Chaotic Polynomial Automorphisms; counterexamples to several conjectures

Arno van den Essen   Engelbert Hubbers

Abstract

We give a polynomial counterexample to a discrete version of the Markus-Yamabe Conjecture and a conjecture of Deng, Meisters and Zampieri, asserting that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map with $\det(JF) \in \mathbb{C}^*$, then for all $\lambda \in \mathbb{R}$ large enough λF is global analytic linearizable. These counterexamples hold in any dimension ≥ 4.

Introduction

In [4] a new approach to the Jacobian Conjecture is introduced. The authors conjecture that if $F : \mathbb{C}^n \to \mathbb{C}^n$ is a polynomial map with $F(0) = 0$ and $JF(0) = I$, then for all $\lambda > 1$, λ large enough there exists an analytic automorphism $\varphi_\lambda : \mathbb{C}^n \to \mathbb{C}^n$ such that $\varphi_\lambda^{-1} \circ \lambda F \circ \varphi_\lambda = \lambda I$ i.e. φ_λ conjugates λF to its linear part. We also say that λF is analytic linearisable to its linear part. We call this conjecture the DMZ-conjecture (after Deng, Meisters and Zampieri). Of course this conjecture, if true, would imply the Jacobian Conjecture since it follows readily that λF and hence F is injective. The local existence of φ_λ is garanteed by the Poincaré-Siegel theorem (cf. [1, section 25, p. 193]) since if $\lambda > 1$ the eigenvalues of λI are non-resonant. Furthermore $\varphi_\lambda(0) = 0$ and φ_λ is unique if we assume that $J\varphi_\lambda(0) = I$, which we can do without loss of generality. It was shown in [4] that φ_λ^{-1} is entire, however the convergence of φ_λ could only be proved in some neighbourhood of 0. Meisters in [8] restricted the problem to polynomial maps of the form $F = X + H$ with H cubic homogeneous and $\det(JF) = 1$ (or equivalently JH nilpotent) and conjectured that for such maps λF can be conjugated to its linear part λI by means of polynomial automorphisms φ_λ, for almost all $\lambda \in \mathbb{C}$, except a finite number of roots of unity. In [5] the first author gave a counterexample to this conjecture for any dimension ≥ 4. On the other hand it was recently shown by Gorni and Zampieri in [7] that this example can be conjugated to its linear part for all λ with $|\lambda| \neq 1$ by means of an analytic automorphism φ_λ! So the DMZ-conjecture remained open.

Another proof of the fact that the counterexample of [5] satisfies the DMZ-conjecture was even more recently given by Deng in [3]. In his very elegant and short paper he proves that an analytic map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F(0) = 0$ can be analytically conjugated to its linear part if and only if F is an analytic automorphism of $\mathbb{C}^n$ and 0 is a global attractor of F (i.e. for every $x \in \mathbb{C}$ the sequence $x, F(x), F^2(x), \ldots$ tends to 0). In the same paper he conjectured that if $F = X + H$ with H cubic homogeneous and JH nilpotent then 0 is a global attractor of $F \circ \lambda$ for all λ with $|\lambda| < 1$. (In fact in the argument he gave to motivate this conjecture he does not use that H is of degree 3.)

A similar kind of question was brought up independently by Cima, Gasull and Mañosas in [2]. They studied the problem that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map with $F(0) = 0$ and such that the eigenvalues of $JF(x)$ are smaller then 1 in absolute value for all $x \in \mathbb{R}^n$, then 0 is a global attractor of F. They call it the discrete Markus-Yamabe Question and show that this problem implies the Jacobian Conjecture and that it is true for triangular maps.

In this paper we give a counterexample to the DMZ-conjecture of the form $F = X + H$, where H is homogeneous of degree 5 in any dimension $n \geq 4$. Furthermore we show that if $0 < \lambda < 1$ λF is a counterexample to the discrete Markus-Yamabe Question.

Acknowledgement

We like to thank Bo Deng for sending us the preprint [3], which formed the starting point of this work. Furthermore, the first author likes to thank the University of Torun and the Jagiellonian University (Krakow) for their warm hospitality during his visit, in which a major part of this paper was prepared.

