



**VSB — TECHNICAL UNIVERSITY OF OSTRAVA**  
**FACULTY OF ECONOMICS**

DEPARTMENT OF FINANCE

Risk Estimation and Backtesting  
Odhad rizika a zpětné testování

Student: Yuling Li  
Supervisor of the diploma thesis: Ing. Aleš Kresta, Ph.D.

Ostrava 2018

## Diploma Thesis Assignment

Student: **Bc. Yuling Li**  
Study Programme: **N6202 Economic Policy and Administration**  
Study Branch: **6202T010 Finance**  
Title: **Risk Estimation and Backtesting**  
**Odhad rizika a zpětné testování**  
The thesis language: **English**

### Description:

1. Introduction
  2. Description of Selected Risk Measures
  3. Description of Backtesting
  4. Estimation and Backtesting of Selected Risk Measures
  5. Conclusion
- Bibliography  
List of Abbreviations  
Declaration of Utilisation of Results from the Diploma Thesis  
List of Annexes  
Annexes

### References:

- DANIÉLSSON, Jón. *Financial risk forecasting: the theory and practice of forecasting market risk, with implementation in R and Matlab*. Chichester: Wiley, 2011. ISBN 978-0-470-66943-3.  
KRESTA, Aleš. *Financial Engineering in Matlab: Selected Approaches and Algorithms*. Ostrava: VŠB-TU Ostrava, 2015. ISBN 978-80-248-3702-4.  
ZMEŠKAL, Z., D. DLUHOŠOVÁ and T. TICHÝ. *Financial Models*. Ostrava: VŠB-TU Ostrava, 2004. ISBN 80-248-0754-8.

Extent and terms of a thesis are specified in directions for its elaboration that are opened to the public on the web sites of the faculty.

Supervisor: **doc. Ing. Aleš Kresta, Ph.D.**

Date of issue: 24.11.2017

Date of submission: 27.04.2018



Ing. Iveta Ratmanová, Ph.D.  
Head of Department



prof. Dr. Ing. Zdeněk Zmeškal  
Dean

The declaration

“Herewith I declare that I elaborated the entire thesis, including all annexes, independently.”

Ostrava dated 24. 04. 2018

Yuling Li 李钰灵

Student's name and surname

# Contents

<b>1</b>	<b>Introduction .....</b>	<b>5</b>
<b>2</b>	<b>Description of Risk Measures .....</b>	<b>7</b>
2.1	History of VaR .....	7
2.2	Market Risk .....	8
2.3	Principles of Risk Measures .....	8
2.4	Value at Risk .....	9
2.5	Conditional Value at Risk .....	11
2.6	Method of Risk Estimation .....	12
2.6.1	Historical Simulation.....	12
2.6.2	Filtered Historical Simulation .....	14
2.6.3	Analytical Solution.....	17
2.6.4	Monte Carlo Simulation .....	22
2.7	Comparing the Four Methods .....	24
<b>3</b>	<b>Backtesting Procedure.....</b>	<b>26</b>
3.1	Kupiec's Unconditional Coverage Test .....	30
3.2	Christoffersen's Conditional Coverage Test .....	33
3.3	The Joint Test of Coverage and Independence .....	35
<b>4</b>	<b>Empirical Results of Statistical Testing .....</b>	<b>36</b>
4.1	Historical Simulation .....	37
4.1.1	Simplified Example.....	37
4.1.2	Empirical Study.....	40
4.2	Filtered Historical Simulation.....	43
4.2.1	Simplified Example.....	43
4.2.2	Empirical Study.....	47
4.3	Normal Distribution.....	50
4.3.1	Simplified Example.....	50

4.3.2	Empirical Study.....	53
4.4	Student's t-distribution.....	56
4.4.1	Simplified Example.....	56
4.4.2	Empirical Study.....	59
<b>5</b>	<b>Conclusion.....</b>	<b>63</b>
	<b>Bibliography .....</b>	<b>65</b>
	<b>List of abbreviations.....</b>	<b>68</b>
	<b>Declaration of Utilisation of Results from the Diploma Thesis</b>	
	<b>List of Annexes</b>	

# 1 Introduction

“All of life is the management of risk, not its elimination.”

– Walter Wriston, former chairman of Citicorp

Financial markets are always changing, and the change can be positive or negative. The prospect of gain or loss is a consequence of change, which therefore leads to risk, and risk (or more precisely, downside risk) is something that is sometimes inevitable in financial markets. The acceptance of risk does not invariably imply the elimination of risk, which clearly is impossible. Nor does it imply that one should not mitigate risks and accept consequent losses fatalistically. It means that risk requires management. Decide on what risks to avoid and how to avoid them. What risks should be accepted, the terms of accepting them and what new risk to take on. (Dowd, 1999) Therefore, risk management should establish a risk quantification method firstly, and secondly, to develop and implement a validation by means of backtesting technique for the Value of Risk (henceforth VaR).

In the thesis, the objective is to verify different VaR estimation approaches by means of backtesting on chosen time series. However, in order to provide an exhaustive description about the backtesting process in the empirical part, we first discuss VaR estimation approaches in general and the theory of backtesting method.

The thesis consists of five chapters. The first one is the introduction, which states the objective of the thesis and the structure of the thesis.

The second chapter describes the basic idea behind VaR and gives some background and history on the subject. Then, it mainly describes the methods of VaR estimation, which consists of description of their main principals, formulations, advantages and disadvantages.

The third chapter concentrates on the description of backtesting procedure. Several statistical tests are presented in detail.

The fourth chapter describes the empirical study of the thesis, which can be considered to be the core of this study. Some of the tests presented in the preceding chapter are applied to actual VaR calculations. In order to show the calculation steps of VaR estimation and backtesting

procedures clearly, we present simplified examples in the beginning of each empirical study. Then, the empirical results are discussed in detail.

The fifth chapter concludes and reviews the most significant results of both theoretical and empirical parts.



## 2 Description of Risk Measures

*“The stock market will fluctuate”*

J. P. Morgan, when asked what the market was going to do.

Financial theory defines *risk* as the dispersion of unexpected outcomes owing to movements in financial variables. Therefore, both positive and negative deviations should be viewed as sources of risk. To measure risk, one has to define first the variables, which could be portfolio value, earnings, capital, or particular cash flows. Since risk needs to be quantified, this chapter lays the approaches of market risk measurement, see Jorion (2006). First, we introduce the background of value at risk, market risk and the principles of risk measures. Then, the value at risk and conditional value at risk are defined. Next, we introduce three methods of value at risk estimation, which is most important in this chapter. Finally, we compare these three methods, and state their advantages, disadvantages, and the situations when they are proper for application.

### 2.1 History of VaR

The problem of risk measurement is an old one in statistics, economics and finance. Financial risk management has been a concern of regulators and financial executives for a long time as well. Retrospective analysis has found some VaR-like concepts in this history. But VaR did not emerge as a distinct concept until the late 1980s. The triggering event was the stock market crash of 1987. This was the first major financial crisis in which a lot of academically-trained quants were in high enough positions to worry about firm-wide survival, see Jorion (2006).

