the use of Jordan theory in this context provides highly explicit formulas for the sections and their invariants. In the final section of this chapter, Section 5.4, we determine the $G^{\mathbb{C}}$-orbit structure of the product manifold $\mathbb{G}(Z) \times \overline{\mathbb{G}}(Z)$ and we prove that the $G^{\mathbb{C}}$-invariants defined in the last section indeed characterize the $G^{\mathbb{C}}$-orbits (Theorem 5.12).

Chapter 6 is devoted to the investigation of manifolds which are defined on the basis of a fundamental equivalence relation on phJTS, namely the Peirce equivalence relation. Moreover, it turns out that these manifolds are in fact smooth algebraic varieties in the sense of D. Mumford [**35**]. The most basic Peirce variety is the *Peirce Grassmannian* defined as the quotient of Z by the Peirce equivalence relation,

$$\mathbb{P} = Z/\approx \quad \text{with} \quad u \approx \tilde{u} \quad \text{if and only if} \quad Z_1^u = Z_1^{\tilde{u}} .$$

In Section 6.1, we apply Godement's theorem and show that this equivalence relation is regular, and hence induces a manifold structure on $\mathbb{P}$. Moreover, an appropriate $K^{\mathbb{C}}$-action on the Peirce Grassmannian turns $\mathbb{P}$ into a hermitian symmetric space of compact type (Theorem 6.1). This result is well-known (cf. [**28**, §5.6b]), however our proof is consistently complex analytic, as it uses the extension of the Peirce equivalence relation to all elements of Z (instead of the set of tripotents), and we thus obtain a complex analytic fibration of the set of rank-j tripotents over the corresponding connected component $\mathbb{P}_j$ of the Peirce Grassmannian. Furthermore, our approach provides new explicit descriptions of chart maps on $\mathbb{P}_j$ and their transition functions (Proposition 6.3). In Section 6.2, we show that different realizations of the Peirce equivalence relation [**28, 18, 16, 2**] yield to isomorphic manifold structures on various objects (Theorem 6.5). We note that the realization of the Peirce Grassmannian $\mathbb{P}$ as a $K^{\mathbb{C}}$-invariant submanifold of the Grassmannian $\mathbb{G}(Z)$, namely

$$\mathbb{P} \hookrightarrow \mathbb{G}(Z),\ [u] \mapsto \left[u : u^{\dagger}\right] ,$$

motivated the placement of this chapter into the context of orbit structures on the Grassmannian. Furthermore, W. Kaup proves in [**18**] the existence of an abstract isomorphism between the connected component of $\mathbb{P}$ containing some element $[u]$ and the Grassmannian corresponding to the phJTS $Z^u_{1/2}$. In connection with the explicit description of charts on the Peirce Grassmannian $\mathbb{P}$, we make this isomorphism explicit by showing that it is given by the restriction of the partial inverse map $j_{u^\dagger}$ to the closure of the Peirce 1/2-space $Z^u_{1/2}$ in $\mathbb{G}(Z)$, cf. Theorem 6.10. In Section 6.3, we use the Godement approach to define line bundles on the Peirce Grassmannian. The corresponding cocycles are given on the basis of a denominator of the quasi-inverse on Z. It is proved explicitly that these line bundles are very ample, and hence $\mathbb{P}$ is a projective variety (Theorem 6.14). This proof is a variation of the corresponding proof for ample line bundles on the Grassmannian $\mathbb{G}(Z)$ given by O. Loos [**28**, §7.10]. The last section of this chapter provides a discussion of the obvious generalization of the Peirce Grassmannian to *Peirce flag varieties*. A brief account on this using the usual Peirce decomposition by tripotents can be found in [**2**]. We base the definition of the Peirce flag variety $\mathbb{P}_J$ on the pre-Peirce flag Z_J discussed in Section 3.3 and the extended Peirce equivalence relation,

$$\mathbb{P}_J = Z_J/\approx \quad \text{with} \quad (u_1,\dots,u_k)\approx(\tilde{u}_1,\dots,\tilde{u}_k) \iff Z_1^{u_i} = Z_1^{\tilde{u}_i} \text{ for all } i.$$

In Section 6.4, we determine in detail the analytic structure of the Peirce flag variety using Godement's Theorem, and describe an atlas on $\mathbb{P}_J$, cf. Theorem 6.15 and Proposition 6.17. By the use of the generalized Peirce decomposition, we also obtain a natural action of the structure group on $\mathbb{P}_J$, which turns the canonical projection of Z_J onto $\mathbb{P}_J$ into a $\mathrm{Str}(Z)$-equivariant map. Using pullbacks, we transfer the $K^{\mathbb{C}}$-equivariant line bundles on the Peirce Grassmannian to the Peirce flag variety. We finally show that an appropriate product of these line bundles is very ample, and hence $\mathbb{P}_J$ is indeed a projective variety (Theorem 6.20). We emphasize that the advantage of this Jordan theoretic discussion in contrast to an abstract Lie theoretic investigation is the gain of explicit formulas e.g. for the chart maps and line bundles.

In Chapter 7, we return to the study of the G- and $K^{\mathbb{C}}$-orbit structures on the Grassmannian $\mathbb{G}(Z)$. Here we assume Z to be simple. The results of the last chapters are gathered to prove the main result of this chapter, which describes the G- and $K^{\mathbb{C}}$-orbits on $\mathbb{G}(Z)$ explicitly in Jordan theoretic terms (Theorem 7.2). As noted above, the G-orbit structure has already been investigated by J. Wolf [**44**], and T. Matsuki proved a one-to-one correspondence between the G- and the $K^{\mathbb{C}}$-orbits [**34**]. It turns out that the number of orbits is $\binom{r+2}{2}$, where r denotes the rank of the triple system, which equals the rank of the real semisimple Lie group G. Furthermore, the G- and the $K^{\mathbb{C}}$-orbits are fibrations over some special K-orbits. J. Wolf shows that the fiber of the G-orbits is the product of two hermitian symmetric spaces of non-compact type [**44**, §9]. In this respect, the results of Section 7.1 are well-known, but we note that our proof is independent of the Lie theoretic considerations and that the strength of the Jordan theoretic approach is that we get explicit formulas.
Our description of the G- and the $K^{\mathbb{C}}$-orbits is based on the two systems of representatives for the elements of $\mathbb{G}(Z)$ as described above. Therefore, we are able to identify[3] the orbits of the Grassmannian with certain subsets of $Z\times\overline{Z}$. We obtain that the related G-, $K^{\mathbb{C}}$-, and K-orbits are given by

$$G_b^a = \{[e+d_e : c+d_c] \,|\, e\in S_{a+b},\, c\in S_a,\, e\geq c,\, d_e\in\mathcal{D}_0^e,\, d_c\in\mathcal{D}_1^c\}\ ,$$

$$\mathbb{K}_b^a = \left\{[u+z : u^\dagger] \,\middle|\, u\in Z_a,\, z\in Z_b,\, u \perp\!\!\!\perp z\right\}\ ,$$

$$K_b^a = \{[e:c] \,|\, e\in S_{a+b},\, c\in S_a,\, e\geq c\} = \{[c+\tilde{c}:c] \,|\, c\in S_a,\, \tilde{c}\in S_b,\, c\perp\!\!\!\perp\tilde{c}\}\ ,$$

with $0\leq a\leq a+b\leq r$, where S_j and Z_j denote the set of rank-j tripotents and rank-j elements. The corresponding fibrations over the K-orbit are

$$\mathcal{D}_0^e\times\mathcal{D}_1^c\to G_b^a\to K_b^a\ ,\quad \varOmega(Z_+^u)\to\mathbb{K}_b^a\to K_b^a\ ,$$

where $\varOmega(Z_+^u)$ denotes the symmetric cone of the euclidean Jordan algebra Z_+^u, cf. Theorem 3.27. The result on the fibration of the $K^{\mathbb{C}}$-orbits is well-known for 'finite' $K^{\mathbb{C}}$-orbits, i.e. orbits contained in $Z\hookrightarrow\mathbb{G}(Z)$, but the extension of this result to orbits at infinity seems to be new. We also note that there is a different (substantially more abstract) Jordan theoretic account on these orbit structures introduced by W. Kaup using a generalized functional calculus [**19**], which is closely related

[3] This identification holds up to a simple equivalence relation on the considered subsets of $Z\times\overline{Z}$, since the involved systems of representatives are not completely unique, cf. Theorem 4.12.

to the Lie theoretic investigation via momentum maps due to R. Bremingan and J. Lorch in [**7**]. Section 7.1 closes with a description of the tangent structures of the various orbits and provides explicit formulas of the G-invariant (pseudo-)hermitian metrics on the open G-orbits. In Section 7.2, we prove standard topological properties of the orbits and determine the topological closures of the orbits (Theorem 7.3). Besides the 'global' description of the orbits, Section 7.3 provides explicit simple formulas for the $K^{\mathbb{C}}$-orbits regarded as subvarieties of some chart domain on the Grassmannian $\mathbb{G}(Z)$. Finally, in Section 7.4, we prove the Matsuki duality between the G- and the $K^{\mathbb{C}}$-orbits by purely Jordan theoretic arguments (Theorem 7.6).

Part 3 is devoted to the discussion of a Jordan theoretic description of generalized flag varieties. We approach the central question of this part by considering the matrix case $Z = \mathbb{C}^{r\times s}$. Here, the Grassmannian $\mathbb{G}(\mathbb{C}^{r\times s})$ just equals the (ordinary) Grassmannian variety $\mathrm{Gr}_s(\mathbb{C}^{r+s})$, which initiated our terminology. Lie theoretically, the Grassmannian is represented by the quotient $\mathrm{Gr}_s(\mathbb{C}^{r+s}) = G^{\mathbb{C}}/P$, where $G^{\mathbb{C}} = \mathrm{SL}(r+s)$, and P is the parabolic subgroup of upper triangular block matrices of type (s) and determinant 1. This Grassmannian variety admits the extensive generalization to *flag varieties*: For any strictly increasing sequence of integers $0 \le i_1 < \ldots < i_m \le r+s$,

$$\mathrm{Gr}_{(i_1,\ldots,i_m)}(\mathbb{C}^{r+s}) = \{0 \subset V_1 \subset \ldots \subset V_m \subset \mathbb{C}^{r+s} \mid \dim V_\ell = i_\ell\}$$

is called the flag variety of type $(i_1,\ldots,i_m)$. It turns out that this is a projective variety [**15**], and the Lie theoretic description is given by

$$\mathrm{Gr}_{(i_1,\ldots,i_m)}(\mathbb{C}^{r+s}) \cong G^{\mathbb{C}}/P' ,$$

where $G^{\mathbb{C}} = \mathrm{SL}(r+s)$, and P' is the parabolic subgroup of all invertible upper triangular block matrices of type $(i_1,\ldots,i_m)$, cf. [**13, 10**]. The question arises whether these flag varieties also admit a Jordan theoretic description by the triple system $Z = \mathbb{C}^{r\times s}$. Since the flag variety $\mathrm{Gr}_{(i_1,\ldots,i_m)}(\mathbb{C}^{r+s})$ does not distinguish between the characteristic numbers of the tripe system, namely r and s, we expect that not all such flag varieties admit a Jordan theoretic realization. However, taking into account that the real form $G = \mathrm{SU}(r,s)$ of $G^{\mathbb{C}}$ preserves the characteristics of Z, it is plausible to expect a Jordan theoretic description of those flag varieties which are represented by quotients $G^{\mathbb{C}}/P$, where $P = Q^{\mathbb{C}}$ is the complexification of some (real) parabolic subgroup Q of G. This formulation immediately transfers to the general case:

Question: *Given a phJTS Z with unit ball $\mathcal{D}$ and a parabolic subgroup Q of the identity component G of the automorphism group* $\mathrm{Aut}(\mathcal{D})$*, is there a Jordan-theoretic realization of the generalized flag variety $G^{\mathbb{C}}/Q^{\mathbb{C}}$?*

Chapter 8 is devoted to the affirmation of this question. In Section 8.1, we briefly recall the Jordan theoretic description of the real parabolic subgroups of G, which is due to O. Loos [**28**, §9], and determine their complexifications (Theorem 8.4). In particular, we show for the matrix case $Z = \mathbb{C}^{r\times s}$ and $G = \mathrm{SU}(r,s)$ that the complexification of a parabolic subgroup $Q \subset G$ is conjugate to a standard parabolic subgroup of $G^{\mathbb{C}} = \mathrm{SL}(r+s)$ of type $\mathcal{I} = (j_1,\ldots,j_k, n-j_k,\ldots,n-j_1)$ with $n = r+s$ and $j_k \le r$, i.e. the corresponding flag variety $G^{\mathbb{C}}/Q^{\mathbb{C}}$ is given by

$$\mathrm{Gr}_{\mathcal{I}}(\mathbb{C}^n) = \{0 \subset E_1 \subset \ldots \subset E_k \subset F_k \subset \ldots \subset F_1 \subset \mathbb{C}^n \mid \dim E_\ell = j_\ell,\ \dim F_\ell = n - j_\ell\} .$$

In Section 8.2, we use this matrix case as a toy model for the construction of the Jordan theoretic description of flag manifolds. The question is (1) how to represent a pair of subspaces $E \subset F \subset \mathbb{C}^n$ with $\dim E = j$ and $\dim F = n - j$ by elements

of $Z = \mathbb{C}^{r\times s}$, such that (2) realizations of the same pair by different elements lead to an equivalence relation on these elements that is compatible with the Jordan triple structure on Z. This is a substantial extension of questions that led to the Jordan theoretic model of the Grassmannian due to O. Loos. The key result in this section is Lemma 8.7, which solves[4] the problem given by (1) and (2). It turns out that the Jordan theoretic model for $\mathrm{Gr}_{(j,n-j)}(\mathbb{C}^n)$ is built on triples (u,z,a) of elements in $\mathbb{C}^{r\times s}$ with $\mathrm{rk}(u)=j$. On the basis of this lemma, it is straightforward to give a generalization to all the flag manifolds mentioned above by using tuples $(u_1,\dots,u_k,z,a)$ with $\mathrm{rk}(u_\ell)=j_\ell$, cf. Lemma 8.10.
In Section 8.3, we turn to the general case and define *Jordan flag varieties* on arbitrary phJTS via an equivalence relation on $\overline{Z}_J\times Z\times\overline{Z}$, namely

$$\mathbb{F}_J := (\overline{Z}_J\times Z\times\overline{Z})/\sim$$

with

$$\big((u_i),z,a\big)\sim\big((\tilde{u}_i),\tilde{z},\tilde{a}\big)\iff\begin{cases}\tilde{u}_i\approx B_{a-\tilde{a},\,z}u_i \text{ for } i=1,\dots,k;\text{ and}\\ \text{there exist } u^{\perp\!\!\perp}\in Z_0^{u_k}\text{ and }\tilde{u}^{\perp\!\!\perp}\in Z_0^{\tilde{u}_k},\\ \text{such that } B_{a-\tilde{a},\,z+u^{\perp\!\!\perp}}\text{ is invertible}\\ \text{and } \tilde{z}+\tilde{u}^{\perp\!\!\perp}=(z+u^{\perp\!\!\perp})^{a-\tilde{a}}.\end{cases}$$

This definition connects the two fundamental equivalence relations, i.e. the Peirce equivalence relation and O. Loos' equivalence relation for the definition of the Grassmannian, in a highly non-trivial way. We note that it is far from obvious that this indeed defines an equivalence relation. The main theorem verifies this and shows in addition using Godement's Theorem that the Jordan flag variety $\mathbb{F}_J$ is a compact complex manifold (Theorem 8.11). In Section 8.4, we investigate the analytic and algebraic structures of the Jordan flag variety, and show that $\mathbb{F}_J$ is indeed a smooth algebraic variety (Proposition 8.14). Furthermore, we define a transitive $G^{\mathbb{C}}$-action on the Jordan flag variety and prove that some stabilizer coincides with the given parabolic subgroup $Q_J^{\mathbb{C}}$, so $\mathbb{F}_J\cong G^{\mathbb{C}}/Q_J^{\mathbb{C}}$, which finishes our project (Theorem 8.20). Finally, in the last section we use the Godement approach to define line bundles on the Jordan flag varieties. In addition we show that these bundles are $G^{\mathbb{C}}$-homogeneous, cf. Proposition 8.23.

In the last chapter of this thesis we give a first application of the Jordan theoretic description of generalized flag varieties. The purpose of this chapter is to generalize the determinant functions introduced by L. Barchini, S.G. Gindikin and H.W. Wong from the matrix case involving ordinary Grassmannian flag varieties to generalized flag varieties [**3, 4**]. In this way, we also demonstrate the ability of Jordan theory to provide simple and explicit formulas. In Section 9.1, we give a brief review on the definition of the Barchini-Gindikin-Wong determinant functions and recall a first application in geometry. Section 9.2 provides a Jordan theoretic account on this topic by using the Godement approach on line bundles to define $G^{\mathbb{C}}$-invariant sections on the product manifold $\mathbb{G}\times\overline{\mathbb{G}}\times\mathbb{F}_J$, the *Jordan determinant functions*. Finally, in Section 9.3, we identify the manifold involved in the definition of the Barchini-Gindikin-Wong determinant functions with an open and dense submanifold of $\mathbb{G}\times\overline{\mathbb{G}}\times\mathbb{F}_J$, and we prove that the vanishing sets of the restricted versions of these determinant functions coincide.

Prospective work. We recall the aims of the program *"Jordan theory and geometric realizations"* introduced above, namely (i) to give a Jordan theoretic

[4]We admit that there is still an open question concerning a technical lemma, which we have not proved so far, see Lemma 8.8. However, we note that this lemma is *not* involved in the general construction of Jordan flag varieties discussed in Section 8.3.

description of generalized flag varieties, (ii) to determine explicitly the G-orbit structure, and (iii) to describe the corresponding representation theory. In this thesis, we completely solved problem (i) for a generalized flag variety $G^{\mathbb{C}}/Q^{\mathbb{C}}$ with real parabolic subgroup $Q \subset G$, and we argued that this is the most general form of a generalized flag variety we could expect to describe in Jordan theoretic terms. In addition, we solved problem (ii) in the hermitian symmetric case $\mathbb{G}(Z)$. The first step towards (ii) and (iii) in the general case is made in the last chapter by the discussion of determinant functions. We briefly outline some of the prospective work:

Orbit structure on Jordan flag varieties. We expect to find a description of the G- and $K^{\mathbb{C}}$-orbit structures on a Jordan flag variety $\mathbb{F}_J$ similar to the description of the orbit structures on the Grassmannian $\mathbb{G}(Z)$ discussed in Chapter 7, i.e. the defining equivalence relation on $\overline{Z}_J \times Z \times \overline{Z}$ might admit two systems of representatives which correspond to the G- and $K^{\mathbb{C}}$-orbit structures.

Conical and spherical functions. The works of H. Upmeier [**41**] and of J. Faraut and A. Korányi [**8**] show that Jordan theory is particularly useful for describing conical and spherical functions on symmetric cones and bounded symmetric domains. Having derived a Jordan theoretic description of more general G- and $K^{\mathbb{C}}$-orbits (Chapter 7), we hope to find similar results for the harmonic analysis on these orbits.

Determinant functions. In Chapter 9, we define Jordan determinant functions which are closely related to the determinant functions on ordinary flag varieties introduced by Barchini-Gindikin-Wong in [**3, 4**]. So far we have investigated some of their connections (Section 9.3) and besides the extension of this investigation, it remains to study their applications in both geometry and representation theory. In particular, Barchini-Gindikin-Wong show how to use determinant functions in the context of Szegő mappings, which are intertwining operators from principal series representations to discrete series representations. We expect that Jordan determinant functions can be used to generalize these results to all simple Lie groups of hermitian type.

Cohomology of the Grassmannian $\mathbb{G}(Z)$. We suppose that the decomposition of the Grassmannian variety $\mathbb{G}(Z)$ into $K^{\mathbb{C}}$-orbits induces a CW-decomposition, which might yield a new approach to the cohomology of the Grassmannian $\mathbb{G}(Z)$. In the following, we summarize the basic ideas behind this statement, the details still need to be worked out. Let Z be a simple phJTS of rank r, and let $\mathbb{K}_b^a$ denote the $K^{\mathbb{C}}$-orbits of $\mathbb{G}(Z)$ for $0 \leq a \leq a+b \leq r$ (cf. Chapter 7). Then,

$$\mathbb{G}(Z) = \dot{\bigcup_{0\leq a\leq a+b\leq r}} \mathbb{K}_b^a = \dot{\bigcup_{0\leq a\leq r}} \mathcal{K}^a \quad \text{with} \quad \mathcal{K}^a = \dot{\bigcup_{0\leq b\leq r-a}} \mathbb{K}_b^a \,.$$

For $a = 0$, we obtain the open and dense subset $\mathcal{K}^0 = Z \subset \mathbb{G}(Z)$, the main 'cell' of this decomposition. For $a > 0$, $\mathcal{K}^a$ is given by

$$\mathcal{K}^a = \left\{ \left[u + z : u^\dagger\right] \middle| \, u \in Z_a,\ z \in Z_0^u \right\}$$

and can be regarded as a vector bundle on the Peirce Grassmannian $\mathbb{P}_a$ via the projection map

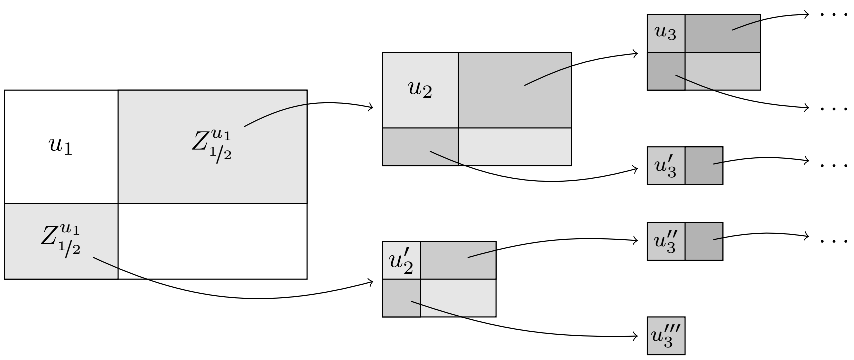
$$\mathcal{K}^a \to \mathbb{P}_a,\ \left[u + z : u^\dagger\right] \mapsto \left[u : u^\dagger\right] \,.$$

Since the Peirce Grassmannian $\mathbb{P}_a$ can be identified (explicitly) with the Grassmannian $\mathbb{G}(Z^u_{1/2})$ of the Peirce $1/2$-space of some $u \in Z_a$, each $\mathcal{K}^a$

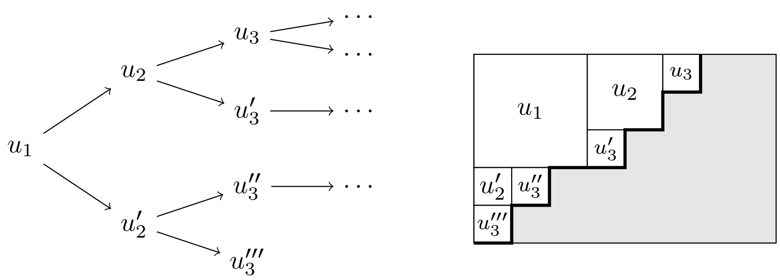
decomposes according to the orbit decomposition of $\mathbb{G}(Z^u_{1/2})$. Again we obtain $Z^u_{1/2}$ as the main 'cell' of $\mathbb{G}(Z^u_{1/2})$ and we obtain additional subsets which can be identified with vector bundles on certain Peirce Grassmannians, which again are identified with Grassmannians of appropriate Peirce subspaces, and so forth, until we arrive at a space consisting of a single point. In the special case $Z = \mathbb{C}^{1\times n}$ this yields the well-known CW-decomposition of the complex projective space,

$$\mathbb{CP}^n = \mathbb{C}^n \uplus \mathbb{CP}^{n-1} = \mathbb{C}^n \uplus \mathbb{C}^{n-1} \uplus \mathbb{CP}^{n-2} = \ldots = \mathbb{C}^n \uplus \mathbb{C}^{n-1} \uplus \ldots \uplus \mathbb{C} \uplus \{\mathrm{pt}\}\,.$$

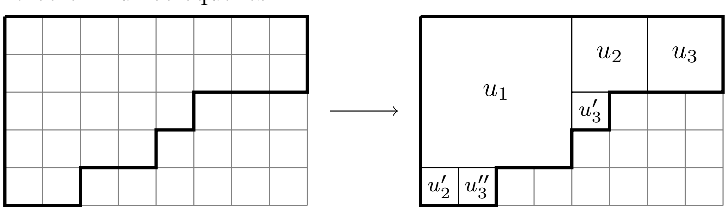
We note that in general, the Peirce space $Z^u_{1/2}$ and hence the Grassmannian $\mathbb{G}(Z^u_{1/2})$ need not be simple. In this case, the subsequent decompositions split into several parts according to the decomposition of $\mathbb{G}(Z^u_{1/2})$ into simple Grassmannians. For the matrix case $Z = \mathbb{C}^{r\times s}$, this procedure is illustrated in the following diagram:



By construction, the allocation of a cell is given by a tree of elements, where one elements sits in the Peirce 1/2-space of all its predecessors, i.e.



This provides a one-to-one correspondence beween the cells and Young diagrams sitting inside an $(r\times s)$-grid, which are also used in the classical treatment of this subject to identify cells in the CW-decomposition of the Grassmannian variety $\mathrm{Gr}_s(\mathbb{C}^{r+s}) = \mathbb{G}(\mathbb{C}^{r\times s})$ [**13, 10**]. We note that the converse direction of this correspondence decomposes a Young diagram into a tree of Durfee squares:



In the classical treatment, the multiplicative structure of the cohomology of the Grassmannian $\mathrm{Gr}_r(\mathbb{C}^{r+s})$ is determined by Pieri's formula and the

Littlewood-Richardson rule [**13**]. Using the Jordan triple structure on Z, the CW-decomposition described above and the correspondence between its cells and 'trees of elements' might yield a new approach to these issues, which might also yield new results on the cohomology of the Grassmannians $\mathbb{G}(Z)$ besides the matrix case. Here, also E. Nehers work on the grid approach to Jordan triple systems seems to be relevant [**36**].

Basic notions and notations. Throughout this thesis, Jordan triple systems are denoted by the letter Z, if necessary with additional labels. We denote the Jordan triple product by $\{x,\, y,\, z\}$ for $x, y, z \in Z$. To avoid cumbersome terms in lengthy calculations, we omit to indicate the antilinearity of the second entry with an additional label. Concerning the diverse operators defined on the basis of the triple product, we follow the convention also used by W. Kaup, H. Upmeier et al. by setting

$$\{x,\, y,\, z\} = (x \,\square\, y)z = Q_{x,z}y = \tfrac{1}{2}\big(Q_{x+z}y - Q_x y - Q_z y\big)\ ,\ \ B_{x,\,y} = \mathrm{Id} - 2\,x \,\square\, y - Q_x Q_y\ .$$

With these definitions, the admissible eigenvalues of the box operator $e \,\square\, e$ with respect to a tripotent $e \in Z$, are given by 1, $1/2$, 0, and the corresponding Peirce decomposition is denoted by $Z = Z_1^e \oplus Z_{1/2}^e \oplus Z_0^e$.

We primarily discuss real or complex analytic manifolds in this thesis. Therefore, the term 'manifold' without further qualification refers to an analytic manifold, the context determines field over which the manifold is defined. For a complex manifold M, we denote by $\overline{M}$ the same manifold endowed with the conjugate complex structure, and we refer to $\overline{M}$ as the (*complex*) *conjugate* of M. In the special case of a complex vector space Z (e.g. a Jordan triple system), the identity map induces an antilinear map from Z to $\overline{Z}$. The restriction of this map to open subsets provides the method for constructing chart maps on $\overline{M}$ from given chart maps on M.

Notions of algebraic geometry are used in the sense of D. Mumford's 'Red book of Varieties and Schemes' [**35**]. In particular, the term 'algebraic variety' (or just 'variety') refers to the definitions of the first chapter in [**35**]. To show that a given manifold is a smooth algebraic variety, we note the following sufficient condition: If M is covered by finitely many smooth maps $\varphi_i : \mathbb{C}^n \to M$ such that (1) all transition maps $\varphi_j^i = \varphi_j \circ \varphi_i^{-1}$ are birational maps on $\mathbb{C}^n$ and (2) for any two points x, y of M there exist an i, such that x and y are contained in the image of φ_i, then M is a smooth algebraic variety.

Part 1

Methods for Jordan theoretic varieties

CHAPTER 1

Jordan algebras

This chapter is a review of the basic notions of Jordan algebras. Primarily we follow the exposition of [**8**]. For a detailed account on Jordan algebras and even more general structures associated with them we refer to [**6**]. In this thesis, Jordan algebras appear as substructures of Jordan triple systems, which we describe in the next chapter. In particular, unital Jordan algebras occur in a natural way as Peirce 1-spaces of Jordan triple systems, which gives us the additional structure of invertible elements. Furthermore, the symmetric cone of a euclidean Jordan algebra will reappear in later chapters as the fiber of some real analytic fiber bundle, cf. Theorem 3.27.

1.1. Basic structure

We consider real or complex Jordan algebras of finite dimension. Let $\mathbb{k}$ be the field of real or complex numbers. Then a Jordan $\mathbb{k}$-algebra is a finite dimensional $\mathbb{k}$-vector space A together with a commutative multiplication $(x,y) \mapsto xy$ such that

$$x^2(xy) = x(x^2y)$$

holds for all $x, y \in A$, where we used $x^2 = xx$ for abbreviation. This is called the *Jordan identity* for Jordan algebras, it weakens the associativity usually considered in algebras.

The Jordan identity implies a series of relations which can be stated most efficiently by using the following operators. For $x \in A$ let

$$L_x : A \to A,\ y \mapsto xy \quad \text{and} \quad P_x : A \mapsto A,\ y \mapsto (2L_x^2 - L_{x^2})(y) = 2x(xy) - x^2y$$

be *left multiplication* and the *quadratic map* with respect to x. Then the Jordan identity is equivalent to $[L_x, L_{x^2}] = 0$, where $[f,g] = f \circ g - g \circ f$ is the commutator of endomorphisms on A. We define inductively $x^n = xx^{n-1}$ for $n \in \mathbb{N}$. One can show [**8**, II.1.2] that the Jordan identity implies

$$[L_{x^p}, L_{x^q}] = 0 \quad \text{and} \quad x^p x^q = x^{p+q} \tag{1.1}$$

for all $p, q \in \mathbb{N}$, $x \in A$. The second property turns A into a *power-associative* algebra, i.e. each subalgebra generated by an element $x \in A$ is associative. By the method of polarization [**8**, II.1] one proves the relation

$$2\,[L_u, L_{uv}] = [L_{u^2}, L_v] \tag{1.2}$$

for all $u, v \in A$.

Example 1.1. The standard example of a Jordan algebra is the space of $(n \times n)$-matrices $A = \mathbb{k}^{n\times n}$ with Jordan product

$$x \circ y = \tfrac{1}{2}(xy + yx)\ . \tag{1.3}$$

Here, we have to distinguish between the associative product of matrices (as it is used on the right hand side) and the Jordan product. More generally, on any associative $\mathbb{k}$-algebra A, the product (1.3) defines a Jordan algebra structure on A.

1.2. Idempotents and Peirce decomposition

An *idempotent* is an element $c \in A$ satisfying $c^2 = c$. In this case, the Jordan identity implies

$$2{L_c}^3 - 3{L_c}^2 + L_c = 0 \tag{1.4}$$

for left multiplication by c. Therefore, the minimal polynomial of L_c divides $2X^2 - 3X^2 + X$, so the solely feasible eigenvalues of L_c are 1, $1/2$ and 0. The linear operators

$$\pi_1 = 2{L_c}^2 - L_c = P_c \quad , \quad \pi_{1/2} = -4{L_c}^2 + 4L_c \quad , \quad \pi_0 = 2{L_c}^2 - 3L_c + \mathrm{Id} \ . \tag{1.5}$$

map A onto the respective eigenspace of L_c, and satisfy the relations $\pi_\mu \pi_\nu = \delta_{\mu\nu}\pi_\mu$ and $\pi_1 + \pi_{1/2} + \pi_0 = \mathrm{Id}$. Therefore,

$$A = A_1(c) \oplus A_{1/2}(c) \oplus A_0(c) \quad \text{with} \quad A_\nu(c) = \pi_\nu(A) \ . \tag{1.6}$$

This is called the *Peirce decomposition* of A with respect to the idempotent $c \in A$. The *Peirce spaces* $A_\nu = A_\nu(c)$ satisfy the following *Peirce rules*:

$$A_{1/2}A_{1/2} \subset A_1 + A_0 \quad , \quad A_\nu A_\nu \subset A_\nu \quad , \quad A_\nu A_{1-\nu} = 0 \quad , \quad A_\nu A_{1/2} \subset A_{1/2} \tag{1.7}$$

for $\nu = 0, 1$. In particular, A_1 and A_0 form subalgebras of A, which annihilate each other.

Two idempotents $e, c \in A$ are said to be (*strongly*) *orthogonal*, if $ec = 0$. In this case, we denote $e \perp\!\!\!\perp c$. Due to (1.2), the multiplication operators L_e and L_c of orthogonal idempotents commute, and therefore, the corresponding Peirce decompositions are compatible: A decomposes into the joint eigenspaces of L_e and L_c. More generally, for a system of pairwise orthogonal idempotents $e_1, \ldots, e_n \in A$, we obtain the *joint Peirce decomposition* of A, given by

$$A = \bigoplus_{0 \le i \le j \le n} A_{ij} \quad \text{with} \quad A_{ij} = \left\{ x \in A \,\middle|\, e_k x = \tfrac{1}{2}(\delta_{ki} + \delta_{kj})x \text{ for } k = 1, \ldots, n \right\} \ .$$

The Peirce rules imply the following relations: For $i, j \in \{1, \ldots, n\}$,

$$\begin{aligned} A_{ii} &= A_1(e_i) \ , & A_{ij} &= A_{1/2}(e_i) \cap A_{1/2}(e_j) \ , \\ A_{0i} &= A_{1/2}(e_i) \cap \bigcap_{j \ne i} A_0(e_j) \ , & A_{00} &= A_0(e_1) \cap \ldots \cap A_0(e_n) \ . \end{aligned}$$

The set of idempotent elements is partially ordered by

$$e < c \iff c = e + e' \text{ for some idempotent } e' \ne 0 \text{ with } e' \perp\!\!\!\perp e.$$

The non-trivial minimal elements with respect to this partial order are called *primitive* idempotents. Since a system of orthogonal idempotents in A is linear independent, and since A is assumed to be finite dimensional, each idempotent admits a decomposition $c = c_1 + \ldots + c_r$ into a sum of pairwise orthogonal primitive idempotents. Each such decomposition has the same number of summands, which defines the *rank* of the idempotent c. The rank of the Jordan algebra A is defined by

$$\mathrm{rk}(A) \coloneqq \max\{\mathrm{rk}(c) \,|\, c \text{ idempotent}\} \le \dim A \ . \tag{1.8}$$

A *frame* is a maximal system of pairwise orthogonal primitive idempotents.

1.3. Jordan algebras with unit element

Let A be a Jordan algebra with *unit element* $e \in A$, i.e. $ex = x$ for all $x \in A$. Then, for fixed $x \in A$, the subalgebra of A generated by x and e is given by

$$\mathbb{k}[x] \coloneqq \{p(x) \,|\, p \in \mathbb{k}[X]\} \ . \tag{1.9}$$

It is a commutative and associative subalgebra, and we have

$$\mathbb{k}[x] \cong \mathbb{k}[X]/I_x \ ,$$

where I_x is a principle ideal in $\Bbbk[X]$ generated by some non-vanishing uniquely determined monic polynomial p_x, called the *minimal polynomial* of x. If $m(x)$ denotes the rank of the minimal polynomial, then the rank of the Jordan algebra satisfies

$$\mathrm{rk}(A) = \max\{m(x) \,|\, x \in A\} \ .$$

An element $x \in A$ is said to be *regular* if $m(x) = \mathrm{rk}(A)$.

Proposition 1.2. [**8**, II.2.1] *Let A be a unital Jordan algebra of rank r. The set of regular elements is open and dense in A. There exist polynomials $a_1, \ldots, a_r$ on A such that the minimal polynomial of every regular element x is given by*

$$f(\lambda; x) = \lambda^r - a_1(x)\lambda^{r-1} + a_2(x)\lambda^{r-2} + \ldots + (-1)^r a_r(x) \ .$$

The polynomials $a_1, \ldots, a_r$ are unique and a_j is homogeneous of degree j.

The coefficient $a_r(x)$ is called the *(Jordan algebra) determinant*, we denote

$$\Delta(x) = a_r(x) \ . \tag{1.10}$$

This notation fits into the notation used for Jordan triple systems and their determinant functions, cf. Section 2.9.

An element x is said to be *invertible* if there exists an element $y \in \Bbbk[x]$ such that $xy = e$. Since $\Bbbk[x]$ is associative, y is unique. It is called the *inverse* of x and is denoted by $y = x^{-1}$. We have the following characterization of invertible elements. Recall from Section 1.1 that $P_x \in \mathrm{End}(A)$ denotes the quadratic map corresponding to x.

Proposition 1.3. *Let x be an element of a unital Jordan algebra A. Then the following are equivalent:*

(i) *x is invertible,* (ii) $\Delta(x) \neq 0$, (iii) *P_x is invertible.*

If one of these holds, then

$$P_x x^{-1} = x \ , \quad (P_x)^{-1} = P_{x^{-1}} \ , \quad x^{-1} = \frac{\nu(x)}{\Delta(x)} \ , \tag{1.11}$$

where ν is an A-valued polynomial on A of degree $r-1$.

More generally, a polynomial function $\delta : A \to \mathbb{C}$, normalized such that $\delta(e) = 1$, is called a *denominator* of the inverse, if (i) $\delta(x) \neq 0$ if and only if x is invertible, and (ii) $\nu(x) := \delta(x)\cdot x^{-1}$ is a A-valued polynomial map, called the *numerator* of x^{-1} (with respect to δ). By Proposition 1.3, the map $\delta(x) = \mathrm{Det}\, P_x$ is a denominator of the inverse, and $\nu(x) = P_x^{\#} x$, where $(\ )^{\#}$ denotes the adjoint matrix, is the respective numerator. The Jordan algebra determinant is the unique minimal denominator obtained by canceling all common factors of δ and ν.

The deviation for an element $x \in A$ of being invertible is measures by the *rank* defined by

$$\mathrm{rk}(x) := \max\{j \,|\, a_j(x) \neq 0\} \ .$$

We immediately obtain the following characterization of invertibility:

Corollary 1.4. *Let A be a unital Jordan algebra of finite rank r. An element $x \in A$ is invertible if and only if* $\mathrm{rk}(x) = r$.

1.4. Euclidean Jordan algebras

A *euclidean* Jordan algebra A is a *real* Jordan algebra, such that there exists a positive definite bilinear form on A satisfying the *symmetry* condition $\langle L_x u | v\rangle = \langle u | L_x v\rangle$ for all $x, u, v \in A$. In addition we assume A to be unital with unit element $e \in A$. A system of orthogonal idempotents $e_1, \ldots, e_r$ is *complete*, if $e_1 + \ldots + e_r = e$.

Theorem 1.5 (Spectral theorem). *For $x \in A$ there exist unique real numbers $\lambda_1, \dots, \lambda_r$ all distinct, and a unique complete system of orthogonal idempotents $e_1, \dots, e_r$ such that*

$$x = \lambda_1 e_1 + \dots + \lambda_r e_r \ .$$

For each $j = 1, \dots, r$, we have $e_j \in \mathbb{R}[x]$. The numbers λ_j are said to be the eigenvalues *and $\sum \lambda_j e_j$ the* spectral decomposition *of x. Furthermore,*

$$\Delta(x) = \lambda_1 \cdot \ldots \cdot \lambda_r \ ,$$

and more generally,

$$a_j(x) = \sum_{1 \le i_1 < \ldots < i_j \le r} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_j} \ ,$$

where a_j, $1 \le j \le r$, is the polynomial defined in Proposition 1.2.

Corollary 1.6. *The rank* $\operatorname{rk}(x)$ *of an element $x \in A$ equals the number of non-zero eigenvalues of x (with multiplicities counted).*

We recall that each euclidean Jordan algebra contains a symmetric cone: Let Q be the set of all squares, $Q := \{x^2 \,|\, x \in A\}$, then

$$\Omega(A) := \text{interior of } Q \tag{1.12}$$

is a symmetric cone [**8**, III.2.1]. This is the starting point for the theory which relates (real) symmetric cones with Jordan algebras in a one-to-one correspondence. Finally, we recall different Jordan theoretic characterizations of Ω given in [**8**]:

Proposition 1.7. *Let A be a euclidean Jordan algebra with unit element $e \in A$, and let Ω be defined as in* (1.12)*. The following are equivalent*

(i) $x \in \Omega(A)$,
(ii) $x \in A$ *invertible and* $x = y^2$ *for some* $y \in A$,
(iii) $x = \exp(y)$ *for some* $y \in A$,
(iv) $x = \sum \lambda_i e_i$ *(spectral decomposition) with* $\lambda_i > 0$ *for all* i,
(v) $x \in A$ *with* L_x *positive definite.*

Furthermore, $\Omega(A)$ equals the connected component of the set of invertible elements.

The various characterizations of the symmetric cone are used in Sections 2.5, 3.4, and 7.1.

CHAPTER 2

Jordan triple systems

This chapter is a review of basic notions in the theory of Jordan triple systems. In Sections 2.1 through 2.4 we follow the usual treatment, as it can be found in more detail e.g. in [**8, 28**]. Starting with Section 2.5, we deviate from the usual presentation by introducing a generalized Peirce decomposition we adopt from [**18**]. Usually the Peirce decomposition is defined solely for *tripotent* elements. This concept can be extended to arbitrary elements of the Jordan triple system if one introduces the pseudo-inverse of elements. In the finite dimensional setting, every element of the Jordan triple admits a pseudo-inverse. The benefit of this concept comes in when we study the action of the structure group on various objects, e.g. sets of certain Peirce spaces (Lemma 2.32) or the compactification of Z (Chapter 7). Since the set of tripotents is not invariant under the structure group, the usual Peirce decomposition is not the appropriate one for these group actions.

Using the generalized Peirce decomposition in Section 2.6, we extend E. Neher's equivalence relation on the set of tripotents[1] to an equivalence relation on all of Z. Sections 2.7 through 2.10 pick up the threads of the usual presentation. However, we have to reformulate and prove again some of the results to fit them into the concepts of generalized Peirce decompositions and pseudo-inverses. In addition, these sections contain some technical results which are needed in later chapters but can be proved by elementary methods.

2.1. Basic structure

Let Z be a complex vector space. Then we denote by $\overline{Z}$ the complex conjugate vector space, i.e. $\overline{Z} = Z$ as real vector spaces and scalar multiplication on $\overline{Z}$ by complex scalars $\lambda \in \mathbb{C}$ is defined by $\lambda \cdot z = \overline{\lambda} z$, where $\overline{\lambda} z$ denotes scalar multiplication in Z. Therefore an antilinear map from Z into some complex vector space W can be considered as a linear map from $\overline{Z}$ to W. Since $\overline{Z}$ and Z coincide as sets, we often do not distinguish between $z \in Z$ being an element of Z or $\overline{Z}$. The distinction is of importance only in cases when we consider the properties of maps such as complex (anti-)linearity or, in later chapters, (anti-)holomorphy. Then saying that a map is antilinear (resp. antiholomorphic) in $z \in Z$ is equivalent to saying that the map is linear (resp. holomorphic) in $z \in \overline{Z}$.

A *hermitian Jordan triple system* is a pair $(Z, \{\ ,\ ,\ \})$ of a complex vector space Z and a complex trilinear map

$$\{\ ,\ ,\ \} : Z \times \overline{Z} \times Z \to Z,\ (u, v, w) \mapsto \{u, v, w\}\ ,$$

such that for all $x, y, z, u, v \in Z$ the following holds:

(i) *Symmetry*:
$$\{x, y, z\} = \{z, y, x\}\ ,$$

(ii) *Jordan identity*:
$$\{u, v, \{x, y, z\}\} - \{x, y, \{u, v, z\}\} = \{\{u, v, x\}, y, z\} - \{x, \{v, u, y\}, z\}\ .$$

[1] Two tripotents are equivalent if their Peirce decompositions coincide, see [**36**].

Fixing some entries in the *Jordan triple product* $\{\,,\,,\,\}$ yields the following definitions: For $u, v \in Z$ let

$$u \,\Box\, v : Z \to Z,\ z \mapsto u \,\Box\, v(z) := \{u, v, z\}\ , \quad Q_u : \overline{Z} \to Z,\ z \mapsto Q_u z := \{u, z, u\}$$

be the *box operator* and the *quadratic operator*. We also use the following polarization of the quadratic operator:

$$Q_{u,v} := \tfrac{1}{2}(Q_{u+v} - Q_u - Q_v)\ .$$

The Jordan identity is equivalent to

$$[u \,\Box\, v, x \,\Box\, y] = \{u, v, x\} \,\Box\, y - x \,\Box\, \{v, u, y\} \tag{2.1}$$

for all $x, y, u, v \in Z$, where $[T, S] = T \circ S - S \circ T$ denotes the ordinary commutator of endomorphisms $T, S \in \mathrm{End}(Z)$. The quadratic operator satisfies the *fundamental formula*

$$Q_{Q_x y} = Q_x Q_y Q_x \tag{2.2}$$

for all $x, y \in Z$. Besides these relations there is a whole list of (unnamed) relations satisfied by combinations of the Jordan triple product, the box and the quadratic operator. Appendix A is an adaption of O. Loos' list of identities given in [**28**]. In the sequel we refer to the single identities by JT1 to JT35.

Inductively we define the *odd powers* of an elements $z \in Z$ by

$$z^{(1)} := z \quad \text{and} \quad z^{(2n+1)} := Q_z z^{(2n-1)}\ . \tag{2.3}$$

A hermitian Jordan triple system Z is *positive*, if $\{z, z, z\} = \lambda z$ with $\lambda \in \mathbb{C}$ and $z \neq 0$ implies $\lambda > 0$. This is equivalent [**28**, §3.16] to the requirement that the sesquilinear map

$$\langle\ |\ \rangle : Z \times Z \to \mathbb{C},\ (u, v) \mapsto \mathrm{Tr}(u \,\Box\, v) \tag{2.4}$$

is positive definite and therefore defines a scalar product on Z. Here $\mathrm{Tr}(u \,\Box\, v)$ denotes the trace of the box operator $u \,\Box\, v$. Due to the Jordan identity, this scalar product is *associative*, i.e.

$$\langle \{u, v, w\} | z \rangle = \langle u | \{v, w, z\} \rangle \tag{2.5}$$

for all $u, v, w, z \in Z$. By the symmetry of the Jordan triple product, this is equivalent to the statement, that the adjoint of a box operator $u \Box v$ is given by $(u \Box v)^* = v \Box u$. The associativity also implies that the quadratic operators satisfy the following relation:

$$\langle Q_u x | y \rangle = \langle Q_u y | x \rangle\ . \tag{2.6}$$

In this thesis we are engaged exclusively with **finite dimensional positive hermitian Jordan triple systems** (**phJTS**).

Remark 2.1. As O. Loos indicates in the introduction to [**28**], a phJTS can be regarded as a composed object, namely as a Jordan pair with positive hermitian involution. More explicitly, if Z denotes the phJTS, then $(Z, \overline{Z})$ is the associated pair with involution being the identity on Z (which is antilinear when regarded as a map from Z to $\overline{Z}$). Some concepts used in phJTS become more transparent when considered from the point of view of Jordan pairs (such as the structure group, quasi-invertibility, duality and even the construction of the compactification of Z). Other concepts rely heavily on the identification of Z with $\overline{Z}$ via the involution, so there is no distinction between elements of Z and elements of $\overline{Z}$ (e.g. the definition of tripotents, pseudo-inverses and the Peirce decomposition[2]). Since one of the main concepts used in this thesis involve pseudo-inverses and Peirce decompositions, we prefer the point of view of phJTS, so the main focus lies on the single object Z, and the complex conjugate $\overline{Z}$ of Z is merely used to describe antilinearity of certain maps. Nevertheless, this usage indicates that there is an underlying concept for general Jordan pairs. By duality of Jordan pairs, we can interchange Z and $\overline{Z}$ and thus obtain the Jordan triple system $\overline{Z}$, which is identical to Z except that the complex structure is conjugate.

We notice that each phJTS is *semisimple* in the following sense [**28**, §4.10]: An ideal in Z is a subset $I \subset Z$, such that $\{I, Z, Z\} + \{Z, I, Z\} \subset I$. A Jordan triple system Z is *simple*, if all quadratic operators are non-trivial, and if it contains no proper ideals. A Jordan triple system is *semisimple*, if it is decomposable into a direct sum of simple Jordan triple systems.

Example 2.2. The main example of a phJTS is given by the vector space of complex matrices, $Z = \mathbb{C}^{r\times s}$ with $r \leq s$, equipped with the triple product

$$\{x, y, z\} = \tfrac{1}{2}(xy^*z + zy^*x) , \tag{2.7}$$

where $y^* = \overline{y}^\top$ denotes the transposed and complex conjugated matrix. In this case, the scalar product is given by

$$\langle x|y\rangle = \frac{r+s}{2}\,\mathrm{Tr}(xy^*) . \tag{2.8}$$

In the sequel we refer to this example with the term '*matrix case*'. Often it is usefull to identify the elements $z \in \mathbb{C}^{r\times s}$ with linear maps $(\eta \to z\eta)$ from $\eta \in \mathbb{C}^s$ to $\mathbb{C}^r$. Then, y^* is interpreted as the adjoint map of y with respect to the standard scalar products given on $\mathbb{C}^s$ and $\mathbb{C}^r$. Unless otherwise stated, we regard vectors in $\mathbb{C}^s$ and $\mathbb{C}^r$ as columns.

2.2. Conjugate Jordan triple systems

Let Z be a phJTS and let $\overline{Z}$ denote the complex conjugate vector space as described in Section 2.1. Since swapping Z and $\overline{Z}$ in the definition of the triple product does not change its properties (trilinearity, symmetry, Jordan identity and positivity), $\overline{Z}$ can be regarded as a phJTS in its own right. Since $\overline{Z}$ and Z coincide as sets, the corresponding triple systems are essentially the same, except from their opposite complex analytic structures. We call $\overline{Z}$ the *conjugate* phJTS of Z. As already mentioned in Remark 2.1, the concept of duality/conjugation becomes more transparent when discussed within the theory of Jordan pairs.

2.3. Connection to Jordan algebras

Given a phJTS Z and an arbitrary element $a \in \overline{Z}$, the product $x \circ_a y = \{x, a, y\}$ induces on Z the structure of a complex Jordan algebra. To distinguish the algebra structure from the triple structure, we often denote this Jordan algebra by $Z^{(a)}$. If there is no danger of ambiguity, we write $x \circ y$ for the Jordan algebra product. Left multiplication and the quadratic map are given by

$$L_x^{(a)} = x \,\square\, a \quad \text{and} \quad P_x^{(a)} = Q_x Q_a . \tag{2.9}$$

[2]The Peirce decomposition plays a special role in this context. It can be defined on arbitrary Jordan pairs (Z_+, Z_-) and leads to decompositions of Z_+ and Z_- separately. In the theory of Jordan pairs with positive hermitian involution (or equivalently in the theory of phJTS) one compares these decompositions via the involution and demands them to coincide. This is valid for Peirce decompositions with respect to *tripotents* and can be generalized to arbitrary elements by introducing the concept of *pseudo-inverses*. See Section 2.5

There is an analogous construction using $a \in Z$ and the dual phJTS $\overline{Z}$, resulting in the complex conjugate Jordan algebra $\overline{Z}^{(a)}$. In general these Jordan algebras are neither euclidean nor unital. Different elements $a, b \in \overline{Z}$ induce different Jordan algebra structures on $Z = Z^{(a)} = Z^{(b)}$. In some cases there are naturally given homomorphisms between these structures.

Lemma 2.3. *Let* $(a,b) \in Z \times \overline{Z}$ *be related by some* $x \in Z$ *via* $a = Q_x b$. *Then*

$$Q_x : \overline{Z}^{(a)} \to Z^{(b)},\ z \mapsto Q_x z \tag{2.10}$$

is a complex linear homomorphism of Jordan algebras. If $\overline{Z}^{(a)}$ *is unital with unit element* e, *then* $Z^{(b)}$ *is also unital with unit element* $Q_x e$, *and* Q_x *is a complex linear isomorphism of unital Jordan algebras.*

Proof. Due to the fundamental formula we have

$$Q_x(u \circ_a u) = Q_x Q_u a = Q_x Q_u Q_x b = Q_{Q_x u} b = (Q_x u) \circ_b (Q_x u)\ .$$

By polarization this implies $Q_x(u \circ_a v) = (Q_x u) \circ_b (Q_x v)$ for all $u, v \in \overline{Z}^{(a)}$. Now let e be the unit element of $\overline{Z}^{(a)}$. Then e is invertible in $\overline{Z}^{(a)}$, and by (1.11) and (2.9) this implies the invertibility of Q_e and Q_a. Again using the fundamental formula $Q_a = Q_x Q_b Q_x$, it follows that Q_x is also invertible and hence an antilinear isomorphism of unital Jordan algebras. □

2.4. Tripotents and the spectral theorem

A *tripotent* $e \in Z$ is an element satisfying $\{e, e, e\} = e$. Two tripotents $e, c \in Z$ are (*strongly*) *orthogonal*, $e \perp\!\!\!\perp c$, if one of the following equivalent conditions is satisfied [**28**, §3.9]

$$\text{(i) } c \,\square\, e = 0 \qquad \text{(ii) } e \,\square\, c = 0 \qquad \text{(iii) } \{c, c, e\} = 0 \qquad \text{(iv) } \{e, e, c\} = 0\ .$$

Strongly orthogonal tripotents are orthogonal in the sense of the scalar product on Z. In the following the term 'orthogonal tripotents' always means 'strongly orthogonal tripotents'. The set $S \subset Z$ of tripotents is partially ordered by

$$e < c \iff c = e + e' \text{ for some tripotent } e' \neq 0 \text{ with } e' \perp\!\!\!\perp e\ .$$

The non-trivial minimal tripotents are called *primitive*. Since a family of orthogonal tripotents in Z is linearly independent and since Z is assumed to be finite dimensional, every tripotent $c \in S$ can be decomposed into a sum $e = e_1 + \ldots + e_r$ of orthogonal primitive idempotents. Each such decomposition has the same number of summands, this defines the *rank* of c, denoted by $\mathrm{rk}(e) = r$. The *rank* of the Jordan triple system Z is defined by

$$\mathrm{rk}(Z) := \max\{\mathrm{rk}(e)\,|\,e \text{ tripotent}\} \leq \dim_{\mathbb{C}} Z\ . \tag{2.11}$$

Let S_k denote the set of rank-k-tripotents, then

$$S = \bigcup_{k=1}^{r} S_k \qquad \text{with} \qquad S_k = \{e \in S\,|\,\mathrm{rk}(e) = k\}\ , \quad r = \mathrm{rk}(Z) \tag{2.12}$$

is a disjoint decomposition of the set of tripotents. If Z is simple, one can show ([**28**, §5.12]) that the S_k are exactly the connected components of S. In Chapter 3 we discuss the analytic properties of these subsets. A *frame* is a maximal system of orthogonal primitive tripotents. Each frame has $r = \mathrm{rk}(Z)$ elements.

Example 2.4. In the matrix case $Z = \mathbb{C}^{r\times s}$, an element $e \in Z$ is tripotent if and only if $e = ee^*e$. This is the condition characterizing *parital isometries.* The standard example of a tripotent element is

$$e = \begin{pmatrix} \mathbf{1}_j & 0_{j,s-j} \\ 0_{r-j,j} & 0_{r-j,s-j} \end{pmatrix} \in \mathbb{C}^{r\times s} \,, \tag{2.13}$$

where $\mathbf{1}_j$ denotes the $(j \times j)$-identity matrix and $0_{p,q}$ the $(p \times q)$-vanishing matrix. Moreover, two tripotents e, c are orthogonal if and only if $ec^* = 0$ and $c^*e = 0$. For e as in 2.13 this is equivalent to the condition that c has is of the form $\left(\begin{smallmatrix} 0 & 0 \\ 0 & \star \end{smallmatrix}\right)$, where the entries denote block matrices of the same size as in (2.13).

The following spectral theorem generalizes the singular value decomposition of matrices: Recall that for each matrix $z \in \mathbb{C}^{r\times s}$ with $r \le s$ there exist unitary matrices $U_1 \in U(r)$ and $U_2 \in U(s)$ such that $z = U_1 \Lambda U_2$ with a unique 'diagonal' matrix

$$\Lambda = \left(\begin{array}{ccc|c} \lambda_1 \mathbf{1}_{j_1} & & & \\ & \ddots & & 0_{\rho,s-\rho} \\ & & \lambda_n \mathbf{1}_{j_n} & \\ \hline & 0_{r-\rho,\rho} & & 0_{r-\rho,s-\rho} \end{array}\right) \in \mathbb{C}^{r\times s} \quad \text{with} \quad \lambda_1 > \ldots > \lambda_n > 0 \,,$$

where $\rho = j_1 + \ldots + j_k$ is the rank of Λ. Let E_i be the $(r \times s)$-matrix obtained from Λ by setting λ_i to 1 and all others to 0. Then we obtain

$$z = \lambda_1 e_1 + \ldots + \lambda_r e_r \quad \text{with} \quad e_i = U_1 E_i U_2 \,.$$

It is a simple calculation to verify that the e_i form an orthogonal system of tripotents in the Jordan triple system $Z = \mathbb{C}^{r\times s}$ as defined in Example 2.2.

Theorem 2.5 (Spectral theorem)**.** *Let Z be a phJTS. Then every element $z \in Z$ admits a unique decomposition*

$$z = \lambda_1 e_1 + \ldots + \lambda_n e_n \tag{2.14}$$

where the e_i are pairwise orthogonal non-zero tripotents which are real linear combinations of powers of z, and the λ_i satisfy

$$\lambda_1 > \ldots > \lambda_n > 0 \,. \tag{2.15}$$

We call (2.14) *the* spectral decomposition *and the λ_i the* spectral values *of z. Moreover, setting $|z| := \lambda_1$ defines the* spectral norm *on Z.*

For a proof we refer to [**28**, §3.12 and §3.17]. Using the spectral theorem we define

$$\boldsymbol{\epsilon} : Z \to S, \; z = \sum \lambda_i e_i \mapsto \epsilon(z) := \sum e_i \,. \tag{2.16}$$

This is a projection onto the set of tripotents. The element $\boldsymbol{\epsilon}(z) \in S$ is called the *base-tripotent* of z. We also define the *rank* of an arbitrary element $z \in Z$ by

$$\mathrm{rk}(z) := \mathrm{rk}(\boldsymbol{\epsilon}(z)) \quad \text{and set} \quad Z_k := \{z \in Z \,|\, \mathrm{rk}(z) = k\} \,. \tag{2.17}$$

In this way we obtain a partition of Z into the subsets Z_k of constant rank elements. Again, if Z is simple, one can show that these subsets are connected. The restriction of ϵ to Z_k is a projection onto S_k, the set of rank-k tripotents. The analytic properties of Z_k and $\boldsymbol{\epsilon}$ are discussed in Section 3.4. In the matrix case $Z = \mathbb{C}^{r\times s}$ the Jordan triple rank coincides with the ordinary rank of matrices.

2.5. Pseudo-inverse elements and generalized Peirce decompositions

Let Z be a phJTS. An *idempotent* of Z is a pair $(a,b) \in Z \times \overline{Z}$ that satisfies $a = Q_a b$ and $b = Q_b a$. In this case, a is an idempotent of the Jordan algebra $Z^{(b)}$ and b is an idempotent of the Jordan algebra $Z^{(a)}$.[3] The corresponding Peirce decompositions of $Z = Z^{(b)}$ and $Z = Z^{(a)}$ are given by

$$Z = Z_1^{a,b} \oplus Z_{1/2}^{a,b} \oplus Z_0^{a,b} \quad \text{and} \quad Z = Z_1^{b,a} \oplus Z_{1/2}^{b,a} \oplus Z_0^{b,a} \,, \tag{2.18}$$

where $Z_\nu^{a,b}$ and $Z_\nu^{b,a}$ denote the eigenspaces of $L_a^{(b)} = a \,\square\, b$ and $L_b^{(a)} = b \,\square\, a$ to the eigenvalue ν. We notice, that an $e \in Z$ is a tripotent if and only if the the pair (e,e) is idempotent, and then the decompositions 2.18 coincide. For a general idempotent (a,b) we have

$$Z_\nu^{a,b} = Z_\nu^{b,a} \text{ for } \nu = 1, 1/2, 0 \iff a \,\square\, b = b \,\square\, a \,.$$

Since $(a \square b)^* = b \square a$, this is equivalent to the selfadjointness of $a \square b$. In this case, the corresponding Peirce spaces form an orthogonal decomposition and the orthogonal projections are given by

$$\begin{aligned} \pi_1 &= a \,\square\, b \circ (2\, a \,\square\, b - \mathrm{Id}) = Q_a Q_b \,, \\ \pi_{1/2} &= 4\, a \,\square\, b \circ (\mathrm{Id} - a \,\square\, b) \,, \\ \pi_0 &= (2\, a \,\square\, b - \mathrm{Id}) \circ (a \,\square\, b - \mathrm{Id}) = \mathrm{Id} - 2\, a \,\square\, b + Q_a Q_b \,. \end{aligned} \tag{2.19}$$

We notice that $Z_1^{a,b}$ forms a unital Jordan algebra with product $x \circ_b y = \{x, b, y\}$ and unit element a, and similarly $Z_1^{b,a}$ forms a unital Jordan algebra with product $x \circ_a y = \{x, a, y\}$ and unit element b.

Theorem 2.6 (pseudo-inverses). *Let Z be a phJTS. Then for every element $a \in Z$ there exists a unique element $a^\dagger \in Z$, such that $(a, a^\dagger)$ is an idempotent and the corresponding Peirce decompositions coincide. This element is called the* pseudo-inverse *of a. It is uniquely determined by the relations*

$$a^\dagger = Q_{a^\dagger} a \,, \quad a = Q_a a^\dagger \,, \quad a \,\square\, a^\dagger = a^\dagger \,\square\, a \,, \tag{2.20}$$

and satisfies

$$a^\dagger = Q_a a^{-1} \qquad \textit{and} \qquad (a^\dagger)^\dagger = a \,, \tag{2.21}$$

where a^{-1} is the inverse of a in the unital Jordan algebra $Z_1^{a^\dagger, a}$. If $a = \sum \lambda_i e_i$ is the spectral decomposition of a, then $a^\dagger$ is given by $a^\dagger = \sum \frac{1}{\lambda_i} e_i$.

Proof. First we note that the identities (2.20) are equivalent to the statement that $(a, a^\dagger)$ is an idempotent such that the corresponding Peirce decompositions coincide. For the moment, we call *any* element satisfying (2.20) a pseudo-inverse of a. Using the properties of orthogonal tripotents, it is a simple calculation to show that $a^\dagger = \sum \frac{1}{\lambda_i} e_i$ indeed is a pseudo-inverse of a. For uniqueness, it suffices to prove the first equation in (2.21). By assumption we have $Z_1^{a,a^\dagger} = Z_1^{a^\dagger,a} =: Z_1$, and due to (2.19) this yields $Q_a Q_{a^\dagger} = Q_{a^\dagger} Q_a = \mathrm{Id}_{Z_1}$ in the restriction to Z_1. Therefore Q_a and $Q_{a^\dagger}$ are invertible on Z_1 with $Q_a^{-1} = Q_{a^\dagger}$, and we obtain

$$a^{-1} = {P_a^{(a)}}^{-1} a = Q_a^{-1} Q_a^{-1} a = Q_a^{-1} Q_{a^\dagger} a = Q_a^{-1} a^\dagger \,,$$

where we used (1.11) and the defining relations (2.20). Finally, the relation $(a^\dagger)^\dagger = a$ follows from the symmetry of a and $a^\dagger$ in the defining relations. $\square$

[3]Identifying Z with the Jordan pair $(Z, \overline{Z})$ one could be more precise and say that a is an idempotent of the Jordan algebra $\overline{Z}^{(b)}$, cf. Remark 2.1.

Now let $a \in Z$ be an arbitrary element. Then we define the (*generalized*) *Peirce decomposition with respect to* a to be the Peirce decomposition with respect to $(a, a^\dagger)$, and obtain

$$Z = Z_1^a \oplus Z_{1/2}^a \oplus Z_0^a \quad \text{with} \quad Z_\nu^a := Z_\nu^{a,a^\dagger} = Z_\nu^{a^\dagger,a} \, . \tag{2.22}$$

Since this decomposition can be considered as a Peirce decomposition of the Jordan pair $(Z, \overline{Z})$, we can adapt several results from [**27**]. In particular we obtain

Theorem 2.7 (Peirce rules)**.** *Let Z be a phJTZ and let $Z = Z_1^a \oplus Z_{1/2}^a \oplus Z_0^a$ be the Peirce decomposition with respect to $a \in Z$. Then*

$$\left\{Z_\alpha^a, Z_\beta^a, Z_\gamma^a\right\} \subset Z_{\alpha-\beta+\gamma}^a \, , \quad \{Z_1^a, Z_0^a, Z\} = \{0\} \, , \quad \{Z_0^a, Z_1^a, Z\} = \{0\}$$

for all $\alpha, \beta, \gamma \in \{0, 1/2, 1\}$ and $Z_\nu^a := \{0\}$ for $\nu \notin \{0, 1/2, 1\}$. Each Peirce space Z_ν^a forms a subtriple of Z and therefore is a phJTS on its own.

Remark 2.8. The generalized Peirce decomposition introduced in this section is also discussed by W. Kaup in [**18**] — even in the infinite dimensional setting. Doing this, one has to restrict to the set of (von Neumann) regular elements, since just those elements admit a pseudo-inverse [**18**, Lem.3.2]. In the finite dimensional setting, all elements are (von Neumann) regular. However, one has to be more careful about the relation of this Peirce decomposition to the action of the structure group. This action is not as simple as described in [**18**, p.573], we take on this point in Section 2.8 (Example 2.31 and Lemma 2.32).

It should be noted that there is yet another Peirce decomposition defined by W. Kaup and D. Zaitsev in [**20**], but this one differs from ours, though it looks very similar. For example, their Peirce 1-space is the 1-eigenspace of $Q_{a,a^\dagger}^2$, whereas ours is the 1-eigenspace of $Q_a Q_{a^\dagger}$.

Example 2.9. In the matrix case $Z = \mathbb{C}^{r\times s}$, $r \le s$, the pseudo-inverse is exactly the conjugate transpose of the *Moore-Penrose pseudo-inverse* defined in [**37**]. We note that any $(r \times s)$-matrix a of rank j can be decomposed into $z = xy$ with $x \in \mathbb{C}^{r\times j}$ and $y \in \mathbb{C}^{j\times s}$ of full rank.[4] Then, it is a straightforward computation to show that the pseudo-inverse is given by

$$a^\dagger = x(x^* x)^{-1}(yy^*)^{-1} y \, . \tag{2.23}$$

If a is of maximal rank r, this reduces to $a^\dagger = (aa^*)^{-1}a$. If in addition, $s = r$, then a is invertible and we obtain $a^\dagger = a^{-*}$. If $a \in \mathbb{C}^{r\times s}$ is of the block form

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{C}^{r\times s} \quad \text{with invertible } \alpha \in \mathbb{C}^{j\times j}, \text{ then} \quad a^\dagger = \begin{pmatrix} \alpha^{-*} & 0 \\ 0 & 0 \end{pmatrix} , \tag{2.24}$$

and the Peirce decomposition with respect to a is given by

$$\left.\begin{aligned} Z_1^a &= \left\{\left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right) \,\middle|\, A \in \mathbb{C}^{j\times j}\right\} , \\ Z_{1/2}^a &= \left\{\left(\begin{smallmatrix} 0 & B \\ C & 0 \end{smallmatrix}\right) \,\middle|\, B \in \mathbb{C}^{j\times(s-j)},\ C \in \mathbb{C}^{(r-j)\times j}\right\} , \\ Z_0^a &= \left\{\left(\begin{smallmatrix} 0 & 0 \\ 0 & D \end{smallmatrix}\right) \,\middle|\, D \in \mathbb{C}^{(r-j)\times(s-j)}\right\} , \end{aligned}\right\} \quad \rightsquigarrow \quad \begin{array}{|c|c|} \hline Z_1^a & Z_{1/2}^a \\ \hline Z_{1/2}^a & Z_0^a \\ \hline \end{array}$$

The right hand is used as a schematic diagram of the Peirce decomposition.

The next step is to generalize this Peirce decomposition to systems of orthogonal elements.

---

4 E.g., if $a = U_1 \Lambda U_2$ denotes the spectral decomposition of a as discussed in the last section, then let x be the matrix consisting of the first j columns of $U_1 \Lambda$, and let y be the matrix constisting of the first j rows of U_2.

Joint Peirce decomposition. First we have to define the notion of (strong) orthogonality for arbitrary elements $a, b \in Z$. The following lemma and its proof is an adaptation of [**28**, §3.9].

Lemma 2.10. *Let a, b be elements of Z. Then the following are equivalent:*

(i) $a \square b = 0$ (ii) $b \square a = 0$ (iii) $\left\{a,\, a^\dagger,\, b\right\} = 0$ (iv) $\left\{b,\, b^\dagger,\, a\right\} = 0$.

*If one of these holds, then $a \square a^\dagger$ and $b \square b^\dagger$ commute, and a and b are said to be (*strongly*)* orthogonal *to each other, denoted by $a \perp\!\!\!\perp b$.*

Proof. First we notice that by the properties of the pseudo-inverse we have $a \square a^\dagger = a^\dagger \square a$ and $b \square b^\dagger = b^\dagger \square b$. Therefore (i) implies (iv) and (ii) implies (iii). Assume that (iv) holds. Then by JT1,

$$\{b,\, a,\, b\} = \left\{b,\, a,\, Q_b b^\dagger\right\} = (b \square a) Q_b b^\dagger = Q_b (a \square b) b^\dagger = Q_b \left\{a,\, b^\dagger,\, b\right\} = 0 \,,$$

and using the Jordan identity we obtain on the one hand

$$[b \square b^\dagger, b \square a] = \left\{b,\, b^\dagger,\, b\right\} \square a - b \square \left\{b,\, b^\dagger,\, a\right\} = b \square a$$

and on the other hand

$$[b \square b^\dagger, b \square a] = -[b \square a, b \square b^\dagger] = -\{b,\, a,\, b\} \square b^\dagger + b \square \left\{a,\, b,\, b^\dagger\right\} = 0 \,.$$

Therefore (iv) implies (ii). Interchanging a and b in these calculations also shows that (iii) implies (i), hence proving the equivalence of all four conditions. Finally,

$$[a \square a^\dagger, b \square b^\dagger] = \left\{a,\, a^\dagger,\, b\right\} \square b^\dagger - a \square \left\{b,\, a^\dagger,\, b^\dagger\right\} = 0 \,,$$

thus $a \square a^\dagger$ and $b \square b^\dagger$ commute. □

Remark 2.11. Since the defining properties of the pseudo-inverse allow to interchange a and $a^\dagger$ in condition (iii) of Lemma 2.10, we can exchange a by $a^\dagger$ in conditions (i), (ii) and (iv). Similarly, it is possible to exchange b by $b^\dagger$ in (i) to (iii). This proves that orthogonality is equivalent among the pairs (a, b), $(a^\dagger, b)$, $(a, b^\dagger)$ and $(a^\dagger, b^\dagger)$.

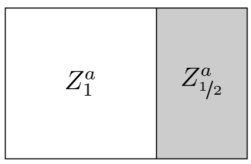
Remark 2.12. The concept of (strong) orthogonality provides some relations between the Peirce spaces of some Peirce decomposition. Let U be some subspace of Z and denote by $U^\perp$ the (ordinary) orthogonal complement of U with respect to the scalar product of Z. In addition, we set

$$U^{\perp\!\!\!\perp} := \{z \in Z \,|\, z \perp\!\!\!\perp u \text{ for all } u \in U\} \;. \tag{2.25}$$

Then is is easy to see that $U^{\perp\!\!\!\perp} \subset U^\perp$ and $U \subset (U^{\perp\!\!\!\perp})^{\perp\!\!\!\perp}$. In general, equality does not hold. Now let $Z = Z^u_1 \oplus Z^u_{1/2} \oplus Z^u_0$ be the Peirce decomposition with respect to some element $u \in Z$. Due to the Peirce rules we obtain

$$Z^a_0 = (Z^a_1)^\perp \,, \quad Z^a_{1/2} = (Z^a_1 \oplus (Z^a_1)^{\perp\!\!\!\perp})^\perp \,, \quad (Z^a_0)^{\perp\!\!\!\perp} \supset Z^a_1 \;. \tag{2.26}$$

The first two equations imply that the Peirce decomposition is uniquely determined by Z^a_1, which is important for the study of Peirce varieties, see Chapter 6. Concerning the inclusion, it should be noted that in general this is not an equality. For instance, consider the matrix case $Z = \mathbb{C}^{r \times s}$ with r strictly less than s. Then the Peirce decomposition with respect to $a = (\mathbf{1}_r\ 0)$ is given by



Therefore, $Z^a_0 = \{0\}$, which implies $(Z^a_0)^{\perp\!\!\!\perp} = Z$, but the Peirce 1-space Z^a_1 is a proper subset of Z.

Now let $a_1, \dots, a_n$ be an *orthogonal system of elements* $a_i \in Z$, i.e. $a_i \perp\!\!\!\perp a_j$ for $i \neq j$. Then by Lemma 2.10, the box operators $a_i \,\square\, a_i^\dagger$ can be diagonalized simultaneously, therefore we obtain the *joint Peirce decomposition*

$$Z = \bigoplus_{0 \le i \le j \le n} Z_{ij} \quad \text{with} \quad Z_{ij} = \left\{ z \in Z \;\middle|\; \begin{array}{l} \{a_k,\, a_k^\dagger,\, z\} = \frac{1}{2}(\delta_{ki} + \delta_{kj})z \\ \text{for } k = 1, \dots, n \end{array} \right\} . \tag{2.27}$$

Usually, one sets $Z_{ij} = Z_{ji}$. As in the (ordinary) joint Peirce decomposition with respect to an orthogonal system of tripotents, one obtains the *joint Peirce rules*

$$\{Z_{ij},\, Z_{jk},\, Z_{k\ell}\} \subset Z_{i\ell}$$

and all other types of products are zero. If $I \subset \{1, \dots, n\}$ is a subset, and $a_I = \sum_{i \in I} a_i$, then the Peirce spaces of a_I are given by

$$Z_1^{a_I} = \sum_{i,j \in I} Z_{ij} \;, \quad Z_{1/2}^{a_I} = \sum_{i \in I, j \notin I} Z_{ij} \;, \quad Z_0^{a_I} = \sum_{i,j \notin I} Z_{ij} \;. \tag{2.28}$$

To obtain a direct sum decomposition one has to be more careful about the pairs (i, j) occurring in the sum. These results follow immediately from [**28**, §3.14] by application of Corollary 2.21.

Example 2.13. In the matrix case $Z = \mathbb{C}^{r \times s}$, a standard example of an orthogonal system of elements and their corresponding joint Peirce decomposition is obtained from the following arrangement of invertible block matrices α_1, α_2, α_3:

$$\begin{array}{|c|c|c|c|} \hline \alpha_1 & & & \\ \hline & \alpha_2 & & \\ \hline & & \alpha_3 & \\ \hline & & & \\ \hline \end{array} \quad \rightsquigarrow \quad \begin{array}{|c|c|c|c|} \hline Z_{11} & Z_{12} & Z_{13} & Z_{01} \\ \hline Z_{12} & Z_{22} & Z_{23} & Z_{02} \\ \hline Z_{13} & Z_{23} & Z_{33} & Z_{03} \\ \hline Z_{01} & Z_{02} & Z_{03} & Z_{00} \\ \hline \end{array}$$

If a_i denotes the matrix obtained from the left hand side by setting all blocks but the α_i-block to zero, then the elements $a_1, a_2, a_3 \in \mathbb{C}^{r \times s}$ are an orthogonal system of tripotents, whose joint Peirce decomposition is illustrated on the right hand side.

Peirce 1-space revisited. Let Z_1^a be the Peirce 1-space of some element $a \in Z$. There are three canonical ways to define on Z_1^a the structure of a unital Jordan algebra. On the one hand we have $Z_1^a = Z_1^{a,a^\dagger} = Z_1^{a^\dagger,a}$, so the products $x \circ y = \{x,\, a,\, y\}$ and $x \circ_\dagger y = \{x,\, a^\dagger,\, y\}$ turn Z_1^a into a unital Jordan algebra with unit element $a^\dagger$ and a, respectively. On the other hand, the base-tripotent $e := \boldsymbol{\epsilon}(a) = \boldsymbol{\epsilon}(a^\dagger)$ satisfies the relations $e^\dagger = e$ and $e \,\square\, e = a \,\square\, a^\dagger$, so we have $Z_1^a = Z_1^e$ and the product $x \bullet y = \{x,\, e,\, y\}$ also establishes on Z_1^a the structure of a unital Jordan algebra, with unit element e. In the following we always equip Z_1^a with the product $x \circ y = \{x,\, a,\, y\}$ and address the other structures by using the identity $Z_1^a = Z_1^{a^\dagger} = Z_1^e$ and referring to $Z_1^{a^\dagger}$ or Z_1^e.

Due to Lemma 2.3, these structures are related via naturally given isomorphisms: Let $a = \sum \lambda_i e_i$ be the spectral decomposition of a and set $\sqrt{a} := \sum \sqrt{\lambda_i}\, e_i$. Then $a^\dagger = \sum \lambda_i^{-1} e_i$ and $\sqrt{a}^\dagger = \sum \lambda_i^{-1/2} e_i$, and we have the relations

$$\begin{aligned} e &= Q_e e = Q_{\sqrt{a}} a^\dagger = Q_{\sqrt{a}^\dagger} a \;, \\ a &= Q_e a = Q_a a^\dagger = Q_{\sqrt{a}} e \;, \\ a^\dagger &= Q_e a^\dagger = Q_{a^\dagger} a = Q_{\sqrt{a}^\dagger} e \;. \end{aligned} \tag{2.29}$$

Therefore, we obtain the following diagram of Jordan algebra isomorphisms

$$\begin{tikzcd}
 & & \overline{Z}_1^e \arrow[d, leftrightarrow, "Q_e"'] \arrow[dddrr, bend left=40, "Q_{\sqrt{a}}"] & & \\
 & & Z_1^e \arrow[dl, bend right, "Q_{\sqrt{a}^\dagger}"'] & & \\
 & \overline{Z}_1^a \arrow[drrr, bend right, "Q_a"'] & & \overline{Z}_1^{a^\dagger} \arrow[ul, bend right, "Q_{\sqrt{a}^\dagger}"'] \arrow[dlll, bend left, "Q_{a^\dagger}"] \arrow[dr, leftrightarrow, "Q_e"] & \\
Z_1^a \arrow[uuurr, bend left=40, "Q_{\sqrt{a}}"] \arrow[ur, leftrightarrow, "Q_e"] & & & & Z_1^{a^\dagger}
\end{tikzcd}$$

Using the fundamental formula one can show that this diagram is commutative.

Identifying the complex conjugate $\overline{Z}^{(a)}$ with $Z^{(a)}$ as real Jordan algebras, the map Q_e becomes a non-trivial automorphism of order 2. We set

$$()^{\#} : Z_1^a \to Z_1^a,\ z \mapsto z^{\#} := Q_e z \tag{2.30}$$

and consider the decomposition of Z_1^a,

$$Z_1^a = Z_+^a \oplus Z_-^a \quad \text{with} \quad Z_\pm^a = \left\{ z \in Z_1^a \,\middle|\, z^{\#} = \pm z \right\} , \tag{2.31}$$

into the eigenspaces of the involution $()^{\#}$. This decomposition is orthogonal with respect to the *real* scalar product on Z defined by $\langle x|y\rangle_{\mathbb{R}} = \operatorname{Re}\langle x|y\rangle$. Indeed, for $x \in Z_+^a$ and $y \in Z_-^a$ using equation (2.6) we have $\langle x|y\rangle = \langle Q_e x|y\rangle = \langle Q_e y|x\rangle = -\langle y|x\rangle$, and so $\langle x|y\rangle_{\mathbb{R}} = (\langle x|y\rangle + \langle y|x\rangle)/2 = 0$. The fundamental formula shows that Z_+^a forms a real Jordan triple system: For $x, y \in Z_+^a$ we have

$$Q_e Q_y x = Q_e Q_y Q_e x = Q_{Q_e y} x = Q_y x ,$$

i.e. $Q_y x$ is also an element of Z_+^a. Therefore by polarization, Z_+^a is invariant under triple products. Due to (2.29), a and $a^\dagger$ are elements of Z_+^a, and since $()^{\#}$ is homomorphic, the eigenspace Z_+^a is a real Jordan algebra with unit element $a^\dagger$. In fact, the next proposition shows that Z_+^a is even a euclidean Jordan algebra.

Proposition 2.14. *Let Z be a phJTS with scalar product $\langle\,|\,\rangle$. Let Z_1^a be the Peirce 1-space of some element $a \in Z$, and let $Z_1^a = Z_+^a \oplus Z_-^a$ be its decomposition into eigenspaces of the involution $z \mapsto z^{\#} = Q_e z$ with $e = \epsilon(a)$. Then*

$$\langle x|y\rangle_a := \langle x|Q_a y\rangle \quad \text{for} \quad x, y \in Z_+^a \tag{2.32}$$

defines a scalar product on Z_+^a, which turns Z_+^a into a euclidean Jordan algebra.

Proof. It is immediate that (2.32) defines a (complex) bilinear form on Z. Symmetry follows immediately from equation (2.6). We have to show that $\langle x|y\rangle_a$ is real for all $x, y \in Z_+^a$. For this we use equation (2.6), the fundamental formula and the relation $a = Q_e a$:

$$\langle x|Q_a y\rangle = \langle Q_e x|Q_a Q_e y\rangle = \langle Q_e Q_a Q_e y|x\rangle = \langle Q_{Q_e a} y|x\rangle = \langle Q_a y|x\rangle \ .$$

By a similar argument we prove positive definiteness on Z_+^a, using in addition the identities $a = Q_{\sqrt{a}} e$ and $\sqrt{a} = Q_e \sqrt{a}$ (similar to (2.29)):

$$\langle x|Q_a x\rangle = \left\langle x \middle| (Q_{\sqrt{a}} Q_e Q_{\sqrt{a}})(Q_e x)\right\rangle = \left\langle x \middle| Q_{\sqrt{a}}(Q_{\sqrt{a}} x)\right\rangle = \left\langle Q_{\sqrt{a}} x \middle| Q_{\sqrt{a}} x\right\rangle \ .$$

Finally, by JT1 and associativity of the scalar product on Z we obtain

$$\langle x \circ y | z \rangle_a = \langle \{x,\, a,\, y\} | Q_a z \rangle = \langle x | \{a,\, y,\, Q_a z\} \rangle = \langle x | Q_a \{y,\, a,\, z\} \rangle = \langle x | y \circ z \rangle_a \ .$$

Therefore $\langle x|y\rangle_a$ is an associative scalar product on Z_+^a. □

For calculations with elements of Z_+^a and Z_-^a the following lemma is important.

Lemma 2.15. *Let $Z = Z_1^a \oplus Z_{1/2}^a \oplus Z_0^a$ be the Peirce decomposition with respect to $a \in Z$, let $e = \boldsymbol{\epsilon}(a)$ be the base-tripotent of a, and let $(\,)^\# : Z_1^a \to Z_1^a$ be as in* (2.30)*. Then*

$$z \,\square\, e = e \,\square\, z^\#$$

for all $z \in Z_1^a$. In particular, $z \,\square\, e = \pm e \,\square\, z$ for $z \in Z_\pm^a$.

For the proof see [**28**, §9.13]. For later use we finally note a lemma, which relates invertible elements of Peirce 1-spaces with their pseudo-inverses.

Lemma 2.16. *Let Z be a phJTS and Z_1^a be the Perice 1-space of some element $a \in Z$. Then $b \in Z_1^a$ is invertible (with respect to the Jordan algebra product $x \circ y = \{x,\, a,\, y\}$) if and only if the restriction of Q_b to Z_1^a is invertible. In this case,*

$$b^{-1} = Q_{a^\dagger} b^\dagger \quad \text{and} \quad b^\dagger = Q_a b^{-1} \ ,$$

where b^{-1} denotes the Z_1^a-inverse of b and $b^\dagger$ is the pseudo-inverse of b.

Proof. An element of a Jordan algebra is invertible if and only if the corresponding quadratic map is invertible. Therefore b is invertible in Z_1^a if and only if $P_b{}^{(a)} = Q_b Q_a$ is invertible on Z_1^a. Since Q_a is invertible on Z_1^a with inverse $Q_{a^\dagger}$, this is equivalent to the invertibility of Q_b on Z_1^a. Due to (1.11) and the relation $b = Q_b b^\dagger$, the inverse of b is given by $b^{-1} = Q_a^{-1} Q_b^{-1} b = Q_{a^\dagger} b^\dagger$. The second identity follows from this by applying Q_a on both sides. □

Finally, we compare the spectral decomposition of the phJTS Z with the spectral decomposition given in a euclidean Jordan algebra. First, we note that if A is a euclidean Jordan algebra with unit element e, the Jordan algebraic spectral decomposition $z = \sum \mu_j c_j$ of some element $z \in A$ might contain a vanishing term $0 \cdot c_s$ with $\mu_s = 0$. This is required by the condition that $c_1 + \ldots + c_\ell = e$. This makes the formulation of the following result slightly more complicated than one might expect at first.

Proposition 2.17. *Let e be a tripotent element, and let z be an element of the euclidean Jordan algebra Z_+^e. Let*

$$z = \sum_{i=1}^{k} \lambda_i e_i \quad \text{and} \quad z = \sum_{j=1}^{\ell} \mu_j c_j$$

be the spectral decompositions of z with respect to the phJTS Z and with respect to the euclidean Jordan algebra Z_+^e. Then

$$\ell = \begin{cases} k & \text{if } \mu_j \neq 0 \text{ for all } j, \\ k+1 & \text{if } \mu_j = 0 \text{ for some } j, \end{cases}$$

and there exists a permutation $\sigma \in \mathfrak{S}_\ell$, such that

$$\lambda_i = |\mu_{\sigma(i)}| \quad \text{and} \quad e_i = \operatorname{sign}(\mu_{\sigma(i)})\, c_{\sigma(i)} \quad \text{for } i = 1, \ldots, k.$$

Moreover, the spectral decompositions coincide if and only if z is an element of the symmetric cone $\Omega(Z_+^e) \subset Z_+^e$.

Proof. We first consider the spectral decomposition with respect to the euclidean Jordan algebra Z_1^e. By assumption, the c_j satisfy $\sum c_j = e$, $\{c_j, e, c_j\} = c_j$ and $\{c_i, e, c_j\} = 0$ for all $i \neq j$. We claim that the c_j form a system of orthogonal tripotents in Z: Using JT12 we obtain for $i \neq j$

$$0 = \{c_i, e, c_j\} = \{Q_{c_i} e, e, c_j\} = 2\,\{c_i, e, \{c_j, e, c_j\}\} - Q_{c_i}\{e, c_j, e\} = -Q_{c_i} c_j\ .$$

Here we also used Lemma 2.15 for the identity $\{e, c_j, e\} = \{e, e, c_j\} = c_j$. Therefore, $\{c_j, c_i, c_j\}$ vanishes for $i \neq j$, and thus $\{c_j, c_j, c_j\} = \{c_j, e, c_j\} = c_j$ shows that c_j is a tripotent. Similarly, using JT9 it follows

$$0 = \{c_j, c_i, c_j\} = \{c_j, Q_e c_i, c_j\} = 2\,(c_j \,\square\, e)^2(c_i) - (c_j \,\square\, Q_e c_j)(c_i) = -\{c_j, c_j, c_i\}\ .$$

Therefore, the c_j are pairwise orthogonal. Now assume that all μ_j are non-zero. We show that the c_j are elements of the real linear span of the odd powers of z. This is a consequence of the decomposition $z = \sum \mu_j c_j$ with all μ_j distinct, and follows by a Vandermonde argument: We claim that $\left\langle z, \ldots, z^{(2\ell-1)}\right\rangle = \langle c_1, \ldots, c_\ell\rangle$, where $\langle\cdots\rangle$ denotes the real linear span. The inclusion '$\subset$' is trivial, so it suffices to show that $z, \ldots, z^{(2\ell-1)}$ are linearly independent, since the c_j are linearly independent. By orthogonality, we have $z^{2i-1} = \sum \mu_j^{2i-1} c_j$, and therefore

$$\alpha_1 z + \ldots + \alpha_k z^{(2k-1)} = 0 \quad \text{if and only if} \quad \sum_i \alpha_i \mu_j^{2i-1} = 0 \text{ for all } j.$$

This linear system on the α_i is just trivially solvable, since

$$\begin{vmatrix} \mu_1 & \cdots & \mu_1^{2\ell-1} \\ \vdots & & \vdots \\ \mu_\ell & \cdots & \mu_\ell^{2\ell-1} \end{vmatrix} = \mu_1 \cdot \ldots \cdot \mu_\ell \cdot \prod_{1 \le i < j \le \ell} (\mu_i^2 - \mu_j^2) \neq 0\ .$$

In the case that one of the μ_j is zero, say $\mu_s = 0$, this argument still works for the other c_j for $j \neq s$, we just omit the $(0 \cdot c_s)$-term in the sum $z = \sum \mu_j c_j$. Therefore, in any case we proved that the Jordan algebraic decomposition $z = \sum \mu_j c_j$ is also a decomposition into a sum of pairwise orthogonal tripotents, which are linear combinations of (triple-)powers of z. By uniqueness of the spectral decomposition on triple systems, we conclude that the decomposition

$$z = \sum_{\mu_j \neq 0} |\mu_j| \left(\tfrac{\mu_j}{|\mu_j|}\, c_j \right)$$

coincides with the decomposition $z = \sum_{i=1}^k \lambda_i e_i$ up to the order of the terms. This proves the main part of Proposition 2.17. Finally, we note that by Proposition 1.7 an element $z \in Z_+^e$ belongs to the symmetric cone if and only if its eigenvalues μ_j are positive. This implies the last assertion. □

Corollary 2.18. *The rank of an element $u \in Z$ coincides with the rank of the unital Jordan algebra Z_1^u.*

Proof. Proposition 2.17 shows that the spectral decompositions of u in Z coincides with the spectral decomposition of u in euclidean Jordan algebra Z_+^e with $e = \epsilon(u)$. By Corollary 1.6, this implies that the rank of u in Z equals the rank of u in Z_+^e. Since u is invertible in Z_+^e, it is also invertible in the complexification Z_1^e and in Z_1^u. Therefore, Corollary 1.4 yields $\operatorname{rk}(u) = \operatorname{rk}(Z_1^u)$. □

2.6. Peirce equivalence

We introduce an equivalence relation on Z, which generalizes the idea of *associated* tripotents introduced by E. Neher in [**36**]. Two elements $u, v \in Z$ are said to be *Peirce equivalent* if they induce the same Peirce decomposition, i.e.

$$u \approx v \iff Z_\nu^u = Z_\nu^v \quad \text{for } \nu = 1, 1/2, 0. \tag{2.33}$$

Clearly, this defines an equivalence relation on Z. We denote the set of equivalence classes by $\mathbb{P}$. In Section 3.1, we introduce a complex analytic structure on $\mathbb{P}$, which we investigate in more detail in Chapter 6. There are several equivalent descriptions of the Peirce equivalence relation.

Proposition 2.19 (Peirce equivalence)**.** *Let Z be an hermitian Jordan triple system and $u, v \in Z$. The following are equivalent:*

(i) *u and v are Peirce equivalent,*
(ii) $u \,\square\, u^\dagger = v \,\square\, v^\dagger$,
(iii) $Z_1^u = Z_1^v$,
(iv) *$u \in Z_1^v$ and $v \in Z_1^u$,*
(v) *$u \in Z_1^v$ invertible or $v \in Z_1^u$ invertible.*

Here, invertibility is to be understood in the sense of the respective Jordan algebras.

Proof. The equivalence of (i) and (ii) is evident, since self-adjoint operators coincide if and only if their eigenvalues and the respective eigenspaces coincide. Also, (i) immediately implies (iii). The converse is stated and proved in Remark 2.12. (iii) implies (iv). For the converse, we first note that a Peirce 1-space always satisfies the relation $Z_1^u = Q_u Z$, since $Q_u Q_{u^\dagger}$ is the orthogonal projection onto Z_1^u. Therefore, $u \in Z_1^v$ can be written as $u = Q_v x$ for some $x \in Z$. Using the quadratic formula, we obtain $Z_1^u = Q_u Z = Q_{Q_v x} Z = Q_v Q_x Q_v Z \subset Q_v Z = Z_1^v$. Applying to $v \in Z_1^u$ the same argument yields the converse inclusion, and hence the two Peirce 1-spaces coincide. It remains to prove the equivalence of (v) to the others. Assume (i)-(iv). By (iv), v is an element of Z_1^u. Due to Lemma 2.16, the invertibility of v in Z_1^u is equivalent to the invertibility of Q_v on Z_1^u. By (ii), the Peirce spaces Z_1^u and Z_1^v coincide, so the inverse $Q_{v^\dagger}$ of Q_v on Z_1^v is also the inverse of Q_v on Z_1^u. Finally, assume (v), i.e. without restriction, we assume v to be invertible in Z_1^u. Then, again by Lemma 2.16, Q_v is invertible on Z_1^u. In particular, Q_v is a surjection onto Z_1^u. Therefore, u can be written as $u = Q_v x$ for some $x \in Z_1^u$. Since $Z_1^v = Q_v Z$, this implies that u is also an element of Z_1^v, and hence (iv) is satisfied. □

Remark 2.20. Due to the fifth characterization of the Peirce decomposition, the Peirce equivalence class $[u]$ of an element $u \in Z$ is given by $[u] = (Z_1^u)^\times$, the invertible elements of the unital Jordan algebra Z_1^u. By Corollary 1.4, this implies that the equivalence class $[u]$ is contained within the set of constant rank elements Z_j with $j = \mathrm{rk}(u)$. Thus, the partition of Z into subsets of constant rank, $Z = \dot{\bigcup} Z_j$, is compatible with the Peirce equivalence relation. Therefore, the set of Peirce equivalence classes $\mathbb{P}$ decomposes into

$$\mathbb{P} = \dot{\bigcup_{j=0}^{r}} \mathbb{P}_j \quad \text{with} \quad \mathbb{P}_j = Z_j / \approx \,, \tag{2.34}$$

where r denotes the rank of Z.