1 A counterexample to the discrete Markus-Yamabe Question

Let $n \geq 4$ and consider the polynomial ring $\mathbb{R}[X] := \mathbb{R}[X_1, \ldots, X_n]$. In $\mathbb{R}[X]$ define the element

$$d(X) := X_3 X_1 + X_4 X_2$$

Theorem 1.1 *Let $n \geq 4$ and $m \in \mathbb{N}, m \geq 1$. Define the polynomial automorphism*

$$F = (X_1 + X_4 d(X)^2, X_2 - X_3 d(X)^2, X_3 + X_4^m, X_4, \ldots, X_n).$$

Then for each $0 < \lambda < 1$ λF is a counterexample to the discrete Markus-Yamabe Question. More precisely, if $0 < \lambda < 1$ and $a \in \mathbb{R}$ is such that $a\lambda > 1$ then the first component of $(\lambda F)^k(a, a, \ldots, a)$ tends to infinity if k tends to infinity.

Definition 1.2 For each $\lambda > 0$ and $a > 0$ we put $(\lambda F)^k(a) := (\lambda F)^k(a, a, \ldots, a)$ and denote the first component of this vector by $f_k(\lambda, a)$. So

$$f_k(\lambda, a) := ((\lambda F)^k(a))_1,$$

for all $k \geq 1$. Furthermore we put

$$d_k(\lambda, a) := d((\lambda F)^k(a)),$$

for all $k \geq 1$.

Lemma 1.3 *i). $d(\lambda F(X)) = \lambda^2[X_4^{m+1} d(X)^2 + d(X) + X_4^m X_1]$*

ii). $d_{k+1}(\lambda, a) \geq \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2$, for all $k \geq 1$.

iii). $f_{k+1}(\lambda, a) \geq \lambda^{k+1} a (d_k(\lambda, a))^2$, for all $k \geq 1$.

Proof. i) is easy to verify. Consequently, since all monomials in $d(\lambda F(X))$ have positive coefficients, we get

$$\begin{aligned} d_{k+1}(\lambda, a) &= d((\lambda F)(\lambda F)^k(a)) \\ &\geq \lambda^2((\lambda F)^k(a))_4^{m+1} d((\lambda F)^k(a))^2 \\ &= \lambda^2(\lambda^k a)^{m+1}(d_k(\lambda, a))^2 \end{aligned}$$

since the fourth component of $(\lambda F)^k(a)$ equals $\lambda^k a$. This proves ii). Finally

$$\begin{aligned} f_{k+1}(\lambda, a) &= (\lambda F)_1((\lambda F)^k(a)) \\ &\geq \lambda((\lambda F)^k(a))_4 d((\lambda F)^k(a)) \end{aligned}$$

(using that $(\lambda F)_1 = \lambda X_4 d(X)^2 + \lambda X_1$). So $f_{k+1}(\lambda, a) \geq \lambda^{k+1} a (d_k(\lambda, a))^2$, which proves iii). □

Proposition 1.4 *We have:*

$$\begin{aligned} f_k(\lambda, a) &\geq \lambda^{p_k} a^{p_k+(2m+1)(k-1)+4} \\ d_k(\lambda, a) &\geq \lambda^{p_k+m(k-1)+1} a^{p_k+(2m+1)(k-1)+m+4} \end{aligned}$$

for all $k \geq 1$, *where* $p_1 = 1$ *and* $p_{k+1} = 2p_k + (2m+1)(k-1) + 4$ *for all* $k \geq 1$.

Proof. Use induction on k. Details are left to the reader. □

Proof of theorem 1.1. It follows immediately from the estimation of $f_k(\lambda, a)$ in proposition 1.4 that $\lim_{k\to\infty} f_k(\lambda, a) = \infty$ if $\lambda a > 1$. Furthermore one easily verifies that $\lambda F = \lambda X + H$ with JH nilpotent. So for all $x \in \mathbb{R}^n$ the eigenvalues of $JF(x)$ are equal to λ. □

Corollary 1.5 *Let* $m = 5$ *and* $0 < \lambda < 1$. *Put* $\tilde{F} := \lambda F \lambda^{-1}$. *Then* $\tilde{F} = X + H$ *with* H *homogeneous of degree 5 and* JH *is nilpotent. However 0 is not a global attractor of* $\tilde{F} \circ \lambda$ *(*$= \lambda F$*).*

2 A counterexample to the DMZ-conjecture

Let $n \geq 4$ and consider the polynomial ring $\mathbb{C}[X] := \mathbb{C}[X_1, \ldots, X_n]$. In $\mathbb{C}[X]$ define the element $d(X) := X_3 X_1 + X_4 X_2$.

Theorem 2.1 *Let* $n \geq 4$ *and* $m \geq 3$, m *odd. Define the polynomial automorphism*

$$F = (X_1 + X_4 d(X)^2, X_2 - X_3 d(X)^2, X_3 + X_4^m, X_4, \ldots, X_n).$$

Then F *is a counterexample to the* DMZ*-conjecture. More precisely, for every* $\lambda > 0$, $\lambda \neq 1$, λF *is not global analytic linearisable to* λX.