After the collapse of Herstatt Bank, the central bank governors of G10 countries established the Basel Committee. In order to strengthen the stability of the international banking system to withstand shocks, the Basel Committee announced the Basel I Accord in 1988, published a set of minimum capital requirements for banks. Duo to the introduction of Basel I Accord, banks became broad-based providers of financial services and the trading activities in bonds, equities, foreign exchange and commodities began playing a more significant role. However, the risk arising from these trading activities were not only credit risk, but market risk, which was a new risk category separated from credit risk. So, the Basel Committee in 1996 issued the Market Risk Amendment,

that focused on the effect of a bank's positions in various market-traded financial assets—foreign exchange, debt securities, equities, commodities, and derivatives, see Apostolik (2009). In particular, the VaR methods invented by J.P. Morgan, that is quantitative analysis of market risk were allowed to apply. In 1997, the U.S. Securities and Exchange Commission ruled that public corporations must disclose quantitative information about their derivatives activity. Major banks and dealers chose to implement the rule by including VaR information in the notes to their financial statements. Worldwide adoption of the Basel II Accord, beginning in 1999 and nearing completion today, gave further impetus to the use of VaR. VaR is the preferred measure of market risk, and concepts similar to VaR are used in other parts of the accord, see Jorion (2006).

## **2.2 Market Risk**

According to Apostolik (2009), market risk is exposure to the uncertain market value of an asset or a portfolio. Usually, market risk includes two components: the general market risk and specific risk. The former affects financial market and the latter only affects individual financial assets. According to the different objectives of the risk occurrence, the market risk is distinguished: foreign exchange risk, interest rate risk, equity risk and commodity risk. Foreign exchange risk is the potential loss due to an adverse change in foreign exchange rates and applies to all exchange rate-related products whose positions are valued in a currency that differs from the bank's reporting currency. Interest rate risk is the potential loss due to adverse changes in interest rates. Equity risk is the potential loss due to an adverse change in the price of stocks and applies to all instruments that use equity prices as part of their valuation—for example, derivative products such as futures contracts. Commodity risk is the potential loss from an adverse change in commodity prices. This applies to all commodity positions and any derivative commodity positions such as futures contracts.

## **2.3 Principles of Risk Measures**

Generally speaking, the development of risk measurement goes through three stages: firstly, the traditional risk measurement stage with variance and risk factors as the main indicators. In the

thesis, the risk factor is regarded as price of selected assets. Secondly, the modern risk measurement stage represented by the VaR; and finally risk measurement stage represented by Conditional VaR (henceforth CVaR). In the thesis, we focus on VaR and CVaR method.

Artzner et al. (1997) explain the principle of risk measurement, they regarded the risk quantity as a measure of capital needs, that is, in a certain period of time, at least the amount of capital that an enterprise should reserve to meet the security requirements. In this thesis, the VaR of single financial asset is measured. So, for a single asset  $X$ , quantity of risk measure  $\rho(X)$  should satisfy following principle:

1. Subadditivity:  $\rho(X + Y) \leq \rho(X) + \rho(Y)$ . This formula shows that the overall risk of the portfolio does not exceed the sum of all individual risks;
2. Monotonicity: if  $X \leq Y, \rho(X) \geq \rho(Y)$ . If the value of asset  $X$  is less or equal to the value of asste  $Y$ , the risk of  $X$  should be greater or equal to the risk of  $Y$ . In other words, the risk of good assets should be less than the risk of inferior assets.
3. Positive homogeneity:  $\rho(\alpha \cdot X) = \alpha \cdot \rho(X)$ . This formula shows that for financial risk measurement, it should not be affected by the unit of measurement.
4. Translation invariance:  $\rho(X + \alpha) = \rho(X) - \alpha$ . This equation implies that if the amount of cash  $\alpha$  is added into the asset  $X$ , it offset corresponding risk of  $X$ .

## 2.4 Value at Risk

VaR is defined as the predicted loss at a specific confidence level over a given period of time, see Glyn, A. (2009). That is to say, for a given period and probability, the VaR measure purports to indicate an amount of money such that there is that probability of the asset not losing more than that amount of money over that period. To make it clear, we need to determine the following three factors: the length of the holding period, the size of the confidence interval and the period of the observation.

First, the holding period is to determine how long the maximum loss of assets need to be measured. That means whether the managers are concerned about the value at risk of the assets within one day or one week or one month. The investors choose the length of holding period by

means of the characteristics of the assets, such as liquidity. For example, for high liquid assets, risk, returns VaR are often calculated based on daily basis, such as OTC derivatives or stocks. While, for longer maturities assets, such as pension funds, the holding period can be month or year.

Second, the confidence level is the frequency of possible confidence intervals that contain the true value of their corresponding parameter, see Cox D.R. (1974). For example, if the support rate for someone in a general election is 55%, and a 0.95 confidence interval is 50% to 60%. Then his true support rate falls on interval between 50% and 60% with a probability of 95%. So, his support rate is less than 50% with a probability of 2.5% (assuming the distribution is symmetrical). In this case, we call the 0.95 as “confidence level”, which reflects a significance level of 0.05. Usually, we defined the significance level as  $\alpha$ , due to the frequent usage in calculation of VaR and its backtesting. Applied in single financial asset, if an asset has a one-day 5% VaR of \$1 million, that means there is a 0.05 probability that the asset will fall in value by more than \$1 million over a one-day period if there is no trading. Informally, a loss of \$1 million or more on this asset is expected on 1 day out of 20 days (because of 5% probability). In this thesis, a loss which exceeds the VaR is called exception.

The third is the observation period. The observation period is the overall length of time for the observations, sometimes referred as the historical window. For example, we can choose observation period over the previous 6 months, or 1 year to consider the weekly returns volatility of an asset. In order to avoid the influence of business cycle, the longer the historical data, the better. However, the longer the period, the greater the possibility of structural changes in the market. As the result, the historical data is harder to reflect future real performance.

According to the definition of VaR, it can be expressed as follows,

$$VaR_{\alpha}(X) = \inf\{x \in R: F_X(x) \geq \alpha\}, \quad (2.1)$$

where  $\alpha$  is significance level,  $X$  is the underlying (e.g. profit of asset), the  $F_X$  is cumulative distribution function. Therefore, under the specified probability level  $\alpha$ , if the profit we regard as a negative loss, the probability with which the observed loss exceed estimated loss can be expressed as follows,

$$Pr(X \leq -VaR_\alpha(X)) = \alpha. \quad (2.2)$$

From the formula (2.1), we can see the advantages of VaR model. VaR model is simple and easy to comprehend for market risk measurement, because the risk can be quantified and expressed as a particular number. In addition, it can be calculated in advance to reduce market risk. That means the managers will know the market risk of assets before investment. However, VaR methodology still have several defects.

First, VaR model have no subadditivity. Follow the principle that diversification of investment can reduce the risk, risk of portfolio should lower than the sum of risk of particular assets. But in the VaR methodology, the VaR of portfolio may exceed the sum of VaR of its asset, which is contrary to the principle.