Corollary 2.21. *Every element $z \in Z$ is Peirce equivalent to some tripotent. More precisely, if $z = \sum_k \lambda_k e_k$ is the spectral decomposition of z, then z is Peirce equivalent to its base-tripotent $\boldsymbol{\epsilon}(z) = \sum e_k$.*

Finally, we generalize the Peirce equivalence to a partial order on Z, defined by

$$u \subset \tilde{u} \iff Z_1^u \subset Z_1^{\tilde{u}} \,. \tag{2.35}$$

We call a tuple $(u_1, \dots, u_k)$ of elements in Z *Peirce ordered*, if $u_1 \subset \dots \subset u_k$ holds, i.e. if $Z_1^{u_1} \subset \dots \subset Z_1^{u_k}$. This partial order plays a particular role in this thesis, since it forms the base for the definition of pre-Peirce manifolds (Section 3.3), Peirce flag varieties (Section 6.4) and Jordan flag varieties (Section 8.3).

2.7. Bergman operators and the quasi-inverse

One of the most important operators in Jordan theory is the *Bergman operator*, defined by

$$B_{z,w} := \mathrm{Id} - 2\, z \,\Box\, w + Q_z Q_w \in \mathrm{End}(Z) \tag{2.36}$$

for any pair $(z,w) \in Z \times \overline{Z}$. The Bergman operator is closely related to the geometry, which is given by the phJTS Z, see Section 3.5. The pair (z,w) is said to be *quasi-invertible* if $B_{z,w}$ is invertible. In this case, the element

$$z^w := B_{z,w}^{-1}(z - Q_z w) \tag{2.37}$$

is the *quasi-inverse* of (z,w). The pair (z,w) is quasi-invertible if and only if z is quasi-invertible in the unital extension $\mathbb{C} \cdot 1 \oplus Z^{(w)}$ of the Jordan algebra $Z^{(w)}$, i.e. $1 - z$ is invertible, and $(1-z)^{-1} = 1 + z^w$. This justifies the term 'quasi-inverse', see also [**27**, §3]. A pair (z,w) is said to be *nilpotent*, if z is a nilpotent element of the Jordan algebra $Z^{(w)}$. Let $z^{(n,w)}$ denote the n-th power of z in $Z^{(w)}$.

Lemma 2.22. *If (z,w) is nilpotent, then (z,w) is quasi-invertible, and in this case*

$$z^w = \sum_{n=1}^{\infty} z^{(n,w)} \; . \tag{2.38}$$

In particular, let $u \in Z$ be some element of Z, and let $z \in Z^u_{1/2} \oplus Z^u_0$ and $w \in Z^u_1$. Then, the pair (z,w) is nilpotent, and $z^w = z + Q_z w$.

For the proof of the first part, we refer to [**27**, §3.8], the second part follows immediately from the Peirce rules. For calculations with quasi-inverses the following well-known formulas are essential:

Symmetry formula. Let $(z,w) \in Z \times \overline{Z}$. Then (z,w) is quasi-invertible if and only if (w,z) is quasi-invertible, and in this case,

$$z^w = z + Q_z(w^z) \; .$$

Shifting formulas. Let (u,v), $(z,w) \in Z \times \overline{Z}$.

(a) $(z, Q_v u)$ is quasi-invertible if and only if $(Q_v z, u)$ is quasi-invertible, and in this case,

$$Q_v\big(z^{Q_v u}\big) = \big(Q_v z\big)^u \; .$$

(b) $(z, B_{v,u} w)$ is quasi-invertible if and only if $(B_{u,v} z, w)$ is quasi-invertible, and in this case,

$$B_{u,v}\big(z^{B_{v,u} w}\big) = \big(B_{u,v} z\big)^w \; .$$

Addition formulas. Let $(z,w) \in Z \times \overline{Z}$ be quasi-invertible and let $(u,v) \in Z \times \overline{Z}$.

(a) $(z, w+v)$ is quasi-invertible if and only if (z^w, v) quasi-invertible, and in this case,

$$z^{w+v} = (z^w)^v \; .$$

(b) $(z+u, w)$ is quasi-invertible if and only if (u, w^z) is quasi-invertible, and in this case,

$$(z+u)^w = z^w + B_{z,w}^{-1}\big(u^{(w^z)}\big) \; .$$

We note that the addition formulas rely on the following relations for the Bergman operator:

$$B_{z,w} B_{z^w,v} = B_{z,w+v} \; , \quad B_{z,w^u} B_{u,w} = B_{z+u,w} \; , \quad B_{z,w}^{-1} = B_{z^w,-w} = B_{-z,w^z} \; .$$

We refer to these formulas by JT33 to JT35, cf. the list of identities in Appendix A.

Next we recall the notions of denominators and corresponding numerators of the quasi-inverse, cf.[**28**, §7.4]. Clearly the map $(z,w) \to z^w$ is a rational map from $Z \times \overline{Z}$ to Z. A polynomial function

$$\delta : Z \times \overline{Z} \to \mathbb{C}$$

is called a *denominator* of the quasi-inverse, if it is normalized such that $\delta(0,0) = 1$, and if it satisfies

(i) $\delta(z,w) \neq 0$ if and only if (z,w) is quasi-invertible, and
(ii) $\nu(z,w) := \delta(z,w) \cdot z^w$ is a Z-valued polynomial map, called the *numerator* of z^w (with respect to δ).

For example, we can take $\delta(z,w) = \det B_{z,w}$ and $\nu(z,w) = (B_{z,w})^{\mathrm{ad}}(z - Q_z w)$, where $(\;)^{\mathrm{ad}}$ denotes the adjoint matrix. The unique minimal denominator obtained by canceling all common factors of δ and ν is called the *generic norm* or (as we will do) the *Jordan triple determinant* of Z, we denote $\Delta(z,w)$. Any pair of denominator and corresponding numerator satisfies some fundamental relations:

Lemma 2.23. [**28**, §7.5] *Let δ be a denominator of the quasi-inverse and ν the corresponding numerator. Then*

$$\begin{aligned} \delta(tx,y) &= \delta(x,ty) \;, \\ \delta(x,y)\delta(x^y,z) &= \delta(x,y+z) \;, \\ \nu(tx,y) &= t\,\nu(x,ty) \;, \\ \delta(x,y)\nu(x^y,z) &= \nu(x,y+z) \;, \end{aligned}$$

for all $x \in Z$, y, $z \in \overline{Z}$, $t \in \mathbb{C}$.

In the same way as the relations of Lemma 2.23 one proves the identities

$$\delta\left(x, B_{v,u}y\right) = \delta\left(B_{u,v}x, y\right) \;, \quad \delta\left(Q_x y, v\right) = \delta\left(Q_x v, y\right) \tag{2.39}$$

for all $(x,y),(u,v) \in Z \times \overline{Z}$.

Example 2.24. In the matrix case $Z = \mathbb{C}^{r\times s}$, the Bergman operator $B_{z,w}$ satisfies

$$B_{z,w}x = (1 - zw^*)x(1 - w^*z) \;. \tag{2.40}$$

Therefore, by a standard result of linear algebra, the Bergman operator is invertible if and only if the matrix $1 - zw^*$ (or equivalently the matrix $1 - w^*z$) is invertible. In this case, the quasi-inverse of (z,w) is given by

$$z^w = (1 - zw^*)^{-1}z = z(1 - w^*z)^{-1} \;. \tag{2.41}$$

From this it follows that $\Delta(z,w) = \det(1 - zw^*)$ is the Jordan triple determinant.

The following lemma relates the quasi-inverse with Peirce decompositions even in the case, where (z,w) is not nilpotent.

Lemma 2.25. *Let $z = z_1 + z_{1/2} + z_0$ and $w = w_1 + w_{1/2} + w_0$ be the components of $z, w \in Z$ in the Peirce decomposition with respect to $u \in Z$. Then for $\nu = 0$ and 1:*

(a) *(z_ν, w) is quasi-invertible if and only if (z_ν, w_ν) is quasi-invertible, and then*

$$z_\nu{}^w = z_\nu{}^{w_\nu} \;.$$

In particular, $z_\nu{}^w \in Z^u_\nu$ for all $w \in Z$.

(b) *$(z_1 + z_0, w_\nu)$ is quasi-invertible if and only if (z_ν, w_ν) is quasi-invertible, and then*

$$(z_1 + z_0)^{w_\nu} = z_\nu{}^{w_\nu} + z_{1-\nu} \;.$$

PROOF. Recall from Section 2.5 that the orthogonal projections of Z onto the Peirce spaces Z_1^u and Z_0^u are given by $Q_uQ_{u^\dagger}$ and $\mathrm{Id} - 2\,u\square u^\dagger - Q_uQ_{u^\dagger}$. We note that the second one coincides with the Bergman operator $B_{u,\,u^\dagger}$. Therefore, $z_1 = Q_uQ_{u^\dagger}z$ and $z_0 = B_{u,\,u^\dagger}z$. In addition, the defining properties of the pseudo-inverse yield the relations $B_{u,\,u^\dagger} = B_{u^\dagger,\,u}$ and $Q_uQ_{u^\dagger} = Q_{u^\dagger}Q_u$. Hence, part (a) follows from the shifting formulas. For (b), we consider the case $\nu = 1$. Due to the symmetry formula and part (a), we obtain

$$(z_1 + z_0)^{w_1} = z_1 + z_0 + Q_{z_1+z_0}(w_1^{z_1+z_0}) = z_1 + z_0 + Q_{z_1+z_0}(w_1^{z_1}) \,.$$

Since $w_1^{z_1}$ is an element of Z_1^u, the Peirce rules imply that $Q_{z_1+z_0}$ can be replaced by Q_{z_1}. Applying the symmetry formula once again, it follows

$$(z_1 + z_0)^{w_1} = z_1 + z_0 + Q_{z_1}(w_1^{z_1}) = z_1 + z_0 + z_1^{w_1} - z_1 = z_1^{w_1} + z_0 \,.$$

The corresponding statement with $\nu = 0$ is proved analogously. □

The next (quite technical) lemma gives a criterion for the quasi-invertibility of pairs of the form $(u, u^\dagger - z)$, it is used in several subsequent sections. In some ways, it relates quasi-invertibility in Z with invertibility in the unital Jordan algebra Z_1^u. This lemma is an essential tool in the proof of Theorem 4.12.

LEMMA 2.26. *Let $z = z_1 + z_{1/2} + z_0$ be the components of $z \in Z$ in the Peirce spaces corresponding to $u \in Z$. Then:*

(a) *$B_{u,\,u^\dagger-z}$ is invertible if and only if z_1 is invertible as an element of the Jordan algebra Z_1^u (with product $x \circ y = \{x,\,u,\,y\}$). In this case,*

$$B_{u,\,u^\dagger-z}^{-1} = B_{u,\,u^\dagger-(z_1^{-1}-z_1^{-1}\circ z_{1/2})} \quad \textit{and} \quad u^{u^\dagger-z} = {z_1}^{\dagger} = Q_u z_1^{-1} \,.$$

(b) *If $z \in Z_1^u$, then the Peirce spaces Z_ν^u are invariant under $B_{u,\,u^\dagger-z}$, more precisely*

$$B_{u,\,u^\dagger-z} = Q_uQ_z\big|_{Z_1^u} \oplus\ 2\,u\,\square\,z\big|_{Z_{1/2}^u} \oplus \mathrm{Id}\big|_{Z_0^u} \,,$$

and $B_{u,\,u^\dagger-z}$ is invertible if and only if Q_z is invertible on Z_1^u; in this case, $2\,u\,\square\,z^{-1}$ is the inverse of $2\,u\,\square\,z$ on $Z_{1/2}^u$.

PROOF. First we note that by Lemma 2.25, the Bergman operator $B_{u,\,u^\dagger-z}$ is invertible if and only if $B_{u,\,u^\dagger-z_1}$ is invertible. For this operator we obtain by using the Peirce rules and the Jordan identity the relation

$$B_{u,\,u^\dagger+z_1}v = v_0 - 2\,u\,\square\,z_1(v_{1/2}) + Q_uQ_{z_1}v_1 \quad \text{for} \quad v = v_1 + v_{1/2} + v_0 \,.$$

This proves the first part of (b). Let $B_{u,\,u^\dagger-z}$ be invertible. Since Q_u is invertible on Z_1^u, it follows that also Q_{z_1} is invertible on Z_1^u, and hence Lemma 2.16 implies that z_1 is invertible in Z_1^u. Conversely, assume the invertibility of z_1 in Z_1^u, then it remains to show that $2\,u\square z_1$ is invertible on $Z_{1/2}^u$. Using JT13, the Peirce rules and Lemma 2.16, we obtain for $v \in Z_{1/2}^u$

$$2\,u\,\square\,z_1^{-1} \circ u\,\square\,z_1(v) = (Q_u z_1^{-1})\,\square\,z_1(v) = \left\{z_1^\dagger,\,z_1,\,v\right\} \,.$$

By Proposition 2.19, we have $z_1^\dagger\,\square\,z_1 = u\,\square\,u^\dagger$, thus $4\,u\,\square\,z_1^{-1} \circ u\,\square\,z_1(v) = v$ for all $v \in Z_{1/2}^u$. Since $Z_{1/2}^u$ is finite dimensional, we conclude that $2\,u\,\square\,z_1$ is invertible on $Z_{1/2}^u$ with inverse given by $2\,u\,\square\,z_1^{-1}$. It remains to determine the formulas for $B_{u,\,u^\dagger-z}^{-1}$ and $u^{u^\dagger-z}$. Due to Lemma 2.16, we obtain

$$\begin{aligned} u^{u^\dagger-z} &= u^{u^\dagger-z_1} = B_{u,\,u^\dagger-z_1}^{-1}(u - Q_u(u^\dagger - z_1)) = B_{u,\,u^\dagger-z_1}^{-1}Q_u z_1 \\ &= (Q_uQ_{z_1})^{-1}Q_u z_1 = Q_{z_1}^{-1}z_1 = z_1^\dagger = Q_u z_1^{-1} \,. \end{aligned}$$

To determine the Bergman operator $B_{u,\,u^\dagger - z} = B_{-u,\,(u^\dagger - z)^u}$, we use the symmetry formula and calculate $(u^\dagger - z)^u$ modulo Z_0^u, since due to the Peirce rules, terms in Z_0^u do not contribute to the Bergman operator. Using also the relation $z_1^\dagger \,\square\, z_1 = u \,\square\, u^\dagger$ once again, we get

$$\begin{aligned}
(u^\dagger - z)^u &= u^\dagger - z + Q_{u^\dagger - z}(u^{u^\dagger - z}) \\
&= u^\dagger - z + Q_{u^\dagger - z} z_1^\dagger \\
&= u^\dagger - z + Q_{u^\dagger} z_1^\dagger - 2\,\{u^\dagger,\, z_1^\dagger,\, z\} + Q_z z_1^\dagger \\
&\equiv u^\dagger - z_1 - z_{1/2} + z_1^{-1} - 2\{u^\dagger,\, z_1^\dagger,\, z_1\} - 2\{u^\dagger,\, z_1^\dagger,\, z_{1/2}\} + 2\{z_{1/2},\, z_1^\dagger,\, z_1\} \\
&\quad + \{z_1,\, z_1^\dagger,\, z_1\} \\
&= -u^\dagger + z_1^{-1} - 2\{u^\dagger,\, Q_u z_1^{-1},\, z_{1/2}\} \\
&= -u^\dagger + z_1^{-1} - \{z_1^{-1},\, u,\, z_{1/2}\}\ .
\end{aligned}$$

For the last step, we used the Jordan identity. Finally, we note that the Bergman operator admits the shifting of the sign from the first to the second entry. This completes the proof of Lemma 2.26. □

2.8. Morphisms and the structure group

In this section, we briefly recall the basic notions and result on homomorphisms of positive hermitian Jordan triple systems. Let Z and Z' be phJTS. Then a *homomorphism* of Z to Z' is $\mathbb{C}$-linear map $f : Z \to Z'$ satisfying

$$f(\{x,\, y,\, z\}) = \{f(x),\, f(y),\, f(z)\}' \quad \text{for all} \quad x, y, z \in Z\ , \tag{2.42}$$

where $\{\ ,\ ,\ \}$ and $\{\ ,\ ,\ \}'$ denote the triple products on Z and Z'. If f is invertible, then f is called an *isomorphism*, and f^{-1} is an isomorphism of Z' onto Z. More generally, a *structure homomorphism* is a pair (f, g) of $\mathbb{C}$-linear maps $f, g : Z \to Z'$ satisfying

$$f(\{x,\, y,\, z\}) = \{f(x),\, g(y),\, f(z)\}' \quad \text{for all} \quad x, y, z \in Z\ , \tag{2.43}$$

If f is invertible, then g is invertible, and $g = f^{-*}$, where $f^{-*} = (f^*)^{-1}$, and f^* denotes the adjoint map of f with respect to the scalar products on Z and Z' defined in (2.4). In this case, f is called a *structure isomorphism* of Z and Z'.

Specializing to the case $Z = Z'$, we denote by $\mathrm{Aut}(Z)$ the group of automorphisms of Z, called the *automorphism group*, and by $\mathrm{Str}(Z)$ the group of structure automorphisms on Z, called the *structure group*. A structure automorphism f is an automorphism if and only if $f^{-*} = f$, i.e. if f is a unitary map with respect to the scalar product on Z. Therefore, $\mathrm{Aut}(Z)$ is a closed subgroup of the unitary group on Z. Moreover, $\mathrm{Aut}(Z)$ is a compact real form of $\mathrm{Str}(Z)$. Let K denote the identity component of $\mathrm{Aut}(Z)$, then the identity component of $\mathrm{Str}(Z)$ is just the complexification of K [**28**, §3.2],

$$K = \mathrm{Aut}(Z)^0 \quad \text{and} \quad K^{\mathbb{C}} = \mathrm{Str}(Z)^0\ . \tag{2.44}$$

Elements of K are usually denoted by k and elements of $K^{\mathbb{C}}$ by h. A *derivation* on Z is an endomorphism $\delta \in \mathrm{End}(Z)$, such that

$$\delta(\{x,\, y,\, z\}) = \{\delta(x),\, y,\, z\} + \{x,\, \delta(y),\, z\} + \{x,\, y,\, \delta(z)\} \tag{2.45}$$

for all $x, y, z \in Z$. Let $\mathrm{Der}(Z)$ denote the set of all derivations on Z. It is a real vector subspace of the endomorphisms on Z, which is invariant under the commutator $[f, g] = f \circ g - g \circ f$ on $\mathrm{End}(Z)$, thus $\mathrm{Der}(Z)$ is a Lie algebra. The Lie algebra $\mathfrak{k}$ of K is identified with $\mathrm{Der}(Z)$. Accordingly, the Lie algebra $\mathfrak{k}^{\mathbb{C}}$ of $K^{\mathbb{C}}$ is identified with the complexified derivation algebra $\mathrm{Der}(Z)^{\mathbb{C}}$ [**28**, §3.2]. Therefore, the elements of $\mathfrak{k}^{\mathbb{C}}$ are characterized by the relation

$$\delta(\{x,\, y,\, z\}) = \{\delta(x),\, y,\, z\} - \{x,\, \delta^*(y),\, z\} + \{x,\, y,\, \delta(z)\} \tag{2.46}$$

for all $x, y, z \in Z$. In addition, we have

$$\mathfrak{k} = \left\{ \delta \in \mathfrak{k}^{\mathbb{C}} \,\middle|\, \delta = -\delta^* \right\} .$$

In particular, it follows that

$$x \,\Box\, y \in \mathfrak{k}^{\mathbb{C}} \quad \text{and} \quad i\, x \,\Box\, x,\; x \,\Box\, y - y \,\Box\, x \in \mathfrak{k}$$

for all $x, y \in Z$. The elements $x \Box y$ are called *inner derivations*, and it turns out [**28**, §8.9], that $\mathfrak{k}^{\mathbb{C}}$ is generated by the inner derivations. Analogously, JT26 implies that if $(x, y) \in Z \times \overline{Z}$ is quasi-invertible, then the Bergman operator $B_{x,y}$ is a structure automorphism of Z, called an *inner automorphism*, and it turns out that the set of all inner automorphisms generate the identity component $K^{\mathbb{C}}$ of the structure group $\mathrm{Str}(Z)$ [**28**, §8.9].

Example 2.27. In the matrix case $Z = \mathbb{C}^{r\times s}$, the identity component $K^{\mathbb{C}}$ of the structure group $\mathrm{Str}(Z)$ is isomorphic to the projective group $\mathrm{P}(\mathrm{GL}_r \times \mathrm{GL}_s)$, which acts on Z via

$$\mathrm{P}(\mathrm{GL}_r \times \mathrm{GL}_s) \times Z \to Z, \big([A, D], z\big) \mapsto AzD^{-1} . \tag{2.47}$$

One often considers instead of the projective group the finite cover $\mathrm{S}(\mathrm{GL}_r \times \mathrm{GL}_s)$. The kernel of the canonical projection onto $\mathrm{P}(\mathrm{GL}_r \times \mathrm{GL}_s)$ is the set of all $(\lambda\ \mathrm{Id}, \lambda\ \mathrm{Id})$ with $\lambda^{r+s} = 1$, thus

$$K^{\mathbb{C}} \cong \mathrm{P}(\mathrm{GL}_r \times \mathrm{GL}_s) \cong \mathrm{S}(\mathrm{GL}_r \times \mathrm{GL}_s)/\langle\zeta\rangle , \tag{2.48}$$

where ζ is a primitive $(r+s)$-th root of unity. Since $\langle x|y\rangle = \frac{r+s}{2}\,\mathrm{Tr}(xy^*)$, the adjoint of a structure automorphism $h = (A, D)$ is given by $h^{-*} = (A^{-*}, D^{-*})$, where A^{-*} denotes the inverse of the complex conjugate of A. Therefore, the identity component K of the automorphism group $\mathrm{Aut}(Z)$ is given by

$$K \cong \mathrm{P}(U_r \times U_s) \cong \mathrm{S}(U_r \times U_s)/\langle\zeta\rangle .$$

Next we recall some results on structure homomorphisms, which are used in several parts of this thesis.

Proposition 2.28 ([**28**, §7.3])**.** *Let Z and Z' be phJTS. If (f, g) is a structure homomorphism, and if $(z, w) \in Z \times \overline{Z}$ is quasi-invertible, then $(f(z), g(w))$ is also quasi-invertible, and*

$$f(z^w) = (f(z))^{g(z)} .$$

In particular, if $Z = Z'$ this holds for structure automorphisms $(f, g) = (h, h^{-*})$.

Applying Proposition 2.28 to the canonical imbedding of a subtriple $W \subset Z$ into Z yields the following result:

Corollary 2.29. *Let W be a subtriple of Z and $z, w \in W$. Then (z, w) is quasi-invertible in W if and only if (z, w) is quasi-invertible in Z.*

A further consequence of Proposition 2.28 is the $\mathrm{Str}(Z)$-invariance of a denominator of the quasi-inverse.

Corollary 2.30. *Let δ be a denominator of the quasi-inverse, and let $h \in \mathrm{Str}(Z)$ be a structure automorphism. Then,*

$$\delta(hz, h^{-*}w) = \delta(z, w)$$

for all $(z, w) \in Z \times \overline{Z}$.

Next we investigate the compatibility of the generalized Peirce decomposition with respect to an element $u \in Z$ with the action of the automorphism group and the structure group. For an automorphism $k \in \mathrm{Aut}(Z)$ it is straightforward to show that

$$(ku)^\dagger = k\,u^\dagger\ , \tag{2.49}$$

i.e. the pseudo-inverse map $u \mapsto u^\dagger$ is $\mathrm{Aut}(Z)$-equivariant. It immediately follows that $(ku) \,\square\, (ku)^\dagger = k \circ (u \,\square\, u^\dagger) \circ k^{-1}$. Therefore, we obtain for the Peirce spaces with respect to ku the relations

$$Z_1^{ku} = kZ_1^u\ , \quad Z_{1/2}^{ku} = kZ_{1/2}^u\ , \quad Z_0^{ku} = kZ_0^u\ . \tag{2.50}$$

In the case of a structure automorphism $h \in \mathrm{Str}(Z)$, the situation is considerably more complicated. A first candidate for the pseudo-inverse of hu seems to be $h^{-*}u^\dagger$ (cf. **[18]**), since the pair $(hu, h^{-*}u^\dagger)$ is an idempotent, i.e. satisfies the relations $hu = Q_{hu}(h^{-*}u)$ and $h^{-*}u = Q_{h^{-*}u}(hu)$. However, the third defining relation of the pseudo-inverse, $(hu) \,\square\, (hu)^\dagger = (hu)^\dagger \,\square\, (hu)$, is not satisfied in general. We obtain the following condition:

$$(hu)^\dagger = h^{-*}u^\dagger \iff [h^*h,\, u \,\square\, u^\dagger] = 0\ , \tag{2.51}$$

i.e. the Peirce spaces Z_ν^u need to be h^*h-invariant.

Example 2.31. The following is a typical example, in which $(hu)^\dagger$ differs from $h^{-*}u^\dagger$: Take $h = \exp(2\,v \,\square\, u^\dagger)$ with $v \in Z_{1/2}^u$. Then, $h^{-*} = \exp(2\,u^\dagger \,\square\, v)$, and by the Peirce rules this yields

$$hu = u + v + \left\{x,\, u^\dagger,\, v\right\} \quad \text{and} \quad h^{-*}u^\dagger = u^\dagger\ . \tag{2.52}$$

Since $hu \neq u$ for $v \neq 0$, this implies $h^{-*}u^\dagger \neq (hu)^\dagger$.

Even though the pseudo-inverse of hu admits no simple formula involving h and $u^\dagger$, the following lemma states a crucial relation between these terms. In addition, using this relation we prove a simple relation between the Peirce spaces of u and hu..

Lemma 2.32. *Let $u \in Z$ and $h \in \mathrm{Str}(Z)$. Then*

$$u^\dagger = (h^*(hu)^\dagger)_1\ , \quad Z_1^{hu} = hZ_1^u \quad \text{and} \quad Z_0^{hu} = h^{-*}Z_0^u\ .$$

Here $(h^*(hu)^\dagger)_1$ denotes the Z_1^u-component of $h^*(hu)^\dagger$. In particular, two elements $u, \tilde{u} \in Z$ are Peirce equivalent if and only if hu and $h\tilde{u}$ are Peirce equivalent.

Proof. According to one of the defining relations for the pseudo-inverse we have $\{hu,\, (hu)^\dagger,\, hu\} = hu$, and since h is an element of the structure group, this implies $\{u,\, h^*(hu)^\dagger,\, u\} = u$. Applying $Q_{u^\dagger}$ to this equation yields

$$Q_{u^\dagger}Q_u(h^*(hu)^\dagger) = u^\dagger\ ,$$

this is the first assertion. For $v \in Z_1^u$ we calculate

$$\left(hu \,\square\, (hu)^\dagger\right) hv = \left\{hu,\, (hu)^\dagger,\, hv\right\} = h\left\{u,\, h^*(hu)^\dagger,\, v\right\} = h\left\{u,\, u^\dagger,\, v\right\} = hv\ ,$$

where we replaced $h^*(hu)^\dagger$ due to the first assertion and the Peirce rules by $u^\dagger$. Therefore hZ_1^u is a subset of Z_1^{hu}. The reverse inclusion follows from this by the substitution $h \to h^{-1}$, $u \to hu$. The last equation follows from the relation $Z_0^{hu} = (Z_1^{hu})^{\perp\!\!\perp}$ and the general assertion, that for any subspace $U \subset Z$ we have $(hU)^{\perp\!\!\perp} = h^{-*}U^{\perp\!\!\perp}$, which can easily be verified. The additional statement about Peirce equivalence is a simple application of the identity $Z_1^{hu} = hZ_1^u$ and Proposition 2.19. $\square$

REMARK 2.33. We note that there is not such a simple relation between the Peirce $1/2$-spaces of u and hu as it is given for the Peirce 1- and the Peirce 0-spaces. However, by orthogonality we have

$$Z_{1/2}^{hu} = (Z_1^{hu} \oplus Z_0^{hu})^\perp = (hZ_1^u \oplus h^{-*} Z_0^u)^\perp ,$$

see also (2.26). In addition, it should be noted that in general the Peirce projections $Q_{hu}Q_{(hu)^\dagger}$ and $B_{hu,(hu)^\dagger}$ onto Z_1^{hu} and Z_0^{hu} do not coincide with the terms $hQ_uQ_{u^\dagger}h^{-1}$ and $h^{-*}B_{u,u^\dagger}h^*$. Even though the latter terms define projections onto the Peirce spaces Z_1^{hu} and Z_0^{hu}, in general they are not self-adjoint, and hence, these projections differ from the orthogonal ones.

As a first application of Lemma 2.32, we prove that the subsets of constant rank elements are invariant under the action of the structure group. Further application can be found e.g. in Sections 2.9, 3.3, 6.2 and 8.3.

PROPOSITION 2.34. *Let Z be a hermitian Jordan triple system. The subsets $Z_j \subset Z$ of rank-j elements are invariant under the action of the structure group* $\mathrm{Str}(Z)$.

PROOF. We note that the rank of an element $u \in Z$ equals to the rank of the Jordan algebra Z_1^u, and since u and its pseudo-inverse $u^\dagger$ have the same rank, this also equals to the rank of the Jordan algebra $Z_1^{u^\dagger}$. We claim that for $h \in \mathrm{Str}(Z)$, the map

$$h : Z_1^{u^\dagger} \to Z_1^{(hu)^\dagger} ,\ x \mapsto hx$$

is an isomorphism of Jordan algebras. Indeed, due to Lemma 2.32 and using the Peirce rules, we have $hQ_xu^\dagger = hQ_xh^*(hu)^\dagger = Q_{hx}(hu)^\dagger$ for all $x \in Z_1^{u^\dagger}$. Polarization now implies that $h(x \circ_{u^\dagger} y) = (hx) \circ_{(hu)^\dagger} (hy)$ for all $x, y \in Z_1^{u^\dagger}$. Therefore, we obtain

$$\mathrm{rk}\, u = \mathrm{rk}\, u^\dagger = \mathrm{rk}\, Z_1^{u^\dagger} = \mathrm{rk}\, Z_1^{(hu)^\dagger} = \mathrm{rk}(hu)^\dagger = \mathrm{rk}\, hu .$$

This completes the proof. □

2.9. Induced Jordan algebra denominators

Recall from Section 2.5 that the Peirce 1-space Z_1^u is a unital Jordan algebra with product $x \circ y = \{x, u, y\}$ and unit element $u^\dagger$, the pseudo-inverse of u. Choosing a different element $\tilde{u} \in Z$ which is Peirce equivalent to u, yields a different Jordan algebra structure on $Z_1^{\tilde{u}} = Z_1^u$. The next proposition shows that any denominator of the quasi-inverse on Z induces denominators of the Jordan algebras Z_1^u and relates denominators of Jordan algebras defined by Peirce equivalent elements.

PROPOSITION 2.35. *Let Z be a phJTS, and let δ be a denominator of the quasi-inverse with corresponding nominator ν. For any $u \in Z$, define*

$$\delta^u : Z \to \mathbb{C},\ z \mapsto \delta(u^\dagger - z, u) .$$

Then, the restriction of δ^u to the unital Jordan algebra Z_1^u is a denominator of the inverse, and the restriction of δ^u to $Z_{1/2}^u \oplus Z_0^u$ is the vanishing map. We call δ^u the induced Jordan algebra denominator *corresponding to δ and u. Moreover, if u and $\tilde{u}$ are Peirce equivalent element, then*

$$\delta^u(z) = \delta^{\tilde{u}}\big(u^\dagger\big)^{-1} \cdot \delta^{\tilde{u}}(z) \tag{2.53}$$

for all $z \in Z$.

PROOF. For any $x \in Z_1^u$, it follows by the properties of δ that $\delta^u(x)$ is non-vanishing if and only if the pair $(u^\dagger - x, u)$ is quasi-invertible. Due to Lemma 2.26, this is equivalent to the invertibility of x in Z_1^u. Furthermore, Lemma 2.26 yields that

$$\nu^u(x) = \delta^u(x) \cdot x^{-1} = \delta^u(x)\, Q_{u^\dagger} u^{u^\dagger - z} = Q_{u^\dagger}\delta(u,\, u^\dagger - z)\, u^{u^\dagger - z} = Q_{u^\dagger}\nu(u,\, u^\dagger - x)$$

is a well-defined (complex) polynomial in $x \in Z_1^u$. Finally, δ^u is also normalized, since $\delta^u\big(u^\dagger\big) = \delta(0, u) = 1$. Therefore, δ^u is a denominator of the inverse in Z_1^u. For $z \in Z_{1/2}^u \oplus Z_0^u$ we obtain $\delta^u(z) = 0$, since $(u - z, u^\dagger)$ is not quasi-invertible. Now let u and $\tilde{u}$ be Peirce equivalent. Again using Lemma 2.26, this implies that the pair $(\tilde{u},\, \tilde{u}^\dagger - u^\dagger)$ is quasi-invertible with quasi-inverse $\tilde{u}^{\tilde{u}^\dagger - u^\dagger} = u$. Therefore, using Lemma 2.23 we obtain

$$\begin{aligned}\delta^{\tilde{u}}(z) &= \delta(\tilde{u}^\dagger - z,\, \tilde{u}) = \delta(\tilde{u}^\dagger - u^\dagger + (u^\dagger - z),\, \tilde{u})\\ &= \delta(\tilde{u}^\dagger - u^\dagger,\, u) \cdot \delta(u^\dagger - z,\, \tilde{u}^{\tilde{u}^\dagger - u^\dagger}) = \delta^{\tilde{u}}\big(u^\dagger\big) \cdot \delta(u^\dagger - z,\, u) = \delta^{\tilde{u}}\big(u^\dagger\big) \cdot \delta^u(z)\end{aligned}$$

for all $z \in Z$. This completes the proof. □

COROLLARY 2.36. *Let Δ denote the Jordan triple determinant of Z, and for $u \in Z$ let Δ^u denote the Jordan algebra determinant of the unital Jordan algebra Z_1^u. Then*

$$\Delta^u(x) = \Delta\left(u^\dagger - x, u\right)$$

for all $x \in Z_1^u$.

Next, we investigate the properties of the induced Jordan algebra denominators concerning the action of the structure group.

PROPOSITION 2.37. *Let Z be a phJTS, δ be a denominator of the quasi-inverse. Then, for $u \in Z$ and a structure automorphism $h \in \mathrm{Str}(Z)$, the induced Jordan algebra determinant satisfies*

$$\delta^u(hz) = \delta^{h^* u}(z) \tag{2.54}$$

for all $z \in Z$. Moreover, if $h^* u \in Z_1^u$, then

$$\delta^u(hz) = \delta^u\big(hu^\dagger\big) \cdot \delta^u(z)$$

for all $z \in Z$.

PROOF. To prove the first relation, recall from Corollary 2.30 that δ is invariant under the action of h^{-1}. Therefore,

$$\delta^u(hz) = \Delta\left(u^\dagger - hz, u\right) = \Delta\left(h^{-1}u^\dagger - z, h^* u\right)\ .$$

By Proposition 2.35, this term just depends on the $Z_1^{h^* u}$-component of $h^{-1}u^\dagger - z$. Therefore, Lemma 2.32 implies that $h^{-1}u^\dagger$ can be replaced by $(h^* u)^\dagger$, since $h^{-1}u^\dagger = h^{-1}(h^{-*}(h^* u))^\dagger$. This yields

$$\delta^u(hz) = \Delta\left(h^{-1}u^\dagger - z, h^* u\right) = \Delta\left((h^* u)^\dagger - z, h^* u\right) = \delta^{h^* u}(z)\ .$$

Now, assume that $h^* u$ is an element of Z_1^u. Since h^* preserves the rank of elements, $h^* u$ is invertible in Z_1^u, and hence $h^* u$ is Peirce equivalent to u. From Proposition 2.35 it follows

$$\delta^u(hz) = \delta^{h^* u}(z) = \delta^u\big((h^* u)^\dagger\big)^{-1} \cdot \delta^u(z) = \delta^{h^* u}\big(u^\dagger\big) \cdot \delta^u(z)\ ,$$

where we used the relation $\delta^u\big(\tilde{u}^\dagger\big)^{-1} = \delta^{\tilde{u}}\big(u^\dagger\big)$ obtained from (2.53) by setting $z = \tilde{u}^\dagger$. Finally, applying (2.54) to the first factor yields the assertion. □

REMARK 2.38. The second formula of Proposition 2.37 is a well-known formula in the theory of Jordan algebras [**8**, VIII §5]. It induces a character of the structure group $\mathrm{Str}(Z_1^u)$ of the unital Jordan algebra, given by $h \mapsto \Delta^u\big(hu^\dagger\big)$. This character has applications to the harmonic analysis on symmetric cones [**1**].

2.10. Simple Jordan triple systems and their classification

In this section we briefly recall the classification of simple phJTS. Recall that any semisimple phJTS is the direct sum of simple phJTS. One of the main properties of simple phJTS is the transitivity of the K-action on the set of frames on Z [**28**, §5.9]. This makes it possible to define to following *characteristic multiplicities*: If $(e_1, \dots, e_r)$ is a frame of Z ($\mathrm{rk}\, Z = r$), then the Peirce spaces of the corresponding joint Peirce decomposition satisfy

$$\left.\begin{array}{rcll} Z_{00} & = & \{0\} & \\ Z_{ii} & = & \mathbb{C}\, e_i & 1 \le i \le r \\ \dim_{\mathbb{C}} Z_{ij} & = & \mathsf{a} & 1 \le i < j \le r \\ \dim_{\mathbb{C}} Z_{0j} & = & \mathsf{b} & 1 \le j \le r \end{array}\right\} \quad \text{for fixed } \mathsf{a}, \mathsf{b} \in \mathbb{N}. \tag{2.55}$$

By the transitivity of the K-action on the frames, the multiplicities $\mathsf{a}, \mathsf{b} \in \mathbb{N}$ are independent of the choice of the frame, and thus indeed are characteristic for Z. If b vanishes, Z is said to be of *tube type*. The *genus* of Z is defined by

$$\mathsf{p} = 2 + \mathsf{a}(r-1) + \mathsf{b}\,. \tag{2.56}$$

If d denotes the dimension of Z, we have

$$\frac{d}{r} = 1 + \frac{\mathsf{a}}{2}(r-1) + \mathsf{b} = \frac{1}{2}(\mathsf{p} + \mathsf{b})\,,$$

and Z is of tube type if and only if $\frac{\mathsf{p}}{2} = \frac{d}{r}$. Again by counting, it follows that the dimensions of the Peirce spaces of some element $u \in Z$ are related to the rank of u via

$$\left.\begin{array}{l} \dim Z_1^u = k + \mathsf{a}\,\dfrac{k(k-1)}{2} \\ \dim Z_{1/2}^u = \mathsf{a}\,k\,(r-k) + \mathsf{b}\,k \\ \dim Z_0^u = (r-k) + \mathsf{a}\,\dfrac{(r-k)(r-k-1)}{2} + \mathsf{b}\,(r-k) \end{array}\right\} \quad \text{for } k = \mathrm{rk}(u).$$

A simple calculation shows that each of these identities can be solved for the rank of u, hence we obtain the following result.

COROLLARY 2.39. *If Z is simple, then the rank of an element $u \in Z$ is uniquely determined by the dimension of one of its Peirce spaces.*

Another fundamental fact on simple phJTS is the following relation between the Jordan triple determinant and the Bergman operator.

THEOREM 2.40 ([**27**, §17.3])**.** *Let Z be a phJTS. Then, the Jordan triple determinant Δ is irreducible, and satisfies*

$$\mathrm{Det}\, B_{x,y} = \Delta\,(x,y)^{\mathsf{p}}\,, \tag{2.57}$$

where p denotes the genus of Z.

We refer to **[27]** for the proof that the following list of simple phJTS is complete:

type	Z	triple product $\{x, y, z\}$	$\dim_{\mathbb{C}} Z$	$\mathrm{rk}(Z)$	a[5]	b[5]
$I_{r,s},\ r \le s$	$\mathbb{C}^{r\times s}$	$\frac{1}{2}(xy^*z + zy^*x)$	$r \cdot s$	r	2	$s - r$
$II_n,\ n$ even	$\mathbb{C}^{n\times n}_{\text{asym}}$	$\frac{1}{2}(xy^*z + zy^*x)$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}n$	4	0
$II_n,\ n$ odd	$\mathbb{C}^{n\times n}_{\text{asym}}$	$\frac{1}{2}(xy^*z + zy^*x)$	$\frac{1}{2}n(n-1)$	$\frac{1}{2}(n-1)$	4	2
III_n	$\mathbb{C}^{n\times n}_{\text{sym}}$	$\frac{1}{2}(xy^*z + zy^*x)$	$\frac{1}{2}n(n+1)$	n	1	0
IV_n	$\mathbb{C}^n$	$\langle x\vert y\rangle\, z + \langle z\vert y\rangle\, x - \langle x\vert\overline{z}\rangle\,\overline{y}$	n	2	$n-2$	0
V	$\mathbb{O}^{1\times 2}$	$Q_x y = x(y^* x)$	16	2	6	4
VI	$H_3(\mathbb{O})$	$Q_x y = \frac{1}{2}(x \circ (x \circ \overline{y}) - x^2 \circ \overline{y})$	27	3	8	0

In type IV_n, the term $\overline{z}$ denotes complex conjugation in each variable, and in type VI, the product $x \circ y$ is Jordan algebra product given by $x \circ y = \frac{1}{2}(xy + yx)$. The only isomorphisms among the types listed above are the following [**28**, §4.18]:

$$I_{1,1} \cong II_1 \cong III_1 \cong IV_1\ , \quad IV_2 \cong IV_1 \times IV_1\ , \quad I_{1,3} \cong II_3\ ,$$
$$III_2 \cong IV_3\ , \quad I_{2,2} \cong IV_4\ , \quad II_4 \cong IV_6\ .$$

The main ideas of this thesis are demonstrated on phJTS of type $I_{r,s}$. We call this type the 'matrix-case', even though phJTS of type II_n and III_n also consist of matrices (with additional conditions).

[5]For type II, III and IV, these entries are valid just with the following restrictions: in type II_n for $n \ge 4$, in type III_n for $n \ge 2$, and in type IV_n for $n \ge 3$. For lower indices consider the isomorphisms quoted below the table.

CHAPTER 3

Analytic structures

So far we have described the algebraic structure of Jordan algebras and positive hermitian Jordan triple systems. Now we prepare the study of analytic aspects in Jordan theory. In the first section, we recall some basic definitions and results concerning submanifolds.

The second section deals with equivalence relations and their analytic structures. This is of particular importance since there are naturally defined equivalence relations on Jordan triple systems. This section provides a method for proving that the corresponding sets of equivalence classes carry appropriate analytic structures (Godement's Theorem). The resulting manifolds are called quotient manifolds. We indicate how a local description is obtained from the global definition via equivalence classes and extend the "Godement approach" to a treatment of suitable vector bundles on quotient manifolds.

In the third section we investigate the analytic structures of naturally defined subsets of a positive hermitian Jordan triple system, namely the set of tripotent elements S and the sets of constant-rank elements Z_j. One of the main results of this chapter is the extension of Z_j and its analytic properties to certain flags of constant rank elements. This seems to be a new concept in Jordan theory, and it is based on the generalized Peirce decomposition. We obtain the so called pre-Peirce flag manifolds, which form the basis of the definition of *Peirce flag varieties* in Chapter 6.

Finally, in the last section we recall the basic notions of the functional calculus defined on a hermitian Jordan triple system and extend a well-known result on functions on open discs in Z to functions on constant rank elements. As an application we conclude that the projection map of constant rank elements onto corresponding tripotent elements and the pseudo-inverse map are indeed real analytic maps on the sets of constant-rank. We determine their derivatives.

3.1. Manifolds and substructures

In this thesis we are concerned with manifolds over the real or complex numbers that are smooth or even analytic manifolds. We recall some basic results on these manifolds. For a detailed treatment and for the proofs, we refer to **[38, 26]**.

Let $\mathbb{k}$ be the real or complex numbers, and let M be smooth (or analytic) manifold over $\mathbb{k}$ or dimension $n = \dim M$, called an *n-manifold*. An *imbedded k-submanifold* S of M is a subset $S \subset M$, such that for each $p \in S$ there exists a neighborhood $U \subset M$ of p and a chart map

$$\varphi : U \to V \subset \mathbb{k}^n \quad \text{such that} \quad U \cap S = \varphi^{-1}(V \cap (\mathbb{k}^k \times \{0\}^{n-k})) \,. \tag{3.1}$$

Such a submanifold is a manifold on its own right, and the topology of S coincides with the topology induced by M. Therefore, the canonical inclusion $\iota : S \hookrightarrow M$ is a *smooth imbedding*, i.e. ι is an immersion and a topological homeomorphism onto its image. S need not be closed in M.

Theorem 3.1. *Imbedded submanifolds are precisely the images of smooth (analytic) imbeddings.*

More generally, a subset $S \subset M$ is said to be an *immersed k-submanifold*, if it is the image of an injective immersion. By definition, an imbedded submanifold is also an immersed submanifold, but in general, the topology of an immersed submanifold may differ from the topology induced by M. The following proposition provides a criterion for an immersed submanifold to be actually an imbedded submanifold.

Proposition 3.2. *Let M, N be smooth (analytic) manifolds, and let $F : M \to N$ be an injective immersion. Suppose, one of the following conditions holds,*

(a) M is compact, *(b) F is a proper map,*

then, F is a smooth (analytic) imbedding with closed image.

A typical example of an immersed submanifold is the orbit of a Lie group action on a manifold. For a compact Lie group, Proposition 3.2 yields the following result.

Proposition 3.3. *Let M be a smooth (analytic) manifold, and let K be a compact Lie group acting on M smoothly (analytically). Then the orbits of K are smooth (analytic) imbedded submanifolds of M.*

For the general case, the following proposition characterizes imbedded submanifolds as subsets which locally look like level sets of appropriate submersions. In Section 3.3, we use this criterion to prove that certain substructures of phJTS (rank-j elements and more general pre-Peirce flags) are imbedded submanifolds.

Proposition 3.4. *Let S be a subset of an n-manifold M. Then S is an imbedded k-submanifold of M if and only if every point $p \in S$ has a neighborhood U in M such that $U \cap S$ is a level set of a submersion $\Phi : U \to \mathbb{k}^{n-k}$. Moreover, the tangent space of S in p is then given by $T_pS = \ker D_p\Phi$, where $D_p\Phi$ denotes the derivative of Φ in p.*

In the next section, we discuss equivalence relations on manifolds and describe a criterion for the possibility to impose the set of equivalence classes with a manifold structure. In this context, the more restrictive *imbedded* submanifolds are of particular importance.

Unless otherwise stated, in the following the term 'submanifold' always refers to an 'imbedded submanifold'. By abuse of language we also call the disjoint union of (sub-)manifolds (of possibly different dimensions) a (sub-)manifold. So, each connected component of a (sub-)manifold must be a k-(sub-)manifold for an appropriate k.

Vector fields on Z. Considering a Jordan triple system (or more generally a $\mathbb{k}$-vector space) Z as a manifold, we identify the tangent space T_zZ in $z \in Z$ with Z itself, so we identify the tangent bundle TZ with $Z \times Z$. Therefore a vector field on Z is given by a map

$$\widehat{\zeta} : Z \to Z \times Z,\ z \mapsto (z, \zeta(z))\ . \tag{3.2}$$

Most of the time we identify the map $\zeta : Z \to Z$ with the corresponding vector field $\widehat{\zeta}$. The commutator of two vector fields ζ and η is given by

$$[\zeta, \eta](z) = D_z\eta\big(\zeta(z)\big) - D_z\zeta\big(\eta(z)\big)\ . \tag{3.3}$$

Given a submanifold $S \subset Z$, we identify the tangent bundle TS with the corresponding vector bundle

$$TS \equiv \{(z, v) \,|\, z \in S,\ v \in T_zS \subset Z\} \subset S \times Z\ , \tag{3.4}$$

and vector fields on S are represented by maps

$$\widehat{\zeta} : S \to S \times Z,\ z \mapsto (z, \zeta(z)) \quad \text{with} \quad \zeta(z) \in T_zS \quad \text{for all } z \in S. \tag{3.5}$$

Again we use ζ and $\widehat{\zeta}$ synonymously.

3.2. Quotient manifolds

Let M be a real or complex n-manifold, $\Bbbk = \mathbb{R}$ or $\mathbb{C}$, correspondingly, and let $R \subset M \times M$ be an equivalence relation on M. We also denote the equivalence relation by

$$x \sim_R y \iff (x,y) \in R\,, \tag{3.6}$$

and an equivalence class by $[x] = \{\tilde{x} \in M \,|\, x \sim_R \tilde{x}\}$. The question arises under what conditions the quotient space M/R (equipped with the quotient topology) carries a manifold structure. We refer to [**38**] for proofs of the following two results.

Theorem 3.5. *If there exists a manifold structure on M/R such that the canonical projection $\pi : M \to M/R$ is a submersion then this manifold structure is unique.*

A criterion for the existence of such a manifold structure is due to Godement:

Theorem 3.6 (Godement). *The following are equivalent:*

(1) *$X = M/R$ is a manifold and $\pi : M \to M/R$ is a submersion,*
(2) *the equivalence relation R is* regular, *i.e.*
  (i) *R is a submanifold of $M \times M$,*
  (ii) *the projection $\mathrm{pr}_1 : R \to M$ is a submersion.*

The manifold $X = M/R$ is often called a quotient manifold.

Remark 3.7. The charts of X can be described as follows: For $x \in M$ consider $[x] \in X$ as a subset of M. Since the projection $\pi : M \to X$ is a submersion, $[x]$ is a submanifold of M. Let $\varphi : U \to V$ be a chart of M, which flattens $[x]$, as described in (3.1). Then $W := \varphi^{-1}(V \cap (\{0\}^k \times \Bbbk^{n-k}))$ is a submanifold of M, which is *minimally transversal* to $[x]$ in $x \in M$, i.e. $T_xM = T_x[x] \oplus T_xW$. Then the inverse function theorem states that the restriction of π to W is a local diffeomorphism, so we obtain a chart around $[x] \in X$. A submanifold N of M is said to be *minimally transversal*, if for *all* $x \in N$, the submanifold $[x]$ is minimally transversal to N in x. In this case, the same argument as above shows that the restriction of π to N is a local diffeomorphism. We call a family $\{N_\lambda\}$ of minimally transversal submanifolds a *transversal covering* of $X = M/R$, if the family $\{\pi(N_\lambda)\}$ covers X.

Vector fields. Let M be a manifold, R a regular equivalence relation, and M/R the corresponding quotient manifold. Often it is more convenient to describe vector fields on M than on M/R, and sometimes it is possible to lift vector field on M/R to vector fields on M. Let $\pi(x) = [x]$ denote the canonical projection of M onto M/R. A vector field $\hat{\zeta}$ on M is said to be the *lift* of a vector field ζ on M/R, if

$$(D_x\pi)\hat{\zeta}(x) = \zeta(\pi(x)) \quad \text{for all } x \in M. \tag{3.7}$$

A vector field $\hat{\zeta}$ on M is said to be *projectable*, if

$$(D_x\pi)\hat{\zeta}(x) = (D_{\tilde{x}}\pi)\hat{\zeta}(\tilde{x}) \quad \text{for all } (x,\tilde{x}) \in R. \tag{3.8}$$

In this case, $\zeta([x]) := \hat{\zeta}(x)$ defines a vector field on M/R, and $\hat{\zeta}$ is the lift of ζ. If $\hat{\zeta}$ is smooth (analytic) then ζ is smooth (analytic).

Vector bundles. Using Godement's Theorem we can give a description of suitable vector bundles (especially line bundles) on the quotient manifold $X = M/R$.

THEOREM 3.8. *Let M be a manifold, R a regular equivalence relation on M, and $X = M/R$. Let E be a vector space over $\Bbbk$ and $\phi : R \to \mathrm{GL}(E)$ a smooth cocycle, i.e. a smooth map such that*

$$\phi^x_{\tilde{x}} = \phi^{\hat{x}}_{\tilde{x}} \circ \phi^x_{\hat{x}} \quad \textit{with} \quad \phi^x_{\tilde{x}} := \phi(x, \tilde{x}) \tag{3.9}$$

for all pairwise equivalent $x, \tilde{x}, \hat{x} \in M$. Then

$$(x, v) \sim_{R_\phi} (\tilde{x}, \tilde{v}) \iff x \sim_R \tilde{x} \textit{ and } \tilde{v} = \phi^x_{\tilde{x}} v \tag{3.10}$$

defines a regular equivalence relation R_ϕ on $M \times E$, and $\mathcal{E} = (M \times E)/R_\phi$ is a vector bundle on $X = M/R$ with fiber E.

PROOF. Using the cocycle condition (3.9) it is easily verified that (3.10) indeed defines an equivalence relation on $M \times E$. To show that R_ϕ is regular, consider the map

$$\varphi : R \times E \to (M \times E)^2,\ (x, \tilde{x}, v) \mapsto (x, v, \tilde{x}, \phi^x_{\tilde{x}} v)\ .$$

Since R is regular, this is a smooth imbedding and hence $\varphi(R \times E) = R_\phi$ is a submanifold of $(M \times E)^2$. Furthermore, since $\mathrm{pr}_1 : R \to M$ is a submersion, so is $\mathrm{pr}_1 : R_\phi \to M \times E$ with $\mathrm{pr}_1((x, v), (\tilde{x}, \tilde{v})) = (x, v)$. Therefore R_ϕ is regular and $\mathcal{E} = (M \times E)/R_\phi$ has a well-defined manifold structure. To prove that $\mathcal{E}$ is a vector bundle over X with fiber E, it remains to show that

(i) $\pi : \mathcal{E} \to X,\ [x, v] \mapsto [x]$ is a smooth projection,
(ii) $\mathcal{E}_{[x]} := \pi^{-1}([x])$ is a vector space,
(iii) for all $[x] \in X$ there exists a neighborhood $U \subset X$ of $[x]$ and a diffeomorphism $\varphi : \pi^{-1}(U) \to U \times E$ such that for all $[\tilde{x}] \in U$ the restriction to $\mathcal{E}_{[\tilde{x}]}$ is a linear isomorphism onto $\{[\tilde{x}]\} \times E \cong E$.

The projection $\mathrm{pr}_1 : M \times E \to M^2$ is the lift of π with respect to the canonical projections $\pi_{\mathcal{E}} : M \times E \to \mathcal{E}$ and $\pi_X : M \to X$, i.e. $\pi \circ \pi_{\mathcal{E}} = \pi_X \circ \mathrm{pr}_1$. Since pr_1 is smooth and $\pi_{\mathcal{E}}$, π_X are submersions, π is also a smooth projection. This proves (i). For (ii), we have $\mathcal{E}_{[x]} = \{[x, v] \,|\, v \in E\}$ and

$$[x, v_1] + [x, v_2] = [x, v_1 + v_2] \quad , \quad \lambda \cdot [x, v_1] = [x, \lambda v_1]$$

for $v_1, v_2 \in E$ and $\lambda \in \Bbbk$ is a well-defined vector space structure on $\mathcal{E}_{[x]}$. To verify (iii), fix $x_0 \in [x]$ and let W be a submanifold of M such that $x_0 \in W$ and the restriction of π_X to W is a diffeomorphism (cf. Remark 3.7). Set $U := \pi_X(W)$ and let $\psi : U \to W$ denote the inverse of $\pi_X\big|_W$. We claim that $W \times E \subset M \times E$ is minimally transversal to the submanifold $[x, v]$ for all $(x, v) \in W \times E$. Indeed, let (x_t, v_t) be a smooth curve in $(W \times E) \cap [x, v]$ with $(x_0, v_0) = (x, v)$, then $x_t = x$ for all t, since W is minimally transversal to $[x]$, and so $v_t = \phi^{x_t}_x v = \phi^x_x v = v$ for all t. Therefore $T_{(v,x)}(W \times E) \cap T_{(v,x)}[v, x] = \{0\}$, and comparing dimensions we obtain

$$T_{(v,x)}(M \times E) = T_{(v,x)}(W \times E) \oplus T_{(v,x)}[v, x]\ .$$

Thus the restriction of $\pi_{\mathcal{E}}$ to $W \times E$ is a diffeomorphism onto $\mathcal{E}_U = \pi^{-1}(U)$, and using its inverse and the diffeomorphism $\psi : U \to W$ we obtain

$$\varphi : \mathcal{E}_U \to U \times E,\ [x, v] \mapsto \big([x], \phi^x_{\psi([x])} v\big)$$

as a well-defined local trivialization of $\mathcal{E}$. The linearity condition is obvious. □

REMARK 3.9. The transition maps between local trivializations are given as follows: Let W_1 and W_2 be two submanifolds of M, which are minimally transversal to the equivalence classes in M, and let $\psi_1 : U_1 \to W_1$ and $\psi_2 : U_2 \to W_2$ denote the corresponding charts of X. Then the transition map from $\mathcal{E}_{U_1}$ to $\mathcal{E}_{U_2}$ is given by

$$(U_1 \cap U_2) \times E \to (U_1 \cap U_2) \times E,\ ([x], v) \mapsto \Big([x],\ \phi^{\psi_1([x])}_{\psi_2([x])} v\Big)\ .$$

Using the manifolds $W_1' = \psi(U_1 \cap U_2)$ and $W_2' = \psi(U_1 \cap U_2)$ instead of U_1 and U_2, we obtain

$$W_1' \times E \to W_2' \times E,\ (x, v) \mapsto \left(\psi_2 \circ {\psi_1}^{-1}(x),\ \phi^x_{\psi_2 \circ {\psi_1}^{-1}(x)} v\right) . \tag{3.11}$$

This gives an explicit description of the transition maps on $\mathcal{E}$.

REMARK 3.10. If $\mathcal{E}_1$ and $\mathcal{E}_2$ are two vector bundles on $X = M/R$ defined by cocycles ϕ_1 and ϕ_2, then the usual constructions such as the Whitney sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ and the tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ have obvious analogues on the cocycle side.

In the case that there is a Lie group G acting transitively on a manifold X, there is a well-known correspondence between homogeneous[1] vector bundles on X and finite dimensional representations of the stabilizer subgroup $H = \mathrm{Stab}_G(x)$ of some $x \in X$, given as follows (see e.g. [**23**]):
If $\mathcal{E}$ is a homogeneous vector bundle on X, then the restriction of the G-action on $\mathcal{E}$ to H leaves the vector space $\mathcal{E}_x$ invariant, and so induces a representation $\rho_{\mathcal{E}} : H \to \mathrm{GL}(\mathcal{E}_x)$. Conversely, let ρ be some representation on a finite dimensional vector space E, then

$$G \times_\rho E := (G \times E)/\sim_\rho \quad \text{with} \quad (g, v) \sim_\rho (\tilde{g}, \tilde{v}) \iff \begin{cases} \tilde{g} = gh \text{ and } \tilde{v} = \rho(h)^{-1} v \\ \text{for some } h \in H \end{cases}$$

defines a vector bundle on $X = G/H$ with projection $\pi([g, v]) = gH$. In the situation of a quotient manifold $X = M/R$ as discussed above we can give a third characterization.

PROPOSITION 3.11. *Let M, R, X, E, ϕ, R_ϕ and $\mathcal{E}$ be as in Theorem 3.8. Assume in addition that there is a Lie group G acting smoothly on M, such that*

(i) R is G-invariant, i.e. if $(x, \tilde{x}) \in R$ then $(gx, g\tilde{x}) \in R$ for all $g \in G$,
(ii) the induced G-action on $X = M/R$ is transitive,
(iii) ϕ is G-invariant, i.e. $\phi(x, \tilde{x}) = \phi(gx, g\tilde{x})$ for all $(x, \tilde{x}) \in R$ and $g \in G$.

Then for fixed $x \in M$ the map

$$\rho_x : H \to \mathrm{GL}(E),\ h \mapsto \phi^{hx}_x \tag{3.12}$$

defines a representation of $H := \mathrm{Stab}_G([x])$ on E, and

$$\varPhi : G \times_{\rho_x} E \to \mathcal{E},\ [g, v] \mapsto [gx, v] \tag{3.13}$$

is an isomorphism of homogeneous line bundles. For a different choice of $\tilde{x} \in [x]$, we have $\rho_x(h) = \phi^{\tilde{x}}_x \circ \rho_{\tilde{x}}(h) \circ \phi^x_{\tilde{x}}$ for all $h \in H$, i.e. the representations are equivalent and the corresponding line bundles are isomorphic to each other.

PROOF. Using the G-invariance of ϕ and the cocycle condition we obtain for $h_1, h_2 \in H$ the relation

$$\rho_x(h_1 h_2) = \phi^{h_1 h_2 x}_x = \phi^{h_2 x}_{h_1^{-1} x} = \phi^x_{h_1^{-1} x} \circ \phi^{h_2 x}_x = \phi^{h_1 x}_x \circ \phi^{h_2 x}_x = \rho_x(h_1)\rho_x(h_2) .$$

This shows that ρ_x is a representation. The same argument yields the relation between ρ_x and $\rho_{\tilde{x}}$ for $\tilde{x} \in [x]$:

$$\rho_x(h) = \phi^{hx}_x = \phi^{\tilde{x}}_x \circ \phi^{hx}_{\tilde{x}} = \phi^{\tilde{x}}_x \circ \phi^x_{h^{-1}\tilde{x}} = \phi^{\tilde{x}}_x \circ \phi^{\tilde{x}}_{h^{-1}\tilde{x}} \circ \phi^x_{\tilde{x}} = \phi^{\tilde{x}}_x \circ \phi^{h\tilde{x}}_{\tilde{x}} \circ \phi^x_{\tilde{x}} = \phi^{\tilde{x}}_x \circ \rho_{\tilde{x}}(h) \circ \phi^x_{\tilde{x}} .$$

Next we prove that $\varPhi$ is well-defined. Let $[g, v] = [gh, \rho_x(h)^{-1} v]$. Since h preserves $[x]$ and R is G-invariant, we have $[ghx] = [gx]$ and therefore

$$[ghx,\ \rho_x(h)^{-1} v] = [gx,\ \phi^{ghx}_{gx} \left(\phi^{hx}_x\right)^{-1} v] = [gx,\ \phi^{hx}_x \left(\phi^{hx}_x\right)^{-1} v] = [gx,\ v] .$$

[1] i.e. there is a G action on the vector bundle, such that the canonical projection onto X is G-equivariant.

Thus $\Phi([g,v]) = \Phi([gh, \rho_x(h)^{-1}v])$, i.e. Φ is well-defined. Now let $[gx,v] = [\tilde{g}x,\tilde{v}]$. Then $[gx] = [\tilde{g}x]$ and $\tilde{v} = \phi^{gx}_{\tilde{g}x}v$. By the G-invariance of R and ϕ, this implies

$$[\tilde{g}^{-1}gx] = [x] \quad \text{and} \quad \tilde{v} = \phi_x^{\tilde{g}^{-1}gx}v\ .$$

Therefore $\tilde{g}^{-1}g$ is an element of H, so $\tilde{g} = gh$ and $\tilde{v} = \phi_x^{h^{-1}x}v = \rho_x(h)^{-1}v$. Hence $[g,v] = [\tilde{g},\tilde{v}]$. This shows that Φ is injective. Surjectivity follows from the transitivity of the G-action on X. Finally, by the same sort of argument, it is easy to show, that

$$G \ltimes \mathcal{E} \to \mathcal{E},\ (g,[x,v]) \mapsto [gx,v]$$

defines a G-action on $\mathcal{E}$ and that Φ is G-equivariant. Since all group actions are smooth, Φ also respects the analytic structures and hence is an isomorphism of vector bundles. □

The next proposition describes sections on a quotient manifold M/R, which are induced by maps on M.

Proposition 3.12. *Let M, R, X, E, ϕ, R_ϕ and $\mathcal{E}$ be as in Theorem 3.8. Let $\hat{\sigma}$ be a map $\hat{\sigma} : M \to E$ satisfying*

$$\hat{\sigma}(\tilde{x}) = \phi^x_{\tilde{x}}\big(\hat{\sigma}(x)\big) \quad \textit{for all} \quad (x,\tilde{x}) \in R_\phi\ .$$

Then, $\sigma([x]) := (x, \sigma(x))$ defines a section in $\mathcal{E}$. Moreover, if $\{N_\lambda\}$ is a transversal covering[2] *of $X = M/R$, and if the restriction of $\hat{\sigma}$ to each N_λ is smooth (analytic), then σ is smooth (analytic).*

Proof. Due to the relation (3.10), σ obviously defines a section in $\mathcal{E}$, and since transversal coverings correspond to an atlas on X (cf. Remark 3.7, also the second statement is obvious. □

We note that the map $\hat{\sigma}$ need not be smooth or analytic along the equivalence classes $[x]$ in order to induce a smooth or analytic section on X. In Section 6.3 we discuss the case of complex manifolds M, $X = M/R$ and a *real* analytic map $\hat{\sigma}$, which nevertheless induces a *holomorphic* section σ on X.

3.3. Analytic structures on Jordan triple systems

Let Z be a hermitian Jordan triple system and let r denote its rank.

Tripotent elements. For $0 \le j \le r$ let S_j be the set of tripotents of rank j. Recall from Section 2.5, that the Peirce 1-space Z^u_1 of some element $u \in Z$ has a real decomposition $Z^u_1 = Z^u_+ \oplus Z^u_-$ into the eigenspaces of the involution $z \mapsto z^\# = Q_e z$ on Z^u_1, where $e = \boldsymbol{\epsilon}(u)$ is the base-tripotent of u. For a tripotent $e \in S_j$ we have $\boldsymbol{\epsilon}(e) = e$. The following is a classical result. For a proof, we refer to [**28**, §5.6].

Theorem 3.13. *Let Z be a phJTS of rank r and fix $0 \le j \le r$. Then $S_j \subset Z$, the set of rank-j tripotents, is a compact real analytic submanifold of Z. The tangent space of S_j in $e \in S_j$ is given by*

$$T_e S_j = Z^e_- \oplus Z^e_{1/2}\ . \tag{3.14}$$

Moreover, S_j is invariant under the action of the automorphism group $\mathrm{Aut}(Z)$*, and the identity component K of* $\mathrm{Aut}(Z)$ *acts transitively on each connected component of S_j. In the case of Z being simple, S_j is connected and hence K-homogeneous.*

[2] see Remark 3.7.

We briefly discuss an atlas on S_j. For fixed $e \in S_j$ consider the map

$$\varphi_e : Z_-^e \oplus Z_{1/2}^e \to S_j,\ v \mapsto \exp(v \,\square\, e - e \,\square\, v)e\ . \tag{3.15}$$

Its derivative along $v = v_- + v_{1/2}$ is given by

$$D_0\varphi(v) = v \,\square\, e(e) - e \,\square\, v(e) = 2\,v_- + \tfrac{1}{2}\,v_{1/2}\ . \tag{3.16}$$

Therefore, $D_0\varphi$ is an isomorphism of $Z_-^e \oplus Z_{1/2}^e$ onto T_eS_j, hence φ is a diffeomorphism of a neighborhood of 0 onto an open subset of S_j, and its inverse defines a chart of S_j around e.

Constant rank elements. For $0 \le j \le r$ let Z_j be the set of all elements $z \in Z$ of rank j. We prove that this subset forms a complex submanifold of Z, and it is invariant under structure automorphisms of Z. Before we can do that, we recall some basic properties concerning Peirce decompositions and structure automorphisms. Let $Z = Z_1^u \oplus Z_{1/2}^u \oplus Z_0^u$ be the Peirce decomposition of Z with respect to some element $u \in Z$. Then Z_1^u is a unital Jordan algebra with product $x \circ y = \{x, u, y\}$ and unit element $u^\dagger$, the pseudo-inverse of u. For an invertible element $z \in Z_1^u$, its inverse z^{-1} is related to its pseudo-inverse $z^\dagger$ by

$$z^\dagger = Q_{u^\dagger} z^{-1}\ , \tag{3.17}$$

see Lemma 2.26. In the following we need some special elements of the structure group, the so called Frobenius transformations (cf. [**8**], VI.3).

Lemma 3.14. *For $u \in Z$ and $z \in Z_1^u$, $y \in Z_{1/2}^u$, the structure automorphism*

$$\tau_{z,y} := \exp(2\,y \,\square\, z^\dagger) \in K^{\mathbb{C}} \tag{3.18}$$

is called a Frobenius transformation *(with respect to u). It satisfies*

$$\tau_{z,y} = B_{y,-z^\dagger} \quad \text{and} \quad \tau_{z,y}z = z + y + Q_y z^\dagger = Q_{y+z} z^\dagger\ .$$

Any two Frobenius transformations with respect to u commute.

Proof. The equality of $\tau_{z,y}$ and $B_{y,-z^\dagger}$ follows using the Peirce rules and JT13:

$$\tau_{z,y} = \exp(2\,y \,\square\, z^\dagger) = \mathrm{Id} + 2\,y \,\square\, z^\dagger + 2(y \,\square\, z^\dagger) \circ (y \,\square\, z^\dagger) = \mathrm{Id} + 2\,y \,\square\, z^\dagger + Q_y Q_{z^\dagger} = B_{y,-z^\dagger}\ .$$

A simple calculation verifies the second equality. Now let $\tau_{\tilde z,\tilde y}$ be another Frobenius transformation with respect to u, i.e. $\tilde z \in Z_1^u$ and $\tilde y \in Z_{1/2}^u$. Then the Jordan identity and the Peirce rules imply

$$[y \,\square\, z^\dagger, \tilde y \,\square\, \tilde z^\dagger] = \left\{y,\, z^\dagger,\, \tilde y\right\} \square\, z^\dagger - \tilde y \,\square \left\{z^\dagger,\, y,\, \tilde z^\dagger\right\} = 0\ .$$

Therefore $\tau_{z,y} \circ \tau_{\tilde z,\tilde y} = \exp(2\,(y \,\square\, z^\dagger + \tilde y \,\square\, \tilde z^\dagger)) = \tau_{\tilde z,\tilde y} \circ \tau_{z,y}$. □

According to Theorem 3.13 we obtain:

Theorem 3.15. *Let Z be a phJTS of rank r, and fix $0 \le j \le r$. Then $Z_j \subset Z$, the set of all rank-j-elements of Z, is a complex submanifold of Z. The tangent space of Z_j in $u \in Z_j$ is given by*

$$T_u Z_j = Z_1^u \oplus Z_{1/2}^u\ . \tag{3.19}$$

Moreover, Z_j is invariant under the action of the structure group $\mathrm{Str}(Z)$*, and the identity component $K^{\mathbb{C}}$ of* $\mathrm{Str}(Z)$ *acts transitively on each connected component of Z_j. In case of Z being simple, Z_j is connected and hence $K^{\mathbb{C}}$-homogeneous.*

PROOF. The invariance of Z_j under the action of the structure group is already proved in Section 2.8, Proposition 2.34. To show that Z_j is a submanifold of Z we prove that it is locally the level set of a submersion. This proof extends the one given for the matrix case in [**26**], Ex. 8.14. Fix $u \in Z_j$ and let $z = z_1 + z_{1/2} + z_0$ denote the Peirce decomposition of elements $z \in Z$ with respect to u. Let U be the set of all $z \in Z$ such that z_1 is invertible in the unital Jordan algebra Z_1^u. Since z_1 is invertible in Z_1^u if and only if the Jordan algebra determinant $\Delta^u(z)$ does not vanish, U is an open set of Z. Then for $z \in U$, due to Lemma 3.14 and the relations of the pseudo-inverse, we obtain

$$\tau_{-z_1, z_{1/2}} z = B_{z_{1/2}, z_1^\dagger} z = z_1 + z_0 - Q_{z_{1/2}} z_1^\dagger = z_1 + z_0 - Q_{z_{1/2}} Q_{u^\dagger} z_1^{-1} . \tag{3.20}$$

Since the Frobenius transformation is an element of the structure group, it preserves the rank of z, and since $\operatorname{rk} z_1 = \operatorname{rk} u = j$ and $z_0 - Q_{z_{1/2}} z_1^\dagger \in Z_0^u$, it follows that $z \in U$ has rank j if and only if $z_0 - Q_{z_{1/2}} {z_1}^\dagger$ vanishes. Therefore,

$$Z_j \cap U = \Phi^{-1}(0) \quad \text{with} \quad \Phi : U \to Z_0^u, z \mapsto z_0 - Q_{z_{1/2}} Q_{u^\dagger} z_1^{-1} .$$

For $t \in \mathbb{R}$ let z_t be the curve $z_t = z + t v_0$ in U with $v_0 \in Z_0^u$, then $\frac{d}{dt} \Phi(z_t)\big|_{t=0} = v_0$, and thus Φ is a submersion. We conclude with Proposition 3.4 that Z_j is a submanifold of Z of dimension $\dim(Z_1^u \oplus Z_{1/2}^u)$. Moreover, for $z = u$ we obtain $D_u \Phi(\dot{z}) = \dot{z}_0$, and therefore

$$T_u Z_j = \ker D_u \Phi = Z_1^u \oplus Z_{1/2}^u .$$

Now consider for $t \in \mathbb{R}$ and $v \in Z_1^u \oplus Z_{1/2} u$ the curve $h_t = \exp(t v \,\Box\, u^\dagger)$ in $K^{\mathbb{C}}$. Due to the $K^{\mathbb{C}}$-invariance of Z_j, the curve $u_t = h_t u$ stays in Z_j, and its derivative in $t = 0$ is given by $\dot{u}_0 = v \,\Box\, u^\dagger(u) = v_1 + \frac{1}{2} v_{1/2}$. Therefore, the $K^{\mathbb{C}}$-orbits are open and closed in Z_j, and hence $K^{\mathbb{C}}$ acts transitively on the connected components of Z_j. Finally, suppose that Z is simple. It is well-known that in this case K acts transitively on tripotent elements, and therefore it suffices to show that for any $z \in Z$ there exists an element of $h \in K^{\mathbb{C}}$ such that hz is tripotent. Let $z = \sum \lambda_i e_i$ be the spectral decomposition of z and set $x = \sum (1 - 1/\sqrt{\lambda_i}) e_i$ and $y = \sum e_i$. Then (x, y) is quasi-invertible, $h = B_{x,y}$ is an element of $K^{\mathbb{C}}$ and satisfies $B_{x,y} z = y$. □

This proof contains some important identities for certain rank-j elements, which we denote separately:

COROLLARY 3.16. *For $u \in Z_j$ let $z = z_1 \oplus z_{1/2} \oplus z_0$ denote the Peirce decomposition of elements of Z with respect to u, and let $U \subset Z$ be the set of elements $z \in Z$, such that z_1 is invertible in Z_1^u. Then*

$$z \in U \cap Z_j \iff z = z_1 + z_{1/2} + Q_{z_{1/2}} z_1^\dagger = \tau_{z_1, z_{1/2}} z_1 .$$

In particular, z is uniquely determined by its Z_1^u- and $Z_{1/2}^u$-components.

REMARK 3.17. As noted in Section 2.1, each positive hermitian Jordan triple system is semisimple, so Z is decomposable into a direct sum of simple phJTS:

$$Z = Z^{(1)} \oplus \ldots \oplus Z^{(s)} . \tag{3.21}$$

Let $Z_j^{(\ell)}$ denote the submanifold of rank-j elements of $Z^{(\ell)}$, then the Spectral Theorem 2.5 immediately implies that the manifold of rank-j elements of Z decomposes into

$$Z_j = \dot{\bigcup_{j_1 + \ldots + j_s = j}} Z_{j_1}^{(1)} \oplus \ldots \oplus Z_{j_s}^{(s)} . \tag{3.22}$$

These are the connected components of Z_j.

In addition to the imbedded structure of $Z_j \subset Z$ we describe Z_j intrinsically via charts. This is an application of the Frobenius transformations on Z_j. For $u \in Z_j$ consider

$$\phi_u : (Z_1^u)^\times \times Z_{1/2}^u \to Z_j,\ (z, y) \mapsto \tau_{z,y} z\ . \tag{3.23}$$

Here $(Z_1^u)^\times$ denotes the (open) subset of invertible elements in Z_1^u. As in (3.20) we obtain

$$\phi_u(z, y) = \tau_{z,y} z = \exp(2\, y \,\Box\, z^\dagger) z = z + y + Q_y Q_{u^\dagger} z^{-1}\ , \tag{3.24}$$

and since the Frobenius transformation is an element of the structure group, the image of ϕ_u is a subset of Z_j. It is injective, since z and y are uniquely determined by the Peirce decomposition of $\phi_u(z, y)$ with respect to u. Moreover, ϕ_u is holomorphic, since $z \mapsto z^{-1}$ is holomorphic, it is even a rational map. Its derivative in (z, y) is given by

$$D_{(z,y)}\phi_u(\dot z, \dot y) = \dot z + \dot y + 2\left\{\dot y,\, Q_{u^\dagger} z^{-1},\, y\right\} - Q_y Q_{u^\dagger} \dot z\ . \tag{3.25}$$

Injectivity of $D_{(z,y)}\phi_u$ is evident, so by comparing dimensions it follows that ϕ_u is a (global) diffeomorphism of $(Z_1^u)^\times \times Z_{1/2}^u$ onto its image. The prove of Proposition 3.15 shows that the image of ϕ_u consists exactly of those elements in Z_j, whose Peirce 1-component is invertible in Z_1^u. Therefore it is an open and dense subset of Z_j. Composing ϕ_u with the exponential map $\exp_u$ of the Jordan algebra Z_1^u, we finally obtain the map

$$\varphi_u : Z_1^u \oplus Z_{1/2}^u \to Z_j,\ (x, y) \mapsto \phi_u(\exp_u(x), y)\ , \tag{3.26}$$

that defines a chart for the open and dense subset of all $z \in Z_j$, such that z_1 is invertible in Z_1^u. Notice, that $\varphi_u(0) = u^\dagger$, since $u^\dagger$ is the unit element in Z_1^u, and therefore $\exp_u(0) = u^\dagger$.

Flags of constant rank. Now we generalize the set of rank-j elements to flags of constant rank. The resulting manifolds are called *pre-Peirce flag manifolds* (*pre-Peirce flags*), since they serve as the background for the definition of *Peirce flags* in Section 6.4.

Recall from Section 2.6 that a tuple $(u_1, \ldots, u_k)$ is *Peirce ordered*, if the corresponding Peirce 1-spaces form a (not necessarily strictly) increasing sequence of subspaces of Z, i.e. if $Z_1^{u_1} \subset \ldots \subset Z_1^{u_k}$ holds. We denote this by $u_1 \subset \ldots \subset u_k$. The tuple $(\operatorname{rk} u_1, \ldots, \operatorname{rk} u_k)$ is called the *type* of the Peirce ordered tuple (u_i). The main result of this section is the following.

Theorem 3.18. *Let Z be a phJTS of rank r, and fix $J = (j_1, \ldots, j_k) \in \mathbb{Z}^k$ with $0 \le j_1 \le \ldots \le j_k \le r$. Then, the set Z_J of Peirce ordered tuples of type J,*

$$Z_J := \{(u_1, \ldots, u_k) \,|\, u_1 \subset \ldots \subset u_k,\ \operatorname{rk} u_i = j_i\}\ ,$$

is a complex submanifold of $Z_{j_1} \times \ldots \times Z_{j_k}$, called the pre-Peirce flag manifold of type J. *The dimension of the connected component of Z_J containing $(u_i) \in Z_J$ is*

$$\sum_{i=1}^{k-1} \dim\left(Z_1^{u_i} \oplus (Z_{1/2}^{u_i} \cap Z_1^{u_{i+1}})\right) + \dim(Z_1^{u_k} \oplus Z_{1/2}^{u_k})\ .$$

Moreover, Z_J is invariant under the action of the structure group $\operatorname{Str}(Z)$. *If Z is simple, then Z_J is connected.*

Remark 3.19. At first sight, it seems to be more natural to define Z_J by using the strict partial order on Z given by

$$(*) \quad u \le \tilde u \iff \tilde u = u + u' \text{ for some } u' \text{ with } u \perp\!\!\!\perp u'.$$

This is the natural extension of the partial order relation on tripotents (cf. Section 2.4). The benefit of our approach comes from the action of the structure group on

Z_J. The more restrictive order relation $(*)$ does not admit such an action, indeed for u and u' with $u \perp\!\!\!\perp u'$ we have $u + u' \leq u$, but for $h = \tau_{u,y}$ with $y \in Z^u_{1/2}$ we obtain $hu = u + y + Q_y u^\dagger$ and $h(u + u') = hu + u'$, so $h(u + u') \leq hu$ if and only if $hu \perp\!\!\!\perp u'$, i.e.

$$u' \,\Box\, (u + y + Q_y u^\dagger) = u' \,\Box\, y + u' \,\Box\, (Q_y u^\dagger) \overset{!}{=} 0 \,.$$

In particular, $\{u',\, y,\, u\} = Q_{u+u'} y \overset{!}{=} 0$. But for generic $y \in Z^u_{1/2} \cap Z^{u+u'}_1$ this term does not vanish unless u' is trivial. It should be noted that the action of the structure group on Z_J is not transitive. For later applications of Theorem 3.18 it is also noteworthy that there is no necessity for the tuple $J = (j_1, \ldots, j_k)$ to be strictly increasing. If some of the j_i are equal, then the corresponding elements of $(u_i) \in Z_J$ induce the same Peirce decomposition.

The proof of Theorem 3.18 requires some preparation. First we define some open and dense subsets of Z and Z_j, which already appeared in the proof of Theorem 3.15 and will be used later to prove that Z_J is locally the level set of some submersion (cf. Proposition 3.4).

Lemma 3.20. *For $u \in Z_j$ let $z = z_1 \oplus z_{1/2} \oplus z_0$ denote the Peirce decomposition of $z \in Z$ with respect to u. Then the sets*

$$\mathcal{I}^u := \{z \in Z \,|\, z_1 \text{ invertible in } Z^u_1\} \quad \text{and} \quad \mathcal{I}^u_j := \mathcal{I}^u \cap Z_j$$

are open and dense in Z and in the connected component of Z_j containing u, respectively. Moreover, $\tilde{u}$ is an element of $\mathcal{I}^u_j$ if and only if u is an element of $\mathcal{I}^{\tilde{u}}_j$.

Proof. The set $\mathcal{I}^u$ is open and dense in Z, since its complement is the vanishing set of the Jordan algebra determinant Δ^u, which is a non-trivial complex polynomial on Z. The subset of rank-j elements Z_j is open and dense in its closure $\mathrm{cl}(Z_j)$, which in turn equals the affine variety of Z given by

$$\mathrm{cl}(Z_j) = \mathcal{V}\big(\,\{\Delta^v \,|\, v \in Z,\ \mathrm{rk}\, v > j\}\,\big)\,.$$

Therefore, Z_j and $\mathcal{I}^u \cap \mathrm{cl}(Z_j)$ are Zariski open in $\mathrm{cl}(Z_j)$, and since u is an element of both, $\mathcal{I}^u$ and Z_j, the intersection $\mathcal{I}^u \cap Z_j$ is non-empty, and thus dense in the irreducible component of $\mathrm{cl}(Z_j)$ containing u. Since the closure of the connected component of Z_j containing u coincides with the irreducible component of $\mathrm{cl}(Z_j)$ containing u (cf. Remark 3.17), this proves the first part of the assertion. Now let $\tilde{u}$ be an element of $\mathcal{I}^u_j$. We have to show that $\Delta^{\tilde{u}}(u) \neq 0$. Let $\tilde{u} = \tilde{u}_1 \oplus \tilde{u}_{1/2} \oplus \tilde{u}_0$ be the Perice decomposition of $\tilde{u}$ with respect to u. Using Corollary 3.16 we have $\tilde{u} = \tau_{\tilde{u}_1, \tilde{u}_{1/2}} \tilde{u}_1$. Due to Proposition 2.37 and the Peirce rules this implies

$$\Delta^{\tilde{u}}(u) = \Delta^{\tau_{\tilde{u}_1, \tilde{u}_{1/2}} \tilde{u}_1}(u) = \Delta^{\tilde{u}_1}\Big(\tau^*_{\tilde{u}_1, \tilde{u}_{1/2}} u\Big) = \Delta^{\tilde{u}_1}\Big(B_{-\tilde{u}_1^\dagger,\, \tilde{u}_{1/2}} u\Big) = \Delta^{\tilde{u}_1}(u) \neq 0\,,$$

since by assumption, $\tilde{u}_1$ is invertible in Z^u_1. Therefore, u is an element of $\mathcal{I}^{\tilde{u}}_j$. The converse follows by symmetry of u and $\tilde{u}$. □

Next we describe a joint Peirce decomposition with respect to a Peirce ordered tuple and introduce some notation. Let $(u_i) = (u_1, \ldots, u_k)$ be a Peirce ordered tuple, then for all $i \geq j$ the Jordan identity implies

$$[u_i \,\Box\, u_i^\dagger, u_j \,\Box\, u_j^\dagger] = \big\{u_i,\, u_i^\dagger,\, u_j\big\} \,\Box\, u_j^\dagger - u_j \,\Box\, \big\{u_i^\dagger,\, u_i,\, u_j\big\} = u_j \,\Box\, u_j^\dagger - u_j \,\Box\, u_j^\dagger = 0\,,$$

and hence all these box operators commute. Therefore the tuple (u_i) induces a joint Peirce decomposition similar to the usual one defined by an orthogonal system (cf. Section 2.5):

$$Z = \bigoplus_{(\nu_1, \ldots, \nu_k) \in \{1, 1/2, 0\}^k} Z^{u_1 \cdots u_k}_{\nu_1 \cdots \nu_k} \quad \text{with} \quad Z^{u_1 \cdots u_k}_{\nu_1 \cdots \nu_k} = \bigcap_{i=1}^k Z^{u_i}_{\nu_i}\,. \tag{3.27}$$

The corresponding orthogonal projections of Z onto the subspaces are denoted by

$$\llbracket\ \rrbracket^{u_1\cdots u_k}_{\nu_1\cdots\nu_k} : Z \mapsto Z^{u_1\cdots u_k}_{\nu_1\cdots\nu_k},\ z \mapsto \llbracket z\rrbracket^{u_1\cdots u_k}_{\nu_1\cdots\nu_k}\ . \tag{3.28}$$

These definitions also make sense for subtuples $(u_{i_1},\dots,u_{i_\ell})$ with $0 < i_1 \le \dots \le i_\ell \le k$ of a given Peirce ordered tuple (u_i). For $\ell = 1$, we obtain the ordinary Peirce decomposition with respect to u_i.

Lemma 3.21. *Let $(u,\tilde u) \in Z^2$ be a Peirce ordered tuple of type (j,k), i.e. $u \subset \tilde u$ and $\operatorname{rk} u = j$, $\operatorname{rk}\tilde u = k$. Then for $a \in \mathcal{I}^u_j$ and $b \in \mathcal{I}^{\tilde u}_k$ it follows*

$$a \in Z^b_1 \iff \llbracket a\rrbracket^{u\tilde u}_{\frac12\frac12} = 2\left\llbracket\left\{\llbracket b\rrbracket^{\tilde u}_{\frac12},\ \left(\llbracket b\rrbracket^{\tilde u}_1\right)^\dagger,\ \llbracket a\rrbracket^{\tilde u}_1\right\}\right\rrbracket^{u\tilde u}_{\frac12\frac12}\ . \tag{3.29}$$

Setting $z_\nu = \llbracket z\rrbracket^{\tilde u}_\nu$ and $z_{\mu\nu} = \llbracket z\rrbracket^{u\tilde u}_{\mu\nu}$ for $z \in Z$, the left-hand side of (3.29) *becomes*

$$a_{\frac12\frac12} = 2\left\{b_{1/2},\ b_1^\dagger,\ a_1\right\}_{\frac12\frac12}\ . \tag{3.30}$$

Proof. During this proof we use the abbreviated notation z_ν and $z_{\mu\nu}$ for the components of the corresponding Peirce decompositions, i.e. $z_\nu \in Z^{\tilde u}_\nu$ and $z_{\mu\nu} \in Z^u_\mu \cap Z^{\tilde u}_\nu$. We set $h := \tau^{-1}_{b_1,b_{1/2}} = \tau_{-b_1,b_{1/2}}$. Then Corollary 3.16 yields $hb = b_1$, and therefore

$$a \in Z^b_1 \iff ha \in hZ^b_1 = Z^{hb}_1 = Z^{b_1}_1 = Z^{\tilde u}_1\ .$$

From Lemma 3.14 and the Peirce rules we obtain

$$\begin{aligned} ha &= B_{b_{1/2},b_1^\dagger}a = a - 2\left\{b_{1/2},\ b_1^\dagger,\ a\right\} + Q_{b_{1/2}}Q_{b_1^\dagger}a \\ &= a_1 \oplus \left(a_{1/2} - 2\left\{b_{1/2},\ b_1^\dagger,\ a_1\right\}\right) \oplus \left(a_0 - 2\left\{b_{1/2},\ b_1^\dagger,\ a_{1/2}\right\} + Q_{b_{1/2}}Q_{b_1^\dagger}a_1\right)\ . \end{aligned}$$

By assumption, a_1 is invertible in Z^u_1. Therefore, ha is an element of $\mathcal{I}^u_j$, and it remains to show that an element z of $\mathcal{I}^u_j$ is also an element of $Z^{\tilde u}_1$ if and only if $z_{\frac12\frac12} = 0$. Due to Corollary 3.16, $z \in \mathcal{I}^u_j$ can be written as

$$z = \llbracket z\rrbracket^u_1 + \llbracket z\rrbracket^u_{1/2} + Q_{\llbracket z\rrbracket^u_{1/2}}\llbracket z\rrbracket^u_1\ ,$$

and using the joint Peirce decomposition with respect to the tuple $(u,\tilde u)$, we obtain

$$z = \left(z_{11} + z_{\frac12 1} + \{z_{\frac12 1},\ z_{11},\ z_{\frac12 1}\}\right) \oplus \left(z_{\frac12\frac12} + 2\{z_{\frac12\frac12},\ c_{11},\ c_{\frac12 1}\}\right) \oplus \{z_{\frac12\frac12},\ c_{11},\ z_{\frac12\frac12}\}\ .$$

Here we arranged the terms according to the decomposition $z = z_1 \oplus z_{1/2} \oplus z_0$. Now, if z is an element of $Z^{\tilde u}_1$, then $z_{1/2}$ (and z_0) must vanish, in particular this implies $z_{\frac12\frac12} = 0$. Conversely, if $z_{\frac12\frac12}$ vanishes, then $z_{1/2}$ and z_0 also vanish, and therefore $z = z_1$ is an element of $Z^{\tilde u}_1$. □

Now we are prepared for the proof of the main theorem of this paragraph.

Proof of Theorem 3.18. First we note that Z_J is invariant under the action of the structure group, since by Theorem 3.18, the submanifolds Z_{j_i} are invariant, and by Lemma 2.32, the Peirce order of the tuples is also respected by this action. According to Proposition 3.4, we have to show that Z_J is locally given as the level set of some appropriate submersion. Fix $(u_i) \in Z_J$ and set $U := \mathcal{I}^{u_1}_{j_1} \times \dots \times \mathcal{I}^{u_k}_{j_k}$. Due to Lemma 3.20, U is open and dense in $Z_{j_1} \times \dots \times Z_{j_k}$. We use the notation of the joint Peirce decomposition with respect to (u_i) defined above. Consider the map $\Phi : U \to Z^{u_1u_2}_{\frac12\frac12} \times \dots \times Z^{u_{k-1}u_k}_{\frac12\frac12}$ defined by

$$(z_i) \mapsto \left(\llbracket z_i\rrbracket^{u_iu_{i+1}}_{\frac12\frac12} - 2\left\llbracket\left\{\llbracket z_{i+1}\rrbracket^{u_{i+1}}_{\frac12},\ \left(\llbracket z_{i+1}\rrbracket^{u_{i+1}}_1\right)^\dagger,\ \llbracket z_i\rrbracket^{u_{i+1}}_1\right\}\right\rrbracket^{u_iu_{i+1}}_{\frac12\frac12}\right)_{i=1,\dots,k-1}\ .$$

This is a holomorphic map, since Peirce projections are holomorphic and we have $v^\dagger = Q_{u^\dagger_{i+1}}v^{-1}$ for invertible $v \in Z^{u_{i+1}}_1$, where $v \mapsto v^{-1}$ is also holomorphic. Lemma 3.21

shows, that $U \cap Z_J = \Phi^{-1}(0)$, i.e. Z_J is locally (even densely) given as the level set of Φ. It remains to show that Φ is a submersion. Fix $(z_i) \in Z_{j_1} \times \ldots \times Z_{j_k}$ and consider the curve

$$(z_i(t)) = (\tau_{z_i, tv_i} z_i) \in Z_{j_1} \times \ldots \times Z_{j_k} \quad \text{with} \quad (v_1, \ldots, v_k) \in Z^{u_1 u_2}_{\frac{1}{2}\frac{1}{2}} \times \ldots \times Z^{u_{k-1} u_k}_{\frac{1}{2}\frac{1}{2}} \,.$$

Then by Lemma 3.14, it is $z_i(t) = z_i + tv_i + t^2 Q_{v_i} z_i^\dagger$, and we obtain

$$D_{(u_i)}\Phi\big((v_i)\big) = \tfrac{d}{dt}\Phi\big((z_i(t))\big)\Big|_{t=0} = \Big(v_i - 2\,\big[\!\big[\{v_{i+1},\, z_{i+1}^\dagger,\, z_i\}\big]\!\big]^{u_i u_{i+1}}_{\frac{1}{2}\frac{1}{2}}\Big)_{i=1,\ldots,k-1}$$

Now setting $v_2 = \ldots = v_k = 0$ shows that $Z^{u_1 u_2}_{\frac{1}{2}\frac{1}{2}}$ is in the image of $D_{(u_i)}\Phi$, and inductively it follows that all $Z^{u_i u_{i+1}}_{\frac{1}{2}\frac{1}{2}}$ lie in the image of $D_{(u_i)}\Phi$. Therefore, Φ is a submersion. Finally, comparing dimensions yields the stated formula. $\square$

Remark 3.22. Due to Proposition 3.4, the tangent space of Z_J at (u_i) is given by the kernel of the derivative of Φ at (u_i). This is a subspace of the tangent space of $Z_{j_1} \times \ldots \times Z_{j_k}$ at (u_i), which is given by

$$T_{(u_i)}\big(Z_{j_1} \times \ldots \times Z_{j_k}\big) = \big(Z^{u_1}_1 \oplus Z^{u_1}_{1/2}\big) \times \ldots \times \big(Z^{u_k}_1 \oplus Z^{u_k}_{1/2}\big) \,,$$

see Theorem 3.15. It is straightforward to compute the derivative of Φ at (u_i):

$$D_{(u_i)}\Phi\big((\dot u_i)\big) = \Big([\![\dot u_i]\!]^{u_i u_{i+1}}_{\frac{1}{2}\frac{1}{2}} - 2\,\Big[\!\Big[\Big\{[\![\dot u_{i+1}]\!]^{u_{i+1}}_{\frac{1}{2}}\,,\, u_{i+1}^\dagger,\, u_i\Big\}\Big]\!\Big]^{u_i u_{i+1}}_{\frac{1}{2}\frac{1}{2}}\Big)_{i=1,\ldots,k-1} \,. \tag{3.31}$$

It immediately follows that $Z^{u_1}_1 \times \ldots \times Z^{u_k}_1$ is in the kernel and hence a subspace of the tangent space of Z_J at (u_i). Unfortunately, (3.31) imposes non-trivial relations on the Peirce 1/2-spaces except for $Z^{u_k}_{1/2}$. Therefore, there is no simple direct sum decomposition of the tangent spaces of Z_J, as compared to the case of Z_j.