The proof of this theorem is based on the following observation which is due to Bo Deng (cf [3]).

Lemma 2.2 *Let $F : \mathbb{C}^n \to \mathbb{C}^n$ be an analytic map with $F(0) = 0$. Put $A := JF(0)$ and suppose that the eigenvalues of A are smaller than 1 in absolute value. If F is global analytic linearisable to its linear part A then 0 is a global attractor of F.*

Proof. Let $x \in \mathbb{C}^n$ and let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ be the analytic automorphism of $\mathbb{C}^n$ such that $\varphi^{-1}F\varphi = A$. Then $F = \varphi A \varphi^{-1}$ and hence $F^k(x) = \varphi A^k \varphi^{-1}(x)$, for all $k \geq 1$. By the hypothesis on the eigenvalues of A it follows that $A^k\varphi^{-1}(x) \to 0$ if $k \to \infty$. Consequently $F^k(x) = \varphi(A^k\varphi^{-1}(x)) \to 0$ if $k \to \infty$. $\square$

Proof of theorem 2.1. i). From lemma 2.2 and theorem 1.1 it follows that λF is not analytic linearisable if $0 < \lambda < 1$.

ii). Now let $\lambda > 1$. Suppose that λF is analytic linearisable. We derive a contradiction. Then $(\lambda F)^{-1} = F^{-1} \circ \lambda^{-1}$ is also analytic linearisable. Put $\mu := \lambda^{-1}$ and $G := F^{-1}$. So $G \circ \mu$ is analytic linearisable. One easily verifies that

$$G = (X_1 - X_4\tilde{d}(X)^2, X_2 + (X_3 - X_4^m)\tilde{d}(X)^2, X_3 - X_4^m, X_4, \ldots, X_n) \quad (1)$$

where

$$\tilde{d}(X) := d(X) - X_4^m X_1. \quad (2)$$

Since $0 < \mu < 1$ it follows from lemma 2.2 that 0 is a global attractor of $G \circ \mu$. However we will show below (corollary 2.6) that for every $0 < \mu < 1$ 0 is not a global attractor of $G \circ \mu$. Hence we have derived a contradiction.

$\square$

So it remains to show that 0 is not a global attractor of $G \circ \mu$. First we show that 0 is not a global attractor of μG if $0 < \mu < 1$. To prove this we need some lemmas. So let G and $\tilde{d}(X)$ be as in (1) resp. (2).

For each $a > 0$ let $a^* := (a, -a, a, -a, a, \ldots, a) \in \mathbb{R}^n$. Then we define for each $a > 0$ and $\mu > 0$:

$$\begin{aligned} g_k(\mu, a) &:= ((\mu G)^k(a^*))_1 \\ \tilde{d}_k(\mu, a) &:= \tilde{d}((\mu G)^k(a^*)) \end{aligned}$$

for all $k \geq 1$.

Lemma 2.3 *i).* $d(G(X)) = \tilde{d}(X)$.

ii). $\tilde{d}((\mu G)(X)) = \mu^2\tilde{d}(X) - \mu^{m+1}X_4^m X_1 + \mu^{m+1}X_4^{m+1}\tilde{d}(X)^2$.

iii). $\tilde{d}_{k+1}(\mu,a) = (\mu^{k+1}a)^{m+1}(\tilde{d}_k(\mu,a))^2 + \mu^2\tilde{d}_k(\mu,a) + \mu(\mu^{k+1}a)^m g_k(\mu,a)$ *for all* $k \geq 1$.

iv). $g_{k+1}(\mu,a) = \mu^{k+1}a(\tilde{d}_k(\mu,a))^2 + \mu g_k(\mu,a)$ *for all* $k \geq 1$.

Proof. The proofs of i) and ii) are straightforward and left to the reader. From ii) we deduce that

$$\begin{aligned}
\tilde{d}_{k+1}(\mu,a) &= \tilde{d}((\mu G)^{k+1}(a^*)) \\
&= \tilde{d}((\mu G)((\mu G)^k(a^*))) \\
&= \mu^2\tilde{d}((\mu G)^k(a^*)) - \mu^{m+1}(((\mu G)^k(a^*))_4)^m((\mu G)^k(a^*))_1 \\
&\quad + \mu^{m+1}(((\mu G)^k(a^*))_4)^{m+1}\tilde{d}((\mu G)^k(a^*))^2
\end{aligned}$$