Second, VaR methodology is hard to measure the tail risk, which means it cannot measure the loss under black swan event. There is an assumption of VaR model that the risk of assets obeys normal distribution. However, in reality, the loss of the assets does not obey normal distribution based on history data. The VaR model just measure the maximum loss under 99% or 95% probability, but cannot answer the maximum loss if the black swan event happens with 1% or 5% probability. So, the VaR model only focus conventional risk with close to center distribution, but ignore tail risk. For example, there is a bet about the coin tossing. The coin will be throwed seven times. If the coin faces up under 7 times, the player wins a bet. But if the coin faces up 7 times, the player losses 127 times bet. In terms of probability, the bet is safe, because there is only 0.78% probability that the coin faces up 7 times. That is to say, the player will win the bet with 99% probability. However, there still exist probability of loss with probability of 1%. Once the player loses, the loss is 127 times as much as the player earns.

## 2.5 Conditional Value at Risk

In order to overcome the VaR defects, Rockafeller and Uryasev (2000, 2002) proposed the Conditional Value at Risk (henceforth CVaR). It is also called expected shortfall. CVaR is an alternative to value at risk that is more sensitive to the shape of the tail of the loss distribution. Therefore, the “CVaR at  $\alpha\%$  level” means the expected return on the assets in the worst  $\alpha\%$  of

cases. The  $\alpha$  represents significance level and  $1 - \alpha$  represents confidence level. For high values of  $1 - \alpha$ , it ignores the most profitable but unlikely possibilities, while for small values of  $\alpha$ , it focuses on the worst losses. On the other hand, unlike the discounted maximum loss, even for lower values of  $\alpha$ , the CVaR does not consider only the single most catastrophic outcome, see Rockafellar, Uryasev (2000). CVaR is measure of financial assets risk. It requires a quantile-level  $\alpha\%$ , and is defined to be the expected loss of assets value given that a loss is occurring at or below the quantile.

According to the definition of CVaR, we can know properties of CVaR, and its difference between VaR. First, the  $CVaR_\alpha$  increases as  $\alpha$  decreases. Second, the 100%-quantile CVaR equals the expected value of the assets. Third, for a given asset, the CVaR is greater than or equal to the VaR at the same  $\alpha$  level.

If the underlying distribution for  $X$  is a continuous distribution, then the CVaR at probability of  $\alpha$  level is equivalent to the tail conditional expectation defined by:

$$CVaR_\alpha(X) = -E[X|X < -VaR_\alpha(X)], \quad (2.3)$$

## 2.6 Method of Risk Estimation

*“One of the most difficult aspects of calculating VaR is selecting among the many types of VaR methodologies and their associated assumptions.”* (Minnich, 1998)

In this section, we introduce four methods for risk estimation. There are historical simulation, filtered historical simulation, analytical solution and Monte Carlo simulation. The historical simulation and filtered historical simulation are based on the history data without parameters. So, both are non-parametric approach. The analytical solution is a parameters method, and assumes that assets returns follow a chosen probability distribution, such as chi-squared distribution, Student's t distribution, normal distribution and so on. Monte Carlo simulation is a process that random numbers produced by computer to simulate the value of assets returns, which is more accurate but more complex. Then, we introduce the four methods in detail, respectively.

### 2.6.1 Historical Simulation

Historical simulation method is a simple, non-parametric approach. If there are numbers of

historical data of assets missing, we cannot get a complete data easily. We can collect the change of its risk factors over the past period based on historical data, and simulate the future return or loss distribution of assets. Then, we can calculate the VaR under a specified significance level. Obviously, historical simulation method does not need to assume that the risk factor must be in line with the specific distribution. The past change and distribution of risk factor directly represents the future distribution. So, historical simulation method does not need to estimate the parameter, which is non-parameters method.

The procedure of historical simulation is as follows: First, we should find the risk factors which effects the return of asset. We utilize the following equation to express the relationship between value of asset and risk factors:

$$V_t = V(f_1(t), f_2(t), f_3(t), \dots, f_n(t)), \quad (2.4)$$

where,  $V_t$  represents the value of asset,  $f_i(t)$  means risk factors,  $i = 1, 2, 3, \dots, n$ ;  $t = \pm 1, 2, 3, \dots, T$ ,  $t < 0$  means past,  $t = 0$  means present,  $t > 0$  means future. Usually, the risk factors can be returns of assets. We select past return of assets for  $T + 1$  trading days as historical window and calculate the change rate of return of adjacent two days.

$$\Delta f_i(-t) = f_i(-t + 1) - f_i(-t). \quad (2.5)$$

Second, assuming the future change of returns will be same as past changes, the future value of returns can be as follows:

$$f_i(t) = f_i(0) + \Delta f_i(-t), \quad (2.6)$$

then, we can get the change of value of asset in the future,

$$\Delta V_t = V_t - V_0. \quad (2.7)$$

Third, the future returns are ranked in increasing order, and calculate VaR corresponding to the quantile as specified significance level. The  $\Delta V_t$  are ranked form smallest to biggest for  $t = 1, 2, 3, \dots, T$  and regard as  $k_1, k_2, k_3, \dots, k_T$ . The equation shows as follows:

$$\Delta V(k_1) \geq \Delta V(k_2) \geq \dots \geq \Delta V(k_{t-1}) \geq \Delta V(k_t) \geq \dots \geq \Delta V(k_T). \quad (2.8)$$

The significance level is  $\alpha$  and the  $[T_\alpha]$  means integral part of quantile  $T_\alpha$ . Then, we can calculate the VaR as follows:

$$VaR_{R,\alpha} = \Delta V(k_{[T_\alpha]+1}). \quad (2.9)$$

According to the principal of historical simulation method, we describe advantages and disadvantages as follows. First, historical simulation method is simple and easy to calculate. So, it is acceptable for managers. Second, we do not need the assumption or simulation about parameters of returns. It can more accurately reflect the probability distribution of risk factors. For example, the thick tail and skewness of general asset returns may be expressed through historical simulation. Third, due to the non-parameters characteristic, the problem of estimation errors is avoided. Moreover, the historical data reflected the volatility and correlation of return on asset. Therefore, compared with other methods, historical simulation method is less affected by model risk.

However, this method still exists disadvantages, which reflect as follows: First, the historical simulation method needs too much continuous historical data for getting the future probability distribution of returns. But, it is hard to collect the large and complete historical data of assets. Due to lack of historical data, the loss of extreme events will be not easy to simulate. However, it is not that the more the history data, the better. The too old historical data cannot predict accurate present value. Both of two extreme situations can lead to a low degree of precision in the value of risk. Second, there is an assumption of historical simulation, that the future change of risk factors will be the same as past performance. This assumption is clearly contrary to realism. So that, we introduce the filtered historical simulation, which overcomes this disadvantage.

### **2.6.2 Filtered Historical Simulation**

As we mentioned before, we need to know the probability distribution of each time series to estimate the VaR. But we know that the actual distribution of the returns cannot conform to any known distribution. The filtered historical simulation (henceforth FHS) methodology was proposed and developed by Giovanni Barone-Adesi and Kostas Giannopoulos (1998). They allowed for the volatility of asset return to differ from the volatility of options, which conforms to reality more. Therefore, we need to introduce the GARCH model at first.

GARCH model (Generalized autoregressive conditional heteroskedasticity) is an extension of the ARCH model (Autoregressive conditional heteroskedasticity model) developed by Bollerslev



(1986). ARCH Model was proposed by Engle (1982), which solved the problem caused by traditional econometrics assumption of homoscedasticity of time series variables. This model is one of the econometric achievement and won the 2003 Nobel Economics Prize.