We extend the description of the pre-Peirce flag manifold Z_J by constructing an atlas of charts for Z_J. Fix $(u_i) \in Z_J$ and consider the map

$$\phi_{(u_i)} : \big((Z^{u_1}_1)^\times \times Z^{u_1 u_2}_{\frac{1}{2}1}\big) \times \ldots \times \big((Z^{u_{k-1}}_1)^\times \times Z^{u_{k-1} u_k}_{\frac{1}{2}1}\big) \times \big((Z^{u_k}_1)^\times \times Z^{u_k}_{1/2}\big) \to Z_J$$

defined by

$$\big((z_i, y_i)\big)_{i=1,\ldots,k} \mapsto \big(\tau_{z_k, y_k} \circ \ldots \circ \tau_{z_i, y_i} z_i\big)_{i=1,\ldots,k} \,. \tag{3.32}$$

This map is well-defined, since the Frobenius transformations preserve the rank and the Peirce order of the elements, so we have

$$\big(\tau_{z_k, y_k} \circ \ldots \circ \tau_{z_i, y_i} z_i\big) \subset \big(\tau_{z_k, y_k} \circ \ldots \circ \tau_{z_{i+1}, y_{i+1}} z_{i+1}\big) \iff \tau_{z_i, y_i} z_i \subset z_{i+1} \,,$$

and since y_i is in particular an element of $Z^{u_{i+1}}_1$, $\tau_{z_i, y_i} z_i$ remains in $Z^{u_{i+1}}_1$. The map $\phi_{(u_i)}$ is the natural extension of (3.23), and with the same arguments as above one can show that $\phi_{(u_i)}$ is a biholomorphic map onto the subset

$$\mathcal{I}^{(u_i)}_J := \big(\mathcal{I}^{u_1}_{j_1} \times \ldots \times \mathcal{I}^{u_k}_{j_k}\big) \cap Z_J \,, \tag{3.33}$$

which is open and dense in the connected component of Z_J containing (u_i). Again using the exponential maps of the Jordan algebras $Z^{u_i}_1$, we obtain the map

$$\varphi_{(u_i)} : \big(Z^{u_1}_1 \oplus Z^{u_1 u_2}_{\frac{1}{2}1}\big) \times \ldots \times \big(Z^{u_{k-1}}_1 \oplus Z^{u_{k-1} u_k}_{\frac{1}{2}1}\big) \times \big(Z^{u_k}_1 \oplus Z^{u_k}_{1/2}\big) \to Z_J$$

given by $\varphi_{(u_i)}\big((x_i, y_i)\big) = \phi_{(u_i)}\big((\exp_{u_i}(x_i), y_i)\big)$, that defines a chart. Considering that the unit element of $Z^{u_i}_1$ is $u_i^\dagger$, we have $\varphi_{(u_i)}(0) = (u_1^\dagger, \ldots, u_k^\dagger)$.

3.4. Functional calculus

Let Z be a phJTS of rank r. The Spectral Theorem on Z (Thm. 2.5) admits the definition of a functional calculus on Z. Recall that any element $z \in Z$ has a unique decomposition

$$z = \lambda_1 e_1 + \ldots + \lambda_n e_n \quad \text{with} \quad \lambda_1 > \ldots > \lambda_n > 0\ ,$$

and the e_i are pairwise orthogonal non-zero tripotents which are real linear combinations of powers of z. Now let $f : \mathbb{R} \to \mathbb{C}$ be an odd function, $f(-t) = -f(t)$, then

$$\mathbf{f} : Z \to Z,\ z = \sum \lambda_i e_i \mapsto \mathbf{f}(z) := \sum f(\lambda_i) e_i \tag{3.34}$$

is a well-defined $\mathrm{Aut}(Z)$-equivariant map on Z, i.e. $\mathbf{f}(kz) = k\mathbf{f}(z)$ for all $z \in Z$ and $k \in \mathrm{Aut}(Z)$.[3] If $g, h : \mathbb{R} \to \mathbb{C}$ are also odd functions, then obviously we have

$$(\mathbf{f} + \mathbf{g})(z) = \mathbf{f}(z) + \mathbf{g}(z)\ , \quad (\mathbf{f}\overline{\mathbf{g}}\mathbf{h})(z) = \{\mathbf{f}(z), \mathbf{g}(z), \mathbf{h}(z)\}\ .$$

In addition, if $g(\mathbb{R}) \subset \mathbb{R}$, then the composition $f \circ g$ also yields

$$(\mathbf{f} \circ \mathbf{g})(z) = \mathbf{f}(\mathbf{g}(z))\ .$$

If $f \in \mathbb{R}[t]$ is an odd polynomial, $f(t) = \sum a_i t^{2i+1}$, then $\mathbf{f}(z) = \sum a_i z^{(2i+1)}$ is continuous on Z. Here $z^{(2i+1)}$ denotes the odd powers of z defined in Section 2.1. Applying the Weierstrass approximation theorem, we obtain that for any continuous odd function $f : \mathbb{R} \to \mathbb{C}$, the corresponding map $\mathbf{f}$ is continuous on Z. For example, the most simple non-trivial odd polynomial $c(t) = t^3$ with corresponding map

$$\mathbf{c} : Z \to Z,\ \mathbf{c}(z) = \{z, z, z\} = z^{(3)} \tag{3.35}$$

induces a homeomorphism on Z with continuous inverse defined by $c^{-1}(t) = t^{1/3}$. The map $\mathbf{c}$ is called the *cubic map* on Z. Its iterations $\mathbf{c}^n$ for $n \in \mathbb{Z}$ are used in the study of certain orbit structure on the compactification $\mathbb{G}(Z)$ of Z, see Chapter 7.

Obviously it is possible to restrict the defining function $f(t)$ to symmetric intervals $(-a, a)$ about 0. Then the corresponding function $\mathbf{f}$ is defined on $\mathcal{D}(a) = \{z \in Z \mid |z| < a\}$. In this case, continuity of f on $(-a, a)$ implies continuity of $\mathbf{f}$ on $\mathcal{D}(a)$ by the same argument as above. The following proposition shows that if f is real analytic around 0 then $\mathbf{f}$ is also real analytic around 0, cf. [**28**, §3.19].

Proposition 3.23. *If $f(t)$ is real analytic for $|t| < \rho$, then the function $\mathbf{f}(z)$ is real analytic on the domain $\mathcal{D}(\rho) = \{z \in Z \mid |z| < \rho\}$.*

Example 3.24. As an example, we consider $f(t) = \tanh(t)$. This function yields a real analytic diffeomorphism of Z onto $\mathcal{D} = \mathcal{D}(1)$,

$$\mathbf{tanh} : Z \xrightarrow{\cong} \mathcal{D},\ \mathbf{tanh}(z) = \sum \tanh \lambda_i e_i \quad \text{for} \quad z = \sum \lambda_i e_i\ . \tag{3.36}$$

Another example is given by $f(t) = \tan(t)$. This defines a real analytic diffeomorphism of $\mathcal{D}(\frac{\pi}{2})$ onto Z, given by

$$\mathbf{tan} : \mathcal{D}(\tfrac{\pi}{2}) \xrightarrow{\cong} Z,\ \mathbf{tan}(z) = \sum \tan \lambda_i e_i \quad \text{for} \quad z = \sum \lambda_i e_i\ . \tag{3.37}$$

In these cases we prefer to denote the corresponding diffeomorphisms in normal style letters instead of boldface letters, i.e. we write $\tanh(z)$ and $\tan(z)$. These functions can be used to describe some geometric data on Z, in particular certain geodesics, cf. Theorem 3.29.

[3] In this definition, f could be restricted to $[0, \infty)$, since the spectral values of $z \in Z$ are non-negative. It is straightforward to check that in any case, $\mathbf{f}$ is an odd function. Therefore it is natural to define f as an odd function on $\mathbb{R}$. The transfer of analytic properties from f to $\mathbf{f}$ requires this extended domain of f.

So far, we discussed functions $f(t)$ which have nice analytic properties around 0. There are important examples, in which this is not the case. Consider the function $f : \mathbb{R} \to \mathbb{C}$ defined by $f(t) = 1/t$ for $t \neq 0$ and $f(0) = 0$. Due to Theorem 2.6, the corresponding function $\mathbf{f}(z)$ maps z onto its pseudo-inverse $z^\dagger$. We denote this by

$$\psi(t) = \begin{cases} 1/t & , t \neq 0 \\ 0 & , t = 0 \end{cases} \qquad \text{yields} \qquad \boldsymbol{\psi}(z) = z^\dagger \; . \tag{3.38}$$

For another example, take $f(t) = \operatorname{sign}(t)$ with $\operatorname{sign}(0) = 0$. This corresponds to the projection map of Z onto the set of tripotents we defined in Section 2.4, i.e.

$$\epsilon(t) = \operatorname{sign}(t) \qquad \text{yields} \qquad \boldsymbol{\epsilon}(z) = \sum e_i \quad \text{for} \quad z = \sum \lambda_i e_i \; . \tag{3.39}$$

We claim that the restrictions of these maps to the sets of constant rank elements $Z_j \subset Z$ have nice analytic properties.

Proposition 3.25. *If $f : \mathbb{R} \to \mathbb{C}$ is odd and real analytic on $\mathbb{R} \setminus \{0\}$, then for all $0 \leq j \leq r$ the restriction of the function $\mathbf{f}(z)$ to Z_j is real analytic.*

Before proving this proposition, we examine the connection between spectral values and the subsets Z_j. For this we note that the spectral decomposition $z = \sum \lambda_i e_i$ of some element $z \in Z$ can be refined by decomposing the tripotents e_i into the sum of pairwise orthogonal primitive tripotents. The set of tripotents we thus obtain can be completed to a frame of Z. Therefore we obtain

$$z = \sigma_1 c_1 + \ldots \sigma_r c_r \quad \text{with} \quad \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$$

This decomposition is no longer unique, since a non-primitive tripotent can be written in infinitely many different way as the sum of primitive tripotents. Nevertheless, the r-tuple $(\sigma_1, \ldots, \sigma_r)$ is uniquely determined by the spectral values λ_i of z and their multiplicities $\mu_i = \operatorname{rk}(e_i)$. We set

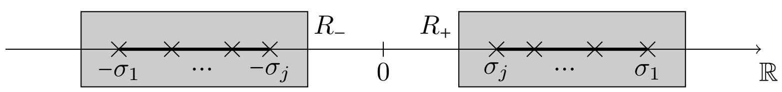
$$\sigma(z) := (\sigma_1, \ldots, \sigma_r) = (\lambda_1, \ldots, \lambda_1, \ldots, \lambda_k, \ldots, \lambda_k, 0, \ldots, 0) \in \mathbb{R}^r \; , \tag{3.40}$$

where each λ_i occurs μ_i-times. This defines the *spectral map* $\sigma : Z \to \mathbb{R}^r$. Strictly speaking, σ maps Z onto the set of non-negative, non-decreasing r-tuples, $\sigma(Z) = \{(\sigma_i) \in \mathbb{R}^r \,|\, \sigma_1 \geq \ldots \geq \sigma_r \geq 0\}$. For $0 \leq j \leq r$, we obtain

$$Z_j = \sigma^{-1}\big(\mathbb{R}^{>j} \times \{0\}^{r-j}\big) \; .$$

Here $\mathbb{R}_>$ denotes the set of positive real numbers. It is well-known that the spectral map σ is continuous.

Proof of Proposition 3.25. This proof is based on the ideas of the proof of Proposition 3.23 given by O. Loos in [**28**, §3.19]. Fix j and $z_0 \in Z_j$ with spectral values $\sigma_1 \geq \ldots \geq \sigma_j > 0$. Since f is real analytic on $\mathbb{R} \setminus \{0\}$, it can be extended holomorphically to some open neighborhood of the real interval $[\sigma_j, \sigma_1]$ in $\mathbb{C}$. Let $R_+ \subset \mathbb{C}$ be a rectangle within this neighborhood, parallel to the real axis and containing the closed interval $[\sigma_j, \sigma_1]$. By symmetry, f also extends holomorphically to some neighborhood of $R = R_+ \cup R_-$ with $R_- = -R_+$. The following diagram illustrates the situation:



Let $I = R^0 \cap \mathbb{R}$ be the real points of the interior R^0 of R. Then by Cauchy's integral formula and the involved symmetry, we obtain for all $t \in I$

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \int_{\partial R} \frac{f(\zeta)}{\zeta - t}\, d\zeta \\ &= \frac{1}{4\pi i} \left(\int_{\partial R} \frac{f(\zeta)}{\zeta - t}\, d\zeta - \int_{\partial R} \frac{f(-\zeta)}{-\zeta - t}\, d\zeta \right) \\ &= \frac{1}{4\pi i} \int_{\partial R} f(\zeta) \left(\frac{1}{\zeta - t} - \frac{1}{\zeta + t} \right) d\zeta \\ &= \frac{1}{2\pi i} \int_{\partial R} f(\zeta)\, \frac{t}{\zeta^2 - t^2}\, d\zeta\ . \end{aligned}$$

Set $U_j = \sigma^{-1}(I^j \times \{0\}^{r-j})$. This is an open subset of Z_j (with respect to the induced topology on $Z_j \subset Z$), since $U = \sigma^{-1}(I^j \times \mathbb{R}^{r-j})$ is open in Z and $U_j = U \cap Z_j$. Thus for $z \in U_j$ with $z = \sum \lambda_\ell e_\ell$ we have

$$\mathbf{f}(z) = \frac{1}{2\pi i} \int_{\partial R} f(\zeta) \sum_\ell \frac{t}{\zeta^2 - \lambda_\ell^2} e_\ell\, d\zeta\ .$$

Now consider Z as real vector space, denoted by $Z^{\mathbb{R}}$, and let $\widetilde{Z}$ be its complexification. Then the real polynomial map $z \mapsto B_{z,z}$ from $Z^{\mathbb{R}}$ to $\mathrm{End}(Z^{\mathbb{R}})$ has a complex extension $\widetilde{B} : \widetilde{Z} \to \mathrm{End}(\widetilde{Z})$. Furthermore, let $z \mapsto z^{[3]}$ denote the complex extension of the real polynomial map $z \mapsto z^{(3)}$. The key observation in this proof is the following relation, which is easily verified using the orthogonality of the involved tripotents e_ℓ:

$$\sum_\ell \frac{\lambda_\ell}{\zeta^2 - \lambda_\ell^2} e_\ell = \zeta^{-1} \widetilde{B}^{-1}_{\zeta^{-1} z} \left(\zeta^{-1} z - (\zeta^{-1} z)^{[3]} \right)\ . \tag{3.41}$$

Firstly we notice that this identity holds for all $z \in Z^{\mathbb{R}}$ and $\zeta \in \mathbb{C}^\times$, such that $\zeta \neq \pm\lambda_\ell$ for all spectral values λ_ℓ of z. In particular, equation (3.41) is valid for $z \in U_j$ and $\zeta \in \partial R$. Secondly, the right hand side of (3.41) is defined on the open subset of $\widetilde{Z} \times \mathbb{C}^\times$ consisting of all $z \in \widetilde{Z}$ and $\zeta \in \mathbb{C}^\times$, such that $\widetilde{B}_{\zeta^{-1} z}$ is invertible. In addition, the right hand side is holomorphic in z. Therefore, the function $z \mapsto \mathbf{f}(z)$ has a holomorphic extension to a neighborhood of U_j given by

$$\tilde{\mathbf{f}}(z) = \frac{1}{2\pi i} \int_{\partial R} f(\zeta) \widetilde{B}^{-1}_{\zeta^{-1} z} \left(\zeta^2 z - z^{[3]} \right) \frac{d\zeta}{\zeta^4}\ .$$

Since Z_j (and thus also U_j) is a holomorphic submanifold of Z, and hence a real analytic submanifold of $Z^{\mathbb{R}}$, this completes the proof. □

Before applying Proposition 3.25 to the base-tripotent map $\boldsymbol{\epsilon}$ and the pseudo-inverse map $\boldsymbol{\psi}$, we prove a lemma that uses the K-equivariance of $\mathbf{f}$ to determine parts of its derivative.

Lemma 3.26. *Let U be a K-invariant subset of Z, and let $\mathbf{f} : U \to Z$ be a K-equivariant map, i.e. $\mathbf{f}(ku) = k\,\mathbf{f}(u)$. Then for $u \in U$, the directional derivative $D_u\mathbf{f}(v)$ along $v = v_- \oplus v_{1/2} \in Z^u_- \oplus Z^u_{1/2}$ exists and is given by*

$$D_u\mathbf{f}(v) = (\mathbf{f}(u) \,\Box\, e)(u \,\Box\, e)^{-1} v_- + 4\,(\mathbf{f}(u) \,\Box\, e)(u^\dagger \,\Box\, e) v_{1/2}$$

Here, we set $e = \boldsymbol{\epsilon}(u)$.

Proof. For $w = w_- \oplus w_{1/2} \in Z^u_- \oplus Z^u_{1/2}$ consider the curve $k_t = \exp t(w \Box e - e \Box w) \in K$ with $t \in \mathbb{R}$. By the K-invariance of U, this curve stays in U. The Peirce rules and Lemma 2.15 imply

$$\begin{aligned} \tfrac{d}{dt}\Big|_{t=0}(k_t u) &= (w \,\square\, e - e \,\square\, w)u \\ &= \{w_-, e, u\} - \{e, w_-, u\} + \left\{w_{1/2}, e, u\right\} \\ &= 2\,\{u, e, w_-\} + \left\{u, e, w_{1/2}\right\} . \end{aligned}$$

The box operator $u \,\square\, e$ is invertible on $Z^u_{1/2}$ with inverse $4\,u^\dagger \,\square\, e$, since JT9 yields

$$4\,(u \,\square\, e)(u^\dagger \,\square\, e)(z) = 2\,Q_{u,u^\dagger} Q_e z + 2\,u \,\square\, (Q_e u^\dagger)(z) = 2\,u \,\square\, u^\dagger(z) = z$$

for all $z \in Z^u_{1/2}$. Using the joint Peirce decomposition with respect to e_i with $u = \sum \lambda_i e_i$ one easily verifies that the box operator $u \,\square\, e$ is also invertible on Z^u_1. It can be regarded as the left multiplication by u within the Jordan algebra Z^e_1. By JT12 and Lemma 2.15 we obtain on Z^u_1 the relation

$$Q_e(u \,\square\, e) = -(u \,\square\, e)Q_e + 2\,Q_{e,u} = -(e \,\square\, u)Q_e + 2\,(u \,\square\, e)Q_e = (u \,\square\, e)Q_e \;.$$

Therefore, the decomposition $Z^u_1 = Z^u_+ \oplus Z^u_-$ is invariant under the action of $u \,\square\, e$. Now set $w_- = \frac{1}{2}(u \,\square\, e)^{-1}(v_-)$ and $w_{1/2} = 4\left\{u^\dagger, e, v_{1/2}\right\}$ for the given $v = v_- + v_{1/2} \in Z^u_- \oplus Z^u_{1/2}$. Then we obtain $\frac{d}{dt}\big|_{t=0}(k_t u) = v$. In particular this shows that Z^u_- and $Z^u_{1/2}$ are subspaces of $T_u U$. By a similar computation as above, we obtain

$$\begin{aligned} D_u \mathbf{f}(v) = \tfrac{d}{dt}\Big|_{t=0}\mathbf{f}(k_t u) &= \tfrac{d}{dt}\Big|_{t=0} k_t(\mathbf{f}(u)) \\ &= (w \,\square\, e - e \,\square\, w)\mathbf{f}(u) \\ &= 2\,\{\mathbf{f}(u), e, w_-\} + \left\{\mathbf{f}(u), e, w_{1/2}\right\} \\ &= \left\{\mathbf{f}(u), e, (u \,\square\, e)^{-1} v_-\right\} + 4\left\{\mathbf{f}(u), e, \left\{u^\dagger, e, v_{1/2}\right\}\right\} . \end{aligned}$$

Replacing the Jordan triple product by the corresponding box operators yields the sought-after formula. ☐

Theorem 3.27. *Let Z be a phJTS of rank r. Fix $0 \le j \le r$.*

(a) *The base-tripotent map $\boldsymbol{\epsilon}$ defined by $z = \sum \lambda_i e_i \mapsto \sum e_i$ (using the spectral theorem) is a real analytic K-equivariant fibration of Z_j onto S_j. The fiber of $e \in S_j$ is given by*

$$\boldsymbol{\epsilon}^{-1}(e) = \Omega(Z^e_+) \,,$$

the symmetric cone of the euclidean Jordan algebra Z^e_+. Identifying $T_u Z_j$ with $Z^u_1 \oplus Z^u_{1/2}$ and $Z_e S_j$ with $Z^e_- \oplus Z^e_{1/2}$, the derivative of $\boldsymbol{\epsilon}$ in $u \in Z_j$ is given by

$$D_u \boldsymbol{\epsilon}(v) = (u \,\square\, e)^{-1}(v_-) + 2\,e \,\square\, u^\dagger(v_{1/2}) \,.$$

Here $e = \boldsymbol{\epsilon}(u)$, and $v = v_+ \oplus v_- \oplus v_{1/2}$ is the decomposition of v with respect to $Z^u_1 \oplus Z^u_{1/2} = Z^u_+ \oplus Z^u_- \oplus Z^u_{1/2}$.

(b) *The pseudo-inverse map $\boldsymbol{\psi}(z) = z^\dagger$ is a real analytic K-equivariant diffeomorphism on Z_j. Identifying $T_u Z_j$ and $T_{u^\dagger} Z_j$ with $Z^u_1 \oplus Z^u_{1/2} = Z^{u^\dagger}_1 \oplus Z^{u^\dagger}_{1/2}$, the derivative of $\boldsymbol{\psi}$ in $u \in Z_j$ is given by*

$$D_u \boldsymbol{\psi}(v) = -Q_{u^\dagger} v_1 + 2\left\{u^\dagger, u^\dagger, v_{1/2}\right\} .$$

Here $v = v_1 \oplus v_{1/2}$ is the decomposition of v with respect to $Z^u_1 \oplus Z^u_{1/2}$.

Proof. Proposition 3.25 shows that $\boldsymbol{\epsilon}$ and $\boldsymbol{\psi}$ are real analytic, since the defining functions $\epsilon(t) = 1$ and $\psi(t) = 1/t$ (for $t \neq 0$, $\epsilon(0) = \psi(0) = 0$) are real analytic functions on $\mathbb{R} \setminus \{0\}$. As $\epsilon(t)$ and $\psi(t)$ do not vanishing for $t \neq 0$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\psi}$ are rank-preserving maps. Next we determine the fibers of $\boldsymbol{\epsilon}$. Fix $e \in S_j$, and let $z = \sum \lambda_i e_i$

be the spectral decomposition of $z \in \boldsymbol{\epsilon}^{-1}(e)$, i.e. $\sum e_i = e$. By orthogonality of the tripotents, we obtain the identities

$$Q_e z = z\ , \quad Q_e e_i = e_i\ , \quad \{e_i, e, e_i\} = e_i\ , \quad \{e_i, e, e_j\} = 0 \quad \text{for all } i \neq j\ .$$

The first two identities show that z and the tripotents e_i are elements of the euclidean Jordan algebra Z_+^e. From this point of view, the last two equations imply that the e_i form an orthogonal system of idempotents in Z_+^a, and in combination with $\sum e_i = e$, this system is complete. Therefore, $z = \sum \lambda_i e_i$ is also the spectral decomposition within the euclidean Jordan algebra Z_+^e as defined in Theorem 1.5. Since all λ_i are positive, Proposition 1.7 implies that z is an element of the symmetric cone $\Omega(Z_+^e)$. This proves $\boldsymbol{\epsilon}^{-1}(e) \subset \Omega(Z_+^e)$. The converse immediately follows from Proposition 2.17: For $z \in \Omega(Z_+^e)$, the spectral decomposition $z = \sum \lambda_i e_i$ coincides with the spectral decomposition of z within the euclidean Jordan algebra Z_+^e, and hence $\sum e_i = e$, i.e. $z \in \boldsymbol{\epsilon}^{-1}(e)$.
Next we determine the derivative of $\boldsymbol{\epsilon}$ in $u \in Z_j$. Since the fiber through u, i.e. $\Omega(Z_+^e)$ with $e = \boldsymbol{\epsilon}(u)$, is an open subset of $Z_+^e = Z_+^u$, the derivative of $\boldsymbol{\epsilon}$ vanishes in the direction of Z_+^u. In addition, since Z_j is K-invariant and $\boldsymbol{\epsilon}$ is K-equivariant, Lemma 3.26 yields the asserted formula on $Z_-^u \oplus Z_{1/2}^u$.
Finally, we determine the derivative of the pseudo-inverse map $\boldsymbol{\psi}(u) = u^\dagger$. By definition of the pseudo-inverse, we have $u = Q_u \boldsymbol{\psi}(u)$ and $u \,\square\, \boldsymbol{\psi}(u) = \boldsymbol{\psi}(u) \,\square\, u$. Taking derivatives yields

$$v = 2\left\{u, u^\dagger, v\right\} + Q_u \psi' \quad \text{and} \quad v \,\square\, u^\dagger + u \,\square\, \psi' = \psi' \,\square\, u + u^\dagger \,\square\, v\ .$$

Here for brevity we set $\psi' = D_u \boldsymbol{\psi}(v)$ with $v \in Z_1^u \oplus Z_{1/2}^u$. Let $\psi' = \psi_1' \oplus \psi_{1/2}'$ be the decomposition according to $Z_1^u \oplus Z_{1/2}^u$. Applying $Q_{u^\dagger}$ to the first equation and using the Peirce rules implies $\psi_1' = -Q_{u^\dagger} v_1$. The second identity applied to $u^\dagger$ yields

$$\left\{v, u^\dagger, u^\dagger\right\} + \left\{u, \psi', u^\dagger\right\} = \left\{\psi', u, u^\dagger\right\} + \left\{u^\dagger, v, u^\dagger\right\}\ .$$

Therefore, again using the Peirce rules, the $Z_{1/2}^u$-component of ψ' is given by

$$(\psi')_{1/2} = 2\ \left\{u^\dagger, u^\dagger, v_{1/2}\right\}\ .$$

In summation this proves the assertion. □

REMARK 3.28. For $u = \boldsymbol{\epsilon}(u) = e \in S_j$, the derivative of $\boldsymbol{\epsilon}$ in e just becomes

$$D_e \boldsymbol{\epsilon}(v) = v_- + v_{1/2}\ ,$$

i.e. $D_e \boldsymbol{\epsilon}$ is the (real) orthogonal projection of $Z_1^e \oplus Z_{1/2}^e$ onto $Z_-^e \oplus Z_{1/2}^e$. This stays in accordance with the fact that $\boldsymbol{\epsilon}$ is a projection of Z_j onto S_j leaving S_j fixed.

The decomposition of the differential of $\boldsymbol{\psi}$ with respect to the Peirce spaces of u shows that $\boldsymbol{\psi}$ is holomorphic in the $Z_{1/2}^u$-direction and anti-holomorphic in the Z_1^u-direction. For a tripotent $e = u = u^\dagger$ we obtain

$$D_e \boldsymbol{\psi}(v) = -Q_e v_1 + v_{1/2} = -v_1^\# + v_{1/2}\ .$$

Therefore, $D_e \boldsymbol{\psi}(v)$ acts identically on $Z_-^e \oplus Z_{1/2}^e$. This was to be expected, since $Z_-^e \oplus Z_{1/2}^e$ represents the tangent space of the set of tripotents S at e, which coincides with the fixed point set of $\boldsymbol{\psi}$.

3.5. Bounded symmetric domains

We review the basic results on bounded symmetric domains associated with positive hermitian Jordan triple systems. For a detailed account, we refer to **[28]**.

Theorem 3.29. *Let Z be a phJTS, and let $\mathcal{D}$ be the open unit ball with respect to the spectral norm on Z,*

$$\mathcal{D} = \{z \in Z \,|\, |z| < 1\} \ . \tag{3.42}$$

Then, $\mathcal{D}$ is a bounded symmetric domain. Moreover,

(i) the hermitian metric on $\mathcal{D}$ is given by

$$h_z(u,v) = \left\langle B_{z,z}^{-1} u \middle| v \right\rangle$$

for all $z \in \mathcal{D}$, $u, v \in T_z\mathcal{D} \cong Z$, and the curvature at 0 is

$$R_0(u,v)w = -\{u,\, v,\, w\} + \{v,\, u,\, w\}$$

for all $u, v, w \in T_0\mathcal{D} \cong Z$,

(ii) the symmetry around 0 is given by $s(z) = -z$,

(iii) the stabilizer of $0 \in \mathcal{D}$ in the automorphism group $\mathrm{Aut}(\mathcal{D})$ of $\mathcal{D}$ coincides with the automorphism group $\mathrm{Aut}(Z)$ of Z,

(iv) the Lie algebra $\mathfrak{g}$ of $\mathrm{Aut}(\mathcal{D})$ decomposes according to the Cartan involution Ad_s into

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \textit{with} \quad \mathfrak{k} = \mathrm{Der}(Z),\ \mathfrak{p} = \{\zeta_v(z) = v - Q_z v \,|\, v \in Z\} \cong_{\mathbb{R}} Z\ ,$$

where the elements of $\mathfrak{p}$ are represented as vector fields on $\mathcal{D}$,

(v) the exponential map $\mathrm{Exp}_0 : Z \to \mathcal{D}$ of the metric at 0 coincides with the real analytic diffeomorphism given by $\mathrm{Exp}_0(v) = \tanh(v)$.

We usually denote by G the identity component of $\mathrm{Aut}(\mathcal{D})$, then Theorem 3.29 implies that

$$\mathcal{D} = G/K \quad \text{with} \quad G = \mathrm{Aut}(\mathcal{D})^0,\ K = \mathrm{Aut}(Z)^0\ , \tag{3.43}$$

and K is a maximal compact subgroup of G.

Example 3.30. In the matrix case $Z = \mathbb{C}^{r\times s}$, the unit ball $\mathcal{D}$ is given by

$$\mathcal{D} = \{z \in \mathbb{C}^{r\times s} \,|\, zz^* \ll 1\}\ , \tag{3.44}$$

since the eigenvalues of the matrix zz^* are just the squares of the singular values of z. The identity component of the automorphism group of $\mathcal{D}$ is

$$G = \mathrm{P}(\mathrm{U}(r,s)) = \left\{g \in \mathbb{C}^{(r+s)\times(r+s)} \,\middle|\, g^* \left(\begin{smallmatrix} 1_r & 0 \\ 0 & -1_s \end{smallmatrix}\right) g = \left(\begin{smallmatrix} 1_r & 0 \\ 0 & -1_s \end{smallmatrix}\right)\right\} / \mathbb{C}^\times\ , \tag{3.45}$$

action on $\mathcal{D}$ as Möbius transformations

$$g(z) = (az+b)(cz+d)^{-1} \quad \text{with} \quad g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\ . \tag{3.46}$$

As in the case of the automorphism group $\mathrm{Aut}(Z)$ of Z, we often prefer to consider the finite cover $\mathrm{SU}(r,s)$ of G, and by abuse of notation we write $G = \mathrm{SU}(r,s)$. We note, that a simple calculation shows that the stabilizer of 0 in G indeed coincides with $K = \mathrm{S}(U(r) \times U(s))$, regarded as the subgroup of block diagonal matrices in G. The Lie algebra of G is represented by vector fields of the form

$$\zeta(z) = Az - zD + B - zB^*z \quad \text{with} \quad A^* = -A,\ D^* = -D,\ B \in Z\ . \tag{3.47}$$

This follows from the derivative of $g(z)$ in (3.46) with respect to g.

Remark 3.31. We note that the converse of Theorem 3.29 is also true: Any bounded symmetric domain is isomorphic to the open unit ball of some positive hermitian Jordan triple system [**28**, §2]. The Jordan triple product can either be extracted from the curvature tensor or by the Bergman kernel function associated to bounded domains. This connection between bounded symmetric domains and positive hermitian Jordan triple systems extends to a correspondence of irreducible components of the symmetric domain on the one hand, and simple ideals of the phJTS on the other hand. We therefore obtain a classification of bounded symmetric domains by simple phJTS [**28**, §4]. More generally, since any hermitian symmetric space of non-compact type admits a Harish-Chandra realization as a bounded symmetric domain [**14**, VIII, §7], this also yields a classification of hermitian symmetric spaces of non-compact type. Finally, there is a duality between hermitian symmetric spaces of non-compact type to those of compact type, so we obtain the following chain of bijections (up to isomorphisms), which also refines to a bijection of simple phJTS and corresponding irreducible objects:

$$\{\text{phJTS}\} \leftrightarrow \left\{ \begin{matrix} \text{bd. symm.} \\ \text{domains} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{herm. symm. spaces} \\ \text{of non-cpt. type} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \text{herm. symm. spaces} \\ \text{of cpt. type} \end{matrix} \right\}$$

On the Lie theoretic level [**44**], the compact dual of a hermitian symmetric space $\mathcal{D} = G/K$ of non-compact type is given by $X = G^c/K$, where G^c is a compact real from of the complexified Lie group $G^{\mathbb{C}}$ such that $K \subset G^c$. For the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^c$ for G and G^c, considered as real subalgebras of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathrm{Lie}(G^{\mathbb{C}})$, this amounts to

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{and} \quad \mathfrak{g}^c = \mathfrak{k} \oplus i\,\mathfrak{p} \,. \tag{3.48}$$

Example 3.32. For the matrix case $Z = \mathbb{C}^{r\times s}$, the complexification of $G = \mathrm{SU}(r,s)$ is given by $G^{\mathbb{C}} = \mathrm{SL}(r+s)$, and thus we obtain the compact real form $G^c = \mathrm{SU}(r+s)$. Therefore, the compact dual X of the bounded symmetric domain $\mathcal{D}$ is

$$X = \mathrm{SU}(r+s)/\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(s)) \cong \mathbb{G}_s(\mathbb{C}^{r+s}) \,. \tag{3.49}$$

Indeed, the special unitary group acts on the Grassmannian manifold $\mathbb{G}_s(\mathbb{C}^{r+s})$ transitively, since it even acts transitively on the set of orthonormal bases. Moreover, the stabilizer of the subspace $\{0\}^r \times \mathbb{C}^s$ of $\mathbb{C}^{r+s}$ is $\mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(s))$.

In the second part of this thesis we give a Jordan theoretic description of the compact dual G^c/K of the bounded symmetric domain $\mathcal{D} = G/K$.

Part 2

Orbit structures on the symmetric Grassmannian variety

CHAPTER 4

Grassmannian variety

The object of this chapter is the compact dual of a bounded symmetric domain $\mathcal{D}$, which is defined via the spectral norm of a positive hermitian Jordan triple system Z. Initiated by the matrix case $Z = \mathbb{C}^{r\times s}$, we call the compact dual the Grassmannian variety of Z and denote it by $\mathbb{G}(Z)$. The work of O. Loos [**28, 29**] provides an inventive Jordan theoretic description of the Grassmannian, which we recall in Section 4.1. The fundamental construction is an equivalence relation on $Z \times \overline{Z}$, for which the set of equivalence classes - the Grassmannian - yields a compactification of the triple system Z. The identification of the Grassmannian with the compact dual stems from an appropriate group action on the Grassmannian (Section 4.2). We use the Godement approach to describe certain vector and line bundles on the Grassmannian. The Jordan theoretic treatment of the compact dual is distinctive algebraic geometric.

In Section 4.3, we extend the notion of partial Cayley mappings, which are usually defined with respect to *tripotent* elements, to partial Cayley mappings which admit arbitrary elements as reference point. We use these maps in Chapter 6 to describe chart maps for submanifolds of the Grassmannian "at infinity".

In Section 4.4, we take advantage of the definition of the Grassmannian via equivalence classes to provide two new descriptions of $\mathbb{G}(Z)$ by introducing two different sets of representatives. It turns out that these descriptions are closely related to the two orbit structures of the Grassmannian which are studied in Chapter 7.

4.1. Loos' construction

Basic definition. Let Z be a phJTS and $\mathcal{D}$ its unit ball with respect to the spectral norm, $\mathcal{D} = \{z \in Z \,|\, |z| < 1\}$, as discussed in Section 3.5. In this section we review the Jordan theoretic description of the compact dual of $\mathcal{D}$ given by O. Loos in [**28**].

We first examine the matrix case $Z = \mathbb{C}^{r\times s}$, $r \le s$. According to Examples 3.30 and 3.32, we have

$$\mathcal{D} = \{z \in \mathbb{C}^{r\times s} \,|\, 1 - zz^* \gg 0\} = G/K \quad \text{with} \quad G = \mathrm{SU}(r,s),\ K = S(U(r)\times U(s))\,,$$

and via Lie theory, the compact dual of $\mathcal{D}$ is given by

$$\mathbb{G} = \mathrm{Gr}_s(\mathbb{C}^{r+s}) = G^c/K \quad \text{with} \quad G^c = \mathrm{SU}(r+s)\,.$$

Therefore, we need a Jordan theoretic description of s-planes in $\mathbb{C}^{r+s}$. The key observation for this is the following result. Let $n = r+s$, and for $(z,a) \in Z \times \overline{Z}$ set

$$C_{z,a} := \begin{pmatrix} z \\ 1_s - a^* z \end{pmatrix} \in \mathbb{C}^{n\times s} \quad \text{and} \quad E_{z,a} := \text{column space of } C_{z,a}\,. \tag{4.1}$$

Lemma 4.1. *For $(z,a) \in Z \times \overline{Z}$ let $C_{z,a}$ and $E_{z,a}$ by as in* (4.1)*. Then,*

(a) *$E_{z,a}$ is an s-dimensional subspace of $\mathbb{C}^n$.*
(b) *Each subspace $E \subset \mathbb{C}^n$ with $\dim E = s$ has a representation $E = E_{z,a}$ with $(z,a) \in Z \times \overline{Z}$.*
(c) *Two subspaces $E_{z,a}$ and $E_{\tilde{z},\tilde{a}}$ coincide if and only if $(z, a - \tilde{a})$ is quasi-invertible and $\tilde{z} = z^{a-\tilde{a}}$, i.e. $1_s - (a-\tilde{a})^*z \in \mathbb{C}^{s\times s}$ is invertible and $\tilde{z} = z(1_s - (a-\tilde{a})^*z)^{-1}$.*

Proof. For (a) we have to prove that $C_{z,a}$ has rank s. Therefore we show that the linear map $(x \mapsto C_{z,a}x)$ with $x \in \mathbb{C}^s$ is one-to-one: if $C_{z,a}x = 0$ then $zx = 0$ and $(1 - a^*z)x = 0$, that is $zx = 0$ and $x = a^*zx = 0$.
Now let E be an arbitrary s-dimensional subspace in $\mathbb{C}^n$. We choose a basis $b_1, \dots, b_s \in \mathbb{C}^n$ of E and set $B := (b_1, \dots, b_s) \in \mathbb{C}^{n\times s}$. Since the column space of two matrices $A, B \in \mathbb{C}^{n\times s}$ are equal if and only if $B = Ag$ for an element $g \in \mathrm{GL}(\mathbb{C}^s)$, we have to show the existence of elements $(z,a) \in Z \times \overline{Z}$ and $g \in \mathrm{GL}(\mathbb{C}^s)$ with $B = C_{z,a}g$. We decompose $B = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ into block matrices $x \in \mathbb{C}^{r\times s}$ and $y \in \mathbb{C}^{s\times s}$. Then $B = C_{z,a}g$ is equivalent to $x = zg$ and $y = g - a^*x$. Now it suffices to find an element $a \in \mathbb{C}^{r\times s}$ such that $g = y + a^*x$ is invertible, since then we can set $z = xg^{-1}$ to get the assertion. Let $x^1, \dots, x^r$ and $y^1, \dots, y^s$ be the rows of x and y, respectively. We choose an index set $J \subset \{1, \dots, s\}$ such that $\{y^j\}_{j\in J}$ is a basis of the row space of y, we set $q := |J|$. Since $\operatorname{rk} B = s$ there are $s-q$ indices $I \subset \{1, \dots, r\}$ such that the span $\langle x^i, y^j \,|\, i \in I, j \in J\rangle$ equals $\mathbb{C}^s$. Without loss of generality we assume $J = \{1, \dots, q\}$ and $I = \{1, \dots, s-q\}$ (otherwise we have to insert an appropriate permutation matrix into the following). Now take $a^* = \left(\begin{smallmatrix} 0 & 0 \\ 1_{s-q} & 0 \end{smallmatrix}\right)$, then using shear invariance of the determinant, we obtain

$$y + a^*x = \begin{pmatrix} y^1 \\ \vdots \\ y^q \\ y^{q+1} + x^1 \\ \vdots \\ y^s + x^{s-q} \end{pmatrix} \quad \text{and} \quad \det(y + a^*x) = \det \begin{pmatrix} y^1 \\ \vdots \\ y^q \\ x^1 \\ \vdots \\ x^{s-q} \end{pmatrix} \neq 0\,,$$

This proves (b). Finally we have to show the equivalence in (c): $E_{z,a} = E_{\tilde{z},\tilde{a}}$ if and only if $C_{z,a} = C_{\tilde{z},\tilde{a}}g$ for some $g \in \mathrm{GL}(\mathbb{C}^s)$, and this is equivalent to

$$z = \tilde{z}g \quad \text{and} \quad 1 - a^*z = (1 - \tilde{a}^*\tilde{z})g\,.$$

Plugging the first equation into the second we obtain $g = 1 - (a-\tilde{a})^*z$ and in conjunction with the first one this gives the assertion. □

Remark 4.2. In the matrix case $Z = \mathbb{C}^{r\times s}$ it is possible to use either row or column spaces in the Jordan theoretic construction of the compact dual. O. Loos uses the row space of $A'_{z,a} = (1_r - za^*, z) \in \mathbb{C}^{r\times n}$ for a model of r-dimensional subspaces in $\mathbb{C}^n$ and thus gets $\mathrm{Gr}_r(\mathbb{C}^n)$ instead of $\mathrm{Gr}_s(\mathbb{C}^n)$ as the compact dual of $\mathcal{D}$. We prefer the construction via column spaces of $C_{z,a}$ as in Lemma 4.1, since here the group actions of G, K, $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$ on the compact dual is simply given by left multiplication by g (instead of right multiplication by g^*) and matches the ordinary Möbius transformation on the affine realization given by

$$\mathbb{C}^{r\times s} \hookrightarrow \mathrm{Gr}_r(\mathbb{C}^n),\ z \mapsto \text{column space of } \begin{pmatrix} z \\ 1 \end{pmatrix}.$$

For further information on these group actions see Example 4.9 in the next section.

We now turn to the general case. Let Z be an arbitrary phJTS. Using the properties of the quasi-inverse one can easily check that

$$(z,a) \sim (\tilde z,\tilde a) \iff \begin{cases} (z,a-\tilde a) \text{ is quasi-invertible} \\ \text{and } \tilde z = z^{a-\tilde a}\,. \end{cases} \tag{4.2}$$

defines an equivalence relation on $Z \times \overline{Z}$. Inspired by the matrix-case we call

$$\mathbb{G}(Z) := (Z \times \overline{Z})/\sim \tag{4.3}$$

the *Grassmannian* of Z. If there is no danger of ambiguity, we will write $\mathbb{G}$ instead of $\mathbb{G}(Z)$. Let $[z:a]$ denote the elements of $\mathbb{G}$, i.e. the equivalence class of (z,a). The next proposition states that $\mathbb{G}$ is a smooth algebraic variety containing Z as an open and dense subset, and hence $\mathbb{G}$ is a compactification of Z.

Proposition 4.3 ([**28**, §7.7]). *For every $a \in \overline{Z}$ let $\mathbb{G}^{(a)} = \{[z:a] \mid z \in Z\} \subset \mathbb{G}$. Then the map $\varphi_a : \mathbb{G}^{(a)} \to Z$, $[z:a] \mapsto z$, is bijective, and the $\mathbb{G}^{(a)}$ form a covering of $\mathbb{G}$. There exists a unique structure of a smooth algebraic variety on $\mathbb{G}$ such that each $\mathbb{G}^{(a)}$ is an open affine subvariety, isomorphic to Z under φ_a. In particular, $Z = \mathbb{G}^{(0)}$ is open and dense in $\mathbb{G}$. Every finite subset of $\mathbb{G}$ is contained in one of the $\mathbb{G}^{(a)}$.*

Tangential structure. The transition function $\varphi^a_{\tilde a}$ of the chart maps φ_a and $\varphi_{\tilde a}$ of $\mathbb{G}$ is defined on the open and dense subset $\varphi_a(\mathbb{G}^{(a)} \cap \mathbb{G}^{(\tilde a)}) = \{z \in Z \mid (z, a-\tilde a) \text{ is quasi-invertible}\}$ and satisfies

$$\varphi^a_{\tilde a}(z) = z^{a-\tilde a} \qquad \text{and} \qquad d\varphi^a_{\tilde a}(z) = B^{-1}_{z,\,a-\tilde a}\,. \tag{4.4}$$

As Proposition 4.3 states, the transition functions of $\mathbb{G}$ are birational mappings on Z. Let $\chi = [z:a] = [\tilde z:\tilde a]$ be an element of $\mathbb{G}$, and for a tangent vector $X \in T_\chi\mathbb{G}(Z)$ set $X^{(a)} := d\varphi_a(X)$ and $X^{(\tilde a)} := d\varphi_{\tilde a}(X)$. Then, identifying T_zZ and $T_{\tilde z}Z$ with Z, we obtain

$$X^{(a)} = B_{z,\,a-\tilde a}X^{(\tilde a)}\,.$$

More generally, a vector field $\zeta : \mathbb{G} \to T\mathbb{G}$ on $\mathbb{G}$ is described locally by

$$\zeta^{(a)} : Z \to Z,\; z \mapsto d\varphi_a([z:a])\zeta([z:a])\,.$$

Here the transformation rule

$$\zeta^{(a)}(z) = B_{z,\,a-\tilde a}\,\zeta^{(\tilde a)}(z^{a-\tilde a}) \tag{4.5}$$

holds on the open and dense subset $\{z \in Z \mid (z, a-\tilde a) \text{ is quasi-invertible}\} \subset Z$.

Godement approach. Since $\mathbb{G}$ is defined via the equivalence relation (4.2), we could also use Godement's Theorem to establish a manifold structure on $\mathbb{G}$ by showing that (4.2) is a *regular* equivalence relation. Provided that the canonical projection $\pi : Z \times \overline{Z} \to \mathbb{G}$ is a submersion, Theorem 3.5 implies that such a manifold structure coincides with the one given by Proposition 4.3. Therefore, we are content with showing that π is indeed a submersion: For fixed $(z,a) \in Z \times \overline{Z}$ we have $(\varphi_a \circ \pi)(\tilde z,\tilde a) = \tilde z^{\tilde a - a}$. Using the identities $\frac{d}{dx}(x^y) = B^{-1}_{x,y}$ and $\frac{d}{dy}(x^y) = Q_xB^{-1}_{y,x}$ (see e.g. [**28**, §7.7]), we obtain

$$D_{(z,a)}(\varphi_a \circ \pi) : Z \times \overline{Z} \to Z,\; (\dot z,\dot a) \mapsto \dot z + Q_z\dot a\,. \tag{4.6}$$

This shows that π is a submersion. Now we can use Godement's Theorem conversely, and we conclude that (4.2) is indeed a regular equivalence relation. As a submanifold of $(Z \times \overline{Z}) \times (Z \times \overline{Z})$, it is given by

$$R_{\mathbb{G}} = \big\{\big((z,a),(\tilde z,\tilde a)\big) \,\big|\, (z,a-\tilde a) \text{ quasi-invertible},\; \tilde z = z^{a-\tilde a}\big\} \subset (Z \times \overline{Z})^2\,. \tag{4.7}$$

In accordance with Remark 3.7 and Proposition 4.3 we note that the submanifolds $Z \times \{a\} \subset Z \times \overline{Z}$ are minimally transversal to the equivalence classes in $Z \times \overline{Z}$, since from 4.6 we obtain

$$T_{(z,a)}\,[z:a] = \left\{(-Q_z v, v) \in Z \times \overline{Z} \,\middle|\, v \in \overline{Z}\right\} \,, \tag{4.8}$$

and therefore $T_{(z,a)}\,[z:a] \oplus T_{(z,a)}(Z \times \{a\}) = Z \times \overline{Z}$. Considering $\mathbb{G}$ as a quotient manifold is particularly useful when we discuss line bundles on $\mathbb{G}$.

Vector bundles. We use the Godement approach to describe vector bundles on $\mathbb{G}$ via Theorem 3.8. Let $R_{\mathbb{G}}$ be as in (4.7). Then the tangent bundle on $\mathbb{G}$ is given by the cocycle

$$\phi : R_{\mathbb{G}} \to \mathrm{GL}(Z),\ \big((z,a),(\tilde{z},\tilde{a})\big) \mapsto \phi^{(z,a)}_{(\tilde{z},\tilde{a})} = B^{-1}_{z,\,a-\tilde{a}} \,. \tag{4.9}$$

Indeed, this is well-defined, since for pairwise equivalent $(x,a),\ (y,b),\ (z,c) \in Z \times \overline{Z}$ we have due to JT33

$$\phi^{(y,b)}_{(z,c)} \circ \phi^{(x,a)}_{(y,b)} = B^{-1}_{y,\,b-c} B^{-1}_{x,\,a-b} = \left(B_{x,\,a-b} B_{x^{a-b},\,b-c}\right)^{-1} = B^{-1}_{x,\,a-c} = \phi^{(x,a)}_{(z,c)} \,.$$

According to Remark 3.9, we obtain on the $Z \times \{a\},\ Z \times \{\tilde{a}\} \subset Z \times \overline{Z}$ the transition function $(z,v) \mapsto (z^{a-\tilde{a}}, B^{-1}_{z,\,a-\tilde{a}} v)$, which indeed coincides with the transition function of the tangent bundle in the corresponding coordinate patches. By the same argument it follows that

$$\phi^{\#} : R \to \mathrm{GL}(\overline{Z}),\ \big((z,a),(\tilde{z},\tilde{a})\big) \mapsto B_{a-\tilde{a},\,z} \tag{4.10}$$

describes the cotangent bundle $T^{\#}\mathbb{G}$, and the canonical line bundle $\mathcal{K}$ on $\mathbb{G}$ is given by the cocycle

$$\phi_{\mathcal{K}} : R \to \mathrm{GL}(\mathbb{C}),\ \big((z,a),(\tilde{z},\tilde{a})\big) \mapsto \mathrm{Det}\, B^{-1}_{z,\,a-\tilde{a}} \,. \tag{4.11}$$

In addition, by Lemma 2.23 also the Jordan triple determinant $\varDelta$ satisfies the cocycle condition, and therefore defines a line bundle

$$\mathcal{L} = (Z \times \overline{Z} \times \mathbb{C})/\sim \quad \text{with} \quad (z,a,\lambda) \sim (\tilde{z},\tilde{a},\tilde{\lambda}) \iff \begin{cases} \tilde{z} = z^{a-\tilde{a}}, \\ \tilde{\lambda} = \varDelta\,(z, a-\tilde{a})\ \lambda \,. \end{cases} \tag{4.12}$$

More generally, due to Lemma 2.23 any denominator δ of the quasi-inverse gives rise to a line bundle $\mathcal{L}_\delta$. Any such line bundle can be used to show that $\mathbb{G}$ is a projective variety, i.e. admits an appropriate imbedding into some projective space.

Theorem 4.4. *The line bundles $\mathcal{L}_\delta$ are very ample. The imbedding defined by $\mathcal{L}_\delta$ is closed and hence $\mathbb{G}$ is a projective variety.*

For a proof we refer to [**28**, §7.10], where this statement is proved for more general Jordan pairs. If Z is simple, Theorem 2.40 implies that the line bundle $\mathcal{L} = \mathcal{L}_\varDelta$ is related to the canonical line bundle by $\mathcal{L}^{\mathsf{p}} = \mathcal{K}^{-1}$, where p denotes the genus of Z. One can show [**29**] that $\mathcal{L}$ even generates the Picard group of $\mathbb{G}$. We denote by $\mathcal{L}_\delta^{-1}$ the inverse of $\mathcal{L}_\delta$, i.e. the line bundle defined by the inverse cocycle, $\mathcal{L}_\delta^{-1} = \mathcal{L}_{\delta^{-1}}$.

Finally, we note that all these vector bundles are trivialized on open and dense subsets of $\mathbb{G}$, namely on $\mathbb{G}^{(a)} = \iota_a(Z)$ for $\iota_a(z) = [z:a]$. For a vector bundle $\mathcal{E}$ with defining cocycle $\phi : R_{\mathbb{G}} \to \mathrm{GL}(E)$ and canonical projection $\pi : \mathcal{E} \to \mathbb{G}$, set $\mathcal{E}^{(a)} = \pi^{-1}(\mathbb{G}^{(a)})$. Then $\mathcal{E}^{(a)}$ is open and dense in $\mathcal{E}$, and the transition functions are described by

$$\begin{tikzcd} & Z \times E \arrow[dl, "\varphi_a"'] \arrow[dd, "\varphi_b^a"] \\ \mathcal{E}^{(a)} \cap \mathcal{E}^{(b)} & \\ & Z \times E \arrow[ul, "\varphi_b"] \end{tikzcd} \quad \text{with} \quad \begin{cases} \varphi_a(z,v) = [z,a,v]\,, \\ \varphi_b(\tilde{z},\tilde{v}) = [\tilde{z},b,\tilde{v}]\,, \\ \varphi_b^a(z,v) = \left(z^{a-b},\ \phi_{(\tilde{z},\tilde{a})}^{(z,a)}\, v\right). \end{cases}$$

Loos' construction revisited. The construction of the Grassmannian $\mathbb{G}$ via equivalence classes admits the following interpretation: Regard the product $Z \times \overline{Z}$ as the product of some *base* Z and a *parameter space* $\overline{Z}$ parametrizing the chart maps φ_a given in Proposition 4.3. Then, the equivalence relation on $Z \times \overline{Z}$ corresponds exactly to the gluing procedure of two different bases $Z \times \{a\}, Z \times \{\tilde{a}\} \subset Z \times \overline{Z}$ via the transition map $\varphi_{\tilde{a}}^{a}$. The advantage of using equivalence classes comes in when we choose different sets of representatives for the elements of the Grassmannian $\mathbb{G}$. In this case, not only the structure of the base (e.g. its topology) is available, but also the structure of the parameter space. The canonical projection

$$\pi : Z \times \overline{Z} \to \mathbb{G},\ (z,a) \mapsto [z:a]$$

is a morphism of smooth algebraic varieties.

Conjugate Grassmannian. Let $\overline{Z}$ be the conjugate phJTS of Z as defined in Section 2.2. The Grassmannian $\mathbb{G}(\overline{Z})$ of $\overline{Z}$ is called the *conjugate Grassmannian*. For shorthand we set $\overline{\mathbb{G}} = \mathbb{G}(\overline{Z})$. As in the case of the conjugation of Z, the conjugate Grassmannian $\overline{\mathbb{G}}$ coincides with the Grassmannian $\mathbb{G}$ as sets, but inherits the opposite complex structure. Various structures on $\mathbb{G}$ can be transferred to $\overline{\mathbb{G}}$ by complex conjugation. For example, the Jordan triple determinant $\overline{\Delta}$ of $\overline{\mathbb{G}}$ is the complex conjugate of the Jordan triple determinant of $\mathbb{G}$, i.e. $\overline{\Delta}(z,a) = \overline{\Delta(z,a)}$. If $\mathcal{E}$ is a holomorphic vector bundle on $\mathbb{G}$, then we denote by $\overline{\mathcal{E}}$ the corresponding holomorphic vector bundle on $\overline{\mathbb{G}}$ obtained for $\mathcal{E}$ by taking the conjugate complex structure on the fibers of $\mathcal{E}$. In particular, we denote by $\overline{\mathcal{L}}$ the line bundle on $\overline{\mathbb{G}}$ corresponding to the Jordan triple product $\overline{\Delta}$.

Example 4.5. In the matrix case $Z = \mathbb{C}^{r\times s}$, $r \geq s$, we identify the conjugate Grassmannian $\overline{\mathbb{G}}$ with the Grassmannian manifold $\mathrm{Gr}_r(\mathbb{C}^n)$ with $n = r+s$, so we set $\overline{\mathrm{Gr}_s(\mathbb{C}^n)} = \mathrm{Gr}_r(\mathbb{C}^n)$. This is justified by the antiholomorphic map

$$\Gamma : \mathrm{Gr}_s(\mathbb{C}^n) \to \mathrm{Gr}_r(\mathbb{C}^n),\ E \mapsto E^\perp\,. \tag{4.13}$$

Indeed, it is straightforward to see that elements of $\mathrm{Gr}_r(\mathbb{C}^n)$ can be represented by elements $(z,a) \in \overline{Z} \times Z$ via

$$\overline{C}_{z,a} := \begin{pmatrix} 1 - az^* \\ -z^* \end{pmatrix} \in \mathbb{C}^{n\times r} \quad \text{and} \quad \overline{E}_{z,a} := \text{column space of } \overline{C}_{z,a}\,. \tag{4.14}$$

As in the case of $\mathrm{Gr}_s(\mathbb{C}^n)$, this induces the same equivalence relation on the set $\overline{Z} \times Z$ as in (4.2), so $\mathrm{Gr}_r(\mathbb{C}^n) \cong (\overline{Z} \times Z)/\sim$. For fixed $a \in Z$, we obtain the chart map $\overline{\varphi}_a([z:a]) = z$. Moreover, with $C_{z,a}$ and $E_{z,a}$ as in (4.1), we have $E_{z,a}^\perp = \overline{E}_{z,a}$, since

$$(C_{z,a})^* \overline{C}_{z,a} = \begin{pmatrix} z^* & 1 - z^* a \end{pmatrix} \begin{pmatrix} 1 - az^* \\ -z^* \end{pmatrix} = z^* - z^* a z^* - z^* + z^* a z^* = 0\,.$$

Therefore, for fixed a, the map Γ in (4.13) is locally given by

$$\overline{\varphi}_a \circ \Gamma \circ \varphi_a^{-1} : Z \to \overline{Z},\ z \mapsto z\,.$$

In particular, Γ is anti-holomorphic, since $\overline{Z}$ carries the complex conjugate structure of Z. In addition, we note that there is a second important identification of the conjugate Grassmannian $\overline{\mathbb{G}}$ with $\mathrm{Gr}_r(\mathbb{C}^n)$ defined by $(z,a) \mapsto \overline{E}'_{z,a}$ with

$$\overline{C}'_{z,a} := \begin{pmatrix} 1 - az^* \\ z^* \end{pmatrix} \in \mathbb{C}^{n\times r} \quad \text{and} \quad \overline{E}'_{z,a} := \text{column space of } \overline{C}_{z,a} \, . \tag{4.15}$$

The only (but crucial) difference to (4.14) is the change of sign in the lower part of $\overline{C}'_{z,a}$. The isomorphism between these two different realizations of $\overline{\mathbb{G}}$ is given by the symmetry s around $[0:0] \in \overline{\mathbb{G}}$ (cf. Example 4.9), and the corresponding antiholomorphic map Γ' identifying $\mathrm{Gr}_s(\mathbb{C}^n)$ with $\mathrm{Gr}_r(\mathbb{C}^n)$ is given by $E \mapsto E^{\perp'}$, where $\perp'$ denotes the orthogonal complement of E with respect to the pseudo-hermitian inner product on $\mathbb{C}^n$ given by

$$\langle x|y\rangle' = x^* \left(\begin{smallmatrix} 1_r & 0 \\ 0 & -1_s \end{smallmatrix}\right) y$$

for $x, y \in \mathbb{C}^n$.

Sub-Grassmannians. Let $W \subset Z$ be a subtriple of a phJTS. By Corollary 2.29, quasi-invertibility transfers from $W \times \overline{W}$ to $Z \times \overline{Z}$, and therefore the canonical injection $W \times \overline{W} \hookrightarrow Z \times \overline{Z}$ quotients to an injection of the Grassmannian $\mathbb{G}(W)$ of W into the Grassmannian $\mathbb{G}(Z)$ of Z. Since W is open and dense in $\mathbb{G}(W)$, we can identify $\mathbb{G}(W)$ with the closure of W in $\mathbb{G}(Z)$.

4.2. Automorphism group

Let $\mathrm{Aut}(\mathbb{G})$ denote the automorphism group of $\mathbb{G}(Z)$ consisting of all biholomorphic maps from $\mathbb{G} = \mathbb{G}(Z)$ to itself. We summarize the basic facts about $\mathrm{Aut}(\mathbb{G})$ and its inner structure. See [**28**, §8,§9] for a detailed account.

Since $\mathbb{G}$ is a smooth projective variety, it follows by Chow's Lemma, that the automorphisms on $\mathbb{G}$ are even algebraic. Therefore, the action of $\mathrm{Aut}(\mathbb{G})$ on the Zariski-open subset $Z \subset \mathbb{G}$ is realized by birational maps. Furthermore, $\mathrm{Aut}(\mathbb{G})$ has the structure of a semi-simple complex Lie group. As before, let G denote the identity component of the automorphism group of the bounded symmetric domain $\mathcal{D} = \{z \in Z \,|\, |z| < 1\}$. It turns out that the identity component of $\mathrm{Aut}(\mathbb{G})$ is just the complexification $G^{\mathbb{C}}$ of G. Its Lie algebra is 3-graded, denoted by

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{u}^+ \, . \tag{4.16}$$

In the following we describe the corresponding subgroups and their action on $\mathbb{G}$. Recall from Section 2.8 that $K = \mathrm{Aut}(Z)^0$ is the identity component of the automorphism group of Z, and that its complexification $K^{\mathbb{C}}$ coincides with the identity component of the structure group, $K^{\mathbb{C}} = \mathrm{Str}(Z)^0$.

Structure automorphisms. For $h \in K^{\mathbb{C}}$ we define

$$h\,[z:a] := [hz : h^{-*}a] \, .$$

This is well-defined, since $h(z^w) = (hz)^{h^{-*}w}$, see Proposition 2.28. The Lie algebra of $K^{\mathbb{C}}$ is represented by the vector fields $\zeta^{(0)}(z) = \delta(z)$ on $Z \subset \mathbb{G}$, where $\delta \in \mathrm{Der}(Z)$ is a derivation. On $Z \cong \mathbb{G}^{(a)}$, this vector field is given by

$$\zeta^{(a)}(z) = \delta z - Q_z \delta^* a \, .$$

For a proof of this relation, consider the curve $h_t = \exp(t\delta)$ in $K^{\mathbb{C}}$. Then $h_t\,[z:a] = [h_t z : h_t^{-*}a] = [(h_t z)^{h_t^{-*}a - a} : a]$, i.e. we obtain

$$\zeta^{(a)}(z) = \tfrac{d}{dt}\big|_0 \big((h_t z)^{h_t^{-*}a-a}\big) = \delta z - Q_z \delta^* a \, ,$$

Here we used the well-known identities $\frac{d}{dx}(x^y) = B^{-1}_{x,y}$ and $\frac{d}{dy}(x^y) = Q_x B^{-1}_{y,x}$.

Quasi-translations. For $v \in \overline{Z}$ set
$$\tilde{t}_v\,[z:a] := [z:a+v]\ .$$
A simple substitution shows that this is well-defined: The elements $[z:a]$ and $[\tilde{z}:b]$ coincide if and only if $B_{a,\,a-b}$ is invertible and $\tilde{z} = z^{a-b}$, or equivalently if $B_{z,\,(a+v)-(b+v)}$ is invertible and $\tilde{z} = z^{(a+v)-(b+v)}$, i.e. $[z:a+v] = [\tilde{z}:b+v]$. This defines an injective group homomorphism from $(\overline{Z},+)$ to $G^{\mathbb{C}}$, the image is denoted by U^-. The elements of the Lie algebra $\mathfrak{u}^-$ of U^- correspond to the following vector fields $\tilde{v}$ on Z:
$$\tilde{v}(z) = Q_z(v) = \{z,\,v,\,z\}\ .$$
On $Z \cong \mathbb{G}^{(a)}$, this vector field is also given by
$$\zeta^{(a)}(z) = B_{z,\,a}Q_{z^a}v \overset{\mathsf{JT28}}{=} Q_z v\ .$$
The elements $\tilde{t}_v$ of U^- are called *quasi-translations.*
Translations. For $u \in Z$ set
$$t_u\,[z:0] := [z+u:0]\ .$$
This action of Z on the open and dense subset $Z \subset \mathbb{G}(Z)$ can be extended smoothly to all of $\mathbb{G}(Z)$. The proof ([**28**, §8.4]) essentially uses the fact, that the corresponding vector field $\zeta(z) = u$ has a smooth extension to $\mathbb{G}(Z)$, namely,
$$\zeta^{(a)}(z) = B_{z,\,a-b}\zeta^{(b)}(z^{a-b}) = B_{z,\,a-b}B_{z^{a-b},\,b}u \overset{\mathsf{JT33}}{=} B_{z,\,a}u\ .$$
Therefore, the map $Z \to G^{\mathbb{C}}$, $u \mapsto t_u$, defines an injective group homomorphism onto U^+. The elements t_u of U^+ are called *translations.*

The following proposition describes $G^{\mathbb{C}}$ by generators and relations:

Theorem 4.6. *Let Z be a phJTS, and let $G^{\mathbb{C}}$ be the identity component of the automorphism group of the Grassmannian $\mathbb{G} = \mathbb{G}(Z)$.*

(a) *$G^{\mathbb{C}}$ is generated by U^+ and U^-. Any element $g \in G^{\mathbb{C}}$ can be written as*
$$g = \tilde{t}_v t_u h\,\tilde{t}_w = t_{\tilde{u}}\tilde{h}\,\tilde{t}_{\tilde{v}}t_{\tilde{w}}\ ,$$
i.e. $G^{\mathbb{C}} = U^-U^+K^{\mathbb{C}}U^- = U^+K^{\mathbb{C}}U^-U^+$ as sets.

(b) *For $h \in K^{\mathbb{C}}$, $\tilde{t}_v \in U^-$, $t_u \in U^+$ with quasi-invertible pair $(u,v) \in Z \times \overline{Z}$ we have*
$$ht_uh^{-1} = t_{hu}\ ,\quad h\tilde{t}_vh^{-1} = \tilde{t}_{h^{-*}v}\ ,\quad \tilde{t}_vt_u = t_{u^v}B^{-1}_{u,\,v}\tilde{t}_{v^u}\ .$$

(c) *Let Γ be a group, let $f_0 : K^{\mathbb{C}} \to \Gamma$, $f_\pm : U^\pm \to \Gamma$ be homomorphisms. Then f_0, f_+, f_- extend to a (unique) homomorphism $f : G^{\mathbb{C}} \to \Gamma$ if and only if they are compatible with the relations in* (b), *i.e.*
$$f_+(t_{hu}) = f_0(h)f_+(t_u)f_0(h)^{-1}\ ,$$
$$f_-(\tilde{t}_{h^{-*}v}) = f_0(h)f_-(\tilde{t}_v)f_0(h)^{-1}\ ,$$
$$f_-(\tilde{t}_v)f_+(t_u) = f_+(t_{u^v})f_0(B^{-1}_{u,v})f_-(\tilde{t}_{v^u})$$
for all $h \in K^{\mathbb{C}}$, $\tilde{t}_v \in U^-$, $t_u \in U^+$ with quasi-invertible $(u,v) \in Z \times \overline{Z}$.

Using this theorem, one can show that there exists a unique complex conjugation σ on $G^{\mathbb{C}}$ such that

$$\sigma(h) = h^{-*}\ ,\quad \sigma(t_u) = \tilde{t}_{-u}\ ,\quad \sigma(\tilde{t}_v) = t_{-v}\ . \tag{4.17}$$

The fixed point set of σ turns out to be isomorphic to the identity component G of the automorphism group of the bounded symmetric domain $\mathcal{D} \subset Z$, cf. Section 3.5.

This justifies the statement (and our notation) that $G^{\mathbb{C}}$ is the complexification of G. From this is follows that the automorphisms of the unit ball $\mathcal{D}$ extend uniquely to automorphisms of the Grassmannian $\mathbb{G}$.

There is a second complex conjugation on $G^{\mathbb{C}}$, defined by $\theta = \sigma \circ \operatorname{Int}(s)$, where $\operatorname{Int}(s)$ denotes conjugation on $G^{\mathbb{C}}$ by the symmetry transformation s, which acts on Z as $s(z) = -z$. By Theorem 4.6, this complex conjugation is uniquely determined by

$$\theta(h) = h^{-*}\ , \quad \theta(t_u) = \tilde{t}_u\ , \quad \theta(\tilde{t}_v) = t_v\ . \tag{4.18}$$

The fixed point set of θ is denoted by G^c, it is the compact real form of $G^{\mathbb{C}}$ and therefore the compact dual of G.

The actions of $G^{\mathbb{C}}$ and G^c on $\mathbb{G}$ are transitive. The stabilizer subgroup of $0 = [0:0] \in \mathbb{G}$ in $G^{\mathbb{C}}$ is $K^{\mathbb{C}}U^-$, isomorphic to the semi-direct product $K^{\mathbb{C}} \ltimes U^-$. The corresponding stabilizer subgroup of 0 in G^c is K. We summarize:

Theorem 4.7. *Let Z be a phJTS and $\mathbb{G} = \mathbb{G}(Z)$ the Grassmannian of Z. Then*

$$\mathbb{G} \cong G^{\mathbb{C}}/K^{\mathbb{C}}U^- \cong G^c/K\ . \tag{4.19}$$

Therefore, $\mathbb{G}$ is a realization of the compact dual of the hermitian symmetric domain $\mathcal{D} \subset Z$ of non-compact type.

Remark 4.8. As in the case of the bounded symmetric domain $\mathcal{D} \subset Z$, the G^c-invariant hermitian metric h on $\mathbb{G}$ also admits a Jordan theoretic description. On the affine part $Z \subset \mathbb{G}$, this metric is given by

$$h_{[z:0]}(u,v) = \left\langle B_{z,-z}^{-1}u \middle| v\right\rangle \quad \text{for} \quad u,v \in T_{[z:0]}\mathbb{G} \cong Z\ , \tag{4.20}$$

where $\langle\,|\,\rangle$ denotes the intrinsic scalar product of the phJTS Z. Using the transformation rule (4.4), we obtain for an arbitrary element $[z:a]$ of $\mathbb{G}$ the formula

$$\begin{aligned} h_{[z:a]}(u,v) &= \left\langle B_{z^a,-z^a}^{-1} B_{z,a}^{-1} u \middle| B_{z,a}^{-1} v\right\rangle \\ &= \left\langle B_{a,z}^{-1} B_{z^a,-z^a}^{-1} B_{z,a}^{-1} u \middle| v\right\rangle \\ &= \left\langle (B_{z,a} B_{z^a,-z^a} B_{a,z})^{-1} u \middle| v\right\rangle \end{aligned}$$

for all $u,v \in T_{[z:a]}\mathbb{G} \cong Z$. Indeed, in Section 5.3 we show that $B_{z,a}B_{z^a,-z^a}B_{a,z}$ is well-defined and positive definite for all $(z,a) \in Z \times \overline{Z}$, see Remark 5.11

Example 4.9. For the matrix case $Z = \mathbb{C}^{r\times s}$ we have $\mathbb{G} = \mathrm{Gr}_s(\mathbb{C}^{r+s})$ and elements of $\mathbb{G}$ are represented as column spaces of $(n\times s)$-matrices of rank s with $n = r+s$, cf. Lemma 4.1. The corresponding groups are[1]

$$G^{\mathbb{C}} = \mathrm{SL}(r+s)\ , \quad G^c = \mathrm{SU}_{r+s}\ , \quad G = \mathrm{SU}(r,s)\ , \quad K = \mathrm{S}(\mathrm{U}(r)\times\mathrm{U}(s))\ .$$

The $G^{\mathbb{C}}$-action on $\mathbb{G}$ extends the natural $G^{\mathbb{C}}$-action on $\mathbb{C}^n$ given by $g \mapsto g\eta$, where $\eta \in \mathbb{C}^n$ is regarded as a column space, i.e. we have $g\langle A\rangle = \langle gA\rangle$, where $\langle A\rangle$ denotes the column space of the $(n\times s)$-matrix A. In particular, on the affine part $Z \hookrightarrow \mathbb{G}$ given by $z \mapsto [z:a]$, we obtain

$$g\left\langle\begin{pmatrix} z \\ 1\end{pmatrix}\right\rangle = \left\langle\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} z \\ 1\end{pmatrix}\right\rangle = \left\langle\begin{pmatrix} az+b \\ cz+d\end{pmatrix}\right\rangle = \left\langle\begin{pmatrix} (az+b)(cz+d)^{-1} \\ 1\end{pmatrix}\right\rangle \tag{4.21}$$

[1] cf. Example 3.30.

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, i.e. $G^{\mathbb{C}}$ acts on Z by the usual Möbius transformations, see also Example 3.30. Actions of the structure group, translations, quasi-translations and the symmetry s around $[0:0] \in \mathbb{G}$ are given by the following matrices:

$$h \equiv \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} , \quad t_u \equiv \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} , \quad \tilde{t}_v \equiv \begin{pmatrix} 1 & 0 \\ -v^* & 1 \end{pmatrix} , \quad s \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{4.22}$$

Furthermore, and the complex conjugations σ and θ on $G^{\mathbb{C}}$ read

$$\sigma \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^* & c^* \\ b^* & a^* \end{pmatrix} , \quad \theta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d^* & -c^* \\ -b^* & a^* \end{pmatrix} . \tag{4.23}$$

Finally, we consider the two realizations of the conjugate Grassmannian $\overline{\mathbb{G}}$ on the Grassmannian manifold $\mathrm{Gr}_r(\mathbb{C}^n)$ defined in Example 4.5, and compare the $G^{\mathbb{C}}$ action on $\overline{\mathbb{G}}$ with the natural $G^{\mathbb{C}}$ action on $\mathrm{Gr}_r(\mathbb{C}^n)$ given by $g\,\langle A\rangle = \langle gA\rangle$. Recall that $\overline{\mathbb{G}}$ is realized as $\mathrm{Gr}_r(\mathbb{C})$ by either of the maps $\phi, \phi' : \overline{\mathbb{G}} \to \mathrm{Gr}_r(\mathbb{C}^n)$ with

$$\phi\,[z:a] = \overline{E}_{z,a} = \left\langle \begin{pmatrix} 1 - az^* \\ -z^* \end{pmatrix} \right\rangle , \quad \phi'\,[z:a] = \overline{E}'_{z,a} = \left\langle \begin{pmatrix} 1 - az^* \\ z^* \end{pmatrix} \right\rangle . \tag{4.24}$$

Now, it is a straightforward calculation to verify on the generators of $G^{\mathbb{C}}$, that

$$\phi \circ g = \theta(g) \circ \phi \quad \text{and} \quad \phi' \circ g = \sigma(g) \circ \phi' \quad \text{for all } g \in G^{\mathbb{C}}. \tag{4.25}$$

Therefore, we call ϕ the θ-realization and ϕ' the σ-realization of the conjugate Grassmannian $\overline{\mathbb{G}}$. These relations are of particular importance in Section 5.1, when we investigate the $G^{\mathbb{C}}$-orbit structure on the product manifold $\mathbb{G} \times \overline{\mathbb{G}}$.

Vector bundles revisited. In the last section, we defined various vector and line bundles on $\mathbb{G}$. Having described $\mathbb{G}$ as a homogeneous space, the question arises whether these bundles are homogeneous, i.e. whether there is a $G^{\mathbb{C}}$-action on them such that the corresponding canonical projection onto $\mathbb{G}$ is $G^{\mathbb{C}}$-equivariant.

For the tangent bundle and the canonical bundle this certainly is true. Regarded as quotient manifolds via Godement, we have

$$T\mathbb{G} = (Z \times \overline{Z} \times Z)/\sim_{T\mathbb{G}} \quad \text{with} \quad (z, a, \zeta) \sim_{T\mathbb{G}} (\tilde{z}, \tilde{a}, \tilde{\zeta}) \iff \begin{cases} \tilde{z} = z^{a-\tilde{a}}, \\ \tilde{\zeta} = B^{-1}_{z,\,a-\tilde{a}}\zeta \,, \end{cases}$$

and it is straightforward to show that the corresponding $G^{\mathbb{C}}$-action is defined on generators by

$$h\,[z,a,\zeta] = [hz,\, h^{-*}a,\, h\zeta] \,, \quad \tilde{t}_v\,[z,a,\zeta] = [z,\, a+v,\, \zeta] \,, \quad t_u\,[z,0,\zeta] = [z+u,\, 0,\, \zeta] \,.$$

On the open and dense subset $Z \times Z \hookrightarrow T\mathbb{G}$ given by $(z,\zeta) \mapsto [z,0,\zeta]$, this implies

$$g\,[z,0,\zeta] = [g(z),\, 0,\, D_z g(\zeta)] \quad \text{for all } g \in G^{\mathbb{C}} \text{ with } g(z) \in Z \subset \mathbb{G}. \tag{4.26}$$

Similarly, for the canonical bundle $\mathcal{K}$, which is defined by

$$\mathcal{K} = (Z \times \overline{Z} \times \mathbb{C})/\sim_{\mathcal{K}} \quad \text{with} \quad (z, a, \lambda) \sim_{\mathcal{K}} (\tilde{z}, \tilde{a}, \tilde{\lambda}) \iff \begin{cases} \tilde{z} = z^{a-\tilde{a}}, \\ \tilde{\zeta} = (\operatorname{Det} B^{-1}_{z,\,a-\tilde{a}})\,\lambda \,, \end{cases}$$

we obtain the corresponding $G^{\mathbb{C}}$-action from the one on $T\mathbb{G}$ by taking determinants:

$$\begin{aligned} h\,[z,a,\lambda] &= [hz,\, h^{-*}a,\, (\operatorname{Det} h)\lambda] \,, \\ \tilde{t}_v\,[z,a,\lambda] &= [z,\, a+v,\, \lambda] \,, \\ t_u\,[z,0,\lambda] &= [z+u,\, 0,\, \lambda] \,. \end{aligned} \tag{4.27}$$

On the open and dense subset $Z \times \mathbb{C} \hookrightarrow \mathcal{K}$, we have the closed formula

$$g\,[z,0,\lambda] = [g(z),\, 0,\, (\operatorname{Det} D_z g)\,\lambda] \quad \text{for all } g \in G^{\mathbb{C}} \text{ with } g(z) \in Z \subset \mathbb{G}. \tag{4.28}$$

By these definitions, it is immediate that the projections of $T\mathbb{G}$ and $\mathcal{K}$ onto $\mathbb{G}$ are $G^{\mathbb{C}}$-equivariant. Analogously, one defines a $G^{\mathbb{C}}$-equivariant structure on all vector bundles, that are derived from the tangent bundle. These bundles stay in correspondence with the representations of the stabilizer subgroup $P^- := K^{\mathbb{C}}U^-$ of $G^{\mathbb{C}}$, as noted in 3.2. For example, the tangent bundle and the canonical bundle correspond to the representations $p \mapsto D_0 p$ and $p \mapsto \operatorname{Det} D_0 p$.

Now consider the line bundle $\mathcal{L}$ defined by the Jordan triple product Δ. If Z is simple, we obtain from Proposition 2.40 the relation $\mathcal{L}^{\mathsf{p}} = \mathcal{K}^{-1}$, where p is the genus of Z. In view of (4.27), the definition of a suitable $G^{\mathbb{C}}$-action on $\mathcal{L}$ must contain the p-th root of $\operatorname{Det} h$. Since $K^{\mathbb{C}}$ is generated by the Bergman operators $B_{x,y}$ with quasi-invertible pairs (x, y), we have

$$\operatorname{Det} h = \prod_i \operatorname{Det} B_{x_i, y_i} = \prod_i \Delta(x_i, y_i)^{\mathsf{p}} ,$$

so it is tempting to set $(\operatorname{Det} h)^{1/\mathsf{p}} = \prod_i \Delta(x_i, y_i)$. Unfortunately, this depends on the choice of the decomposition of h, and by these means we just obtain a *projective* representation of P^-. It seems that $\mathcal{L}$ does not admit a $G^{\mathbb{C}}$-action and hence is not a $G^{\mathbb{C}}$-equivariant line bundle.

4.3. Partial Cayley mappings

One of the most important families of automorphisms of the Grassmannian $\mathbb{G}(Z)$ are the so called partial Cayley mappings, which generalize the ordinary Cayley map known from complex analysis. They relate the bounded symmetric domain $\mathcal{D}$ with Siegel domains of various types [**28**, §10.8]. Whereas in the usual treatment partial Cayley mappings are defined with respect to tripotent element, we generalize this notion to arbitrary elements by an appropriate use of pseudo-inverses. In later chapters, we use the square of the partial Cayley mappings - the so called partial inverse mappings - to relate the Grassmannian of a Peirce $1/2$-space with certain Peirce varieties, see Chapter 6.

Lemma 4.10. *Let $u \in Z$ be an arbitrary element and $u^\dagger$ be its pseudo-inverse. Then there exists a unique homomorphism $f_u : \mathrm{SL}_2(\mathbb{C}) \to G^{\mathbb{C}}$, such that*

$$f_u \left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) = t_{\alpha u^\dagger} , \quad f_u \left(\begin{smallmatrix} 1 & 0 \\ -\alpha & 1 \end{smallmatrix}\right) = \tilde{t}_{\overline{\alpha} u} , \quad f_u \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} = B_{u^\dagger, (1-\overline{\mu})u} \tag{4.29}$$

for all $\alpha \in \mathbb{C}$, $\mu \in \mathbb{C}^*$. Let θ be the involution on $G^{\mathbb{C}}$ defined in (4.18), and let θ' denote the involution on $\mathrm{SL}_2(\mathbb{C})$ defined by $\theta' \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{smallmatrix}\right)$, then

$$f_u \circ \theta' = \theta \circ f_{u^\dagger} . \tag{4.30}$$

In particular, f_u commutes with the involutions if and only if u is a tripotent.

Proof. We follow the proof in [**28**, §9.7]. Define a homomorphism on the Lie algebra level by

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto u^\dagger \in \mathfrak{u}^+ , \quad \left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right) \mapsto \widetilde{u} \in \mathfrak{u}^- , \quad \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \mapsto 2\, u^\dagger \,\square\, u \in \mathfrak{k}^{\mathbb{C}} .$$

It is straightforward to check that this is indeed a Lie algebra homomorphism. Since $\mathrm{SL}_2(\mathbb{C})$ is simply connected, it induces a group homomorphism f_u, which clearly satisfies the first two relations of (4.29), since $t_{\alpha u^\dagger} = \exp(\alpha u^\dagger)$ and $\tilde{t}_{\overline{\alpha} u} = \exp(\alpha \widetilde{u})$. The third one follows from $2\, u^\dagger \,\square\, u(z) = 2\nu\, z$ and $B_{u^\dagger, (1-\overline{\mu})u} z = \mu^{2\nu} z$ for $z \in Z^u_\nu$. Equation (4.30) is easily verified on the generators of $\mathrm{SL}_2(\mathbb{C})$. Finally we note that if f_u commutes with the involutions, then in particular $f_u \circ \theta' \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) = \theta \circ f_u \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, i.e. $t_{u^\dagger} = t_u$, and therefore $u^\dagger = t_{u^\dagger}(0) = t_u(0) = u$, hence showing that u is a tripotent. The converse is immediate. □

For $u \in Z$ we set

$$\gamma_u := \exp \tfrac{\pi}{4}(u^\dagger + \widetilde{u}) \quad \text{and} \quad j_u := \gamma_u^2 \,, \tag{4.31}$$

and call γ_u the *partial Cayley mapping*, and j_u the *partial inverse mapping* with respect to u. In case of a tripotent $u \in Z$, these definitions coincide with those given in [**28**, §10.1]. We notice, that γ_u is an element of the compact real form $G^c \subset G^{\mathbb{C}}$ if and only if u is tripotent. More generally,

$$\gamma_u \circ \theta = \theta \circ \gamma_{u^\dagger} \,. \tag{4.32}$$

According to Lemma 4.10 we get

$$\gamma_u = f_u\left(\exp \tfrac{\pi}{4} \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) = f_u\left(\tfrac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)\right) = f_u\left(\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2}^{-1} \end{pmatrix}\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\right) .$$

This yields the following explicit formula:

$$\gamma_u = t_{u^\dagger} \circ B_{u^\dagger,\,(1-\sqrt{2})u} \circ \tilde{t}_u = t_{u^\dagger} \circ B^{1/2}_{u^\dagger,\,-u} \circ \tilde{t}_u \,, \tag{4.33}$$

where the last equation follows from $B_{u,\,-u^\dagger} z = 2^{2\nu} z$ for $z \in Z^u_\nu$. Similarly, using

$$\left(\tfrac{1}{\sqrt{2}} \left(\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right)\right)^2 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$$

and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)^2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we obtain

$$j_u = t_{u^\dagger} \circ \tilde{t}_u \circ t_{u^\dagger} = \tilde{t}_u \circ t_{u^\dagger} \circ \tilde{t}_u \,, \quad j_u^2 = B_{u^\dagger,\,2u} \quad \text{and} \quad j_u^4 = \mathrm{Id} \,. \tag{4.34}$$

Furthermore one can easily verify, that

$$\gamma_0 = \mathrm{Id} \quad \gamma_u^{-1} = \gamma_{-u} \,, \quad k\gamma_u k^{-1} = \gamma_{ku} \,, \quad \gamma_u \gamma_v = \gamma_{u+v} \tag{4.35}$$

for all $k \in K$ and $u, v \in Z$ with $u \perp\!\!\!\perp v$. The next proposition generalizes [**28**, §10.3], and provides explicit formulas for $\gamma_u(z)$ and $j_u(z)$ using the Peirce decomposition of z with respect to u.

Proposition 4.11. *Let $z = z_1 \oplus z_{1/2} \oplus z_0$ be the components of $z \in Z$ in the Peirce spaces of $u \in Z$. Then $\gamma_u(z)$ is an element of $Z \subset \mathbb{G}(Z)$ if and only if $u^\dagger - z_1$ is invertible in the Jordan algebra Z^u_1 (with product $x \circ y = \{x,\, u,\, y\}$), and then*

$$\gamma_u(z) = (u^\dagger + z_1) \circ (u^\dagger - z_1)^{-1} \oplus 2^{3/2} z_{1/2} \circ (u^\dagger - z_1)^{-1} \oplus \left(z_0 + P_{z_{1/2}}(u^\dagger - z_1)^{-1}\right) .$$

Also, $j_u(z)$ is an element of $Z \subset \mathbb{G}(Z)$ if and only if z_1 is invertible in Z^u_1, and then

$$j_u(z) = \left(-z_1^{-1}\right) \oplus \left(-2\, z_{1/2} \circ z_1^{-1}\right) \oplus \left(z_0 - P_{z_{1/2}} z_1^{-1}\right) .$$

Here $P_{z_{1/2}} = Q_{z_{1/2}} Q_u$ is the quadratic mapping of the Jordan algebra Z^u_1.

Proof. First we notice that $\gamma_u(z) = t_{u^\dagger} \circ B_{u^\dagger,\,(1-\sqrt{2})u} \circ \tilde{t}_u$ is an element of Z if and only if $\tilde{t}_u$ lies in Z, since Z is invariant under $t_{u^\dagger}$ and $B_{u^\dagger,\,(1-\sqrt{2})u}$. Therefore, $\gamma_u(z) \in Z$ if and only if the pair (z, u) is quasi-invertible. According to the symmetry formula of the quasi-inverse, this is equivalent to the quasi-invertibility of (u, z), i.e. the invertibility of $B_{u,\,z}$. Using Lemma 2.26, $B_{u,\,z} = B_{u,\,u^\dagger - (u^\dagger - z)}$ is invertible if and only if $u^\dagger - z_1$ is invertible in the Jordan algebra Z^u_1, as asserted. The addition formula for the quasi-inverse yields

$$z^u = (z_1 + z_{1/2} + z_0)^u = z_1^u + B^{-1}_{z_1,\,u}\left((z_{1/2} + z_0)^w\right)$$

with $w = u^{z_1}$. Due to Lemma 2.26, we have $w = u^{u^\dagger - (u^\dagger - z_1)} = Q_u (u^\dagger - z_1)^{-1}$ and

$$B^{-1}_{z_1,\,u} = \left(B^{-1}_{u,\,u^\dagger - (u^\dagger - z)}\right)^* = (Q_{u^\dagger - z_1} Q_{u^\dagger})^{-1}\big|_{Z^u_1} \oplus\, 2\,(u^\dagger - z_1)^{-1} \,\square\, u\big|_{Z^u_{1/2}} \oplus \mathrm{Id}\big|_{Z^u_0} \,.$$

Since $(z_{1/2}+z_0, w)$ is nilpotent (see Lemma 2.22), it is $(z_{1/2}+z_0)^w = z_{1/2}+z_0+Q_{z_{1/2}+z_0}w$, and by means of the symmetry formula for the quasi-inverse, we obtain

$$z_1^u = z_1 + Q_{z_1} w = z_1 + P_{z_1}(u^\dagger - z_1)^{-1} = z_1 + z_1^2 \circ (u^\dagger - z_1)^{-1} = z_1 \circ (u^\dagger - z_1)^{-1}\ .$$

The last relation is calculated in the commutative associative subalgebra $\langle u^\dagger, z_1\rangle \sqsubset Z_1^u$ generated by $u^\dagger$ and z_1. In summary, this yields

$$z^u = z_1 \circ (u^\dagger - z_1)^{-1} \oplus 2\, z_{1/2} \circ (u^\dagger - z_1)^{-1} \oplus \big(z_0 + P_{z_{1/2}}(u^\dagger - z_1)^{-1}\big)\ . \tag{4.36}$$

Finally, using $B_{u^\dagger,(1-\sqrt{2})u}(w_\nu) = 2^\nu w_\nu$ for $w_\nu \in Z_\nu^u$ and some additional calculation in $\langle u^\dagger, z_1\rangle \sqsubset Z_1^u$, we obtain the assertion for $\gamma(z) = t_u \circ B_{u^\dagger,(1-\sqrt{2})u}(z^u)$. The formula for the partial inverse mapping follows from $j_u = t_{u^\dagger} \circ \tilde{t}_u \circ t_{u^\dagger}$ and equation (4.36) by replacing z_1 by $u^\dagger + z_1$:

$$j_u(z) = (u^\dagger - (u^\dagger + z_1)\circ z_1^{-1}) \oplus (-2\, z_{1/2} \circ z_1^{-1}) \oplus \big(z_0 - P_{z_{1/2}} z_1^{-1}\big)\ .$$

Finally, a short calculation in $\langle u^\dagger, z_1\rangle$ simplifies the Z_1^u-term to $-z_1^{-1}$. □

4.4. Representatives of elements of the Grassmannian

By definition, elements of the Grassmannian $\mathbb{G}(Z)$ are equivalence classes in $Z \times \overline{Z}$. In this chapter we describe two systems of representatives for $\mathbb{G}(Z)$, which are used in next chapter to determine the fine structure of $\mathbb{G}(Z)$.

We recall that two elements $u, \tilde{u} \in Z$ are Peirce equivalent, if they induce the same Peirce decomposition (cf. Section 2.6). For $u \in Z$ with corresponding Peirce decomposition $Z = Z_1^u \oplus Z_{1/2}^u \oplus Z_0^u$ let

$$\mathcal{D}_\nu^u := \mathcal{D} \cap Z_\nu^u \quad , \ \nu = 1, 1/2, 0$$

be the intersection of the symmetric domain $\mathcal{D} = \{z \in Z \,|\, |z| < 1\}$ with each Peirce space. We denote by $\mathrm{cl}(A)$ the topological closure of some subset $A \subset Z$.

Theorem 4.12. *Let Z be a phJTS and $\mathbb{G}(Z)$ be the Grassmannian of Z. Then, any element $\chi \in \mathbb{G}(Z)$ is representable as*

(i) $\chi = \big[u + z : u^\dagger\big]$ *with* $u, z \in Z$, $u \perp\!\!\!\perp z$,

(ii) $\chi = \big[v + d_0 : v^\dagger + d_1\big]$ *with* $v \in Z$, $d_0 \in cl(\mathcal{D}_0^v)$, $d_1 \in \mathcal{D}_1^v$.

These representatives are unique up to Peirce equivalence in u and in v.

Remark 4.13. In later chapters it is often useful to modify the representative $(v + d_0,\, v^\dagger + d_1)$ of χ as follows: firstly, due to Corollary 2.21, we may exchange v by a Peirce equivalent tripotent $e \in S$, i.e. $v \approx e$ and $e = e^\dagger$. Secondly, we use the spectral theorem to decompose d_0 into the sum of a tripotent $e' \in S$ with $e' \perp\!\!\!\perp e$ and an element d_0' of $\mathcal{D}_{e+e'}^0$. Finally we set $c := e + e'$, $d_e := d_0'$ and $d_c := d_1$, and obtain the following representative:

$$\chi = [e + d_e : c + d_c] \quad \text{with} \quad e, c \in S,\ c \le e,\ d_e \in \mathcal{D}_0^e,\ d_c \in \mathcal{D}_1^c\ .$$

In later applications, this is the preferred representative since the ranks of the tripotent parts e and c play a major role. The disadvantage of this presentation is that the uniqueness statement becomes more obscure, since c is 'contained' in e.

Before proving Theorem 4.12, we denote a lemma, which provides a criterion for two element with joint spectral decomposition to be equal.

Lemma 4.14. *Let $e_1, \dots, e_r$ be a frame of Z, and let x, y, x' and y' be elements of the complex linear span of the e_k with coefficients λ_k, μ_k, λ_k' and μ_k', respectively. Then $[x : y] = [x' : y']$ if and only if*

$$1 - (\mu_k - \mu_k')^*\lambda_k \ne 0 \quad \text{and} \quad \lambda_k' = \frac{\lambda}{1 - (\mu_k - \mu_j')^*\lambda_k} \quad \text{for all } k = 1, \dots, r.$$

One can prove this Lemma using the Peirce rules by direct calculation. Instead, we notice that in the background of this calculation, the following injective homomorphism of Jordan triple systems is of importance:

$$\phi : \mathbb{C}^k \to Z,\ \lambda = (\lambda_1, \dots, \lambda_k) \mapsto \sum \lambda_k e_k \ .$$

Here, the triple product on $\mathbb{C}^k$ is given by $\{\lambda_i, \mu_i, \eta_i\} = \lambda_i \cdot \mu_i^* \cdot \eta_i$, i.e. $\mathbb{C}^k$ is the k-fold direct sum of the simple triple system $\mathbb{C}$ with itself. Due to Corollary 2.29, this homomorphism has a well-defined continuation to an imbedding of $\mathbb{G}(\mathbb{C}^k)$ into $\mathbb{G}(Z)$, and it is $\phi([\lambda : \mu]) = [\phi(\lambda) : \phi(\mu)]$ for $\lambda, \mu \in \mathbb{C}^k$. Therefore it remains to discuss the identity $[\lambda : \mu] = [\lambda' : \mu']$ with $\lambda, \mu, \lambda', \mu' \in \mathbb{C}^k$. Since $\mathbb{G}(\mathbb{C}^k) = \mathbb{G}(\mathbb{C})^k = \overline{\mathbb{C}}^k$, this identity is solved in each component independently, which yields the stated relations.