Now observe that $((\mu G)^k(a^*))_4 = \mu^k(-a)$, hence since m is odd $((\mu G)^k(a^*))_4^m = -(\mu^k a)^m$. So we get

$$\begin{aligned}
\tilde{d}_{k+1}(\mu,a) &= \mu^2\tilde{d}_k(\mu,a) + \mu^{m+1}(\mu^k a)^m g_k(\mu,a) + \mu^{m+1}(\mu^k a)^{m+1}(\tilde{d}_k(\mu,a))^2 \\
&= (\mu^{k+1}a)^{m+1}(\tilde{d}_k(\mu,a))^2 + \mu(\mu^{k+1}a)^m g_k(\mu,a) + \mu^2 d_k(\mu,a)
\end{aligned}$$

which proves iii). Finally

$$\begin{aligned}
g_{k+1}(\mu,a) &= ((\mu G)^{k+1}(a^*))_1 \\
&= (\mu G)_1((\mu G)^k(a^*)) \\
&= \mu((\mu G)^k(a^*))_1 - \mu((\mu G)^k(a^*))_4(\tilde{d}((\mu G)^k(a^*)))^2 \\
&= \mu g_k(\mu,a) - \mu\cdot\mu^k(-a)(\tilde{d}_k(\mu,a))^2 \\
&= \mu g_k(\mu,a) + \mu^{k+1}a(\tilde{d}_k(\mu,a))^2
\end{aligned}$$

which proves iv). □

Corollary 2.4 *i).* $\tilde{d}_{k+1}(\mu,a) \geq (\mu^{k+1}a)^{m+1}(\tilde{d}_k(\mu,a))^2$ *for all* $k \geq 1$.

ii). $g_{k+1}(\mu,a) \geq \mu^{k+1}a(\tilde{d}_k(\mu,a))^2$ *for all* $k \geq 1$.

Proof. By induction on k one readily verifies that for all $k \geq 1$ both $\tilde{d}_k(\mu,a)$ and $g_k(\mu,a)$ are polynomials in μ and a with coefficients in $\mathbb{N}$. Then the result follows from lemma 2.3 iii) an iv). □

Proposition 2.5 *We have:*

$$\begin{aligned} g_k(\mu, a) &\geq \mu^{q_k(m+1)+k} a^{(q_k+2k)(m+1)+1} \\ \tilde{d}_k(\mu, a) &\geq \mu^{(q_k+k)(m+1)} a^{(q_k+2k+1)(m+1)} \end{aligned}$$

for all $k \geq 1$, *where* $q_1 = 0$ *and* $q_{k+1} = 2q_k + 2k$ *for all* $k \geq 1$.

Proof. Use induction on k. □

Corollary 2.6 *If* $\mu a > 1$ *and* $a > 1$ *then* $\lim_{k\to\infty}((G \circ \mu)^k(G(a^*)))_1 = \infty$. *So 0 is not a global attractor of* $G \circ \mu$.

Proof. Observe that $(G\mu)^k(G(a^*)) = \mu^{-1}(\mu G)^{k+1}(a^*)$. Then apply proposition 2.5. □

References

[1] V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Grundlehren Math. Wiss., vol. 250, Springer-Verlag, New York Heidelberg Berlin, 1983, Chapter 5: Resonances, Poincaré's Theorem, and Siegel's Theorem.

[2] A. Cima, A. Gasull, and F. Mañosas, private communication, 1995.

[3] B. Deng, *Automorphic conjugation, global attractor, and the Jacobian conjecture*, preprint 1995.

[4] B. Deng, G.H. Meisters, and G. Zampieri, *Conjugation for polynomial mappings*, to appear in Z. Angew. Math. Phys. ZAMP.

[5] A.R.P. van den Essen, *A counterexample to a conjecture of Meisters*, In *Automorphisms of Affine Spaces* [6], Proceedings of the conference 'Invertible Polynomial maps', pp. 231–234.

[6] A.R.P. van den Essen (ed.), *Automorphisms of Affine Spaces*, Curaçao, Caribbean Mathematics Foundation, Kluwer Academic Publishers, July 4–8 1995, Proceedings of the conference 'Invertible Polynomial maps'.

[7] G. Gorni and G. Zampieri, *On the existence of global analytic conjugations for polynomial mappings of Yagzhev type*, preprint Univ. of Udine, Italy, July 1995.

[8] G.H. Meisters, *Polyomorphisms conjugate to Dilations*, In van den Essen [6], Proceedings of the conference 'Invertible Polynomial maps', pp. 67–88.

Department of Mathematics
University of Nijmegen
6525 ED Nijmegen
The Netherlands

essen@sci.kun.nl
hubbers@sci.kun.nl

October 31, 1995