The ARCH model is appropriate when the error variance in a time series follows an autoregressive (AR) model; if an autoregressive moving average model (ARMA) model is assumed for the error variance, the model is a GARCH model. For time series model, let  $\varepsilon_t$  represents the estimated residuals. So, the GARCH model are listed below:

$$\varepsilon_t = \sigma_t q_t, \quad (2.10)$$

$$q_t \sim N(0,1) \text{ or } q_t \sim t_v(0,1), \quad (2.11)$$

where,  $q_t$  is random number of normal distribution or t-Student distribution, which is a strong white noise process (independent and identically distributed, henceforth i.i.d.). and  $\sigma_t$  is standard deviation, which should satisfy following form:

$$\sigma_t^2 = a_0 + \sum_{i=1}^p a_i \cdot \sigma_{t-i}^2 + \sum_{i=1}^q \beta_i \cdot \varepsilon_{t-i}^2, \quad (2.12)$$

where  $a_0$ ,  $a_i$  and  $\beta_i$  are parameters needed to be estimated,  $a_0$  is a constant in variance equation,  $p$  is the lag parameter of the observed variable,  $q$  is the number of lags of the error term. The parameters of this model are limited for ensuring the non-negativity and stationarity of the variance process. The conditions are  $a_i > 0$ ,  $\beta_i > 0$ , and  $\sum_{i=1}^p a_i + \sum_{i=1}^q \beta_i < 1$ . Within a class of autoregressive processes with white noises having conditional heteroscedastic variances, we try to find reasonable models of  $\{y_t\}$ . The  $\{y_t\}$  is called an autoregressive process of order  $k$  with an GARCH noise of order  $p$ ,  $q$ , in short AR (k)-GARCH (p, q) process, which is expressed as follows:

$$y_t = \mu_0 + \sum_{i=1}^k \mu_i \cdot y_{t-i} + \varepsilon_t, \quad (2.13)$$

where,  $\mu_0$  is the mean constant and  $\mu_i$  is parameter,  $\varepsilon_t$  is the error term. Based on the GARCH model, the short AR (1)-GARCH (1, 1) model for one-period forecasting, is expressed as follows:

$$y_t = \mu_0 + \mu_1 y_{t-1} + \varepsilon_t, \quad (2.14)$$

$$\sigma_t^2 = a_0 + \alpha_1 \sigma_{t-1}^2 + \beta \varepsilon_{t-1}^2. \quad (2.15)$$

The residuals should close to a stationary i.i.d. distribution, so that they are suitable for historical simulation. In the thesis, the daily returns  $R_t$  of an asset are regarded as  $\varepsilon_t$ . We divide the residual  $R_t$  by the corresponding daily standard deviation (volatility) estimate  $\sigma_t$ , so the standardized residual returns  $e_t$  can be defined as,

$$e_t = \frac{R_t}{\sigma_t}. \quad (2.16)$$

These random standardized residuals are scaled by the deterministic volatility forecast one day ahead. So, we can get the one day ahead random return  $\tilde{y}_{t+1}$  as follows:

$$\tilde{y}_{t+1} = \tilde{\mu}_0 + \tilde{\mu}_1 y_t + \tilde{\varepsilon}_{t+1}. \quad (2.17)$$

According to formula (2.15), the volatility depends on the return simulated on the first day. For obtaining random residuals  $\tilde{\varepsilon}_{t+1}$ , we utilize standardized residual  $e_t$  to multiply estimation of volatility one day ahead  $\sigma_{t+1}$ :

$$\sigma_{t+1} = \sqrt{a_0 + a_1 \sigma_t^2 + \beta \varepsilon_t^2}. \quad (2.18)$$

Here, we suppose that the mean value of return is zero, and there is no autocorrelation of returns. Then, we can estimate VaR one day ahead with  $\alpha$  probability level as follows:

$$VaR_{t+1} = \sigma_{t+1} \cdot q_\alpha. \quad (2.19)$$

To summarize, the process of FHS method for one day ahead VaR estimation can be following steps:

1. Selecting a GARCH model, such as AR (1)-GARCH (1,1), and using formula (2.10) and (2.12) to obtain parameters terms, estimates of the residuals and the variance.
2. Using formula (2.16) to obtain  $e_t$ .
3. Calculating estimation of volatility one day ahead  $\sigma_{t+1}$  based on formula (2.18).
4. Determining quantile with probability level  $\alpha$ , and estimating one day ahead VaR, according formula (2.19).

From the statistical perspective, the FHS is a semi-parametric model. The price series is not forced to conform to any kind of probability distribution. Thus, it allows for fat tails, volatility clusters and changing means. This provides a more accurate estimation in the tails of possible price

realizations.

### 2.6.3 Analytical Solution

As we mentioned before, historical simulation method and FHS methods are belong to non-parametric or semi-parametric method. Another method for risk estimation, analytical solution is a parametric method. The analytical solution to calculate the Value at Risk calculates the mean, or expected value, and standard deviation of an asset.

The analytical solution computes an asset's maximum loss based on the price movements of asset over a past period within a probability level. It is based on assumption that the changes in market parameter and asset returns follow a probability distribution.

For usage of analytical solution, we need to know the concept of variance. The variance is the square of the standard deviation, and the standard deviation measure that is utilized to quantify the amount of variation or dispersion of a set of data values. A low standard deviation indicates that the data points tend to be close to the expected value (also called the mean) of the asset, while a high standard deviation indicates that the data points are spread out over a wider range of values. We can say that standard deviation is a measure of the volatility of the investment as well. In this thesis, the return of assets  $R_i$  can be defined by:

$$R_i = \frac{P_{i,t}}{P_{i,t-1}} - 1, \quad (2.20)$$

where  $P_{i,t}$  is asset price at time  $t$ ,  $P_{i,t-1}$  is asset price at  $t - 1$ .

The expected return  $E(R_i)$  can be written as,

$$E(R_i) = \frac{1}{N} \cdot \sum_{t=1}^N R_{i,t}, \quad (2.21)$$

where for single asset,  $N$  is the observation period of the asset. The variance  $\sigma_i^2$  and standard deviation  $\sigma_i$  of returns can be expressed as,

$$\sigma_i^2 = \frac{1}{N} \cdot \sum_{t=1}^N [R_{i,t} - E(R_i)]^2, \quad (2.22)$$

$$\sigma_i = \sqrt{\frac{1}{N} \cdot \sum_{t=1}^N [R_{i,t} - E(R_i)]^2} = \sqrt{\sigma_i^2}. \quad (2.23)$$

After the parameters introduced, the probability distribution of asset returns should be distinguished. In this thesis, the normal distribution and Student's  $t$ -distribution are described as follows.

In probability theory, the normal distribution, also called Gaussian distribution, is a very common continuous probability distribution. A random variable with a Gaussian distribution is said to be normally distributed and is called a normal deviate. The normal distribution is useful because of the central limit theorem. In its most general form, under some conditions (which include finite variance), it states that averages of samples of observations of random variables independently drawn from independent distributions converge in distribution to the normal, that is, become normally distributed when the number of observations is sufficiently large. Physical quantities that are expected to be the sum of many independent processes (such as measurement errors) often have distributions that are nearly normal, see Lyon, A. (2014).