Proof of Theorem 4.12. The proof is split into several parts. First we prove the *existence of the representatives.* We show that χ contains a representative of the first kind and deduce from this the representative of the second kind. In each case we give conditions for χ being an element of $Z \subset \mathbb{G}(Z)$.

Claim 1. *In each equivalence class $\chi \in \mathbb{G}(Z)$ there exists an element $(u+z,\, u^\dagger)$ with $u \perp\!\!\!\perp z$. Moreover, χ is an element of $Z \subset \mathbb{G}(Z)$ if and only if u vanishes.*

Proof. Since Z is a open and dense subset of $\mathbb{G}(Z)$, there exists a sequence z_n in Z converging to χ. Let $(e_1, \dots, e_r)$ be a fixed frame in Z and let $W := \langle e_1, \dots, e_r \rangle$ be the real span of this frame. Since the identity component K of the automorphism group acts transitively on the set of frames, there exists a sequence $k_n \in K$, such that $k_n z_n \in W$ for all n. By compactness of K, we assert without restriction that the sequence k_n converges to some $k \in K$. Therefore, the sequence $\tilde{z}_n := k^{-1} k_n z_n$ converges to $k^{-1} k \chi = \chi$, i.e. we can assume without restriction that $z_n \in W$ for all n. Thus χ is an element of the closure $\mathrm{cl}(W)$ in $\mathbb{G}(Z)$, and due to $z_n = \sum \lambda_j^{(n)} e_j$, we obtain sequences $(\lambda_j^{(n)})_{n \in \mathbb{N}}$ for $j = 1, \dots, r$. By orthogonality of the frame elements, these sequences converge within $\overline{\mathbb{R}} \subset \overline{\mathbb{C}}$, $\lambda_j^{(n)} \to \lambda_j$ for $n \to \infty$. After renaming the frame elements, we can assume $\lambda_1 \geq \ldots \geq \lambda_r$. We set $s := \max\{j \,|\, \lambda_j = \infty\}$, and since $[\lambda e : 0] = \left[e : (1 - \frac{1}{\lambda}) e\right]$, we obtain

$$z_n = \left[\sum_{j \leq s} e_j + \sum_{j > s} \lambda_j^{(n)} e_j : \sum_{j \leq s} \left(1 - \tfrac{1}{\lambda_j^{(n)}}\right) e_j \right] \xrightarrow{n \to \infty} \left[\sum_{j \leq s} e_j + \sum_{j > s} \lambda_j e_j : \sum_{j \leq s} e_j \right] .$$

Here we used the continuity of the canonical projection of $Z \times \overline{Z}$ onto $\mathbb{G}(Z)$. With $u = \sum_{j \leq s} e_j$ and $z = \sum_{j > s} \lambda_j e_j$, this leads to the asserted representative $(u+z, u^\dagger) \in \chi$. Here we used in addition that u is a tripotent, so we have $u = u^\dagger$.
To the second statement: $\left[u + z : u^\dagger\right]$ is an element of $Z \subset \mathbb{G}(Z)$ if and only if the Bergman operator $B_{u+z,\, u^\dagger}$ is invertible. Due to the Peirce rules, $B_{u+z,\, u^\dagger} = B_{u,\, u^\dagger}$, and since this is the operator of orthogonal projection onto Z_0^u, the invertibility of $B_{u,\, u^\dagger}$ is equivalent to the identity $Z_0^u = Z$, i.e. to the vanishing of u. □

Claim 2. *Each equivalence class $\chi \in \mathbb{G}(Z)$ contains an element $(v + d_0,\ v^\dagger + d_1)$ with $v \in Z$, $d_0 \in cl(\mathcal{D}_0^v)$, $d_1 \in \mathcal{D}_1^v$. Moreover, χ is an element of $Z \subset \mathbb{G}(Z)$ if and only if d_1 is invertible in the Jordan algebra Z_1^v. In this case, $\chi = d_0 - d_1^\dagger$.*

Proof. By Claim 1, χ can be represented as $\chi = \left[u + z : u^\dagger\right]$ with $u \perp\!\!\!\perp z$. Let $z = \sum \lambda_j e_j$ be the spectral decomposition of z, then setting

$$d_0 := \sum_{\lambda_j \leq 1} \lambda_j e_j \ , \quad d_1 := -\sum_{\lambda_j > 1} \tfrac{1}{\lambda_j}\, e_j \ , \quad v := u + \sum_{\lambda_j > 1} e_j$$

and using the relation $[\lambda e : 0] = \left[e : (1 - \frac{1}{\lambda})e\right]$, we obtain the prospected representative. Furthermore, $\left[v + d_0 : v^\dagger + d_1\right]$ is an element of Z if and only if the Bergman operator $B_{v+d_0,\, v^\dagger+d_1}$ is invertible. Due to the Peirce rules, $B_{v+d_0,\, v^\dagger+d_1} = B_{v,\, v^\dagger+d_1}$. From Lemma 2.26, this operator is invertible if and only if d_1 is invertible in Z_1^v, and then

$$\chi = (v + d_0)^{v^\dagger+d_1} = d_0 + v^{v^\dagger+d_1} = d_0 - d_1^\dagger \,.$$

Here we also used Lemma 2.25. □

Now we turn our attention to the *uniqueness of the representatives* (up to Peirce equivalence). We have to show that $[u+z : u^\dagger]$ and $[\tilde{u}+\tilde{z} : \tilde{u}^\dagger]$ (resp. $[v+d_0 : v^\dagger+d_1]$ and $[\tilde{v} + \tilde{d}_0 : \tilde{v}^\dagger + \tilde{d}_1]$) coincide if and only if u is Peirce equivalent to $\tilde{u}$ and $z = \tilde{z}$ (resp. v is Peirce equivalent to $\tilde{v}$ and $d_0 = \tilde{d}_0$, $d_1 = \tilde{d}_1$). The 'if'-part of this assertion can be generalized as follows:

Claim 3. *Let u and $\tilde{u}$ be Peirce equivalent elements, then*

$$\left[u + z_0 : u^\dagger + z_1\right] = \left[\tilde{u} + z_0 : \tilde{u}^\dagger + z_1\right]$$

for all $z_0 \in Z_0^u$ and $z_1 \in Z_1^u$.

Proof. Since $\tilde{u}^\dagger$ is invertible in Z_1^u, it follows from Lemma 2.26 that the Bergman operator $B_{u+z_0,\, (u^\dagger+z_1)-(\tilde{u}^\dagger+z_1)} = B_{u,\, u^\dagger-\tilde{u}^\dagger}$ is also invertible. Therefore we have

$$(u + z_0)^{(u^\dagger-\tilde{u}^\dagger)} = u^{(u^\dagger-\tilde{u}^\dagger)} + z_0 = (\tilde{u}^\dagger)^\dagger + z_0 = \tilde{u} + z_0 \,.$$

Here we also used Lemma 2.25. □

Finally we have to show the 'only if'-part of the uniqueness statement above. We first consider the slightly more general situation $[u+z_0 : u^\dagger+z_1] = [\tilde{u}+\tilde{z}_0 : \tilde{u}^\dagger+\tilde{z}_1]$ with $z_\nu \in Z_\nu^u$, $\tilde{z}_\nu \in Z_\nu^{\tilde{u}}$, and then specialize to the two representatives. Generally, the equation $[x : y] = [\tilde{x} : \tilde{y}]$ implies $\tilde{x} \in Z_1^x$ and $x \in Z_1^{\tilde{x}}$. This is an application of Lemma 2.25 to $\tilde{x} = x^{y-\tilde{y}}$ and $x = \tilde{x}^{\tilde{y}-y}$. Therefore, Proposition 2.19 implies that x and $\tilde{x}$ are Peirce equivalent, i.e. we have $Z_\nu^x = Z_\nu^{\tilde{x}}$ for $\nu = 1, 1/2, 0$. Application of this to the situation above yields

$$Z_1 := Z_1^{u+z_0} = Z_1^{\tilde{u}+\tilde{z}_0} \quad \text{and} \quad u, \tilde{u}, z_0, \tilde{z}_0, z_1, \tilde{z}_1 \in Z_1 \,.$$

By assumption, the following Bergman operator is invertible:

$$B_{u+z_0,\, (u^\dagger+z_1)-(\tilde{u}^\dagger+\tilde{z}_1)} = B_{u+z_0,\, (u^\dagger+z_0^\dagger)-(\tilde{u}^\dagger+\tilde{z}_1+z_0^\dagger-z_1)} = B_{u+z_0,\, (u+z_0)^\dagger-(\tilde{u}^\dagger+\tilde{z}_1+z_0^\dagger-z_1)} \,.$$

Here we used the relation $u^\dagger + z_0^\dagger = (u + z_0)^\dagger$, which follows from the (strong) orthogonality of u and z_0. Now, Lemma 2.26 implies the invertibility of $\tilde{u}^\dagger+\tilde{z}_1+z_0^\dagger-z_1$ within the Jordan algebra Z_1^{u+z}, and it is

$$(\tilde{u}^\dagger + \tilde{z}_1 + z_0^\dagger - z_1)^\dagger = (u + z_0)^{(u^\dagger+z_0^\dagger)-(\tilde{u}^\dagger+\tilde{z}_1+z_0^\dagger-z_1)} = (u + z_0)^{(u^\dagger+z_1)-(\tilde{u}^\dagger+\tilde{z}_1)} = \tilde{u} + \tilde{z}_0 \,.$$

Taking pseudo-inverses of both sides, we obtain $\tilde{u}^\dagger + \tilde{z}_1 + z_0^\dagger - z_1 = (\tilde{u} + \tilde{z}_0)^\dagger = \tilde{u}^\dagger + \tilde{z}_0^\dagger$, i.e.

$$(*) \qquad z_0^\dagger - z_1 = \tilde{z}_0^\dagger - \tilde{z}_1 \,.$$

In general, this equation does not imply the equality of z_ν and $\tilde{z}_\nu$, but with the additional assumptions of the two representatives, we can prove this equality: In the first case, we have $z_0 = z$, $\tilde{z}_0 = \tilde{z}$ and $z_1 = 0 = \tilde{z}_1$, i.e. $(*)$ implies $z_0 = \tilde{z}_0$ and $z_1 = \tilde{z}_1$. In the second case, we have a restriction to the spectral norm of the involved elements. Comparison of the terms yields $z_0 = d_0$, $\tilde{z}_0 = \tilde{d}_0 \in \mathrm{cl}(\mathcal{D}_0^u)$ and $z_1 = d_1$, $\tilde{z}_1 = \tilde{d}_1 \in \mathcal{D}_1^u$. Therefore, the spectral values of z_1 and $\tilde{z}_1$ are less than 1 and the spectral values of $z_0^\dagger$ and $\tilde{z}_0^\dagger$ are greater or equal to 1. Since by $(*)$, the spectral decompositions of $z_0^\dagger - z_1$ and $\tilde{z}_0^\dagger - \tilde{z}_1$ coincide, it follows that $z_0 = \tilde{z}_0$ and $z_1 = \tilde{z}_1$. It remains to show the Peirce equivalence of u and $\tilde{u}$. Since $Z_1^{\tilde{u}+\tilde{z}_0} = Z_1^{u+z_0}$ and since $\tilde{u} \perp\!\!\!\perp \tilde{z}_0 = z_0$, it follows that $\tilde{u}$ is an element of $Z_1^{u+z_0} \cap Z_0^z = Z_1^u$. Analogously we conclude that u is an element of $Z_1^{\tilde{u}}$. Therefore u and $\tilde{u}$ are Peirce equivalent. This completes the proof of Theorem 4.12. □

Example 4.15. In the matrix case $Z = \mathbb{C}^{r\times s}$, $r \le s$, the Grassmannian coincides with the ordinary Grassmannian variety $\mathrm{Gr}_s(\mathbb{C}^n)$ with $n = r+s$. Theorem 4.12 states, that any s-dimensional subspace E of $\mathbb{C}^n$ is representable both as

$$E = \text{column space of } \begin{pmatrix} u+z \\ 1-(u^\dagger)^*(u+z) \end{pmatrix} = \left\langle \begin{pmatrix} u+z \\ 1-(u^\dagger)^* u \end{pmatrix} \right\rangle$$

with $u, z \in Z$, $u \perp\!\!\!\perp z$, i.e. $u^* z = 0$ and $z u^* = 0$, which also implies $(u^\dagger)^* z = 0$, and as

$$E = \text{column space of } \begin{pmatrix} v+d_0 \\ 1-(v^\dagger+d_1)^*(v+d_0) \end{pmatrix} = \left\langle \begin{pmatrix} v+d_0 \\ 1-(v^\dagger)^* v + d_1^* v) \end{pmatrix} \right\rangle$$

with $v \in Z$, $d_0 \in \mathrm{cl}(\mathcal{D}_0^v)$, $d_1 \in \mathcal{D}_1^v$. To get more insight, we may assume that u and v are of block diagonal form, which is always achievable by a suitable K-action. Since in addition, u and v determined up to Peirce equivalence, we assume that $u = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $v = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with blocks of appropriate size. Then

$$u^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, \quad E = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & d \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

with no further condition on d, and

$$v^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad d_0 = \begin{pmatrix} 0 & 0 \\ 0 & \delta_0 \end{pmatrix}, \quad d_1 = \begin{pmatrix} \delta_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \delta_0 \\ \delta_1^* & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

with $\delta_0 \delta_0^* \le 1$ and $\delta_1 \delta_1^* \ll 1$. Applying the K-action more intensive, we also may assume that d, δ_0 and δ_1 are 'diagonal' matrices according to their spectral decomposition. Reintroducing the K-action, we conclude that an arbitrary s-dimensional subspace E of $\mathbb{C}^n$ is representable as

$$E = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \left\langle \begin{pmatrix} 1 & 0 \\ 0 & d \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \right\rangle = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \delta_0 \\ \delta_1^* & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

for unitary matrices $A \in U(r)$ and $D \in U(s)$ with $\mathrm{Det}(A) \cdot \mathrm{Det}(D) = 1$.

CHAPTER 5

Complexified Grassmannian and diagonal imbedding

In this chapter we consider the Grassmannian $\mathbb{G}$ of a phJTS Z as a *real analytic* submanifold of some *complexification* of $\mathbb{G}$. More precisely, we consider the diagonal imbedding of $\mathbb{G}$ into the product manifold $\mathbb{G} \times \overline{\mathbb{G}}$, where $\overline{\mathbb{G}}$ denotes the conjugate Grassmannian. In addition, we define a $G^{\mathbb{C}}$-action on $\mathbb{G} \times \overline{\mathbb{G}}$, for which the diagonal imbedding becomes G-equivariant. It turns out that it is quite simple to describe the $G^{\mathbb{C}}$-orbit structure on $\mathbb{G} \times \overline{\mathbb{G}}$ via invariants. The crucial observation of this chapter is that the restriction to the G-action on the diagonal provides a refinement of the invariants such that we obtain invariants for the G-action on the Grassmannian $\mathbb{G}$. This explains the appearance of this chapter in the context of the discussion of G-orbit structures.

5.1. Motivating example

In this section, we demonstrate the main results of this chapter in the special situation $Z = \mathbb{C}^{r\times s}$ (matrix case) with $r \le s$. The stated results are established using geometric arguments instead of Jordan theoretic considerations. In addition derive Jordan theoretic terms by which these results can be described. This serves as a motivation for the general case we discuss in the next sections. From Example 4.5, we recall that the Grassmannian $\mathbb{G}$ of Z and its conjugate $\overline{\mathbb{G}}$ can be identified with ordinary Grassmannian varieties,

$$\mathbb{G} \cong \mathrm{Gr}_s(\mathbb{C}^n) \quad \text{and} \quad \overline{\mathbb{G}} \cong \mathrm{Gr}_r(\mathbb{C}^n)\ .$$

with $n = r + s$. For the latter, we have two different identifications (the θ- and the σ-realization of $\overline{\mathbb{G}}$), what becomes important below, when we transfer the following results into Jordan theory. The groups involved in the matrix case $Z = \mathbb{C}^{r\times s}$ are[1]

$$G^{\mathbb{C}} = \mathrm{SL}(r+s)\ , \quad G^c = \mathrm{SU}(r+s)\ , \quad G = \mathrm{SU}(r,s)\ , \quad K = \mathrm{S}(\mathrm{U}(r)\times\mathrm{U}(s))\ .$$

First we determine the $G^{\mathbb{C}}$-orbit structure on the product manifold $\mathrm{Gr}_s(\mathbb{C}^n) \times \mathrm{Gr}_r(\mathbb{C}^n)$.

Proposition 5.1. *Let $G^{\mathbb{C}}$ act on $M = \mathrm{Gr}_s(\mathbb{C}^n)\times\mathrm{Gr}_r(\mathbb{C}^n)$ by $g(E,F) = (gE, gF)$. Then, M decomposes into $r+1$ different $G^{\mathbb{C}}$-orbits. More precisely,*

$$M = \dot{\bigcup_{j=0}^{r}} \mathcal{O}_j \quad \text{with} \quad \mathcal{O}_j = \{(E,F)\in M \,|\, \dim E\cap F = j\}\ .$$

The orbit $\mathcal{O}_0$ is open and dense in M, and for $E_0 = \{0\}^r \times \mathbb{C}^s$ and $F_0 = \mathbb{C}^r \times \{0\}^s$, the stabilizer of $(E_0, F_0) \in \mathcal{O}_0$ is given by $K^{\mathbb{C}} = \mathrm{S}(\mathrm{GL}(r)\times\mathrm{GL}(s))$.

Proof. Since $g(E\cap F) = gE \cap gF$, it follows that the subsets $\mathcal{O}_j$ are $G^{\mathbb{C}}$-invariant. Moreover, since $G^{\mathbb{C}}$ acts transitively on the set of bases (up to a constant), it also acts transitively on $\mathcal{O}_j$. To prove that $\mathcal{O}_0$ is open and dense in M, we first note that $(E,F)\in\mathcal{O}_0$ if and only if $E\oplus F = \mathbb{C}^n$, and if $E = \langle A\rangle$, $F = \langle B\rangle$ are

[1] cf. Example 3.30.

represented as column spaces of the matrices $A \in \mathbb{C}^{n\times s}$, $B \in \mathbb{C}^{n\times r}$, then $E \oplus F = \mathbb{C}^n$ if and only if $\det(A, B) \neq 0$. Consider the chart map

$$\varphi : \mathbb{C}^{r\times s} \times \mathbb{C}^{s\times r} \to M,\ (x, y) \mapsto \left(\left\langle\begin{pmatrix} x \\ \mathbf{1}_s \end{pmatrix}\right\rangle, \left\langle\begin{pmatrix} \mathbf{1}_r \\ y \end{pmatrix}\right\rangle\right).$$

It is well-known that the image of φ is open and dense in M. Furthermore, $\varphi(x, y)$ is in $\mathcal{O}_0$ if and only if $\det\left(\begin{smallmatrix} x & 1_s \\ 1_r & y \end{smallmatrix}\right) \neq 0$. Therefore, $\mathcal{O}_0$ is open and dense in the image of φ, and hence also open and dense in M. The form of the stabilizer of (E_0, F_0) follows from

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ \mathbf{1}_s \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \mathbf{1}_r \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}.$$

Hence, $g(E_0) = E_0$ implies $b = 0$, and $g(F_0) = F_0$ implies $c = 0$. □

Next we motivate the Jordan theoretic generalization of Proposition 5.1 by using the Jordan theoretic description of the Grassmannians $\mathrm{Gr}_r(\mathbb{C}^n)$ and $\mathrm{Gr}_s(\mathbb{C}^n)$ and determining the intersection $E \cap F$ explicitly. Recall from Lemma 4.1 and Example 4.5 that the Grassmannians $\mathbb{G}$ and $\mathrm{Gr}_s(\mathbb{C}^n)$ are related via

$$\mathbb{G} \to \mathrm{Gr}_s(\mathbb{C}^n),\ [x : a] \mapsto E_{x,a} = \left\langle\begin{pmatrix} x \\ 1 - a^*x \end{pmatrix}\right\rangle, \tag{5.1}$$

and that $\phi_+, \phi_- : \overline{\mathbb{G}} \to \mathrm{Gr}_r(\mathbb{C}^n)$ with

$$\phi_\pm\,[y : b] = F^\pm_{y,b} = \left\langle\begin{pmatrix} 1 - by^* \\ \pm y^* \end{pmatrix}\right\rangle \tag{5.2}$$

define isomorphisms of the conjugate Grassmannian $\overline{\mathbb{G}}$ and $\mathrm{Gr}_s(\mathbb{C}^n)$. Now, to determine the intersections $E_{x,a} \cap F^\pm_{y,b}$, we note that for $\eta \in \mathbb{C}^r$ and $\xi \in \mathbb{C}^s$, a simple calculation shows

$$\begin{pmatrix} x \\ 1 - a^*x \end{pmatrix}\eta = \begin{pmatrix} 1 - by^* \\ \pm y^* \end{pmatrix}\xi \iff \begin{cases} \eta = (a^* - (a^*b \mp 1)y^*)\xi\,, \\ ((1 - xa^*)(1 - by^*) \mp xy^*)\xi = 0\,. \end{cases}$$

Therefore, the intersection $E_{x,a} \cap F^\pm_{y,b}$ is completely determined by the kernel of the map $\xi \mapsto ((1 - xa^*)(1 - by^*) \mp xy^*)\xi$. In particular, we have

$$\dim E_{x,a} \cap F^\pm_{y,b} = r - \mathrm{rk}((1 - xa^*)(1 - by^*) \mp xy^*)\,, \tag{5.3}$$

and hence it follows that the $G^{\mathbb{C}}$-orbits of M are characterized by the rank of the matrix $((1 - xa^*)(1 - by^*) \mp xy^*)$, where the elements of M are represented by tuples $(E_{x,a}, F^\pm_{y,b})$. The crucial observation with respect to a full Jordan theoretic description of this is the following identity for the Bergman operator

$$B_{x,a}B_{x^a,\pm y^b}B_{y,b}(z) = \big((1 - xa^*)(1 - by^*) \mp xy^*\big)\, z\, \big((1 - y^*b)(1 - a^*x) \mp y^*x\big)\,,$$

which can easily be verified. This relates the rank of $((1 - xa^*)(1 - by^*) \mp xy^*)$ with the rank of the combination of Bergman operators $B_{x,a}B_{x^a,\pm y^b}B_{y,b}$, so we obtained a description in purely Jordan theoretic terms.

G-orbits on $\mathrm{Gr}_s(\mathbb{C}^n)$ and diagonal imbedding. Next we investigate the G-orbit structure on the Grassmannian $\mathrm{Gr}_s(\mathbb{C}^n)$. Classically, one considers the G-invariant scalar product on $\mathbb{C}^n$ given by

$$\langle x|y\rangle_G = x^*\left(\begin{smallmatrix} 1_r & 0 \\ 0 & -1_s \end{smallmatrix}\right) y\,,$$

and notes that the restriction of $\langle\,|\,\rangle_G$ to a subspace $E \in \mathrm{Gr}_s(\mathbb{C}^n)$ yields a (possibly degenerate) hermitian form on E. Let (a, b) denote the signature of this form, i.e. a (resp. b) is the number of positive (resp. negative) eigenvalues of a matrix representation of the hermitian form on E. By the G-invariance of the used scalar product on $\mathbb{C}^n$, this signature is an invariant for the G-action on $\mathrm{Gr}_s(\mathbb{C}^n)$, and by

Witt's Theorem it follows that this invariant completely characterizes the G-orbits on $\mathrm{Gr}_s(\mathbb{C}^n)$. This proves:

PROPOSITION 5.2. *The $G = \mathrm{SU}(r,s)$-orbits on $\mathrm{Gr}_s(\mathbb{C}^n)$ are given by*

$$\mathcal{O}_{a,b} = \left\{E \in \mathrm{Gr}_s(\mathbb{C}^n) \,\middle|\, \mathrm{sign}\left(\langle\,|\,\rangle_G \big|_{E\times E}\right) = (a,b)\right\} .$$

There are exactly $\binom{r+2}{2}$ different orbits according to the choices of (a,b) with $0 \le a \le r$, $0 \le b \le s$ and $a+b \le s$.

Using the Jordan theoretic description of E, the restriction of the G-invariant scalar product to E is given by

$$\langle x|y\rangle_G = \eta^* \begin{pmatrix} x^* & 1-x^*a\end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} \begin{pmatrix} x \\ 1-a^*x\end{pmatrix} \xi = \eta^* \big(x^*x - (1-x^*a)(1-a^*x)\big)\,\xi$$

for $x = \left(\begin{smallmatrix} x \\ 1-a^*x\end{smallmatrix}\right)\eta$ and $y = \left(\begin{smallmatrix} x \\ 1-a^*x\end{smallmatrix}\right)\xi$ with $\eta,\xi \in \mathbb{C}^s$. Therefore, the signature of the E-restricted scalar product satisfies

$$\mathrm{sign}\left(\langle\,|\,\rangle_G \big|_{E\times E}\right) = \mathrm{sign}\big(x^*x - (1-x^*a)(1-a^*x)\big) . \tag{5.4}$$

We identify this as a refinement of (5.3) by condisering the following imbedding of $\mathrm{Gr}_s(\mathbb{C}^n)$ into the product manifold:

$$\iota_\sigma : \mathrm{Gr}_s(\mathbb{C}^n) \to \mathrm{Gr}_s(\mathbb{C}^n) \times \mathrm{Gr}_r(\mathbb{C}^n),\ E \mapsto \left(E, E^{\perp_G}\right) ,$$

where $E^{\perp_G}$ denotes the orthogonal complement of E with respect to the G-invariant scalar product defined above. By construction, this is a G-equivariant map, and using the Jordan theoretic description of E, we obtain

$$E = E_{x,a} = \left\langle\!\!\left\langle \begin{pmatrix} x \\ 1-a^*x\end{pmatrix} \right\rangle\!\!\right\rangle , \quad E^{\perp_G} = F^+_{x,a} = \left\langle\!\!\left\langle \begin{pmatrix} 1-ax^* \\ x^*\end{pmatrix} \right\rangle\!\!\right\rangle .$$

Therefore, (5.3) yield exactly the dimension of the null-subspace in E corresponding to the restricted scalar product.

5.2. Group action and vector bundles

Let Z be a phJTS, let $\mathbb{G} = \mathbb{G}(Z)$ be its Grassmannian and let $\overline{\mathbb{G}} = \mathbb{G}(\overline{Z})$ be the conjugate Grassmannian, see Section 4.1. In this chapter we investigate the product manifold $\mathbb{G}\times\overline{\mathbb{G}}$ and the diagonal imbedding of $\mathbb{G}$ into $\mathbb{G}\times\overline{\mathbb{G}}$,

$$\mathbb{G} \to \mathbb{G}\times\overline{\mathbb{G}},\ \chi \mapsto (\chi,\chi) .$$

Therefore, $\mathbb{G}$ is considered as a *real* analytic submanifold of complex manifold $\mathbb{G}\times\overline{\mathbb{G}}$. As before, let $G^{\mathbb{C}}$ be the identity component of the automorphism group of $\mathbb{G}$, and let σ be the complex conjugation on $G^{\mathbb{C}}$ defined in (4.17) and having $G = \mathrm{Aut}(\mathcal{D})^0$ as fixed point set. Then

$$g\big(\chi,\,\eta\big) = \big(g\chi,\ \sigma(g)\,\eta\big) \quad \text{for} \quad g \in G^{\mathbb{C}} \tag{5.5}$$

defines a $G^{\mathbb{C}}$-action on $\mathbb{G}\times\overline{\mathbb{G}}$. In the restriction to $G \subset G^{\mathbb{C}}$, this action turns the diagonal imbedding $\mathbb{G} \hookrightarrow \mathbb{G}\times\overline{\mathbb{G}}$ into a G-equivariant map.

REMARK 5.3. We note that instead of using σ in the definition of the $G^{\mathbb{C}}$-action on $\mathbb{G}\times\overline{\mathbb{G}}$, one could also use the complex conjugation θ on $G^{\mathbb{C}}$ defined in (4.18) and having G^c as fixed point set,

$$g\big(\chi,\,\eta\big) = \big(g\chi,\ \theta(g)\,\eta\big) \quad \text{for} \quad g \in G^{\mathbb{C}} \tag{5.6}$$

In this case, the diagonal imbedding of $\mathbb{G}$ into $\mathbb{G}\times\overline{\mathbb{G}}$ becomes a G^c-equivariant map. Since we are primarily interested in the G-action on $\mathbb{G}$, we investigate the $G^{\mathbb{C}}$-action (5.5) in detail, and briefly state the corresponding results for the G^c-action on $\mathbb{G}$, which are obtained by slight changes in the given arguments.

Let $\mathcal{E}$ be a vector bundle on $\mathbb{G}$, and $\mathcal{F}$ be a vector bundle on $\overline{\mathbb{G}}$. Then we denote by $\mathcal{E} \boxtimes \mathcal{F}$ the vector bundle on $\mathbb{G} \times \overline{\mathbb{G}}$ given by the tensor products of the fibers of $\mathcal{E}$ and $\mathcal{F}$, i.e.

$$(\mathcal{E} \boxtimes \mathcal{F})_{([x:a],[y:b])} = \mathcal{E}_{[x:a]} \otimes \mathcal{F}_{[y:b]} \,. \tag{5.7}$$

Furthermore, we denote by $\overline{\mathcal{E}}$ the conjugate vector bundle on $\overline{\mathbb{G}}$, obtained from $\mathcal{E}$ by taking the conjugate complex structure on the fibers. We describe two examples in more detail:

(1) Let $\mathcal{L}_\delta$ denote the line bundle on $\mathbb{G}$ defined by a denominator δ of the quasi-inverse,

$$\mathcal{L}_\delta = (Z \times \overline{Z} \times \mathbb{C})/\sim \quad \text{with} \quad [z, a, \lambda] = [z^{a-\tilde{a}},\, \tilde{a},\, \delta(z, a-\tilde{a})\,\lambda]\,,$$

and let $\mathcal{L}_\delta^{-1}$ denote the corresponding inverse line bundle defined by

$$[z, a, \lambda] = [z^{a-\tilde{a}},\, \tilde{a},\, \delta(z, a-\tilde{a})^{-1}\,\lambda]\,.$$

Then $\mathcal{L}_\delta^{-1} \boxtimes \overline{\mathcal{L}_\delta^{-1}}$ is a holomorphic line bundle on $\mathbb{G}\times\overline{\mathbb{G}}$, on which the transition maps are described by

$$\big[(x,a),(z,b),\lambda\big] = \Big[(x^{a-\tilde{a}},\tilde{a}),\ (y^{b-\tilde{b}},\tilde{b}),\ \delta(x,a-\tilde{a})^{-1}\,\overline{\delta(y,b-\tilde{b})^{-1}}\,\lambda\Big]\,.$$

We recall from Section 4.2 that for $\delta(x,y) = \operatorname{Det} B_{x,y}$, the line bundle $\mathcal{L}_\delta^{-1}$ coincides with the canonical bundle $\mathcal{K}$, which is a $G^{\mathbb{C}}$-homogeneous line bundle on $\mathbb{G}$. Therefore, $\mathcal{K} \boxtimes \overline{\mathcal{K}}$ is also a $G^{\mathbb{C}}$-homogeneous line bundle on $\mathbb{G} \times \overline{\mathbb{G}}$.

(2) If $T\mathbb{G}$ denotes the tangent bundle on $\mathbb{G}$, then $T\mathbb{G} \boxtimes \overline{T\mathbb{G}}$ is a vector bundle on $\mathbb{G} \times \overline{\mathbb{G}}$, whose fiber $T_{[x:a]}\mathbb{G} \otimes T_{[y:b]}\mathbb{G}$ can be identified with $Z \otimes \overline{Z}$, and hence with linear maps from $\overline{Z}^{\#}$ to Z, where $\overline{Z}^{\#}$ denotes the vector space dual to $\overline{Z}$. The description of $T\mathbb{G} \boxtimes \overline{T\mathbb{G}}$ via equivalence classes is given by

$$\big[(x,a),(z,b),T\big] = \Big[(x^{a-\tilde{a}},\tilde{a}),\ (y^{b-\tilde{b}},\tilde{b}),\ B^{-1}_{x,\,a-\tilde{a}}\,T\,B^{-1}_{b-\tilde{b},\,y}\Big]$$

with $T \in \operatorname{Hom}(\overline{Z}^{\#}, Z)$. Since $T\mathbb{G}$ is a $G^{\mathbb{C}}$-homogeneous vector bundle on $\mathbb{G}$, the bundle $T\mathbb{G} \boxtimes T\overline{\mathbb{G}}$ also carries a $G^{\mathbb{C}}$-action turning it into a homogeneous vector bundle on $\mathbb{G}\times\overline{\mathbb{G}}$. Using the intrinsic scalar product of the phJTS Z, we often identify $Z^{\#}$ with Z. This yields a complex linear isomorphism of $\overline{Z}^{\#}$ to Z, and elements $T \in \operatorname{Hom}(\overline{Z}^{\#}, Z)$ can be interpreted as endomorphisms on Z, but doing this, we have to keep in mind that elements of the domain transform as $(v \mapsto B_{b-\tilde{b},\,y} v)$, and elements of the image transform as $(u \mapsto B^{-1}_{x,\,a-\tilde{a}} u)$.

In the next sections we describe naturally defined sections on $\mathcal{L}_\delta^{-1} \boxtimes \overline{\mathcal{L}_\delta^{-1}}$ and $T\mathbb{G} \boxtimes \overline{T\mathbb{G}}$, which have nice properties concerning $G^{\mathbb{C}}$-actions.

5.3. Equivariant sections

Let δ be a denominator of the quasi-inverse. The first aim of this section is to extend $\delta : Z \times \overline{Z} \to \mathbb{C}$ to a map on $\mathbb{G} \times \overline{\mathbb{G}}$. Recall that $Z \times \overline{Z}$ imbeds into $\mathbb{G} \times \overline{\mathbb{G}}$ as an open and dense subset via

$$Z \times \overline{Z} \hookrightarrow \mathbb{G} \times \overline{\mathbb{G}},\ (x,y) \mapsto \big([x:0],[y:0]\big)\,.$$

Since $\mathbb{G}\times\overline{\mathbb{G}}$ is compact, and δ is holomorphic, it is clear that such an extension must be an extension into a line bundle on $\mathbb{G} \times \overline{\mathbb{G}}$. The suitable line bundle turns out to be $\mathcal{L}_\delta^{-1} \boxtimes \overline{\mathcal{L}_\delta^{-1}}$. In particular, this procedure provides an extension of the Jordan triple determinant $\delta = \mathit{\Delta}$ to a section $\boldsymbol{\mathit{\Delta}}$ on $\mathbb{G} \times \overline{\mathbb{G}}$.

Proposition 5.4. *Let δ be a denominator of the quasi-inverse.*

(a) *For $(x,a) \in Z \times \overline{Z}$ and $(y,b) \in \overline{Z} \times Z$ set*

$$\widehat{\delta}\big((x,a),(y,b)\big) = \delta(x,a)\,\delta(x^a,y^b)\,\overline{\delta(y,b)}\,.$$

Then $\widehat{\delta}$ is a complex polynomial function on $(Z \times \overline{Z}) \times (\overline{Z} \times Z)$.

(b) *The map $\boldsymbol{\delta} : \mathbb{G} \times \overline{\mathbb{G}} \to \mathcal{L}_\delta^{-1} \boxtimes \overline{\mathcal{L}_\delta^{-1}}$ given by*

$$\boldsymbol{\delta}([x:a],[y:b]) = \big[(x,a),\ (y,b),\ \widehat{\delta}\big((x,a),(y,b)\big)\big]$$

is a section of $\mathcal{L}_\delta^{-1} \boxtimes \overline{\mathcal{L}_\delta^{-1}}$.

(c) *The vanishing set of $\boldsymbol{\delta}$ is $G^{\mathbb{C}}$-invariant and independent of the choice of the denominator δ. If $\mathcal{L}_\delta^{-1} = \mathcal{K}$ is the canonical bundle, then $\boldsymbol{\delta}$ is a $G^{\mathbb{C}}$-equivariant section.*

Proof. The proof of (a) is due to O. Loos [**31**]: From Lemma 2.23, we obtain

$$\delta(x^a,y^b) = \delta(x,a)^{-1}\delta(x,a+y^b) = \delta(x^a+b,y)\overline{\delta(y,b)}^{-1}\,. \tag{5.8}$$

The first identity implies that $\widehat{\delta}$ is polynomial in $(x,a) \in Z \times \overline{Z}$, and the second identity implies that $\widehat{\delta}$ is polynomial in $(y,b) \in \overline{Z} \times Z$, hence $\widehat{\delta}$ is polynomial in all of its variables. For (b), we have to show that if $[x:a] = [\tilde{x}:\tilde{a}]$ and if $[y:b] = [\tilde{y}:\tilde{b}]$, then $\widehat{\delta}$ transforms as

$$\widehat{\delta}\big((\tilde{x},\tilde{a}),(\tilde{y},\tilde{b})\big) = \delta(x,a-\tilde{a})^{-1}\,\widehat{\delta}\big((x,a),(y,b)\big)\,\overline{\delta(y,b-\tilde{b})^{-1}}\,. \tag{5.9}$$

This is a straightforward calculation:

$$\begin{aligned}
\widehat{\delta}((\tilde{x},\tilde{a}),\,(\tilde{y},\tilde{b})) &= \widehat{\delta}((x^{a-\tilde{a}},\,\tilde{a}),(y^{b-\tilde{b}},b))\\
&= \delta(x^{a-\tilde{a}},\tilde{a})\,\delta\big((x^{a-\tilde{a}})^{\tilde{a}},\,(y^{b-\tilde{b}})^{\tilde{b}}\big)\,\overline{\delta(y^{b-\tilde{b}},\tilde{b})}\\
&= \delta(x,a-\tilde{a})^{-1}\delta(x,a)\,\delta\big(x^a,y^b\big)\,\overline{\delta(y,b-\tilde{b})^{-1}\delta(y,b)}\\
&= \delta(x,a-\tilde{a})^{-1}\,\widehat{\delta}((x,y),(y,b))\,\overline{\delta(y,b-\tilde{b})^{-1}}\,.
\end{aligned}$$

To prove (c), we first claim that for different denominators δ and δ', the vanishing sets of $\widehat{\delta}$ and $\widehat{\delta}'$ coincide. It suffices to prove this for an arbitrary δ and $\delta' = \varDelta$, the Jordan triple determinant of Z. Without restriction we assume that Z is simple.[2] Then, δ is a power of $\varDelta$, since $\varDelta$ is irreducible and by definition, the vanishing sets of δ and $\varDelta$ coincide [**25**, I §1.7]. This again implies that $\widehat{\delta}$ is a power of $\widehat{\varDelta}$, so the corresponding vanishing sets indeed coincide. Now, for a proof of (c), it is sufficient to show the $G^{\mathbb{C}}$-equivariance of $\boldsymbol{\delta}$ in the case of $\delta(u,v) = \operatorname{Det} B_{u,v}$, i.e. $\mathcal{L}_\delta^{-1} = \mathcal{K}$. We check this equivariance on the generators t_u and $\tilde{t}_v$ of $G^{\mathbb{C}}$ with $(u,v) \in Z \times \overline{Z}$. By continuity, it suffices to consider the open and dense subset $Z \times \overline{Z}$ in $\mathbb{G} \times \overline{\mathbb{G}}$. Recall from Section 4.2 that t_u and $\tilde{t}_v$ act on $\mathcal{K}$ by

$$\tilde{t}_v\,[z,a,\lambda] = [z,\,a+v,\,\lambda] \quad \text{and} \quad t_u\,[z,0,\lambda] = [z+u,\,0,\,\lambda] \tag{5.10}$$

[2] Otherwise, consider the decomposition Z into simple phJTS Z_i, and replace the single Jordan triple determinant by the product of the Jordan triple determinants on the Z_i.

Due to Section 4.2, it is $\sigma(t_u) = \tilde{t}_{-u}$ and $\sigma(\tilde{t}_v) = t_{-v}$. We compute the fiber coordinate of $\boldsymbol{\delta} \circ t_u$ in $\big([x:0],[y:0]\big)$:

$$\begin{aligned}\widehat{\delta}\big((x+u,0),(y,-u)\big) &= \delta(x+u,0)\,\delta(x+u,y^{-u})\,\overline{\delta(y,-u)}\\ &= \delta(x+u-v,y)\,\overline{\delta(y,-u)^{-1}}\,\overline{\delta(y,-u)}\\ &= \delta(x,y)\\ &= \widehat{\delta}\big((x,0),(y,0)\big)\,.\end{aligned} \tag{5.11}$$

Comparing this with (5.10), yields $\boldsymbol{\delta}\circ t_u = t_u \circ \boldsymbol{\delta}$, i.e. $\boldsymbol{\delta}$ is t_u-equivariant. Analogously, we obtain for the fiber coordinate of $\boldsymbol{\delta} \circ \tilde{t}_v$ in $\big([x:0],[y:0]\big)$:

$$\begin{aligned}\widehat{\delta}\big((x,v),(y-v,0)\big) &= \delta(x,v)\,\delta(x^v,y-v)\,\overline{\delta(y-u,0)}\\ &= \delta(x,v)\,\delta(x,v)^{-1}\,\delta(x,y-v+v)\\ &= \delta(x,y)\\ &= \widehat{\delta}\big((x,0),(y,0)\big)\,.\end{aligned} \tag{5.12}$$

Thus, $\boldsymbol{\delta}$ is also t_u-equivariant, and hence $G^{\mathbb{C}}$-equivariant. □

REMARK 5.5. We notice that for an arbitrary denominator δ, Proposition 5.4 does not assert the $G^{\mathbb{C}}$-equivariance of $\boldsymbol{\delta}$, since in general there is no $G^{\mathbb{C}}$-action on $\mathcal{L}_{\delta}^{-1}$. Nevertheless, the proof of Proposition 5.4 shows that whenever there *is* a $G^{\mathbb{C}}$-action on $\mathcal{L}_{\delta}^{-1}$ satisfying (5.10), then $\boldsymbol{\delta}$ is indeed $G^{\mathbb{C}}$-equivariant. For the problem to define such a $G^{\mathbb{C}}$-action on $\mathcal{L}_{\delta}^{-1}$, see Section 4.2.

Next we consider the restriction of $\boldsymbol{\delta}$ to the diagonal $\mathbb{G} \hookrightarrow \mathbb{G} \times \overline{\mathbb{G}}$. In this case, $\widehat{\delta}$ takes on only real values, since

$$\widehat{\delta}\big((x,a),(x,a)\big) = \delta(x,a)\,\delta(x^a,x^a)\,\overline{\delta(x,a)} = \overline{\widehat{\delta}\big((x,a),(x,a)\big)}\,,$$

and $\boldsymbol{\delta}$ transforms by (5.9) as

$$\widehat{\delta}\big((\tilde{x},\tilde{a}),(\tilde{x},\tilde{a})\big) = |\delta(x,a-\tilde{a})|^{-2}\,\widehat{\delta}\big((x,a),(x,a)\big)\,.$$

Therefore, the signum of $\boldsymbol{\delta}$ is a well defined quantity on the diagonal $\mathbb{G} \hookrightarrow \mathbb{G} \times \overline{\mathbb{G}}$, which is invariant under the action of G, since G is connected.

COROLLARY 5.6. *Let δ be a denominator of the quasi-inverse, and let $\boldsymbol{\delta}$ be as in Proposition 5.4. The map*

$$\operatorname{sign}\boldsymbol{\delta} : \mathbb{G} \to \{-1,0,1\},\ [x:a] \mapsto \operatorname{sign}\boldsymbol{\delta}([x:a],[x:a]) \tag{5.13}$$

is well-defined and G-invariant.

REMARK 5.7. Using (5.6) instead of (5.5) as definition of the $G^{\mathbb{C}}$-action on $\mathbb{G} \times \overline{\mathbb{G}}$, Proposition 5.4 and Corollary 5.6 remain true, when $\widehat{\delta}$ is replaced by

$$\widehat{\delta}_\theta\big((x,a),(y,b)\big) = \delta(x,a)\,\delta(x^a,-y^b)\,\overline{\delta(y,b)} = \widehat{\delta}\big((x,a),(-y,-b)\big)\,,$$

and G is replaced by G^c. The difference in sign comes in, since we have $\theta(t_u) = \tilde{t}_u$ and $\theta(\tilde{t}_v) = t_v$ instead of $\sigma(t_u) = \tilde{t}_{-u}$ and $\sigma(\tilde{t}_v) = t_{-v}$. By Theorem 4.7, G^c acts transitively on $\mathbb{G}$. Therefore, the sign function $\operatorname{sign}\boldsymbol{\delta}_\theta$ has to be constant. Since $\operatorname{sign}\boldsymbol{\delta}_\theta([0:0]) = 1$, we conclude that $\boldsymbol{\delta}_\theta$ is strictly positive on the restriction to $\mathbb{G}$.

Rank map. So far we discussed bundles and sections that are induced by a denominator of the quasi-inverse, and obtained a quite broad $G^{\mathbb{C}}$-invariant on $\mathbb{G}\times\overline{\mathbb{G}}$ and an equally broad G-invariant on $\mathbb{G}$. The second aim of this section is to refine these invariants by considering a bundle and a section that are induced by Bergman operators (instead of their determinants as in the case of the canonical bundle).

We consider the $G^{\mathbb{C}}$-homogeneous vector bundle $T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$ on $\mathbb{G}\times\overline{\mathbb{G}}$. As described in the last section, the fibers can be identified with endomorphisms of Z, and the transition maps are defined by

$$[(x,a),(y,b),T] = \big[(x^{a-\tilde{a}},\tilde{a}),\ (y^{b-\tilde{b}},\tilde{b}),\ B^{-1}_{x,\,a-\tilde{a}}\,T\,B^{-1}_{b-\tilde{b},\,y}\big] \tag{5.14}$$

We define the *rank* of an element $[(x,a),(y,b),T]\in T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$ to be

$$\operatorname{rk}[(x,a),(y,b),T] := \operatorname{rk} T\ .$$

Due to (5.14), this is well-defined. The action of $G^{\mathbb{C}}$ on $T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$ is generated by the actions of t_u and $\tilde{t}_v$ for $(u,v)\in Z\times\overline{Z}$. On the open and dense subset $TZ\boxtimes T\overline{Z}\subset T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$ these actions are given by

$$t_u[(x,0),(y,0),T] = [(x+u,0),\ (y,-u),\ T]$$

and

$$\tilde{t}_v[(x,0),(y,0),T] = [(x,v),\ (y-v,0),\ T]\ .$$

Therefore, $G^{\mathbb{C}}$ preserves the rank of the element on $TZ\boxtimes T\overline{Z}$. By semi-continuity of the rank map $T\mapsto \operatorname{rk}T$, and by the invertibility of the action of each $g\in G^{\mathbb{C}}$ on $T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$, we conclude:

Lemma 5.8. *The action of $G^{\mathbb{C}}$ on the vector bundle $T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$ is rank-preserving.*

The next proposition defines a $G^{\mathbb{C}}$-equivariant section, which is used in the next section to determine the $G^{\mathbb{C}}$-orbits on $\mathbb{G}\times\overline{\mathbb{G}}$. This is a natural extension of the section $\boldsymbol{\delta}$ discussed above.

Proposition 5.9. *Let Z be a phJTS.*

(a) *For $(x,a)\in Z\times\overline{Z}$ and $(y,b)\in\overline{Z}\times Z$ set*

$$\widehat{\beta}\big((x,a),(y,b)\big) = B_{x,\,a}B_{x^a,\,y^b}B_{b,\,y}\ .$$

Then $\widehat{\beta}$ depends polynomially on $((x,a),(y,b))\in(Z\times\overline{Z})\times(\overline{Z}\times Z)$.

(b) *The map $\boldsymbol{\beta}:\mathbb{G}\times\overline{\mathbb{G}}\to T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$ given by*

$$\boldsymbol{\beta}([x:a],[y:b]) = \big[(x,a),\ (y,b),\ \widehat{\beta}\big((x,a),(y,b)\big)\big]$$

is a $G^{\mathbb{C}}$-equivariant section of $T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$.

(c) *The map*

$$\operatorname{rk}\boldsymbol{\beta}:\mathbb{G}\times\overline{\mathbb{G}}\to\mathbb{N},\ ([x:a],[y:b])\mapsto \operatorname{rk}\widehat{\beta}\big((x,a),(y,b)\big)$$

is well-defined and $G^{\mathbb{C}}$-invariant. We call $\operatorname{rk}\boldsymbol{\beta}$ the rank map *on $\mathbb{G}\times\overline{\mathbb{G}}$.*

Proof. Part (a) follows by application of the identities JT33 and JT34:

$$B_{x,\,a}B_{x^a,\,y^b}B_{b,\,y} = B_{x,\,y^b+a}B_{b,\,y} = B_{x,\,a}B_{x^a+b,\,y}\ . \tag{5.15}$$

The first identity show that $\widehat{\beta}$ depends polynomially on $(x,a)\in Z\times\overline{Z}$, and the second one implies the polynomial dependence on $(y,b)\in\overline{Z}\times Z$. For (b), we first verify the correct transformation rule for sections in $T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$. So let $[x:a]=[\tilde{x}:\tilde{a}]$ and if $[y:b]=[\tilde{y}:\tilde{b}]$, then

$$\begin{aligned}\widehat{\beta}((\tilde{x},\tilde{a}),\,(\tilde{y},\tilde{b})) &= \widehat{\beta}((x^{a-\tilde{a}},\,\tilde{a}),(y^{b-\tilde{b}},b))\\ &= B_{x^{a-\tilde{a}},\,\tilde{a}}\,B_{(x^{a-\tilde{a}})^{\tilde{a}},\,(y^{b-\tilde{b}})^{\tilde{b}}}\,B_{\tilde{b},\,y^{b-\tilde{b}}}\\ &= B^{-1}_{x,\,a-\tilde{a}}B_{x,\,a}B_{x^a,\,y^b}B_{b,\,y}B^{-1}_{b-\tilde{b},\,y}\\ &= B^{-1}_{x,\,a-\tilde{a}}\,\widehat{\beta}((x,y),(y,b))\,B^{-1}_{b-\tilde{b},\,y}\,.\end{aligned} \tag{5.16}$$

Since the Bergman operator satisfies the same relations as a denominator δ of the quasi-inverse, this calculation is essentially the same as in the proof of Proposition 5.4. We only have to be aware of the non-commutativity of the Bergman operators. Our calculation shows that the terms have the appropriate order. Similarly, exchanging δ in (5.11) and (5.12) by the corresponding Bergman operator yields the identities $\boldsymbol{\beta}\circ t_u=t_u\circ\boldsymbol{\beta}$ and $\boldsymbol{\beta}\circ\tilde{t}_v=\tilde{t}_v\circ\boldsymbol{\beta}$ for all $(u,v)\in Z\times\overline{Z}$. Therefore, $\boldsymbol{\beta}$ is a $G^{\mathbb{C}}$-equivariant section. Finally, we note that the $G^{\mathbb{C}}$-equivariance of $\boldsymbol{\beta}$ and Lemma 5.8 imply the $G^{\mathbb{C}}$-invariance of $\boldsymbol{\beta}$. □

Restriction to $\mathbb{G}$. As in the case of the line bundle induced by a denominator of the quasi-inverse, we consider the restriction of $\boldsymbol{\beta}$ to the diagonal $\mathbb{G}\hookrightarrow\mathbb{G}\times\overline{\mathbb{G}}$. In this restriction, the fiber $T_{[x:a]}\mathbb{G}\otimes T_{[x:a]}\overline{\mathbb{G}}$ can be identified with sesquilinear forms on $T^{\#}_{[x:a]}\mathbb{G}$. Therefore, by means of the scalar product on Z, the canonical fiber of $T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$ in the restriction to $\mathbb{G}$ can also be identified with sesquilinear forms on Z. Since

$$\widehat{\beta}\big((x,a),(x,a)\big)=B_{x,\,a}B_{x^a,\,x^a}B_{a,\,x}=\widehat{\beta}\big((x,a),(x,a)\big)^*\,,$$

the section $\boldsymbol{\beta}$ takes on its values in the subspace of *hermitian* forms, which is invariant under coordinate change due to (5.16). Therefore, according to Sylvester's law for hermitian forms, the *signature*

$$\sigma\big(\boldsymbol{\beta}([x:a]\,,[x:a])\big):=\text{signature of }\widehat{\beta}\big((x,a),(x,a)\big)=(\sigma_+,\sigma_-)\in\mathbb{N}^2$$

is a well-defined quantity. Here σ_+ and σ_- denote the dimensions of the maximally positive and maximally negative subspaces of Z with respect to $\widehat{\beta}((x,a),(x,a))$.

Corollary 5.10. *Let $\boldsymbol{\beta}$ be as in Proposition 5.9. The map*

$$\sigma(\boldsymbol{\beta}):\mathbb{G}\to\mathbb{N}^2,\ [x:a]\mapsto\sigma\big(\boldsymbol{\beta}([x:a]\,,[x:a])\big) \tag{5.17}$$

is well-defined and G-invariant. We call $\sigma(\boldsymbol{\beta})$ the signature map *on $\mathbb{G}$.*

Proof. It remains to prove the G-invariance. Consider the open and dense subset $Z\hookrightarrow\mathbb{G}$. Due to (4.26), $G^{\mathbb{C}}$ acts on $T\mathbb{G}$ by

$$g\,[z,0,\zeta]=[g(z),\,0,\,D_zg(\zeta)]\quad\text{for all }g\in G^{\mathbb{C}}\text{ with }g(z)\in Z\subset\mathbb{G}.$$

Therefore, we obtain for the G-action on $T\mathbb{G}\boxtimes T\overline{\mathbb{G}}$ the relation

$$g\,[(z,0),(z,0),T]=[(g(z),0),\,(g(z),0),\,(D_zg)\,T\,(D_zg)^*)]$$

for all $g\in G$ with $g(z)\in Z\subset\mathbb{G}$. Therefore, on $Z\subset\mathbb{G}\hookrightarrow\mathbb{G}\times\overline{\mathbb{G}}$ the signature of $\boldsymbol{\beta}$ is invariant under the action of all $g\in G$ with $g(z)\in Z$. Since Z is open and dense in $\mathbb{G}$, and since the set of all $g\in G$ with $g(z)\in Z$ is also open and dense in G, we conclude by semi-continuity of the components of the signature map **[12]**, that $\sigma(\boldsymbol{\beta})$ is invariant under the action of g for all $g\in G$ and on all of $\mathbb{G}$. □

Corollary 5.10 shows that the signature $\sigma(\boldsymbol{\beta})$ is a G-invariant on $\mathbb{G}$. In Section 7.1 we use this invariant to determine the G-orbit structure on $\mathbb{G}$. In fact, we will show that for a simple phJTS Z of rank r, the Grassmannian $\mathbb{G}$ decomposes into exactly $\binom{r+2}{2}$ disjoint G-orbits, which can be characterized by their signature.

In particular, this shows that $\sigma(\boldsymbol{\beta})$ takes on only $\binom{r+2}{2}$ different values. Furthermore, we will see that on $r+1$ of these orbits the signature is non-degenerate, and therefore, the section $\boldsymbol{\beta}$ induces G-invariant pseudo-Riemannian metrics on these orbits, cf. Section 7.1.

REMARK 5.11. As in Remark 5.7, we briefly discuss the results obtained by using (5.6) instead of (5.5) as definition of the $G^{\mathbb{C}}$-action on $\mathbb{G}\times\overline{\mathbb{G}}$. It is straightforward to check that Proposition 5.9 and Corollary 5.10 remain true, when $\widehat{\beta}$ is replaced by

$$\widehat{\beta}_\theta\big((x,a),(y,b)\big) = B_{x,a}B_{x^a,-y^b}B_{b,y} = \widehat{\beta}\big((x,a),(-y,-b)\big)\,.$$

and G is replaced by G^c. The restriction of $\widehat{\beta}_\theta$ to the diagonal $\mathbb{G}\subset\mathbb{G}\times\overline{\mathbb{G}}$ yields a G^c-equivariant section on $\mathbb{G}$, whose signature map $\operatorname{rk}\boldsymbol{\beta}_\theta$ is constant, since G^c acts transitively on $\mathbb{G}$. Moreover, since $\widehat{\beta}_\theta\big((0,0),(0,0)\big)=\mathrm{Id}$, we conclude that $B_{x,a}B_{x^a,-x^a}B_{a,x}$ is positive definite for all $(x,a)\in Z\times\overline{Z}$. Now consider the map

$$h([x:a],[x:a]) = \big[(x,a),\ (x,a),\ (B_{x,a}B_{x^a,x^a}B_{a,x})^{-1}\big]\,.$$

It is easily checked that h defines a G^c-invariant hermitian metric on $\mathbb{G}$. Since the evaluation of h on the affine part $Z\subset\mathbb{G}$ reduces to the well-known formula $h_x(u,v)=\big\langle B^{-1}_{x,-x}u\big|v\big\rangle$, we conclude that h indeed coincides with the G^c-invariant hermitian metric described in Remark 4.8.

5.4. Orbit structure

We determine the $G^{\mathbb{C}}$-orbit structure of $\mathbb{G}\times\overline{\mathbb{G}}$. Let $\boldsymbol{\Delta}$ be the section on $\mathbb{G}\times\overline{\mathbb{G}}$ extending the Jordan triple determinant Δ on $Z\times\overline{Z}$ as described in Proposition 5.4. Furthermore, let $\boldsymbol{\beta}$ be the rank map on $\mathbb{G}\times\overline{\mathbb{G}}$ defined in Proposition 5.9.

THEOREM 5.12. *Let Z be a simple phJTS of rank r.*

(a) *$\mathbb{G}\times\overline{\mathbb{G}}$ decomposes into exactly $r+1$ different $G^{\mathbb{C}}$-orbits, which are characterized by the rank map $\operatorname{rk}\boldsymbol{\beta}$. More precisely, for any flag $0=e_0<e_1<\ldots<e_r$ of tripotents in Z, the orbits are given by*

$$\mathcal{O}_j := G^{\mathbb{C}}\big(\,[e_j:0]\,,[e_j:0]\,\big) = (\operatorname{rk}\boldsymbol{\beta})^{-1}\big(\dim Z_0^{e_j}\big)$$

for $0\le j\le r$.

(b) *The only open and dense orbit $\mathcal{O}_0=G^{\mathbb{C}}(0,0)$ is also characterized by*

$$\mathcal{O}_0 = \big\{(\chi,\eta)\in\mathbb{G}\times\overline{\mathbb{G}}\,\big|\,\boldsymbol{\Delta}(\chi,\eta)\neq 0\big\}\,.$$

The stabilizer subgroup of $(0,0)$ in $G^{\mathbb{C}}$ is $K^{\mathbb{C}}$, therefore $\mathcal{O}_0\cong G^{\mathbb{C}}/K^{\mathbb{C}}$.

PROOF. First we show that the $G^{\mathbb{C}}$-orbit of any element $([x:a],[y:b])$ of $\mathbb{G}\times\overline{\mathbb{G}}$ contains an element $([e:0],[e:0])$ for a suitable tripotent $e\in Z$. Since $G^{\mathbb{C}}$ acts transitively on $\mathbb{G}$, there exists an element $g\in G^{\mathbb{C}}$, such that $g([x:a],[y:b]) = ([\tilde{x}:\tilde{a}],[0,0:)]$. Due to Theorem 4.12 and Corollary 2.21, $[\tilde{x}:\tilde{a}]$ can be represented as $[\tilde{x}:\tilde{a}]=[e+z:e]$ for some tripotent e and an element $z\in Z$ with $z\perp\!\!\!\perp e$. Therefore,

$$\begin{aligned} t_{-z}\tilde{t}_{-e}g([x:a],[y:b]) &= t_{-z}\tilde{t}_{-e}([e+z:e],[0:0]) \\ &= t_{-z}([e+z:0],[e:0]) \\ &= ([e:0],[e:z]) \\ &= ([e:0],[e:0])\,. \end{aligned}$$

In the last step we used Lemma 2.25. By simplicity of Z, there exists an element $k\in K$ such that $ke=e_j$ for $j=\operatorname{rk}(e)$. Since $\sigma(k)=k$, we finally obtain

$$kt_{-z}\tilde{t}_{-e}g([x:a],[y:b]) = ([e_j:0],[e_j:0])\,.$$

This proves that the subsets $\mathcal{O}_j$ for $0 \leq j \leq r$ indeed form the $G^{\mathbb{C}}$-orbit decomposition of $\mathbb{G} \times \overline{\mathbb{G}}$. By Proposition 5.9, the rank map is constant on each $\mathcal{O}_j$. It is

$$\operatorname{rk} \boldsymbol{\beta}([e:0],[e:0]) = \operatorname{rk} B_{e,e} = \dim Z_0^e \,, \tag{5.18}$$

since $B_{e,e}$ can be regarded as the orthogonal projection onto Z_0^e. This completes the proof of (a). For (b), we recall from Proposition 5.4 that the vanishing set of $\boldsymbol{\Delta}$ is $G^{\mathbb{C}}$-invariant, so it is a union of $G^{\mathbb{C}}$-orbits. Since a Bergman operator $B_{x,y}$ is invertible if and only if $\Delta(x,y) \neq 0$, it follows by (5.18) that the vanishing set is precisely the union of the orbits $\mathcal{O}_j$ with $j = 1, \ldots, r$. Therefore, $\mathcal{O}_0$ coincides with the non-vanishing set of $\boldsymbol{\Delta}$, and hence is the only open and dense $G^{\mathbb{C}}$-orbit of $\mathbb{G} \times \overline{\mathbb{G}}$. It remains to determine the $G^{\mathbb{C}}$-stabilizer of $(0,0)$. Due to Theorem 4.6, every $g \in G^{\mathbb{C}}$ can be written as $g = \tilde{t}_v t_u h \tilde{t}_w$ with $u \in Z$, $v, w \in \overline{Z}$ and $h \in K^{\mathbb{C}}$. By (4.17), the conjugate of this transformation is $\sigma(g) = t_{-v} \tilde{t}_{-u} h^{-*} t_{-w}$. We determine the conditions on u, v, w and h for stabilizing $[0:0]$. Applying the first transformation to $[0:0]$ yields $g\,[0:0] = [u:v]$, so u must vanish. Therefore, the second transformation is given by $\sigma(g)\,[0:0] = [-h^{-*}w - v : 0]$, which implies that w and v also vanish. Thus we derived the necessary condition $g = h \in K^{\mathbb{C}}$. This is also sufficient, since $K^{\mathbb{C}}$ stabilizes $(0,0)$ in $\mathbb{G} \times \overline{\mathbb{G}}$. $\square$

CHAPTER 6

Peirce varieties

There are naturally defined smooth varieties associated with a Jordan triple systems Z. One of these is the variety $\mathbb{P}$ of all Peirce 1-spaces in Z, which we call the *Peirce Grassmannian.* In the infinite dimensional setting an appropriate variant of this variety has been studied in some detail by W. Kaup [**18**] and by J. Isidro and L.L. Stacho [**16**]. The finite dimensional setting is discussed by J. Arazy and H. Upmeier [**2**] and briefly by O. Loos [**28**]. Due to the use of generalized Peirce decompositions, our approach to the Peirce Grassmannian is essentially complex analytic (Section 6.1). Besides the global viewpoint, we give a local description of the Peirce Grassmannian via charts and explicit formulas for the transition maps, thus showing that the Peirce Grassmannian is a smooth algebraic variety.

In Section 6.2, we compare different realizations of the Peirce Grassmannian. One of the main results of this chapter is the identification of the connected components of $\mathbb{P}$ with closed $K^{\mathbb{C}}$-orbits in the Grassmannian, which in turn can be identified explicitly with the Grassmannian of some Peirce $1/2$-space. This connection of the Peirce Grassmannian $\mathbb{P}$ to the orbit structure on the Grassmannian $\mathbb{G}(Z)$ is the justification for the discussion of the Peirce varieties within this part of the thesis. Furthermore, in the next chapter it turns out that each G-orbit of the Grassmannian $\mathbb{G}(Z)$ is fibered over some extension of a corresponding Peirce Grassmannian.

In Section 6.3, we use the Godement approach to define line bundles on the Peirce Grassmannian. The corresponding cocycles are given on the basis of a denominator of the quasi-inverse on Z. It is proved explicitly that these line bundles are very ample, and hence $\mathbb{P}$ is a projective variety.

The last section provides a discussion of the obvious generalization of the Peirce Grassmannian to Peirce flag varieties. A brief account of this can be found in [**2**]. Again, our approach has the advantage of being complex analytic, so it is straightforward to extend the proofs of the analytic and algebraic structures of Peirce Grassmannian to the case of the Peirce flag varieties.[1] Peirce flag varieties will form the essential ingredient in the definition of Jordan flag varieties in the third part of this thesis.

6.1. Peirce Grassmannians

Let Z be a phJTS of rank r. In Section 2.6 we introduced the notion of *Peirce equivalence* on Z,

$$u \equiv v \iff u \text{ and } v \text{ induce the same Peirce decomposition of } Z.$$

In this section we investigate the set of equivalence classes $\mathbb{P} := Z/\equiv$. We prove that this gives a compact manifold, called the *Peirce Grassmannian* of Z, whose connected components are compact hermitian symmetric spaces. Therefore, by

[1] It should be noted that the approach via flags of tripotents [**2**] misses an appropriate $K^{\mathbb{C}}$-action, so the fiber projection just becomes a real analytic instead of a complex analytic map, cf. also Remark 3.19.

duality of symmetric spaces and the classification of non-compact hermitian symmetric spaces, each component of $\mathbb{P}$ is related to some phJTS. In Section 6.2, this relation is made explicit. The following theorem is an analogue of [**28**, §5.6b].

Global structure. Recall from Proposition 2.19 that for the Peirce equivalence of elements it suffices to demand the equality of the corresponding Peirce 1-spaces. Moreover, a necessary condition for u and $\tilde{u}$ to be Peirce equivalent is that their ranks coincide, so we can restrict the equivalence relation to the submanifolds Z_j of rank-j elements in Z.

Theorem 6.1. *Let Z be a phJTS, Z_j the manifold of rank-j elements of Z and let $R_{\mathbb{P}_j}$ be the Peirce equivalence relation on Z_j*

$$(u,\tilde{u}) \in R_{\mathbb{P}_j} \quad \textit{if and only if} \quad Z^u_\nu = Z^{\tilde{u}}_\nu \ \textit{for}\ \nu = 1, 1/2, 0.$$

Then

(a) *$R_{\mathbb{P}_j}$ is a regular equivalence relation. The quotient manifold $\mathbb{P}_j = Z_j/R_{\mathbb{P}_j}$ is called* Peirce Grassmannian of type j. *Its tangent space at $[u] \in \mathbb{P}_j$ can be identified with*
$$T_{[u]}\mathbb{P}_j \cong Z^u_{1/2}\,.$$
The canonical projection $\pi : Z_j \to \mathbb{P}_j$ is a complex analytic submersion.

(b) *The naturally given $K^{\mathbb{C}}$-action on $\mathbb{P}_j$ turns the canonical projection π into a $K^{\mathbb{C}}$-equivariant fibration. The fiber through $u \in Z_j$ is $(Z^u_1)^\times$, the set of invertible elements in Z^u_1.*

(c) *The $K^{\mathbb{C}}$-action on $\mathbb{P}_j$ and even its restriction to $K \subset K^{\mathbb{C}}$ is transitive on each connected component of $\mathbb{P}_j$ and imposes on $\mathbb{P}_j$ the structure of a (not connected) hermitian symmetric space of compact type. The symmetry around $\pi(u)$ is induced by the* Peirce reflection
$$s_u = B_{u,\,2u^\dagger} = \exp(2\pi i\, u \,\Box\, u^\dagger) \quad \left(= (-1)^{2\nu}\ \mathrm{Id}\ \ \textit{on}\ Z^u_\nu \right).$$
If Z is simple, then $\mathbb{P}_j$ is connected.

Proof. For (a) we use Godement's Theorem and therefore show that $R_{\mathbb{P}_j}$ is a submanifold of $Z_j \times Z_j$, and the projection pr_1 of $R_{\mathbb{P}_j}$ onto the first component is a submersion. The first statement follows from Theorem 3.18, since $R_{\mathbb{P}_j}$ equals the pre-Peirce flag $Z_{(j,j)}$. Hence, $R_{\mathbb{P}_j}$ is a $K^{\mathbb{C}}$-invariant submanifold of $Z_j \times Z_j$ of dimension $\dim R_{\mathbb{P}_j} = 2\dim Z^u_1 + \dim Z^u_{1/2}$ for $u \in Z_j$. For the second statement, fix $(u,\tilde{u}) \in R_{\mathbb{P}_j}$ and consider $h_t = \exp(t\, v \,\Box\, u^\dagger) \in K^{\mathbb{C}}$ with $v \in Z^u_1 \oplus Z^u_{1/2}$. By $K^{\mathbb{C}}$-invariance, the tuple $(h_t u, h_t \tilde{u})$ defines a curve in $R_{\mathbb{P}_j}$. Its derivative at $(u,\tilde{u})$ is given by

$$\tfrac{d}{dt}\,\mathrm{pr}_1(h_t u, h_t\tilde{u})\big|_{t=0} = \tfrac{d}{dt}\exp(t\, v \,\Box\, u^\dagger)u\big|_{t=0} = \left\{v,\, u^\dagger,\, u\right\} = v_1 \oplus \tfrac{1}{2}\, v_{1/2}\,.$$

Therefore, pr_1 is a submersion of $R_{\mathbb{P}_j}$ onto Z_j, and the connected component of $\mathbb{P}_j = Z_j/R_{\mathbb{P}_j}$ containing $[u]$ has dimension $\dim\mathbb{P}_j = 2\dim Z_j - \dim R_{\mathbb{P}_j} = \dim Z^u_{1/2}$.
To prove (b), it suffices to notice that by the $K^{\mathbb{C}}$-invariance of the equivalence relation, $h \star [u] := [hu]$ is a well-defined $K^{\mathbb{C}}$-action on $\mathbb{P}_j$, which commutes with the canonical projection π. The fiber through $u \in Z_j$ equals the equivalence class of u, and due to Proposition 2.19, this is the set of invertible elements in Z^u_1.
For (c), we recall that any element $u \in Z_j$ is Peirce equivalent to some tripotent $e \in S_j$ (Corollary 2.21), and that K acts transitively each connected component of S_j (Theorem 3.13). Since K is compact, this also implies the compactness of $\mathbb{P}_j$. From (a) and (b) it follows that the canonical projection $\pi : Z_j \to \mathbb{P}_j$ induces a vector space isomorphism between $Z^u_{1/2}$ and the tangent space $T_{\pi(u)}\mathbb{P}_j$. By transferring the scalar product on $Z^u_{1/2}$ (i.e. the restriction of the scalar product on Z to $Z^u_{1/2}$)

to the tangent space, we obtain a K-invariant hermitian metric on each connected component of $\mathbb{P}_j$. Since the Peirce projection s_u is an element of $K^{\mathbb{C}}$, it defines a diffeomorphism on $\mathbb{P}_j$. The formula $s_u(z_\nu) = (-1)^{2\nu}$ for $z_\nu \in Z^u_\nu$ shows that s_u is metric preserving and has $\pi(u)$ as isolated fixed point. Therefore $\mathbb{P}_j$ is a hermitian symmetric space of compact type. If Z is simple, then K acts transitively on S_j and hence transitively on $\mathbb{P}_j$. So the connectedness of K transfers to $\mathbb{P}_j$. $\square$

REMARK 6.2. The manifold structure on the Peirce Grassmannian $\mathbb{P}$ could also be derived from the transitive $K^{\mathbb{C}}$-action on the connected components of $\mathbb{P}$. Instead, taking the Godement approach, enables us due to Theorem 3.8 and Proposition 3.11 to define line bundles on $\mathbb{P}$ via cocycles on Z_j, see Section 6.3.

Local structure. Next we determine an atlas on $\mathbb{P}_j$. We claim that certain restrictions of the chart maps of Z_j are appropriate for this. Recall from Lemma 3.20 that for $u \in Z$ with $\mathrm{rk}(u) = j$, the set

$$\mathcal{I}^u_j = \{z \in Z_j \,|\, Q_u Q_{u^\dagger} z \text{ invertible in } Z^u_1\}$$

is an open and dense subset of the connected component of Z_j containing u. Furthermore, for $y \in Z^u_{1/2}$ let $\tau_y = \exp(2\, y \,\square\, u^\dagger)$ be a Frobenius transformation with respect to u.

PROPOSITION 6.3. *For $0 \le j \le r$ let $\mathbb{P}_j$ be the Peirce manifold of type j and $\pi : Z_j \to \mathbb{P}_j$ the canonical projection. Then for $u \in Z_j$ the map*

$$\varphi_u : Z^u_{1/2} \to \mathbb{P}_j,\ y \mapsto [\tau_y u]$$

is a diffeomorphism of $Z^u_{1/2}$ onto the open and dense subset $\mathbb{P}^{(u)}_j := \pi(\mathcal{I}^u_j)$ of the connected component of $\mathbb{P}_j$ containing $[u]$. Moreover, for $u, \tilde{u} \in Z_j$ in the same connected component, the transition map $\varphi^{\tilde{u}}_u = \varphi^{-1}_u \circ \varphi_{\tilde{u}}$ is a birational map from $Z^{\tilde{u}}_{1/2}$ onto $Z^u_{1/2}$, and $\mathbb{P}_j$ is a smooth algebraic variety.

PROOF. As described in Remark 3.7, it suffices to show that the image of $\tilde{\varphi}_u : Z^u_{1/2} \to Z_j$, defined by $\tilde{\varphi}_u(y) = \tau_y u$, is a submanifold, which is minimally transversal to the equivalence classes corresponding to the elements of $\mathbb{P}_j$. We note that $\tilde{\varphi}_u$ is just a restriction of the chart map ϕ_u of Z_j defined in (3.23) to the submanifold $\{u\} \times Z^u_{1/2}$. Therefore the image of $\tilde{\varphi}_u$ is a submanifold of Z_j. For fixed $y \in Z^u_{1/2}$, set $z = \tau_y u$, then we have to show that

$$T_z Z_j = D_y \tilde{\varphi}_u(Z^u_{1/2}) \oplus T_z[z]\ .$$

Since $[z]$ is a $(\dim Z^u_1)$-dimensional submanifold, it suffices to show that $D_y\tilde{\varphi}_u(Z^u_{1/2})$ and $T_z[z]$ intersect trivially. Due to (3.25) we have

$$D_y \tilde{\varphi}_u(\dot{y}) = \dot{y} + 2\,\left\{\dot{y},\, u^\dagger,\, y\right\}\ ,$$

and by Theorem 6.1, it is

$$T_z[z] = Z^z_1 = \tau_y Z^u_1 = \left\{\dot{u} + 2\,\{u^\dagger,\, y,\, \dot{u}\} + Q_y Q_{u^\dagger}\dot{u} \,\middle|\, \dot{u} \in Z^u_1\right\}\ .$$

Now let z be in the intersection $D_y\tilde{\varphi}_u(Z^u_{1/2}) \cap T_z[z]$, i.e.

$$z = \dot{y} + 2\,\left\{\dot{y},\, u^\dagger,\, y\right\} = \dot{u} + 2\,\{u^\dagger,\, y,\, \dot{u}\} + Q_y Q_{u^\dagger}\dot{u}$$

for some $\dot{y} \in Z^u_{1/2}$ and $\dot{u} \in Z^u_1$. Comparing the Z^u_1-components yields $\dot{u} = 0$ and hence $z = 0$. Since $\tilde{\varphi}$ is one-to-one, this shows that $\varphi_u = \pi \circ \tilde{\varphi}_u$ is a diffeomorphism onto its image. To prove that this image equals $\pi(\mathcal{I}^u_j)$, we have to show that for all $\tilde{u} \in \mathcal{I}^u_j$ there exists an element $y \in Z^u_{1/2}$, such that $\tilde{u}$ is Peirce equivalent to $\tau_y u$, or equivalently, $(\tilde{u}, \tau_y u) \in R_{\mathbb{P}_j}$. Using the identification of $R_{\mathbb{P}_j}$ and the pre-Peirce flag $Z_{(j,j)}$, this is satisfied due to Lemma 3.21 by choosing $y = 2\,\left\{\tilde{u}_{1/2},\, \tilde{u}^\dagger_1,\, u\right\}$, where

$\tilde{u} = \tilde{u}_1 + \tilde{u}_{1/2} + \tilde{u}_0$ denotes the Peirce decomposition of $\tilde{u}$ with respect to u. The same argument finally provides a proof for the assertion on the transition function $\varphi_u^{\tilde{u}}$ for arbitrary $u, \tilde{u} \in Z_j$. Since the images of φ_u and $\varphi_{\tilde{u}}$ are open and dense in the same connected component of $\mathbb{P}_j$, the transition function is defined on the open and dense subset $\tilde{\varphi}_{\tilde{u}}^{-1}(\mathcal{I}_j^{\tilde{u}} \cap \mathcal{I}_j^u)$ of $Z_{1/2}^{\tilde{u}}$. Due to Lemma 3.21 we have $\varphi_{\tilde{u}}(\tilde{y}) = \varphi_u(y)$ if and only if

$$y = 2\left\{\left(\tilde{\varphi}_{\tilde{u}}(\tilde{y})\right)_{1/2}, \left(\tilde{\varphi}_{\tilde{u}}(\tilde{y})\right)_1^{\dagger}, u\right\} \quad \text{with} \quad \tilde{\varphi}_{\tilde{u}}(\tilde{y}) = \tilde{u} + \tilde{y} + Q_{\tilde{y}}\tilde{u}^{\dagger} . \tag{6.1}$$

Here $(z)_{1/2}$ and $(z)_1$ denote the Z_1^u- and $Z_{1/2}^u$-components of some element $z \in Z$. Since $\left(\tilde{\varphi}_{\tilde{u}}(\tilde{y})\right)_1$ is assumed to be invertible in Z_1^u, we have

$$\left(\tilde{\varphi}_{\tilde{u}}(\tilde{y})\right)_1^{\dagger} = Q_{u^{\dagger}}\left(\tilde{\varphi}_{\tilde{u}}(\tilde{y})\right)_1^{-1} , \tag{6.2}$$

where the inverse is calculated in the Jordan algebra Z_1^u. Since $\varphi_{\tilde{u}}(\tilde{y})$ is polynomial in y, this term is a rational map in $\overline{y}$ (due to the quadratic operator $Q_{u^{\dagger}}$ it is antiholomorphic in y). Since (6.2) only occurs inside the antiholomorphic component of the Jordan triple product in (6.1), we conclude that the transition map $\varphi_u^{\tilde{u}}(\tilde{y}) = y$ is birational. The domain of $\varphi_u^{\tilde{u}}$ is precisely the set of $\tilde{y} \in Z_1^{\tilde{u}}$ such that $\left(\tilde{\varphi}_{\tilde{u}}(\tilde{y})\right)_1$ is invertible in Z_1^u. Using Jordan algebra determinants, this is

$$\operatorname{dom}\varphi_u^{\tilde{u}} = \left\{\tilde{y} \in Z_{1/2}^{\tilde{u}} \,\middle|\, \Delta^u\left(\tilde{u} + \tilde{y} + Q_{\tilde{y}}\tilde{u}^{\dagger}\right) \neq 0\right\} .$$

This completes the proof of Proposition 6.3. □

REMARK 6.4. One could extend the investigation of the charts on $\mathbb{P}$ given in the proof of Proposition 6.3 and show that (1) finitely many charts suffice to cover $\mathbb{P}$, and (2) for any finite number of points in $\mathbb{P}$ there exist a chart containing them. By definition this would prove that $\mathbb{P}$ has the structure of an algebraic variety. We omit this reasoning, since in the next section we show in a quite explicit way that the connected component of $\mathbb{P}$ containing an element $[u] \in \mathbb{P}$ can be identified with the Grassmannian of the phJTS $Z_{1/2}^u$. Due to Theorem 4.4 this implies that $\mathbb{P}$ is even a projective variety.

6.2. Realizations of the Peirce Grassmannian

In this section we give different realizations of the Peirce Grassmannian $\mathbb{P}$ originating from various descriptions of the Peirce equivalence. Proposition 2.19 and Theorem 4.12 provide the following:

> *Two elements $u, v \in Z$ are Peirce equivalent, $u \equiv v$, if one of the following equivalent conditions is satisfied:*
>
> *(i)* $u \,\square\, u^{\dagger} = v \,\square\, v^{\dagger}$, *(ii)* $Z_1^u = Z_1^v$, *(iii)* $\left[u : u^{\dagger}\right] = \left[v : v^{\dagger}\right]$.

This motivates the following definitions:

$$\mathbb{D} := \left\{i\, u \,\square\, u^{\dagger} \,\middle|\, u \in Z\right\} \subset \operatorname{Der}(Z) ,$$

$$\mathbb{U} := \{U \subset Z \,|\, U \text{ Peirce 1-space}\} \subset \operatorname{Gr}(Z) ,$$

$$\mathbb{K} := \left\{[u : u^{\dagger}] \,\middle|\, u \in Z\right\} \subset \mathbb{G}(Z) .$$

Here $\operatorname{Gr}(Z)$ denotes the ordinary Grassmannian manifold consisting of all subspaces of Z; this motivates our terminology for $\mathbb{P}$. Each of these spaces carries its own topological structure induced by the ambient spaces. They obviously decompose into the following components:

$$\mathbb{D}_j = \left\{i\, u \,\square\, u^{\dagger} \,\middle|\, \operatorname{rk} u = j\right\} , \quad \mathbb{U}_j = \{U \subset \mathbb{P} \,|\, \operatorname{rk} U = j\} , \quad \mathbb{K}_j = \left\{[u : u^{\dagger}] \,\middle|\, \operatorname{rk} u = j\right\}$$

for $j = 0, \dots, r$. The characterization of Peirce equivalence given above yields the following component preserving bijections:

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{D} \arrow[ddl, leftrightarrow, bend right] \arrow[d, leftrightarrow] \arrow[ddr, leftrightarrow, bend left] & \\
 & \mathbb{P} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\mathbb{U} \arrow[rr, leftrightarrow, bend right] & & \mathbb{K}
\end{tikzcd}
\qquad \textit{with} \qquad
\begin{tikzcd}[column sep=small]
 & i\,u \,\square\, u^\dagger \arrow[ddl, leftrightarrow, bend right] \arrow[d, leftrightarrow] \arrow[ddr, leftrightarrow, bend left] & \\
 & {[u]} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
Z_1^u \arrow[rr, leftrightarrow, bend right] & & {[u : u^\dagger]}
\end{tikzcd}
\tag{6.3}
$$

In the following we investigate the analytic and algebraic structure of these realizations and their relationship. On an abstract level it is easy to see that $\mathbb{D}$, $\mathbb{U}$ and $\mathbb{K}$ are equivalent to $\mathbb{P}$ as real analytic manifolds: Consider the natural K-actions on $\mathrm{Der}(Z)$, $\mathrm{Gr}(Z)$ and $\mathbb{G}(Z)$ given by

$$k \star \delta = k\delta k^{-1}\ , \quad k \star U = kU\ , \quad k \star [x : y] = [kx : ky]\ .$$

Then $\mathbb{D}$, $\mathbb{U}$ and $\mathbb{K}$ are K-invariant subsets, and the bijections (6.3) are all K-equivariant. Since K acts transitively on each of the connected components, we have $\mathbb{D} \cong \mathbb{U} \cong \mathbb{K} \cong \mathbb{P}$ as real analytic manifolds, and the bijections are real analytic maps. By Proposition 3.3, the topology of these manifold structures coincide with the induced topologies of the ambient spaces.

For $\mathbb{D}$ this is the most we can expect, since $\mathbb{D}$ is a *compact* submanifold of $\mathrm{Der}(Z)$ (it is $\|i\,u \,\square\, u^\dagger\| = 1$ for all $u \neq 0$), and so there is no induced complex analytic structure on $\mathbb{D}$. Thus in the following we concentrate on the relationship between $\mathbb{U}$, $\mathbb{K}$ and $\mathbb{P}$. We can extend the argument above to obtain *complex* analytic equivalence of $\mathbb{U}$, $\mathbb{K}$ and $\mathbb{P}$:

Theorem 6.5. *Let $h \star U = hU$ and $h[x : y] = [hx : h^{-*}y]$ be the natural $K^{\mathbb{C}}$-actions on $\mathrm{Gr}(Z)$ and $\mathbb{G}(Z)$, respectively. Then $\mathbb{U}$ and $\mathbb{K}$ are $K^{\mathbb{C}}$-invariant subsets, and the bijections*

$$\mathbb{U} \leftrightarrow \mathbb{P} \leftrightarrow \mathbb{K} \quad \textit{given by} \quad Z_1^u \leftrightarrow [u] \leftrightarrow \left[u : u^\dagger\right] \tag{6.4}$$

are $K^{\mathbb{C}}$-equivariant. Therefore $\mathbb{U}$, $\mathbb{K}$ and $\mathbb{P}$ are biholomorphically equivalent to each other, and the bijections are biholomorphic maps.

Proof. Due to Lemma 2.32 we have $hZ_1^u = Z_1^{hu}$ and $(h^*(hu)^\dagger)_1 = u^\dagger$, where $(h^*(hu)^\dagger)_1$ denotes the Z_1^u-component of $h^*(hu)^\dagger$. The first equation shows the $K^{\mathbb{C}}$-invariance of $\mathbb{U}$, as well as the $K^{\mathbb{C}}$-equivariance of the bijection between $\mathbb{U}$ and $\mathbb{P}$. We use the second equation to prove the invariance of $\mathbb{K}$. For this we assert $[hu : h^{-*}u^\dagger] = [hu : (hu)^\dagger]$. This is equivalent to $[u : u^\dagger] = [u : h^*(hu)^\dagger]$, i.e. we have to show, that $(u, u^\dagger - h^*(hu)^\dagger)$ is quasi-invertible and

$$u^{u^\dagger - h^*(hu)^\dagger} = u\ .$$

According to Lemma 2.25 the pair $(u, u^\dagger - h^*(hu)^\dagger)$ is quasi-invertible if and only if $(u, u^\dagger - (h^*(hu)^\dagger)_1) = (u, u^\dagger - u^\dagger) = (u, 0)$ is quasi-invertible, and $u^{u^\dagger - h^*(hu)^\dagger} = u^0 = u$. This proves our assertion, and in addition it shows that the bijection between $\mathbb{K}$ and $\mathbb{P}$ is $K^{\mathbb{C}}$-equivariant. Since the natural action of $K^{\mathbb{C}}$ on $\mathrm{Gr}(Z)$ and $\mathbb{G}(Z)$ is holomorphic and transitive on each connected component, it follows that the bijections are biholomorphic isomorphisms of $\mathbb{U}$, $\mathbb{K}$ and $\mathbb{P}$. $\square$

Remark 6.6. We note that the correspondence between $\mathbb{P}$ and $\mathbb{K}$ was already known to O. Loos, see [**30**, §2.8], but discussed merely on the level of a bijection rather than on the level of analytic manifolds.

Local structure. So far we used the $K^{\mathbb{C}}$-action on $\mathbb{G}(Z)$ to show, on an abstract level, that $\mathbb{U}$ and $\mathbb{K}$ are analytically equivalent to $\mathbb{P}$. Now we investigate this equivalence on a local level using charts. Recall from Proposition 6.3 that a chart of $\mathbb{P}$ is given by

$$\varphi_u : Z^u_{1/2} \to \mathbb{P},\ y \mapsto [\tau_y u] .$$

Here τ_y denotes the Frobenius transformation $\tau_y = \exp(2\, y \,\square\, u^\dagger)$. The image of φ_u is an open and dense subset of the connected component of $\mathbb{P}$ containing $[u]$. Now, since the Frobenius transformation is an element of $K^{\mathbb{C}}$, and since the bijections between $\mathbb{U}$, $\mathbb{K}$ and $\mathbb{P}$ are $K^{\mathbb{C}}$-equivariant biholomorphic maps, this also shows that the following maps are charts on $\mathbb{U}$ and $\mathbb{K}$ with open and dense images in the corresponding connected components:

$$\varphi^{\mathbb{U}}_u : Z^u_{1/2} \to \mathbb{U},\ y \mapsto Z_1^{\tau_y u} , \quad \varphi^{\mathbb{K}}_u : Z^u_{1/2} \to \mathbb{K},\ y \mapsto \left[\tau_y u : (\tau_y u)^\dagger\right] .$$

Conversely, one could start with these atlases and then prove that the bijections above are biholomorphic. Indeed, in [**18**] the charts $\varphi^{\mathbb{U}}_u$ are used to prove that $\mathbb{U}$ is a complex analytic submanifold of $\mathrm{Gr}(Z)$. Next we give a different description of the charts on $\mathbb{K}$ by relating the Frobenius transformation to the partial inverse map introduced in Section 4.3.

Lemma 6.7. *For $u \in Z$ and $y \in Z^u_{1/2}$ let τ_y be the Frobenius transformation $\tau_y = \exp(2\, y \,\square\, u^\dagger)$, and let $j_{u^\dagger}$ be the partial inverse map with respect to $u^\dagger$. Then*

(a) $\tau_y = B_{y,-u^\dagger}$ *and* $\tau_y u = u + y + Q_y u^\dagger = Q_{y+u} u^\dagger$,

(b) $j_{u^\dagger}(y) = \left[\tau_y u : (\tau_y u)^\dagger\right] = \tau_y \left[u : u^\dagger\right] = \left[\tau_y u : u^\dagger\right]$.

Proof. Part (a) follows immediately from Lemma 3.14. To prove the first equation of (b), let $\lambda \neq 0$ be a complex number, then by Proposition 4.11, it is

$$j_{u^\dagger}(\lambda u + y) = -\tfrac{1}{\lambda}\, u - 2\, \tfrac{1}{\lambda} \left\{y,\, u^\dagger,\, u\right\} - \tfrac{1}{\lambda}\, Q_y Q_{u^\dagger} u = -\tfrac{1}{\lambda}\, Q_{y+u} u^\dagger = -\tfrac{1}{\lambda}\, \tau_y u .$$

For an arbitrary element $w \in Z$, Lemma 4.14 implies the relation $[\frac{1}{\lambda} w : 0] = [w : (1-\lambda^*) w^\dagger]$, and using continuity of the canonical projection $Z \times \overline{Z} \to \mathbb{G}(Z)$, we have $\lim_{\lambda\to 0}[\frac{1}{\lambda} w : 0] = [w : w^\dagger]$. Setting $w = \tau_y u$, this implies the first equation of (b). The second one follows from Theorem 6.5. For the last equation, we use $j_{u^\dagger} = \tilde{t}_{u^\dagger} \circ t_u \circ \tilde{t}_{u^\dagger}$, and obtain for $[j_u(y) : 0] = [w : u^\dagger]$ the relation

$$w = (j_{u^\dagger}(y))^{-u^\dagger} = \tilde{t}_{-u^\dagger} \circ \tilde{t}_{u^\dagger} \circ t_u \circ \tilde{t}_{u^\dagger}(y) = u + y^{u^\dagger} = u + y + Q_y u^\dagger = Q_{y+u} u^\dagger ,$$

where we used the nilpotence of the pair $(y, u^\dagger)$ according to Lemma 2.22. □

In combination with Proposition 6.3 and Theorem 6.5 this lemma yields the following result.

Proposition 6.8. *Let $\mathbb{K}_j \subset \mathbb{G}(Z)$ be the subset $\mathbb{K}_j = \left\{[u : u^\dagger] \,\middle|\, u \in Z_j\right\}$. Then the partial inverse map $j_{u^\dagger}$ with respect to an element $u \in Z_j$ maps $Z^u_{1/2}$ biholomorphically onto an open subset of $\mathbb{K}_j$. The image of j_u is dense in the connected component of $\mathbb{K}_j$ containing $[u : u^\dagger]$. If Z is simple, then $\mathbb{K}_j$ is connected, so $j_{u^\dagger}$ defines a chart of an open and dense subset of $\mathbb{K}_j$.*

Remark 6.9. Lemma 6.7 also provides a description of $\mathbb{K} \subset \mathbb{G}$ in affine coordinates of $\mathbb{G}$: Identifying Z with $\mathbb{G}^{(u^\dagger)} = \{[z : u^\dagger] \,|\, z \in Z\}$ via the coordinate chart $z \mapsto [z : u^\dagger]$, we obtain

$$j_{u^\dagger}(Z^u_{1/2}) = \left\{Q_{y+u} u^\dagger \in Z \,\middle|\, y \in Z^u_{1/2}\right\} . \tag{6.5}$$

In particular, if Z is simple, then

$$\mathbb{K}_j^{(u^\dagger)} := \mathbb{K}_j \cap \mathbb{G}^{(u^\dagger)} = \left\{\left[Q_{y+u} u^\dagger : u^\dagger\right] \,\middle|\, y \in Z^u_{1/2}\right\} . \tag{6.6}$$

Indeed: the inclusion '$\supset$' follows from (6.5). For the converse inclusion, choose any $[z : u^\dagger]$ in $\mathbb{K}_j^{(u^\dagger)}$. Since by Proposition 6.8, the set $\{[Q_{y+u}u^\dagger : u^\dagger] \,|\, y \in Z^u_{1/2}\}$ is open and dense in $\mathbb{K}_j$, there exists a sequence y_n such that $[Q_{y_n+u}u^\dagger : u^\dagger]$ converges to $[z : u^\dagger]$, hence $Q_{y_n+u}u^\dagger$ converges to z. Therefore, the sequence y_n itself converges, $y_n \to y$, and we obtain $[z : u^\dagger] = [Q_{y+u}u^\dagger : u^\dagger]$. This completes the prove of (6.6).

Regarded as an affine variety in $Z \cong \mathbb{G}^{u^\dagger}$, $\mathbb{K}_j^{(u^\dagger)}$ given by

$$\mathbb{K}_j^{(u^\dagger)} = \mathcal{V}(z_1 - u^\dagger,\ z_0 + Q_{z_{1/2}}u)\ ,$$

where $z = z_1 + z_{1/2} + z_0$ denotes the coordinates of $Z = Z^u_1 \oplus Z^u_{1/2} \oplus Z^u_0$.

Global structure. The partial inverse mappings used to describe the coordinate charts of $\mathbb{K}$ are extendable to the compactification of $Z^u_{1/2}$ in $\mathbb{G}(Z)$. This provides an explicit isomorphism of $\mathbb{K}$ with the Grassmannian of the phJTS $Z^u_{1/2}$, whereby the structure of $\mathbb{P}_j$ as a compact hermitian symmetric space is recovered within $\mathbb{G}(Z)$. More precisely, we obtain the following result:

Theorem 6.10. *Let Z be a phJTS with Grassmannian $\mathbb{G}(Z)$, and for $u \in Z$ let $\mathbb{G}(Z^u_{1/2})$ denote the Grassmannian of the phJTS $Z^u_{1/2}$.*

(a) *The topological closure of $Z^u_{1/2}$ in $\mathbb{G}(Z)$ is canonically isomorphic to $\mathbb{G}(Z^u_{1/2})$.*

(b) *The partial inverse mapping $j_{u^\dagger}$ induces an isomorphism of $\mathbb{G}(Z^u_{1/2})$ and the connected component of $\mathbb{K}$ containing $[u : u^\dagger]$ as smooth algebraic varieties. In case of a tripotent u, this is also an isomorphism of compact hermitian symmetric spaces.*

If Z is simple, the components $\mathbb{K}_j$ for $0 \le j \le r$ are the connected components.