The probability density of the normal distribution is defined as follows,

$$f(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad (2.24)$$

where  $\mu$  the mean or expectation of the distribution,  $\sigma$  is the standard deviation and  $\sigma^2$  is the variance. The simplest case of a normal distribution is known as the standard normal distribution. This is a special case when  $\mu = 0$ ,  $\sigma = 1$ , and it is described by this probability density function:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}, \quad (2.25)$$

where the factor  $\frac{1}{\sqrt{2\pi}}$  in this expression ensures that the total area under the curve  $\varphi(x)$  is equal to one.

In figure 2.1 and 2.2, we can see the probability density function and cumulative distribution function of normal distribution. The blue curve represents the standard normal distribution, which the mean value is zero and standard deviation is one.

Figure 2.1 Probability density function of normal distribution with different means and standard deviations.

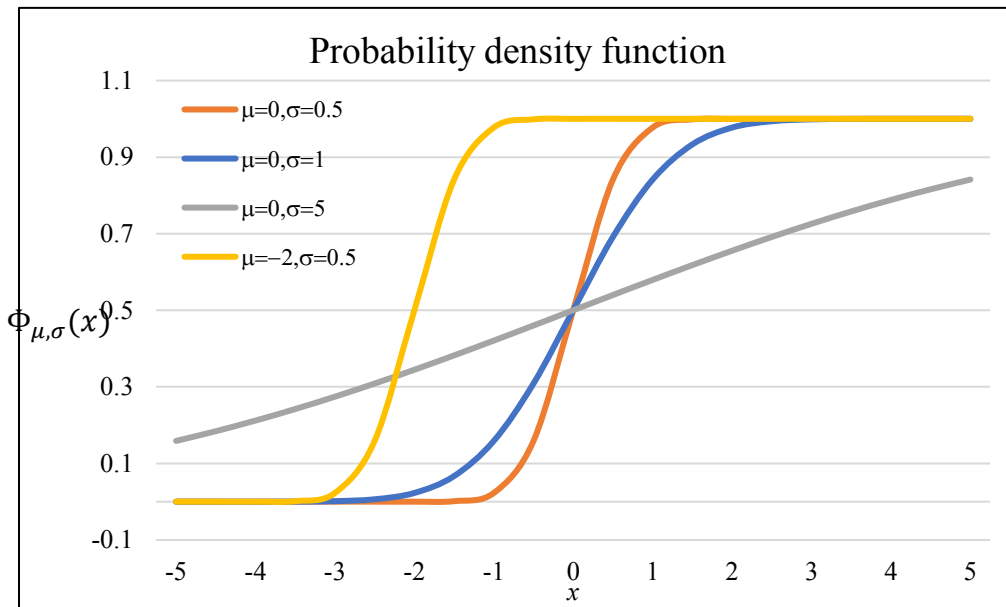
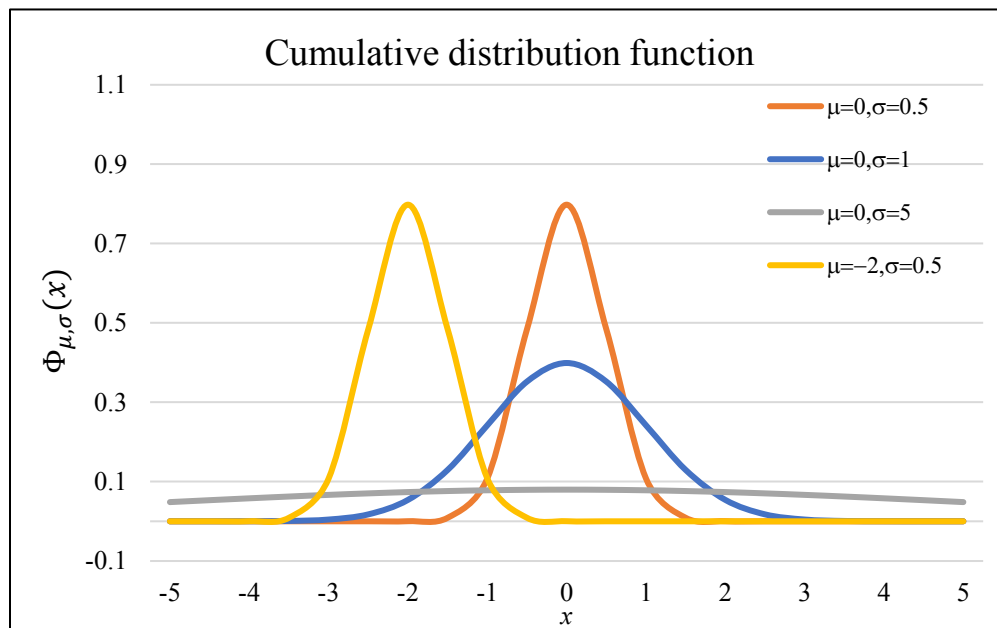


Figure 2.2 Cumulative distribution function of normal distribution with different means and standard deviations.



In probability and statistics, Student's *t*-distribution is any member of a family of continuous probability distributions that arises when estimating the mean of a normally distributed population in situations where the sample size is small and population standard deviation is unknown. It was

developed by William Sealy Gosset (1908). If we take a sample of  $n$  observations from a normal distribution, then the  $t$ -distribution with  $\nu = n - 1$  degrees of freedom can be defined as the distribution of the location of the sample mean relative to the true mean, divided by the sample standard deviation, after multiplying by the standardizing term  $\sqrt{n}$ . In this way, the  $t$ -distribution can be utilized to say how confident you are that any given range would contain the true mean.

The  $t$ -distribution has heavier tails, meaning that it is more prone to producing values that fall far from its mean. This makes it useful for understanding the statistical behavior of certain types of ratios of random quantities, in which variation in the denominator is amplified and may produce outlying values when the denominator of the ratio falls close to zero, see Richardson (1994).

The probability density function is symmetric, and its shape is similar to the bell shape of a standard normally distributed, except that it is a bit lower and wider. As the number of degrees of freedom grows, the  $t$ -distribution approaches the normal distribution with mean of 0 and variance of 1. The figure 2.3 and 2.4 shows the density of the  $t$ -distribution for increasing values of  $\nu$ . Therefore, when the degree of freedom is close to infinity, the Student  $t$ -distribution curve is close to the normal distribution. The blue curves show the situation in figure 2.3 and 2.4.

Figure 2.3 Probability density function of Student's  $t$ -distribution with different degrees of freedom.

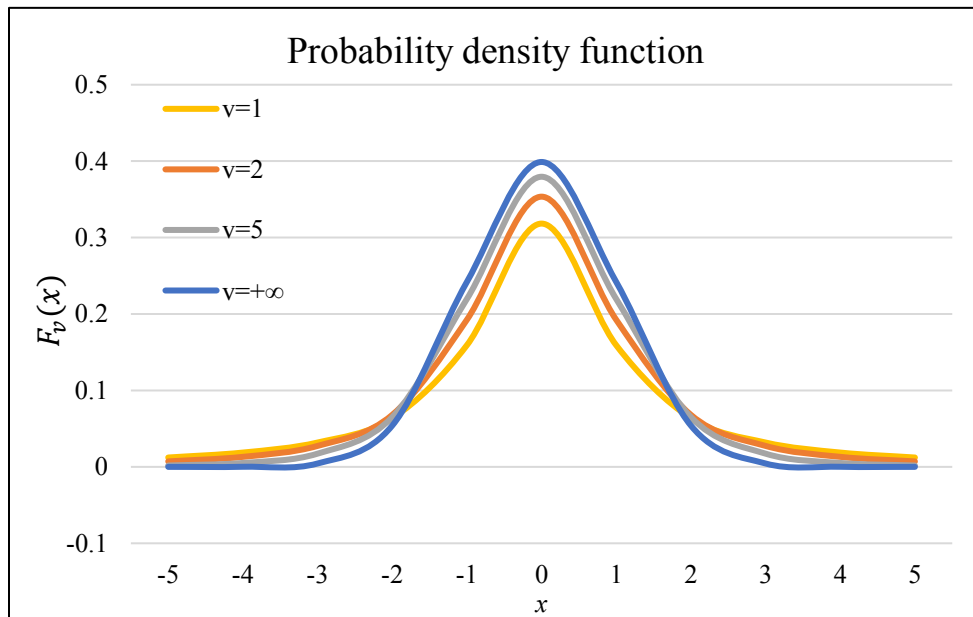
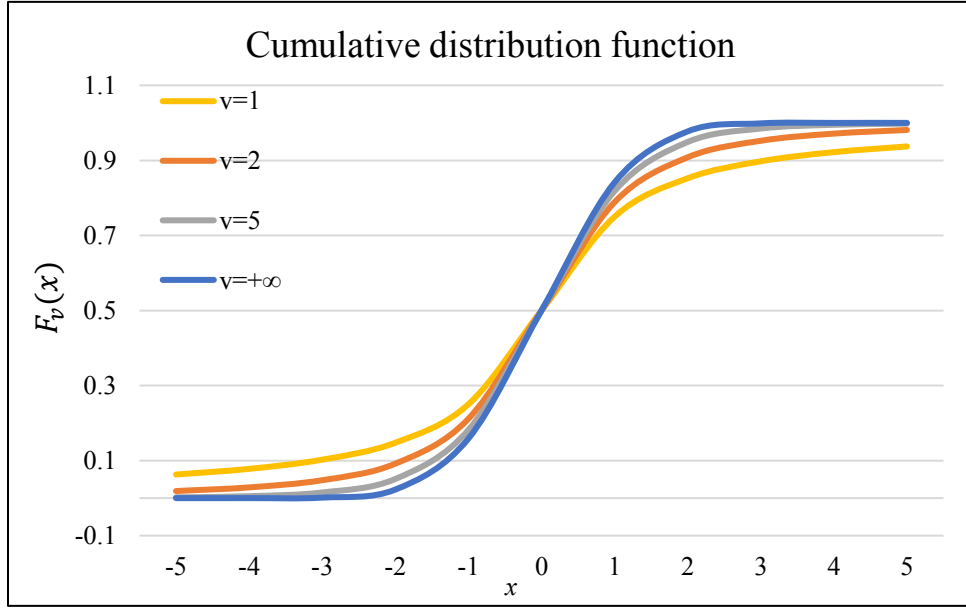


Figure 2.4 Cumulative distribution function of Student's  $t$ -distribution with different degrees of freedom.



Now, we utilize analytical solution to calculate the VaR. Here we assume that the increment in value of an asset can be formulated by returns as

$$X = V_t \cdot R_i, \quad (2.26)$$

where  $R_i$  is asset return,  $V_t$  is the initial asset value. The returns of asset are normal distribution. According to the formula (2.2), we know that

$$Pr(X \leq -VaR_\alpha(X)) = \alpha.$$

Then, let  $g = X + VaR$  and rearrange, we get,

$$Pr(g \leq 0) = \alpha. \quad (2.27)$$

After normalization, we apply expected value  $E(g)$  and standard deviation  $\sigma_g$ :

$$Pr\left(\frac{g - E(g)}{\sigma_g} \leq \frac{0 - E(g)}{\sigma_g}\right) = \alpha, \quad (2.28)$$

where, according to our assumption, the  $\frac{g - E(g)}{\sigma_g}$  represents asset return, which is standard normal distribution. Thus, the cumulative density function  $\Phi_\alpha$  can be defined as follows,

$$\Phi_\alpha(x) = \Phi(x; 0, 1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt. \quad (2.29)$$

By means of cumulative density function of standard normal distribution, we can write its inverse function  $\Phi_s^{-1}$ ,

$$\Phi_\alpha\left(\frac{0 - E(g)}{\sigma_g}\right) = \alpha, \quad (2.30)$$

$$\frac{0 - E(g)}{\sigma_g} = \Phi_\alpha^{-1}(\alpha). \quad (2.31)$$

Then, putting  $g$  back and substitute for  $X + VaR$ , we obtain,

$$\frac{0 - E(X + VaR)}{\sigma_{X+VaR}} = \Phi_\alpha^{-1}(\alpha), \quad (2.32)$$

$$VaR = -\Phi_\alpha^{-1}(\alpha) \cdot \sigma(x) - E(x). \quad (2.33)$$

Usually, we suppose that expected return is zero, especially for short-term returns. The  $-\Phi_s^{-1}(\alpha)$  can be written as  $\Phi_s^{-1}(1 - \alpha)$ . The standard deviation can be calculated by formula (2.23). So, we can simply as follows:

$$VaR = \Phi_\alpha^{-1}(1 - \alpha) \cdot \sigma(x). \quad (2.34)$$

#### 2.6.4 Monte Carlo Simulation

Monte Carlo simulation is another method of risk measures. The basis of the Monte Carlo simulation method of calculating VaR that generates random movements in the risk factors by using past events to model future return distribution of an asset.

Financial assets are typical by their random evolution in time. The random evolution of variable in time is referred to as the stochastic process. In this respect, crucial terms are the Geometric Brownian motion (also known as Wiener process). In particular, Geometric Brownian motion is utilized in mathematical finance to model stock prices in the Black–Scholes model. The Wiener process or Brownian motion is regarded as  $It\hat{o}$  process, which can be defined as follows:

$$dS = \alpha(S, t) \cdot dt + b(S, t) \cdot d\omega, \quad (2.35)$$

where  $S_t$  means a variable for  $t \geq 0$ ,  $\alpha(\cdot)$  is drift of random variable and  $b(\cdot)$  is its standard deviation. In addition,  $d\omega = \varepsilon \cdot dt$ ,  $\varepsilon$  is normal random variable with a mean of 0 as well as variance of 1. Particularly, when  $\alpha(S, t) = \mu S$ ,  $b(S, t) = \sigma S$ , the geometric Brownian motion can be defined by:



$$dS = \mu \cdot S \cdot dt + \sigma \cdot S \cdot d\omega. \quad (2.36)$$

Then, it can be reformulated to interpret clearly:

$$\frac{dS}{S} = \mu \cdot dt + \sigma \cdot d\omega. \quad (2.37)$$

The Brownian motion is applied on simulation of future change of stock price. First, the continuous Brownian motion should be discretized. Let  $t$  and  $T$  represent the initial date and due date, and the interval  $[t, T]$  is divided into  $n$  equal small intervals. So, the length of each small interval should be as follows:

$$\Delta t = \frac{T - t}{n}. \quad (2.38)$$

For any initial value  $S_0$ , the price return with formula (dS) can expressed as follows:

$$S_t = S_0 \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) \cdot t + \sigma \cdot d\omega\right). \quad (2.39)$$