Proof. For brevity we set $W := Z^u_{1/2}$ and let $\mathbb{K}'$ denote the connected component of $\mathbb{K}$ containing $[u : u^\dagger]$. We can identify $\mathbb{G}(W)$ with the topological closure of W in $\mathbb{G}(W)$, since Corollary 2.29 ensures that, the imbedding of W into Z extends to an imbedding of $\mathbb{G}(W)$ into $\mathbb{G}(Z)$. This proves (a). For (b), we note on the one hand that $\mathbb{K}'$ is a closed submanifold of $\mathbb{G}(Z)$, since K acts transitivity on $\mathbb{K}'$. On the other hand, due to Proposition 6.8, $j_{u^\dagger}$ maps W onto an open and dense subset of $\mathbb{K}'$. Therefore, since $j_{u^\dagger}$ is a biholomophic mapping on $\mathbb{G}(Z)$, the closure of W maps onto the closure of $j_{u^\dagger}(W)$, i.e. onto $\mathbb{K}'$. Since j_u is an (algebraic) automorphism of $\mathbb{G}(Z)$, the algebraic and complex analytic structures of $\mathbb{G}(W)$ and $\mathbb{K}_j$ coincide. This proves the first part of (b). Finally, if u is a tripotent, the partial inverse mapping j_u is an element of G^c, so even the Riemannian structure is preserved. □

In all subsequent chapters we identify $\mathbb{P}$ with $\mathbb{K}$ via Theorem 6.5. By this means, Theorem 6.10 relates the Peirce Grassmannian $\mathbb{P}_j$ of Z to the Grassmannian $\mathbb{G}(Z^u_{1/2})$ of some triple system $Z^u_{1/2} \subset Z$ with $\mathrm{rk}(u) = j$.

Corollary 6.11. *If Z is a simple phJTS, then for any element $u \in Z_j$ the Peirce Grassmannian $\mathbb{P}_j$ of type j is isomorphic (as smooth algebraic variety) to the Grassmannian $\mathbb{G}(Z^u_{1/2})$. In particular, $\mathbb{P}_j$ is a projective variety.*

6.3. Line bundles

In this section, fix $0 \le j \le \mathrm{rk}\, Z$, and let $\mathbb{P}_j$ be the Peirce Grassmannian of type j. We regard $\mathbb{P}_j$ as the quotient manifold $\mathbb{P}_j = Z_j / R_{\mathbb{P}_j}$ with

$$R_{\mathbb{P}_j} = \left\{ (u, \tilde{u}) \in Z_j \times Z_j \,\middle|\, Z^u_\nu = Z^{\tilde{u}}_\nu \text{ for } \nu = 1, 1/2, 0 \right\}\ ,$$

and use Theorem 3.8 and Proposition 3.11 to define (homogeneous) line bundles on $\mathbb{P}_j$ via cocycles.

We recall from Section 2.9 some formula for induced Jordan algebra denominators. Let δ be a denominator of the quasi-inverse in Z. Then, for any $u \in Z$, the Peirce 1-space Z_1^u is a unital Jordan algebra with product $x \circ_u y = \{x, u, y\}$ and unit element $u^\dagger$. Furthermore,

$$\delta^u(z) = \delta(u^\dagger - z, u) \quad \text{for} \quad z \in Z \tag{6.7}$$

is a denominator of the inverse in Z_1^u, normalized by $\delta^u(u^\dagger) = 1$. If $\tilde{u} \in Z$ is Peirce equivalent to u, and if h is a structure automorphism of Z, then

$$\delta^u(z) = \delta^{\tilde{u}}\big(u^\dagger\big)^{-1} \delta^{\tilde{u}}(z) \quad \text{and} \quad \delta^u(hz) = \delta^{h^* u}(z) \tag{6.8}$$

for all $z \in Z$. If in addition h satisfies $h_u^* u \in Z_1^u$, then

$$\delta^u(hz) = \delta^u\big(hu^\dagger\big)\,\delta^u(z) \tag{6.9}$$

for all $z \in Z$. Using these formulas, we are able to define a cocycle on $R_{\mathbb{P}_j}$, and thus obtain a line bundle on $\mathbb{P}_j$.

Proposition 6.12. *Let δ be a denominator of the quasi-inverse of Z. Then,*

$$\phi : R_{\mathbb{P}_j} \to \mathbb{C}^\times,\ (u, \tilde{u}) \mapsto \phi_{\tilde{u}}^u = \overline{\delta^{\tilde{u}}(u^\dagger)}\ . \tag{6.10}$$

is a $K^{\mathbb{C}}$-invariant holomorphic cocycle. Let $\mathcal{L}_\delta$ denote the corresponding $K^{\mathbb{C}}$-homogeneous line bundle on $\mathbb{P}_j$. For fixed $[u] \in \mathbb{P}_j$ and $U = Z_1^u$, let $K_U^{\mathbb{C}}$ denote the stabilizer of U with respect to the $K^{\mathbb{C}}$-action on Z. Then

$$\chi : K_U^{\mathbb{C}} \to \mathbb{C}^\times,\ h \mapsto \overline{\delta^u((hu)^\dagger)} = \overline{\delta^u(h^* u^\dagger)}^{-1} \tag{6.11}$$

is a character of $K_U^{\mathbb{C}}$, and the induced homogeneous line bundle $K^{\mathbb{C}} \times_\chi \mathbb{C}$ on $\mathbb{P}_j = K^{\mathbb{C}}/K_U^{\mathbb{C}}$ is isomorphic to $\mathcal{L}_\delta$.

Proof. The cocycle condition follows by applying the first equation of (6.8) twice and using the normalization $\delta^{\hat{u}}\big(\hat{u}^\dagger\big) = 1$:

$$\phi_{\tilde{u}}^{\hat{u}} \cdot \phi_{\hat{u}}^u = \overline{\delta^{\tilde{u}}(\hat{u}^\dagger)\delta^{\hat{u}}(u^\dagger)} = \overline{\delta^{\hat{u}}(\tilde{u}^\dagger)^{-1}}\,\overline{\delta^{\hat{u}}(u^\dagger)} = \overline{\delta^{\tilde{u}}(u^\dagger)} = \phi_{\tilde{u}^\dagger}^u\ .$$

For the $K^{\mathbb{C}}$-invariance of ϕ, we have to show that $\delta^{h\tilde{u}}\big((hu)^\dagger\big) = \delta^{\tilde{u}}\big(u^\dagger\big)$ holds for all Peirce equivalent $u, \tilde{u} \in Z_j$ and $h \in K^{\mathbb{C}}$. Due to equation (6.9), we have $\delta^{h\tilde{u}}\big((hu)^\dagger\big) = \delta^{\tilde{u}}\big(h^*(hu)^\dagger\big)$, and since $\delta^{\tilde{u}}$ only depends on the Peirce 1-component of $h^*(hu)^\dagger$ with respect to $\tilde{u}$ or u (by Peirce equivalence), Lemma 2.32 yields

$$\delta^{h\tilde{u}}\big((hu)^\dagger\big) = \delta^{\tilde{u}}\big(h^*(hu)^\dagger\big) = \delta^{\tilde{u}}\big(u^\dagger\big)\ .$$

Next we prove that ϕ is a holomorphic map. By Proposition 2.35, we have $\delta^u\big(\tilde{u}^\dagger\big) = \delta(u^\dagger - \tilde{u}^\dagger, u)$. Since the pseudo-inverse map is real analytic on Z_j, we conclude that ϕ is a real analytic map, and it suffices to show that its derivative along $R_{\mathbb{P}_j}$ is complex linear. By $K^{\mathbb{C}}$-invariance, we just have to consider derivatives along the fibers $[u] \times [\tilde{u}] \subset R_{\mathbb{P}_j}$, and even more restrictively, it suffices to consider the derivatives along one of the components. Therefore, consider the holomorphic chart map $y \mapsto \exp(y \,\square\, u^\dagger)u$ of $[u]$ with $y \in Z_1^u$. Then, using the relation $x^\dagger = Q_{u^\dagger} x^{-1}$ in the unital Jordan algebra Z_1^u, we conclude that

$$y \mapsto (\exp(y \,\square\, u^\dagger)u)^\dagger = Q_u(\exp(y \,\square\, u^\dagger)u)^{-1}$$

is an anti-holomorphic map, and hence, ϕ is holomorphic along each fiber $[u] \times \{\tilde{u}\}$. The remaining statements of Proposition 6.12 are immediate consequences of Theorem 6.1 and Proposition 3.11. The identity $\delta^u\big((hu)^\dagger\big) = \delta^u\big(h^* u^\dagger\big)^{-1}$ is verified in the proof of Proposition 2.37. $\square$

REMARK 6.13. As noted in Remark 3.10, we obtain the k-th power of $\mathcal{L}_\delta$ by taking the k-th power of the cocycle of $\mathcal{L}_\delta$, i.e.

$$\phi_{\mathcal{L}_\delta^k}(u,\tilde{u}) = (\phi_{\mathcal{L}_\delta}(u,\tilde{u}))^k \quad \text{for all} \quad (u,\tilde{u}) \in R_{\mathbb{P}_j} \ .$$

If E is a complex vector space, then the cocycle of the vector bundle[2] $\mathcal{L}_\delta \otimes E$ is given by $\phi_{\mathcal{L}_\delta \otimes E}(u,\tilde{u}) = \phi_{\mathcal{L}_\delta}(u,\tilde{u}) \cdot \mathrm{Id}_E$.

Once again we prove that $\mathbb{P}_j$ is a projective variety:

THEOREM 6.14. *The line bundle $\mathcal{L}_\delta$ defined in Proposition 6.12 is very ample, and hence $\mathbb{P}_j$ is a projective variety.*

PROOF. This proof is based on an idea of [**28**, §7.10]. Let δ^u denote the induced Jordan algebra denominator of Z_1^u, and let ν^u be the corresponding numerator of the inverse map on Z_1^u, see Section 2.9 for more details. We note that the inverse x^{-1} depends on the choice of u, since for different u, the unit element $u^\dagger$ differs. However, by Lemma 2.16 we obtain for the pseudo-inverse the relation

$$x^\dagger = Q_u x^{-1} = \frac{Q_u\big(\nu^u(x)\big)}{\overline{\delta^u(x)}} =: \frac{\mathfrak{N}^u(x)}{\overline{\delta^u(x)}} \ . \tag{6.12}$$

Now the numerator $\mathfrak{N}^u$ is a $\overline{Z}_1^u$-valued polynomial of the same degree as ν^u. The pair $(\delta^u, \mathfrak{N}^u)$ can be regarded as a $\mathbb{C} \times \overline{Z}$ valued polynomial on Z. For Peirce equivalent $u, \tilde{u} \in Z_j$ the corresponding polynomials are related due to (6.8) and (6.12) by

$$\delta^u(x) = \delta^{\tilde{u}}\big(u^\dagger\big)^{-1} \delta^{\tilde{u}}(x) \quad \text{and} \quad \mathfrak{N}^u(x) = \overline{\delta^{\tilde{u}}(u^\dagger)^{-1}}\, \mathfrak{N}^{\tilde{u}}(x) \ . \tag{6.13}$$

Let N_δ be the maximum of the degrees of δ^u with varying $u \in Z_j$, which is bounded by the degree of δ, and let N_ν be correspondingly the maximum of the degrees of ν^u. Now let $\overline{E}$ be the finite dimensional vector space of all polynomial functions (φ, f) on Z with values in $\overline{\mathbb{C}} \times Z$ such that $\varphi : Z \to \overline{\mathbb{C}}$ is of degree at most N_δ and $f : Z \to Z$ is of degree at most N_ν, i.e.

$$\overline{E} = \big\{(\varphi, f) : Z \to \overline{\mathbb{C}} \times Z \,\big|\, (\varphi, f) \text{ polynomial, } \deg\varphi \le N_\delta, \ \deg f \le N_\nu\big\} \ .$$

Consider the map

$$\sigma : Z_j \to \overline{E}, \ u \mapsto \sigma_u(x) = \big(\overline{\delta^u(x)}, \ \overline{\mathfrak{N}^u(x)}\big) \ .$$

Due to Proposition 3.12, this map defines a section s in $\mathcal{L} \otimes \overline{E}$, since by (6.13) it is

$$\sigma_{\tilde{u}}(x) = \big(\overline{\delta^{\tilde{u}}(x)}, \ \overline{\mathfrak{N}^{\tilde{u}}(x)}\big) = \overline{\delta^{\tilde{u}}(u^\dagger)^{-1}} \cdot \big(\overline{\delta^{u^\dagger}(x)}, \ \overline{\mathfrak{N}^{u^\dagger}(x)}\big) = \overline{\delta^u(\tilde{u}^\dagger)} \cdot \sigma_u(x) \ .$$

We claim that s is a holomorphic section. By Proposition 3.12, it suffices to show that σ is holomorphic along a transversal covering of $\mathbb{P}_j$, i.e. by Proposition 6.3 along the images of the maps $y \mapsto \tau_{u,y} u$ for $y \in Z_{1/2}^u$, where $\tau_{u,y}$ is the Frobenius transformation given by $\tau_{u,y} = \exp(2\, y \,\square\, u^\dagger) = B_{y,-u^\dagger}$. Applying this to the first component of σ and using (6.9), we obtain the map

$$y \mapsto \overline{\delta^{\tau_{u,y} u}(x)} = \overline{\delta^u\big(B_{-u^\dagger, y} x\big)} \ ,$$

which is obviously holomorphic in y. Furthermore, since $\mathfrak{N}^u(x) = \overline{\delta^u(x)} \cdot x^\dagger$, this shows that also the second component of σ is holomorphic along a transversal covering of $\mathbb{P}_j$, and hence s is a holomorphic section on $\mathbb{P}_j$.
Since $\sigma_u(u^\dagger) = (1, u)$ for all $u \in Z_j$, the section s is non-vanishing and therefore defines a morphism $\varSigma : \mathbb{P}_j \to \mathrm{P}(\overline{E})$, where $\mathrm{P}(\overline{E})$ is the projective space on $\overline{E}$. We

[2] More precisely, if $\mathcal{E}$ denotes the trivial bundle $\mathbb{P}_J \times E$, we consider $\mathcal{L}_\delta \otimes \mathcal{E}$.

claim that Σ is one-to-one: Let $[\overline{\delta^u}, \overline{\mathfrak{N}^u}] = [\overline{\delta^{\tilde{u}}}, \overline{\mathfrak{N}^{\tilde{u}}}]$ for some $u, \tilde{u} \in Z_j$, i.e. there exists a $\lambda \in \mathbb{C}^\times$, such that

$$\delta^u(x) = \lambda \cdot \delta^{\tilde{u}}(x) \ , \quad \mathfrak{N}^u(x) = \overline{\lambda} \cdot \mathfrak{N}^{\tilde{u}}(x) \quad \text{for all} \quad x \in Z \ .$$

Setting $x = u^\dagger$, the first relation yields $\lambda = \delta^{\tilde{u}}\big(u^\dagger\big)^{-1}$, and we obtain by (6.12)

$$u = \frac{\overline{\mathfrak{N}^u(u^\dagger)}}{\overline{\delta^u(u^\dagger)}} = \frac{\overline{\mathfrak{N}^{\tilde{u}}(u^\dagger)}}{\overline{\delta^{\tilde{u}}(u^\dagger)}} = Q_{\tilde{u}}\big((u^\dagger)_1\big)^{-1} \ ,$$

where the index $(\ )_1$ denotes the $Z_1^{\tilde{u}}$-component and the inverse is taken in the unital Jordan algebra $Z_1^{\tilde{u}}$. Since $Q_{\tilde{u}}$ maps Z onto $Z_1^{\tilde{u}}$, we conclude that u is an element of $Z_1^{\tilde{u}}$, and since $\mathrm{rk}(u) = \mathrm{rk}(\tilde{u})$, this implies that u and $\tilde{u}$ are indeed Peirce equivalent. Therefore, Σ is a holomorphic injection. Finally we note, that by compactness of $\mathbb{P}_j$, Σ is also a proper map, and hence defines an imbedding of $\mathbb{P}_j$ into $\mathrm{P}(E)$. □

6.4. Peirce flag varieties

In this section we extend the notion of Peirce manifolds to Peirce flags. This is done by the passage from Z_j to the pre-Peirce flag manifolds Z_J introduced in Section 3.3, Theorem 3.18. Let $J = (j_1, \ldots, j_k)$ be a strictly increasing[3] sequence of integers, $0 \le j_1 < \ldots < j_k \le r$. Then the Peirce equivalence relation on Z_j has a natural generalization to Z_J: Two Peirce ordered tuples (u_i), $(\tilde{u}_i)$ of type J are Peirce equivalent, if for all i, u_i is Peirce equivalent to $\tilde{u}_i$. It is evident that this indeed defines an equivalence relation on Z_J. By abuse of language we also call it *Peirce equivalence relation.*

Theorem 6.15. *Let Z be a phJTS of rank r, $J = (j_1, \ldots, j_k)$ an increasing sequence of integers, $0 \le j_1 < \ldots < j_k \le r$, and Z_J the pre-Peirce flag manifold of type J. Let $R_{\mathbb{P}_J}$ be the Peirce equivalence relation*

$$\big((u_i), (\tilde{u}_i)\big) \in R_{\mathbb{P}_J} \quad \textit{if and only if} \quad Z_\nu^{u_i} = Z_\nu^{\tilde{u}_i} \ \textit{for all}\ \nu = 1, 1/2, 0 \ \textit{and}\ i = 1, \ldots, k.$$

Then

(a) *$R_{\mathbb{P}_J}$ is a regular equivalence relation. The quotient manifold $\mathbb{P}_J = Z_J / R_{\mathbb{P}_J}$ is called* Peirce flag manifold of type J. *The connected component containing $[u_i] \in \mathbb{P}_J$ is a complex manifold of dimension*

$$\dim \mathbb{P}_J = \sum_{i=1}^{k-1} \dim\big(Z_{1/2}^{u_i} \cap Z_1^{u_{i+1}}\big) + \dim Z_{1/2}^{u_k} \ .$$

The canonical projection $\pi : Z_J \to \mathbb{P}_J$ is a complex analytic submersion.

(b) *The naturally given $K^{\mathbb{C}}$-action on $\mathbb{P}_j$ turns the canonical projection π into a $K^{\mathbb{C}}$-equivariant fibration. The fiber through $(u_i) \in Z_J$ is*

$$\pi^{-1}\big([(u_i)]\big) = (Z_1^{u_1})^\times \times \ldots \times (Z_1^{u_k})^\times \ .$$

(c) *The $K^{\mathbb{C}}$-action on $\mathbb{P}_J$ and even its restriction to $K \subset K^{\mathbb{C}}$ is transitive, hence $\mathbb{P}_J$ is a compact homogeneous manifold.*

Proof. We essentially follow the proof of Theorem 6.1. For (a), we first identify $R_{\mathbb{P}_J}$ with the pre-Peirce flag $Z_{\tilde{J}}$ of type $\tilde{J} = (j_1, j_1, \ldots, j_k, j_k)$ by a simple permutation of the components. Therefore, $R_{\mathbb{P}_J}$ is a $K^{\mathbb{C}}$-invariant submanifold of $(Z_{j_1} \times \ldots \times Z_{j_k})^2$ with $\dim R_{\mathbb{P}_J} = \dim Z_{\tilde{J}}$. Since $R_{\mathbb{P}_J}$ is contained in $Z_J \times Z_J$, Theorem 3.1 implies that $R_{\mathbb{P}_J}$ is even a submanifold of $Z_J \times Z_J$. Next we show that

[3]In the context of Peirce flags it is no longer meaningful to allow some j_i to coincide.

the projection pr_1 of $R_{\mathbb{P}_J}$ onto the first component Z_J is a submersion. For fixed $\big((u_i),(\tilde{u}_i)\big) \in R_{\mathbb{P}_J}$ and

$$(v_1,\dots,v_k) \in \big(Z_1^{u_1} \oplus (Z_{1/2}^{u_1} \cap Z_1^{u_2})\big) \times \dots \times \big(Z_1^{u_{k-1}} \oplus (Z_{1/2}^{u_{k-1}} \cap Z_1^{u_k})\big) \times (Z_1^{u_k} \oplus Z_{1/2}^{u_k})$$

consider $h_i(t) \coloneqq \exp(tv_k \square u_k^\dagger) \circ \dots \circ \exp(tv_i \square u_i^\dagger)$ and the curve $\big((h_i(t)u_i),(h_i(t)\tilde{u}_i)\big)$. By $K^{\mathbb{C}}$-invariance this is a curve in $R_{\mathbb{P}_J}$. The derivative of the first component is given by

$$\tfrac{d}{dt}(h_i(t)u_i)\big|_{t=0} = \big(\{v_k,\, u_k^\dagger,\, u_i\} + \dots + \{v_i,\, u_i^\dagger,\, u_i\}\big)_{i=1,\dots,k} \ .$$

Now it is straightforward to show by induction that the map

$$(v_1,\dots,v_k) \mapsto \big(\{v_k,\, u_k^\dagger,\, u_i\} + \dots + \{v_i,\, u_i^\dagger,\, u_i\}\big)_{i=1,\dots,k} \in T_{(u_i)} Z_J$$

is a linear injection. Comparing dimensions then implies that this actually is an isomorphism, so we conclude that pr_1 is a submersion. Therefore, $R_{\mathbb{P}_J}$ is a regular equivalence relation on Z_J, and the quotient manifold $\mathbb{P}_J = Z_J / R_{\mathbb{P}_J}$ is of dimension

$$\dim \mathbb{P}_J = 2 \dim Z_J - \dim R_{\mathbb{P}_J} = \sum_{i=1}^{k-1} \dim(Z_{1/2}^{u_i} \cap Z_1^{u_{i+1}}) + \dim Z_{1/2}^{u_k} \ .$$

This proves (a). For (b), it again suffices to note that by the $K^{\mathbb{C}}$-invariance of the equivalence relation, $h \star [(u_i)] \coloneqq [(hu_i)]$ is a well-defined $K^{\mathbb{C}}$-action on $\mathbb{P}_J$, for which the canonical projection is $K^{\mathbb{C}}$-equivariant. The fiber through (u_i), i.e. the equivalence class $[(u_i)] \subset Z_J$, is given by

$$[(u_i)_{i=1,\dots,k}] = ([u_i])_{i=1,\dots,k} = (Z_1^{u_1})^\times \times \dots \times (Z_1^{u_k})^\times \ ,$$

as stated. To prove (c), we show that any Peirce ordered tuple (u_i) is Peirce equivalent to a tuple (e_i), where $e_1 < \dots < e_k$ are (strictly) ordered tripotents. Indeed, just take e_1 as a maximal tripotent in $Z_1^{u_1}$ and complete it inductively to tripotents e_i of $Z_1^{u_i}$. By maximality, u_i is Peirce equivalent to e_i. Since K acts transitively on the set of frames on Z, is also acts transitively on the set of (strictly) ordered tripotents $(e_1,\dots,e_k)$ with $\mathrm{rk}\, e_i = j_i$. This completes the proof. ☐

REMARK 6.16. It is often convenient to identify the elements of the Peirce flag manifold via Theorem 6.5 with increasing sequences of Peirce 1-spaces, i.e.

$$\mathbb{P}_J = \{(U_1,\dots,U_k) \,|\, U_i \subset Z \text{ Peirce 1-space of rank } j_i,\ U_1 \subset \dots \subset U_k\} \ .$$

As in the case of $\mathbb{P}_j$, it is possible to describe an atlas on $\mathbb{P}_J$ by appropriate restrictions of the charts of Z_J. Recall from Section 3.3 that for $(u_i) \in Z_J$ with $\mathrm{rk}(u) = j$, the set

$$\mathcal{I}_J^{(u_i)} \coloneqq (\mathcal{I}_{j_1}^{u_1} \times \dots \times \mathcal{I}_{j_k}^{u_k}) \cap Z_J$$

is open and dense in the connected component of Z_J containing (u_i). This is the image of a corresponding chart map of Z_J described at the end of Section 3.3.

PROPOSITION 6.17. *Let $\mathbb{P}_J$ be the Peirce flag variety of type $J = (j_1,\dots,j_k)$, and let $\pi : Z_J \to \mathbb{P}_J$ be the canonical projection. Then for $(u_i) \in Z_J$ the map*

$$\varphi_{(u_i)} : (Z_{1/2}^{u_1} \cap Z_1^{u_2}) \times \dots \times (Z_{1/2}^{u_{k-1}} \cap Z_1^{u_k}) \times Z_{1/2}^{u_k} \to \mathbb{P}_J$$

defined by

$$(y_i) \mapsto \big[\tau_{u_k,y_k} \circ \dots \circ \tau_{u_i,y_i} u_i\big]_{i=1,\dots,k}$$

is a diffeomorphism onto the open and dense subset $\mathbb{P}_J^{(u_i)} \coloneqq \pi\big(\mathcal{I}_J^{(u_i)}\big)$ of the connected component of $\mathbb{P}_J$ containing $[u_i]$. Moreover, for $(u_i), (\tilde{u}_i) \in Z_J$, the transition map of $\varphi_{(u_i)}$ and $\varphi_{(\tilde{u}_i)}$ is a birational map, and $\mathbb{P}_J$ is a smooth algebraic variety.

Proof. We first note that $\varphi_{(u_i)} = \pi\circ\tilde\varphi_{(u_i)}$, where $\tilde\varphi_{(u_i)}$ is just the restriction of the chart map of the pre-Peirce flag Z_J to $U := (Z^{u_1}_{1/2}\cap Z^{u_2}_1)\times\ldots\times(Z^{u_{k-1}}_{1/2}\cap Z^{u_k}_1)\times Z^{u_k}_{1/2}$, see Section 3.3, (3.32). Therefore, $W := \tilde\varphi_{(u_i)}(U)$ is a submanifold of Z_J, and according to Remark 3.7, we have to show that this submanifold is transversal to the equivalence classes. For brevity we set $\tilde\varphi := \tilde\varphi_{(u_i)}$. Now fix $(y_i)\in U$. By Theorem 6.15, we have

$$V := \big[\tilde\varphi\big((y_i)\big)\big] = \bigtimes_{i=1}^{k} \tau_{u_k,y_k}\circ\ldots\circ\tau_{u_i,y_i}(Z^{u_i}_1)^\times\,.$$

Since $\dim V + \dim W = \dim Z_J$, it suffices to show that the corresponding tangent spaces intersect trivially. So let $(\dot y_i)$ be an element of $T_{(y_i)}U = U$ such that $D\tilde\varphi((\dot y_i))$ is an element of $T_{\tilde\varphi((y_i))}V$. We prove by induction on k that $\dot y_i = 0$ for all i. In the case of $k = 1$, this is done in the proof of Proposition 6.3. For $k > 1$, the k-th component of $\tilde\varphi$ is given by $\tau_{u_k,y_k}u_k$, and the k-th component of the tangent space of V in $\tilde\varphi((y_i))$ is just $\tau_{u_k,y_k}Z^{u_k}_1$. This is the same situation as for $k = 1$, so this implies $\dot y_k = 0$. Therefore, we obtain for the derivative of the lower components

$$\big(D\tilde\varphi\big)_\ell((\dot y_i)) = \tau_{u_k,y_k}D\big(\tau_{u_{k-1},y_{k-1}}\circ\ldots\circ\tau_{u_\ell,y_\ell}u_\ell\big)((y_i)) \quad\text{for}\quad \ell<k\,.$$

Since the Frobenius transformation can also be extracted from each component of $T_{(y_i)}U$, it follows by induction that $\dot y_i = 0$ for all i. Therefore, $\varphi_{(u_i)}$ defines a chart on $\mathbb{P}_J$. In the same way one shows that the image of $\varphi_{(u_i)}$ equals $\pi\big(\mathcal{I}^{(u_i)}_J\big)$ and that the transition maps are birational maps: For the k-th component this is proved in Proposition 6.3, and for the lower components this follows by induction. We note that by compactness of $\mathbb{P}_J$, *finitely* charts suffice to cover $\mathbb{P}_J$. This shows that $\mathbb{P}_J$ is a prevariety in the sense of [**35**]. Finally, to show that $\mathbb{P}_J$ is a smooth algebraic variety, it suffices to show that that any two points of a connected component of $\mathbb{P}_J$ are contained in a domain of some chart [**35**, §6, Prop.6]. More generally, let $(u^1_i),\ldots,(u^s_i)$ of Z'_J be finitely many points of a connected component Z'_J of Z_J, and let $\mathbb{P}'_J = \pi(Z'_J)$ be the corresponding connected component of $\mathbb{P}_J$. Then the set

$$\mathcal{I}' = \mathcal{I}^{(u^1_i)}_J\cap\ldots\cap\mathcal{I}^{(u^s_i)}\cap Z'_J$$

is open and dense in Z'_J, since each $\mathcal{I}^{(u^\ell_i)}$ is open and dense in Z'_J. Now, for any $(u_i)\in\mathcal{I}'$, Lemma 3.20 implies that all (u^ℓ_i) are contained in $\mathcal{I}^{(u_i)}_J$, and hence the points $[u^\ell_i]$ are contained in the image of $\varphi_{(u_i)}$. This completes the proof. □

Line bundles. In the situation of the Peirce flag variety $\mathbb{P}_J$ we do not have to bother with the construction of line bundles from scratch, but we can transfer line bundles from Peirce Grassmannians via pullbacks. Let $\mathbb{P}_J$ be the Peirce flag variety of type $J = (j_1,\ldots,j_k)$. For $1\le\ell\le k$ let $\hat{\mathrm{pr}}_\ell$ and pr_ℓ be the projections

$$\hat{\mathrm{pr}}_\ell : Z_J\to Z_{j_\ell},\ (u_i)\mapsto u_\ell \quad\text{and}\quad \mathrm{pr}_\ell:\mathbb{P}_J\to\mathbb{P}_{j_\ell},\ [u_i]\mapsto[u_\ell]\,.$$

If π_J and π_{j_ℓ} denote the canonical projections of Z_J onto $\mathbb{P}_J$ and of Z_{j_ℓ} onto $\mathbb{P}_{j_\ell}$, respectively, then we immediately obtain

$$\pi_{j_\ell}\circ\hat{\mathrm{pr}}_\ell = \mathrm{pr}_\ell\circ\pi_J\,, \tag{6.14}$$

i.e. $(\mathrm{pr}_\ell,\hat{\mathrm{pr}}_J)$ is a bundle morphism from $(Z_J,\mathbb{P}_J)$ onto $(Z_{j_\ell},\mathbb{P}_{j_\ell})$. Therefore, if $\mathcal{L}$ is a line bundle on $\mathbb{P}_{j_\ell}$ defined by a cocycle ϕ_{j_ℓ} on $R_{j_\ell}\subset Z^2_{j_\ell}$, then the pullback of $\mathcal{L}$ to $\mathbb{P}_J$ is given by the cocycle $\phi_J = \phi_{j_\ell}\circ(\mathrm{pr}_\ell,\mathrm{pr}_\ell)$. Thus, we obtain:

Proposition 6.18. *Let δ be a denominator of the quasi-inverse in Z, and let $\mathcal{L}_{j_\ell}(\delta)$ be the pullback of the line bundle on $\mathbb{P}_{j_\ell}$ defined in Proposition 6.12 to $\mathbb{P}_J$, and set*

$$\mathcal{L}_J(\delta) := \mathcal{L}_{j_1}(\delta)\otimes\ldots\otimes\mathcal{L}_{j_k}(\delta)\,. \tag{6.15}$$

Then, $\mathcal{L}_J(\delta)$ is a $K^{\mathbb{C}}$-homogeneous line bundle on $\mathbb{P}_J$, which is induced by the cocycle

$$\phi_J : R_{\mathbb{P}_J} \to \mathbb{C}^\times,\ \big((u_i),(\tilde{u}_i)\big) \mapsto \phi^{(u_i)}_{(\tilde{u}_i)} = \prod_{i=1}^k \overline{\delta^{\tilde{u}_i}\big(u_i^\dagger\big)}\,. \tag{6.16}$$

If $K^{\mathbb{C}}_{(U_i)}$ denotes the stabilizer of $(U_i) \in \mathbb{P}_J$ in $K^{\mathbb{C}}$, then

$$\chi : K^{\mathbb{C}}_{(U_i)} \to \mathbb{C}^\times,\ h \mapsto \prod_{i=1}^k \overline{\delta^{u_i}\big(h^* u_i^\dagger\big)}^{-1} \tag{6.17}$$

is a character of $K^{\mathbb{C}}_{(U_i)}$, and the corresponding line bundle $K^{\mathbb{C}} \times_\chi \mathbb{C}$ is isomorphic to $\mathcal{L}_J$.

REMARK 6.19. Proposition 6.18 obviously generalizes to any other family of tensor products of the line bundles $\mathcal{L}_{j_\ell}(\delta)$, i.e. to tensor products of the form

$$\mathcal{L} = \mathcal{L}_{j_1}(\delta)^{\mu_1} \otimes \ldots \otimes \mathcal{L}_{j_k}(\delta)^{\mu_k} \quad \text{with} \quad \mu_i \in \mathbb{Z}\,.$$

The next theorem uses $\mathcal{L}_J(\delta)$ to show that $\mathbb{P}_J$ is a projective variety. The proof is an adaption of the proof of Theorem 6.14.

THEOREM 6.20. *The line bundle $\mathcal{L}_J(\delta)$ defined in Proposition 6.18 is very ample, and hence $\mathbb{P}_J$ is a projective variety.*

PROOF. Recall from the proof of Theorem 6.14 the following constructions: For $u \in Z_j$ let δ^u be the induced Jordan algebra denominator of Z_1^u, and let $\mathfrak{N}^u(x) = \overline{\delta^u(x)} \cdot x^\dagger$ be the corresponding numerator of the pseudo-inverse. Let $\overline{E}_j$ be the finite dimensional vector space of polynomial functions (φ, f) from Z to $\mathbb{C} \times Z$, such that $\deg \varphi \le N_\delta$ and $\deg f \le N_\nu$ for appropriate N_δ, N_ν. Then, the map

$$\sigma^j : Z_j \to \overline{E}_j,\ u \mapsto \sigma^j_u(x) = \big(\overline{\delta^u(x)},\ \overline{\mathfrak{N}^u(x)}\big)$$

defines via Proposition 3.12 a non-vanishing, holomorphic section s_j on $\mathbb{P}_j$ in $\mathcal{L}_j(\delta) \otimes E_j$, which provides an imbedding of $\mathbb{P}_j$ into $\mathrm{P}(E_j)$, the projective space of E_j. This construction immediately extends to the Peirce flag variety: Let $\overline{E}_J$ be the tensor product $\overline{E} = \overline{E}_{j_1} \otimes \ldots \otimes \overline{E}_{j_k}$. We identify $\overline{E}$ with

$$\overline{E} = \Big\{(\varphi, f) : Z^k \to \overline{\mathbb{C}} \times Z^{\otimes k} \text{ polynomial} \,\Big|\, \deg_i \varphi \le N_{\delta,i},\ \deg_i f \le N_{\nu,i} \text{ for all } i\Big\}\ ,$$

where $\deg_i$ denotes the degree of φ and f considered as polynomials of the i-th variable of Z^k. In addition, let σ be the map $\sigma : Z_J \to \overline{E}$ defined by

$$\sigma_{(u_i)}(x_1, \ldots, x_k) = \big(\overline{\delta^{u_1}(x_1)} \cdot \ldots \cdot \overline{\delta^{u_k}(x_k)},\ \overline{\mathfrak{N}^{u_1}(x_1)} \otimes \ldots \otimes \overline{\mathfrak{N}^{u_k}(x_k)}\big)\,.$$

The same reasoning as in the proof of Theorem 6.14 shows that this defines a holomorphic section s on $\mathbb{P}_J$ in $\mathcal{L}_J \otimes \overline{E}$, which corresponds to the tensor product of the pullback sections s_{j_i} for $i = 1, \ldots, k$. Since $\sigma_{(u_i)}(u_1^\dagger, \ldots, u_k^\dagger) = (1, u_1 \otimes \ldots \otimes u_k)$, the section s is non-vanishing and therefore defines a holomorphic map Σ of $\mathbb{P}_J$ the projective space $\mathrm{P}(\overline{E})$ of $\overline{E}$. Since $\mathbb{P}_J$ is compact, it remains to show that Σ is one-to-one. Let $\Sigma([u_i]) = \Sigma([\tilde{u}_i])$ for some $(u_i), (\tilde{u}_i) \in Z_J$, i.e. there exists a $\lambda \in \mathbb{C}^\times$, such that

$$\prod_{i=1}^k \delta^{u_i}(x_i) = \lambda \cdot \prod_{i=1}^k \delta^{\tilde{u}_i}(x_i) \quad \text{and} \quad \bigotimes_{i=1}^k \mathfrak{N}^{u_i}(x_i) = \overline{\lambda} \cdot \bigotimes_{i=1}^k \mathfrak{N}^{\tilde{u}_i}(x_i)$$

for all $(x_1, \ldots, x_k) \in Z^k$. Setting $x_i = u_i^\dagger$, the first relation yields $\lambda = \prod_{i=1}^k \delta^{\tilde{u}_i}\big(u_i^\dagger\big)^{-1}$, and we obtain by (6.12) from the second relation

$$u_1 \otimes \ldots \otimes u_k = \bigotimes_{i=1}^k \frac{\overline{\mathfrak{N}^{u_i}(u_i^\dagger)}}{\overline{\delta^{u_i}\big(u_i^\dagger\big)}} = \bigotimes_{i=1}^k \frac{\overline{\mathfrak{N}^{\tilde{u}_i}(u_i^\dagger)}}{\overline{\delta^{\tilde{u}_i}\big(u_i^\dagger\big)}} = \bigotimes_{i=1}^k Q_{\tilde{u}_i}\big((u_i^\dagger)_1\big)^{-1}\,.$$

Therefore, $u_i = \mu_i \cdot Q_{\tilde{u}_i}\big((u_i^\dagger)_1\big)^{-1}$ for some μ_i with $\mu_1 \cdot \ldots \cdot \mu_k = 1$, which implies that u_i is an element of $Z_1^{\tilde{u}_i}$, and since u_i and $\tilde{u}_i$ have the same rank, we conclude that u_i and $\tilde{u}_i$ are Perice equivalent for all i, and hence $[u_i] = [\tilde{u}_i]$. $\square$

CHAPTER 7

Orbit structure of the Grassmannian

In this chapter, we finally gather the results of the previous chapters to describe the G- and the $K^{\mathbb{C}}$-orbit structure of the Grassmannian $\mathbb{G}(Z)$. We note that the G-orbit structure has already been studied by J. Wolf using Lie theoretic methods [**43, 44**], cf. also [**9**]. Our Jordan theoretic approach provides a more explicit description of these orbits. In particular, the fibration of each G-orbit over a corresponding K-orbit with fiber isomorphic to a product of hermitian symmetric spaces becomes quite obvious in this representation (Section 7.1). Similarly, we show that each $K^{\mathbb{C}}$-orbit forms a fiber bundle on the corresponding K-orbit with fiber isomorphic to the real symmetric cone of some appropriate euclidean Jordan algebra.

In Section 7.2, we describe basic topological properties of the orbits, and Section 7.3 provides an affine realization of the $K^{\mathbb{C}}$-orbits. By abstract arguments it follows that the $K^{\mathbb{C}}$-orbits form subvarieties of the Grassmannian, and hence affine varieties in the affine parts of the Grassmannian. We determine explicitly the corresponding polynomials which describe the $K^{\mathbb{C}}$-orbits as affine varieties on the affine parts of $\mathbb{G}(Z)$.

The last part of this chapter is devoted to the connection between the G- and the $K^{\mathbb{C}}$-orbit structure. In a quite more general setting, T. Matsuki worked out that there is a one-to-one correspondence between these orbits, now called Matsuki duality [**32, 33**]. In Section 7.4, we briefly recall this result in the general setting. Specializing to the particular situation of the Grassmannian, we finally work out this duality by explicit calculations.

7.1. Orbit structure

Let Z be a simple phJTS of rank r, and let $\mathbb{G}$ be its Grassmannian. Using Theorem 4.12 and Remark 4.13, we define for $0 \le a \le a+b \le r$ the following subsets of the Grassmannian

$$\begin{aligned} \mathbb{K}_b^a &:= \left\{ [u + z : u^\dagger] \,\middle|\, u \in Z_a,\, z \in Z_b,\, u \perp\!\!\!\perp z \right\} , \\ G_b^a &:= \left\{ [e + d_e : c + d_c] \,\middle|\, e \in S_{a+b},\, c \in S_a,\, e \ge c,\, d_e \in \mathcal{D}_0^e,\, d_c \in \mathcal{D}_1^c \right\} , \end{aligned} \tag{7.1}$$

where Z_a denotes the set of rank a elements, and S_a is the subset of tripotent elements of rank a. Due to Theorem 4.12, these subsets define a partition of the Grassmannian $\mathbb{G}$. The aim of this section is to show that these partitions correspond to the $K^{\mathbb{C}}$- and G-orbits on $\mathbb{G}$. Before proving this, we mention a basic fact on the inner structure of $\mathbb{K}_b^a$ and G_b^a. For this, we define in addition

$$K_b^a := \{ [c + \tilde{c} : c] \,|\, c \in S_a,\, \tilde{c} \in S_b,\, c \perp\!\!\!\perp \tilde{c} \} = \{ [e : c] \,|\, e \in S_{a+b},\, c \in S_a,\, e \ge c \} . \tag{7.2}$$

It is obvious that K_b^a is invariant under the action of K, and since Z is assumed to be simple, K acts transitively on the frames of Z, so it also acts transitively on each K_b^a. Therefore, all K_b^a are K-homogeneous manifolds, and by compactness of K, they are compact submanifolds of $\mathbb{G}(Z)$. We already encountered the special cases

$$K_0^a = \mathbb{K}_0^a = \mathbb{P}_a , \quad K_r^0 = G_r^0 = S_r .$$

Generally, we have

$$K_b^a \subset \mathbb{K}_b^a \cap G_b^a \ .$$

Now consider the following maps:

$$\begin{aligned} \pi &: \mathbb{K}_b^a \to K_b^a,\ \left[u + z : u^\dagger\right] \mapsto \left[\boldsymbol{\epsilon}(u) + \boldsymbol{\epsilon}(z) : \boldsymbol{\epsilon}(u^\dagger)\right] \ , \\ \mathit{\Pi} &: G_b^a \to K_b^a,\ \left[e + d_e : c + d_c\right] \mapsto \left[e : c\right] \ , \end{aligned} \tag{7.3}$$

where $\boldsymbol{\epsilon}$ denotes the base-tripotent map defined in Section 2.4. Since $\boldsymbol{\epsilon}(u) = \boldsymbol{\epsilon}(\tilde{u})$ for Peirce equivalent elements $u, \tilde{u} \in Z$, Theorem 4.12 ensures that these maps are well-defined. It is $\pi^2 = \pi$ and $\mathit{\Pi}^2 = \mathit{\Pi}$, and we note that (1) π and $\mathit{\Pi}$ are K-equivariant, and (2) the fibers of $\chi = [c + \tilde{c} : c] = [e : c] \in K_b^a$ are given by

$$\pi^{-1}(\chi) = \left[c + \mathit{\Omega}(Z_+^{\tilde{c}}) : c\right] \quad \text{and} \quad \mathit{\Pi}^{-1}(\chi) = \left[e + \mathcal{D}_0^e : c + \mathcal{D}_1^c\right] \ . \tag{7.4}$$

Here, $\mathit{\Omega}(Z_+^{\tilde{c}})$ denotes the symmetric cone defined in the euclidean Jordan algebra $Z_+^{\tilde{c}}$, see Section 1.4. Indeed, the K-equivariance is straightforward to check, the identity on the π-fiber follows from Theorem 3.27, and the identity on the $\mathit{\Pi}$-fiber is an application of Theorem 4.12 to the definition of G_b^a. Once we have proved that $\mathbb{K}_b^a$ and G_b^a are the $K^{\mathbb{C}}$- and G-orbits of $\mathbb{G}(Z)$, it follows by Lie theoretic arguments that π and $\mathit{\Pi}$ are in fact real analytic fibrations. Since the fibers are simply connected, the K-orbits K_b^a are strong deformation retracts of the corresponding $K^{\mathbb{C}}$- and G-orbits.

REMARK 7.1. We note that W. Kaup interprets the projections π and $\mathit{\Pi}$ as the backward and forward limit of the dynamical system on the Grassmannian $\mathbb{G}$ given by the cubic map $\mathbf{c}(z) = z^{(3)}$ on Z, cf. [**19**]. For this, Kaup extends the functional calculus on Z to a generalized functional calculus on $\mathbb{G}$, so $\mathbf{c}$ becomes a real analytic diffeomorphism on $\mathbb{G}$. It turns out that the K-orbits are exactly the connected components of the fixed point set of $\mathbf{c}$, and that $\pi = \lim_{n\to-\infty} \mathbf{c}^n$ and $\mathit{\Pi} = \lim_{n\to\infty} \mathbf{c}^n$. It is an owing task to prove the real analyticity of $\mathbf{c}$ on $\mathbb{G}$ by explicit calculations using charts of $\mathbb{G}$, and similarly to show that the restriction of the $\pm\infty$-limit maps of $\mathbf{c}^n$ to $\mathbb{K}_b^a$ and G_b^a stays real analytic. This approach seems to be in close analogy with the Lie theoretic discussion of momentum maps on $\mathbb{G}$ given by R. Bremingan and J. Lorch in [**7**]. We do not pursue these questions within this thesis, but we refer to Section 7.4 for a discussion of the Matsuki duality, which is also closely related to these issues.

THEOREM 7.2. *Let Z be a simple phJTS of rank r and let $\mathbb{G}$ be its Grassmannian. Then the $K^{\mathbb{C}}$- and the G-orbits on $\mathbb{G}$ are given by the partitions*

$$\mathbb{G}(Z) = \bigcupdot_{0\le a\le a+b\le r} \mathbb{K}_b^a \quad \textit{and} \quad \mathbb{G}(Z) = \bigcupdot_{0\le a\le a+b\le r} G_b^a \ . \tag{7.5}$$

In particular, both orbit structures consist of $\binom{r+2}{2}$ orbits.

PROOF OF THEOREM 7.2. Fix $c \in S_a$ and $\tilde{c} \in S_b$ with $c \perp\!\!\!\perp \tilde{c}$ and set $e := c + \tilde{c} \in S_{a+b}$. Then $\mathbb{c} := [e : c]$ is an element of K_b^a. We prove the identities $\mathbb{K}_b^a = K^{\mathbb{C}}\mathbb{c}$ and $G_b^a = G\mathbb{c}$ in two steps. The inclusions '$\subset$' are proved by explicit calculation of certain group actions, and the inclusions '$\supset$' follow by determining invariants for the $K^{\mathbb{C}}$- and the G-action.

To prove the inclusions '$\subset$', consider the K-equivariant fibrations (7.3). Since Z is simple, K acts transitively on the base K_b^a. Therefore it suffices to show that the fibers

$$\pi^{-1}(\mathbb{c}) = \left[c + \mathit{\Omega}(Z_+^{\tilde{c}}) : c\right] \quad \text{and} \quad \mathit{\Pi}^{-1}(\mathbb{c}) = \left[e + \mathcal{D}_0^e : c + \mathcal{D}_1^c\right]$$

are contained in the corresponding orbits.

CLAIM. *For $w \in Z_+^{\tilde{c}}$ and $v = v_1 \oplus v_0 \in Z_1^c \oplus Z_0^e$ let $h \in K^{\mathbb{C}}$ and $g \in G$ be defined by $h = \exp(w \,\square\, \tilde{c})$ and $g = \exp \zeta_v$. Then*

$$h(\mathbb{c}) = [c + \exp(w) : c] \quad \text{and} \quad g(\mathbb{c}) = [e + \tanh(v_0) : c + \tanh(-v_1)] \;. \tag{7.6}$$

Here, $\exp(w)$ is calculated within the unital Jordan algebra $Z_+^{\tilde{c}}$, and $\tanh(v_0)$ and $\tanh(-v_0)$ are calculated in the Jordan triple systems Z_0^e and Z_1^c.

PROOF. For h we have $h(\mathbb{c}) = [hc + h\tilde{c} : h^{-*}c]$. The Peirce rules imply $hc = c$,

$$h\tilde{c} = \exp(w \,\square\, \tilde{c})\, \tilde{c} = \sum \tfrac{1}{n!} (w \,\square\, \tilde{c})^n \tilde{c} = \sum \tfrac{1}{n!}\, w^n = c + \exp(w)$$

and $h^{-*}c = \exp(-\tilde{c} \,\square\, w)c = c$. To prove the identity on the G-action, we first transform the vector field $\zeta_v(z) = v - Q_z v$ to the affine coordinates given by $\varphi_c : Z \to \mathbb{G}^{(c)}$, $\varphi_c(z) = [z : c]$: According to (4.5) and using the Peirce rules we obtain

$$\zeta_v^{(c)}(z) = B_{z,\,c}(v - Q_{z^c} v) = B_{z,\,c} v - Q_z v = v_1 - 2\,\{z,\, c,\, v_1\} + v_0 - Q_z v_0 \;.$$

We claim that the integral curve of $\zeta_v^{(c)}$ with initial condition $z_0 = e$ is given by

$$z_t := \tfrac{1}{2}\big(\exp(-2tv_1) - c\big) + e + \tanh(tv_0) \quad \text{for } t \in \mathbb{R}.$$

Here, exp denotes the exponential map of the Jordan algebra Z_1^c and tanh is calculated via the functional calculus on Z. Indeed, the initial condition is satisfied by z_t, since $\exp(0) = c$ and $\tanh(0) = 0$. Before determining the derivative of z_t, we calculate the derivative of $\tanh(tv_0)$. Let $v_0 = \sum \lambda_i e_i$ be the spectral decomposition of v_0, then

$$\tfrac{d}{dt} \tanh(tv_0) = \tfrac{d}{dt} \sum \tanh(t\lambda_i)\, e_i = \sum \lambda_i (1 - \tanh^2(t\lambda_i))\, e_i = v_0 - Q_{\tanh(tv_0)} v_0 \;.$$

Therefore, it is

$$\dot{z}_t = -\exp(-2tv_1) \circ v_1 + v_0 - Q_{\tanh(tv_0)} v_0 \;,$$

where $u \circ w = \{u,\, c,\, w\}$ denotes the product of the Jordan algebra Z_1^c. Then again, using the Peirce rules and the identity $\{e,\, c,\, v_1\} = \{c,\, c,\, v_1\} = v_1$, we have

$$\begin{aligned}
\zeta_v^{(c)}(z_t) &= v_1 - 2\left\{\tfrac{1}{2}\big(\exp(-2tv_1) - c\big) + e,\, c,\, v_1\right\} + v_0 - Q_{\tanh(tv_0)} v_0 \\
&= v_1 - \exp(-2tv_1) \circ v_1 + v_1 - 2\,\{e,\, c,\, v_1\} + v_0 - Q_{\tanh(tv_0)} v_0 \\
&= -\exp(-2tv_1) \circ v_1 + v_0 - Q_{\tanh(tv_0)} v_0 \\
&= \dot{z}_t \;.
\end{aligned}$$

This proves that z_t is the integral curve of $\zeta_v^{(c)}$ with initial value $z_0 = e$. Thus

$$g(\mathbb{c}) = \left[e + \tanh(v_0) + \tfrac{1}{2}\big(\exp(-2v_1) - c\big) : c\right] \;.$$

It remains to show that this expression coincides with the stated one. Abbreviating $z_0 := \tilde{c} + \tanh(v_0) \in Z_0^{\tilde{c}}$, we have $g(\mathbb{c}) = [z_0 + \frac{1}{2}(\exp(-2v_1) + c) : c]$, and for $x \in Z_1^c$ it is:

$$g(\mathbb{c}) = [c + z_0 : c + x] \quad \text{if and only if} \quad z_0 + \tfrac{1}{2}\big(\exp(-2v_1) + c\big) = (c + z_0)^x \quad (*) \;.$$

Due to Lemma 2.25 and Lemma 2.26, the right hand side is given by $(c + z_0)^x = c^x + z_0 = c^x = Q_c(c - x)^{-1} + z_0$, where the inverse is calculated in the Jordan algebra Z_1^c. Solving these equations for $x \in Z_1^c$, we obtain

$$x = c - \frac{2}{\exp(-2v_1^\#) + c} = \frac{\exp(-v_1^\#) - \exp(v_1^\#)}{\exp(-v_1^\#) + \exp(v_1^\#)} = \tanh(-v_1^\#)$$

with $v_1^\# = Q_c v_1$, cf. Section 2.5. This yields

$$g(\mathbb{c}) = \left[e + \tanh(v_0) : c + \tanh(-v_1^\#)\right] \;.$$

Finally we have to show that $\tanh(-v_1^\#)$, which is calculated in the Jordan algebra Z_1^c, can be exchanged by $\tanh(-v_1)$, which is determined by the functional calculus on the whole Jordan triple Z. Let $v_1 = \sum \lambda_i e_i$ be the spectral decomposition of v_1 in Z. Since the element c is just determined up to Peirce equivalence, we can exchange e (for a short time) by a tripotent c' with $\sum e_i < c'$. Hence we have $v_1^{\#} = Q_{c'} v_1 = v_1$, and the powers of v_1 in the Jordan algebra are given by $v_1^n = \sum \lambda_i^n e_i$. Therefore the v_1-evaluation of the hyperbolic tangent on $Z_1^{c'}$ and the one on Z coincide. $\square$

Now, the claim above implies that the fibers $\pi^{-1}(\mathbb{c})$ and $\Pi^{-1}(\mathbb{c})$ are contained in the corresponding orbits, since due to Proposition 1.7 and Example 3.24, it is

$$\Omega(Z_+^{\tilde{c}}) = \{\exp(w) \,|\, w \in Z_+^{\tilde{c}}\} \quad \text{and} \quad \mathcal{D}_\nu^u = \{\tanh(z) \,|\, z \in Z_\nu^u\}$$

for $u \in Z$, $\nu = 1, 1/2, 0$. This finishes the proof of the inclusions $\mathbb{K}_b^a \subset K^{\mathbb{C}}\mathbb{c}$ and $G_b^a \subset G\mathbb{c}$. Now we turn to the proof of the opposite inclusions.

First we consider the $K^{\mathbb{C}}$-action. Let h be in $K^{\mathbb{C}}$, and let $h\mathbb{c} = [u + z : u^\dagger]$ be the representation of $h\mathbb{c}$ corresponding to Theorem 4.12, i.e. $z, u \in Z$ with $u \perp\!\!\!\perp z$. We have to show that

$$\operatorname{rk}(u + z) = \operatorname{rk}(e) = a + b \quad \text{and} \quad \operatorname{rk}(u) = \operatorname{rk}(c) = a\ .$$

By assumption, it is $h\mathbb{c} = h\,[e : c] = [he : h^{-*}c] \overset{!}{=} [u + z : u^\dagger]$. As in the last part of the proof of Theorem 4.12, this immediately implies $\operatorname{rk}(he) = \operatorname{rk}(u + z)$. Since the rank is invariant under the action of $h \in K^{\mathbb{C}}$, it follows that $\operatorname{rk}(e) = \operatorname{rk}(u + z)$. Furthermore, by assumption it is

$$(u + z)^{u^\dagger - h^{-*}c} = he\ .$$

Using the properties of Bergman operators (in particular JT34), we obtain

$$\begin{aligned} h^{-*} Z_0^c &= h^{-*} B_{c,\,c} Z = h^{-*} B_{c,\,e} Z = B_{h^{-*}c,\,he} Z \\ &= B_{h^{-*}c,\,(u+z)^{u^\dagger - h^{-*}c}} Z = B_{u^\dagger,\,u+z} Z = B_{u^\dagger,\,u} Z = Z_0^u\ . \end{aligned}$$

By Corollary 2.39, this implies $\operatorname{rk}(u) = \operatorname{rk}(c)$. This completes the proof of the inclusion $K^{\mathbb{C}}\mathbb{c} \subset \mathbb{K}_b^a$.

Finally, we show that the inclusion $G\mathbb{c} \subset G_b^a$ follows from Corollary 5.10, which describes a G-invariant on $\mathbb{G}$, the so called signature map $\sigma(\boldsymbol{\beta})$ given by[1]

$$\sigma(\boldsymbol{\beta})\big(\,[x : a]\,\big) = \text{signature of } (u, v) \mapsto \langle B_{x,\,a} B_{x^a,\,x^a} B_{a,\,x} u | v\rangle. \tag{7.7}$$

The inclusion $G_b^a \subset G\mathbb{c}$ implies that $\sigma(\boldsymbol{\beta})$ is constant on G_b^a. We calculate $\sigma(\boldsymbol{\beta})$ on G_b^a. Set $x = e = c + \tilde{c}$ and $a = c + \lambda c$ for any real $\lambda \in (0, 1)$. By definition we have $[x : a] \in G_b^a$, and using Lemma 4.14 and the Peirce rules, we obtain

$$B_{x,\,a} B_{x^a,\,x^a} B_{a,\,x} = B_{c,\,(1+\lambda)c} B_{\tilde{c} - \frac{1}{\lambda}c,\,\tilde{c} - \frac{1}{\lambda}c} B_{(1+\lambda)c,\,c}\ .$$

Let $Z = \bigoplus Z_{ij}$ be the joint Peirce decomposition with respect to $(c, \tilde{c})$. Then it is straightforward to show that

$$B_{x,\,a} B_{x^a,\,x^a} B_{a,\,x} = \operatorname{pr}_{00} + (1 - \lambda)^2 \operatorname{pr}_{11} - (1 - \lambda^2) \operatorname{pr}_{10}\ , \tag{7.8}$$

where pr_{ij} denotes the orthogonal projection onto Z_{ij}. Since $Z_{00} = Z_0^e$, $Z_{11} = Z_1^c$ and $Z_{10} = Z_{1/2}^e \cap Z_{1/2}^c$, we obtain

$$\sigma(\boldsymbol{\beta})\big(\,[e : (1 + \lambda))c]\,\big) = \big(\dim(Z_0^e \oplus Z_1^c),\ \dim(Z_{1/2}^e \cap Z_{1/2}^c)\big)\ .$$

As a passing comment, we note that equation (7.8) also holds for $\lambda \to 0$, i.e. for $[x : a] \to [e : c] = \mathbb{c}$. Therefore, $\boldsymbol{\beta}(\mathbb{c}) = (\mathbb{c}, \mathbb{c}, \operatorname{pr}_{00} + \operatorname{pr}_{11} - \operatorname{pr}_{10})$. Finally we conclude that G_b^a is uniquely characterized by its $\sigma(\boldsymbol{\beta})$-value, since from Corollary 2.39 it

[1] Here we made the abstract isomorphism between Z and its dual space $Z^{\#}$ more explicit than in Section 5.3. We note that u and v transform according to $(x \mapsto B_{a - \tilde{a},\,x} x)$, since they are interpreted as elements of the cotangent space $T^{\#}_{[x:a]}\mathbb{G} \cong Z^{\#} \cong Z$, and not as elements of the tangent space $T_{[x:a]}\mathbb{G}$.

follows that $a = \mathrm{rk}(c)$ and $a + b = \mathrm{rk}(e)$ are uniquely determined by $\dim(Z_0^e \oplus Z_1^c)$ and $\dim(Z_{1/2}^e \cap Z_{1/2}^c)$. This completes the proof of the inclusion $G\mathfrak{c} \subset G_b^a$, and hence the proof of Theorem 7.2. $\square$

Tangent structures. We first encounter the K-orbits K_b^a. Fix (a, b) and an element $\mathfrak{c} = [e : c] = [c + \tilde{c} : c]$ of K_b^a. Since the map

$$K_b^a \to \mathbb{P}_a,\ [e : c] \mapsto [c : c] \tag{7.9}$$

is a K-equivariant fibration with fiber $Z_0^c \cap S_b$, the tangent space of K_b^a at $\mathfrak{c}$ can be identified with

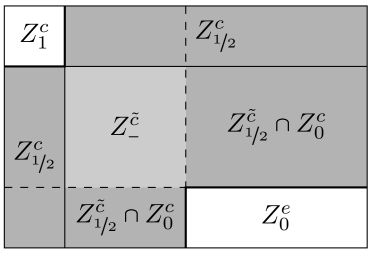
$$T_{\mathfrak{c}} K_b^a \cong Z_{1/2}^c \oplus Z_-^{\tilde{c}} \oplus \left(Z_0^c \cap Z_{1/2}^{\tilde{c}}\right) . \tag{7.10}$$

Due to the fibrations π and Π of the $K^{\mathbb{C}}$-orbit and the G-orbit onto the corresponding K-orbit K_b^a, the tangent space of $\mathbb{K}_b^a$ and G_b^a in $\mathfrak{c}$ decomposes into the direct sum of $T_{\mathfrak{c}} K_b^a$ and the tangent space of the respective fiber which is described in (7.4). Since $T_{\tilde{c}}(Z_+^{\tilde{c}}) \cong Z_+^{\tilde{c}}$ and $T_{(0,0)}(\mathcal{D}_0^e \times \mathcal{D}_1^c) \cong Z_0^e \oplus Z_1^c$, we therefore obtain

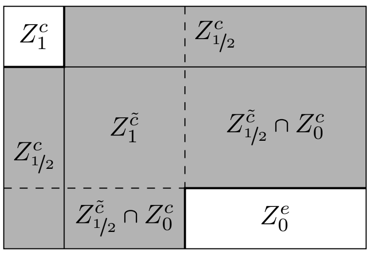
$$\begin{aligned} T_{\mathfrak{c}} \mathbb{K}_b^a &\cong Z_{1/2}^c \oplus Z_1^{\tilde{c}} \oplus \left(Z_0^c \cap Z_{1/2}^{\tilde{c}}\right) \cong Z/(Z_1^c \oplus Z_0^e) , \\ T_{\mathfrak{c}} G_b^a &\cong Z_{1/2}^c \oplus Z_-^{\tilde{c}} \oplus \left(Z_0^c \cap Z_{1/2}^{\tilde{c}}\right) \oplus Z_0^e \oplus Z_1^c \cong Z/Z_+^{\tilde{c}} . \end{aligned} \tag{7.11}$$

Hence, $\dim \mathbb{K}_b^a = \dim Z - \dim Z_1^c - \dim Z_0^e$ and $\dim G_b^a = \dim Z - \dim Z_+^{\tilde{c}}$. For the matrix case $Z = \mathbb{C}^{r \times s}$, the following figures illustrate the corresponding tangent spaces:

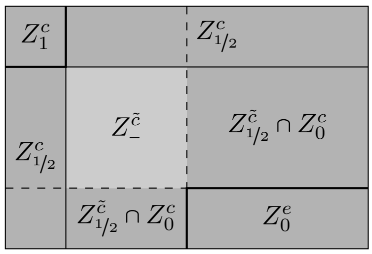
$T_{\mathfrak{c}} K_b^a$



$T_{\mathfrak{c}} \mathbb{K}_b^a$



$T_{\mathfrak{c}} G_b^a$



Invariant metric on open G-orbits. In Section 5.3, we discussed the following G-equivariant section[2] of the vector bundle $T\mathbb{G} \otimes \overline{T\mathbb{G}}$ on $\mathbb{G}$:

$$\boldsymbol{\beta}([x : a]) = \left[(x, a),\ B_{x,a} B_{x^a, x^a} B_{a,x}\right] . \tag{7.12}$$

As in the proof of Theorem 7.2, we interpret $\boldsymbol{\beta}([x : a])$ as a dual hermitian form on $\mathbb{G}$, i.e.

$$\boldsymbol{\beta}([x : a]) : T^{\#}_{[x:a]}\mathbb{G} \times T^{\#}_{[x:a]}\mathbb{G},\ (u, v) \mapsto \langle B_{x,a} B_{x^a, x^a} B_{a,x} u | v \rangle\ ,$$

[2]More precisely, this is the restriction of a corresponding section on the complexification of the Grassmannian $\mathbb{G}$.

where we identified $T^{\#}_{[x:a]}\mathbb{G} \cong Z^{\#}$ with Z via the scalar product $\langle\,|\,\rangle$ on Z. For $[x : a] = \mathbb{c}$ as above, the proof of Theorem 7.2 yields

$$\boldsymbol{\beta}(\mathbb{c})(u,v) = \langle u|v\rangle_{Z_1^c} + \langle u|v\rangle_{Z_0^e} - \langle u|v\rangle_{Z^e_{1/2}\cap Z^c_{1/2}} \tag{7.13}$$

with $\langle u|v\rangle_U := \langle \mathrm{pr}_U\, u|\mathrm{pr}_U\, v\rangle$, where pr_U denotes the orthogonal projection to the subspace $U \subset Z$. Comparing this with (7.11), we conclude that $\boldsymbol{\beta}$ is non-degenerate along G^a_b if and only if $\tilde{c} = 0$, i.e. if and only if $b = 0$. In this case, let $\mathbf{h}$ be the non-degenerate hermitian form on G^a_0 defined by

$$\mathbf{h}([x:a]) : T_{[x:a]}G^a_0 \times T_{[x:a]}G^a_0,\ (u,v) \mapsto \big\langle (B_{x,\,a}B_{x^a,\,x^a}B_{a,\,x})^{-1}u\big|v\big\rangle\ .$$

This time, u and v are elements of $Z \cong T_{[x:a]}G^a_0$ without any further identifications. The G-invariance of $\mathbf{h}$ immediately follows from the G-invariance of $\boldsymbol{\beta}$, and hence $\mathbf{h}$ defines a (pseudo-)hermitian metric on G^a_0 of signature $(\dim(Z^c_1 \oplus Z^c_0), \dim Z^c_{1/2})$. We note, that (7.11) also shows that G^a_0 is an open orbit of $\mathbb{G}$.

7.2. Topology of the orbits

In this section we describe some basic topological properties of the G- and the $K^{\mathbb{C}}$-orbits on $\mathbb{G}$. As before, let Z_j denote the set of all rank-j elements in Z, let S_j be its subset of rank-j tripotents in Z, and let $\mathbb{P}_j$ be the Peirce Grassmannian of Z of type j, considered as a submanifold of $\mathbb{G}$ via the identification given in Theorem 6.5.

Theorem 7.3. *Let Z be a phJTS of rank r, let $\mathbb{G}$ be its Grassmannian, and let*

$$\mathbb{G} = \biguplus_{0\le a\le a+b\le r} \mathbb{K}^a_b = \biguplus_{0\le a\le a+b\le r} G^a_b\ .$$

be the decomposition of $\mathbb{G}$ into $K^{\mathbb{C}}$- and G-orbits.

(a) *In both cases, there are exactly $r+1$ orbits contained in the affine part $Z \subset \mathbb{G}$, namely*

$$\mathbb{K}^0_b = Z_b \qquad \textit{and} \qquad G^0_b = \biguplus_{e\in S_b} e + \mathcal{D}^e_0 \qquad (b = 0,\dots,r)\ .$$

In particular, $G^0_0 = \mathcal{D}$, the unit ball of Z, and $\bigcup_{b=0}^r G^0_b = cl(\mathcal{D})$.

(b) *There are exactly $r+1$ closed $K^{\mathbb{C}}$-orbits. These coincide with the Peirce Grassmannians:*

$$\mathbb{K}^a_0 = \mathbb{P}_a \quad (a = 0,\dots,r)\,.$$

The Shilov boundary $S_r = G^0_r$ is the only closed G-orbit.

(c) *The only open $K^{\mathbb{C}}$-orbit is $\mathbb{K}^0_r = Z_r$. There are exactly $r+1$ open G-orbits, namely*

$$G^a_0 = \biguplus_{[u]\in\mathbb{P}_a} \big[u + \mathcal{D}^u_0 : u^\dagger + \mathcal{D}^u_1\big] \qquad (a = 0,\dots,r)\ .$$

In particular, $G^0_0 = \mathcal{D}$.

(d) *The topological closure of an orbit is given by*

$$cl(\mathbb{K}^a_b) = \biguplus_{\alpha\ge a\,,\,\alpha+\beta\le a+b} \mathbb{K}^\alpha_\beta \quad \textit{and} \quad cl(G^a_b) = \biguplus_{\alpha\le a\,,\,\alpha+\beta\ge a+b} G^\alpha_\beta\ .$$

Therefore, $cl(\mathbb{K}^a_b)$ is the disjoint union of $\frac{1}{2}\,b(b+1)$ orbits, and $cl(G^a_b)$ is the disjoint union of $(a+1)\cdot(r+1-(a+b))$ orbits. The Shilov boundary $S_r = G^0_r$ is contained in the closure of each G-orbits.

PROOF. Part (a) follows from Claim 1 and 2 in the proof of Theorem 4.12. Now it suffices to proof (d) in detail, since (b) and (c) are immediate consequences of (d). First we consider the $K^{\mathbb{C}}$-orbits. Let χ be an element of $\mathrm{cl}(\mathbb{K}^a_b)$, i.e. there exists a sequence χ_n in $\mathbb{K}^a_b$ converging to χ. By Theorem 4.12, each χ_n is representable as $\chi_n = [u_n + z_n : u_n^\dagger]$ with $u_n \in Z_a$, $z_n \in Z_b$ and $u_n \perp\!\!\!\perp z_n$. Moreover, we can assume, that all u_n are tripotent, since any $u_n \in Z_a$ is Peirce equivalent to some tripotent $\epsilon(u_n) \in S_a$, see Corollary 2.21. Let (e_k) be a frame of Z. Due to the spectral theorem of Z, and since K acts transitively on the set of frames, there exists a sequence k_n in K such that

$$k_n u_n = e_1 + \ldots + e_a =: e\ , \quad k_n z_n = \sum_{k=a+1}^{a+b} \lambda_k^{(n)} e_k \quad \text{with} \quad \lambda_a^{(n)} \geq \lambda_2^{(n)} \geq \ldots \geq \lambda_{a+b}^{(n)} \geq 0\ .$$

By compactness of K, we assume that k_n converges to $k \in K$. In addition, we assume that each sequence $\lambda_k^{(n)}$ converges for $n \to \infty$ to $\lambda_k \in [0,\infty]$. Set $s := \#\{\lambda_k = \infty\}$ and $t := \#\{\lambda_k = 0\}$. Due to Lemma 4.14, we obtain

$$\begin{aligned} k_n\chi_n &= \left[e + \sum_{k=a+1}^{a+b} \lambda_k^{(n)} e_k : e\right] = \left[e + \sum_{k=a+1}^{a+s} e_k + \sum_{k=a+s+1}^{a+b} \lambda_k^{(n)} e_k : e + \sum_{k=a+1}^{a+s} \left(1 - \tfrac{1}{\lambda_k^{(n)}}\right) e_k\right] \\ &\xrightarrow{n\to\infty} \left[e + \sum_{k=a+1}^{a+s} e_k + \sum_{k=a+s+1}^{a+b} \lambda_k e_k : e + \sum_{k=a+1}^{a+s} e_k\right] =: \chi' \in \mathbb{K}^{a+s}_{b-s-t}\ , \end{aligned}$$

and therefore

$$\chi = \lim_{n\to\infty} \chi_n = k^{-1} \lim_{n\to\infty} (k_n\chi_n) = k^{-1}\chi' \in \mathbb{K}^{a+s}_{b-s-t}\ .$$

This proves the inclusion $\mathrm{cl}(\mathbb{K}^a_b) \subset \dot\bigcup_{\alpha\geq a\,,\,\alpha+\beta\leq a+b} \mathbb{K}^\alpha_\beta$. Now we turn to the converse inclusion: Let χ be an element of $\mathbb{K}^\alpha_\beta$ with $\alpha \geq a$ and $\alpha+\beta \leq a+b$. It is $\chi = [u+z : u^\dagger]$ with $u \in Z_\alpha$, $z \in Z_\beta$ and $u \perp\!\!\!\perp z$. Since $\alpha \geq a$, there is a decomposition of u into $u = u_1 + u_2$ with $u_1 \in Z_a$, $u_2 \in Z_{\alpha-a}$ and $u_1 \perp\!\!\!\perp u_2$. In addition, he inequality $a + b \geq \alpha + \beta$ implies that the intersection of $Z_{(a+b)-(\alpha+\beta)}$ with Z_0^{u+z} is non-empty. Let $\tilde z$ be an element of $Z_{(b+a)-(\alpha+\beta)} \cap Z_0^{u+z}$. For $\lambda \in \mathbb{R}^\times$, using Lemma 4.14 we thus obtain

$$\mathbb{K}^a_b \ni \left[u_1 + \tfrac{1}{\lambda}\, u_2 + z + \lambda\tilde z : u_1^\dagger\right] \xrightarrow{\lambda\to 0} \left[u_1 + u_2 + z : u_1^\dagger + u_2^\dagger\right] = \left[u + z : u^\dagger\right]\ .$$

Analogously we prove the assertion on the G-orbits: Let χ be an element of $\mathrm{cl}(G^a_b)$, and let χ_n be a sequence in G^a_b converging to χ. Each χ_n is representable as $\chi_n = [c_n + d_{e_n} : c_n + d_{c_n}]$ with $[e_n : c_n] \in K^a_b$ and $d_{e_n} \in \mathcal{D}_0^{e_n}$, $d_{c_n} \in \mathcal{D}_1^{c_n}$. Without restriction, we assume that $[e_n : c_n]$ converges in K^a_b, and the sequences d_{e_n} and d_{c_n} converge in $\mathrm{cl}(\mathcal{D})$,

$$[e_n : c_n] \to [e : c] \in K^a_b\ , \quad d_{e_n} \to d_e \in \mathrm{cl}(\mathcal{D})\ , \quad d_{c_n} \to d_c \in \mathrm{cl}(\mathcal{D})\ .$$

Since the orthogonal projections $\pi_\nu(u)$ onto Peirce spaces Z^u_ν depend continuously on the element $u \in Z$, we obtain $d_e \in \mathrm{cl}(\mathcal{D}^e_0)$ and $d_c \in \mathrm{cl}(\mathcal{D}^c_1)$. Using the spectral theorem, we decompose d_e and d_c according to their spectral values:

$$d_e = \tilde e + \tilde d_e \quad \text{with} \quad \tilde e \in S_s,\ \tilde d_e \in \mathcal{D}_0^{e+\tilde e} \quad \text{and} \quad d_c = \tilde c + \tilde d_c \quad \text{with} \quad \tilde c \in S_t,\ \tilde d_c \in \mathcal{D}_1^{e+\tilde e}\ .$$

Since c is exchangeable by any element, which is Peirce equivalent to c, we choose $c = c' - \tilde c$ with $c' \in S_{a-s}$ and $c' \perp\!\!\!\perp \tilde c$, and obtain

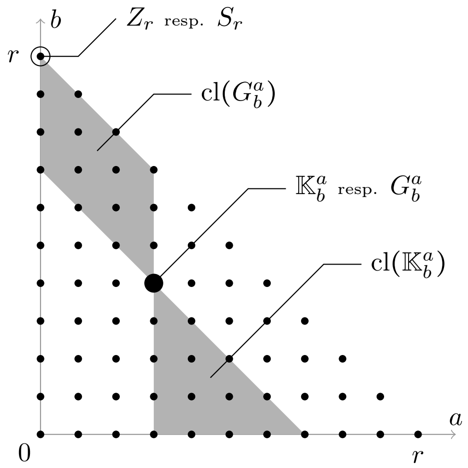
$$\chi = \lim_{n\to\infty} \chi_n = [e + d_e : c + d_c] = \left[e + \tilde e + \tilde d_e : c' + \tilde d_c\right] \in G^{a-s}_{b+s+t}\ .$$

This proves the inclusion $\mathrm{cl}(G^a_b) \subset \dot\bigcup_{\alpha\leq a\,,\,\alpha+\beta\geq a+b} G^\alpha_\beta$. For the opposite inclusion consider $\chi = [e + d_e : c + d_c] \in G^\alpha_\beta$ with $\alpha \leq a$ and $\alpha + \beta \geq a + b$. By assumption, e is decomposable into $e = c + \tilde c_1 + \tilde c_2 + \tilde c_3$ with pairwise orthogonal tripotents $\tilde c_1 \in S_{a-\alpha}$, $\tilde c_2 \in S_b$, $\tilde c_3 \in S_{(a+b)-(\alpha+\beta)}$. Therefore,

$$G^a_b \ni [c + \tilde c_1 + \tilde c_2 + \lambda\,\tilde c_3 + d_e : c + \tilde c_2 - \lambda\,\tilde c_2 + d_c] \xrightarrow{\lambda \to 1} [e + d_e : c + d_c] = \chi\ .$$

The number of orbits contained in the closure of some given orbit is easily determined using the diagrammatic representation of the index set $\{(a,b)\,|\,0 \le a \le a+b \le r\}$. This also shows the assertion on the Shilov boundary. $\square$

(7.14)



7.3. Affine realization of the $K^{\mathbb{C}}$-orbits

Recall from Proposition 4.3, that each $v \in \overline{Z}$ defines an open affine subvariety $\mathbb{G}^{(v)} = \{[z : v]\,|\,z \in Z\}$ of the Grassmannian $\mathbb{G}$, which is isomorphic to Z via the isomorphism

$$\varphi_v : \mathbb{G}^{(v)} \to Z,\ [z : v] \mapsto z\ .$$

The main result of this section states that for a given $K^{\mathbb{C}}$-orbit $\mathbb{K}^a_b$ we can choose $v \in \overline{Z}$ such that the intersection $\mathbb{K}^a_b \cap \mathbb{G}^{(v)}$ is open and dense in $\mathbb{K}^a_b$. Furthermore, the *affine realization* $\varphi_v(\mathbb{K}^a_b \cap \mathbb{G}^{(v)})$ of the $K^{\mathbb{C}}$-orbit has a simple Jordan theoretic description.

Proposition 7.4. *Let Z be a phJTS of rank r, let $\mathbb{K}^a_b$ be a $K^{\mathbb{C}}$-orbit on $\mathbb{G}$. Then, for any $u \in Z_a$, the intersection $\mathbb{K}^a_b \cap \mathbb{G}^{(u^\dagger)}$ is a open and dense subset of $\mathbb{K}^a_b$, and*

$$\varphi_{u^\dagger}\big(\mathbb{K}^a_b \cap \mathbb{G}^{(u^\dagger)}\big) = \big\{Q_{y+u^\dagger}u + z\,\big|\,y \in Z^u_{1/2},\ z \in Z_b \cap Z^u_0\big\}\ . \tag{7.15}$$

Moreover, the restriction of the partial inverse map $j_{u^\dagger}$ to $Z^u_{1/2} \oplus (Z_b \cap Z^u_0)$ is a diffeomorphism onto $\mathbb{K}^a_b \cap \mathbb{G}^{(u^\dagger)}$.

The key observation for the proof of this proposition is the following generalization of Lemma 6.7.

Lemma 7.5. *For $u \in Z$ and $y \in Z^u_{1/2}$ let τ_y be the Frobenius transformation $\tau_y = \exp(2\,y \,\square\, u^\dagger)$, and let $j_{u^\dagger}$ be the partial inverse mapping with respect to $u^\dagger$. Then for $z \in Z^u_0$ it is*

$$j_{u^\dagger}(y + z) = \big[\tau_y u + z : u^\dagger\big] = \big[\tau_y u + B_{\tau_y u,\,(\tau_y u)^\dagger} z : (\tau_y u)^\dagger\big] \tag{7.16}$$

Moreover, the Peirce projection $B_{\tau_y u,\,(\tau_y u)^\dagger}$ maps Z^u_0 isomorphically (as vector spaces) onto $Z^{\tau_y u}_0$ and preserves the rank of elements.

PROOF. We first recall from Lemma 6.7 the identities

$$\tau_y = B_{y,-u^\dagger} \quad \text{and} \quad \tau_y u = u + y + Q_y u^\dagger = Q_{y+u} u^\dagger \ .$$

To determine $j_u(y+z)$, we use the relation $j_{u^\dagger} = \tilde{t}_{u^\dagger} \circ t_u \circ \tilde{t}_{u^\dagger}$ and the nilpotence of the pair $(y+z, u^\dagger)$ (cf. Lemma 2.22), and obtain

$$j_{u^\dagger}(y+z)^{-u^\dagger} = \tilde{t}_{-u^\dagger} \circ j_{u^\dagger}(y+z) = u + (y+z)^{u^\dagger} = u + y + z + Q_{y+z} u^\dagger = Q_{y+u} u^\dagger + z \ .$$

This proves the first equality of (7.16). To show the second one, we set for brevity $v := \tau_{y,u}$. With this notation we have to show that

$$\left[v + z : u^\dagger\right] = \left[v + B_{v,v^\dagger} z : v^\dagger\right] \ .$$

From Lemma 6.7 we have $[v : u^\dagger] = [v : v^\dagger]$. Therefore the Bergman operator $B_{v,u^\dagger - v^\dagger}$ is invertible, and $v = v^{u^\dagger - v^\dagger}$. The addition formula for the quasi-inverse implies

$$(v+z)^{u^\dagger - v^\dagger} = v^{u^\dagger - v^\dagger} + B^{-1}_{v,u^\dagger - v^\dagger}\left(z^w\right) = v + B_{v,v^\dagger - u^\dagger}\left(z^w\right) \quad \text{with} \quad w := (u^\dagger - v^\dagger)^v \ .$$

Applying the symmetry formula to $w := (u^\dagger - v^\dagger)^v$, we obtain

$$w = (u^\dagger - v^\dagger)^v = u^\dagger - v^\dagger + Q_{u^\dagger - v^\dagger} v = u^\dagger + Q_u^\dagger v - 2\left\{u^\dagger, v, v^\dagger\right\} \ .$$

Therefore, by the Peirce rules, w is an element of $Z_1^u \oplus Z_{1/2}^u$, and Lemma 2.25 implies $z^w = z$, since z is an element of Z_0^u. In summary we have

$$(v+z)^{u^\dagger - v^\dagger} = v + B_{v,v^\dagger - u^\dagger} z \ .$$

Again applying the Peirce rules shows that the restriction of the Bergman operator $B_{v,v^\dagger - u^\dagger} \in \mathrm{Str}(Z)$ to Z_0^u coincides with the Peirce projection $B_{v,v^\dagger}$ onto Z_0^v, and therefore $B_{v,v^\dagger}$ is a rank-preserving vector space isomorphism from Z_0^u onto Z_0^v. □

PROOF OF PROPOSITION 7.4. Since the partial inverse map $j_{u^\dagger}$ is an automorphism of the Grassmannian, the restriction of $j_{u^\dagger}$ to $Z_{1/2}^u \oplus (Z_b \cap Z_0^u)$ remains a holomorphic map. Lemma 7.5 implies that the image of this restriction lies in $\mathbb{K}_b^a$ and coincides with the right hand side of (7.15). Recall from Proposition 6.8 that $j_{u^\dagger}$ maps $Z_{1/2}^u$ onto an open and dense subset of $\mathbb{K}_0^a \subset \mathbb{K}_b^a$. Due to Lemma 7.5, this implies that $j_{u^\dagger}$ maps $Z_{1/2}^u \oplus (Z_b \cap Z_0^u)$ onto an open and dense subset of $\mathbb{K}_b^a$, since an element of $[u + z : u^\dagger] \in \mathbb{K}_b^a$ is an element of $j_{u^\dagger}(Z_{1/2}^u \oplus (Z_b \cap Z_0^u))$ if and only if $[u : u^\dagger] \in \mathbb{K}_0^a$ is an element of $j_{u^\dagger}(Z_{1/2}^u)$. This also proves that the left hand side of (7.15) is contained in the right hand side, since by Remark 6.9 this is true for the case $b = 0$. This completes the proof. □

7.4. Matsuki duality

For the moment, let $G^{\mathbb{C}}$ be an arbitrary connected complex semisimple Lie group, let G be a connected real form of $G^{\mathbb{C}}$, and let $K^{\mathbb{C}}$ be the complexification of a maximal compact subgroup K of G. Furthermore, choose any complex parabolic subgroup P of $G^{\mathbb{C}}$ and let $X = G^{\mathbb{C}}/P$ be the corresponding complex flag manifold. In [**32, 33, 34**], T. Matsuki proved that there are only finitely many G- and $K^{\mathbb{C}}$-orbits on X, and there is a natural one-to-one correspondence between these orbits given by

$$G\text{-orbit} \leftrightarrow K^{\mathbb{C}}\text{-orbit} \quad \Longleftrightarrow \quad \begin{cases} (G\text{-orbit}) \cap (K^{\mathbb{C}}\text{-orbit}) \\ \text{is non-empty and compact.} \end{cases} \tag{7.17}$$

In fact, the intersection of dual orbits is a single K-orbit. We call this duality of G- and $K^{\mathbb{C}}$-orbits *Matsuki duality.*

Now we verify this duality in the case of the Grassmannian $\mathbb{G}$ of a phJTS Z by explicit Jordan theoretic arguments. As in the last sections, let $G = \mathrm{Aut}(\mathcal{D})^0$ be the identity component of the automorphism group of the unit ball $\mathcal{D} \subset Z$, let $K = \mathrm{Aut}(Z)$ be the identity component of the automorphism group of Z, which is a maximal compact subgroup of G, and let $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$ be their complexifications. Furthermore, let G^a_b and $\mathbb{K}^a_b$ denote the G- and $K^{\mathbb{C}}$-orbits of $\mathbb{G}$ as described in Section 7.1, and let K^a_b be the corresponding K-bases of the orbits.

Theorem 7.6. *Let Z be a phJTS, $\mathbb{G}$ its Grassmannian, and let G^a_b, $\mathbb{K}^a_b$, K^a_b be the G-, $K^{\mathbb{C}}$- and K-Orbits on $\mathbb{G}$ as described in* (7.1) *and* (7.2)*. Then*

$$\mathbb{K}^a_b \cap G^\alpha_\beta = \begin{cases} \varnothing & \text{, if } a > \alpha \text{ or } a + b < \alpha + \beta \text{ ,} \\ K^a_b & \text{, if } a = \alpha \text{ and } b = \beta \text{ ,} \\ \text{non-compact union of } \infty\text{-many } K\text{-orbits} & \text{, otherwise.} \end{cases}$$

Proof. Fix (a, b), (α, β) and let χ be an element of the intersection of $\mathbb{K}^a_b$ and G^α_β. By definition, χ is representable as

$$\chi = \left[u + z : u^\dagger\right] = [e + d_e : c + d_c] \ ,$$

where $u \in Z_a$, $z \in Z_b$ with $u \perp\!\!\!\perp z$ and $e \in S_{\alpha+\beta}$, $c \in S_\alpha$ with $e \geq c$ and $d_e \in \mathcal{D}^e_0$, $d_c \in \mathcal{D}^c_1$. The relation of these representations is elaborated in the proof of Theorem 4.12 and in Remark 4.13. We recall that (without restriction) c is decomposible into a sum of orthogonal tripotents $c = c_1 + c_2$, where $c_2 = \epsilon(d_c)$ is the base-tripotent of d_c, and e decomposes into $e = c_1 + c_2 + e'$ with $e' < e$. It follows that $[e + d_e : c + d_c] = \left[c_1 + (c_2 + e' + d_e - d_c^\dagger) : c_1\right]$ with $c_1 \perp\!\!\!\perp (c_2 + e' + d_e - d_c^\dagger)$. Now Theorem 4.12 implies

$$u \approx c_1 \quad \text{and} \quad z = c_2 + e' + d_e + d_c^\dagger \ .$$

For the intersection $\mathbb{K}^a_b \cap G^\alpha_\beta$ to be non-empty, the first relation yields the necessary condition $\mathrm{rk}(u) \leq \mathrm{rk}(c)$, i.e. $a \leq \alpha$, and adding c_1 to the second relation, orthogonality implies

$$\begin{aligned} a + b = \mathrm{rk}(u + z) = \mathrm{rk}(c_1 + z) &= \mathrm{rk}(c_1 + c_2 + e' + d_e + d_c^\dagger) \\ &= \mathrm{rk}(c_1) + \mathrm{rk}(c_2 + d_c^\dagger) + \mathrm{rk}(e') + \mathrm{rk}(d_e) \\ &= \mathrm{rk}(c_1) + \mathrm{rk}(c_2) + \mathrm{rk}(e') + \mathrm{rk}(d_e) \\ &= \mathrm{rk}(c_1 + c_2 + e') + \mathrm{rk}(d_e) = \mathrm{rk}(e) + \mathrm{rk}(d_e) \\ &\geq \mathrm{rk}(e) = \alpha + \beta \end{aligned}$$

It remains to show that $\mathbb{K}^a_b \cap G^\alpha_\beta$ consists of a single K-orbit if and only if $a = \alpha$ and $b = \beta$. Assume that $a < \alpha$. Then $c_2 \neq 0$, and hence $d_c \neq 0$. Therefore,

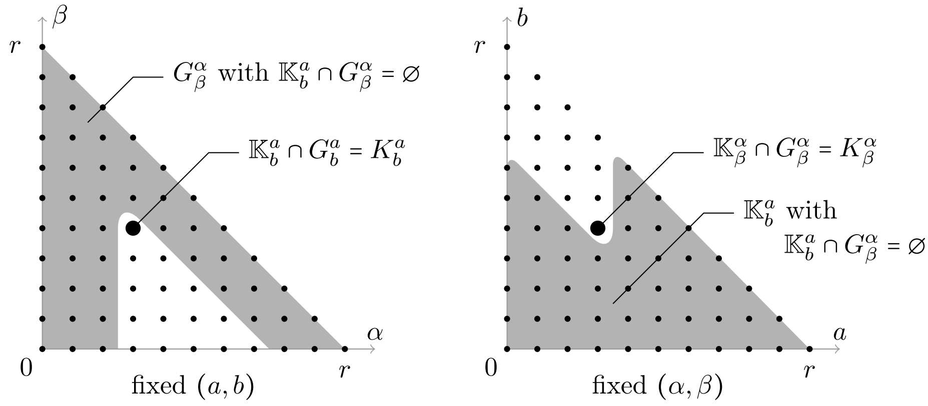
$$\chi_\lambda := [e + d_e : c + \lambda \cdot d_c] = \left[c_1 + (c_2 + e' + d_e - d_c^\dagger/\lambda) : c_1\right] \tag{7.18}$$

is an element of the intersection $\mathbb{K}^a_b \cap G^\alpha_\beta$ for all $\lambda \in (0, 1/|d_c|)$, and each χ_λ belongs to a separate K-orbit, since the K-action respects the spectral values of $\lambda \cdot d_c$. Now assume that $a = \alpha$ and $b > \beta$. In this case, $c_2 = 0$ and $d_c = 0$, so $z = e + d_e$ with $d_e \neq 0$. Again,

$$\chi_\lambda := [e + \lambda \cdot d_e : c] = [c + (e' + \lambda \cdot d_e) : c] \tag{7.19}$$

is an element of the intersection $\mathbb{K}^a_b \cap G^\alpha_\beta$ for all $\lambda \in (0, 1/|d_c|)$, and each χ_λ belongs to a separate K-orbit, since the K-action respects the spectral values of $\lambda \cdot d_e$. In both cases, (7.18) and (7.19), the limit $\lambda \to 1/|d_c|$ yields an element not contained in G^α_β, so the intersection $\mathbb{K}^a_b \cap G^\alpha_\beta$ is non-compact. Finally, assume $a = \alpha$ and $b = \beta$. Then χ is reduced to $\chi = [e : c] = [c + e' : c]$, and any two such elements are joined by an appropriate K-action, hence the intersection consists of a single K-orbit, namely K^a_b. ☐

Remark 7.7. The following two diagrams illustrate the relation between the $K^{\mathbb{C}}$- and the G-orbits on the level of the index set of the orbits.



Comparing these diagrams with figure (7.14), we note that the each complement of the shaded area coincides with the sets of indices (a, b) and (α, β), which describe the closures of $\mathbb{K}^a_b$ and G^{α}_{β}, respectively. Therefore, the intersection of $\mathbb{K}^a_b$ and G^{α}_{β} is non-trivial if and only if $\mathbb{K}^{\alpha}_{\beta}$ is part of the closure of $\mathbb{K}^a_b$, or equivalently if and only if G^a_b is part of the closure of G^{α}_{β}.

Remark 7.8. There is a different Jordan theoretic approach to the proof of the Matsuki duality on the Grassmannian $\mathbb{G}$ of Z using the generalized functional calculus on $\mathbb{G}$, which is describe by W. Kaup in **[19]**, as we already mentioned in Section 7.1, Remark 7.1. One can show that the extended cubic map $\mathbf{c}$ on $\mathbb{G}$ satisfies

$$\mathbf{c}([u + z : u^{\dagger}]) = [u + z^{(3)} : u^{\dagger}] \quad \text{and} \quad \mathbf{c}([e + d_e : c + d_c]) = [e + d_e^{(3)} : c + d_c^{(3)}] \,,$$

where $[u + z : u^{\dagger}]$ and $[e + d_e : c + d_c]$ are given as in Theorem 4.12 and Remark 4.13. Now, the $\pm\infty$-limits of $\mathbf{c}^n(\chi)$ with $\chi = [u + z : u^{\dagger}] = [e + d_e : c + d_c]$ can be used to prove the Matsuki duality similar to the prove given above.

Part 3

Homogeneous Jordan theoretic varieties

CHAPTER 8

Jordan flag varieties

So far, we started with a hermitian Jordan triple system Z and its bounded symmetric domain $\mathcal{D} = \{z \in Z \,|\, |z| < 1\}$, and used the Jordan theoretic description of the compact dual $\mathbb{G}$ of $\mathcal{D}$ to derive a description of the orbit structure of $\mathbb{G}$ under the action of $G = \mathrm{Aut}(\mathcal{D})^0$. As before, let K denote the identity component of the automorphism group $\mathrm{Aut}(Z)$ of Z, and let $G^{\mathbb{C}}$ and $K^{\mathbb{C}}$ be the complexifications of G and K. From the Lie theoretic prospective, we have

$$\mathcal{D} = G/K \quad \text{and} \quad \mathbb{G} = G^{\mathbb{C}}/P \ ,$$

where $P = K^{\mathbb{C}} \ltimes Z^*$ is the semidirect product defined by the action $h \star w := h^{-*}w$ of $h \in K^{\mathbb{C}}$ on $w \in Z^*$. Since the Grassmannian $\mathbb{G}$ is a projective variety, it is also a complete variety, and hence P is a parabolic subgroup of $G^{\mathbb{C}}$.