According to above formula, we can obtain the general stock price evolution and defined the mean value and variance of stock price for which the parameters are constant and independent on other variables:

$$S_t = S \cdot \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) \cdot \Delta t + \varepsilon \cdot \sqrt{\Delta t}\right), \quad (2.40)$$

$$E(S_T) = S_0 \cdot \exp(\mu \cdot \Delta t \cdot n), \quad (2.41)$$

$$\text{Var}(S_T) = S^2 \cdot \exp\left(2\left(\mu - \frac{\sigma^2}{2}\right) \cdot \Delta t \cdot n\right) [\exp(\sigma^2 \cdot \Delta t \cdot n) - 1]. \quad (2.42)$$

By means of the comparison with other methods of VaR estimation, the Monte Carlo simulation has number of advantages. First, by means of Monte Carlo, it produces large number of possible scenarios about risk factors. So, it does not require significant numbers of historical observations. Compared with historical simulation, the calculated results based on this method is more accurate. Second, the non-linearity or non-normality problem can be considered. Because it utilizes possible market scenarios to generate the VaR estimation, instruments, such as options, (whose value varies in a non-linear way) can be included in the final calculation. Third, it can consider a wider range of possible outcomes. Because it is not dependent on historical observations

to determine the VaR estimation, it can consider possible events that could have occurred in the past but didn't happen. However, based on historical simulation, it would not consider the likelihood of an event which had not taken place during the period of observation. Forth, a high number of outcomes can be generated. By using computer technology, the Monte Carlo simulation considers a large number of possible scenarios. So that, the numbers of observed exceptions are large. The distribution of the results is likely to be smoother, giving a more accurate of the VaR estimation.

Meanwhile, the Monte Carlo simulation has number of defects. First, the calculated results by Monte Carlo rely heavily on selected stochastic model and historical observation of parameters estimation. So, it exits the model risk and parameter estimation error. Second, in the model simulations, we often utilize the pseudo random number. The circulation and clustering might happen, which might lead to error or failure simulation. Third, due to the large numbers of risk factors and complicated calculation, it takes a long time to obtain a large number of times for simulation. But if it has not enough times of simulation, the accuracy of result will decline.

## **2.7 Comparing the Four Methods**

The previous four methods have their own characteristics and no one method is universally applicable. Each of them has advantages and disadvantages. Linsmeier and Pearson (1996) argue that the methods differ roughly in four dimensions:

1. The ability of the model to capture risk of options and other non-linear instruments;
2. The relative ease of implementation and that of interpretation by users;
3. Flexibility of the model to incorporate alternative assumptions;
4. Reliability of the results.

The last aspect is a key point of this work as it recognizes the importance of backtesting of VaR. Then, the advantages and disadvantages of them will be list as follows:

If the observation period is long, the analytic solution will not reflect accurate estimated VaR well, because it utilizes the linear expansion to approximately reflect the mapping risk. However, for the option with nonlinear change, if the observation period is longer, the gap between the linear

approximation and the actual change will be bigger and bigger. This is the main reason why the parametric method is utilized to calculate VaR of option-based financial instruments failure.

The historical simulation method is simple, and easy to understand and apply. It does not need assumption of return distribution, also it does not need parameters, such as expected value and variance, which will lead to avoid error of parameter estimation and risk of model selection. However, the Monte Carlo simulation needs knowledge of probability statistics, also the process of calculation is complex and its calculated amount is huge.

The VaR estimates market risk assuming a normal market condition. If the big changes occur in the future, it will need a flexible method, which will enable to correct the parameter estimates according to the risk factor. Because historical simulation method directly relies on historical data, there is no space to add more risk factors in this respect. The analytical solution and Monte Carlo simulation have greater flexibility. The user can ignore the estimation of parameters and choose what they think reasonable parameter values to measure market risk.

The historical simulation method directly depends on the historical data. Therefore, when the selected history data is not representative, the VaR estimated by the historical simulation method cannot reflect the market risk well. The "thick-tailed problem" often occurs when the parametric method is applied. At this point, if the estimation of correlation coefficient and standard deviation are properly corrected, the failure of the VaR estimation can be avoided. Compared with the parametric method, the extremum method gives up the assumption of normal distribution and considers the distribution of the tail. In addition, the parametric methods tend to focus on data features of concentrated distribution, while ignoring the tail of sparse distribution, which is the most important part of VaR estimation. Therefore, the parametric method has some limitations. In addition, historical simulation method cannot estimate the risk of loss or return that exceeds the existence of historical value. Compared with non-parametric methods, historical simulation assume that the volatility of returns is constant. However, we know that is unreality. The filtered historical simulation solved this defect, because it does not impose any assumption about the parametric distribution of returns.

### 3 Backtesting Procedure

“VaR is only as good as its backtest. When someone shows me a VaR number, I don’t ask how it is computed, I ask to see the backtest.” (Brown, 2008).

In the last chapter, we introduced the different VaR estimation approaches. However, not all of methods are able to accurately predict future risk in the specific cases. Therefore, we need to assess whether a model is adequate. One of the tools of assessing accuracy is by backtesting. Other forms include stress testing and scenario analysis, marketing risk reporting, hedging, and so on, see Apostolik (2009). Furthermore, backtesting is a key approach to market risk management as laid out by the Basel Committee on Banking Supervision (1996).

The backtesting is a test of how well the current procedure for estimating VaR would have performed if it had been utilized in the past. For example, we assume the confidence level is 99%. It involves looking at how often the loss in a day would have exceeded the one-day 99% VaR when the latter is calculated using the current procedure. Days when the actual loss exceeds VaR are referred to as exceptions. If exceptions happen on about 1% of the days, we can feel reasonably comfortable with the current methodology for calculating VaR. If they happen on, say, 7% of days, the methodology is suspect and it is likely that VaR is underestimated. From a regulatory perspective, the capital calculated using the current VaR estimation procedure is then too low. On the other hand, if exceptions happen on, say, 0.3% of days it is likely that the current procedure is overestimating VaR and the capital calculated is too high.

In order to express clearly whether the calculating VaR is reasonably comfortable or suspect, we obtain the sequence of logical values corresponding to the fact. If the loss exceeds VaR, it will be recorded as 1. On the other hand, if the loss is lower than VaR, it will be recorded as 0. Denoting the backtested VaR over a fixed time interval as  $[t; t + \Delta t]$ , then define sequence  $I_t$  is defined as follows,

$$I_t = \begin{cases} 1, & \text{if } L_R(t; t + \Delta t) > VaR_{R,\alpha}(t; t + \Delta t) \\ 0, & \text{if } L_R(t; t + \Delta t) \leq VaR_{R,\alpha}(t; t + \Delta t) \end{cases} \quad (3.1)$$

Christoffersen (1998) points out that the problem of determining the accuracy of a VaR

model can be reduced to the problem of determining whether the sequence satisfies two properties: the unconditional coverage property and independence property.