In the matrix case, the Grassmannian $\mathbb{G}(\mathbb{C}^{r\times s})$ just equals the (ordinary) Grassmannian variety $\mathrm{Gr}_s(\mathbb{C}^{r+s})$, which initiated our terminology, and the parabolic subgroup P is conjugate to the subgroup of invertible upper triangular block matrices of type (s). This Grassmannian variety admits the extensive generalization to *flag varieties*: for any strictly increasing sequence of integers $0 \le i_1 < \ldots < i_k \le r+s$,

$$\mathrm{Gr}_{(i_1,\ldots,i_k)}(\mathbb{C}^{r+s}) = \{0 \subset V_1 \subset \ldots \subset V_k \subset \mathbb{C}^{r+s} \,|\, \dim V_\ell = i_\ell\} \ .$$

is called the flag variety of type $(i_1,\ldots,i_k)$. It turns out that this is a projective variety [**15**], and the Lie theoretic description is given by

$$\mathrm{Gr}_{(i_1,\ldots,i_k)}(\mathbb{C}^{r\times s}) \cong G^{\mathbb{C}}/P' \ ,$$

where $G^{\mathbb{C}} = \mathrm{SL}(r+s)$, and P' is the parabolic subgroup of all invertible upper triangular block matrices of type $(i_1,\ldots,i_k)$ [**13, 10**]. The question arises whether these flag varieties also admit a Jordan theoretic description by the triple system $Z = \mathbb{C}^{r\times s}$. Since the flag variety $\mathrm{Gr}_{(i_1,\ldots,i_k)}(\mathbb{C}^{r\times s})$ does not distinguish between the characteristic numbers of the tripe system, namely r and s, we expect that not all such flag varieties admit a Jordan theoretic realization. However, taking into account that the real form G of $G^{\mathbb{C}}$ preserves the characteristics of Z, it is plausible to expect a Jordan theoretic description of those flag varieties, which are represented by quotients $G^{\mathbb{C}}/P$, where $P = Q^{\mathbb{C}}$ is the complexification of some (real) parabolic subgroup Q of G. This formulation immediately transfers to the general case:

Question: *Given a phJTS Z with unit ball $\mathcal{D}$ and a parabolic subgroup Q of the automorphism group $G = \mathrm{Aut}(\mathcal{D})$. Is there a Jordan-theoretic realization of the projective variety $G^{\mathbb{C}}/Q^{\mathbb{C}}$?*

This chapter is devoted to an affirmative answer. In Section 8.1, we briefly recall the Jordan theoretic description of the real parabolic subgroups of G, which is due to O. Loos [**28**], and determine their complexifications. After investigating the matrix case $Z = \mathbb{C}^{r\times s}$ as a toy model (Section 8.2), we describe in Section 8.3 the general Jordan theoretic model of the quotient $G^{\mathbb{C}}/Q^{\mathbb{C}}$, the Jordan flag variety $\mathbb{F}_J$. By definition, $\mathbb{F}_J$ is just a set of equivalence classes. Using Godement's Theorem, we show that the corresponding equivalence relation is regular, so the Jordan flag variety $\mathbb{F}_J$ is a manifold. In addition, it is proved that $\mathbb{F}_J$ is compact. In Section 8.4, we investigate the analytic and algebraic structure of the Jordan flag variety, and show that $\mathbb{F}_J$ is indeed a smooth algebraic variety. Finally, we define a transitive $G^{\mathbb{C}}$-action on the Jordan flag variety and prove that some stabilizer coincides with the given parabolic subgroup $Q^{\mathbb{C}}$, so $\mathbb{F}_J \cong G^{\mathbb{C}}/Q^{\mathbb{C}}$, which finishes our project.

In the last section provides a description of certain homogeneous line bundles on the Jordan flag variety, which are used in the next chapter.

8.1. Parabolic subgroups

Let Z be a simple phJTS of rank r, let G be the identity component of the automorphism group of the unit ball $\mathcal{D}$, and let $G^{\mathbb{C}}$ be its complexification, or equivalently the identity component of the automorphism group of the Grassmannian $\mathbb{G}(Z)$. In this section we give a Jordan theoretic description of the parabolic subgroups $P \subset G$ and their complexifications $P^{\mathbb{C}} \subset G^{\mathbb{C}}$. For this it suffices to describe the corresponding Lie algebras, since parabolic subgroups are uniquely determined by their Lie algebras. We first investigate the maximal parabolic subgroups, since all others are obtained by appropriate intersections of these.

According to [**28**, §9.21], there exists a bijection between the set of proper maximal parabolic subgroups of G and the set of non-zero tripotents. More explicitly, recall from Theorem 7.3 that the boundary of $\mathcal{D}$ decomposes into r disjoint G-orbits, namely

$$\partial\mathcal{D} = \dot{\bigcup_{j=1}^{r}} G_j^0 \quad \text{with} \quad G_j^0 = \dot{\bigcup_{e\in S_j}} e + \mathcal{D}_0^e\ , \quad \mathcal{D}_0^e = \mathcal{D} \cap Z_0^e\ .$$

Now, for each non-zero tripotent $e \in S$ let Q_e be the stabilizer of the boundary component $J_e = e + \mathcal{D}_0^e$ with respect to the G-action. This indeed is a proper maximal parabolic subgroup of G, and the mentioned bijection between non-zero tripotents and proper maximal parabolic subgroups of G is given by $(e \mapsto Q_e)$.

This generalizes to a bijection between the set of *all* parabolic subgroups of G and the set of *flags of tripotents* in Z, where a flag of tripotents is a k-tuple $(e_1, \ldots, e_k)$ of tripotents such that $0 < e_1 < \ldots < e_k$. This should not be confused with the notion of (pre-)Peirce flags, cf. Section 3.3. The bijection is given by

$$(e_1, \ldots, e_k) \mapsto Q_{(e_1,\ldots,e_k)} \coloneqq Q_{e_1} \cap \ldots \cap Q_{e_k}\ . \tag{8.1}$$

Geometrically, the parabolic subgroup $Q_{(e_1,\ldots,e_k)}$ represents the stabilizer of the successive boundary components $(J_{e_1}, \ldots, J_{e_k})$ with $J_{e_{i+1}} \subset \mathrm{cl}(J_{e_i})$. We call the tuple $(\operatorname{rk} c_1, \ldots, \operatorname{rk} c_k)$ the *type* of the parabolic subgroup $Q_{(e_1,\ldots,e_k)}$. Since K acts transitively on the set of frames, two parabolic subgroups $Q_{(e_1,\ldots,e_k)}$ and $Q_{(c_1,\ldots,c_\ell)}$ are conjugate if and only if $k = \ell$ and $\operatorname{rk}(e_i) = \operatorname{rk}(c_i)$ for all i. Therefore, the type characterizes the conjugacy classes of the parabolic subgroups of G.

In the following we fix a tripotent $e \in S$, and examine the corresponding maximal parabolic subgroup Q_e in more detail. Recall from Section 3.5 that the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of G decomposes into

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \quad \text{with} \quad \mathfrak{k} = \mathrm{Lie}(K)\ , \quad \mathfrak{p} = \{\zeta_v \in \mathfrak{g} \,|\, \zeta_v(z) = v - Q_z v,\ v \in Z\}\ .$$

Here and in the following, we identify the elements of all Lie algebras with vector fields on Z. Due to [**28**, §9.14], the weight space decomposition of $\mathfrak{g}$ with respect to $\langle\zeta_e\rangle \subset \mathfrak{g}$ is given by

$$\begin{aligned}\mathfrak{g}_e^0 &= \mathfrak{k}^e \oplus \{\zeta_v \,|\, v \in Z_+^e \oplus Z_0^e\}\,,\\ \mathfrak{g}_e^{\pm 1} &= \{\zeta_v \mp 2(v \,\square\, e - e \,\square\, v) \,|\, v \in Z_{1/2}^e\}\,,\\ \mathfrak{g}_e^{\pm 2} &= \{\zeta_v \mp 2\, v \,\square\, e \,|\, v \in Z_-^e\}\,,\end{aligned}$$

where $\mathfrak{k}^e = \{\delta \in \mathfrak{k} \,|\, \delta(e) = 0\}$, and the Lie algebra of Q_e decomposes into

$$\mathfrak{q}_e := \mathrm{Lie}(Q_e) = \mathfrak{g}_e^0 \oplus \mathfrak{g}_e^1 \oplus \mathfrak{g}_e^2\,.$$

Moreover, the parabolic subgroup Q_e turns out to be

$$Q_e = (K^e \cdot \exp(\mathfrak{g}_e^0 \cap \mathfrak{p})) \ltimes \big(\exp(\mathfrak{g}_e^1) \cdot \exp(\mathfrak{g}_e^2)\big)\,.$$

This also coincides with a Levi decomposition of Q_e, the first term is a Levi factor, and the second term is the unipotent radical of Q_e.

Conjugation. Before discussing the complexification of Q_e, we investigate an appropriate conjugate of Q_e. Let γ be any element of $G^{\mathbb{C}}$, and set $P_e := \mathrm{Ad}_\gamma(Q_e)$, the γ-conjugate of Q_e. A simple calculation shows

$$\mathrm{Ad}_\gamma(Q_e) = \mathrm{Ad}_\gamma(\mathrm{Stab}_G\,(J_e)) = \mathrm{Stab}_{\mathrm{Ad}_\gamma(G)}\,(\gamma(J_e))\,,$$

i.e. P_e is the stabilizer of $\gamma(J_e)$ with respect to the action of the conjugate subgroup $\mathrm{Ad}_\gamma(G)$. This shows the purpose of conjugation: For an appropriate choice of γ, the elements stabilizing $\gamma(J_e)$ become more simple than the elements stabilizing J_e. The Lie algebras of Q_e and P_e are related by

$$\mathrm{Lie}(\mathrm{Ad}_\gamma(Q_e)) = \mathrm{Ad}_\gamma\,\mathrm{Lie}(P_e)\,.$$

Set $\mathfrak{p}_e := \mathrm{Lie}(P_e)$. Conjugation respects the graduation of $\mathfrak{q}_e$, therefore we have

$$\mathfrak{p}_e = \mathfrak{p}_e^0 \oplus \mathfrak{p}_e^1 \oplus \mathfrak{p}_e^2\,, \quad [\mathfrak{p}_e^\alpha, \mathfrak{p}_e^\beta] \subset \mathfrak{p}_e^{\alpha+\beta} \quad \text{with} \quad \mathfrak{p}_e^\alpha = \mathrm{Ad}_\gamma\,\mathfrak{g}^\alpha\,.$$

Representing elements of the Lie algebras as vector fields, conjugation reads

$$(\mathrm{Ad}_\gamma\,\zeta)(z) = \big(D_{\gamma^{-1}(z)}\gamma\big)\,\zeta(\gamma^{-1}(z)) = \big(D_z\gamma^{-1}\big)^{-1}\,\zeta(\gamma^{-1}(z))\,.$$

Now we specialize to the case $\gamma = \gamma_e^{-1} = \gamma_{-e}$, where γ_u denotes the partial Cayley map with respect to u, defined in Section 4.3. Due to Proposition 4.11, $J_e = e + \mathcal{D}_e^0$ is just translated by γ_e to $\mathcal{D}_e^0$, with now is centered at 0. The conjugation of vector fields becomes

$$(\mathrm{Ad}_{\gamma_{-e}}\,\zeta)(z) = (D_z\gamma_e)^{-1}\,\zeta(\gamma_e(z))\,,$$

and using the relation $D_z\gamma_e = B_{e,-e}^{1/2} B_{z,e}^{-1}$, this yields

$$(\mathrm{Ad}_{\gamma_{-e}}\,\zeta)(z) = B_{z,e} B_{e,-e}^{-1/2}\,\zeta(\gamma_e(z))\,. \tag{8.2}$$

We note that $B_{e,-e}$ is a positive definite operator. We conjugate weight spaces $\mathfrak{p}_e$ are determined by using the following formulas [**28**, §10.5][1]

$$\begin{aligned}
&\mathrm{Ad}_{\gamma_{-e}}(\delta) = \delta && \text{for } \delta \in \mathfrak{k}^e,\\
&\mathrm{Ad}_{\gamma_{-e}}(\zeta_v) = \zeta_v && \text{for } v \in Z_0^e,\\
&\mathrm{Ad}_{\gamma_{-e}}(\zeta_v) = -2\, v \,\square\, e && \text{for } v \in Z_+^e,\\
&\mathrm{Ad}_{\gamma_{-e}}(\zeta_v - 2(v \,\square\, e - e \,\square\, v)) = -\sqrt{2}\,(\tilde{v} + 2\, v \,\square\, e) && \text{for } v \in Z_{1/2}^e,\\
&\mathrm{Ad}_{\gamma_{-e}}(\zeta_v - 2v \,\square\, e) = -2\,\tilde{v} && \text{for } v \in Z_-^e.
\end{aligned}$$

[1] We note that [**28**] uses γ_e for conjugation instead of γ_{-e}. By this, J_e is translated to the infinite part of $\mathbb{G}(Z)$, i.e. $\gamma_e(J_e) \subset \mathbb{G}(Z) \smallsetminus Z$. The formulas for the γ_{-e}-conjugation are obtained form the corresponding γ_e-conjugate formulas by using the $G^{\mathbb{C}}$-involution σ, which stabilizes $G \subset G^{\mathbb{C}}$: Since $\sigma(\gamma_e) = \gamma_{-e}$ and $\sigma(\zeta_e) = \zeta_e$, it follows $\mathfrak{p}_e^\alpha = \mathrm{Ad}_{\gamma_{-e}}\,\mathfrak{g}_e^\alpha = \mathrm{Ad}_{\sigma(\gamma_e)}\,\sigma(\mathfrak{g}_e^\alpha) = \sigma(\mathrm{Ad}_{\gamma_e}\,\mathfrak{g}_e^\alpha)$.

This yields

$$\begin{aligned}
\mathfrak{p}_e^0 &= \mathrm{Ad}_{\gamma_{-e}}\left(\mathfrak{g}^0\right) = \mathfrak{k}^e \oplus \{\zeta_v \,|\, v \in Z_0^e\} \oplus \{v \,\square\, e \,|\, v \in Z_+^e\}\,,\\
\mathfrak{p}_e^1 &= \mathrm{Ad}_{\gamma_{-e}}\left(\mathfrak{g}^1\right) = \{\tilde{v} + 2v \,\square\, e \,|\, v \in Z_{1/2}^e\}\,,\\
\mathfrak{p}_e^2 &= \mathrm{Ad}_{\gamma_{-e}}\left(\mathfrak{g}^2\right) = \{\tilde{v} \,|\, v \in Z_-^e\}\,.
\end{aligned}$$

Complexification. Considering $\mathfrak{g}$ as a real Lie subalgebra of the complex Lie algebra of vector fields on Z, we obtain the relation $\mathfrak{g} \cap i\mathfrak{g} = \{0\}$. Hence, the complexification of a subspace $\mathfrak{h} \subset \mathfrak{g}$ is given by $\mathfrak{h}^{\mathbb{C}} = \mathfrak{h} \oplus i\mathfrak{h}$. This also hold for the conjugate subspaces $(\mathrm{Ad}_\gamma \mathfrak{h})^{\mathbb{C}} = \mathrm{Ad}_\gamma \mathfrak{h} \oplus i\, \mathrm{Ad}_\gamma \mathfrak{h}$. We determine the complexification of $\mathfrak{k}^e$ and of $\{\zeta_v \,|\, v \in Z_0^e\}$ explicitly:

$$\begin{aligned}
(\mathfrak{k}^e)^{\mathbb{C}} &= \{\delta_1 + i\delta_2 \,|\, \delta_i \in \mathfrak{k}^e\}\\
&= \{\delta_1 + i\delta_2 \,|\, \delta_i \in \mathfrak{k},\ \delta_i(e) = 0\}\\
&= \{\delta_1 + i\delta_2 \,|\, \delta_i \in \mathfrak{k},\ \delta_i(e) = \delta_i^*(e) = 0\}\\
&= \{\delta \in \mathfrak{k}^{\mathbb{C}} \,|\, \delta(e) = \delta^*(e) = 0\}\,.
\end{aligned} \tag{8.3}$$

$$\begin{aligned}
\{\zeta_v \,|\, v \in Z_0^e\}^{\mathbb{C}} &= \{\zeta_v + i\zeta_w \,|\, v, w \in Z_0^e\}\\
&= \{(v + i\,w) - (\tilde{v} + i\,\tilde{w}) \,|\, v, w \in Z_0^e\}\\
&= \{(v + i\,w) - (\widetilde{v - iw}) \,|\, v, w \in Z_0^e\}\\
&= \{a + \tilde{b} \,|\, a, b \in Z_0^e\}\\
&= \{a \,|\, a \in Z_0^e\} \oplus \{\tilde{b} \,|\, b \in Z_0^e\}\,.
\end{aligned}$$

The complexification of the other subspaces are obtained by similar arguments. In total, the complexification of the weight spaces $\mathfrak{p}_e^\alpha$ is given by

$$\begin{aligned}
\left(\mathfrak{p}_e^0\right)^{\mathbb{C}} &= (\mathfrak{k}^e)^{\mathbb{C}} \oplus \{a \,|\, a \in Z_0^e\} \oplus \{\tilde{b} \,|\, b \in \overline{Z}_0^e\} \oplus \{c \,\square\, e \,|\, c \in Z_1^e\}\,,\\
\left(\mathfrak{p}_e^1\right)^{\mathbb{C}} &= \{\tilde{a} \,|\, a \in \overline{Z}_{1/2}^e\} \oplus \{b \,\square\, e \,|\, b \in Z_{1/2}^e\}\,,\\
\left(\mathfrak{p}_e^2\right)^{\mathbb{C}} &= \{\tilde{a} \,|\, a \in \overline{Z}_1^e\}\,.
\end{aligned} \tag{8.4}$$

These subspaces fit together nicely: Let $\mathfrak{k}^{\mathbb{C}}(e)$ denote the subspace of $\mathfrak{k}^{\mathbb{C}}$ given by

$$\mathfrak{k}^{\mathbb{C}}(e) := \left\{\delta \in \mathfrak{k}^{\mathbb{C}} \,\middle|\, \delta(Z_0^e) \subset Z_0^e,\ \delta^*(Z_1^e) \subset Z_1^e\right\}\,.$$

In fact, this is a complex Lie subalgebra of $\mathfrak{k}^{\mathbb{C}}$.

Proposition 8.1. *For a tripotent $e \in S$ let $Q_e \subset G$ be the stabilizer subgroup of $J_e = e + \mathcal{D}_0^e$, and let $P_e = \mathrm{Ad}_{\gamma_{-e}} Q_e$. Then*

$$\begin{aligned}
\mathfrak{p}_e^{\mathbb{C}} &= \{\tilde{a} \,|\, a \in \overline{Z}\} \oplus \mathfrak{k}^{\mathbb{C}}(e) \oplus \{b \,|\, b \in Z_0^e\}\\
&\cong \overline{Z} \oplus \mathfrak{k}^{\mathbb{C}}(e) \oplus Z_0^e\,,
\end{aligned} \tag{8.5}$$

according to the complex grading $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{u}^+$. For two tripotents $e, c \in S$ the complexified Lie algebras $\mathfrak{p}_e^{\mathbb{C}}$ and $\mathfrak{p}_c^{\mathbb{C}}$ coincide if and only if e and c are Peirce equivalent.

Proof. We just have to show the relation $\mathfrak{k}^{\mathbb{C}}(e) = (\mathfrak{k}^e)^{\mathbb{C}} \oplus \{a \,\square\, e \,|\, a \in Z_1^e \oplus Z_{1/2}^e\}$, since all other terms obviously coincide. We first prove the inclusion '$\supset$': For $\delta \in (\mathfrak{k}^e)^{\mathbb{C}}$ and $v \in Z_0^e$, $w \in Z_1^e$, relation (8.3) implies

$$0 = \delta\,\{e, e, v\} = \{\delta(e), e, v\} - \{e, \delta^*(e), v\} + \{e, e, \delta(v)\} = (e \,\square\, e)(\delta(v))\,,$$

$$\delta^* w = \delta^*(\{e, e, w\}) = \{\delta^*(e), e, w\} - \{e, \delta(e), w\} + \{e, e, \delta^*(w)\} = (e \,\square\, e)(\delta^*(w))\,,$$

i.e. Z_0^e is δ-invariant, and Z_1^e is δ^*-invariant, and hence $\delta \in \mathfrak{k}^{\mathbb{C}}(e)$. For $a \in Z$, the Peirce rules imply $a \,\square\, e(Z_0^e) = \{0\}$ and $e \,\square\, a(Z_1^e) \subset Z_1^e$, so $a \,\square\, e$ is also an element of

$\mathfrak{k}^{\mathbb{C}}(e)$. Now we turn to the proof of the opposite inclusion '$\subset$'. Let δ be an element of $\mathfrak{k}^{\mathbb{C}}(e)$, and let $\delta(e) = \delta_1 + \delta_{1/2} + \delta_0$ be the Peirce decomposition of $\delta(e)$ with respect to e. Since δ is a derivation, we obtain

$$\delta(e) = \delta\{e, e, e\} = 2\{\delta(e), e, e\} - \{e, \delta^*(e), e\} = 2\delta_1 + \delta_{1/2} - Q_e\delta^*(e) \in Z_1^e \oplus Z_{1/2}^e .$$

Therefore, δ_0 vanishes. Now set $a = 2\delta(e) - Q_e^2\delta(e)$. A simple calculation shows $a = \delta_1 + 2\delta_{1/2}$. We claim that $\delta' = \delta - a \,\Box\, e$ is an element of $(\mathfrak{k}^e)^{\mathbb{C}}$. For this, $\delta'(e) = \delta'^*(e) = 0$ must be verified. We obtain

$$\delta'(e) = \delta(e) - a \,\Box\, e(e) = \delta(e) - \delta_1 - \delta_2 = \delta(e) - \delta(e) = 0 ,$$

$$\delta'^*(e) = \delta^*(e) - e \,\Box\, a(e) = \delta^*(e) - Q_e(\delta_1) = \delta^*(e) - Q_e\delta(e) \stackrel{?}{=} 0 .$$

For the prove of the last equation, we note that by assumption $\delta^*(e)$ is an element of Z_1^e, and using the derivation property of δ^*, we obtain

$$\delta^*(e) = \delta^*\{e, e, e\} = 2\{\delta^*(e), e, e\} - \{e, \delta(e), e\} = 2\delta^*(e) - Q_e\delta(e) ,$$

i.e. $\delta^*(e) = Q_e\delta(e)$, and the proof is complete. □

The following lemma shows that the defining relations of $\mathfrak{k}^{\mathbb{C}}(e)$ are redundant.

Lemma 8.2. *Let e be a tripotent, and let δ be an element of $\mathfrak{k}^{\mathbb{C}}$. If Z_1^e is δ^*-invariant, then Z_0^e is δ-invariant. In particular,*

$$\mathfrak{k}^{\mathbb{C}}(e) = \left((\mathfrak{k}^{\mathbb{C}})^{Z_1^e}\right)^* = \left\{\delta \in \mathfrak{k}^{\mathbb{C}} \,\middle|\, \delta^*(Z_1^e) \subset Z_1^e\right\} .$$

Proof. Let v be an element of Z_0^e. We have to show that $\delta(v)$ is also an element of Z_0^e. Since $Z_0^e = (Z_1^e)^{\perp\!\!\perp}$, it is equivalent to show that $\delta(v) \,\Box\, u = 0$ for all $u \in Z_1^u$. Since δ is a derivation, we have

$$0 = \delta \circ (v \,\Box\, u) = \delta(v) \,\Box\, u - v \,\Box\, \delta^*(u) + (v \,\Box\, u) \circ \delta = \delta(v) \,\Box\, u ,$$

since $v \,\Box\, u = v \,\Box\, \delta^*(u) = 0$ by (strong) orthogonality and by the assumption that Z_1^u is δ^*-invariant. □

Remark 8.3. We note that in general, the converse of Lemma 8.2 is not true: The invariance of Z_0^e under δ does not imply in general the invariance of Z_1^e under δ^*. The proof relies on the fact that $Z_0^e = (Z_1^e)^{\perp\!\!\perp}$. Conversely, just the inclusion $(Z_0^e)^{\perp\!\!\perp} \supset Z_1^e$ is valid, cf. Remark 2.12.

Now we return to the general case of a flag of tripotents $(e_1, \ldots, e_k)$. Before we state the main result of this section, we note that due to Proposition 8.1, the complexification $\mathfrak{p}_e^{\mathbb{C}}$ of $\mathfrak{p}_e$ merely depends on the Peirce equivalence class of e. Therefore, we extend the notation to arbitrary elements: For $u \in Z$, set

$$\mathfrak{p}_u^{\mathbb{C}} = \{\tilde{a} \,|\, a \in \overline{Z}\} \oplus \mathfrak{k}^{\mathbb{C}}(u) \oplus \{b \,|\, b \in Z_0^u\} \quad \text{with} \quad \mathfrak{k}^{\mathbb{C}}(u) = \left\{\delta \in \mathfrak{k}^{\mathbb{C}} \,\middle|\, \delta^*(Z_1^u) \subset Z_1^u\right\} .$$

Obviously, $\mathfrak{p}_u^{\mathbb{C}} = \mathfrak{p}_{\tilde{u}}^{\mathbb{C}}$ if and only if u is Peirce equivalent to $\tilde{u}$. More generally, if $(u_1, \ldots, u_k)$ is a pre-Peirce flag of type J, set

$$\mathfrak{p}_{(u_1,\ldots,u_k)}^{\mathbb{C}} := \left\{\tilde{a} \,\middle|\, a \in \overline{Z}\right\} \oplus \mathfrak{k}^{\mathbb{C}}(u_1, \ldots, u_k) \oplus \{b \,|\, b \in Z_0^{u_k}\} , \tag{8.6}$$

with $\mathfrak{k}^{\mathbb{C}}(u_1, \ldots, u_k) = \left\{\delta \in \mathfrak{k}^{\mathbb{C}} \,\middle|\, \delta^*(Z_1^{u_i}) \subset Z_1^{u_i} \text{ for all } i\right\}$. It easily checked that

$$\mathfrak{p}_{(u_1,\ldots,u_k)}^{\mathbb{C}} = \mathfrak{p}_{u_1}^{\mathbb{C}} \cap \ldots \cap \mathfrak{p}_{u_k}^{\mathbb{C}} . \tag{8.7}$$

It follows by a simple inductive argument that $\mathfrak{p}_{(u_1,\ldots,u_k)}^{\mathbb{C}} = \mathfrak{p}_{(\tilde{u}_1,\ldots,\tilde{u}_k)}^{\mathbb{C}}$ if and only if $(u_1, \ldots, u_k)$ and $(\tilde{u}_1, \ldots, \tilde{u}_k)$ are Peirce equivalent. We note that even though the single subalgebras $\mathfrak{p}_{u_i}^{\mathbb{C}}$ are parabolic, it is not obvious that the subalgebra $\mathfrak{p}_{(u_1,\ldots,u_k)}^{\mathbb{C}}$ is also parabolic. So far, we just know that the intersections (8.1) lead to parabolic subalgebras $\mathfrak{q}_{(e_1,\ldots,e_k)} = \mathfrak{q}_{e_1} \cap \ldots \cap \mathfrak{q}_{e_k}$, but since the $\mathfrak{p}_{u_i}^{\mathbb{C}}$ are conjugates of the $\mathfrak{q}_{e_i}^{\mathbb{C}}$

with respect to *different* elements γ_{e_i}, it is not clear that intersections of the $\mathfrak{p}^{\mathbb{C}}_{u_i}$ are parabolic. The next Theorem ensures this fact.

Theorem 8.4. *Let Z be a simple phJTS, let $(e_1,\ldots,e_k)$ be a flag of tripotents, and let $Q_{(e_1,\ldots,e_k)}$ be the corresponding parabolic subgroup of G with Lie algebra $\mathfrak{q}_{(e_1,\ldots,e_k)}$. Then, for any pre-Peirce flag $(u_1,\ldots,u_k)$ such that (u_i) and (e_i) are Peirce equivalent,*

$$\mathfrak{p}^{\mathbb{C}}_{(u_1,\ldots,u_k)} = \mathrm{Ad}_{\gamma_{-e_k}}\mathfrak{q}^{\mathbb{C}}_{(e_1,\ldots,e_k)}\ .$$

In particular, $\mathfrak{p}^{\mathbb{C}}_{(u_1,\ldots,u_k)}$ is a complex parabolic subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $P^{\mathbb{C}}_{(u_1,\ldots,u_k)}$ denote the corresponding parabolic subgroup, then

$$P^{\mathbb{C}}_{(u_1,\ldots,u_k)} = \mathrm{Ad}_{\gamma_{-e_k}}Q^{\mathbb{C}}_{(e_1,\ldots,e_k)}\ .$$

Proof. We first prove that for orthogonal tripotents $e \perp\!\!\!\perp c$, we have $\mathrm{Ad}_{\gamma_c}\mathfrak{p}^{\mathbb{C}}_e = \mathfrak{p}^{\mathbb{C}}_e$. Since Ad_{γ_c} is invertible, it suffices to prove the inclusion '$\subset$'. Due to Proposition 8.1, $\mathfrak{p}^{\mathbb{C}}_e$ decomposes into $\mathfrak{p}^{\mathbb{C}}_e \cong \overline{Z}\oplus\mathfrak{k}^{\mathbb{C}}(e)\oplus Z^e_0$. We discuss each term separately.
(1) For $a\in\overline{Z}$, we have to evaluate $\mathrm{Ad}_{\gamma_c}(\tilde{a})$. Recall that $\gamma_c = t_c B^{1/2}_{c,-c}\tilde{t}_c$. Therefore,

$$\begin{aligned}\mathrm{Ad}_{\gamma_c}(\tilde{a})(z) &= \tfrac{d}{d\tau}\Big|_{\tau=0}\gamma_c\tilde{t}_{\tau a}\gamma_c^{-1} = \tfrac{d}{d\tau}\Big|_{\tau=0}t_cB^{1/2}_{c,-c}\tilde{t}_c\tilde{t}_{\tau a}\tilde{t}_{-c}B^{-1/2}_{c,-c}t_{-c}\\ &= \tfrac{d}{d\tau}\Big|_{\tau=0}t_c\tilde{t}_{\tau B^{-1/2}_{-c,c}a}t_{-c} = Q_{z-c}\big(B^{-1/2}_{-c,c}a\big)\ .\end{aligned}$$

Setting $a' = B^{-1/2}_{-c,c}a$, this decomposes into

$$\mathrm{Ad}_{\gamma_c}(\tilde{a})(z) = Q_za' - 2\,c\,\square\,a'(z) + Q_ca'\ .$$

The first term is the vector field of a quasi-translation, and by the Peirce rules it follows that the second term is an element of $\mathfrak{k}^{\mathbb{C}}(e)$, and the third term belongs to Z^e_0. Hence, $\mathrm{Ad}_{\gamma_c}(\tilde{a})$ is an element of $\mathfrak{p}^{\mathbb{C}}_e$.
(2) For $\delta\in\mathfrak{k}^{\mathbb{C}}(e)$ let $\mathrm{Ad}_{\gamma_c}(\delta) = \tilde{a}+\delta'+b$ be the decomposition with respect to $\mathfrak{g}^{\mathbb{C}}\cong\overline{Z}\oplus\mathfrak{k}^{\mathbb{C}}\oplus Z$. Since $\gamma_c^{-1}(0) = -c$, we have due to (8.2) on the one hand

$$\mathrm{Ad}_{\gamma_c}(\delta)(0) = B^{-1/2}_{-c,c}(\delta(-c))\ ,$$

and on the other hand $(\tilde{a}+\delta'+b)(0) = b$. Since δ and $B^{-1/2}_{-c,c}$ preserve the Peirce 0-space Z^e_0, we conclude that b is an element of Z^e_0. It remains to show that δ' belongs to $\mathfrak{k}^{\mathbb{C}}(e)$, i.e. that $\delta'^*(Z^e_1)\subset Z^e_1$. Applying the complex conjugation θ of $G^{\mathbb{C}}$ to $\mathrm{Ad}_{\gamma_c}(\delta)$, we obtain from (4.18) and (4.32) the relation

$$\mathrm{Ad}_{\gamma_c}(\delta^*) = a+\delta'^*+\tilde{b}\ .$$

Since Z^e_1 is a subspace of Z^c_0, and since γ_c acts identically on Z^c_0, it follows

$$\delta^*(z) = \mathrm{Ad}_{\gamma_c}(\delta^*)(z) = a+\delta'^*(z)+Q_zb = \delta'^*(z)\quad\text{for all}\quad z\in Z^e_1\ ,$$

since choosing $z=0$ yields $a=0$, and by the Peirce rules $Q_zb=0$ for all $z\in Z^e_1$. Therefore, δ'^* coincides in the restriction to Z^e_1 with δ^*, and hence Z^e_1 is also invariant under δ'^*.
(3) For $b\in Z^e_0$, we apply (8.2) and obtain $\mathrm{Ad}_{\gamma_c}b(z) = B_{z,-c}B^{-1/2}_{c,-c}b$. Extending the first Bergman operator and setting $b' = B^{-1/2}_{c,-c}b$ yields

$$\mathrm{Ad}_{\gamma_c}b(z) = b'+2\,b'\,\square\,c(z)+Q_zQ_cb'\ .$$

Since b and c are elements of Z^e_0, it follows that b' also belongs to Z^e_0. Therefore, $(b'\square c)^*(Z^e_1) = c\square b'(Z^e_1) = \{0\}$, and hence $2\,b'\square c$ is an element of $\mathfrak{k}^{\mathbb{C}}(e)$. We conclude that $\mathrm{Ad}_{\gamma_c}b$ belongs to $\mathfrak{p}^{\mathbb{C}}_e$. This finally completes the proof of $\mathrm{Ad}_{\gamma_c}\mathfrak{p}^{\mathbb{C}}_e = \mathfrak{p}^{\mathbb{C}}_e$.
Now we apply this relation to the flag of tripotents $(e_1,\ldots,e_k)$. Since $\gamma_{c+e} = \gamma_c\circ\gamma_e$ for orthogonal tripotents, we obtain

$$\mathfrak{p}^{\mathbb{C}}_{e_i} = \mathrm{Ad}_{\gamma_{e_i-e_k}}\mathfrak{p}^{\mathbb{C}}_{e_i} = \mathrm{Ad}_{\gamma_{e_i-e_k}}\big(\mathrm{Ad}_{\gamma_{-e_i}}\mathfrak{q}_{e_i}\big)^{\mathbb{C}} = \mathrm{Ad}_{\gamma_{e_i-e_k}}\mathrm{Ad}_{\gamma_{-e_i}}\mathfrak{q}^{\mathbb{C}}_{e_i} = \mathrm{Ad}_{\gamma_{-e_k}}\mathfrak{q}^{\mathbb{C}}_{e_i}\ .$$

Here, we used that conjugation and complexification commute. Finally, this implies

$$\mathfrak{p}^{\mathbb{C}}_{(e_1,\ldots,e_k)} = \mathrm{Ad}_{\gamma_{-e_k}} \left(\mathfrak{q}^{\mathbb{C}}_{e_1} \cap \ldots \cap \mathfrak{q}^{\mathbb{C}}_{e_i}\right) = \mathrm{Ad}_{\gamma_{-e_k}} \left(\mathfrak{q}_{e_1} \cap \ldots \cap \mathfrak{q}_{e_k}\right)^{\mathbb{C}} = \mathrm{Ad}_{\gamma_{-e_k}} \mathfrak{q}^{\mathbb{C}}_{(e_1,\ldots,e_k)} .$$

Therefore, $\mathfrak{p}^{\mathbb{C}}_{(e_1,\ldots,e_k)}$ is parabolic. The corresponding relation on the group level follows from the fact that a parabolic subgroup equals the normalizer of its Lie algebra. □

REMARK 8.5. Let $\mathfrak{P}^{\mathbb{C}}_G$ be the set of parabolic subalgebras of $\mathfrak{g}^{\mathbb{C}}$ which are conjugate to the complexification of some parabolic subalgebra of $\mathfrak{g}$. Then, Theorem 8.4 shows that the map

$$\bigcup_J \mathbb{P}_J \to \mathfrak{P}^{\mathbb{C}}_G,\ [u_1,\ldots,u_k] \mapsto \mathfrak{p}^{\mathbb{C}}_{(u_1,\ldots,u_k)} \tag{8.8}$$

is well-defined and injective. The question of surjectivity remains open.

EXAMPLE 8.6. We determine the parabolic subgroups described by Theorem 8.4 for the matrix case $Z = \mathbb{C}^{r\times s}$. Set $n = r \times s$. We have[2]

$$G^{\mathbb{C}} = \mathrm{SL}(n)\ ,\quad G = \mathrm{SU}(r,s)\ ,\quad K^{\mathbb{C}} = \mathrm{S}(\mathrm{GL}(r)\times\mathrm{GL}(s))\ ,\quad K = \mathrm{S}(\mathrm{U}(r)\times\mathrm{U}(s))\ .$$

Therefore, the Lie algebra of $G^{\mathbb{C}}$ is given by the space of $n\times n$-matrices with vanishing trance, and Example 4.9 shows that these matrices correspond to the vector fields

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(r+s)\times(r+s)} \quad \rightleftarrows \quad \zeta(z) = Az - zD + B - zCz\ ,$$

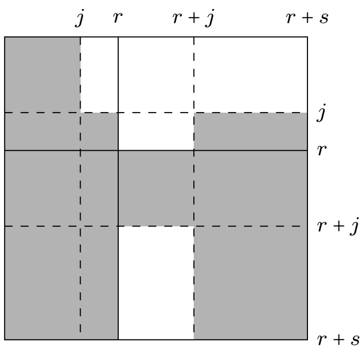
where the matrix is devided into blocks of sizes corresponding to r and s. Hence, the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{u}^+$ is given by

$$\mathfrak{u}^- = \left\{\left(\begin{smallmatrix} 0 & 0 \\ C & 0 \end{smallmatrix}\right) \,\middle|\, C \in \mathbb{C}^{s\times r}\right\}\ ,\quad \mathfrak{u}^+ = \left\{\left(\begin{smallmatrix} 0 & B \\ 0 & 0 \end{smallmatrix}\right) \,\middle|\, B \in \mathbb{C}^{r\times s}\right\}\ ,$$

$$\mathfrak{k}^{\mathbb{C}} = \left\{\left(\begin{smallmatrix} A & 0 \\ 0 & D \end{smallmatrix}\right) \in \mathbb{C}^{n\times n} \,\middle|\, A \in \mathbb{C}^{r\times r},\ D \in \mathbb{C}^{s\times s},\ \mathrm{Tr}\,A + \mathrm{Tr}\,D = 0\right\}\ .$$

Now, it is straightforward to show that for block diagonal $u \in Z$, the parabolic subalgebra $\mathfrak{p}^{\mathbb{C}}_u = \{\tilde{a}\,|\,a \in \overline{Z}\} \oplus \mathfrak{k}^{\mathbb{C}}(u) \oplus \{b\,|\,b \in Z^u_0\}$ is represented by the following matrices, which vanish on the blank area, admit arbitrary entries on the shaded area and have zero trace

$\mathfrak{p}^{\mathbb{C}}_u \cong$



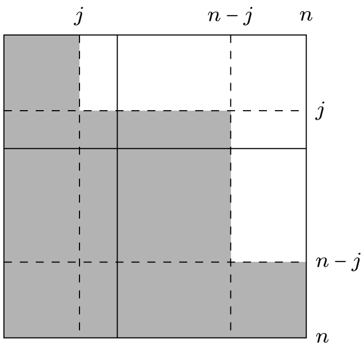
for $u = \left(\begin{array}{c|c} \mathbf{1}_j & 0 \\ \hline 0 & 0 \end{array}\right)$.

Interchanging the last two rows with each other and equally the last two columns (which corresponds to the conjugation by an appropriate permutation matrix T)

[2]cf. Example 3.30.

yields

$$\mathrm{Ad}_T(\mathfrak{p}_u^{\mathbb{C}}) \cong \tag{8.9}$$



Therefore, the parabolic Lie algebra $\mathfrak{p}_u^{\mathbb{C}}$ is conjugate to the set of lower block diagonal matrices (with vanishing trance) of type $(j, n-j)$, and the corresponding parabolic Lie group $P_u^{\mathbb{C}}$ is conjugate to the set of invertible lower block diagonal matrices (with determinant 1) of type $(j, n-j)$. We conclude that

$$G^{\mathbb{C}}/P_u^{\mathbb{C}} \cong \mathrm{Gr}_{(j,n-j)}(\mathbb{C}^n) = \{E \subset F \subset \mathbb{C}^n \,|\, \dim E = j,\ \dim F = n-j\}\ . \tag{8.10}$$

In the general case of a pre-Peirce flag $(u_1, \ldots, u_k)$ of type $(j_1, \ldots, j_k)$ we obtain intersections of (8.9) for different j. Therefore, the parabolic Lie group $P_{(u_i)}^{\mathbb{C}}$ is conjugate to the set of determinant-1 lower block diagonal matrices of type $I = (j_1, \ldots, j_k, n-j_k, \ldots, n-j_1)$, and hence $G^{\mathbb{C}}/P_{(u_i)}^{\mathbb{C}}$ is isomorphic to the Grassmannian variety of type I,

$$\mathrm{Gr}_I(\mathbb{C}^n) = \{E_1 \subset \ldots \subset E_k \subset F_k \subset \ldots \subset F_1 \subset \mathbb{C}^n \,|\, \dim E_\ell = j_\ell,\ \dim F_\ell = n - j_\ell\}\ .$$

In the next section we give a Jordan theoretic description of this variety, which motivates the general definition of Jordan flag varieties in Section 8.3.

8.2. Motivating example

In this section, we concentrate on the matrix case $Z = \mathbb{C}^{r\times s}$ and develope a Jordan theoretic model for the classical Grassmannian flag varieties $\mathrm{Gr}_I(\mathbb{C}^n)$ of type $I = (j_1, \ldots, j_k, n-j_k, \ldots, n-j_1)$ for $n = r+s$ and $0 \le j_1 < \ldots < j_k \le r$. According to Example 8.6, these are exactly the flag varieties obtained from quotiens $G^{\mathbb{C}}/Q^{\mathbb{C}}$ with $G^{\mathbb{C}} = \mathrm{SL}(r+s)$ and a parabolic subgroup $Q \subset G = \mathrm{SU}(r,s)$. In this section, the the elements of $\mathrm{Gr}_I(\mathbb{C}^n)$ are called (*subspace*) *flags of type* I.

We fix some notation: Let J denote the tuple $(j_1, \ldots, j_r)$, and let J^c be

$$J^c := (n-j_k, \ldots, n-j_1)\ .$$

Then, we write $I = J \sqcup J^c = (j_1, \ldots, j_k, n-j_1, \ldots, n-j_k)$. We first examine the case $J = (j)$, i.e. $I = (j, n-j)$ with $0 \le j \le r$. The design of the Jordan theoretic model for the Grassmannian flag variety $\mathrm{Gr}_I(\mathbb{C}^n)$ is a based on O. Loos' construction for the Grassmannian variety $\mathrm{Gr}_s(\mathbb{C}^n)$. For convenience, we recall

Lemma 4.1. *Let $n = r + s$, and for $(z,a) \in Z \times \overline{Z}$ set*

$$C_{z,a} := \begin{pmatrix} z \\ 1 - a^* z \end{pmatrix} \in \mathbb{C}^{n\times s} \quad \textit{and} \quad E_{z,a} := \textit{column space of } C_{z,a}\ .$$

Then:

(a) *$E_{z,a}$ is an s-dimensional subspace of $\mathbb{C}^n$.*

(b) *Each subspace $E \subset \mathbb{C}^n$ with $\dim E = s$ has a representation $E = E_{z,a}$ with $(z,a) \in Z \times \overline{Z}$.*

(c) *Two subspaces* $E_{z,a}$ *and* $E_{\tilde{z},\tilde{a}}$ *coincide if and only if* $(z, a-\tilde{a})$ *is quasi-invertible and* $\tilde{z} = z^{a-\tilde{a}}$, *i.e.* $1_s - (a-\tilde{a})^* z \in \mathbb{C}^{s\times s}$ *is invertible and* $\tilde{z} = z(1-(a-\tilde{a})^* z)^{-1}$.

Before stating the generalization of this lemma, recall some facts about Peirce decompositions in the matrix case: Let u be an element of $Z = \mathbb{C}^{r\times s}$. Then

$$Z_1^u = \{z \in Z \mid z = Tu = uS \text{ for some } T \in \mathbb{C}^{r\times r},\ S \in \mathbb{C}^{s\times s}\}\ , \tag{8.11}$$

In addition, $z \in Z_1^u$ is invertible in Z_1^u if and only if T and S can be chosen invertible.

$$Z_0^u = \{z \in Z \mid uz^* = 0 \text{ and } z^* u = 0\}\ . \tag{8.12}$$

From this it follows, that two elements $z, w \in \mathbb{C}^{r\times s}$ are *equal modulo* Z_0^u if and only if $uz^* = uw^*$ and $z^*u = w^*u$. Furthermore recall, that a pair (z,a) with $z, a \in \mathbb{C}^{r\times s}$ is quasi-invertible if and only if $(1-za^*)$ is invertible (or equivalently, if $(1-a^*z)$ is invertible), and the quasi-inverse of (z,a) is given by

$$z^a = (1-za^*)^{-1} z = z(1-a^*z)^{-1}\ . \tag{8.13}$$

Now we are prepared to state and prove the key observation of this section.

Lemma 8.7 (First generalization)**.** *Let* Z *be the phJTS* $Z = \mathbb{C}^{r\times s}$, *and for* $u, z, a \in Z$ *set*

$$C_{u,z,a} := \begin{pmatrix} z \\ 1-a^*z \end{pmatrix} u^* \in \mathbb{C}^{n\times s} \quad \text{and} \quad D_{u,z,a} := \begin{pmatrix} 1-az^* \\ -z^* \end{pmatrix} u \in \mathbb{C}^{n\times s}$$

with $n = r+s$. *Furthermore let* $E_{u,z,a}$ *be the column space of* $C_{u,z,a}$ *and* $F_{u,z,a}$ *be the orthogonal complement of the column space of* $D_{u,z,a}$, *with respect to the ordinary inner product on* $\mathbb{C}^n$, *i.e.*

$$E_{u,z,a} = \langle C_{u,z,a}\rangle \quad \text{and} \quad F_{u,z,a} = \langle D_{u,z,a}\rangle^\perp\ .$$

Then:

(a) *The pair* $(E_{u,z,a}, F_{u,z,a})$ *is a flag of type* $(\operatorname{rk} u,\ n-\operatorname{rk} u)$.

(b) *For each flag* (E,F) *of type* $(j, n-j)$ *there exist some* $u, z, a \in Z$ *with* $\operatorname{rk} u = j$, *such that* $(E,F) = (E_{u,z,a}, F_{u,z,a})$.

(c) *Two flags* $(E_{u,z,a}, F_{u,z,a})$ *and* $(E_{\tilde{u},\tilde{z},\tilde{a}}, F_{\tilde{u},\tilde{z},\tilde{a}})$ *coincide if and only if*

   (i) $B_{a-\tilde{a},\,z} u$ *and* $\tilde{u}$ *are Peirce equivalent,*

   (ii) there exist elements $u^{\perp\!\!\!\perp} \in Z_0^u$ *and* $\tilde{u}^{\perp\!\!\!\perp} \in Z_0^{\tilde{u}}$ *such that* $(z+u^{\perp\!\!\!\perp}, a-\tilde{a})$ *is quasi-invertible and* $(z+u^{\perp\!\!\!\perp})^{a-\tilde{a}} = \tilde{z} + \tilde{u}^{\perp\!\!\!\perp}$.

Proof. For *(a)*, recall the rank formula $\operatorname{rk}(XY^*) = \operatorname{rk} Y^*$ for matrices $X, Y \in \mathbb{C}^{n\times s}$ with $\operatorname{rk} X = s$. Thus by Lemma 4.1 we have $\operatorname{rk} C_{u,z,a} = \operatorname{rk} u^* = \operatorname{rk} u$, so $\dim E_{u,z,a} = \operatorname{rk} u$. The same argument shows that $D_{u,z,a}$ has the same rank as u, so that $F_{u,z,a}$ has dimension $n - \operatorname{rk} u$. The inclusion $E_{u,z,a} \subset F_{u,z,a}$ follows from

$$\langle C_{u,z,a}x | D_{u,z,a}y\rangle = (D_{u,z,a}y)^*(C_{u,z,a}x) = y^*u^*\big(1-za^*,\ -z\big)\begin{pmatrix} z \\ 1-a^*z \end{pmatrix} u^*x = 0$$

for any $x \in \mathbb{C}^r$, $y \in \mathbb{C}^s$.

To prove *(b)*, we first choose an arbitrary s-dimensional subspace E' with $E \subset E' \subset F$. Using the notation of Lemma 4.1 we can find $z, a \in \mathbb{C}^{r\times s}$ such that $E' = E_{z,a}$. Set

$$C_{z,a} = \begin{pmatrix} z \\ 1-a^*z \end{pmatrix} \in \mathbb{C}^{n\times s} \quad \text{and} \quad D_{z,a} = \begin{pmatrix} 1-az^* \\ -z^* \end{pmatrix} \in \mathbb{C}^{n\times s}\ .$$

Let $\{b_1, \ldots, b_j\}$ be a basis of E, then there exist $t_i \in \mathbb{C}^s$ with $b_i = C_{z,a}t_i$. Complete $(t_1, \ldots, t_j)$ to a basis of $\mathbb{C}^s$ and set $T := (t_1, \ldots, t_s)$. Analogously let $\{b'_1, \ldots, b'_j\}$

be a basis of $F^\perp$, take $s_i \in \mathbb{C}^r$ with $b'_i = D_{z,a}s_i$ and complete this to a basis $S := (s_1, \ldots, s_r)$ of $\mathbb{C}^r$. Now we can choose

$$u := S\begin{pmatrix} 1_j & 0 \\ 0 & 0 \end{pmatrix} T^* ,$$

since $\langle C_{z,a}u^*\rangle = \langle b_1, \ldots, b_j\rangle = E$ and $\langle D_{z,a}u\rangle^\perp = \left\langle b'_1, \ldots, b'_j\right\rangle^\perp = (F^\perp)^\perp = F$.

For *(c)*, the equality of the two flags is equivalent to the existence of matrices $g \in \mathrm{GL}(\mathbb{C}^r)$ and $h \in \mathrm{GL}(\mathbb{C}^s)$ such that

$$C_{u,z,a} = C_{\tilde{u},\tilde{z},\tilde{a}}\,g \quad \text{and} \quad D_{u,z,a} = D_{\tilde{u},\tilde{z},\tilde{a}}\,h .$$

This again is equivalent to the following system of equations

$$\begin{aligned} &(1)\quad uz^* = g^*\tilde{u}\tilde{z}^* , &&(2)\quad u(1 - z^*a) = g^*\tilde{u}(1 - \tilde{z}^*\tilde{a}) , \\ &(3)\quad z^*u = \tilde{z}^*\tilde{u}h , &&(4)\quad (1 - az^*)u = (1 - \tilde{a}\tilde{z}^*)\tilde{u}h . \end{aligned}$$

We first show the 'only if'-part of the assertion. Plugging (1) into (2) and (3) into (4), respectively, we get

$$(2')\quad g^*\tilde{u} = u(1 - z^*(a - \tilde{a})) , \qquad (4')\quad \tilde{u}h = (1 - (a - \tilde{a})z^*)u .$$

For brevity we set $L = 1 - (a - \tilde{a})z^*$ and $R = 1 - z^*(a - \tilde{a})$, so

$$(2')\quad g^*\tilde{u} = uR , \qquad (4')\quad \tilde{u}h = Lu .$$

Multiplying $(2')$ from the left by L and $(4')$ from the right by R, we see that $B_{a-\tilde{a},\,z}u = LuR$ is an element of $Z_1^{\tilde{u}}$ (using (8.11)). To verify *(i)*, we have to show in addition, that $B_{a-\tilde{a},\,z}u$ has the same rank as $\tilde{u}$. This follows from Frobenius' rank inequality[3] and the observation, that $(2')$ and $(4')$ imply

$$\mathrm{rk}(Lu) = \mathrm{rk}(\tilde{u}) = \mathrm{rk}(uR) .$$

For *(ii)*, we defer the first part to the next Lemma, it assures the existence of an element $u^{\perp\!\!\!\perp} \in Z_0^u$, such that, respectively, $L' := 1 - (a - \tilde{a})(z + u^{\perp\!\!\!\perp})^*$ and $R' := 1 - (z + u^{\perp\!\!\!\perp})^*(a - \tilde{a})$ are invertible. Due to (8.12) we can replace z by $z' := z + u^{\perp\!\!\!\perp}$ in all equations above. Plugging $(2')$ into (1) and $(4')$ into (3) , respectively, we obtain

$$(1')\quad uz'^* = uR'\tilde{z}^* , \qquad (3')\quad z'^*u = \tilde{z}^*L'u$$

Multiplying $(1')$ from the left by L' and inserting $R'R'^{-1}$ between u and z'^*, and perform analog operations on $(3')$, it follows

$$(1'')\ (B_{a-\tilde{a},\,z'}u)R'^{-1}z'^* = \tilde{z}^*(B_{a-\tilde{a},\,z'}u) , \quad (3'')\ z'^*L'^{-1}(B_{a-\tilde{a},\,z'}u) = \tilde{z}^*(B_{a-\tilde{a},\,z'}u) .$$

Now since

$$B_{a-\tilde{a},\,z'}u = B_{a-\tilde{a},\,z}u \quad \text{and} \quad ((z')^{a-\tilde{a}})^* = R'^{-1}z'^* = z'^*L'^{-1} ,$$

equations $(1'')$ and $(3'')$ imply, that $(z')^{a-\tilde{a}}$ and $\tilde{z}$ are equal modulo $Z_0^{B_{a-\tilde{a},\,z'}u} = Z_0^{\tilde{u}}$. Thus there exists an element $\tilde{u}^{\perp\!\!\!\perp}$ in $Z_0^{\tilde{u}}$, such that

$$(z + u^{\perp\!\!\!\perp})^{a-\tilde{a}} = \tilde{z} + \tilde{u}^{\perp\!\!\!\perp} .$$

Finally we have to prove the 'if'-part of the assertion. By assumption $B_{a-\tilde{a},\,z}u$ and $\tilde{u}$ are Peirce equivalent, so there are invertible matrices $T \in \mathrm{GL}(\mathbb{C}^r)$ and $S \in \mathrm{GL}(\mathbb{C}^s)$, such that

$$B_{a-\tilde{a},\,z}u = T\tilde{u} \quad \text{and} \quad B_{a-\tilde{a},\,z}u = \tilde{u}S .$$

Now we can choose $g := T(L')^{-1}$ and $h := S(R')^{-1}$ with the notation above, and it is a simple computation to verify $C_{u,z,a} = C_{\tilde{u},\tilde{z},\tilde{a}}\,g$ and $D_{u,z,a} = D_{\tilde{u},\tilde{z},\tilde{a}}\,h$. □

In the last proof we needed the following technical result.

[3]For any matrices A, B, C such that ABC exists, then $\mathrm{rk}\,AB + \mathrm{rk}\,BC \le \mathrm{rk}\,ABC + \mathrm{rk}\,B$.

Lemma 8.8. *Let $x, y, u \in \mathbb{C}^{r\times s}$ be matrices with*

$$\operatorname{rk}(1 - xy^*)u(1 - y^*x) = \operatorname{rk} u\,. \tag{8.14}$$

Then there exists an element $u^{\perp\!\!\perp} \in Z_0^u$ such that $(1 - x(y + u^{\perp\!\!\perp})^*)$ is invertible.

> Unfortunately, up to the day of publication, we have not established a complete proof of this assertion. However, we note that the results outside of this chapter do *not* rely on this assertion. The above lemma is used just in this chapter, which serves as a motivation for the definition of Jordan flag varieties given in the next section.

Partial proof. We first assume that u is of maximal rank. In this case we have $Z_0^u = \{0\}$, and Lemma 8.8 asserts that $(1 - xy^*)$ itself is invertible. This is evident from the assumption, which always implies $\operatorname{rk}(1 - xy^*) \geq \operatorname{rk} u$, and so if u is of maximal rank, then $(1 - xy^*)$ is also of maximal rank, and hence invertible. Also the case $\operatorname{rk} u = 0$ is trivial, since $u = 0$ implies $Z_0^u = Z$, and we may choose $u^{\perp\!\!\perp} = -y$. Now let u be of rank $0 < j < r$. We may assume that u, x and y are given by

$$u = \begin{pmatrix} \mathbf{1}_j & 0 \\ 0 & 0 \end{pmatrix}\,, \quad x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \quad y = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\,. \tag{8.15}$$

Indeed, it is straightforward to check that the statement of Lemma 8.8 is invariant under the exchange of u, x and y by $\tilde{u} = AuD^{-1}$, $\tilde{x} = AxD^{-1}$ and $\tilde{y} = A^{-*}uD^*$ for arbitrary matrices $A \in \mathrm{GL}(r)$, $D \in \mathrm{GL}(s)$. Due to the singular value decomposition of u, the matrices A and D can be chosen to achieve $\tilde{u} = \left(\begin{smallmatrix} \mathbf{1}_j & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Next, we claim that the assumption $\operatorname{rk}(1 - xy^*)u(1 - y^*x) = \operatorname{rk} u$ is equivalent to the statement that

$$\operatorname{rk}(1 - xy^*)u = \operatorname{rk} u\,, \quad \operatorname{rk} u(1 - y^*x) = \operatorname{rk} u\,. \tag{8.16}$$

By the rank inequality $\operatorname{rk} AB \leq \max(\operatorname{rk} A, \operatorname{rk} B)$, we obtain the inequalities $\operatorname{rk}(1 - xy^*)u \leq \operatorname{rk} u$ and $\operatorname{rk} u(1 - y^*x) \leq \operatorname{rk} u$ without any further assumption. Assuming (8.14), the rank inequality also implies the converse inequalities, so we obtain (8.16). Now assume (8.16), then Frobenius' rank inequality[3] implies that

$$\operatorname{rk}(1 - xy^*)u(1 - y^*x) \geq \operatorname{rk}(1 - xy^*)u + \operatorname{rk} u(1 - y^*x) - \operatorname{rk} u = \operatorname{rk} u\,.$$

The converse inequality is obvious since $\operatorname{rk} u = j$. Therefore, (8.14) is indeed equivalent to (8.16). Using (8.15), we therefore obtain the assumption

$$\operatorname{rk}\begin{pmatrix} \mathbf{1}_j - a\alpha^* - b\beta^* \\ -c\alpha^* - d\beta^* \end{pmatrix} = j\,, \quad \operatorname{rk}\begin{pmatrix} \mathbf{1}_j - \alpha^*a - \gamma^*c\,, & -\alpha^*b - \gamma^*d \end{pmatrix} = j\,, \tag{8.17}$$

i.e. these two matrices are of maximal rank. Now the statement to prove is that there exists a δ such that the matrix

$$1 - xy^* = \begin{pmatrix} \mathbf{1}_j - a\alpha^* - b\beta^* & -a\gamma^* - b\delta^* \\ -c\alpha^* - d\beta^* & 1 - c\gamma^* - d\delta^* \end{pmatrix}$$

is invertible. Let $f(\delta) := \det(1 - xy^*)$ be the determinant of this matrix considered as a function on δ. Then, f is a polynomial, and it remains to show that f is not trivial, i.e. different from the zero polynomial. For the special case $r = s = 2$ (and $j = 1$) this can be proved by explicit (but still non-trivial) calculations. So far, the general case remains unsolved. △

Remark 8.9. We note that the statement of Lemma 8.8 immediately generalizes to the following statement on positive hermitian Jordan triple systems:

> Let Z be a Jordan triple system, and let $(x, y) \in Z \times \overline{Z}$ and $u \in Z$. If $\operatorname{rk} B_{x,y}u = \operatorname{rk} u$, then there exists an element $u^{\perp\!\!\perp} \in Z_0^u$, such that $B_{x,\, y+u^{\perp\!\!\perp}}$ is invertible.

It might be easier to give an abstract proof of this statement.

Up to now we derived a Jordan theoretic model for the flag varieties $\mathrm{Gr}_{(j,n-j)}(\mathbb{C}^n)$. Next we investigate the general case. Recall that a tuple $(u_1,\dots,u_k)$ of elements in a Jordan triple system is called a *Peirce flag of type* $J=(j_1,\dots,j_k)$, if $Z_1^{u_1}\subset Z_1^{u_2}\subset\dots\subset Z_1^{u_k}$ and $\operatorname{rk} u_i = j_i$ for all $i=1,\dots,k$. For brevity, we write (u_i) for the tuple $(u_1,\dots,u_k)$.

Lemma 8.10 (Second generalization). *Let Z be the phJTS $Z=\mathbb{C}^{r\times s}$, and for $u,z,a\in Z$ let $C_{u,z,a}$, $D_{u,z,a}$, $E_{u,z,a}$ and $F_{u,z,a}$ be as in Lemma 8.7. Then:*

(a) *For any Peirce flag (u_i) of type J and $z,a\in Z$, the tuple*

$$\mathcal{F}_{(u_i),z,a} := (E_{u_1,z,a},\dots,E_{u_k,z,a},F_{u_k,z,a},\dots,F_{u_1,z,a})$$

is a flag of ype $J\sqcup J^c$.

(b) *For each flag $\mathcal{F}=(E_1,\dots,E_k,F_k,\dots F_1)$ of type $J\sqcup J^c$, there exist a Peirce flag (u_i) of type J and $z,a\in Z$ such that $\mathcal{F}=\mathcal{F}_{(u_i),z,a}$.*

(c) *Two flags $\mathcal{F}_{(u_i),z,a}$ and $\mathcal{F}_{(\tilde u_i),\tilde z,\tilde a}$ coincide if and only if*

  (i) $B_{a-\tilde a,z}u_i$ and $\tilde u_i$ are Peirce equivalent for $i=1,\dots,k$,

  (ii) the exist elements $u^{\perp\!\!\perp}\in Z_0^{u_k}$ and $\tilde u^{\perp\!\!\perp}\in Z_0^{\tilde u_k}$ such that $(z+u^{\perp\!\!\perp},a-\tilde a)$ is quasi-invertible and $(z+u^{\perp\!\!\perp})^{a-\tilde a}=\tilde z+\tilde u^{\perp\!\!\perp}$.

Proof. For *(a)*, it is sufficient to show that for fixed $z,a\in Z$ and $u_1,u_2\in Z$ with $u_1\in Z_1^{u_2}$, we have

$$E_{u_1,z,a}\subset E_{u_2,z,a}\subset F_{u_2,z,a}\subset F_{u_1,z,a}\ .$$

By (8.11), there exist $T\in\mathbb{C}^{r\times r}$ and $S\in\mathbb{C}^{s\times s}$, such that $u_1=Tu_2=u_2S$, so the first inclusion follows from $C_{u_1,z,a}=C_{u_2,z,a}T^*$. The second inclusion is proved in Lemma 8.7, and the third is equivalent to $F^\perp_{u_1,z,a}\subset F^\perp_{u_2,z,a}$, what again follows from $D_{u_1,z,a}=D_{u_2,z,a}S$.

The proof of *(b)* is just a refinement of the respective proof of Lemma 8.7. We only have to choose the vectors $b_i,b_i'\in\mathbb{C}^n$ more carefully: Let $(b_1,\dots,b_{j_1})$ be a basis of E_1, extend this to a basis of E_2, then to a basis of E_3 and so forth, up to a basis of E_k. Similarly start with a basis $(b_1',\dots,b_{j_1}')$ of $F_1^\perp$ and extend this successively to a basis of $F_k^\perp$. Now we choose an r-dimensional subspace $E'\subset\mathbb{C}^n$ with $E_k\subset E'\subset F_k$, and some representation $E'=E_{z,a}$ with $z,a\in Z$ as in Lemma 4.1. Again set

$$C_{z,a}=\begin{pmatrix} z\\ 1-a^*z\end{pmatrix}\in\mathbb{C}^{n\times s}\quad\text{and}\quad D_{z,a}=\begin{pmatrix}1-az^*\\ -z^*\end{pmatrix}\in\mathbb{C}^{n\times s}\ ,$$

and define t_i and s_i by the equations $b_i=C_{z,a}t_i$ and $b_i'=D_{z,a}s_i$, respectively. Completing $(t_1,\dots,t_{j_k})$ and $(s_1,\dots,s_{j_k})$ to respective bases we obtain invertible matrices $T=(t_1,\dots,t_s)\in\mathbb{C}^{s\times s}$ and $S=(s_1,\dots,s_r)\in\mathbb{C}^{r\times r}$. Finally we set

$$u_i := S\begin{pmatrix}1_{j_i} & 0\\ 0 & 0\end{pmatrix}T^*\qquad\text{for } i=1,\dots,k.$$

This indeed defines a Peirce flag (u_i), and since $\langle C_{z,a}u_i^*\rangle=\langle b_1,\dots,b_{j_i}\rangle=E_i$ and $\langle D_{z,a}u_i\rangle^\perp=\langle b_1',\dots,b_{j_i}'\rangle^\perp=(F_i^\perp)^\perp=F_i$, we have $\mathcal{F}=\mathcal{F}_{\mathcal{U},z,a}$.

For *(c)*, we can restrict the given flags $\mathcal{F}_{(u_i),z,a}$ and $\mathcal{F}_{(\tilde u_i),\tilde z,\tilde a}$ to the flags $(E_{u_i,z,a},F_{u_i,z,a})$ and $(E_{\tilde u_i,\tilde z,\tilde a},F_{\tilde u_i,\tilde z,\tilde a})$ of type $(j_i,n-j_i)$ for $i=1,\dots,k$. Using Lemma 8.7 this implies *(i)* and *(ii)*. To prove the converse, we note that $Z_1^u\subset Z_1^{\tilde u}$ implies $Z_0^{\tilde u}\subset Z_0^u$, so *(ii)* remains true if we replace the index k by any $i=1,\dots,k$. So we can apply Lemma 8.7 for each $i=1,\dots,k$ separately, and thus obtain the equality of the flags $(E_{u_i,z,a},F_{u_i,z,a})$ and $(E_{\tilde u_i,\tilde z,\tilde a},F_{\tilde u_i,\tilde z,\tilde a})$. ☐

8.3. Basic definition of Jordan flag varieties

In this section, we provide a general definition of Jordan flag varieties. Let Z be a phJTS of rank r, and let $0 \le j_1 < \ldots < j_k \le r$ be an increasing family of integers; set $J = (j_1, \ldots, j_k)$. As in Section 3.3, let Z_J denote the pre-Peirce flag variety given by

$$Z_J = \{(u_1, \ldots, u_k) \,|\, u_1 \subset \ldots \subset u_k,\ \mathrm{rk}\, u_i = j_i \text{ for } i = 1, \ldots, k\}\ ,$$

where $u_i \subset u_j$ is equivalent to $Z_1^{u_i} \subset Z_1^{u_j}$. Using the conjugate phJTS $\overline{Z}$ instead of Z, we obtain the conjugate pre-Peirce flag variety $\overline{Z}_J$, which coincides with Z_J as set but carries the opposite complex structure. The elements of $\overline{Z}_J$ are denoted by (u_i). Section 8.2 motivates the following construction.

Theorem 8.11. *For elements $((u_i), z, a)$ and $((\tilde{u}_i), \tilde{z}, \tilde{a})$ in $\overline{Z}_J \times Z \times \overline{Z}$ let*

$$\big((u_i), z, a\big) \sim \big((\tilde{u}_i), \tilde{z}, \tilde{a}\big) \iff \begin{cases} \tilde{u}_i \approx B_{a-\tilde{a},\, z} u_i \text{ for } i = 1, \ldots, k; \text{ and} \\ \text{there exist } u^{\perp\!\!\perp} \in Z_0^{u_k} \text{ and } \tilde{u}^{\perp\!\!\perp} \in Z_0^{\tilde{u}_k}, \\ \text{such that } B_{a-\tilde{a},\, z+u^{\perp\!\!\perp}} \text{ is invertible} \\ \text{and } \tilde{z} + \tilde{u}^{\perp\!\!\perp} = (z + u^{\perp\!\!\perp})^{a-\tilde{a}}. \end{cases} \tag{8.18}$$

This defines a regular equivalence relation on $\overline{Z}_J \times Z \times \overline{Z}$. The quotient

$$\mathbb{F}_J = (\overline{Z}_J \times Z \times \overline{Z})/\sim \tag{8.19}$$

is a compact complex manifold, called the Jordan flag variety of type J*. The elements of $\mathbb{F}_J$ are denoted by $[(u_i) : z : a]$.*

Proof. The proof of this theorem takes several steps. First we give an alternative formulation of the relation (8.18). It is easy to see that for fixed $a \in \overline{Z}$ two elements $((u_i), z, a)$ and $((\tilde{u}_i), \tilde{z}, a)$ are related via (8.18) if and only if u_i and $\tilde{u}_i$ are Peirce equivalent for all i, and z equals $\tilde{z}$ modulo $Z_0^{u_k} = Z_0^{\tilde{u}_k}$. This observation indicates the following construction: Let $\overline{\mathbb{P}}_J = Z_J/\approx$ be the (conjugate) Peirce flag manifold of type J as discussed in Section 6.4. We identify the elements of $\overline{\mathbb{P}}_J$ according to Remark 6.16 with increasing sequences of Peirce 1-spaces of appropriate rank, i.e.

$$\overline{\mathbb{P}}_J = \{(U_1, \ldots, U_k) \,|\, U_i \subset Z \text{ Peirce 1-space of rank } j_i,\ U_1 \subset \ldots \subset U_k\}\ .$$

We recall from Remark 2.12 that a Peirce decomposition is uniquely determined by its Peirce 1-space, and in particular, if U denotes the Peirce 1-space, then the corresponding Peirce 0-space is given by $U^{\perp\!\!\perp}$. Let $\overline{\mathcal{E}}_J$ be the vector bundle on $\overline{\mathbb{P}}_J$ defined by

$$\overline{\mathcal{E}}_J = \big\{\big((U_i),\, [z]\big) \,\big|\, (U_1, \ldots, U_k) \in \overline{\mathbb{P}}_J,\ [z] \in Z/U_k^{\perp\!\!\perp}\big\}\ , \tag{8.20}$$

where $Z/U_k^{\perp\!\!\perp}$ is the usual quotient of vector spaces, and $[z] = z + U_k^{\perp\!\!\perp}$ denotes the corresponding equivalence class of $z \in Z$. Furthermore, for any subset $M \subset Z$ and $a \in \overline{Z}$ let M^a be the set $M^a = \{z^a \,|\, z \in M,\ (z, a) \text{ quasi-invertible}\}$, and denote by $\mathrm{cl}(M) \subset Z$ the Zariski closure of M in Z. Since quasi-invertibility is an algebraic condition, the set V^a is dense in $\mathrm{cl}(V^a)$ for any (affine) subvariety $V \subset Z$.

Claim 1. *For elements $((U_i), [z], a)$ and $((\tilde{U}_i), [\tilde{z}], \tilde{a})$ in $\overline{\mathcal{E}}_J \times \overline{Z}$ let*

$$\big((U_i), [z], a\big) \sim \big((\tilde{U}_i), [\tilde{z}], \tilde{a}\big) \iff \begin{cases} B_{a-\tilde{a},\, z} \tilde{U}_i = U_i \text{ for all } i, \\ \text{and } [\tilde{z}] = cl([z]^{a-\tilde{a}})\ . \end{cases} \tag{8.21}$$

Then, two elements $((u_i), z, a)$ and $((\tilde{u}_i), \tilde{z}, \tilde{a})$ in $\overline{Z}_J \times Z \times \overline{Z}$ are related with respect to (8.18) if and only if the corresponding elements $((Z_1^{u_i}), [z], a)$ and $((Z_1^{\tilde{u}_i}), [\tilde{z}], \tilde{a})$ in $\overline{\mathcal{E}}_J \times \overline{Z}$ are related with respect to (8.21).

PROOF. First we notice, that the term $B_{a-\tilde{a},\,z}U_i$ in (8.21) is independent of the choice of $z+u^{\amalg}\in[z]$, since by the Peirce rules we have $B_{a-\tilde{a},\,z+u^{\amalg}}U_i = B_{a-\tilde{a},\,z}U_i$ for any $u^{\amalg}\in U_k^{\amalg}\subset U_i^{\amalg}$. Therefore, (8.21) is a well-defined relation on $\overline{\mathcal{E}}_J\times\overline{Z}$. Now let $((u_i),z,a)$ and $((\tilde{u}_i),\tilde{z},\tilde{a})$ be related elements of $\overline{Z}_J\times Z\times\overline{Z}$. For brevity we set $U_i = Z_1^{u_i}$, and so $U_i^{\amalg} = Z_0^{u_i}$. Consider the set of all $u^{\amalg}\in U_k^{\amalg}$, such that $B_{a-\tilde{a},\,z+u^{\amalg}}$ is invertible. By assumption this set is non-empty, and since invertibility is an algebraic condition, this set is also Zariski open, and therefore dense in $U_k^{\amalg}$. For such an element $u^{\amalg}$, the Bergman operator $B_{a-\tilde{a},\,z+u^{\amalg}}$ is a structure automorphism of Z. By the Peirce rules and Lemma 2.32, we obtain

$$B_{a-\tilde{a},\,z}U_i = B_{a-\tilde{a},\,z+u^{\amalg}}Z_1^{u_i} = Z_1^{B_{a-\tilde{a},\,z+u^{\amalg}}u_i} = Z_1^{B_{a-\tilde{a},\,z}u_i} = Z_1^{\tilde{u}_i} = \tilde{U}_i\ .$$

Secondly we have to show the equality $[\tilde{z}]$ of $\mathrm{cl}([z]^{a-\tilde{a}})$. By assumption there is an element $u_1^{\amalg}\in U_k^{\amalg}$ with $(z+u_1^{\amalg})^{a-\tilde{a}}\in[\tilde{z}]$. Then for the generic element $u_2^{\amalg}$ in $U_k^{\amalg}$, using the addition formula for the quasi-inverse we obtain

$$(z+u_2^{\amalg})^{a-\tilde{a}} = (z+u_1^{\amalg}+(u_2^{\amalg}-u_1^{\amalg}))^{a-\tilde{a}} = (z+u_1^{\amalg})^{a-\tilde{a}} + B^{-1}_{z-u_1^{\amalg},\,a-\tilde{a}}((u_2^{\amalg}-u_1^{\amalg})^{\nu})\ ,$$

where $\nu = (a-\tilde{a})^{z+u_1^{\amalg}}$. By assumption, the first term is an element of $\tilde{U}_k^{\amalg}$. Since $u_2^{\amalg}-u_1^{\amalg}$ is an element of $U_k^{\amalg}$, Lemma 2.25 ensures that $(u_2^{\amalg}-u_1^{\amalg})^{\nu}$ stays in $U_k^{\amalg}$, so again by Lemma 2.32, the second term is also an element of $\tilde{U}_k^{\amalg}$. This proves the inclusion $\mathrm{cl}([z]^{a-\tilde{a}})\subset[\tilde{z}]$. Since the map $(x\mapsto x^{a-\tilde{a}})$ is birational on Z, the reverse inclusion follows by dimension. The converse of the assertion is obvious. □

By Claim 1, it is still not obvious that (8.18) defines an equivalence relation on $\overline{Z}_J\times Z\times\overline{Z}$, even though now it suffices to prove that (8.21) is an equivalence relation on $\overline{\mathcal{E}}_J\times\overline{Z}$. For this we give yet another description of (8.21) by leaving the 'affine picture', which uses Zariski closure in Z, and passing to the compactification, whereby using closures in $\mathbb{G}(Z)$. Recall from Section 4.1 that the Grassmannian $\mathbb{G}(U)$ of a subtriple $U\subset Z$ can be identified with the closure of U in the Grassmannian $\mathbb{G}(Z)$ of Z. By this means (quasi-)translations of $\mathbb{G}(Z)$ can be applied to $\mathbb{G}(U)$.

CLAIM 2. *Two elements $((U_i),[z],a)$ and $((\tilde{U}_i),[\tilde{z}],\tilde{a})$ in $\overline{\mathcal{E}}_J\times\overline{Z}$ are related via* (8.21) *if and only if*

$$B_{a-\tilde{a},\,z}U_i = \tilde{U}_i \quad \text{for all } i, \text{ and} \quad \tilde{t}_a t_z\mathbb{G}(U_k^{\amalg}) = \tilde{t}_{\tilde{a}}t_{\tilde{z}}\mathbb{G}(\tilde{U}_k^{\amalg})\ . \tag{8.22}$$

This is an equivalence relation on $\overline{\mathcal{E}}_J\times\overline{Z}$, and hence (8.18) *is an equivalence relation on $\overline{Z}_J\times Z\times\overline{Z}$.*

PROOF. As in the case of (8.21), we first remark that the translation is independent of the choice of $z+u^{\amalg}\in[z]$, since $\mathbb{G}(U_k^{\amalg})$ is invariant under translation with $u^{\amalg}\in U_k^{\amalg}$. Therefore, (8.22) is well-defined. It is obvious that if two elements of $\mathcal{Z}_J\times\overline{Z}$ are related with respect to (8.21), then they are also related with respect to (8.22). For the converse we have to show that if $\tilde{t}_a t_z\mathbb{G}(U_k^{\amalg}) = \tilde{t}_{\tilde{a}}t_{\tilde{z}}\mathbb{G}(\tilde{U}_k^{\amalg})$, we can find elements $u^{\amalg}\in U_k^{\amalg}$ and $\tilde{u}^{\amalg}\in\tilde{U}_k^{\amalg}$ (i.e. in the *finite* parts of $\mathbb{G}(U_k^{\amalg})$ and $\mathbb{G}(\tilde{U}_k^{\amalg})$, respectively), such that $\tilde{t}_a t_z u^{\amalg} = \tilde{t}_{\tilde{a}}t_{\tilde{z}}\tilde{u}^{\amalg}$. Since $U_k^{\amalg}$ is open and dense in $\mathbb{G}(U_k^{\amalg})$ and $\tilde{t}_a t_z$ is a morphism of $\mathbb{G}(Z)$, the image $\tilde{t}_a t_z U_k^{\amalg}$ is also open and dense in $\tilde{t}_{\tilde{a}}t_{\tilde{z}}\mathbb{G}(\tilde{U}_k^{\amalg})$ and therefore open and dense in $\tilde{t}_{\tilde{a}}t_{\tilde{z}}\tilde{U}_k$.

Finally we show that (8.22) describes an equivalence relation on $\overline{\mathcal{E}}_J\times\overline{Z}$, and hence (8.21) and (8.18) also do. Reflexivity is obvious, since in this case it is $B_{0,\,z} = \mathrm{Id}$. For the symmetry we just have to show that $U_i = B_{\tilde{a}-a,\,\tilde{z}}\tilde{U}_i$ for all i. By assumption there are elements $u^{\amalg}\in U_k^{\amalg}$ and $\tilde{u}^{\amalg}\in\tilde{U}_k^{\amalg}$, such that $\tilde{t}_a t_z u^{\amalg} = \tilde{t}_{\tilde{a}}t_{\tilde{z}}\tilde{u}^{\amalg}$, or equivalently

$$B_{z+u^{\amalg},\,a-\tilde{a}} \text{ is invertible and } (z+u^{\amalg})^{a-\tilde{a}} = \tilde{z}+\tilde{u}^{\amalg}\ .$$

Therefore, the Bergman operator $B^*_{z+u^{\amalg},\,a-\tilde a} = B_{a-\tilde a,\,z+u^{\amalg}}$ is invertible as well, and by using JT35, we obtain

$$U = B^{-1}_{a-\tilde a,\,z+u^{\amalg}}\tilde U = B_{\tilde a-a,\,(z+u^{\amalg})^{a-\tilde a}}\tilde U = B_{\tilde a-a,\,\tilde z+\tilde u^{\amalg}}\tilde U = B_{\tilde a-a,\,\tilde z}\tilde U\ .$$

To prove transitivity, let $((U_i),[z],a)$ be related to $((\tilde U_i),[\tilde z],\tilde a)$ and $((\tilde U_i),[\tilde z],\tilde a)$ be related to $((\hat U_i),[\hat z],\hat a)$; we have to show the relation between $((U_i),[z],a)$ and $((\hat U_i),[\hat z],\hat a)$. By assumption, there are elements $u^{\amalg}\in U^{\amalg}$, $\tilde u^{\amalg}\in\tilde U^{\amalg}$ and $\hat u^{\amalg}\in\hat U^{\amalg}$, such that $\tilde t_a t_z u^{\amalg} = \tilde t_{\tilde a} t_{\tilde z}\tilde u^{\amalg} = \tilde t_{\hat a} t_{\hat z}\hat u^{\amalg}$. Using the Peirce rules, JT34 and JT35, we therefore obtain

$$\begin{aligned}\hat U &= B_{\tilde a-\hat a,\,\tilde z}\tilde U = B_{\tilde a-\hat a,\,\tilde z+\tilde u^{\amalg}}\tilde U = B_{\tilde a-\hat a,\,\tilde z+\tilde u^{\amalg}}B_{a-\tilde a,\,z+u^{\amalg}}U = B_{\tilde a-\hat a,\,\tilde z+\tilde u^{\amalg}}B^{-1}_{\tilde a-a,\,(z+u^{\amalg})^{a-\tilde a}}U\\ &= B_{\tilde a-\hat a,\,\tilde z+\tilde u^{\amalg}}B^{-1}_{\tilde a-a,\,\tilde z+\tilde u^{\amalg}}U = B_{a-\tilde a,\,(\tilde z+\tilde u^{\amalg})^{\tilde a-a}}U = B_{a-\tilde a,\,z+u^{\amalg}}U = B_{a-\tilde a,\,z}U\ .\end{aligned}$$

This completes the proof of Claim 2. □

Now we turn to the proof of the regularity of the equivalence relation (8.18). We have to show (i) that

$$R_{\mathbb F_J} = \{((u_i),z,a),\,((\tilde u_i),\tilde z,\tilde a)\,|\,((u_i),z,a)\sim((\tilde u_i),\tilde z,\tilde a)\}\subset(\overline Z_J\times Z\times\overline Z)^2$$

is a submanifold of $(\overline Z_J\times Z\times\overline Z)^2$ and (ii) that the projection pr_1 of $R_{\mathbb F_J}$ onto the first component is a submersion. In order to keep the notation clear, we first discuss the case $k=1$, i.e. $J=(j)$. For (i), we use Proposition 3.4 and show that $R_{\mathbb F_j}$ is locally given as the level set of a submersion. Recall from Section 3.3 that for $u\in\overline Z_j$, the set

$$\overline{\mathcal I}{}^u_j = \left\{v\in\overline Z_j\,\middle|\,v_1\text{ invertible in }\overline Z{}^u_1\right\}$$

is open and dense in $\overline Z_j$, and any element $v\in\overline{\mathcal I}{}^u_j$ satisfies

$$v = \tau_{v_1,v_{1/2}}v_1\quad\text{with}\quad v = v_1\oplus v_{1/2}\oplus v_0\in\overline Z{}^u_1\oplus\overline Z{}^u_{1/2}\oplus\overline Z{}^u_0\ ,\tag{8.23}$$

where $\tau_{v_1,v_{1/2}} = B_{v_{1/2},-v_1^\dagger}$ is a Frobenius transformation. Due to Lemma 2.32, we therefore obtain

$$Z^v_1 = \tau_{v_1,v_{1/2}}Z^{v_1}_1 = \tau_{v_1,v_{1/2}}Z^u_1\ ,\quad\text{and hence}\quad Z^v_0 = \tau^{-*}_{v_1,v_{1/2}}Z^u_0 = B_{v_1^\dagger,\,v_{1/2}}Z^u_0\ .\tag{8.24}$$

We note that since $v_1^\dagger = Q_u v_1^{-1}$, the Bergman operator $B_{v_1^\dagger,\,v_{1/2}}$ depends holomorphically (even rationally) on $v\in\overline{\mathcal I}{}^u_j$. Now fix some element $\big((u,z,a),(\tilde u,\tilde z,\tilde a)\big)\in R_{\mathbb F_j}$, and fix $u^{\amalg}\in Z^u_0$ and $\tilde u^{\amalg}\in Z^{\tilde u}_0$ with $\tilde z+\tilde u^{\amalg} = (z+u^{\amalg})^{a-\tilde a}$. Consider the set U of all $\big((v,w,b),(\tilde v,\tilde w,\tilde b)\big)$ in $(\overline Z_j\times Z\times\overline Z)^2$ with

$$v\in\overline{\mathcal I}{}^u_j\ ,\quad\tilde v\in\overline{\mathcal I}{}^{\tilde u}_j\ ,\quad(w+B_{v_1^\dagger,\,v_{1/2}}u^{\amalg},\,b-\tilde b)\text{ quasi-invertible},\quad B_{\tilde b-b,\,\tilde w}\tilde v\in\overline{\mathcal I}{}^u_j\ .$$

Since $\big((u,z,a),(\tilde u,\tilde z,\tilde a)\big)\in U$, this set is non-empty, and since the condition of quasi-invertibility is a rational condition, U is open and dense in $(\overline Z_j\times Z\times\overline Z)^2$. Let $\tilde v = \tilde v_1\oplus\tilde v_{1/2}\oplus\tilde v_0$ be the decomposition of $\tilde v$ due to the Peirce decomposition with respect to $\tilde u$.

Claim 3. *The intersection $U\cap R_{\mathbb F_j}$ is described by*

$$\big((v,w,b),(\tilde v,\tilde w,\tilde b)\big)\in U\cap R_{\mathbb F_j}\iff\begin{cases}v_{1/2} = 2\left\{\mathrm{pr}^u_{1/2}B_{\tilde b-b,\,\tilde w}\tilde v,\,\left(\mathrm{pr}^u_1 B_{\tilde b-b,\,\tilde w}\tilde v\right)^\dagger,\,v_1\right\}\ ,\\ B_{-\tilde v_1^\dagger,\,\tilde v_{1/2}}\big(\tilde w-(w+B_{v_1^\dagger,\,v_{1/2}}u^{\amalg})^{b-\tilde b}\big)\in Z^{\tilde u}_0\ .\end{cases}$$

Here, pr^u_ν denotes the orthogonal projection onto the Peirce space Z^u_ν for $\nu = 1,1/2,0$.

PROOF. By definition, two elements (v,w,b) and $(\tilde v,\tilde w,\tilde b)$ are equivalent if and only if (1) $\tilde v$ is Peirce equivalent to $B_{b-\tilde b,w}v$ and (2) $\tilde w+\tilde v^{\amalg}$ equals $(w+v^{\amalg})^{b-\tilde b}$ for some $\tilde v^{\amalg}\in Z_0^{\tilde v}$ and $v^{\amalg}\in Z_0^v$. By symmetry, (1) is equivalent to the Peirce equivalence of v and $B_{\tilde b-b,\tilde w}\tilde v$. Due to Lemma 3.21, this again is equivalent to the first condition of the assertion. Now assume (2). Claim 1 implies that $\tilde w-(w+v^{\amalg})^{b-\tilde b}$ is an element of $Z_0^{\tilde v}$ for *all* $v^{\amalg}\in Z_0^v$ such that $(w+v^{\amalg},b-\tilde b)$ is quasi-invertible. By definition of U and due to (8.24), this holds in particular for $v^{\amalg}=B_{v_1^\dagger,v_{1/2}}u^{\amalg}$. Again applying (8.24) to $\tilde u$ and $\tilde v$, yields $Z_0^{\tilde u}=B^{-1}_{\tilde v_1^\dagger,\tilde v_{1/2}}Z_0^{\tilde v}=B_{-\tilde v_1^\dagger,\tilde v_{1/2}}Z_0^{\tilde v}$, which implies the second condition of the assertion. Now the converse is obvious. □

By Claim 3, we proved so far that $R_{\mathbb{F}_j}$ is locally (and even densely) given as the level set of the holomorphic map $\Phi:U\to\overline{Z}^u_{1/2}\times Z/Z_0^{\tilde u}$ given by

$$\Phi\big((v,w,b),(\tilde v,\tilde w,\tilde b)\big)=\begin{pmatrix} v_{1/2}-2\left\{\mathrm{pr}^u_{1/2}B_{\tilde b-b,\tilde w}\tilde v,\ \left(\mathrm{pr}^u_1 B_{\tilde b-b,\tilde w}\tilde v\right)^\dagger,\ v_1\right\}\\ B_{-\tilde v_1^\dagger,\tilde v_{1/2}}\left(\tilde w-(w+B_{v_1^\dagger,v_{1/2}}u^{\amalg})^{b-\tilde b}\right)\end{pmatrix}. \tag{8.25}$$

It remains to show that this is a submersion. First consider the derivative of Φ with respect to w:

$$D\Phi(\dot w)=\left(0,\ -B_{-\tilde v_1^\dagger,\tilde v_{1/2}}B_{w+B_{v_1^\dagger,v_{1/2}}u^{\amalg},\,b-\tilde b}(\dot w)\right).$$

By the definition of U, the two Bergman operators are invertible, so the image of $D\Phi$ contains the whole tangent space of the second component. Secondly consider the derivative of the the first component of Φ with respect to v:

$$D\Phi_1(\dot v)=\dot v_{1/2}-2\left\{\mathrm{pr}^u_{1/2}B_{\tilde b-b,\tilde w}\tilde v,\ \left(\mathrm{pr}^u_1B_{\tilde b-b,\tilde w}\tilde v\right)^\dagger,\ \dot v_1\right\}$$

Evaluation on $\overline{Z}^u_{1/2}$ implies that the image of $D\Phi$ also contains the whole tangent space of the first component. Therefore, ϕ is a submersion, and we proved that $R_{\mathbb{F}_j}$ is indeed a submanifold of $(\overline{Z}_j\times Z\times\overline{Z})^2$.
Now we prove (ii), i.e. that the projection pr_1 of $R_{\mathbb{F}_j}$ onto its first component is a submersion. For this fix $\big((u,a,z),(\tilde u,\tilde z,\tilde a)\big)$ in $R_{\mathbb{F}_j}$ with corresponding $u^{\amalg}\in Z_0^u$ and $\tilde u^{\amalg}\in Z_0^{\tilde u}$ such that $\tilde z+\tilde u^{\amalg}=(z+u^{\amalg})^{a-\tilde a}$. Then it is straightforward to checked that for $\dot z\in Z$, $\dot a\in\overline{Z}$ and small $t\in\mathbb{R}$, the curves $\gamma_1(t)=\big((u,z,a+t\dot a),(\tilde u,\tilde z,\tilde a+t\dot a)\big)$ and

$$\gamma_2(t)=\big((u,\ z+t\dot z,\ a),\ (B_{a-\tilde a,\,z+t\dot z+u^{\amalg}}B^{-1}_{a-\tilde a,\,z+u^{\amalg}}\tilde u,\ (z+t\dot z+u^{\amalg})^{a-\tilde a},\ \tilde a)\big)$$

stay in $R_{\mathbb{F}_j}$, and satisfy $\gamma_1(0)=\gamma_2(0)=\big((u,a,z),(\tilde u,\tilde z,\tilde a)\big)$ and

$$\tfrac{d}{dt}\Big|_{t=0}(\mathrm{pr}_1\circ\gamma_1)=(0,0,\dot a)\ ,\quad \tfrac{d}{dt}\Big|_{t=0}(\mathrm{pr}_2\circ\gamma_1)=(0,\dot z,0)\ . \tag{8.26}$$

Therefore, the image of the derivative of pr_1 at $\big((u,z,a),(\tilde u,\tilde z,\tilde a)\big)$ contains $\{0\}\times Z\times\overline{Z}$. To handle the first component, we note that $R_{\mathbb{F}_j}$ is a $\mathrm{Str}(Z)$-invariant submanifold.

CLAIM 4. *The structure group* $\mathrm{Str}(Z)$ *acts on* $Z_j\times Z\times\overline{Z}$ *via*

$$h(u,z,a):=(h^{-*}u,hz,h^{-*}a)\quad \textit{for}\quad h\in K^{\mathbb{C}},\ (u,z,a)\in\overline{Z}_j\times Z\times\overline{Z}\ . \tag{8.27}$$

With respect to this action, the equivalence relation $R_{\mathbb{F}_j}$ *is* $\mathrm{Str}(Z)$*-invariant.*

PROOF. It is obvious, that (8.27) defines a $\mathrm{Str}(Z)$-action on $\overline{Z}_j\times Z\times\overline{Z}$. Now assume that (u,z,a) and $(\tilde u,\tilde z,\tilde a)$ are equivalent with respect to $R_{\mathbb{F}_j}$. Then

$$h^{-*}\tilde u\approx h^{-*}B_{a-\tilde a,z}u=B_{h^{-*}a-h^{-*}\tilde a,hz}h^{-*}u\ ,$$

and

$$h\tilde z+h\tilde u^{\amalg}=h(\tilde z+\tilde u^{\amalg})=h(z+u^{\amalg})^{a-\tilde a}=(hz+hu^{\amalg})^{h^{-*}a-h^{-*}\tilde a}\ .$$

From Lemma 2.32 we obtain $hZ_0^u = Z_0^{h^{-*}u}$, and therefore $hu^{\perp\!\!\perp} \in Z_0^{h^{-*}u}$ and $h\tilde{u}^{\perp\!\!\perp} \in Z_0^{h^{-*}\tilde{u}}$. This shows that $(h^{-*}u, hz, h^{-*}a)$ and $(h^{-*}\tilde{u}, h\tilde{z}, h^{-*}\tilde{a})$ are also equivalent with respect to $R_{\mathbb{F}_j}$, and hence $R_{\mathbb{F}_j}$ is a $\mathrm{Str}(Z)$-invariant equivalence relation. □

Now consider the curve $h(t) = \exp(t\dot{u} \,\square\, u^\dagger)$ in $\mathrm{Str}(Z)$ for some $\dot{u} = \dot{u}_1 + \dot{u}_{1/2} \in Z_1^u \oplus Z_{1/2}^u$. By Claim 4, the application of h_t to $\big((u,z,a),(\tilde{u},\tilde{z},\tilde{a})\big)$ yields a curve $\gamma(t)$ in $R_{\mathbb{F}_j}$, and we obtain

$$\tfrac{d}{dt}\Big|_{t=0}(\mathrm{pr}_1 \circ \gamma) = (\dot{u} \,\square\, u^\dagger(u), \cdots, \cdots) = (\dot{u}_1 + \tfrac{1}{2}\,\dot{u}_{1/2}, \cdots, \cdots)\,.$$

Due to (8.26), it is not necessary to determine the second and the third component of this determinant. In combination with (8.26), the first component shows that pr_1 is indeed a submersion, and hence $R_{\mathbb{F}_j}$ is a regular equivalence relation on $\overline{Z}_j \times Z \times \overline{Z}$.

We note that it is straightforward to extend the argument above to the general case $J = (j_1, \ldots, j_k)$. The single elements v, $\tilde{v}$ become tuples (v_i), $(\tilde{v}_i)$, in Claim 3, the first condition on the right hand side is extended to k different equations on v_i and $\tilde{v}_i$, and in the second condition, v and $\tilde{v}$ refer to v_k and $\tilde{v}_k$, respectively. The statement that the corresponding map Φ and the projection of $R_{\mathbb{F}_J}$ onto its first component are submersions can be proved by induction as in the proof of Theorem 3.18. Finally, we have to prove compactness.

Claim 5. *$\mathbb{F}_J$ is compact.*

Proof. Let $((u_i)_\ell, z_\ell, a_\ell)_{\ell\in\mathbb{N}}$ be an arbitrary sequence in $\overline{Z}_J \times Z \times \overline{Z}$. We show that the corresponding sequence $\chi_\ell = \big([(u_i)_\ell : z_\ell : a_\ell]\big)_{\ell\in\mathbb{N}}$ in $\mathbb{F}_J$ contains a convergent subsequence. Consider the sequence $[z_\ell : a_\ell]$ in the Grassmannian $\mathbb{G}(Z)$. Since $\mathbb{G}(Z)$ is a projective variety, it is compact, and so after restriction to a subsequence, we may assume that $[z_\ell : a_\ell]$ converges in $\mathbb{G}(Z)$ to some element $[z : a]$. Thus, after further restriction, we may assume that the pair $(z_\ell, a_\ell - a)$ is quasi-invertible for all ℓ. Therefore, we obtain

$$[(u_i)_\ell : z_\ell : a_\ell] = [(\tilde{u}_i)_\ell : \tilde{z}_\ell : a] \quad \text{with} \quad (\tilde{u}_i)_\ell = B_{a_\ell - a, z_\ell}(u_i)_\ell\,, \quad \tilde{z}_\ell = z_\ell^{a-\tilde{a}}\,,$$

or equivalently, we may assume that $a_\ell = a$ for all ℓ, and $z_\ell \to z$ for $\ell \to \infty$. Finally, we note that each $(u_i)_\ell$ can be replaced by a Peirce equivalent flag $(e_i)_\ell$ consisting of tripotents, and since the set of tripotents is compact, this sequence contains a convergent subsequence. This proves Claim 5, and finishes the proof of Theorem 8.11. □

The proof of Theorem 8.11 yields the following alternative description of the Jordan flag variety $\mathbb{F}_J$:

Corollary 8.12. *Let $\overline{\mathcal{E}}_J$ be as in (8.20), and let $\sim'$ be the equivalence relation on $\overline{\mathcal{E}}_J \times \overline{Z}$ defined in Claim 1 above. Then*

$$\pi : \overline{Z}_J \times Z \times \overline{Z} \to \overline{\mathcal{E}}_J \times \overline{Z},\ \big((u_i), z, a\big) \mapsto \big((Z_1^{u_i}), [z], a\big) \tag{8.28}$$

is a submersion, which respects the equivalence classes and induces an isomorphism of the Jordan flag variety $\mathbb{F}_J = (\overline{Z}_j \times Z \times \overline{Z})/\sim$ and $(\overline{\mathcal{E}}_J \times \overline{Z})/\sim'$ as complex analytic manifolds.

Proof. Let π_1 and π_2 denote the canonical projections of $\overline{Z}_J \times Z \times \overline{Z}$ and $\overline{\mathcal{E}}_J \times \overline{Z}$ onto their sets of equivalence classes. By Claim 1, we can identify these sets with $\mathbb{F}_J$. Due to Theorem 3.5 and Godement's Theorem, the complex analytic structure on $\mathbb{F}_J$ is uniquely determined by the condition that π_1 is a submersion. Since the canonical projection of Z_J onto $\mathbb{P}_J = Z_J/\approx$ is a submersion, π is obviously also a submersion onto $\overline{\mathcal{E}}_J \times \overline{Z}$. Now, since $\pi_1 = \pi_2 \circ \pi$, we conclude that π_2 is also a submersion, and hence the manifold structures on $\mathbb{F}_J$ induced by π_1 and π_2 coincide. □

REMARK 8.13. If the type $J = (j_1, \dots, j_k)$ ends with $j_k = r$, the rank of Z, then the defining equivalence relation of the Jordan flag variety $\mathbb{F}_J$ reduces to

$$\big((u_i), z, a\big) \sim \big((\tilde u_i), \tilde z, \tilde a\big) \iff \begin{cases} \tilde u_i \approx B_{a-\tilde a,\, z} u_i \text{ for } i = 1, \dots, k; \\ \text{and } (z, a - \tilde a) \text{ is quasi-invertible} \\ \text{with } \tilde z = z^{a-\tilde a}, \end{cases} \tag{8.29}$$

since $Z_0^{u_k} = \{0\}$ for all $u_k \in Z_r$. Comparing this with the definition of the Grassmannian $\mathbb{G}$ of Z, we conclude that in this case $\mathbb{F}_J$ is a (non-trivial) fiber bundle on $\mathbb{G}$ with canonical fiber $\overline{\mathbb{P}}_J$, the (conjugate) Peirce flag of type J, and canonical projection

$$\pi : \mathbb{F}_J \to \mathbb{G},\ [(u_i) : z : a] \mapsto [z : a]\ .$$

We note that in the case $J = (r)$, the fiber $\overline{\mathbb{P}}_r$ of this bundle is trivial, if and only if the underlying phJTS Z is of tube type. Then, $\overline{\mathbb{P}}_r$ contains only one element, namely $\overline{\mathbb{P}}_r = \{\overline{Z}\}$.

8.4. Analytic and algebraic structure

Local structure. We give a local description of the Jordan flag variety $\mathbb{F}_J$ in two steps. First we describe a covering of $\mathbb{F}_J$ by smooth algebraic varieties $\mathbb{F}_J^{(a)}$ similar to the covering of the Grassmannian $\mathbb{G}$ of Z by affine varieties $\mathbb{G}^{(a)}$. In the second step we investigate a covering of each $\mathbb{F}_J^{(a)}$ by affine varieties $\mathbb{F}_J^{((u_i),a)}$.

First Step: Due to Corollary 8.12 we have two equivalent models of $\mathbb{F}_J$, namely

$$\mathbb{F}_J = (\overline{Z}_J \times Z \times \overline{Z})/\sim \quad \text{and} \quad \mathbb{F}_J = (\overline{\mathcal{E}}_J \times \overline{Z})/\sim'\ ,$$

where $\overline{\mathcal{E}}_J$ is the vector bundle on the Peirce flag $\overline{\mathbb{P}}_J$ with fiber $(\overline{\mathcal{E}}_J)_{(U_i)} = Z/U_k^{\perp\!\!\perp}$. Therefore, the elements of $\mathbb{F}_J$ are denoted alternatively by

$$[(u_i) : z : a] = [(U_i) : [z] : a] \quad \text{with} \quad U_i = Z_1^{u_i},\ [z] = z + Z_0^{u_k}\ .$$

For any $a \in \overline{Z}$ set

$$\iota_a : \overline{\mathcal{E}}_J \to \mathbb{F}_J, \big((U_i), [z]\big) \mapsto [(U_i) : [z] : a] \quad \text{and} \quad \mathbb{F}_J^{(a)} := \iota_a\big(\overline{\mathcal{E}}_J\big)\ . \tag{8.30}$$

This is the restriction of the canonical projection of $\overline{\mathcal{E}}_J \times \overline{Z}$ to $\overline{\mathcal{E}}_J \times \{a\}$, and hence a holomorphic mapping. Furthermore, ι_a is an injection, since $B_{0,\,z} = \mathrm{Id}$ and $z^0 = z$. Therefore, by a standard result on holomorphic maps [**17**, §46], ι_a maps $\overline{\mathcal{E}}_J$ biholomorphically onto $\mathbb{F}_J^{(a)}$, since $\dim \mathbb{F}_J = \dim \overline{\mathcal{E}}_J$. For two elements $a, \tilde a \in \overline{Z}$, we obtain the following diagram:

$$\begin{tikzcd}
 & \overline{\mathcal{E}}_J \arrow[dl, "\iota_a"'] \arrow[dd, dashed, "\iota^a_{\tilde a} = \iota_{\tilde a}^{-1} \circ \iota_a"] & \big((U_i), [z]\big) \arrow[dd, mapsto] \\
\mathbb{F}_J^{(a)} \cap \mathbb{F}_J^{(\tilde a)} & & \\
 & \overline{\mathcal{E}}_J \arrow[ul, "\iota_{\tilde a}"] & \big((B_{a-\tilde a,\, z} U_i), \mathrm{cl}([z]^{a-\tilde a})\big)
\end{tikzcd} \tag{8.31}$$

We remark that even though $\mathbb{F}_J$ looks locally like the vector bundle $\overline{\mathcal{E}}_J$, diagram (8.31) shows that its global structure does not preserve the vector space structure on the fibers, since $[z] \mapsto [z]^{a-\tilde a}$ is not a linear map.

The next step is to use chart maps of the vector bundle $\overline{\mathcal{E}}_J$ to obtain charts on the Jordan flag variety $\mathbb{F}_J$. We recall from Proposition 6.17 that for fixed $(u_i) \in Z_J$ and $Z_{1/2}^{(u_i)} := (Z_{1/2}^{u_1} \cap Z_1^{u_2}) \times \dots \times (Z_{1/2}^{u_{k-1}} \cap Z_1^{u_k}) \times Z_{1/2}^{u_k}$, the map

$$\varphi_{(u_i)} : Z_{1/2}^{(u_i)} \to \mathbb{P}_J,\ (y_i) \mapsto \big[\tau_{u_k, y_k} \circ \dots \circ \tau_{u_i, y_i} u_i\big]_{i=1,\dots,k}$$

is a diffeomorphism onto the open and dense subset $\mathbb{P}_J^{(u_i)}$ of the connected component of $\mathbb{P}_J$ containing $[u_i]$. We extend this map to a chart of $\overline{\mathcal{E}}_J$.

Proposition 8.14. *Let $\varphi_{(u_i)}$ be a chart of $\mathbb{P}_J$ as defined in Proposition 6.17, and let π denote the canonical projection of the vector bundle $\overline{\mathcal{E}}_J$ onto $\overline{\mathbb{P}}_J$. Then*

$$\tilde{\varphi}_{(u_i)} : \overline{Z}_{1/2}^{(u_i)} \times \left(Z_1^{u_k} \oplus Z_{1/2}^{u_k}\right) \to \overline{\mathcal{E}}_J,\ \left((y_i), z\right) \mapsto \left(\varphi_{(u_i)}(y_i),\, [z]\right) \tag{8.32}$$

is a trivialization of the open and dense subset $\pi^{-1}\left(\mathbb{P}_J^{(u_i)}\right)$ of the connected component of the vector bundle $\overline{\mathcal{E}}_J$ containing $[(Z_1^{u_i}), 0]$. For different (u_i), $(\tilde{u}_i) \in Z_J$, the transition map is a birational map, and $\overline{\mathcal{E}}_J$ is a smooth algebraic variety.

Proof. We first check that $\tilde{\varphi}_{(u_i)}$ is still one-to-one, i.e. that $[z]$ is a non-vanishing element of $Z/Z_0^{\tau_{u_k,y_k} u_k}$ for all $z \in Z_1^{u_k} \oplus Z_{1/2}^{u_k}$ and $(y_i) \in Z_{1/2}^{(u_i)}$. Equivalently, we show that the intersection of $Z_1^{u_k} \oplus Z_{1/2}^{u_k}$ and $Z_0^{\tau_{u_k,y_k} u_k}$ is trivial. Due to Lemma 2.32 and Lemma 3.14, we have

$$Z_0^{\tau_{u_k,y_k} u_k} = \tau_{u_k,y_k}^{-*} Z_0^{u_k} = B_{u_k^\dagger,\, y_k} Z_0^{u_k}\ ,$$

and by the Peirce rules, it is $B_{u_k^\dagger,\, y_k} z_0 = z_0 \oplus (\cdots)_{1/2} \oplus (\cdots)_1$ for all $z_0 \in Z_0^{u_k}$ according to the Peirce decomposition with respect to u_k. Therefore, $Z_1^{u_k} \oplus Z_{1/2}^{u_k}$ and $Z_0^{\tau_{u_k,y_k} u_k}$ intersect trivially, and hence $\tilde{\varphi}_{(u_i)}$ is one-to-one. The analytic properties of $\varphi_{(u_i)}$ proved in Proposition 6.17 transfer immediately to the map $\tilde{\varphi}_{(u_i)}$. We just note that the second component of the transition map equals the orthogonal projection of $Z_1^{u_k} \oplus Z_{1/2}^{u_k}$ onto $Z_1^{\tilde{u}_k} \oplus Z_{1/2}^{\tilde{u}_k}$. In particular, this is a linear map, and so the maps $\tilde{\varphi}_{u_i}$ form indeed local trivializations of the vector bundle $\overline{\mathcal{E}}_J$. It remains to show that (i) finitely many trivializations cover $\overline{\mathcal{E}}_J$, and (ii) any two points of $\overline{\mathcal{E}}_J$ are contained in one single trivialization. Both statements follow from Proposition 6.17, since the base $\mathbb{P}_J$ of the vector bundle $\overline{\mathcal{E}}_J$ satisfies (i) and (ii). □

Remark 8.15. We notice that $\overline{\mathcal{E}}_J$ can also be regarded as quotient manifold $\overline{\mathcal{E}}_J = (\overline{Z}_J \times Z)/\sim$ with

$$((u_i), z) \sim ((\tilde{u}_i), \tilde{z}) \quad \text{if and only if} \quad (u_i) \approx (\tilde{u}_i),\ z - \tilde{z} \in Z_0^{u_k}\ . \tag{8.33}$$

This is just the restriction of the equivalence relation on $\mathbb{F}_J$ to elements with vanishing third component. Furthermore, the canonical projection $\pi : \overline{Z}_J \times Z \to \overline{\mathcal{E}}_J$ turns $\mathcal{E}_J$ into a fiber bundle[4] with fiber through $((u_i), z)$ given by

$$[(u_i), z] = (Z_1^{u_1})^\times \times \ldots \times (Z_1^{u_k})^\times \times Z_0^{u_k}\ ,$$

where $(Z_1^{u_i})^\times$ denote the invertible elements of the unital Jordan algebra $Z_1^{u_i}$.

Now we are prepared to describe charts on the Jordan flag variety $\mathbb{F}_J$.

Proposition 8.16. *For $(u_i) \in \overline{Z}_J$ and $a \in \overline{Z}$ let $\phi_{a,(u_i)}$ be defined as*

$$\phi_{a,(u_i)} := \iota_a \circ \tilde{\varphi}_{(u_i)} : \overline{Z}_{1/2}^{(u_i)} \times \left(Z_1^{u_k} \oplus Z_{1/2}^{u_k}\right) \to \mathbb{F}_J \tag{8.34}$$

and set $\mathbb{F}_J^{(a,(u_i))} := \operatorname{Im}\left(\phi_{a,(u_i)}\right)$. Then $\mathbb{F}_J^{(a,(u_i))}$ is an open and dense subset of the connected component of $\mathbb{F}_J$ containing $[(u_i) : 0 : a]$, and $\phi_{a,(u_i)}$ is a diffeomorphism onto its image. For different (u_i), $(\tilde{u}_i) \in Z_J$ and $a, \tilde{a} \in \overline{Z}$, the transition map is a birational map, and the Jordan flag variety $\mathbb{F}_J$ is indeed a smooth algebraic variety.

[4] More correctly, each connected component of $\overline{Z}_J \times Z$ is a fiber bundle on the corresponding connected component of $\mathcal{E}_J$. If Z is simple, then $\overline{Z}_J \times Z$ is connected.

Proof. We consider the case $J = (j)$, i.e. $k = 1$, and investigate the transition map of $\phi_{a,u}$ and $\phi_{\tilde{a},\tilde{u}}$ with $u, \tilde{u} \in \overline{Z}_j$ and $a, \tilde{a} \in \overline{Z}$. We have

$$\phi_{a,u}(y,z) = [\tau_{u,y}u : z : a] \quad \text{and} \quad \phi_{\tilde{a},\tilde{u}}(\tilde{y},\tilde{z}) = [\tau_{\tilde{u},\tilde{y}}\tilde{u} : \tilde{z} : \tilde{a}]$$

for $(y,z) \in \overline{Z}^u_{1/2} \times (Z^u_1 \oplus Z^u_{1/2})$ and $(\tilde{y},\tilde{z}) \in \overline{Z}^{\tilde{u}}_{1/2} \times (Z^{\tilde{u}}_1 \oplus Z^{\tilde{u}}_{1/2})$. Now, $\phi_{a,u}(y,z) = \phi_{\tilde{a},\tilde{u}}(\tilde{y},\tilde{z})$ if and only if

$$\tau_{\tilde{u},\tilde{y}}\tilde{u} \approx B_{a-\tilde{a},\,z}\tau_{u,y}u \quad \text{and} \quad (z + u^{\perp\!\!\perp})^{a-\tilde{a}} = \tilde{z} + \tilde{u}^{\perp\!\!\perp}$$

for some $u^{\perp\!\!\perp} \in Z^u_0$ and $\tilde{u}^{\perp\!\!\perp} \in Z^{\tilde{u}}_0$. If $B_{a-\tilde{a},\,z}\tau_{u,y}u$ is an element of $\mathcal{I}^{\tilde{u}}_j$ and if $(z, a-\tilde{a})$ is quasi-invertible, Lemma 3.21 and Claim 1 in the proof of Theorem 8.11 imply that this is equivalent to

$$\tilde{y} = 2\left\{\left(B_{a-\tilde{a},\,z}\tau_{u,y}u\right)_{1/2},\, \left(B_{a-\tilde{a},\,z}\tau_{u,y}u\right)^{\dagger}_{1},\, \tilde{u}\right\} \quad \text{and} \quad \tilde{z} = z^{a-\tilde{a}} - (z^{a-\tilde{a}})_0\,,$$

where the index $(\,)_\nu$ corresponds to the ν-th component of the Peirce decomposition with respect to $\tilde{u}$. Therefore we established that the transition map between $\phi_{a,u}(y,z)$ and $\phi_{\tilde{a},\tilde{u}}(\tilde{y},\tilde{z})$ is given by

$$\phi^{a,u}_{\tilde{a},\tilde{u}}(y,z) = \left(2\left\{\left(B_{a-\tilde{a},\,z}\tau_{u,y}u\right)_{1/2},\, \left(B_{a-\tilde{a},\,z}\tau_{u,y}u\right)^{\dagger}_{1},\, \tilde{u}\right\},\, z^{a-\tilde{a}} - (z^{a-\tilde{a}})_0\right)$$

for all $(y,z) \in U$ with

$$U = \left\{(y,z) \in \overline{Z}^u_{1/2} \times (Z^u_1 \oplus Z^u_{1/2}) \,\middle|\, \overline{\Delta}^{\tilde{u}}(B_{a-\tilde{a},\,z}\tau_{u,y}u) \neq 0, \Delta(z, a-\tilde{a}) \neq 0\right\}\,.$$

We recall that $B_{a-\tilde{a},\,z}\tau_{u,y} = B_{a-\tilde{a},\,z}B_{y,\,-u^\dagger}$ is anti-holomorphic in (y,z), since y is taken to be an element of the conjugate of Peirce space $\overline{Z}^u_{1/2}$. Therefore, U the the non-vanishing set of two complex polynomials, and hence a Zariski-open subset of $\overline{Z}^u_{1/2} \times (Z^u_1 \oplus Z^u_{1/2}$. Since u and $\tilde{u}$ are assumed to be elements of the same connected component of $\overline{Z}_j$, the intersection $\mathcal{I}^u_j \cap \mathcal{I}^{\tilde{u}}_j$ is non-empty, and for generic $y \in Z^u_{1/2}$ the element $\tau_{u,y}u$ belongs to this intersection. For such y, we obtain $(y,0) \in U$, so U is non-empty, and hence U is open and dense subset of $\overline{Z}^u_{1/2} \times (Z^u_1 \oplus Z^u_{1/2})$. We also recall from Lemma 2.16 that

$$\left(B_{a-\tilde{a},\,z}\tau_{u,y}u\right)^{\dagger}_{1} = Q_{\tilde{u}^\dagger}\left((B_{a-\tilde{a},\,z}\tau_{u,y}u)_1\right)^{-1}\,,$$

where $(\,)^{-1}$ denotes the inverse of elements in the Jordan algebra $Z^{\tilde{u}}_1$. Therefore, $\phi^{a,u}_{\tilde{a},\tilde{u}}$ indeed defines a birational map from $\overline{Z}^u_{1/2} \times (Z^u_1 \oplus Z^u_{1/2})$ to $\overline{Z}^{\tilde{u}}_{1/2} \times (Z^{\tilde{u}}_1 \oplus Z^{\tilde{u}}_{1/2})$. This also shows that $\mathbb{F}^{(a,u)}_j$ is an open and dense subset of the connected component of $\mathbb{F}_j$ containing $[u : 0 : a]$. The extension of this proof to the general case $J = (j_1, \ldots, j_k)$ is straightforward. One just replaces u, $\tilde{u}$, y and $\tilde{y}$ by the corresponding tuples (u_i), $(\tilde{u}_i)$, (y_i) and $\tilde{y}$, the single Frobenius transformation $\tau_{u,y}$ becomes the chain $\tau_{u_k,y_k} \circ \ldots \circ \tau_{u_i,y_i}$, and in the relations of z, $\tilde{z}$, a and $\tilde{a}$, the involved elements u, $\tilde{u}$, y and $\tilde{y}$ are decorated with the highest index k. We omit the detailed exposition. Finally, we prove that $\mathbb{F}_J$ is a smooth algebraic variety. By Proposition 8.14, the open and dense subsets $\mathbb{F}^{(a)}_J \cong \overline{\mathcal{E}}_J$ are smooth algebraic varieties. Therefore, it suffices to show (i) that finitely many of these do cover $\mathbb{F}_J$, and (ii) that any two points of $\mathbb{F}_J$ are contained in one of these. We claim that (i) and (ii) follow immediately from the corresponding properties of the Grassmannian $\mathbb{G}$ of Z, see Section 4.1: $\mathbb{G}$ admits a cover of the form $\mathbb{G} = \bigcup_\ell \mathbb{G}^{(a_\ell)}$ with finitely many $a_\ell \in \overline{Z}$ and $\mathbb{G}^{(a_\ell)} = \{[z : a_\ell] \,|\, z \in Z\}$. We claim that the same a_ℓ suffice to cover $\mathbb{F}_J$. Let $[(u_i) : z : a]$ be an arbitrary element of $\mathbb{F}_J$. Due to the finite covering of $\mathbb{G}$, there exists an index ℓ, such that $(z, a - a_\ell)$ is quasi-invertible. Therefore,

$$[(u_i) : z : a] = [(B_{a-a_\ell,\,z}u_i) : z^{a-a_\ell} : a_\ell] \in \mathbb{F}^{(a_\ell)}_J\,.$$

This proves (i). For (ii), recall that any finite subset of $\mathbb{G}$ is contained in $\mathbb{G}^{(a)}$ for some $a \in \overline{Z}$. So let $\{[(u_i^\ell) : z_\ell : a_\ell] \,|\, \ell = 1, \ldots, N\}$ be any finite subset of $\mathbb{F}_J$. Consider the finite subset[5] $\{[z_\ell : a_\ell] \,|\, \ell = 1, \ldots, N\}$ in $\mathbb{G}$. Then there exists an element $a \in \overline{Z}$ such that $[z_\ell : a_\ell] \in \mathbb{G}^{(a)}$ for all ℓ, i.e. $(z_\ell, a_\ell - a)$ is quasi-invertible for all ℓ. Therefore,

$$\big[(u_i^\ell) : z_\ell : a_\ell\big] = \big[(B_{a_\ell - a,\, z_\ell} u_i^\ell) : z_\ell^{a_\ell - a} : a\big] \in \mathbb{F}_J^{(a)}$$

for all ℓ. This completes the proof of Proposition 8.16. □

REMARK 8.17. We note that if Z is simple, then $\overline{Z}_J$, $\overline{\mathbb{P}}_J$, $\overline{\mathcal{E}}_J$ and hence $\mathbb{F}_J$ are connected, and so all charts defined above are open and dense in the corresponding varieties. As in the case of the Grassmannian $\mathbb{G}$, some charts are distinguished, namely those with index $a = 0$. It is

$$\mathbb{F}_J^{(0)} = \big\{[(u_i) : z : 0] \,\big|\, (u_i) \in \overline{Z}_J,\ z \in Z\big\} \cong \overline{\mathcal{E}}_J$$

and for fixed $(u_i) \in Z_J$

$$\mathbb{F}_J^{(0,(u_i))} = \left\{[(\tilde u_i) : z : 0] \,\middle|\, (\tilde u_i) \in \overline{\mathcal{I}}_J^{(u_i)},\ z \in Z\right\} \cong \overline{Z}_{1/2}^{(u_i)} \times (Z_1^{u_k} \oplus Z_{1/2}^{u_k}) \,.$$

By density, it often suffices to consider the restriction of maps on $\mathbb{F}_J$ to one of these subsets.

Vector fields. According to the covering of $\mathbb{F}_J$ by the smooth subvarieties $\mathbb{F}_J^{(a)}$, a globally defined vector field ζ on $\mathbb{F}_J$ can be described by a family of vector fields $\{\zeta^{(a)} \,|\, a \in \overline{Z}\}$ on $\overline{\mathcal{E}}_J \cong \mathbb{F}_J^{(a)}$, whereby two vector fields $\zeta^{(a)}$ and $\zeta^{(\tilde a)}$ are connected by the differential of the transition map given in (8.31):

$$\zeta^{(a)} = \big(\iota_a^{\tilde a}\big)_* \zeta^{(\tilde a)}\ , \text{ and more explicitly,} \quad \zeta^{(a)}(\chi) = \big(D_{\iota_{\tilde a}^a(\chi)} \iota_a^{\tilde a}\big)\, \zeta^{(\tilde a)}\big(\iota_{\tilde a}^a(\chi)\big)\,.$$

The same holds for the covering by the smooth subvarieties $\mathbb{F}_J^{(a,(u_i))}$ and a corresponding family of vector fields on $\overline{Z}_{1/2}^{(u_i)} \times (Z_1^{u_k} \oplus Z_{1/2}^{u_k})$. Since the transition maps between these subvarieties are quite complicated, it is often more convenient (if possible) to lift the vector fields to smooth varieties, for which the lifted transition maps are more simple.

In the following, we regard $\overline{\mathcal{E}}_J$ as quotient manifold $\overline{\mathcal{E}}_J = (\overline{Z}_J \times Z)/\sim$ as described in Remark 8.15. Let π be the canonical projection of $\overline{Z}_J \times Z$ onto $\overline{\mathcal{E}}_J$. As also noted in Remark 8.15, this defines a fiber bundle. Since $\overline{Z}_J \times Z$ is a submanifold of $Z^k \times Z$, the scalar product on Z induces a metric on $\overline{Z}_J \times Z$, and hence defines a connection on the fiber bundle $(\overline{Z}_J \times Z, \overline{\mathcal{E}}_J, \pi)$, i.e. a 'horizontal distribution' on $\overline{Z}_J \times Z$: For $((u_i), z) \in \overline{Z}_J \times Z$, the fiber through $((u_i), z)$ is given by $[(u_i), z] = (Z_1^{u_1})^\times \times \ldots \times (Z_1^{u_k})^\times \times Z_0^{u_k}$, so the vertical distribution on $\overline{Z}_J \times Z$ is given by

$$T[(u_i), z] = Z_1^{u_1} \times \ldots \times Z_1^{u_k} \times Z_0^{u_k}\ ,$$

and the horizontal distribution can be taken to be the orthogonal complement of $T[(u_i), z]$ in $T(\overline{Z}_J \times Z)$.

Therefore, any section ζ on $\overline{\mathcal{E}}_J$ admits a *canonical lift* to a section $\hat\zeta$ on $\overline{Z}_J \times Z$. Recall from Section 3.2 that an arbitrary lift of a vector field ζ on $\overline{\mathcal{E}}_J$ is a vector field $\hat\zeta$ on $\overline{Z}_J \times Z$ satisfying the relation $D\pi \circ \hat\zeta = \zeta \circ \pi$. Any two lifts $\hat\zeta_1$, $\hat\zeta_2$ of ζ just differ by a *vertical* section, i.e. a section $\hat\zeta_v$ on $\overline{Z}_J \times Z$ satisfying $D\pi \circ \hat\zeta_v = 0$. In this case we call $\hat\zeta_1$ and $\hat\zeta_2$ *equivalent* and write

$$\hat\zeta_1 \approx \hat\zeta_2\ .$$

[5]We stress that this subset depends on the choice of the representatives u_i^ℓ, z_ℓ, a_ℓ for the elements $[(u_i^\ell) : z_\ell : a_\ell]$, but this does not harm the following argument.

A vector field $\hat{\zeta}$ on $\overline{Z}_J \times Z$ is said to be *projectable*, if the projection $D\pi \circ \hat{\zeta}$ is constant along the fibers. In this case,

$$\zeta([(u_i), z]) := \big(D_{((u_i),z)}\pi\big)\hat{\zeta}\big(((u_i), z)\big)$$

defines a section on $\overline{\mathcal{E}}_J$, and $\hat{\zeta}$ is a lift of ζ. We investigate the conditions on a family of projectable vector fields[6] on $\overline{Z}_J \times Z$ to define a vector field on the Jordan flag variety $\mathbb{F}_J$.

Lemma 8.18. *Let π be the canonical projection of $\overline{Z}_J \times Z$ onto $\overline{\mathcal{E}}_J$, and for $a, \tilde{a} \in \overline{Z}$ let $\iota^a_{\tilde{a}}$ be the birational map on $\overline{Z}_J \times Z$ described in* (8.31)*. Then, the map*

$$\hat{\iota}^a_{\tilde{a}} : \overline{Z}_J \times Z \dashrightarrow \overline{Z}_J \times Z,\ ((u_i), z) \mapsto ((B_{a-\tilde{a},\, z} u_i), z^{a-\tilde{a}})$$

is a birational map, which satisfies $\pi \circ \hat{\iota}^a_{\tilde{a}} = \iota^a_{\tilde{a}} \circ \pi$. On its domain, the derivative of $\hat{\iota}^a_{\tilde{a}}$ in $((u_i), z)$ is given by

$$D\hat{\iota}^a_{\tilde{a}}((\dot{u}_i), \dot{z}) = \big((B_{a-\tilde{a},\, z}\dot{u}_i - 2\,\{a - \tilde{a},\, \dot{z},\, u_i\} + 2\,Q_{a-\tilde{a}}\,\{z,\, u_i,\, \dot{z}\}),\ B^{-1}_{z,\, a-\tilde{a}}\dot{z}\big)\,,$$

where $((\dot{u}_i), \dot{z})$ denotes a tangent vector of $\overline{Z}_J \times Z$ in $((u_i), z)$.

Proof. The map $\hat{\iota}^a_{\tilde{a}}$ is obviously birational on $\overline{Z}_J \times Z$, we just have to be careful about the respective complex structure on $\overline{Z}_J$ and Z. Its domain is given by the condition that $(z, a - \tilde{a})$ is quasi-invertible. The calculation of the derivative is straightforward, so it remains to prove the relation $\pi \circ \hat{\iota}^a_{\tilde{a}} = \iota^a_{\tilde{a}} \circ \pi$. On the one hand we have, due to Lemma 2.32

$$\pi \circ \hat{\iota}^a_{\tilde{a}}(u, z) = ((Z_1^{B_{a-\tilde{a},\, z} u_i}),\ z^{a-\tilde{a}} + Z_0^{B_{a-\tilde{a},\, z} u_k}) = ((B_{a-\tilde{a},\, z} Z_1^{u_i}),\ z^{a-\tilde{a}} + B^{-1}_{z,\, a-\tilde{a}} Z_0^{u_k})\,.$$

On the other hand, we obtain

$$\iota^a_{\tilde{a}} \circ \pi(u, z) = ((B_{a-\tilde{a},\, z} Z_1^{u_i}),\ \mathrm{cl}([z]^{a-\tilde{a}})) = ((B_{a-\tilde{a},\, z} Z_1^{u_i}),\ \mathrm{cl}((z + Z_0^{u_k})^{a-\tilde{a}}))\,.$$

The first components already coincide, and applying the addition formula to the second component of $\iota^a_{\tilde{a}} \circ \pi$, we get

$$\mathrm{cl}((z + Z_0^{u_k})^{a-\tilde{a}}) = z^{a-\tilde{a}} + B^{-1}_{z,\, a-\tilde{a}}\big(\mathrm{cl}((Z_0^{u_k})^{(a-\tilde{a})^z})\big) = z^{a-\tilde{a}} + B^{-1}_{z,\, a-\tilde{a}} Z_0^{u_k}\,,$$

since by Lemma 2.25, the Peirce space $Z_0^{u_k}$ is invariant under the birational map $x \mapsto x^y$. This finishes the proof of the relation $\pi \circ \hat{\iota}^a_{\tilde{a}} = \iota^a_{\tilde{a}} \circ \pi$. ☐

Lemma 8.19. *Let ζ be a vector field on the Jordan flag variety $\mathbb{F}_J$, and let $\{\zeta^{(a)} \mid a \in \overline{Z}\}$ be the corresponding family of vector fields on $\overline{\mathcal{E}}_J$. For each $a \in \overline{Z}$ let $\hat{\zeta}^{(a)}$ be a lift of $\zeta^{(a)}$ to $\overline{Z}_J \times Z$. Then, the $\hat{\zeta}^{(a)}$ are related by*

$$\hat{\zeta}^{(a)} \approx \big(\hat{\iota}^{\tilde{a}}_a\big)_* \hat{\zeta}^{(\tilde{a})}\,,\ \text{i.e.}\quad \hat{\zeta}^{(a)}(\chi) \approx \big(D_{\hat{\iota}^a_{\tilde{a}}(\chi)}\hat{\iota}^{\tilde{a}}_a\big)\,\hat{\zeta}^{(\tilde{a})}\big(\hat{\iota}^a_{\tilde{a}}(\chi)\big) \tag{8.35}$$

for $\chi = ((u_i), z) \in \mathrm{dom}(\hat{\iota}^a_{\tilde{a}})$. More explicitly, let $\hat{\zeta}^{(a)} = \big((\hat{\zeta}^{(a)}_{u_i}),\, \hat{\zeta}^{(a)}_z\big)$ denote the components of $\hat{\zeta}^{(a)}$, and let $\eta = \hat{\iota}^a_{\tilde{a}}(\chi) = ((B_{a-\tilde{a},\, z} u_i),\, z^{a-\tilde{a}})$, then

$$\begin{aligned}\hat{\zeta}^{(a)}_{u_i}\big((u_i), z\big) \equiv{} & B^{-1}_{a-\tilde{a},\, z}\hat{\zeta}^{(\tilde{a})}_{u_i}(\eta)\\ & - 2\,\big\{\tilde{a} - a,\, \hat{\zeta}^{(\tilde{a})}_z(\eta),\, B_{a-\tilde{a},\, z} u_i\big\}\\ & + 2\,Q_{\tilde{a}-a}\big[\big\{z,\, u_i,\, \hat{\zeta}^{(\tilde{a})}_z(\eta)\big\} - \big\{Q_z u_i,\, a - \tilde{a},\, \hat{\zeta}^{(\tilde{a})}_z(\eta)\big\}\big] \mod Z_1^{u_i}\,,\end{aligned}$$

$$\hat{\zeta}^{(a)}_z\big((u_i), z\big) \equiv B_{z,\, a-\tilde{a}}\hat{\zeta}^{(\tilde{a})}_z(\eta) \mod Z_0^{u_k}\,.$$

Conversely, if $\{\hat{\zeta}^{(a)} \mid a \in \overline{Z}\}$ is a family of projectable vector fields on $\overline{Z}_J \times Z$ satisfying (8.35)*, then the projected vector fields $\zeta^{(a)}$ form a family of vector fields on $\overline{\mathcal{E}}_J$ corresponding to a well-defined vector field ζ on the Jordan flag variety $\mathbb{F}_J$.*

[6] cf. Section.

PROOF. By assumption, each $\hat{\zeta}^{(a)}$ is related to $\zeta^{(a)}$ by $D\pi \circ \hat{\zeta}^{(a)} = \zeta^{(a)} \circ \pi$, where π denotes the canonical projection of $\overline{Z}_J \times Z$ onto $\overline{\mathcal{E}}_J$, and $\zeta^{(a)}$ is related to $\zeta^{(\tilde{a})}$ by $\zeta^{(a)} = \big(D\iota_a^{\tilde{a}}\big)\zeta^{(\tilde{a})}$. Due to Lemma 8.18, and using the chain rule, we therefore obtain

$$\begin{aligned} D\pi \circ \hat{\zeta}^{(a)} &= \zeta^{(a)} \circ \pi = \big(D\iota_a^{\tilde{a}}\big)\big(\zeta^{(\tilde{a})} \circ \pi\big) = \big(D\iota_a^{\tilde{a}}\big)\big(D\pi\big)\hat{\zeta}^{(\tilde{a})} \\ &= \big(D(\iota_a^{\tilde{a}} \circ \pi)\big)\hat{\zeta}^{(\tilde{a})} = \big(D(\pi \circ \hat{\iota}_a^{\tilde{a}})\big)\hat{\zeta}^{(\tilde{a})} = \big(D\pi\big)\big(D\hat{\iota}_a^{\tilde{a}}\big)\hat{\zeta}^{(\tilde{a})} . \end{aligned}$$

Therefore, $\hat{\zeta}^{(a)}$ and $\big(D\hat{\iota}_a^{\tilde{a}}\big)\hat{\zeta}^{(\tilde{a})}$ differ by a vertical vector field, and hence are equivalent. The explicit formulas are straightforward to derive from Lemma 8.18 and using the relations

$$B_{\tilde{a}-a,\, z^{a-\tilde{a}}} \overset{\mathsf{JT35}}{=} B^{-1}_{a-\tilde{a},\, z} , \quad \big\{z^{a-\tilde{a}},\, B_{a-\tilde{a},\, z}u_i,\, x\big\} \overset{\mathsf{JT29}}{=} \{z,\, u_i,\, x\} - \{Q_z u_i,\, a-\tilde{a},\, x\} .$$

The converse statement holds, since the same calculation as above yields for the projected vector fields the relation $\zeta^{(a)} = \big(D\iota_a^{\tilde{a}}\big)\zeta^{(\tilde{a})}$, so these vector fields can be glued together to form a vector field on the Jordan flag variety $\mathbb{F}_J$. □

In the next section we will use Lemma 8.19 to show that certain group actions (translations), which are originally defined just on the open and dense subset $\mathbb{F}_J^{(0)}$ of $\mathbb{F}_J$, can be extended to all of $\mathbb{F}_J$.

8.5. Group action

As before, let Z be a phJTS of rank r, let $J = (j_1, \dots, j_k)$ be an increasing family of integers with $0 \le j_i \le r$, and let $\mathbb{F}_J$ be the corresponding Jordan flag variety of type J. Furthermore, let $G^{\mathbb{C}}$ be the identity component of the automorphism group of the Grassmannian $\mathbb{G}(Z)$. We use the characterization of $G^{\mathbb{C}}$ via generators and relations to define a $G^{\mathbb{C}}$-action on the Jordan flag variety $\mathbb{F}_J$, see Section 4.2 for this characterization of $G^{\mathbb{C}}$. For each of the following subgroups of $G^{\mathbb{C}}$, we determine their action on $\mathbb{F}_J$ and the corresponding representation of their Lie algebras as vector fields on $\mathbb{F}_J$. For the latter, we use the description of vector fields given in Section 8.4. Let $[(u_i) : z : a]$ be a fixed element of $\mathbb{F}_J$.

Structure automorphisms. For $h \in K^{\mathbb{C}}$ we define

$$h\,[(u_i) : z : a] \coloneqq [(h^{-*}u_i) : hz : h^{-*}a] .$$

To show that this is well-defined, we refer to Claim 4 in the proof of Theorem 8.11.[7] By abuse of notation, we denote the vector field on $\mathbb{F}_J$ corresponding to an element $\delta \in \mathfrak{k}^{\mathbb{C}}$ also by δ. Let $\delta^{(a)}$ be the family of vector fields on $\overline{\mathcal{E}}_J \cong \mathbb{F}_J^{(a)}$ defined by δ. Since $\mathbb{F}_J^{(0)}$ is $K^{\mathbb{C}}$-invariant, an since this action lifts to the naturally defined $K^{\mathbb{C}}$-action on $\overline{Z}_J \times Z$, the vector field $\delta^{(0)}$ lifts to the vector field

$$\hat{\delta}^{(0)}\big((u_i), z\big) = \big((-\delta^*(u_i)), \delta(z)\big) .$$

Using the relation $[(h_t^{-*}u_i) : h_t z : h^{-*}a] = \Big[(B_{h_t^{-*}a-a,\, h_t z}u_i) : (h_t z)^{h_t^{-*}a-a} : a\Big]$ for $h_t = \exp(t\delta)$ with small $t \in \mathbb{R}$, and taking the derivative with respect to t yields the general formula:

$$\hat{\delta}^{(a)}\big((u_i), z\big) = \big((-\delta^* u_i + 2\,\{\delta a,\, z,\, u_i\}),\ \delta z - Q_u \delta^* a\big) .$$

Quasi-translations. For $w \in \overline{Z}$ set

$$\tilde{t}_v\,[(u_i) : z : a] \coloneqq [(u_i) : z : a + w] .$$

Since in the equivalence relation on $\overline{Z}_J \times Z \times \overline{Z}$ the third components just occur in the difference $a - \tilde{a}$, the quasi-translations are obviously well-defined.

[7] Claim 4 covers the case $k = 1$. For general k, just replace u by (u_i).

For fixed $a \in \overline{Z}$, we have $\tilde{t}_w[(u_i) : z : a] = [(B_{w,z}u_i) : z^w : a]$, and therefore the one-parameter subgroup $\tilde{t}_{tw}$ induces the (lifted) vector field

$$\tilde{w}^{(a)}\big((u_i), z\big) = \big((-2\,\{w,\,z,\,u_i\}), Q_z w\big)\ .$$

We note that these vector fields are in fact independent of a.

Translations. As in the case of the Grassmannian $\mathbb{G}(Z)$, translations are firstly defined on the open and dense subset $\mathbb{F}_J^{(0)} = \{[(u_i) : z : 0] \in \mathbb{F}_J\}$, and secondly it is proved that this action extends to all of $\mathbb{F}_J$. For $v \in Z$ set

$$t_v\,[(u_i) : z : 0] := [(u_i) : z + v : 0]\ .$$

This clearly defines an action of Z on $\mathbb{F}_J^{(0)}$. The one-parameter subgroup t_{tv} induces on the (lifted) vector field on $\overline{Z}_J \times Z$ given by

$$v^{(0)}\big((u_i), z\big) = \big((0), v\big)\ .$$

For the extension of the group action, it suffices to show that the vector field $v^{(0)}$ extends to a smooth vector field on all of $\mathbb{F}_J$, since $\mathbb{F}_J$ is compact. Using Lemma 8.19, we obtain

$$v^{(a)}\big((u_i), z\big) = \big((2\,\{a,\,v,\,B_{a,\,z}u_i\} + 2\,Q_a\,[\{z,\,u_i,\,v\} - \{Q_z u_i,\,a,\,v\}]),\ B_{z,\,a}v\big)$$

Since all of these vector fields are defined on all of $\overline{Z}_J \times Z$, Lemma 8.19 implies that they define a globally defined vector field on $\mathbb{F}_J$, which by construction coincides on $\mathbb{F}_J^{(0)}$ with $v^{(0)}$. Therefore, the action of Z on $\mathbb{F}_J^{(0)}$ by translations extends to the whole Jordan flag variety $\mathbb{F}_J$.

Theorem 8.20. *The actions of structure automorphisms $h \in K^{\mathbb{C}}$, quasi-translations $\tilde{t}_w$ and translations t_v on $\mathbb{F}_J$ defined above induce a $G^{\mathbb{C}}$-action on the Jordan flag variety $\mathbb{F}_J$. Moreover, if Z is* simple*, then:*

(a) *The $G^{\mathbb{C}}$-action on $\mathbb{F}_J$ is transitive: For a fixed $(u_i) \in Z_J$ and any $[(\tilde{u}_i) : z : a]$, we have*

$$[(\tilde{u}_i) : z : a] = \tilde{t}_a t_z h\,[(u_i) : 0 : 0]$$

for some $h \in K^{\mathbb{C}}$ with $h[u_i] = [\tilde{u}_i]$ in the Peirce flag variety $\mathbb{P}_J$.

(b) *The stabilizer subgroup of $[(u_i) : 0 : 0]$ coincides with the parabolic subgroup $P^{\mathbb{C}}_{(u_1,\ldots,u_k)}$ defined in Theorem 8.4, which is conjugate to the complexification of a real parabolic subgroup $Q \subset G$ of type J. In particular, the Jordan flag variety*

$$\mathbb{F}_J \cong G^{\mathbb{C}}/P^{\mathbb{C}}_{(u_1,\ldots,u_k)} \cong G^{\mathbb{C}}/Q^{\mathbb{C}}$$

is a projective variety.

(c) *On the open and dense subset $\mathbb{F}_J^{(0)}$, we have*

$$g\,[(u_i) : z : 0] = [((D_z g)^{-*}u_i) : g(z) : 0]$$

for all $g \in G^{\mathbb{C}}$ with $g(z) \in Z$.

Proof. In view of Theorem 4.6 we have to show first that the actions of h, $\tilde{t}_w$ and t_v satisfy the following relations:

$$(i)\quad h t_v h^{-1} = t_{hv}\ ,\quad (ii)\quad h\tilde{t}_w h^{-1} = \tilde{t}_{h^{-*}w}\ ,\quad (iii)\quad \tilde{t}_w t_v = t_{v^w} B^{-1}_{v,\,w}\tilde{t}_{w^v}\ .$$

In the last relation, the pair $(v, w) \in Z \times \overline{Z}$ is assumed to be quasi-invertible. It suffices to prove these relations on the open and dense subset $\mathbb{F}_J^{(0)}$. The first two relations are straightforward to check:

$$(8.36)\qquad \begin{aligned} ht_v h^{-1}\left[(u_i):z:0\right] &= ht_v\left[(h^*u_i):h^{-1}z:0\right] = h\left[(h^*u_i):h^{-1}z+v:0\right] \\ &= \left[(u_i):z+hv:0\right] = t_{hv}\left[(u_i):z:0\right] , \end{aligned}$$

and

$$(8.37)\qquad \begin{aligned} h\tilde{t}_w h^{-1}\left[(u_i):z:0\right] &= h\tilde{t}_w\left[(h^*u_i):h^{-1}z:0\right] = h\left[(h^*u_i):h^{-1}z:w\right] \\ &= \left[(u_i):z:h^{-*}w\right] = \tilde{t}_{h^{-*}w}\left[(u_i):z:0\right] . \end{aligned}$$

The third relation follows by using the addition formula for the quasi-inverse:

$$(8.38)\qquad \begin{aligned} \tilde{t}_w t_v\left[(u_i):z:0\right] &= \left[(u_i):z+v:w\right] \\ &= \left[(B_{w,\,z+v}u_i):(v+z)^w:0\right] \\ &= \left[(B_{w,\,v}B_{w^v,\,z}u_i):v^w+B_{v,\,w}^{-1}(z^{(w^v)}):0\right] \\ &= t_{v^w}B_{v,\,w}^{-1}\left[(B_{w^v,\,z}u_i):z^{(w^v)}:0\right] \\ &= t_{v^w}B_{v,\,w}^{-1}\left[(u_i):z:w^v\right] \\ &= t_{v^w}B_{v,\,w}^{-1}\tilde{t}_{w^v}\left[(u_i):z:0\right] . \end{aligned}$$

Therefore, the translations, quasi-translations and the action of $K^{\mathbb{C}}$ on $\mathbb{F}_J$ fit together to form a $G^{\mathbb{C}}$-action on the Jordan flag variety $\mathbb{F}_J$. For (a), we just have to note that since Z is simple, $K^{\mathbb{C}}$ acts transitively on the Peirce flag variety $\mathbb{P}_J$, and hence there exists an $h \in K^{\mathbb{C}}$ sending $[u_i]$ to $[\tilde{u}_i]$, and $[(u_i):0:0]$ to $[(\tilde{u}-i):0:0]$.

To prove (b), let P_J denote the stabilizer subgroup of $[(u_i):0:0]$. Since $\mathbb{F}_J$ is a compact complex algebraic variety, it follows that $\mathbb{F}_J$ is a *complete* variety [**35**, §10, Thm.2], and hence P_J is a parabolic subgroup. In addition, this implies that $\mathbb{F}_J = G^{\mathbb{C}}/P_J$ is even a projective variety [**39**, §28.1.4]. So it remains to determine the Lie algebra $\mathfrak{p}_J$ of the stabilizer P_J, and compare this to the results of Theorem 8.4. Elements of the Lie algebra $\mathfrak{g}^{\mathbb{C}}$ are uniquely determined by their representation as vector fields on $\mathbb{F}_J^{(0)}$. Such a vector field $\zeta^{(0)}$ is an element of $\mathfrak{p}_J$ if and only if it vanishes at $[(u_i):0:0]$. In the following, we omit the index $(\,)^{(0)}$, and simply write $\zeta = \zeta^{(0)}$. Due to the decomposition $\mathfrak{g}^{\mathbb{C}} = \mathfrak{u}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{u}^+$, ζ decomposes into $\zeta = \tilde{a}+\delta+b$. As described above, each term lifts to a vector field on $\overline{Z}_J \times Z$, and we obtain

$$\hat{\zeta}\big((\tilde{u}_i),z\big) = \big((-\delta^*(\tilde{u}_i) - 2\,\{a,\,z,\,\tilde{u}_i\}),\; Q_z a + \delta(z) + b\big) .$$

Now, ζ is an element of $\mathfrak{p}_J$ if and only if for all $((\tilde{u}_i),z)$ in $[(u_i),0]$, the vector $\hat{\zeta}\big((\tilde{u}_i),z\big)$ is tangent to the fiber $[(u_i),0]$, i.e. if and only if

$$\hat{\zeta}\big((\tilde{u}_i),z\big) \in \left(\bigtimes_{i=1}^{k} Z_1^{u_i}\right) \times Z_0^{u_k} \quad \text{for all} \quad \big((\tilde{u}_i),z\big) \in \left(\bigtimes_{i=1}^{k} (Z_1^{u_i})^\times\right) \times Z_0^{u_k} .$$

Setting $z = 0$ immediately implies the necessary conditions $b \in Z_0^{u_k}$ and $\delta^*(\tilde{u}_i) \in Z_1^{u_i}$ for all $\tilde{u}_i \in (Z_1^{u_i})^\times$. By density, the second condition is equivalent to $\delta^*(Z_1^{u_i}) \subset Z_1^{u_i}$. These two conditions are also sufficient, since for all $z \in Z_0^{u_k}$, the Peirce rules imply that $\{a,\,z,\,\tilde{u}_i\} = 0$, and for all a, the term $Q_z a$ is an element of $Z_0^{u_k}$. Moreover, from $\delta^*(Z_1^{u_k}) \subset Z_1^{u_k}$ it follows by Lemma 8.2 that $\delta(Z_0^{u_k}) \subset Z_0^{u_k}$. Therefore, we conclude that

$$\mathfrak{p}_J = \overline{Z} \oplus \mathfrak{k}^{\mathbb{C}}(u_1,\ldots,u_k) \oplus Z_0^{u_k}$$

with $\mathfrak{k}^{\mathbb{C}}(u_1,\ldots,u_k) = \left\{\delta \in \mathfrak{k}^{\mathbb{C}} \,\middle|\, \delta^*(Z_1^{u_i}) \subset Z_1^{u_i} \text{ for all } i\right\}$. By Theorem 8.4, this is the Lie algebra of $P^{\mathbb{C}}_{(u_1,\ldots,u_k)}$, and since parabolic subgroups are uniquely determined by their Lie algebras, we conclude that $P_J = P^{\mathbb{C}}_{(u_1,\ldots,u_k)}$. Furthermore, for any flag of tripotents $(e_1,\ldots,e_k)$, which is Peirce equivalent to $(u_1,\ldots,u_k)$, Theorem 8.4

implies that $Q := \mathrm{Ad}_{\gamma_{e_k}} P_J$ is the complexification of the real parabolic subgroup $Q_{(e_1,\ldots,e_k)} \subset G$ of type J described in Section 8.1.

Finally, for (c) it suffices to prove the stated relation for the generators of $G^{\mathbb{C}}$, namely t_v and $\tilde{t}_w$ with $(v,w) \in Z \times \overline{Z}$. For translations t_v, we have $D_z t_v = \mathrm{Id}$, so there is nothing to prove. The derivative of the quasi-translation $\tilde{t}_w$ is given by $D_z\tilde{t}_w = B_{z,w}^{-1}$, and hence we obtain

$$\tilde{t}_w[(u_i) : z : 0] = [(u_i) : z : w] = [(B_{w,z}u_i) : z^w : 0] = [((D_z\tilde{t}_w)^{-*}u_i) : \tilde{t}_w(z) : 0]\,.$$

For an arbitrary element $g \in G^{\mathbb{C}}$, this relation follows by using the chain rule in the first component. This completes the proof of Theorem 8.20. $\square$

8.6. Line bundles

As in the case of the Grassmannian $\mathbb{G}(Z)$ and the Peirce flag varieties $\mathbb{P}_J$, we define line bundles on the Jordan flag variety $\mathbb{F}_J$ by the Godement approach via appropriate cocycles on the equivalence relation. Since the Jordan flag variety locally looks like a vector bundle on the Peirce flag variety $\mathbb{P}_J$, it is not astonishing that these cocycles have similar structure as those used for line bundles on $\mathbb{P}_J$. However, we note that since cocycles are defined globally, it is not sufficient just to use the local description.

As before let Z be a phJTS, and let $\mathbb{F}_J$ be the Jordan flag variety of type $J = (j_1,\ldots,j_k)$. Let $R_{\mathbb{F}_J} \subset (\overline{Z}_J \times Z \times \overline{Z})^2$ denote the defining equivalence relation of $\mathbb{F}_J$. Recall from Section 2.9 that any denominator δ of the quasi-inverse on Z defines for each u a denominator of the inverse on the unital Jordan algebra Z_1^u given by $\delta^u(z) = \delta(u^\dagger - z,\, u)$ for $z \in Z$. By Proposition 6.12, this induced Jordan algebra denominator satisfies

$$\delta^{\tilde{u}}\big(u^\dagger\big) = \delta^u\big(\tilde{u}^\dagger\big)^{-1} \quad \text{and} \quad \delta^{h\tilde{u}}\big((hu)^\dagger\big) = \delta^{\tilde{u}}\big(u^\dagger\big) \tag{8.39}$$

for all Peirce equivalent $u, \tilde{u}$ and $h \in K^{\mathbb{C}}$.

Proposition 8.21. *Let δ be a denominator of the quasi-inverse on Z, and fix $\ell \in \{1,\ldots,k\}$. Then, the map*

$$\phi : R_{\mathbb{F}_J} \to \mathbb{C}^\times \quad \textit{with} \quad \phi_{((\tilde{u}_i),\tilde{z},\tilde{a})}^{((u_i),z,a)} = \delta^{\tilde{u}_\ell}\big((B_{a-\tilde{a},\,z}u_\ell)^\dagger\big)\,. \tag{8.40}$$

is a $K^{\mathbb{C}}$-invariant holomorphic cocycle. Let $\mathcal{L}_{j_\ell}(\delta)$ denote the corresponding line bundle on $\mathbb{F}_J$.

Proof. First we note that ϕ does not depend on u_i, $\tilde{u}_i$ for $i \neq \ell$. Therefore, it is equivalent to prove that $\phi_{(\tilde{u},\tilde{z},\tilde{a})}^{(u,z,a)}$ is a $K^{\mathbb{C}}$-invariant holomorphic cocycle on $\mathbb{F}_{(\ell)}$. First we note that ϕ does not attain 0, since by assumption, $\tilde{u}$ is Peirce equivalent to $B_{a-\tilde{a},\,z}u$, so the corresponding Jordan algebra denominator does not vanish. Now let (u,z,a), $(\tilde{u},\tilde{z},\tilde{a})$, $(\hat{u},\hat{z},\hat{a})$ be pairwise equivalent elements of $\overline{Z}_\ell \times Z \times \overline{Z}$. Without restriction, we assume that $(z, a-\tilde{a})$ and $(\tilde{z}, \tilde{a}-\hat{a})$ are quasi-invertible, otherwise replace in the following z and $\tilde{z}$ by appropriate $z + u^{\perp\!\!\perp}$ and $\tilde{z} + \tilde{u}^{\perp\!\!\perp}$ according to the equivalence relation. Therefore, $B_{\hat{a}-\tilde{a},\,\hat{u}}$ is an element of $K^{\mathbb{C}}$, and by (8.39) we obtain

$$\begin{aligned}
\phi_{(\tilde{u},\tilde{z},\tilde{a})}^{(\hat{u},\hat{z},\hat{a})} \cdot \phi_{(\hat{u},\hat{z},\hat{a})}^{(u,z,a)} &= \delta^{\tilde{u}}\big((B_{\hat{a}-\tilde{a},\,\tilde{z}}\hat{u})^\dagger\big) \cdot \delta^{\hat{u}}\big((B_{a-\hat{a},\,z}u)^\dagger\big) \\
&= \delta^{B_{\hat{a}-\tilde{a},\,\tilde{z}}\hat{u}}\big(\tilde{u}^\dagger\big)^{-1} \cdot \delta^{B_{\hat{a}-\tilde{a},\,\tilde{z}}\hat{u}}\big((B_{\hat{a}-\tilde{a},\,\tilde{z}}B_{a-\hat{a},\,z}u)^\dagger\big)
\end{aligned}$$

By equivalence, we have $\hat{z} = z^{a-\hat{a}}$, and therefore $B_{\hat{a}-\tilde{a},\,\tilde{z}}B_{a-\hat{a},\,z} = B_{a-\tilde{a},\,z}$, and using (2.53), the right hand side becomes $\phi_{(\tilde{u},\tilde{z},\tilde{a})}^{(u,z,a)}$. This proves the cocycle condition. The $K^{\mathbb{C}}$-invariance follows immediately from the second equation of (8.39). Finally we prove that ϕ is holomorphic. Since $\tilde{u}$ and $B_{a-\tilde{a},\,z}u$ are Peirce equivalent, Lemma 2.16 yields

$$(B_{a-\tilde{a},\,z}u)^{\dagger} = Q_{\tilde{u}}(B_{a-\tilde{a},\,z}u)^{-1} .$$

Therefore, ϕ is holomorphic in $a, \tilde{a} \in \overline{Z}$ and $z \in Z$. The argument for the holomorphic dependence is exactly the same as in the proof of Proposition 6.12. We just note that since $\overline{Z}_{(\ell)}$ is taken with the conjugate complex structure, here is no need for complex conjugation of the cocycle. □

REMARK 8.22. Proposition 8.21 obviously generalizes to any other family of tensor products of the line bundles $\mathcal{L}_{j_\ell}(\delta)$, i.e. to tensor products of the form

$$\mathcal{L} = \mathcal{L}_{j_1}(\delta)^{\mu_1} \otimes \ldots \otimes \mathcal{L}_{j_k}(\delta)^{\mu_k} \quad \text{with} \quad \mu_i \in \mathbb{Z} .$$

The defining cocycles are given by the corresponding products of the cocycles for the single line bundles $\mathcal{L}_{j_\ell}(\delta)$. In particular, we define

$$\mathcal{L}_J(\delta) \coloneqq \mathcal{L}_{j_1}(\delta) \otimes \ldots \otimes \mathcal{L}_{j_k}(\delta) . \tag{8.41}$$

In the next chapter we describe sections in $\mathcal{L}_J(\delta)$ of particular importance in harmonic analysis.

Homogeneity. Recall from Theorem 3.8 that the line bundle $\mathcal{L}_{j_\ell}(\delta)$ is given as the quotient manifold of $\overline{Z}_J \times Z \times \overline{Z} \times \mathbb{C}$ by the equivalence relation

$$((u_i), z, a, \lambda) \sim ((\tilde{u}_i), \tilde{z}, \tilde{a}, \tilde{\lambda}) \iff \begin{cases} [(u_i) : z : a] = [(\tilde{u}_i) : \tilde{z} : \tilde{a}] , \\ \tilde{\lambda} = \delta^{\tilde{u}_\ell}\big((B_{a-\tilde{a},\,z}u_\ell)^{\dagger}\big) \cdot \lambda . \end{cases}$$

The elements of $\mathcal{L}_{j_\ell}(\delta)$ are denoted by $[(u_i), z, a, \lambda]$. Let π denote the canonical projection of $\mathcal{L}_{j_\ell}(\delta)$ onto the Jordan flag variety $\mathbb{F}_J$. Since the defining cocycle is $K^{\mathbb{C}}$-invariant,

$$h[(u_i), z, a, \lambda] \coloneqq [(h^{-*}u_i), hz, h^{-*}a, \lambda] \quad \text{for} \quad h \in K^{\mathbb{C}} \tag{8.42}$$

defines a $K^{\mathbb{C}}$-action on $\mathcal{L}_{j_\ell}(\delta)$ such that π is $K^{\mathbb{C}}$-equivariant. We claim that this extends to a $G^{\mathbb{C}}$-action on $\mathcal{L}_{j_\ell}(\delta)$ which turns $\mathcal{L}_{j_\ell}(\delta)$ into a $G^{\mathbb{C}}$-homogeneous line bundle on $\mathbb{F}_J$. For $(v, w) \in Z \times \overline{Z}$ set

$$t_v[(u_i), z, 0, \lambda] \coloneqq [(u_i), z + v, 0, \lambda] , \tag{8.43}$$

and

$$\tilde{t}_w[(u_i), z, a, \lambda] \coloneqq [(u_i), z, a + w, \lambda] . \tag{8.44}$$

As in the definition of the $G^{\mathbb{C}}$-action on $\mathbb{F}_J$, the translations t_u are defined just on an open and dense subset of $\mathcal{L}_{j_\ell}(\delta)$, namely $\pi^{-1}(\mathbb{F}_J^{(0)}) \cong \mathbb{F}_J^{(0)} \times \mathbb{C}$, and by the same method as above one shows that the translations extend to all of $\mathcal{L}_{j_\ell}(\delta)$: The vector field on $\mathbb{F}_J^{(0)} \times \mathbb{C}$ which is induced by the 1-parameter subgroup $(t_{\tau v})_{\tau \in \mathbb{R}}$ lifts to a vector field $\zeta^{(0)}$ on $\overline{Z}_J \times Z \times \overline{Z} \times \mathbb{C}$. Using extensions of Lemma 8.18 and Lemma 8.19 with[8]

$$\hat{\iota}_{\tilde{a}}^{a} : \overline{Z}_J \times Z \times \mathbb{C} \dashrightarrow \overline{Z}_J \times Z \times \mathbb{C}, \ ((u_i), z, \lambda) \mapsto ((B_{a-\tilde{a},\,z}u_i), z^{a-\tilde{a}}, \lambda) ,$$

the vector field $\zeta^{(0)}$ defines a family of smooth vector fields $\{\zeta^{(a)} \mid a \in \overline{Z}\}$ on $\overline{Z}_J \times Z \times \mathbb{C}$. Since $\hat{\iota}_{\tilde{a}}^{a}$ acts identically on the fiber coordinate, these vector fields essentially coincide with the ones in the discussion of translations on $\mathbb{F}_J$. Therefore, they define a complete vector field on $\mathcal{L}_{j_\ell}(\delta)$, which extends the action of the translation to all of $\mathcal{L}_{j_\ell}(\delta)$. It remains to show that the $K^{\mathbb{C}}$-action, the translations and the quasi-translations fit together to build a $G^{\mathbb{C}}$-action on $\mathcal{L}_{j_\ell}(\delta)$, i.e. h, t_v and $\tilde{t}_w$ satisfy

$$(i) \quad h t_v h^{-1} = t_{hv} , \quad (ii) \quad h \tilde{t}_w h^{-1} = \tilde{t}_{h^{-*}w} , \quad (iii) \quad \tilde{t}_w t_v = t_{v^w} B_{v,w}^{-1} \tilde{t}_{w^v} .$$

---

[8] Since $\delta^{B_{a-\tilde{a},\,z}u_\ell}\big((B_{a-\tilde{a},\,z}u_\ell)^{\dagger}\big) = 1$, the transition map $\hat{\iota}_{\tilde{a}}^{a}$ does not change the fiber.

It suffices to prove these relations on the open and dense subset $\mathbb{F}_J^{(0)} \times \mathbb{C} \subset \mathcal{L}_{j_\ell}(\delta)$. The first two relations are proved in exactly the same way as (8.36) and (8.37) above. The crucial relation is the third one, since here the equivalence relation is used. However, from (8.39) it follows

$$\begin{aligned}\big[(u_i), z, a, \lambda\big] &= \Big[(B_{a-\tilde{a},\,z} u_i), z^{a-\tilde{a}}, \tilde{a}, \delta^{B_{a-\tilde{a},\,z} u_\ell}\big((B_{a-\tilde{a},\,z} u_\ell)^\dagger\big) \cdot \lambda\Big] \\ &= \Big[(B_{a-\tilde{a},\,z} u_i), z^{a-\tilde{a}}, \tilde{a}, \lambda\Big]\,.\end{aligned}$$

Therefore, also the third relation is proved by the same calculation as in (8.38). This finally proves the $G^{\mathbb{C}}$-homogeneity of the line bundle $\mathcal{L}_{j_\ell}(\delta)$. We summarize:

Proposition 8.23. *The $K^{\mathbb{C}}$-action* (8.42)*, the translation* (8.43) *and the quasi-translation* (8.44) *define a $G^{\mathbb{C}}$-action on the line bundle $\mathcal{L}_{j_\ell}(\delta)$ which turns $\mathcal{L}_{j_\ell}(\delta)$ into a $G^{\mathbb{C}}$-homogeneous line bundle on $\mathbb{F}_J$. On the open and dense subset $\pi^{-1}\big(\mathbb{F}_J^{(0)}\big) \subset \mathcal{L}_{j_\ell}(\delta)$, the $G^{\mathbb{C}}$-action is given by*

$$g\big[(u_i), z, 0, \lambda\big] = \Big[\big((D_z g)^{-*} u_i\big),\, g(z),\, 0,\, \lambda\Big]$$

for all $g \in G^{\mathbb{C}}$ with $g(z) \in Z$.

Remark 8.24. We note that the defining cocycle of the line bundle $\mathcal{L}_{j_\ell}(\delta)$ is not $G^{\mathbb{C}}$-invariant, since there is no $G^{\mathbb{C}}$-action on the set $\overline{Z}_J \times Z \times \overline{Z}$. Again, Proposition 8.23 immediately generalizes to tensor products of the line bundles $\mathcal{L}_{j_\ell}(\delta)$. In particular, $\mathcal{L}_J(\delta)$ is a $G^{\mathbb{C}}$-homogeneous line bundle.

Projective imbedding. In the case of the Grassmannian $\mathbb{G}(Z)$ and the Peirce flag varieties $\mathbb{P}_J$, suitable line bundles are used to define imbeddings of $\mathbb{G}(Z)$ and $\mathbb{P}_J$ into some projective space, cf. Theorem 4.4 and Theorem 6.20. It turns out that the line bundle $\mathcal{L}_J(\delta)$ defined in (8.41) is not ample enough to define such an imbedding for the Jordan flag variety $\mathbb{F}_J$. In the same way as for the Peirce flag variety, the line bundle $\mathcal{L}_J(\delta)$ defines a morphism of $\mathbb{F}_J$ into some projective space. However, this morphism fails to be injective: loosely speaking, the single line bundle $\mathcal{L}_\ell(\delta)$ just take care for the injectivity corresponding to the u_ℓ-coordinate, but different (z, a)-coordinates are not separated. Therefore, we still need to tensor $\mathcal{L}_J(\delta)$ with an additional line bundle $\mathcal{L}'(\delta)$, which should be closely related to the line bundle $\mathcal{L}_\delta$ defined on the Grassmannian $\mathbb{G}(Z)$. So far, we failed in the construction of a corresponding cocycle – except from the case $\mathrm{rk}(Z) \in J$, where the map $\big[(u_i), z, a\big] \mapsto [z : a]$ is a well-defined fibration of $\mathbb{F}_J$ over $\mathbb{G}(Z)$, and hence $\mathcal{L}'(\delta)$ is defined as the pull-back of $\mathcal{L}_\delta$. For the general situation, this remains an open problem.

Open problem. *Define a line bundle $\mathcal{L}'(\delta)$ on $\mathbb{F}_J$ corresponding to the line bundle $\mathcal{L}_\delta$ on $\mathbb{G}(Z)$ defined in Section 4.1, such that $\mathcal{L}_J(\delta) \otimes \mathcal{L}'(\delta)$ is very ample.*

CHAPTER 9

Determinant functions

In 1996/97, L. Barchini, S.G. Gindikin and H.W. Wong published two papers on certain 'determinant functions' on the full flag manifold, which have both geometric and representation theoretic importance [**3, 4**]. In Section 9.1, we briefly recall the basic definition of these determinant functions together with a first geometric application. Section 9.2 provides a Jordan theoretic account on determinant functions. We introduce so called Jordan determinant functions which are defined using the Godement approach in line bundles. In Section 9.3, we give a first comparison of the determinant functions of Barchini-Gindikin-Wong with the Jordan determinant functions.

9.1. Determinant functions of Barchini-Gindikin-Wong

This section is a review of the construction and the geometric interpretation of the determinant functions introduced by L. Barchini, S.G. Gindikin and H.W. Wong [**3, 4**]. We modify the notation such that it fits into the notation used so far. For fixed $r, s \in \mathbb{N}$ and $n = r + s$ let X be the manifold of complete flags in $\mathbb{C}^n$,

$$X = \{U_1 \subset \ldots \subset U_{n-1} \subset U_n = \mathbb{C}^n \,|\, \dim U_i = i \text{ for all } i\} \ , \tag{9.1}$$

and let Y be the manifold of disjoint subspaces V, W in $\mathbb{C}^n$ of dimension r and s,

$$Y = \{(V, W) \subset \mathbb{C}^n \times \mathbb{C}^n \,|\, \dim V = r,\ \dim W = s,\ V \oplus W = \mathbb{C}^n\} \ . \tag{9.2}$$

Elements of X are abbreviated by $(U_i) \coloneqq (U_1, \ldots, U_n)$. We consider the following semisimple Lie groups: $G = \mathrm{SU}(r, s)$ with maximal compact subgroup $K = \mathrm{S}(\mathrm{U}(r) \times \mathrm{U}(s))$, and let

$$G^{\mathbb{C}} = \mathrm{SL}(r + s) \ , \quad K^{\mathbb{C}} = \mathrm{S}(\mathrm{GL}(r) \times \mathrm{GL}(s)) \tag{9.3}$$

be their complexifications. Furthermore, let B denote the Borel subgroup of $G^{\mathbb{C}}$ consisting of all upper triangular matrices in $\mathbb{C}^{n\times n}$ of determinant 1, then we have

$$X = G^{\mathbb{C}}/B \quad \text{and} \quad Y = G^{\mathbb{C}}/K^{\mathbb{C}} \ . \tag{9.4}$$

Indeed, the natural $G^{\mathbb{C}}$-action on $\mathbb{C}^n$ given by $(g, x) \mapsto gx$ extends to a transitive action on X and Y, and if $e_1, \ldots, e_n$ denotes the canonical basis of $\mathbb{C}^n$ the flag $(U_1, \ldots, U_n)$ with $U_i = \langle e_1, \ldots, e_i\rangle$ is stabilized by B, and the disjoint subspaces

$$(V_0, W_0) = (\langle e_1, \ldots, e_r\rangle,\ \langle e_{r+1}, \ldots, e_{r+s}\rangle) \tag{9.5}$$

are stabilized by $K^{\mathbb{C}}$.

The *determinant functions* $\mathcal{D}_j$ are defined on the product manifold $X \times Y$. It is noted in [**4**, Remark 2.2] that these 'functions' are in fact sections of suitable line bundles on $X \times Y$. For the definition, we need two operators on subspaces of $\mathbb{C}^n$.

For any two subspaces U, V in $\mathbb{C}^n$ let

$$U \sqcap V := \begin{cases} U \cap V & \text{, if } U \text{ and } V \text{ intersect transversely, i.e. } U + V = \mathbb{C}^n, \\ \{0\} & \text{, otherwise.} \end{cases}$$

$$U \sqcup V := \begin{cases} U \oplus V & \text{, if } U \text{ and } V \text{ intersect trivially, i.e. } U \cap V = \{0\}, \\ \{0\} & \text{, otherwise.} \end{cases}$$

We note that in [**3, 4**], subspaces of $\mathbb{C}^n$ are represented due to the Plücker imbedding by elements in the projective space of the exterior algebra $\bigwedge^\bullet \mathbb{C}^n$. Then, $\sqcap$ and $\sqcup$ are defined using the star operator, the wedge product and the interior product of alternating forms. More precisely, if U and V correspond to the forms ω and θ, then $\omega \sqcap \theta = \iota(*\omega)\theta$ and $\omega \sqcup \theta = \omega \wedge \theta$, where $\iota(\,)$ denotes the interior product on the exterior algebra. Furthermore, we can identify a form ω of top degree with a scalar $[\omega]$, by comparing it to the standard top form on $\mathbb{C}^n$ given by $e_1 \wedge \ldots \wedge e_n$. For forms of lower degree, we have $[\omega] = 0$. Above, we reviewed the geometric interpretations of $\sqcap$ and $\sqcup$, and having the correspondence between a subspace U and an appropriate alternating form ω in mind, we also set $[U] = [\omega]$. Therefore, $[U] = 0$ if and only if $\dim U < n$.

The *Barchini-Gindikin-Wong determinant functions* $\mathcal{D}_j$, $j = 0, \ldots, r$, are defined on the product $X \times Y$ as follows:

$$\mathcal{D}_j\big((U_i), (V, W)\big) = \begin{cases} [U_s \sqcup V] & \text{, if } j = 0, \\ [(U_{n-j} \sqcap W) \sqcup U_j \sqcup V] & \text{, if } 1 \le j \le r-1, \\ [U_r \sqcup W] & \text{, if } j = r. \end{cases} \tag{9.6}$$

In addition, let $(V_0, W_0) \in Y$ be as in (9.5), and define the *restricted* determinant functions $\mathcal{D}_j^0$ on X by

$$\mathcal{D}_j^0\big((U_i)\big) := D_j\big((U_i), (V_0, W_0)\big) . \tag{9.7}$$

Remark 9.1. We note that the determinant function $\mathcal{D}_j$ just depends on the subspaces U_j, U_{n-j}, V and W. Therefore, $\mathcal{D}_j$ can also be considered as a function on $\mathrm{Gr}_{(j,n-j)} \times Y$. In the same way, the restricted determinant function $\mathcal{D}_j^0$ can be regarded as a function on $\mathrm{Gr}_{(j,n-j)}$.

For the geometric interpretation of the restricted determinant functions, we define the following subsets of X: for $0 < j < r$,

$$\mathcal{X}_j^0 := \left\{(U_i) \,\middle|\, \mathcal{D}_\ell^0((U_i)) \neq 0 \text{ if } \ell \neq j,\ \mathcal{D}_j^0((U_i)) = 0\right\} ,$$

and for $j = 0$ and $j = r$,

$$\mathcal{X}_0^0 := \left\{(U_i) \,\middle|\, \mathcal{D}_\ell^0((U_i)) \neq 0 \text{ if } \ell > 0,\ \mathcal{D}_0^0((U_i)) = 0,\ [U_{s-1} \sqcap V_0] \neq 0\right\} ,$$

$$\mathcal{X}_r^0 := \left\{(U_i) \,\middle|\, \mathcal{D}_\ell^0((U_i)) \neq 0 \text{ if } \ell < r,\ \mathcal{D}_r^0((U_i)) = 0,\ [U_{r-1} \sqcup W_0] \neq 0,\ [U_{r+1} \sqcap W_0] \neq 0\right\} .$$

With these definitions, Barchini-Gindikin-Wong have proved the following result on $K^{\mathbb{C}}$-orbits on X of codimension at most one [**3**, §1-4].

Proposition 9.2. *Let X and $\mathcal{D}_j^0$ be as defined above, then:*

(a) *The vanishing sets of the determinant functions are $K^{\mathbb{C}}$-invariant.*

(b) *The set*

$$\mathcal{X} = \left\{(U_i) \in X \,\middle|\, \mathcal{D}_j^0((U_i)) \neq 0 \textit{ for all } j\right\}$$

is the unique open $K^{\mathbb{C}}$-orbit in X,

(c) *There are exactly $r+1$ codimensional one $K^{\mathbb{C}}$-orbits in X. They are $\mathcal{X}_j^0$.*

(d) *The Zariski closure $\mathcal{X}_j$ of $\mathcal{X}_j^0$ is the set $\left\{(U_i) \in X \,\middle|\, \mathcal{D}_j^0((U_i)) = 0\right\}$.*

9.2. Jordan theoretic determinant functions

Let Z be a phJTS of rank r, and let $\mathbb{F}_J$ be the Jordan flag variety of type $J = (j_1, \ldots, j_k)$. Recall from Section 8.6 that each denominator δ of the quasi-inverse on Z defines a series of $G^{\mathbb{C}}$-homogeneous line bundles $\mathcal{L}_{j_\ell}(\delta)$, $1 \le \ell \le k$, on $\mathbb{F}_J$ which are given as the quotient manifold of $\overline{Z}_J \times Z \times \overline{Z} \times \mathbb{C}$ by the equivalence relation[1]

$$((u_i), z, c, \lambda) \sim ((\tilde{u}_i), \tilde{z}, \tilde{c}, \tilde{\lambda}) \iff \begin{cases} [(u_i) : z : c] = [(\tilde{u}_i) : \tilde{z} : \tilde{c}] \ , \\ \tilde{\lambda} = \delta^{\tilde{u}_\ell}\big((B_{c-\tilde{c},\, z} u_\ell)^\dagger\big) \cdot \lambda \ . \end{cases}$$

Now, we introduce naturally defined sections of these line bundles, which turn out to generalize the Barchini-Gindikin-Wong determinant functions $\mathcal{D}_j^0$ defined in the last section.

Proposition 9.3. *Let δ be a denominator of the pseudo-inverse on Z, let $J = (j_1, \ldots, j_k)$ and fixed $\ell \in \{1, \ldots, k\}$. The map*

$$\mathbf{D}_{j_\ell}^0 : \mathbb{F}_J \to \mathcal{L}_{j_\ell}(\delta),\ [(u_i) : z : c] \mapsto \big[(u_i),\, z,\, c,\, \delta^{u_\ell}(z - Q_z c)\,\big]$$

is a holomorphic $K^{\mathbb{C}}$-equivariant section on $\mathbb{F}_J$, called the restricted j_ℓ*-th* Jordan determinant function *on $\mathbb{F}_J$. The vanishing set of $\mathbf{D}_{j_\ell}^0$ is independent of the choice of the denominator δ.*

Proof. Without restriction we assume $J = (j_\ell)$ and set $j = j_\ell$, $u = u_\ell$. Due to Proposition 3.12, we first have to show that the map $\hat{\sigma}(u, z, c) = \delta^u(z - Q_z c)$ satisfies for all $[u : z : c] = [\tilde{u} : \tilde{z} : \tilde{c}]$ the relation

$$\hat{\sigma}(\tilde{u}, \tilde{z}, \tilde{c}) = \phi^{(u,z,c)}_{(\tilde{u},\tilde{z},\tilde{c})} \cdot \hat{\sigma}(u, z, c) \quad \text{with} \quad \phi^{(u,z,c)}_{(\tilde{u},\tilde{z},\tilde{c})} = \delta^{\tilde{u}}\big((B_{c-\tilde{c},\, z} u)^\dagger\big) \ .$$

We note that the Peirce rules imply that $\delta^u(z - Q_z c)$ does not depend on the Z_0^u-component of z. Therefore, we assume without restriction that $(z, c - \tilde{c})$ is quasi-invertible and $\tilde{z} = z^{c-\tilde{c}}$, otherwise replace in the following z and $\tilde{z}$ by appropriate $z + u^{\perp\!\!\!\perp}$ and $\tilde{z} + \tilde{u}^{\perp\!\!\!\perp}$ according to the equivalence relation on $\mathbb{F}_{(j)}$. Due to Lemma 2.35 and using JT28, we obtain

$$\begin{aligned} \delta^{\tilde{u}}(\tilde{z} - Q_{\tilde{z}}\tilde{c}) &= \delta^{B_{c-\tilde{c},\, z} u}\big(\tilde{u}^\dagger\big)^{-1} \cdot \delta^{B_{c-\tilde{c},\, z} u}(\tilde{z} - Q_{\tilde{z}}\tilde{c}) \\ &= \delta^{\tilde{u}}\big((B_{c-\tilde{c},\, z} u)^\dagger\big) \cdot \delta^u(B_{z,\, c-\tilde{c}}(\tilde{z} - Q_{\tilde{z}}\tilde{c})) \\ &= \delta^{\tilde{u}}\big((B_{c-\tilde{c},\, z} u)^\dagger\big) \cdot \delta^u\big(B_{z,\, c-\tilde{c}} z^{c-\tilde{c}} - B_{z,\, c-\tilde{c}} Q_{z^{c-\tilde{c}}}\tilde{c}\big) \\ &= \delta^{\tilde{u}}\big((B_{c-\tilde{c},\, z} u)^\dagger\big) \cdot \delta^u(z - Q_z(c - \tilde{c}) - Q_z\tilde{c}) \\ &= \delta^{\tilde{u}}\big((B_{c-\tilde{c},\, z} u)^\dagger\big) \cdot \delta^u(z - Q_z c) \ . \end{aligned}$$

Therefore, $\mathbf{D}_{j_\ell}^0$ is a well-defined section in $\mathcal{L}_{j_\ell}(\delta)$. The $K^{\mathbb{C}}$-equivariance follows from

$$\hat{\sigma}(h^{-*}u, hz, h^{-*}c) = \delta^{h^{-*}u}(hz - Q_{hz}h^{-*}c) = \delta^u\big(h^{-1}(hz - hQ_z c)\big) = \hat{\sigma}(u, z, c)$$

and the definition of the $K^{\mathbb{C}}$-action on $\mathcal{L}_{j_\ell}(\delta)$ given in (8.42). To prove that $\mathbf{D}_{j_\ell}^0$ is holomorphic, we first note that $\mathbf{D}_{j_\ell}^0$ is real analytic, since

$$\hat{\sigma}(u, z, c) = \delta(u^\dagger - (z - Q_z c),\, u) \ , \tag{9.8}$$

and the pseudo-inverse map is real analytic. Therefore, it suffices to show that $\mathbf{D}_{j_\ell}^0$ is holomorphic in each of the variables (u, z, c) separately. Equation (9.8) implies holomorphy for $z \in Z$ and $c \in \overline{Z}$. Concerning the u-coordinate, we recall from

[1] In preparation for Theorem 9.5, we change the symbol used so far for the elements of $\overline{Z}$ and write $[(u_i) : z : c]$ instead of $[(u_i) : z : a]$ for an element of the Jordan flag variety $\mathbb{F}_J$.

Proposition 3.12 that it is sufficient to show holomorphy of (9.8) along a transversal covering of $\mathbb{F}_{(j)}$. Therefore, we just have to show that the map $\hat{\sigma}(\varphi_u(y), z, c)$ with

$$\varphi_u : \overline{Z}^u_{1/2} \to \overline{Z}_j,\ y \mapsto \tau_{u,y}(u) = B_{y,-u^\dagger} u\ .$$

is holomorphic in $y \in \overline{Z}^u_{1/2}$. By Lemma 2.35, we obtain

$$\hat{\sigma}(\varphi_u(y), z, c) = \delta^{B_{y,-u^\dagger} u}(z - Q_z c) = \delta^u\big(B_{-u^\dagger, y}(z - Q_z c)\big)\ .$$

This term is indeed holomorphic in $y \in \overline{Z}^u_{1/2}$. Finally, the independence of the vanishing set of $\mathbf{D}^0_{j_\ell}$ follows by the same argument as in the proof of Proposition 5.4. This completes the proof. □

Remark 9.4. We note that the restricted Jordan determinant function $\mathbf{D}^0_{j_\ell}$ itself depends on the choice of a denominator δ. However, since we are just interested in the vanishing set of $\mathbf{D}^0_{j_\ell}$, which *is* independent of δ, we prefer not to refer to δ in the notation for the restricted Jordan determinant function. In addition, we note that since the fiber coordinate of $\mathbf{D}^0_{j_\ell}$ just depends on (u_ℓ, z, c), it is also possible to consider $\mathbf{D}^0_{j_\ell}$ just as well as a section on the Jordan flag variety $\mathbb{F}_{(j_\ell)}$ of type (j_ℓ).

We note that $\mathbf{D}^0_{j_\ell}$ is not $G^{\mathbb{C}}$-equivariant. Next we extend the Jordan determinant functions to sections on the product manifold $(\mathbb{G}\times\overline{\mathbb{G}})\times\mathbb{F}_J$, where $\mathbb{G}\times\overline{\mathbb{G}}$ denotes the complexification of the Grassmannian $\mathbb{G}$, which we discussed in Chapter 5.

As in Section 5.2, let $\mathcal{L}_\delta$ be the standard line bundle on $\mathbb{G}$ defined by the denominator δ, and let $\mathcal{L}_\delta^{-1}$ denote its corresponding inverse bundle defined by

$$[x, a, \lambda] = [x^{a-\tilde{a}},\ \tilde{a},\ \delta(x, a - \tilde{a})^{-1}\,\lambda]\ .$$

Furthermore, we note that the line bundles on $\mathbb{G}$, $\overline{\mathbb{G}}$ and $\mathbb{F}_J$ can be pulled back to line bundle on the product $(\mathbb{G}\times\overline{\mathbb{G}})\times\mathbb{F}_J$. Let $\mathcal{L}_\delta^{-1}\boxtimes\overline{\mathcal{L}_\delta^{-1}}\boxtimes\mathcal{L}_{j_\ell}(\delta)$ denote the tensor product of the corresponding pull-back bundles. Recall from Proposition 5.4 that

$$\widehat{\delta}\big((x,a),(y,b)\big) = \delta(x,b)\,\delta(x^a, y^b)\,\overline{\delta(y,b)} \tag{9.9}$$

with $(x,a)\in Z\times\overline{Z}$ and $(y,b)\in\overline{Z}\times Z$ is a complex polynomial function on $(Z\times\overline{Z})\times(\overline{Z}\times Z)$, which defines a section $\boldsymbol{\delta}$ of $\mathcal{L}_\delta^{-1}\boxtimes\overline{\mathcal{L}_\delta^{-1}}$ on $\mathbb{G}\times\overline{\mathbb{G}}$.

Theorem 9.5. *Let δ be a denominator of the quasi-inverse on Z, let $\widehat{\delta}$ be as in* (9.9)*, and fix $\ell\in\{1,\dots,k\}$. Then,*

(a) *For $(x,a)\in Z\times\overline{Z}$, $(y,b)\in\overline{Z}\times Z$ and $((u_i),z,c)\in\overline{Z}_J\times Z\times\overline{Z}$ set*

$$\widehat{D}_{j_\ell}\big((x,a),(y,b),((u_i),z,c)\big) =$$
$$= \widehat{\delta}((x,a),(y,b))\cdot\delta^{B_{c,z}u_\ell}\Big(-(x^a - z^c)^{\left((y^b)^{(z^c)}\right)}\Big)\ .$$

Then, $\widehat{D}_{j_\ell}$ depends (complex) polynomially on $x,b,z\in Z$, $a,c,y\in\overline{Z}$.

(b) *The map $\mathbf{D}_{j_\ell}:\mathbb{G}\times\overline{\mathbb{G}}\times\mathbb{F}_J\to\mathcal{L}_\delta^{-1}\boxtimes\overline{\mathcal{L}_\delta^{-1}}\boxtimes\mathcal{L}_{j_\ell}(\delta)$ given by*

$$\mathbf{D}_{j_\ell}\big([\xi],[\eta],[\chi]\big) = \big[\xi,\eta,\chi,\widehat{D}_{j_\ell}(\xi,\eta,\chi)\big]$$

is a holomorphic section, called the j_ℓ-th Jordan determinant function *on $\mathbb{F}_J$. Moreover, $\mathbf{D}_{j_\ell}$ coincides in the restriction to $[0:0]\times[0:0]\times\mathbb{F}_J$ with the restricted Jordan determinant function $\mathbf{D}^0_{j_\ell}$.*

(c) *On the open and dense subset $\mathbb{G}^{(0)}\times\overline{\mathbb{G}}^{(0)}\times\mathbb{F}_J^{(0)}\cong Z\times\overline{Z}\times\overline{\mathcal{E}}_J$, the section $\mathbf{D}_{j_\ell}$ is given by the map*

$$\big(x, y, [(u_i), z]\big)\mapsto\delta(x,y)\cdot\delta^u\big(-(x-z)^{(y^z)}\big)\ .$$

(d) *The vanishing set of* $\mathbf{D}_{j_\ell}$ *is* $G^{\mathbb{C}}$*-invariant and independent of choice of the denominator. Moreover, if* $\delta(x,y) = \operatorname{Det} B_{x,y}$*, then* $\mathbf{D}_{j_\ell}$ *is a* $G^{\mathbb{C}}$*-equivariant section.*

PROOF. Without restriction we assume $J = (j_\ell)$ and set $j = j_\ell$, $u = u_\ell$. For (a), we first note that $\widehat{D}_j$ is rational in the variables $z, x, b \in Z$ and $a, c, y \in \overline{Z}$. In the following, we transform $\widehat{D}_j$ into different expressions, each of which shows that $\widehat{D}_j$ is actually polynomially in (x,a), in (y,b), and in (z,c), respectively. This is the same procedure as in the proof of Proposition 5.4. The addition formula for the quasi-inverse yields the relation

$$(u-v)^{(w^v)} = B_{v,w}(u^w - v^w) \tag{9.10}$$

for all $u, v \in Z$ and $w \in \overline{Z}$ with quasi-invertible (u,w) and (v,w). Therefore,

$$\begin{aligned}\widehat{D}_j &= \widehat{\delta}((x,a),(y,b)) \cdot \delta^{B_{c,z}u}\Big(B_{z^c,y^b}\big((z^c)^{(y^b)} - (x^a)^{(y^b)}\big)\Big)\\ &= \widehat{\delta}((x,a),(y,b)) \cdot \delta^{B_{c,z}u}\Big(z^c - Q_{z^c}y^b - B_{z^c,y^b}\big((x^a)^{(y^b)}\big)\Big) \ .\end{aligned}$$

Since $\delta^{B_{c,z}u}(v) = \delta^u(B_{z,c}v)$, we obtain by using JT28 and JT33

$$\delta^{B_{c,z}u}\Big(z^c - Q_{z^c}y^b - B_{z^c,y^b}\big((x^a)^{(y^b)}\big)\Big) = \delta^u\Big(z - Q_z c - Q_z y^b - B_{z,c+y^b}\big((x^a)^{(y^b)}\big)\Big) \ .$$

Therefore, $\widehat{D}_j$ depends polynomially on $(z,c) \in Z \times \overline{Z}$. We also note that this expression (and hence $\widehat{D}_j$)) does not depend on the Z_0^U-component of z. Next, we apply the relation $\delta^u(v) = \delta\left(u^\dagger - v, u\right)$, the addition formula for the denominator δ, and the identity $\delta\left(x, B_{v,u}y\right) = \delta\left(B_{u,v}x, y\right)$, cf. Lemma 2.23. Using the abbreviation $w = u^\dagger - z + Q_z c + Q_z y^b$, this yields

$$\begin{aligned}\widehat{D}_j &= \widehat{\delta}((x,a),(y,b)) \cdot \delta\left(u^\dagger - z + Q_z c + Q_z y^b + B_{z,c+y^b}\big((x^a)^{(y^b)}\big), u\right)\\ &= \widehat{\delta}((x,a),(y,b)) \cdot \delta\left(B_{z,c+y^b}\big((x^a)^{(y^b)}\big), u^w\right) \cdot \delta\left(w,u\right)\\ &= \widehat{\delta}((x,a),(y,b)) \cdot \delta\left((x^a)^{(y^b)}, B_{c+y^b,z}u^w\right) \cdot \delta\left(w,u\right)\\ &= \delta\left(y,b\right) \cdot \delta\left(x, a + y^b + B_{c+y^b,z}u^w\right) \cdot \delta\left(w,u\right) \ .\end{aligned}$$

This expression is polynomial in $(x,a) \in Z \times \overline{Z}$. Finally, we apply to the original definition of $\widehat{D}_j$ the symmetry formula for the quasi-inverse, the relation $\delta^u(v) = \delta\left(u^\dagger - v, u\right)$, and the identity $\delta\left(Q_v u, w\right) = \delta\left(Q_v w, u\right)$. Setting $w = (B_{c,z}u)^\dagger + x^a - z^b$, we obtain

$$\begin{aligned}\widehat{D}_j &= \widehat{\delta}((x,a),(y,b)) \cdot \delta^{B_{c,z}u}\big(-(x^a - z^c) - Q_{x^a - z^c}(y^b)^{(x^a)}\big)\\ &= \widehat{\delta}((x,a),(y,b)) \cdot \delta\left((B_{c,z}u)^\dagger + x^a - z^c + Q_{x^a-z^c}(y^b)^{(x^a)}, B_{c,z}u\right)\\ &= \widehat{\delta}((x,a),(y,b)) \cdot \delta\left(Q_{x^a-z^c}(y^b)^{(x^a)}, (B_{c,z}u)^w\right) \cdot \delta\left(w, B_{c,z}u\right)\\ &= \widehat{\delta}((x,a),(y,b)) \cdot \delta\left(Q_{x^a-z^c}(B_{c,z}u)^w, (y^b)^{(x^a)}\right) \cdot \delta\left(w, B_{c,z}u\right)\\ &= \delta\left(x,a\right) \cdot \delta\left(b + x^a + Q_{x^a-z^c}(B_{c,z}u)^w, y\right) \cdot \delta\left(w, B_{c,z}u\right) \ .\end{aligned}$$

Therefore, $\widehat{D}_j$ also depends polynomially on $(y,b) \in \overline{Z} \times Z$. To prove (b), we have to show that $\widehat{D}_j$ transforms according to

$$\begin{aligned}\widehat{D}_j\big((\tilde{x},\tilde{a}),(\tilde{y},\tilde{b}),(\tilde{u},\tilde{z},\tilde{c})\big) = \delta\left(x, a - \tilde{a}\right)^{-1} \cdot \delta\left(b - \tilde{b}, y\right)^{-1} \cdot \delta^u\big((B_{c-\tilde{x},z}u)^\dagger\big) \cdot\\ \cdot \widehat{D}_j\big((x,a),(y,b),(u,z,c)\big)\end{aligned}$$

for all $[x:a] = [\tilde{x}:\tilde{a}]$, $[y:b] = [\tilde{y}:\tilde{b}]$, $[u:z:c] = [\tilde{u}:\tilde{z}:\tilde{c}]$. The first two factors emerge from the transformation properties of $\widehat{\delta}$, and since the argument of $\delta^{B_{c,z}}(\cdot)$

is independent of the choice of representatives, it suffices to note that

$$\begin{aligned}\delta^{B_{c,\,z}u}(v) &= \delta^{B_{\tilde{c},\,z^{c-\tilde{c}}}B_{c-\tilde{c},\,z}u}(v)\\ &= \delta^{B_{c-\tilde{c},\,z}u}\big(B_{z^{c-\tilde{c}},\,\tilde{c}}v\big)\\ &= \delta^{\tilde{u}}\big((B_{c-\tilde{c},\,z}u)^{\dagger}\big)^{-1}\,\delta^{\tilde{u}}(B_{\tilde{z},\,\tilde{c}}v)\\ &= \delta^{\tilde{u}}\big((B_{c-\tilde{c},\,z}u)^{\dagger}\big)^{-1}\,\delta^{B_{\tilde{c},\,\tilde{z}}u}(v)\ .\end{aligned}$$

Here, we assumed that $(z, c-\tilde{c})$ is quasi-invertible and $\tilde{z} = z^{c-\tilde{c}}$, otherwise replace[2] z and $\tilde{z}$ by appropriate $z+u^{⫫}$ and $\tilde{z}+\tilde{u}^{⫫}$ according to the equivalence relation on $\mathbb{F}_{(j)}$. We conclude that $\mathbf{D}_{j_\ell}$ is a well-defined section in $\mathcal{L}_\delta^{-1} \boxtimes \overline{\mathcal{L}_\delta^{-1}} \boxtimes \mathcal{L}_j(\delta)$. Furthermore, due to (a), $\widehat{D}_j$ is holomorphic in the variables $x, b, z \in Z$, $a, c, y \in \overline{Z}$, and since the pseudo-inverse map is real analytic, $\widehat{D}_j$ is also real analytic in the u-variable. By the same argument as in the proof of Proposition 9.3, it follows that $\widehat{D}_j$ is in fact holomorphic along the restriction of the u-variable to a transversal covering of $\mathbb{F}_{(j)}$. Then by Proposition 3.12, it follows that $\mathbf{D}_j$ is a holomorphic section. In addition, since

$$\widehat{D}_j((0,0),(0,0),(u,z,c)) = \delta^{B_{c,\,z}u}(z^c) = \delta^u(B_{z,\,c}z^c) = \delta^u(z - Q_z c)\ ,$$

the restriction of $\mathbf{D}_j$ to $[0:0]\times[0:0]\times\mathbb{F}_J$ coincides with $\mathbf{D}_j^0$. This proves (b). Part (c) is proved just by setting a, b and c equal to zero. To prove (d), we first note that the same argument as in the proof of Proposition 5.4 shows that for different denominators δ, δ', the vanishing sets of $\mathbf{D}_j$ and $\mathbf{D}'_j$ coincide. Therefore, it suffices to show the $G^{\mathbb{C}}$-equivariance of $\mathbf{D}_j$ in the case of $\delta(x,y) = \operatorname{Det} B_{x,\,y}$. Moreover, it suffices to consider the action of the generators t_v and $\tilde{t}_w$ of $G^{\mathbb{C}}$ for $(v,w) \in Z\times\overline{Z}$ on the open and dense subset $Z\times\overline{Z}\times\overline{\mathcal{E}}_{(j)} \subset \mathbb{G}\times\overline{\mathbb{G}}\times\mathbb{F}_J$. We have

$$\begin{aligned}t_v([x:0],[y:0],[u:z:0]) &= ([x+v:0],[y:-v],[u:z+v:0])\ ,\\ \tilde{t}_w([x:0],[y:0],[u:z:0]) &= ([x:w],[y-v:0],[u:z:w])\ ,\end{aligned}$$

and therefore obtain for the fiber coordinate of $\mathbf{D}_j \circ t_v$ the term

$$\delta^{B_{0,\,z+v}u}\Big(-((x+v)-(z+v))^{(y^{-v})^{(z+v)}}\Big) = \delta^u\Big(-(x-z)^{(y^{-v+z+v})}\Big) = \delta^u\big(-(x-z)^{(y^z)}\big)\ .$$

Using relation (9.10) twice, the fiber coordinate of $\mathbf{D}_j \circ \tilde{t}_w$ determines to

$$\begin{aligned}\delta^{B_{w,\,z}u}\Big(-(x^w - z^w)^{(y-w)^{(z^w)}}\Big) &= \delta^u\big(-B_{z,\,w}B_{z^w,\,y-w}\big((x^w)^{y-w} - (z^w)^{y-w}\big)\big)\\ &= \delta^u(-B_{z,\,y}(x^y - z^y))\\ &= \delta^u\big(-(x-z)^{(y^z)}\big)\ .\end{aligned}$$

Recall from (4.27), (8.43), (8.44) that the translation t_v and the quasi-translation $\tilde{t}_w$ act identically on the fiber coordinate. Thus we conclude that $\mathbf{D}_j \circ t_v = t_v \circ \mathbf{D}_j$ and $\mathbf{D}_j \circ \tilde{t}_w = \tilde{t}_w \circ \mathbf{D}_j$, hence $\mathbf{D}_j$ is $G^{\mathbb{C}}$-equivariant. □

Remark 9.6. As in Remark 9.4, we note that the Jordan determinant function depends on the choice of a denominator δ, but since we are just interested in the vanishing set of $\mathbf{D}_{j_\ell}$, we omit a corresponding additional label. Furthermore, since the fiber coordinate of $\mathbf{D}_{j_\ell}$ does not depend on u_i for $i \neq \ell$, so it is possible to consider $\mathbf{D}_{j_\ell}$ just as well as a section on the product $\mathbb{G}\times\overline{\mathbb{G}}\times\mathbb{F}_{(j_\ell)}$.

[2] As noted above, $\widehat{D}_j$ is independent of the Z_0^u-component of z, so this replacement is indeed well-behaved.

9.3. Comparison of the determinant functions

In this section, we consider the matrix case $Z = \mathbb{C}^{r\times s}$ and compare the Jordan determinant functions $\mathbf{D}_j$ with the determinant functions $\mathcal{D}_j$ of Barchini-Gindikin-Wong, which are defined in Section 9.1. First, we have to identify the domains of these functions. Due to Remark 9.6, $\mathbf{D}_j$ is a section on $\mathbb{G}\times\overline{\mathbb{G}}\times\mathbb{F}_{(j)}$, and in the matrix case this can be identified with $\mathrm{Gr}_s(\mathbb{C}^n)\times\mathrm{Gr}_r(\mathbb{C}^n)\times\mathrm{Gr}_{j,n-j}(\mathbb{C}^n)$ for $n = r+s$ via the isomorphism (cf. Example 4.9 and Section 8.2)

$$([x:a],[y:b],[u:z:c]) \mapsto \big(E_{x,a},\, F_{y,b},\, (U_{u,z,c},\, U'_{u,z,c})\big)$$

with

$$E_{x,a} = \left\langle\begin{pmatrix} x \\ 1-a^*x\end{pmatrix}\right\rangle, \qquad F_{y,b} = \left\langle\begin{pmatrix} 1-by^* \\ y^*\end{pmatrix}\right\rangle,$$

and

$$U_{u,z,c} = \left\langle\begin{pmatrix} z \\ 1-c^*z\end{pmatrix}u^*\right\rangle, \qquad U'_{u,z,c} = \left\langle\begin{pmatrix} 1-cz^* \\ -z^*\end{pmatrix}u\right\rangle^{\perp}.$$

The Barchini-Gindikin-Wong determinant functions $\mathcal{D}_j$ can be considered as functions on $\mathrm{Gr}_{j,n-j}(\mathbb{C}^n)\times Y$, where

$$Y = \{(V,W)\subset\mathbb{C}^n\times\mathbb{C}^n \mid \dim V = r,\ \dim W = s,\ V\oplus W = \mathbb{C}^n\} \cong G^{\mathbb{C}}/K^{\mathbb{C}}$$

with $G^{\mathbb{C}} = \mathrm{SL}(r+s)$ and $K^{\mathbb{C}} = \mathrm{S}(\mathrm{GL}(r)\times\mathrm{GL}(s))$. By Theorem 5.12, Y can be identified with the open and dense $G^{\mathbb{C}}$-orbit $\mathcal{O}_0\subset\mathbb{G}\times\overline{\mathbb{G}}$. We use the imbedding

$$Y \hookrightarrow \mathrm{Gr}_s(\mathbb{C}^n)\times\mathrm{Gr}_r(\mathbb{C}^n),\ (V,W)\mapsto(V^{\perp},W^{\perp})$$

to identify Y with $\mathcal{O}_0$. In this way, we obtain for $V_0 = \mathbb{C}^r\times\{0\}^{n-r}$ and $W_0 = \{0\}^{n-s}\times\mathbb{C}^s$ the identification

$$Y \ni (V_0,W_0)\mapsto \left(\left\langle \begin{smallmatrix}0\\ \mathbf{1}_s\end{smallmatrix}\right\rangle, \left\langle \begin{smallmatrix}\mathbf{1}_r\\ 0\end{smallmatrix}\right\rangle\right) = \big(E_{0,0},\, F_{0,0}\big)\mapsto\big([0:0],\,[0:0]\big)\in\mathbb{G}\times\overline{\mathbb{G}}\,.$$

With these identifications, the main problem in the comparison of the Jordan determinant functions with the Barchini-Gindikin-Wong determinant functions is to show that

$$\mathbf{D}_j\big([x:a],[y:b],[u:z:c]\big) = 0 \iff \mathcal{D}_j\big((U_{u,z,c},U'_{u,z,c}),(E^{\perp}_{x,a},F^{\perp}_{y,b})\big) = 0$$

for all $(([x:a],[y:b]),[u:z:a])\in\mathcal{O}_0\times\mathbb{F}_{(j)}$ and all $j = 0,\ldots,r$. We note that this comparison is independent of the choice of the denominator δ used for the definition of the Jordan determinant functions since by Theorem 9.5, the vanishing set of $\mathbf{D}_j$ is independent of the choice of δ. We finish this thesis with the proof of a restricted version of this comparison.

Proposition 9.7. *For $0 < j < r$, the restricted Jordan determinant function $\mathbf{D}^0_j$ vanishes in $[u:z:0]$ if and only if the restricted Barchini-Gindikin-Wong determinant function $\mathcal{D}^0_j$ vanishes in $(U_{u,z,0},U'_{u,z,0})$.*

Proof. As denominator of the quasi-inverse we choose the Jordan triple determinant, which is given in the matrix case by $\Delta(x,y) = \det(1-xy^*)$. Therefore, $\mathbf{D}^0_j([u:z:0])$ vanishes if and only if $\Delta^u(z) = \Delta\big(u^{\dagger}-z,u\big) = \det(1-u^{\dagger}u^*+zu^*)$ vanishes. Due to the $K^{\mathbb{C}}$-invariance, we may assume that u and z are given by

$$u = \begin{pmatrix}\mathbf{1}_j & 0\\ 0 & 0\end{pmatrix},\qquad z = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}.$$

In this case, we have $u^{\dagger} = u$, and thus obtain

$$\Delta^u(z) = \det\begin{pmatrix}\alpha & 0\\ 0 & \mathbf{1}_{r-j}\end{pmatrix} = \det(\alpha)\,. \tag{9.11}$$

This is the term describing the restricted Jordan determinant function and must be compared to $[(U'_{u,z,0} \sqcap W_0) \sqcup U_{u,z,0} \sqcup V_0]$ as in (9.6) with

$$V_0 = \left\langle\!\!\left\langle \begin{pmatrix} \mathbf{1}_r \\ 0 \end{pmatrix} \right\rangle\!\!\right\rangle , \quad W_0 = \left\langle\!\!\left\langle \begin{pmatrix} 0 \\ \mathbf{1}_s \end{pmatrix} \right\rangle\!\!\right\rangle , \quad U_{u,z,0} = \left\langle\!\!\left\langle \begin{pmatrix} z \\ \mathbf{1}_s \end{pmatrix} u^* \right\rangle\!\!\right\rangle , \quad U'_{u,z,0} = \left\langle\!\!\left\langle \begin{pmatrix} \mathbf{1}_r \\ -z^* \end{pmatrix} u \right\rangle\!\!\right\rangle^{\perp} .$$

We first determine $U_{u,z,0} \sqcup V_0$. We have $U_{u,z,0} \cap V_0 = \{0\}$, since for $v \in U_{u,z,0} \cap V_0$ there exist $\xi, \eta \in \mathbb{C}^r$ with

$$v = \begin{pmatrix} zu^*\xi \\ u^*\xi \end{pmatrix} = \begin{pmatrix} \eta \\ 0 \end{pmatrix} ,$$

which implies $u^*\xi = 0$, $\eta = 0$ and hence $v = 0$. Therefore,

$$U_{u,z,0} \sqcup V_0 = U_{u,z,0} \oplus V_0 = \left\langle\!\!\left\langle \begin{pmatrix} \mathbf{1}_r & zu^* \\ 0 & u^* \end{pmatrix} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & u^* \end{pmatrix} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \begin{pmatrix} \mathbf{1}_r & 0 \\ 0 & \mathbf{1}_j \\ 0 & 0 \end{pmatrix} \right\rangle\!\!\right\rangle . \tag{9.12}$$

Before we determine $U'_{u,z,0} \sqcap W_0$, we claim that

$$U'_{u,z,0} = \left\langle\!\!\left\langle \begin{pmatrix} \mathbf{1}_r \\ -z^* \end{pmatrix} u \right\rangle\!\!\right\rangle^{\perp} = \left\langle\!\!\left\langle \begin{pmatrix} z & \mathbf{1}_r - u^\dagger u^* \\ \mathbf{1}_s & 0 \end{pmatrix} \right\rangle\!\!\right\rangle . \tag{9.13}$$

Indeed, the relation $Q_u u^\dagger = u$ for the pseudo-inverse implies $u^* u^\dagger u^* = u^*$, and hence

$$\begin{pmatrix} u^* & -u^* z \end{pmatrix} \begin{pmatrix} z & \mathbf{1}_r - u^\dagger u^* \\ \mathbf{1}_s & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix} ,$$

which shows the inclusion "$\supset$" of (9.13). The converse inclusion follows from the comparison of the dimensions: Since $\mathbf{1}_r - u^\dagger u^* = \left(\begin{smallmatrix} 0 & 0 \\ 0 & \mathbf{1}_{r-j} \end{smallmatrix}\right)$, the dimension of the vector space on the left hand side of (9.13) is $s + r - j = n - j$ and therefore coincides with the dimension of $U'_{u,z,0}$. We conclude that

$$U'_{u,z,0} = \left\langle\!\!\left\langle \begin{pmatrix} \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & \mathbf{1}_{r-j} \\ \mathbf{1}_j & 0 & 0 & 0 \\ 0 & \mathbf{1}_{s-j} & 0 & 0 \end{pmatrix} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \begin{pmatrix} \alpha & \beta & 0 \\ 0 & 0 & \mathbf{1}_{r-j} \\ \mathbf{1}_j & 0 & 0 \\ 0 & \mathbf{1}_{s-j} & 0 \end{pmatrix} \right\rangle\!\!\right\rangle . \tag{9.14}$$

We now turn to the calculation of $U'_{u,z,0} \sqcap W_0$. It follows immediately form (9.13) that $U'_{u,z,0} + W_0 = \mathbb{C}^n$ if and only if the matrix $(\alpha\ \beta) \in \mathbb{C}^{j \times s}$ is of maximal rank. In this case,

$$U'_{u,z,0} \sqcap W_0 = U'_{u,z,0} \cap W_0 = \left\{ \begin{pmatrix} 0 \\ \xi \end{pmatrix} \,\middle|\, \xi \in \mathbb{C}^s \text{ such that } (\alpha\ \beta)\xi = 0 \right\} = \begin{pmatrix} 0 \\ \ker(\alpha\ \beta) \end{pmatrix} .$$

Since $(\alpha\ \beta)$ is of maximal rank, this is an $(s-j)$-dimensional subspace. Let $\xi_1, \ldots, \xi_{s-j}$ be a basis of $\ker(\alpha\ \beta)$, and set

$$\begin{pmatrix} \kappa_1 \\ \kappa_2 \end{pmatrix} = \begin{pmatrix} \xi_1, \ldots, \xi_{s-j} \end{pmatrix} \in \mathbb{C}^{s \times (s-j)} \quad \text{with} \quad \kappa_1 \in \mathbb{C}^{j \times (s-j)},\ \kappa_2 \in \mathbb{C}^{(s-j) \times (s-j)} .$$

Then, if $(\alpha\ \beta)$ is of maximal rank,

$$U'_{u,z,0} \sqcap W_0 = \left\langle\!\!\left\langle \begin{pmatrix} 0 \\ \kappa_1 \\ \kappa_2 \end{pmatrix} \right\rangle\!\!\right\rangle .$$

Connecting this with (9.12), we conclude that $(U'_{u,z,0} \sqcap W_0) \sqcup U_{u,z,0} \sqcup V_0$ is a subspace of at most dimension n, and

$$\mathcal{D}^0_j(U_{u,z,0}, U'_{u,z,0}) = [(U'_{u,z,0} \sqcap W_0) \sqcup U_{u,z,0} \sqcup V_0] \doteq \det \begin{pmatrix} \mathbf{1}_r & 0 & 0 \\ 0 & \mathbf{1}_j & \kappa_1 \\ 0 & 0 & \kappa_2 \end{pmatrix} = \det(\kappa_2) ,$$

where $\doteq$ denotes equality up to multiplication by a non-zero scalar. Therefore, in the comparison with (9.11), it remains to show that

$$\det(\alpha) \neq 0 \quad \text{if and only if} \quad \begin{cases} (\alpha\ \beta) \text{ is of maximal rank,} \\ \text{and } \det(\kappa_2) \neq 0\,. \end{cases}$$

Assume first that $\det(\alpha) \neq 0$, i.e. α is invertible. Then $(\alpha\ \beta)$ is of maximal rank, and the columns of $\left(\begin{smallmatrix}\kappa_1' \\ \kappa_2'\end{smallmatrix}\right)$ with $\kappa_1' = -\alpha^{-1}\beta$ and $\kappa_2' = \mathbf{1}_{s-j}$ form a basis of $\ker(\alpha\ \beta)$. Since $\kappa_2 = \kappa_2' g$ for some invertible $g \in \mathbb{C}^{(s-j)\times(s-j)}$, this implies that $\det(\kappa_2)$ is non-vanishing. Conversely, assume that $(\alpha\ \beta)$ is of maximal rank and $\det(\kappa_2) \neq 0$. If $\alpha\eta = 0$ for some $\eta \in \mathbb{C}^j$, then $\left(\begin{smallmatrix}\eta \\ 0\end{smallmatrix}\right)$ is an element of the kernel of $(\alpha\ \beta)$. Since the columns of $\left(\begin{smallmatrix}\kappa_1 \\ \kappa_2\end{smallmatrix}\right)$ form a basis of this kernel, there exists an element $\zeta \in \mathbb{C}^{s-j}$ such that $\eta = \kappa_1\zeta$ and $0 = \kappa_2\zeta$. The invertibility of κ_2 now implies that $\zeta = 0$, and hence $\eta = 0$. Therefore, the map $\eta \mapsto \alpha\eta$ is injective, and by finite dimensionality we conclude that α is invertible. $\square$

REMARK 9.8. We note that a comparison of the non-restricted Barchini-Gindikin-Wong determinant functions $\mathcal{D}_j$ with the non-restricted Jordan determinant functions $\mathbf{D}_j$ might be obtained in the following way:

(1) Extend the result of Proposition 9.7 to a comparison of $\mathbf{D}_j^0$ and $\mathcal{D}_j^0$ on all of $\mathbb{F}_j \cong \mathrm{Gr}_{j,n-j}(\mathbb{C}^n)$. This might be done either by similar arguments as above for elements $[u : z : c]$ with non-vanishing c, or by determining an explicit formula for $\mathcal{D}_j^0$ using the original definition of $\mathcal{D}_j^0$ by alternating forms, and comparing this formula with $\Delta^u(z - Q_c z)$.

(2) Show that the vanishing set of $\mathcal{D}_j$ is $G^{\mathbb{C}}$-invariant. In this case, the correspondence of the vanishing sets of $\mathcal{D}_j$ and $\mathbf{D}_j$ follow from (1), since $G^{\mathbb{C}}$ acts transitively on Y.

An alternative approach to (1) is to investigate the $K^{\mathbb{C}}$-orbit structure of the Jordan flag variety, and to prove a result on the connection of codimension one orbits and the vanishing sets of $\mathbf{D}_j^0$ similar to the result of Proposition 9.2. This is prospective work.

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APPENDIX A

List of identities for Jordan triple systems

Let Z be a Jordan triple system with triple product

$$\{\,,\,,\,\} : Z \times \overline{Z} \times Z \to Z,\ (x,y,z) \mapsto \{x,\, y,\, z\}\ .$$

We set[1]

$$\begin{aligned}
&x \,\Box\, y : Z \to Z,\ z \mapsto \{x,\, y,\, z\} && \text{(box operator)},\\
&Q_x : \overline{Z} \to Z,\ y \mapsto \{x,\, y,\, x\} && \text{(quadratic map)},\\
&Q_{x,z} : \overline{Z} \to Z,\ y \mapsto \{x,\, y,\, z\}\ , &&\\
&B_{x,\,y} = \operatorname{Id} - 2\, x \,\Box\, y + Q_x Q_y && \text{(Bergman operator)},\\
&x^y = B_{x,\,y}^{-1}(x - Q_x y) && \text{(quasi-inverse)}.
\end{aligned}$$

Then the following identities hold:[2]

JT1 $\quad (x \,\Box\, y) Q_x = Q_x (y \,\Box\, x)\ ,$

JT2 $\quad (Q_x y) \,\Box\, y = x \,\Box\, (Q_y x)\ ,$

JT3 $\quad Q_{Q_x y} = Q_x Q_y Q_x\ ,$

JT4 $\quad (x \,\Box\, y) Q_x = Q_{x, Q_x y}\ ,$

JT5 $\quad 2\, Q_{x,z}(y \,\Box\, x) + Q_x (y \,\Box\, z) = 2\, Q_{x, \{x,\, y,\, z\}} + Q_{z, Q_x y} = 2\,(x \,\Box\, y) Q_{x,z} + (z \,\Box\, y) Q_x\ ,$

JT6 $\quad 2\, x \,\Box\, \{y,\, x,\, z\} + Q_x Q_{y,z} = 2\,(x \,\Box\, z)(x \,\Box\, y) + Q_x y \,\Box\, z = 2\,(x \,\Box\, y)(x \,\Box\, z) + Q_x z \,\Box\, y\ ,$

JT7 $\quad 2\, \{x,\, y,\, z\} \,\Box\, y = z \,\Box\, (Q_y x) + x \,\Box\, (Q_y z)\ ,$

JT8 $\quad 2\, x \,\Box\, \{y,\, x,\, z\} = (Q_x y) \,\Box\, z + (Q_x z) \,\Box\, y\ ,$

JT9 $\quad 2\,(x \,\Box\, y)(z \,\Box\, y) = Q_{x,z} Q_y + x \,\Box\, (Q_y z)\ ,$

JT10 $\quad 2\, Q_{x,z}(y \,\Box\, x) = Q_{Q_x y, z} + (z \,\Box\, y) Q_x\ ,$

JT11 $\quad 2\,(x \,\Box\, y) Q_{x,z} = Q_{Q_x y, z} + Q_x (y \,\Box\, z)\ ,$

JT12 $\quad (x \,\Box\, y) Q_z + Q_z (y \,\Box\, x) = 2\, Q_{z, \{x,\, y,\, z\}}\ ,$

JT13 $\quad 2\,(x \,\Box\, y)(x \,\Box\, z) = (Q_x y) \,\Box\, z + Q_x Q_{y,z}\ ,$

JT14 $\quad \{x,\, y,\, \{u,\, v,\, z\}\} - \{u,\, v,\, \{x,\, y,\, z\}\} = \{\{x,\, y,\, u\},\, v,\, z\} - \{u,\, \{y,\, x,\, v\},\, z\}\ ,$

JT15 $\quad [x \,\Box\, y, u \,\Box\, v] = \{x,\, y,\, u\} \,\Box\, v - u \,\Box\, \{y,\, x,\, v\}\ ,$

JT16 $\quad \{\{x,\, y,\, u\},\, v,\, z\} - \{u,\, \{y,\, x,\, v\},\, z\} = \{x,\, \{v,\, u,\, y\},\, z\} - \{\{u,\, v,\, x\},\, y,\, z\}\ ,$

[1] We note that there are (at least) two different conventions concerning these operators. We follow the convention based on the polarization formula $Q_{x,y} = \frac{1}{2}(Q_{x+y} - Q_x - Q_y)$. Omitting the factor $\frac{1}{2}$ in this formula yields the other convention perfered e.g. by O. Loos [**27, 28**].

[2] This is an adaption of the list of identities given in [**28**]. According to the different polarization formula, we replace the operator $D(x,y)$ by $2 \cdot x \,\Box\, y$, the operator $Q(x,y)$ by $2 \cdot Q_{x,y}$ and finally $\{xyz\}$ by $2 \cdot \{x,\, y,\, z\}$. All other operators remain unchanged.

JT17 $((Q_x y) \square z) Q_x = Q_x (y \square (Q_x z))$,

JT18 $2\,((Q_x y) \square z)(x \square y) = Q_x Q_y (x \square z) + x \square (Q_y Q_x z)$,

JT19 $Q_{Q_x y,\, 2\{x,\, y,\, z\}} = Q_x Q_y Q_{x,z} + Q_{x,z} Q_y Q_x$,

JT20 $Q_{2\,\{x,\, y,\, z\}} + 2\, Q_{Q_x y, Q_z y} = Q_x Q_y Q_z + Q_z Q_y Q_x + 4\, Q_{x,z} Q_y Q_{x,z}$,

JT21 $Q_{2\,\{x,\, y,\, z\}} + 2\, Q_{Q_x Q_y z, z} = Q_x Q_y Q_z + Q_z Q_y Q_x + 4\,(x \square y) Q_z (y \square x)$,

JT22 $Q_{Q_x Q_y z,\, 2\{x,\, y,\, z\}} = Q_x Q_y Q_z (y \square x) + (x \square y) Q_z Q_y Q_x$,

JT23 $B_{x,\, y} Q_x = Q_x B_{y,\, x} = Q_{x - Q_x y}$,

JT24 $B_{Q_x y,\, y} = B_{x,\, Q_y x} = B_{x,\, y} B_{x,\, -y}$,

JT25 $B^2_{x,\, y} = B_{2x - Q_x y,\, y} = B_{x,\, 2y - Q_y x}$,

JT26 $Q_{B_{x,\, y} z} = B_{x,\, y} Q_z B_{y,\, x}$,

JT27 $Q_{B_{x,\, y} z,\, x - Q_x y} = B_{x,\, y} (Q_{x,z} - (z \square y) Q_x) = (Q_{x,z} - Q_x (y \square z)) B_{y,\, x}$,

JT28 $B_{x,\, y} Q_{x^y} = Q_{x^y} B_{y,\, x} = Q_x$,

JT29 $B_{x,\, y} Q_{x^y, z} + Q_x (y \square z) = Q_{x^y, z} B_{y,\, x} + (z \square y) Q_x = Q_{x,z}$,

JT30 $B_{x,\, y} (x^y \square z) = x \square z - Q_x Q_{y,z}$,

JT31 $(z \square y^x) B_{y,\, x} = z \square x - Q_{y,z} Q_x$,

JT32 $x^y \square (y - Q_y x) = (x - Q_x y) \square y^x = x \square y$,

JT33 $B_{x,\, y} B_{x^y,\, z} = B_{x,\, y+z}$,

JT34 $B_{z,\, x^y} B_{y,\, x} = B_{y+z,\, x}$,

JT35 $B^{-1}_{x,\, y} = B_{x^y,\, -y} = B_{-x,\, y^x}$.

Furthermore, the quasi-inverse satisfies

Symmetry formula

$$z^w = z + Q_z (w^z) ,$$

Shifting formulæ

$$Q_v \big(z^{Q_v u} \big) = \big(Q_v z \big)^u ,$$
$$B_{u,\, v} \big(z^{B_{v,\, u} w} \big) = \big(B_{u,\, v} z \big)^w ,$$

Addition formulæ

$$z^{w+v} = (z^w)^v ,$$
$$(z+u)^w = z^w + B^{-1}_{z,\, w} \big(u^{(w^z)} \big) .$$

APPENDIX B

Deutsche Zusammenfassung

Die geometrische Realisierung von irreduziblen unitären Darstellungen von Liegruppen durch symmetrische bzw. homogene Räume ist eine der grundlegenden Aufgaben der harmonischen Analysis. Im Fall von nilpotenten Liegruppen liefert Kirillovs Orbitmethode eine geometrische Realisierung der irreduziblen unitären Darstellungen auf koadjungierten Orbits [**21**], und für kompakte Liegruppen erhält man gemäß der Borel-Weil-Bott Theorie eine eineindeutige Beziehung zwischen dem unitären Dual und gewissen Geradenbündeln auf entsprechenden Fahnenmannigfaltigkeiten [**11**]. Gegenstand diese Arbeit ist die Untersuchung von Orbitstrukturen, die durch halbeinfache, nicht kompakte Liegruppen hervorgerufen werden. Es ist wohlbekannt, dass in diesem allgemeinen Rahmen die Orbitmethode wesentlich erweitert werden muss, und dass hierbei noch einiges ungeklärt ist. Eine allgemeine Konstruktion von Orbits und den zugehörigen Darstellungen stammt z.B. von J. Wolf (partiell holomorphe Kohomologie), allerdings rein im Rahmen der Lietheorie, die eher eine abstrakt-geometrische Realisierung liefert als eine konkrete Beschreibung der zugrundeliegenden Geometrie. Nur wenige Fälle sind explizit ausgearbeitet.

Im hermiteschen Fall, d.h. für die Automorphismengruppe $G = \mathrm{Aut}(\mathcal{D})$ eines beschränkten symmetrischen Gebietes $\mathcal{D} \subset \mathbb{C}^N$, gibt die Borel-Einbettung von $\mathcal{D}$ in seinen kompakten Dual $X = G^c/K$ Anlass zum Studium von G-Orbits auf dem kompakten Dual. Im Rahmen der Lietheorie wurden diese Orbits von J. Wolf klassifiziert und einige ihrer geometrischen Eigenschaften bestimmt [**44**]. Allerdings ist diese Beschreibung wiederum eher abstrakter Natur als konkret gegeben. Im hermiteschen Fall wird die Lietheoretische Beschreibung durch einen Jordantheoretischen Zugang ergänzt, der von M. Koecher und O. Loos gefunden wurde [**24, 28**]. Ein grundlegender Unterschied zwischen Lie- und Jordantheorie zeigt sich in der Art, wie lokale Strukturen mit globalen Strukturen verbunden werden [**5**]: sind in der Lietheorie die Karten des kompakten Duals $X = G^c/K$ durch die Exponentialabbildung beschrieben, so erhält man im Jordantheoretischen Modell von X (wie es O. Loos beschreibt) einen algebraisch geometrischen Zugang, da die Übergangsfunktionen durch elementare birationale Abbildungen gegeben sind (wie z.B. Determinanten und Quasi-Inverse). Somit liefert die Jordantheorie eine elementare Beschreibung des kompakten Duals als algebraische Varietät im Sinne von Mumford [**35**], und insbesondere wird die beteiligte Jordanstruktur selbst mit einer offenen und dichten Teilmenge des kompakten Duals identifiziert. Deshalb ist es naheliegend zu erwarten, dass die Jordantheorie wesentlich explizitere Beschreibungen der G-Orbits liefert als der entsprechende Lietheoretische Zugang. Im Fall der Randkomponenten von $\mathcal{D}$ wurde dies von O. Loos nachgewiesen [**28**], und der Ausgangspunkt der vorliegenden Arbeit liegt in der Frage, wie die von O. Loos gegebene explizite Realisierung auch auf die übrigen G-Orbits ausgedehnt werden kann. Im nächsten Schritt werden dann zusätzliche Strukturen wie z.B. Geradenbündel und G-invariante Maße untersucht, mit deren Hilfe schließlich Darstellungen von G konstruiert werden. Im Fall des beschränkten symmetrischen Gebiets $\mathcal{D}$ und seiner Randorbits wurde dies z.B. von J. Faraut and A. Koranyi (im Tubenfall, [**8**]) und von H. Upmeier et al. (im nicht-Tubenfall, [**1, 42**]) durchgeführt.

In Erweiterung dieser Fragestellung geht es bei dem Programm „Jordantheorie und geometrische Realisierungen" im weitesten Sinne darum, (i) eine Jordan theoretische Beschreibung verallgemeinerter Fahnenvarietäten $G^{\mathbb{C}}/P$, $P \subset G^{\mathbb{C}}$ parabolisch, zu geben, (ii) die G-Orbitstruktur explizit zu bestimmen, und (iii) die zugehörigen Darstellungen zu beschreiben. Selbst im Fall des kompakten Duals X ist diese Aufgabe von großer Komplexität, da z.B. auch die Ausnahme-Geometrien mit erfasst werden. Dementsprechend ist ein erstes Hauptresultat dieser Arbeit die vollständige konkrete Realisierung aller G-Orbits des kompakten Duals, sowie ihrer Matsuki-dualen Orbits (Theorem 7.2). Grundlage hierfür ist die Jordan theoretischen Beschreibung des kompakten Duals, wie sie von O. Loos gegeben wurde.

Für die Untersuchung allgemeiner Fahnenvarietäten ist zunächst zu bemerken, dass die Existenz einer Jordantheoretischen Beschreibung nicht auf der Hand liegt: Die Jordanstruktur, die zur Beschreibung des beschränkten symmetrischen Gebietes $\mathcal{D} = G/K$ dient, trägt die charakteristischen Größen der reellen Liegruppe G in sich, allerdings verlieren diese charakteristischen Größen beim Übergang zur Fahnenvarietät $G^{\mathbb{C}}/P$ ihre Bedeutung. Tatsächlich weist dieser Umstand darauf hin, dass nicht alle Fahnenvarietäten mit Hilfe der Jordantheorie beschrieben werden können. Stattdessen geben wir als eine zweites Hauptresultat dieser Arbeit eine Jordantheoretische Beschreibung der Fahnenvarietäten $G^{\mathbb{C}}/P$ an, für welche $P = Q^{\mathbb{C}}$ die Komplexifizierung einer reell parabolischen Untergruppe $Q \subset G$ ist (Theorem 8.11 und Theorem 8.20). Es sei angemerkt, dass diese Einschränkung für die Belange der Darstellungstheorie immer noch ausreichend ist, da z.B. die Hauptreihe von G auf dem Quotienten $G^{\mathbb{C}}/Q^{\mathbb{C}}$ mit minimal parabolischem $Q \subset G$ realisiert ist [**22, 4**].

Bezüglich des vorgestellten Programms zur Jordantheorie und geometrischen Realisierungen stellt diese Arbeit einen großen Teil des geometrischen Hintergrunds zur Verfügung, auf dem die darstellungstheoretischen Fragen dieses Programms diskutiert werden können. Außerdem beschreiben wir mittels Jordantheoretischer Methoden fundamentale Geradenbündel auf den verallgemeinerten Fahnenvarietäten, und geben als erste Anwendung (und als ein drittes Hauptresultat dieser Arbeit) eine Verallgemeinerung wir die *Determinantenfunktionen* an, welche von L. Barchini, S.G. Gindikin and H.W. Wong auf gewöhnlichen Fahnenvarietäten definiert wurden. Für eine Beschreibung der besonderen Bedeutung dieser Funktionen sowohl im Bereich geometrischer als auch darstellungstheoretischer Fragestellungen verweisen wir auf [**3, 4**].

Parallel zur G-Orbitstruktur des kompakten Duals bestimmen wir seine $K^{\mathbb{C}}$-Orbitstruktur. In den 80er Jahren des 20. Jahrhunderts hat T. Matsuki eine eins-zu-eins Korrespondenz zwischen diesen Orbitstrukturen entdeckt, die nun *Matsuki-Dualität* genannt wird. Unter der Verwendung von Jordantheoretischen Argumenten weisen wir diese Dualität durch explizite Berechnungen nach. Die Bedeutung der Matsuki-Dualität für die Darstellungstheorie ist durch seine enge Verknüpfung mit der Theorie der „Cycle Spaces" gegeben, welche wiederum wesentliche Beiträge zur geometrischen Realisierung von Darstellungen halbeinfacher Liegruppen geliefert hat [**9**].

Methoden und Resultate. Im Folgenden geben wir eine Übersicht der wesentlichen Methoden, die in dieser Arbeit Anwendung finden, und der aus ihnen gewonnenen Resultate. Außerdem geben wir eine kurze Beschreibung wichtiger Konzepte an.

In den Kapiteln 1 und 2 führen wir die gundlegenden algebraischen Strukturen ein, Jordanalgebren und Jordantripelsysteme. Der Überblick über Jordanalgebren ist rein klassisch und kann in jedem Standardwerk über Jordanalgebren nachgelesen werden [**6, 8**], in ihm werden einige Notationen festgelegt. Dasselbe gilt für die Abschnitte 2.1 bis 2.4, in denen wir an die Grundlagen zu positiv hermitesche Jordantripelsysteme (phJTS) erinnern. Im folgenden wird ein phJTS mit Z bezeichnet. Die Vor- und Nachteile der Verwendung von phJTS im Gegensatz zu Jordanpaaren mit positiv hermitescher Involution diskutieren wir in Bemerkung 2.1. Beginnend mit Abschnitt 2.5 weichen wir von der üblichen Darstellung von phJTS ab, indem wir *Pseudionverse* und eine *verallgemeinerte Peircezerlegung* einführen. Wir übernehmen beide Konzepte aus der Arbeit von W. Kaup in [**18**]. Die systematische Anwendung dieser Konzepte ist allerdings neu. Insbesondere sei auf das nicht-triviale Zusammenwirken der Strukturgruppe mit Pseudoinversen und Peircezerlegungen hingewiesen.

Zu Pseudoinversen. Das Pseudoinverse $a^\dagger$ eines Elements $a \in Z$ ist eindeutig bestimmt durch die Relationen

$$Q_a a^\dagger = a\ , \quad Q_{a^\dagger} a = a^\dagger\ , \quad Q_a Q_{a^\dagger} = Q_{a^\dagger} Q_a\ ,$$

wobei Q_x den quadratischen Operator des Jordantripelsystems Z bezeichnet. Dies verallgemeinert die Moore-Penrose Inverse von Matrizen [**37**]. Die Eigenräume des Boxoperators $a \square a^\dagger$ definieren die *verallgemeinerte Peircezerlegung*

$$Z = Z_1^a \oplus Z_{1/2}^a \oplus Z_0^a\ .$$

Für ein Tripotent $e \in Z$ stimmt diese Zerlegung mit der gewöhnlichen Peircezerlegung überein, da in diesem Fall $e^\dagger = e$ gilt. Neben der Peircezerlegung verallgemeinern wir auch andere Konzepte durch die systematische Verwendung von Pseudoinversen, z.B. bestimmte Jordanalgebra-Strukturen auf Peirce-1-Räumen (Proposition 2.14), Peirce-Äquivalenz (Abschnitt 2.6), Frobenius-Transformationen (Lemma 3.14) und partielle Cayley-Abbildungen (Abschnitt 4.3). Darüber hinaus erhalten wir in Lemma 2.26 die Relation

$$a^{a^\dagger - z} = {z_1}^\dagger = Q_a z_1^{-1} \quad \text{mit} \quad z_1 = Q_a Q_{a^\dagger} z\ ,$$

wodurch Quasiinverse, Pseudoinverse und Inverse der unitalen Jordanalgebra Z_1^a miteinander in Verbindung gebracht werden. Diese Relation ist ebenfalls in der folgenden Gleichung enthalten, die einen Nenner des Quasiinversen mit einem Nenner des Inversen in Z_1^a zueinander in Relation setzt,

$$\delta(a^\dagger - z, a) = \delta^a(z) \quad \text{für alle } z \in Z_1^a.$$

Wir wenden diese Relationen an, um z.B. zu beweisen, dass gewisse Abbildungen, die das Pseudoinverse verwenden, komplexanalytisch sind (Theorem 3.18), oder auch um Geradenbündel über verschiedenen Mannigfaltigkeiten zu definieren (siehe Abschnitt 6.3).

Der entscheidende Vorteil bei der Verwendung von Pseudoinversen wird deutlich, wenn wir die Operation der Strukturgruppe $\mathrm{Str}(Z)$ auf verschiedenen Objekten studieren, die mit Hilfe von Pseudoinversen definiert werden können. Da die Menge der Tripotenten unter der Operation der Strukturgruppe nicht invariant ist, existiert in der üblichen Behandlung dieser Themen für diese Gruppenoperation keine Entsprechung. Beispielsweise zeigen wir, dass für Elemente $a \in Z$ und $h \in \mathrm{Str}(Z)$ folgende Relation der Peirceräume gilt (Lemma 2.32)

$$Z_1^{ha} = hZ_1^a\,, \quad Z_0^{ha} = h^{-*}Z_0^a\,.$$

Es sei bemerkt, dass das Zusammenspiel von Strukturautomorphismen und Pseudoinversen von hoher Komplexität ist: Für ein Gegenbeispiel der Relation $(ha)^\dagger = h^{-*}a^\dagger$, wie sie in [**18**] angegeben ist, sei auf Abschnitt 2.8 verwiesen. Stattdessen erhält man lediglich die Gleichung (Lemma 2.32)

$$a^\dagger = Q_a Q_{a^\dagger} h^* (ha)^\dagger\,.$$

Von daher enthalten die Resultate über die Operation der Strukturgruppe auf verschiedenen Objekten, wie z.B. auf Mengen von Peirceräumen (siehe oben), auf Peirce-Varietäten (Theorem 6.5) und auf dem kompakten Dual (Abschnitt 7.1), stets nicht-triviale algebraische Berechnungen.

Neben dem algebraischen Nutzen der Verwendung von Pseudoinversen ergeben sich auch Vorteile im analytischen Bereich: Im Gegensatz zur Menge der Tripotenten, die eine reell analytische Mannigfaltigkeit bildet, erhalten wir auf der Menge der Elemente vom Rang j eine komplex-analytische Struktur. Durch die Verallgemeinerung von Konzepten, die ursprünglich mit Hilfe von Tripotenten beschrieben werden, zu Konzepten, bei denen beliebige Elemente zugelassen sind, lassen sich nun reell analytische Abbildungen von der Menge der Tripotenten durch komplex-analytische Abbildungen auf der Menge der Elemente von Rank j ersetzen, wie z.B. die kanonische Projektion auf Peirce-Grassmannsche (Theorem 6.1).

In Abschnitt 2.6 verwenden wir die verallgemeinerte Peircezerlegung, um E. Nehers Äquivalenzrelation auf der Menge der Tripotenten[1] zu einer Äquivalenzrelation auf ganz Z zu erweitern, welche wir *Peirce-Äquivalenzrelation* nennen. Die Abschnitte 2.7 bis 2.10 folgen der üblichen Darstellung dieser Themen, die vorgestellten Resultate werden allerdings an das Konzept von Pseudoinversen und der verallgemeinerten Peircezerlegung angepasst und entsprechend neu bewiesen. Insbesondere werden in diesen Abschnitten die oben beschriebenen Relationen gezeigt.

In Kapitel 3 bereiten wir das Studium analytischer Aspekte der Jordantheorie vor. Abschnitt 3.1 enthält eine kurze Zusammenfassung bekannter Aussagen über eingebettete und immergierte Untermannigfaltigkeiten. In dieser Arbeit bezieht sich der Ausdruck „Untermannigfaltigkeit“ ohne weiteren Zusatz stets auf eine eingebettete Untermannigfaltigkeit. In Abschnitt 3.2 beschäftigen wir uns mit Äquivalenzrelationen und ihrem Bezug zu analytischen Strukturen. Wir beschreiben ein bekanntes Kriterium dafür, wann der Quotient M/R einer Mannigfaltigkeit M über einer Äquivalenzrelation $R \subset M \times M$ auf M selbst die Struktur einer Mannigfaltigkeit trägt (Godements Theorem). Das Hauptresultat dieses Abschnitts ist die globale Beschreibung von Vektorbündeln über solchen *Quotientenmannigfaltigkeiten* mit Hilfe von *Kozykeln* auf der Äquivalenzrelation R (Theorem 3.8). Wir nennen dies den *Godement-Zugang* (*Godement approach*) zu analytischen Strukturen auf dem Quotient M/R. Da die meisten Mannigfaltigkeiten, die in dieser Arbeit untersucht werden, über Jordantheoretisch beschriebene Äquivalenzrelationen definiert sind, hat der Godement-Zugang in dieser Arbeit eine besondere Bedeutung. Desweiteren

[1] Zwei Tripotente sind äquivalent, wenn die von ihnen induzierten Peircezerlegungen übereinstimmen, siehe [**36**].

zeigen wir, wie sich aus dieser globalen Sichtweise von Quotientenmannigfaltigkeiten und ihren Vektorbündeln auch lokale Beschreibungen dieser Strukturen ableiten lassen.

Zum Godement-Zugang. In Lehrbüchern wie in [**38, 40**] wird Godements Theorem verwendet, um das klassische Resultat zu beweisen, dass der Quotient einer Liegruppe über einer abgeschlossenen Untergruppe eine Mannigfaltigkeit bildet (vgl. homogene Räume). Wir wenden Godements Theorem auf Äquivalenzrelationen an, die *nicht* durch Gruppenoperationen beschrieben werden. Auf Jordantripelsystemen gibt es zwei grundlegende Äquivalenzrelationen, auf denen alle Mannigfaltigkeiten, die in dieser Arbeit diskutiert werden, aufgebaut werden. Zum einen erhalten wir die *Peirce-Äquivalenzrelation*, die auf der Menge Z_j der Element vom Rang j durch

$$u \approx \tilde{u} \quad \text{genau dann, wenn} \quad Z_1^u = Z_1^{\tilde{u}}$$

definiert ist. Zum anderen gibt es die von O. Loos beschriebene Äquivalenzrelation auf $Z \times \overline{Z}$,

$$(z, a) \sim (\tilde{z}, \tilde{a}) \iff \begin{cases} (z, a - \tilde{a}) \text{ ist quasi-invertierbar} \\ \text{und } \tilde{z} = z^{a-\tilde{a}} \,. \end{cases}$$

Beide Äquivalenzrelationen sind *regulär*, d.h. die entsprechenden Quotienten tragen eine Mannigfaltigkeitsstruktur. Einerseits erhalten wir somit die Peirce-Grassmannsche $\mathbb{P}_j = Z_j/\approx$ vom Typ j (vgl. Kaptiel 6) und andererseits die Grassmannsche $\mathbb{G}(Z) = (Z \times \overline{Z})/\sim$ des phJTS Z (vgl. Kapitel 4).

Durch die Abschwächung der Peirce-Äquivalenzrelation zu Inklusionen erhalten wir eine partielle Ordnung auf Z,

$$u \subset \tilde{u} \quad \text{genau dann, wenn} \quad Z_1^u \subset Z_1^{\tilde{u}} \,.$$

Damit verallgemeinern wir die Mannigfaltigkeit Z_j der Rang-j Elemente zur *prä-Peirce-Fahnenmannigfaltigkeit* Z_J,

$$Z_J := \{(u_1, \dots, u_k) \,|\, u_1 \subset \dots \subset u_k, \ \mathrm{rk}\, u_i = j_i\} \ ,$$

wobei $J = (j_1, \dots, j_k)$ der *Typ* der prä-Peirce-Fahnenmanigfaltigkeit genannt wird. In Abschnitt 3.3 zeigen wir, dass dies tatsächlich eine komplex-analytische Untermannigfaltigkeit von Z^k ist. Durch die offensichtliche Erweiterung der Peirce-Äquivalenzrelation auf Elemente in Z_J erhalten wir hiermit die *Peirce-Fahnenmannigfaltigkeiten* $\mathbb{P}_J = Z_J/\approx$, auf die der Godement-Zugang Anwendung findet (Theorem 6.15).

Schließlich weisen wir darauf hin, dass ein hochgradig nichttriviales Zusammenspiel der Peirce-Äquivalenzrelation und der Loos'schen Äquivalenzrelation die Grundlage für die Definition von Jordan-Fahnenvarietäten bildet (siehe unten). Auch hier wird der Godement-Zugang angewendet. Auf all diesen Mannigfaltigkeiten werden Geradenbündel mit Hilfe von Kozykeln definiert, die unter Verwendung eines Nenners des Quasiinversen durch prägnante Formeln beschrieben werden können (siehe Abschnitte 4.1, 6.3 und 8.6).

In Abschnitt 3.3 wenden wir die bislang vorgestellten Methoden an, um (1) zu zeigen, dass die Menge Z_j der Elemente vom Rang j eine komplex-analytische

Untermannigfaltigkeit von Z bildet (Theorem 3.15), und um (2) diese Mannigfaltigkeit auf die prä-Peirce-Fahnenmannigfaltigkeiten Z_J zu verallgemeinern (Theorem 3.19), welche die Grundlage zur Definition der Peirce-Fahnenmannigfaltigkeiten bildet, siehe Kapitel 6. Es sei bemerkt, dass (1) auch aus abstrakten Argumenten heraus folgt, da die Strukturgruppe $\mathrm{Str}(Z)$ transitiv eine komplexe algebraische Gruppe ist, die transitiv auf den Zusammenhangskomponenten von Z_j operiert. Unser Beweis verwendet explizite Berechnungen, die zudem eine tiefere Einsicht in die Struktur von Z_j liefern, vgl. Korollar 3.16. Dieser explizite Zugang scheint neu zu sein. Für den Beweis von (2) kann das abstrakte Argument nicht verallgemeinert werden, da die Strukturgruppe in diesem Fall nicht mehr transitiv auf den Komponenten von Z_J operiert. In Abschnitt 3.4 erinnern wir zunächst an den Funktionalkalkül auf dem phJTS Z, wie er z.B. in [**28**] eingeführt wird. Wir modifizieren ein bekanntes Resultat über reell-analytische Funktionen um $0 \in Z$ zu einer entsprechenden Aussage über reell-analytische Funktionen auf $\mathbb{R} \setminus \{0\}$, siehe Proposition 3.25. Dadurch können wir beweisen, dass die Pseudoinversen-Abbildung $z \mapsto z^\dagger$ auf den Untermannigfaltigkeiten Z_j reell analytisch ist, und wir können ihre Ableitung explizit bestimmen. Es sei bemerkt, dass dies selbst im Matrixfall $Z = \mathbb{C}^{r\times s}$, bei dem das Pseudoinverse der Moore-Penrose-Inversen entspricht, eine bedeutende Aussage ist (Theorem 3.27). Gleichermaßen zeigen wir, dass die Projektion von Z_j auf die Menge der Tripotenten vom Rang j ebenfalls reell-analytisch ist, und bestimmen ihre Ableitung. Im letzten Abschnitt von Kapitel 3 gehen wir kurz auf den allgemein bekannten Zusammenhang von phJTS und beschränkten symmetrischen Gebieten ein. Wir verwenden in dieser Arbeit durchgängig die folgende Notation:

$$\mathcal{D} = \{z \in Z \,|\, |z| < 1\}\ , \quad \mathcal{D} = G/K \quad \text{with} \quad G = \mathrm{Aut}(\mathcal{D})^0\ , \quad K = \mathrm{Aut}(Z)^0\ ,$$

wobei der Index 0 die Einschränkung auf die Zusammenhangskomponente des neutralen Elements der jeweiligen Gruppe bezeichnet. Außerdem notieren wir mit $G^{\mathbb{C}}$ bzw. $K^{\mathbb{C}}$ die Komplexifizierungen der Gruppen G bzw. K, und verwenden G^c für eine kompakte reelle Form von $G^{\mathbb{C}}$, die K enthält. Dies schließt den ersten Teil dieser Arbeit ab.

Das Ziel von Teil 2 ist die Beschreibung der G- und $K^{\mathbb{C}}$-Orbitstrukturen auf dem kompakten Dual eines beschränkten symmetrischen Gebietes $\mathcal{D} = G/K$. Kapitel 4 liefert das Jordantheoretische Modell des kompakten Duals, welches von O. Loos angegeben wurde [**28**]. Durch das Studium des Matrixfalls $Z = \mathbb{C}^{r\times s}$ begründet definiert man auch im allgemeinen Fall die *Grassmann-Varietät* $\mathbb{G}(Z)$ durch

$$\mathbb{G}(Z) = (Z \times \overline{Z})/\sim \quad \text{mit} \quad (z,a) \sim (\tilde{z},\tilde{a}) \iff \begin{cases} (z, a-\tilde{a}) \text{ ist quasi-invertierbar} \\ \text{und } \tilde{z} = z^{a-\tilde{a}}\,. \end{cases}$$

Die Identifizierung der Grassmann-Varietät $\mathbb{G}(Z)$ mit dem kompakten Dual von $\mathcal{D}$ wird erst mit Hilfe der Definition der Gruppenoperation von $G^{\mathbb{C}}$ auf $\mathbb{G}(Z)$ und der Bestimmung der Stabilisatorgruppe eines Elements begründet (vgl. Theorem 4.7). Eine detaillierte Ausarbeitung dieses Vorgehens ist in [**28**, §§7ff.] zu finden. In Abschnitt 4.1 erinnern wir an die Konstruktion von Loos und untersuchen sie erneut aus Sicht des Godement-Zugangs (siehe oben). Wir beschreiben Vektor- und Geradenbündel der Grassmannschen mit Hilfe von Kozykeln auf der Äquivalenzrelation. In Abschnitt 4.2 stellen wir bekannte Aussagen über die Automorphismengruppe der Grassmannschen zusammen, wie z.B. dass ihre Zusammenhangskomponente des neutralen Elements mit der Komplexifizierung von G übereinstimmt, d.h. $\mathrm{Aut}(\mathbb{G}(Z))^0 = G^{\mathbb{C}}$. Hierbei legen wir u.a. die Notation für (Quasi-)Translationen fest. In Abschnitt 4.3 verallgemeinern wir das Konzept der partiellen Cayley-Abbildungen und partiellen Inversen-Abbildungen zu Konzepten, in denen statt nur tipotente, nun *beliebige* Elemente aus Z zugelassen sind (siehe auch den Einschub zu Pseudoinversen). Die partiellen Inversenabbildungen sind im Zusammenhang der Peirce-Varietäten von besonderer Bedeutung (siehe unten). Das Hauptresultat dieses Kapitels ist die Beschreibung von zwei verschiedenen Repräsentantensystemen für die Elemente der Grassmannschen (Theorem 4.12). Zusätzlich illustrieren wir diese Repräsentantensysteme im Matrixfall $Z = \mathbb{C}^{r\times s}$. In Kapitel 7 zeigt sich, dass diese Repräsentantensysteme passgenau die G- und $K^{\mathbb{C}}$-Orbitstrukturen der Grassmannschen widerspiegeln.

> *Zu Repräsentanten von Elementen der Grassmannschen.* Da die Grassmannsche $\mathbb{G}(Z)$ über eine Äquivalenzrelation auf $Z \times \overline{Z}$ definiert ist (siehe oben), sind ihre Elemente durch Äquivalenzklassen beschrieben, die wir mit $[z : a]$ bezeichnen. Da diese Äquivalenzrelation regulär ist, folgt aus Godements Theorem, dass die kanonische Projektion von $Z \times \overline{Z}$ auf $\mathbb{G}(Z)$ ein Submersion ist. Für festes $a \in \overline{Z}$ liefert die Einschränkung dieser Projektion auf $Z \times \{a\}$ gerade die (Jordantheoretischen) Karten der Grassmannschen. Auf diese Weise kann der Faktor $\overline{Z}$ als Parameterraum für die Karten auf $\mathbb{G}(Z)$ betrachtet werden. Anders betrachtet stellt $Z \times \{a\}$ für jedes feste $a \in \overline{Z}$ ein Teil-Repräsentatensystem für die Elemente der Grassmannschen dar.
>
> Offensichtlich lassen sich auch beliebige andere Repräsentantensysteme wählen, so dass sich die Frage stellt, ob man für eine gegebene Problemstellung zur Grassmannschen ein geeignetes Repräsentantensystem wählen kann, welches diese Problemstellung möglichst einfach beantwortet. Dieses ist ein weiterer Vorteil des Godement-Zugangs (siehe oben). Theorem 4.12 beantwortet die Frage nach der G- und $K^{\mathbb{C}}$-Orbitstrukturen der Grassmannschen auf genau diese Weise. In Verbindung mit Bemerkung 4.13 besagt es, dass jedes Element $\chi \in \mathbb{G}(Z)$ dargestellt werden kann durch[2]
>
> (i) $\chi = [e + d_e : c + d_c]$ mit $e, c \in S,\ c \leq e,\ d_e \in \mathcal{D}_0^e,\ d_c \in \mathcal{D}_1^c$,
>
> (ii) $\chi = [u + z : u^\dagger]$ mit $u, z \in Z,\ u \perp\!\!\!\perp z$.
>
> Diese Repräsentanten sind eindeutig bestimmt bis auf Peirce-Äquivalenz in c bzw. in u. In der Übersicht zu Kapitel 7 beschreiben wir, wie diese Repräsentantensysteme bei der Frage der Orbitstrukturen auf der Grassmannschen Anwendung finden.

Kaptiel 5 und 6 bilden Zwischenstationen auf dem Weg zur Bestimmung der G- und $K^{\mathbb{C}}$-Orbitstrukturen auf der Grassmannschen $\mathbb{G}(Z)$. In diesem Zusammenhang liefert Kapitel 5 eine G-Invariante auf $\mathbb{G}(Z)$, welche die G-Orbits klassifiziert (Korollar 5.10), und Kapitel 6 gibt eine Beschreibung der besonders wichtigen abgeschlossenen $K^{\mathbb{C}}$-Orbits, welche mit Peirce-Grassmannschen identifiziert werden können (Theorem 6.5).

Die Herleitung der G-Invarianten auf $\mathbb{G}(Z)$ in Kapitel 5 geschieht auf indirekte Weise. Wir betten die Grassmannsche diagonal als reelle Untermannigfaltigkeit in das Produkt $\mathbb{G}(Z) \times \overline{\mathbb{G}}(Z)$ ein, wobei $\overline{\mathbb{G}}(Z)$ die konjugierte Grassmannsche bezeichnet (siehe Abschnitt 4.1), und untersuchen eine $G^{\mathbb{C}}$-Gruppenoperation auf diesem Produkt, die in der Einschränkung auf $G \subset G^{\mathbb{C}}$ entlang der Diagonalen mit der üblichen G-Operation auf $\mathbb{G}(Z)$ übereinstimmt. Der Vorteil dieser Vorgehensweise

[2]Hierbei ist S die Menge der Tripotenten, $c \leq e$ bezeichnet die übliche partielle Ordnung von Tripotenten, $\mathcal{D}_\nu^e$ stellt das beschränkte symmetrische Gebiet des phJTS Z_ν^c dar, d.h. $\mathcal{D}_\nu^e = \mathcal{D} \cap Z_\nu^c$, und $u \perp\!\!\!\perp z$ bedeutet starke Orthogonalität der Elemente u und z, d.h. $u \,\square\, z = 0$.

liegt darin, dass eine $G^{\mathbb{C}}$-Operation wesentlich einfacher zu beschreiben ist (über Erzeuger und Relationen) als eine G-Operation. Wir übernehmen diese Idee aus der Theorie der „Cycle Spaces", in der eine entsprechende Diagonaleinbettung des beschränkten symmetrischen Gebiets $\mathcal{D} = G/K$ in die komplexe Mannigfaltigkeit $G^{\mathbb{C}}/K^{\mathbb{C}}$ studiert wird, vgl. [**9**]. Wir zeigen in Theorem 5.12, dass $G^{\mathbb{C}}/K^{\mathbb{C}}$ eine offen und dichte Teilmenge des Produkts $\mathbb{G}(Z)\times\overline{\mathbb{G}}(Z)$ ist. In Abschnitt 5.1 motivieren wir die Ergebnisse dieses Abschnitts durch das Studium des Matrixfalls $Z = \mathbb{C}^{r\times s}$ mit Hilfe von geometrischen Argumenten. In Abschnitt 5.2 betrachten wir das Produkt $\mathbb{G}(Z)\times\overline{\mathbb{G}}(Z)$, definieren auf ihm die $G^{\mathbb{C}}$-Operation und beschreiben Vektorbündel über ihm. Der dritte Abschnitt bildet das Zentrum dieses Kapitels. Hierin definieren wir $G^{\mathbb{C}}$-äquivariante Schnitt auf $\mathbb{G}(Z)\times\overline{\mathbb{G}}(Z)$ und beschreiben die zugehörigen Invarianten (Propositionen 5.4 und 5.9). Im Fall der Einschränkung auf G und auf die Diagonale $\mathbb{G}(Z) \hookrightarrow \mathbb{G}(Z)\times\overline{\mathbb{G}}(Z)$ zeigen wir zudem, dass sich die Invarianten noch weiter verfeinern lassen (Korollare 5.6 und 5.10). Es sei betont, dass die Jordantheoretische Beschreibung in diesem Zusammenhang äußerst explizite Formeln für die Schnitte und ihre Invarianten liefert. Im abschließenden Abschnitt bestimmen wir die $G^{\mathbb{C}}$-Orbitstruktur des Produkts $\mathbb{G}(Z)\times\overline{\mathbb{G}}(Z)$, und wir zeigen, dass die $G^{\mathbb{C}}$-Invarianten des letzten Abschnittes die $G^{\mathbb{C}}$-Orbits eindeutig charakterisieren.

Kapitel 6 ist der Untersuchung von Mannigfaltigkeiten gewidmet, die auf der Basis der Peirce-Äquivalenzrelation definiert sind. Es zeigt sich, dass diese Mannigfaltigkeiten sogar die Struktur von glatten algebraischen Varietäten im Sinne von D. Mumford [**35**] tragen. Der einfachste Fall einer Peirce-Varietät ist die *Peirce-Grassmannsche*, die als Quotient von Z über der Peirce-Äquivalenzrelation definiert ist,

$$\mathbb{P} = Z/\approx \quad \text{mit} \quad u \approx \tilde{u} \quad \text{genau dann, wenn} \quad Z_1^u = Z_1^{\tilde{u}}\ .$$

In Abschnitt 6.1 wenden wir Godements Theorem an und zeigen, dass diese Äquivalenzrelation regular ist und somit eine Mannigfaltigkeitsstruktur auf $\mathbb{P}$ definiert. Zudem zeigen wir, dass die natürlich gegebene $K^{\mathbb{C}}$-Operation auf Z auch eine $K^{\mathbb{C}}$-Operation auf $\mathbb{P}$ induziert, wodurch die Peirce-Grassmannsche zu einem hermitesch symmetrischen Raum nicht-kompakten Typs wird (Theorem 6.1). Dies ist ein bekanntes Resultat (vgl. [**28**, §5.6b]), allerdings sei bemerkt, dass sich unser Beweis vollständig im Rahmen der komplex-analytischen Theorie bewegt, was erst durch die Erweiterung der Peirce-Äquivalenz von der Menge der Tripotenten auf ganz Z möglich wird. Dadurch erhalten wir zudem eine komplex-analytische Faserung der Mannigfaltigkeit Z_j, der Elemente vom Rang j, über der entsprechenden Zusammenhangskomponente $\mathbb{P}_j$ der Peirce-Grassmannschen. Desweiteren liefert unser Zugang neuer explizite Beschreibungen der Karten auf $\mathbb{P}_j$ und ihrer Übergangsfunktionen (Proposition 6.3). In Abschnitt 6.2 zeigen wir, dass verschiedene Realisierungen der Peirce-Äquivalenzrelation [**28, 18, 16, 2**] zu isomorphen Mannigfaltigkeiten führen (Theorem 6.5). Es sei bemerkt, dass die Realisierung der Peirce-Grassmannschen $\mathbb{P}$ als $K^{\mathbb{C}}$-invariante Untermannigfaltigkeit der Grassmannschen $\mathbb{G}(Z)$,

$$\mathbb{P} \hookrightarrow \mathbb{G}(Z),\ [u] \mapsto \left[u : u^{\dagger}\right]\ ,$$

die Positionierung dieses Kapitels im Kontext der Diskussion von Orbitstrukturen auf der Grassmannschen rechtfertigt. Nach W. Kaup [**18**] kann die Peirce-Grassmannsche zudem mit der Grassmannschen eines geeigneten phJTS identifiziert werden, genauer: Die Zusammenhangskomponente von $\mathbb{P}$, die ein Element $[u]$ enthält, ist isomorph zu Grassmannschen des phJTS $Z^u_{1/2}$. Dieser Isomorphismus wird in [**18**] abstrakt begründet. Im Zusammenhang der expliziten Beschreibung der Karten von $\mathbb{P}$ können wir diesen Isomorphismus explizit bestimmen. Wir zeigen, dass er durch die Einschränkung der partiellen Inversen-Abbildung $j_{u^{\dagger}}$ auf den Abschluss der Peirce $1/2$-Raumes $Z^u_{1/2}$ in $\mathbb{G}(Z)$ gegeben ist (Theorem 6.10). In Abschnitt 6.3 verwenden wir den Godement-Zugang zur Definition von Geradenbündeln auf der Peirce-Grassmannschen. Die entsprechenden Kozykeln basieren hierbei auf den Relationen eines Nenners der Quasi-Inversen auf Z. Wir beweisen durch konkrete Rechnungen, dass diese Geradenbündel sehr ampel sind, und somit $\mathbb{P}$ eine projektive Varietät darstellt (Theorem 6.14). Dieser Beweis ist eine Variation des entsprechenden Beweises für ein sehr amples Geradenbündel auf der Grassmannschen $\mathbb{G}(Z)$, wie ihn O. Loos durchführt [**28**, §7.10]. Im letzten Abschnitt dieses Kapitels diskutieren wir die offensichtliche Verallgemeinerung der Peirce-Grassmannschen zu *Peirce-Fahnenvarietäten*. Diese Varietäten wurden in [**2**] auf Grundlage der üblichen Peircezerlegung durch Tripotente eingeführt. Unsere Definition von Peirce-Fahnenvarietäten basiert auf der Anwendung der verallgemeinerten Peirce-Äquivalenzrelation auf der prä-Peirce-Fahnenmannigfaltigkeit Z_J, die in Abschnitt 3.3 diskutiert wird,

$$\mathbb{P}_J = Z_J/\approx \quad \text{mit} \quad (u_1,\ldots,u_k) \approx (\tilde{u}_1,\ldots,\tilde{u}_k) \iff Z_1^{u_i} = Z_1^{\tilde{u}_i} \text{ für alle } i.$$

In Abschnitt 6.4 bestimmen wir durch die Anwendung von Godements Theorem die analytische Struktur dieser Peirce-Fahnenvarietäten, und beschreiben einen Atlas von $\mathbb{P}_J$ (Theorem 6.15 und Proposition 6.17). Durch die Verwendung der verallgemeinerten Peircezerlegung erhalten wir außerdem eine natürliche Operation der Strukturgruppe auf $\mathbb{P}_J$, wodurch die kanonische Projektion von Z_J auf $\mathbb{P}_J$ zu einer $\mathrm{Str}(Z)$-äquivarianten Abbildung wird. Mit Hilfe von geeigneten Projektionen auf Peirce-Grassmannsche können wir $K^{\mathbb{C}}$-äquivariante Geradenbündel von den Peirce-Grassmannschen auf die Peirce-Fahnenvarietät zurückziehen. Schließlich beweisen wir, dass ein geeignetes Produkt dieser Geradenbündel sehr ampel ist, wodurch gezeigt ist, dass $\mathbb{P}_J$ eine projektive Varietät ist (Theorem 6.20). Es sei betont, dass einer der Vorteile dieser Jordantheoretischen Beschreibung der Peirce-Grassmannschen (im Gegensatz zu einer Lietheoretischen Beschreibung) darin besteht, dass wir explizite Formeln z.B. für die Karten und für die Geradenbündel erhalten.

In Kapitel 7 kehren wir zurück zur Untersuchung von G- und $K^{\mathbb{C}}$-Orbitstrukturen auf der Grassmannschen $\mathbb{G}(Z)$. Wir setzen voraus, dass Z einfach ist. Die Ergebnisse der letzten Kapitel werden für den Beweis des Hauptresultats dieses Kapitels zusammengenommen, welches die G- und $K^{\mathbb{C}}$-Orbitstruktur in Jordantheoretischen Begriffen explizit bestimmt (Theorem 7.2). Wie oben angemerkt, wurde die G-Orbitstruktur schon von J. Wolf untersucht [**44**], und T. Matsuki ist der Beweis einer eins-zu-eins Korrespondenz zwischen den G- und der $K^{\mathbb{C}}$-Orbits zuzuschreiben [**34**]. Es zeigt sich, dass die Anzahl der Orbits durch $\binom{r+2}{2}$ gegeben ist, wobei r der Rank des Jordantriples Z bzw. der Rang der reellen halbeinfachen Liegruppe G ist. Desweiteren fasern sowohl die G- als auch die $K^{\mathbb{C}}$-Orbits über bestimmten K-Orbits. J. Wolf zeigt, dass die Faser der G-Orbits aus dem Produkt zweier hermitesch symmetrischer Räume nicht-kompakten Typs besteht [**44**, §9]. Diesbezüglich sind die Ergebnisse aus Abschnitt 7.1 allgemein bekannt, allerdings sei darauf verwiesen, dass unser Beweis unabhängig von dem Lietheoretischen Beweis geführt wird, und dass die Stärke des Jordantheoretischen Zugangs darin besteht, dass die resultierenden Beschreibungen der Orbits sehr explizit sind.
Unsere Beschreibung der G- und $K^{\mathbb{C}}$-Orbits basiert auf den beiden Repräsentantensystemen für die Elemente in $\mathbb{G}(Z)$, die wir oben beschrieben haben (Theorem 4.12). Somit können wir die Orbits der Grassmannschen mit gewissen Teilmengen von

$Z \times \overline{Z}$ identifizieren.[3] Wir erhalten für die entsprechenden G-, $K^{\mathbb{C}}$- und K-Orbits die folgende Beschreibung:

$$\begin{aligned} G_b^a &= \{[e + d_e : c + d_c] \,|\, e \in S_{a+b},\, c \in S_a,\, e \geq c,\, d_e \in \mathcal{D}_0^e,\, d_c \in \mathcal{D}_1^c\} \;, \\ \mathbb{K}_b^a &= \left\{[u + z : u^\dagger] \,\middle|\, u \in Z_a,\, z \in Z_b,\, u \perp\!\!\!\perp z\right\} \;, \\ K_b^a &= \{[e : c] \,|\, e \in S_{a+b},\, c \in S_a,\, e \geq c\} = \{[c + \tilde{c} : c] \,|\, c \in S_a,\, \tilde{c} \in S_b,\, c \perp\!\!\!\perp \tilde{c}\} \end{aligned}$$

mit $0 \leq a \leq a + b \leq r$, wobei S_j bzw. Z_j die Mengen der Tripotenten von Rang j bzw. die Menge aller Elemente vom Rang j beschreiben. Die Faserungen über den K-Orbits sind gegeben durch

$$\mathcal{D}_0^e \times \mathcal{D}_1^c \to G_b^a \to K_b^a \;, \quad \Omega(Z_+^u) \to \mathbb{K}_b^a \to K_b^a \;,$$

wobei $\Omega(Z_+^u)$ der symmetrische Kegel der euklidischen Jordanalgebra Z_+^u ist, vgl. Theorem 3.27. Die Aussage über die Faserung der $K^{\mathbb{C}}$-Orbits ist für die „endlichen“ Orbits bekannt, d.h. für Orbits, die in $Z \hookrightarrow \mathbb{G}(Z)$ enthalten sind, die Erweiterung dieses Resultats auch auf die Orbits im Unendlichen scheint neu zu sein. Es sei bemerkt, dass es noch einen zweiten (wesentlich abstrakteren) Jordantheoretischen Zugang zur Untersuchung dieser Orbitstrukturen gibt, der von W. Kaup in [**19**] beschrieben wird. Er verwendet einen verallgemeinerten Funktionalkalkül und ist eng verbunden mit dem Lietheoretischen Zugang über Impulsabbildungen, wie er von R. Bremingan und J. Lorch in [**7**] vorgestellt wird. Abschnitt 7.1 schließt mit einer Beschreibung der Tangentialstrukturen der verschiedenen Orbits und liefert explizite Formeln für G-invariante (pseudo-)hermitesche Metriken auf den offenen G-Orbits. In Abschnitt 7.2 beweisen wir topologische Eigenschaften der Orbits und bestimmen den jeweiligen topologischen Abschluss (Theorem 7.3). Neben der „globalen“ Beschreibung der Orbits wie sie oben angegeben ist, erhalten wir in Abschnitt 7.3 explizite, einfache Formeln zur Beschreibung der $K^{\mathbb{C}}$-Orbits als Untervarietäten bestimmter Kartengebieten der Grassmannschen $\mathbb{G}(Z)$. Schließlich beweisen wir in Abschnitt 7.4 die Matsuki-Dualität zwischen den G- und den $K^{\mathbb{C}}$-Orbits mittels rein Jordantheoretischer Argumente (Theorem 7.6).

Teil 3 beschäftigt sich mit der Jordantheoretischen Beschreibung verallgemeinerter Fahnenvarietäten. Wir nähern uns der zentralen Frage dieses Teils, indem wir näher auf den Matrixfall $Z = \mathbb{C}^{r \times s}$ eingehen. In diesem Fall kann die Grassmannsche $\mathbb{G}(\mathbb{C}^{r \times s})$ mit der gewöhnlichen Grassmannschen $\mathrm{Gr}_s(\mathbb{C}^{r+s})$ identifiziert werden. Lietheoretisch ist die Grassmannsche über den Quotienten $\mathrm{Gr}_s(\mathbb{C}^{r+s}) = G^{\mathbb{C}}/P$ bestimmt, wobei $G^{\mathbb{C}} = \mathrm{SL}(r+s)$ gilt und P die parabolische Untergruppe der oberen Dreiecks-Blockmatrizen vom Typ (s) und Determinante 1 ist. Diese Grassmannsche lässt sich stark verallgemeinern, indem man Fahnenvarietäten betrachtet: Für eine streng ansteigende Folge natürlicher Zahlen $0 \leq i_1 < \ldots < i_m \leq r + s$ definiert

$$\mathrm{Gr}_{(i_1,\ldots,i_m)}(\mathbb{C}^{r+s}) = \{0 \subset V_1 \subset \ldots \subset V_m \subset \mathbb{C}^{r+s} \,|\, \dim V_\ell = i_\ell\}$$

die Fahnenvarietät vom Typ $(i_1, \ldots, i_m)$. Es zeigt sich, dass dies stets eine projektive Varietät darstellt [**15**], und dass die Lietheoretische Beschreibung durch

$$\mathrm{Gr}_{(i_1,\ldots,i_m)}(\mathbb{C}^{r+s}) \cong G^{\mathbb{C}}/P'$$

gegeben ist, wobei $G^{\mathbb{C}} = \mathrm{SL}(r + s)$ gilt und P' die parabolische Untergruppe der oberen Dreiecks-Blockmatrizen vom Typ $(i_1, \ldots, i_m)$ und Determinante 1 ist, vgl. [**13, 10**]. Es stellt sich nun die Frage, ob auch diese Fahnenvarietäten eine Jordantheoretische Beschreibung über das Jordantriplesystem $Z = \mathbb{C}^{r \times s}$ zulassen. Da

[3] Diese Identifikation gilt bis auf eine einfache Äquivalenzrelation auf den betrachteten Teilmengen von $Z \times \overline{Z}$, da die beteiligten Repräsentantensysteme nicht vollständig eindeutig sind, vgl. Theorem 4.12.

aus der Fahnenvarietät $\mathrm{Gr}_{(i_1,\ldots,i_m)}(\mathbb{C}^{r+s})$ im allgemeinen nicht auf die charakteristischen Größen des Triplesystems, nämlich r und s, zurückgeschlossen werden kann, erwarten wir, dass es nicht für alle solche Fahnenvarietäten eine Jordantheoretische Beschreibung geben wird. Berücksichtigt man aber, dass die reelle Form $G = \mathrm{SU}(r,s)$ von $G^{\mathbb{C}}$ durch dieselben charakteristischen Größen beschrieben ist wie Z, scheint es plausibel zu erwarten, dass eine Jordantheoretische Beschreibung derjenigen Fahnenvarietäten $G^{\mathbb{C}}/P$ möglich ist, deren zugehörige parabolische Untergruppe P die Komplexifizierung einer reell parabolischen Untergruppe Q von G ist, $P = Q^{\mathbb{C}}$. Diese Formulierung ist unabhängig vom Spezialfall $Z = \mathbb{C}^{r\times s}$, und überträgt sich direkt auf den allgemeinen Fall:

Frage: *Sei Z eine phJTS mit beschränktem symmetrischen Gebiet $\mathcal{D}$, und sei P eine parabolische Untergruppe der Einheitskomponente G der Automorphismengruppe* $\mathrm{Aut}(\mathcal{D})$*. Gibt es eine Jordantheoretische Beschreibung der Fahnenvarietät $G^{\mathbb{C}}/Q^{\mathbb{C}}$?*

In Kapitel 8 wird diese Frage positiv beantwortet. In Abschnitt 8.1 erinnern wir an die Jordantheoretische Beschreibung der reell parabolischen Untergruppen von G, wie sie durch O. Loos angegeben wird [**28**, §9], und bestimmen ihre Komplexifizierungen (Theorem 8.4). Insbesondere zeigen wir im Matrixfall $Z = \mathbb{C}^{r\times s}$ und $G = \mathrm{SU}(r,s)$, dass die Komplexifizierung einer parabolischen Untergruppe $Q \subset G$ zu einer Standard-parabolischen Untergruppe von $G^{\mathbb{C}} = \mathrm{SL}(r+s)$ vom Typ $\mathcal{I} = (j_1,\ldots,j_k,n-j_k,\ldots,n-j_1)$ mit $n = r+s$ und $j_k \le r$ konjugiert ist, d.h. die entsprechende Fahnenvarietät $G^{\mathbb{C}}/Q^{\mathbb{C}}$ wird beschrieben durch

$$\mathrm{Gr}_{\mathcal{I}}(\mathbb{C}^n) = \{0 \subset E_1 \subset \ldots \subset E_k \subset F_k \subset \ldots \subset F_1 \subset \mathbb{C}^n \mid \dim E_\ell = j_\ell,\ \dim F_\ell = n - j_\ell\}\ .$$

In Abschnitt 8.2 verwenden wir den Matrixfall als Modell zur Konstruktion einer Jordantheoretischen Beschreibung der Fahnenvarietäten. Hierbei stellt sich die Frage (1) wie man ein Paar von Unterräumen $E \subset F \subset \mathbb{C}^n$ mit $\dim E = j$ und $\dim F = n - j$ durch Elemente aus $Z = \mathbb{C}^{r\times s}$ realisieren kann, so dass (2) die Realisierung ein und desselben Paars durch verschiedene Elemente zu einer Äquivalenzrelation auf diesen Elementen führt, die mit der Jordantripelstruktur von Z verträglich ist. Dies ist eine wesentliche Erweiterung der Fragen, die zum Jordantheoretischen Modell der Grassmannschen führten, wie es O. Loos beschreibt. Das Hauptergebnis in diesem Abschnitt wird in Lemma 8.7 beschrieben, welches das durch (1) und (2) gestellte Problem löst.[4] Es zeigt sich, dass ein Jordantheoretisches Modell der Fahnenvarietät $\mathrm{Gr}_{(j,n-j)}(\mathbb{C}^n)$ mit Hilfe von Tripeln (u,z,a) von Elementen in $\mathbb{C}^{r\times s}$ mit $\mathrm{rk}(u) = j$ konstruiert werden kann. Mit Hilfe dieses Lemmas ist im Anschluss die Verallgemeinerung auf alle oben beschriebenen Fahnenvarietäten leicht möglich. Hierbei werden die Tripel (u,z,a) durch Tupel $(u_1,\ldots,u_k,z,a)$ mit $\mathrm{rk}(u_\ell) = j_\ell$ ersetzt, siehe Lemma 8.10.

In Abschnitt 8.3 wenden wir uns dem allgemeinen Fall zu und definieren *Jordan-Fahnenvarietäten* auf beliebigen phJTS über eine Äquivalenzrelation auf $\overline{Z}_J \times Z \times \overline{Z}$, genauer:

$$\mathbb{F}_J := (\overline{Z}_J \times Z \times \overline{Z})/\sim$$

mit

$$\big((u_i),z,a\big) \sim \big((\tilde{u}_i),\tilde{z},\tilde{a}\big) \iff \begin{cases} \tilde{u}_i \approx B_{a-\tilde{a},\,z}u_i \text{ für } i = 1,\ldots,k; \text{ und} \\ \text{es existieren } u^{\perp\!\!\perp} \in Z_0^{u_k} \text{ und } \tilde{u}^{\perp\!\!\perp} \in Z_0^{\tilde{u}_k}, \\ \text{so dass } B_{a-\tilde{a},\,z+u^{\perp\!\!\perp}} \text{ invertierbar ist} \\ \text{und } \tilde{z} + \tilde{u}^{\perp\!\!\perp} = (z + u^{\perp\!\!\perp})^{a-\tilde{a}}. \end{cases}$$

[4] Wir weisen darauf hin, dass ein technischer Teil des Beweises noch offen bleibt, siehe Lemma 8.8. Diese Lemma hat jedoch *keine* Relevanz für die allgemeine Konstruktion von Jordan-Fahnenvarietäten, die unabhängig hiervon in Abschnitt 8.3 definiert und diskutiert werden.

Diese Definition verbindet die beiden grundlegenden Äquivalenzrelationen eines phJTS, d.h. die Peirce-Äquivalenzrelation und die Loos'sche Äquivalenzrelation zur Beschreibung der Grassmannschen, in hochgradig nicht-trivialer Weise. Das Hauptresultat dieses Kapitels (Theorem 8.11) zeigt, dass diese Relation tatsächlich eine Äquivalenzrelation ist und gemäß Godements Theorem eine Mannigfaltigkeitsstruktur auf $\mathbb{F}_J$ definiert. In Abschnitt 8.4 untersuchen wir die analytische und die algebraische Struktur der Jordan-Fahnenvarietät näher und zeigen, dass $\mathbb{F}_J$ eine kompakte glatte algebraische Varietät ist (Proposition 8.14). Zudem definieren wir eine $G^{\mathbb{C}}$-Operation auf $\mathbb{F}_J$ und beweisen auf diesem Weg, dass die Jordan-Fahnenvarietät $\mathbb{F}_J$ tatsächlich ein Modell des Quotienten $G^{\mathbb{C}}/Q^{\mathbb{C}}$ für eine reell parabolische Untergruppe $Q \subset G$ vom Typ J darstellt (Theorem 8.20). Schließlich verwenden wir den Godement-Zugang, um Geradenbündel auf den Jordan-Fahnenvarietäten zu definieren. Wir zeigen, dass diese Geradenbündel $G^{\mathbb{C}}$-homogen sind (Proposition 8.23).

Im letzen Kapitel dieser Arbeit zeigen wir eine erste Anwendung der Jordantheoretischen Beschreibung von verallgemeinerten Fahnenvarietäten. Das Ziel dieses Kapitels ist die Verallgemeinerung der Determinantenfunktionen, wie sie von L. Barchini, S.G. Gindikin and H.W. Wong für gewöhnliche Fahnenvarietäten definiert wurden [**3, 4**], auf allgemeine Jordan-Fahnenvarietäten. Hierbei zeigt sich erneut die Stärke der Jordantheorie, einfache und explizite Formeln zu generieren. In Abschnitt 9.1 geben wir die Definition der Barchini-Gindikin-Wong Determinantenfunktionen wieder und erinnern an eine erste Anwendung in der Geometrie. Abschnitt 9.2 liefert den Jordantheoretischen Zugang zu diesem Thema. Durch den Godement-Zugang zu Geradenbündeln definieren wir $G^{\mathbb{C}}$-invariante Schnitte auf $\mathbb{G} \times \overline{\mathbb{G}} \times \mathbb{F}_J$, die *Jordan-Determinantenfunktionen.* Schließlich identifizieren wir in Abschnitt 9.3 die Mannigfaltigkeiten, die bei der Definition der Barchini-Gindikin-Wong Determinantenfunktionen eine Rolle spielen mit einer offenen und dichten Teilmenge von $\mathbb{G} \times \overline{\mathbb{G}} \times \mathbb{F}_J$, und wir beweisen dass die Nullstellenmengen der eingeschränkten Versionen dieser Determinantenfunktionen übereinstimmen.

Ausstehende Arbeit. Wir erinnern an die Ziele des Programms „Jordantheorie und geometrische Realisierungen", das oben bereits vorgestellt wurde, (i) eine Jordan theoretische Beschreibung verallgemeinerter Fahnenvarietäten zu geben, (ii) die G-Orbitstruktur explizit zu bestimmen, und (iii) die zugehörigen Darstellungen zu beschreiben. In dieser Arbeit haben wir Problem (i) für verallgemeinerte Fahnenvarietäten $G^{\mathbb{C}}/P^{\mathbb{C}}$ mit reell parabolischer Untergruppe $P \subset G$ vollständig gelöst und begründet, dass dies die allgemeinste Form von Fahnenvarietäten ist, die sich Jordantheoretisch beschreiben lässt. Desweiteren haben wir Problem (ii) im hermitesch symmetrischen Fall $\mathbb{G}(Z)$ gelöst. Einen ersten Schritt in Richtung (ii) und (iii) im allgemeinen Fall haben wir im letzten Kapitel durch die Diskussion von Determinantenfunktionen getan. Im Folgenden skizzieren wir einige Felder der ausstehenden Arbeit:

Orbitstruktur auf Jordan-Fahnenvarietäten. Wir erwarten, eine Beschreibung der G- und $K^{\mathbb{C}}$-Orbitstruktur einer Jordan-Fahnenmannigfaltigkeit zu finden, die der Beschreibung der Orbitstrukturen auf der Grassmannschen $\mathbb{G}(Z)$ aus Kapitel 7 ähnelt, d.h. wir vermuten, dass es für die definierende Äquivalenzrelation auf $\overline{Z}_J \times Z \times \overline{Z}$ zwei Repräsentantensysteme gibt, die der G- und der $K^{\mathbb{C}}$-Orbitstruktur entsprechen.

Konische und sphärische Funktionen. Die Arbeiten von H. Upmeier [**41**], und von J. Faraut and A. Korányi [**8**] zeigen, dass Jordantheorie insbesondere bei der Beschreibung von konischen und symmetrischen Funktionen auf symmetrischen Kegeln und beschränkten symmetrischen Gebieten hilfreich ist. Durch die Jordantheoretische Beschreibung allgemeinerer G- und

$K^{\mathbb{C}}$-Orbits (Kapitel 7) hoffen wir, ähnliche Resultate für die harmonische Analysis auf diesen Orbits zu erhalten.

Determinantenfunktionen. In Kapitel 9 haben wir Jordan-Determinantenfunktionen definiert, die in enger Verbindung zu den Determinantenfunktionen auf gewöhnlichen Fahnenvarietäten stehen, wie sie von Barchini-Gindikin-Wong in **[3, 4]** eingeführt wurden. Die Korrelation der Jordan-Determinantenfunktionen mit den Barchini-Gindikin-Wong Determinantenfunktionen haben wir in Teilen in Abschnitt 9.3 studiert. Neben der weitergehenden Untersuchung dieser Korrelation steht das Studium der Jordan-Determinantenfunktionen in den Bereichen Geometrie und Darstellungstheorie noch aus. Insbesondere zeigen Barchini-Gindikin-Wong, wie ihre Determinantenfunktionen im Zusammenhang mit Szegő-Abbildungen verwendet werden können, welche Hauptreihen-Darstellungen mit Darstellungen der diskreten Reihe in Verbindung setzen. Es ist zu erwarten, dass Jordan-Determinantenfunktionen dieses Resultat auf alle einfachen Liegruppen hermiteschen Typs verallgemeinern können.

Kohomologie der Grassmannschen $\mathbb{G}(Z)$. Wir vermuten, dass die Zerlegung der Grassmannschen $\mathbb{G}(Z)$ in $K^{\mathbb{C}}$-Orbits eine CW-Zerlegung induziert, die möglicherweise einen neuen Zugang zur Kohomologie der Grassmannschen liefern kann. Im Folgenden fassen wir die grundlegenden Ideen hinter dieser Aussage zusammen, deren Details noch ausgearbeitet werden müssen. Sei Z ein einfaches phJTS vom Rang r, und seien die $K^{\mathbb{C}}$-Orbits mit $\mathbb{K}_b^a$ für $0 \leq a \leq a+b \leq r$ bezeichnet (vgl. Kapitel 7). Dann gilt

$$\mathbb{G}(Z) = \biguplus_{0\leq a\leq a+b\leq r} \mathbb{K}_b^a = \biguplus_{0\leq a\leq r} \mathcal{K}^a \quad \text{mit} \quad \mathcal{K}^a = \biguplus_{0\leq b\leq r-a} \mathbb{K}_b^a\,.$$

Für $a = 0$ erhalten wir als $\mathcal{K}^0$ die offene und dichte Teilmenge $Z \subset \mathbb{G}(Z)$, die 'Hauptzelle' der Zerlegung. Für $a > 0$ is $\mathcal{K}^a$ gegeben durch

$$\mathcal{K}^a = \left\{ \left[u + z : u^\dagger\right] \,\middle|\, u \in Z_a,\ z \in Z_0^u \right\}$$

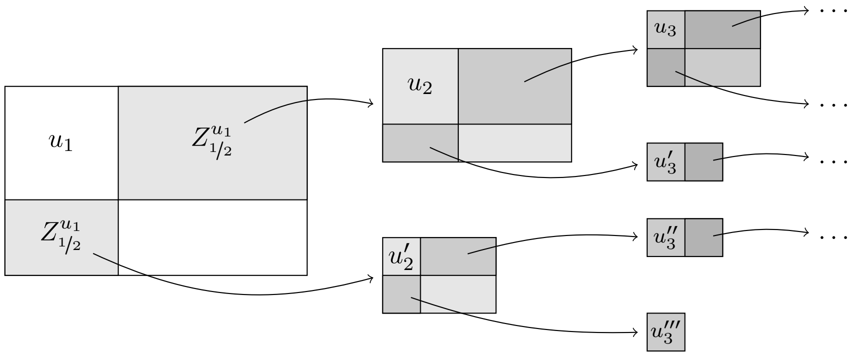
und kann als Vektorbündel über der Peirce-Grassmannschen $\mathbb{P}_a$ betrachtet werden, wobei die zugehörige Projektion durch

$$\mathcal{K}^a \to \mathbb{P}_a,\ \left[u + z : u^\dagger\right] \mapsto \left[u : u^\dagger\right]$$

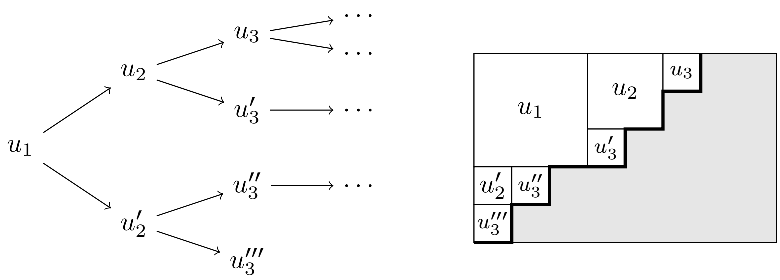
beschrieben ist. Da die Peirce-Grassmannsche selbst (explizit) mit der Grassmannschen $\mathbb{G}(Z_{1/2}^u)$ des Peirce $1/2$-Raumes eines Elements $u \in Z_a$ identifiziert werden kann, wird jedes $\mathcal{K}^a$ für $a > 0$ erneut gemäß der Orbitstruktur von $\mathbb{G}(Z_{1/2}^u)$ zerlegt. Wiederum erhält man $Z_{1/2}^u$ als 'Hauptzelle' von $\mathbb{G}(Z_{1/2}^u)$, und die anderen Orbits können als Vektorbündel über gewissen Peirce-Grassmannschen betrachtet werden, welche erneut mit Grassmannschen entsprechender Peirce $1/2$-Räume identifiziert werden usw., bis wir einen Raum erreichen, der nun noch aus einem Punkt besteht. Im Spezialfall $Z = \mathbb{C}^{1\times n}$ erhalten wir auf diese Weise die allgemein bekannte CW-Zerlegung des komplexen projektiven Raums,

$$\mathbb{CP}^n = \mathbb{C}^n \uplus \mathbb{CP}^{n-1} = \mathbb{C}^n \uplus \mathbb{C}^{n-1} \uplus \mathbb{CP}^{n-2} = \ldots = \mathbb{C}^n \uplus \mathbb{C}^{n-1} \uplus \ldots \uplus \mathbb{C} \uplus \{\mathrm{pt}\}\,.$$

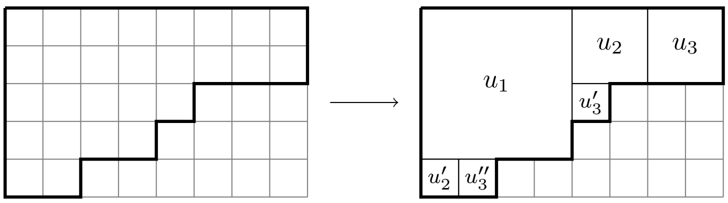
Im Allgemeinen jedoch braucht der Peirce-Raum $Z_{1/2}^u$ und somit auch die Grassmannsche $\mathbb{G}(Z_{1/2}^u)$ nicht einfach zu sein, wodurch sich die nachfolgenden Zerlegungen gemäß der Zerlegung von $\mathbb{G}(Z_{1/2}^u)$ in einfache Grassmannsche in verschiedene Teile aufgegliedern. Für den Matrixfall $Z = \mathbb{C}^{r\times s}$ ist diese Prozedur im folgenden Diagramm beschrieben:



Nach Konstruktion wird eine Zelle der CW-Zerlegung durch einen Baum von Elementen beschrieben, wobei jedes Element im Peirce $1/2$-Raum seines Vorgängers liegt, d.h.



Dies liefert eine eins-zu-eins Korrespondenz zwischen den Zellen und Young-Diagrammen, die innerhalb eines $(r \times s)$-Gitters liegen und schon in der klassischen Theorie der CW-Zerlegung der Grassmannschen $\mathrm{Gr}_s(\mathbb{C}^{r+s})$ mit entsprechenden Zellen identifiziert wurden [**13, 10**]. Es sei aber bemerkt, dass die Umkehrung der Korrespondenz zwischen den hier diskutierten Zellen und Young-Diagrammen ein gegebenes Young-Diagramm in einen Baum von Durfee-Quadraten zerlegt:



In der klassischen Theorie ist die multiplikative Struktur der Kohomologie auf der Grassmannschen $\mathrm{Gr}_s(\mathbb{C}^{r+s})$ durch Pieris Formel und die Littlewood-Richardson-Regel bestimmt [**13**]. Durch die Verwendung der Jordantriplestruktur auf Z kann die oben beschriebene CW-Zerlegung und die Identifikation von Zellen mit 'Bäumen von Elementen' möglicherweise einen neuen Zugang zu diesen Regeln liefern, und zudem entsprechende Resultate über die gewöhnliche Grassmannsche $\mathrm{Gr}_r(\mathbb{C}^{r+s})$ hinaus auf allgemeinere Grassmannsche $\mathbb{G}(Z)$ für andere phJTS Z verallgemeinern. In diesem Zusammenhang scheint auch E. Nehers Arbeit zum Gitter-Zugang zu Jordantripelsystemen [**36**] Bedeutung zu haben.

APPENDIX C

Danksagung

Ich danke Herrn Prof. Upmeier für die gute Betreuung meiner Arbeit, für die viele Zeit, die er in meine Forschung investiert hat und die stets offene Tür. Ohne Sie wäre diese Promotion nicht so einfach möglich gewesen. Mein besonderer Dank gilt Herrn Prof. Hilgert für die Übernahme der Zweitkorrektur, die vielen wertvollen Anmerkungen zur Arbeit und die Perspektive, die er mir für die Zukunft eröffnet hat. Herrn Prof. Loos danke ich für den interessanten Gedankenaustausch über Teilprobleme meiner Arbeit. Außerdem danke ich Julia Becker-Bender für die Einführung in die Tücken englischer Kommasetzung.

APPENDIX D

Lebenslauf

Diese Seite enthält persönliche Daten und ist deshalb nicht Bestandteil der Online-Veröffentlichung.