The first is unconditional coverage property. The probability of realizing a loss in excess of the reported  $VaR_{R,\alpha}$ , must be precisely  $\alpha \times 100\%$  or in terms of the previous notation,  $Pr(I_{t+1}(\alpha) = 1) = \alpha$ . If it is the case that losses in excess of the reported VaR occur more frequently than  $\alpha \times 100\%$  of the time, then this would suggest that the reported VaR measure systematically understates the asset's actual level of risk. The opposite finding of too few VaR violations would alternatively signal an overly conservative VaR measure, which means VaR is overestimated.

The second is independence property. The unconditional coverage property places a restriction on how often VaR violations may occur. The independence property places a strong restriction on the ways in which these violations may occur. Specifically, any two elements of the sequence must be independent from each other. Intuitively, this condition requires that the previous history of VaR violations must not convey any information about whether or not an additional VaR violation, will occur. If previous VaR violations presage a future VaR violation then this points to a general inadequacy in the reported VaR measure. In general, a clustering of VaR violations represents a violation of the independence property that signals a lack of responsiveness in the reported VaR measure as changing market risks fail to be fully incorporated into the reported VaR measure thereby making successive runs of VaR violations more likely, see Campbell (2005).

Suppose that the significance level for a one-day VaR is  $\alpha$ . If the VaR model utilized is accurate, the probability of the VaR being exceeded on any given day is  $\pi_{ex} = \alpha$ . The observed probability of exceptions' occurring is recorded to as  $\pi_{obs}$ . Based on sequence  $I_t$ , the formula (3.1), the observed number of exceptions will be counted, which we refer to as  $n_1$ . The  $n$  is referred to as length of the time series backtested. For VaR estimation, we need historical data of  $m$  days, which is referred to as historical window. Note that the length of time series backtested  $n$  should not include the historical window  $m$ . That means for obtaining the observed number of exceptions  $n_1$ , the backtesting procedure should start form days  $m + 1$ . We hope that the observed

probability of exceptions' occurring  $\pi_{obs}$  is equal to the expected probability of exceptions' occurring  $\pi_{ex}$ . Therefore, we can consider the null hypothesis and alternative hypothesis as follows,

$$H_0: \pi_{obs} = \pi_{ex} \quad (3.2)$$

$$H_1: \pi_{obs} \neq \pi_{ex} \quad (3.3)$$

We want to know, at a given confidence level, whether  $n_1$  is too small or too large under the null hypothesis  $\pi_{obs} = \pi_{ex}$  in a sample of size  $n$ . Note that this test makes no assumption about the return distribution. The distribution could be normal, or skewed, or with heavy tails, or time varying. The setup for this test is the classic testing framework for a sequence of success and failures, also called *Bernoulli trials*. Under the null hypothesis that the model is correctly calibrated, the number of exceptions  $n_1$  follows a binomial probability distribution:

$$\Pr(n_1 | \pi_{ex}, n) = \sum_{n_1=m+1}^n \frac{n!}{n_1! (n - n_1)!} \pi_{ex}^{n_1} (1 - \pi_{ex})^{n-n_1} \quad (3.4)$$

By utilizing this binomial distribution, we can examine the accuracy of the VaR model. However, when conducting a statistical backtesting that either accepts or rejects a null hypothesis (of the model being 'good'), there is a tradeoff between two types of errors. Type 1 error refers to the possibility of rejecting a correct model and type 2 error to the possibility of not rejecting an incorrect model. A statistically powerful test would efficiently minimize both of these probabilities. (Jorion, 2001)

In order to clearly state the relationship between these two types of error, we make the table 3.1. Table 3.1 summarizes these situations and decision. For backtesting purpose, we need to balance type I errors and type II errors. Ideally, one would want to set a low type I error rate and then have a test that creates a very low type II error rate, in which case the test is said to be powerful. It should be noted that the choice of the confidence level for decision rule is not related to the quantitative level  $\pi_{ex}$  selected for VaR. The confidence level refers to the decision rule to reject the model, see (Jorion, 2006)

So, for example, we describe the probabilities of observing exactly  $n_1$  exceptions (second

column),  $n_1$  or less exceptions (third column) and more than  $n_1$  exceptions (forth column) out of 250 backtesting days and assuming  $\pi_{ex} = 0.01$  and 0.03, which are shown as the table A-1 and A-2 in Annex A.

Table 3.1 Decision errors

Decision making	Null hypothesis is true	Null hypothesis is false
Reject null hypothesis	Type I error	Correct
Accept null hypothesis	Correct	Type II error

Figure 3.1 describes the distribution when the model is calibrated correctly, that is, when  $\pi_{ex} = 0.01$  and with one year of data,  $n = 250$ . Assuming that the model is correct (that is, the actual coverage of the model is 99%), the expected number of days when losses exceed VaR estimates is  $250 \times 0.01 = 2.5$ . One may set the cut-off level for rejecting a model, for instance, to 5 exceptions. In this case, the probability of committing a type I error is 10.8%. On the other hand, if the model has an incorrect coverage of 97%, the expected number of exceptions is  $250 \times 0.03 = 7.5$ . There is now a 12.8% probability of committing a type 2 error, that is, accepting an inaccurate model. The horizontal axis represents number of exceptions, and the vertical axis represents frequency.

Figure 3.1 Distribution of exceptions when model is correct.

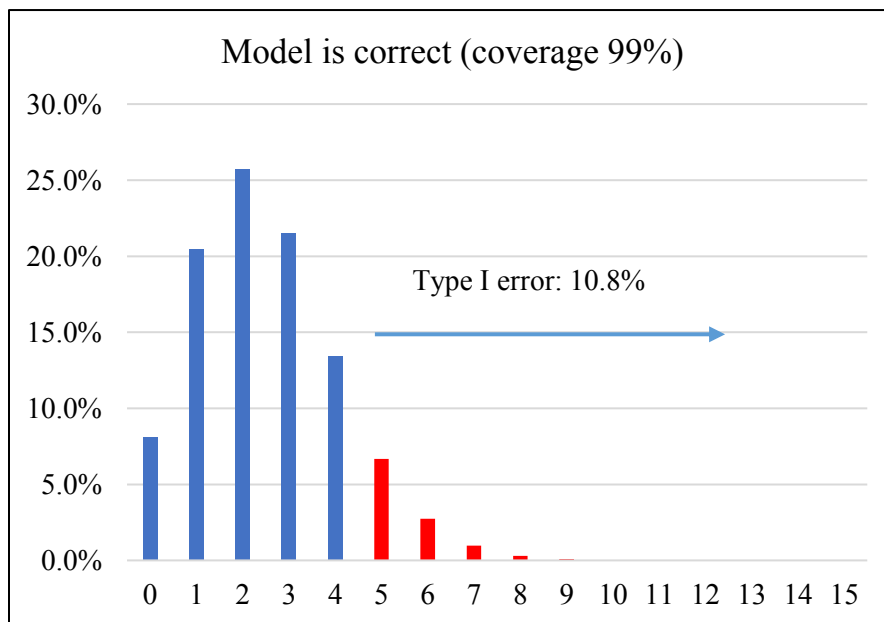
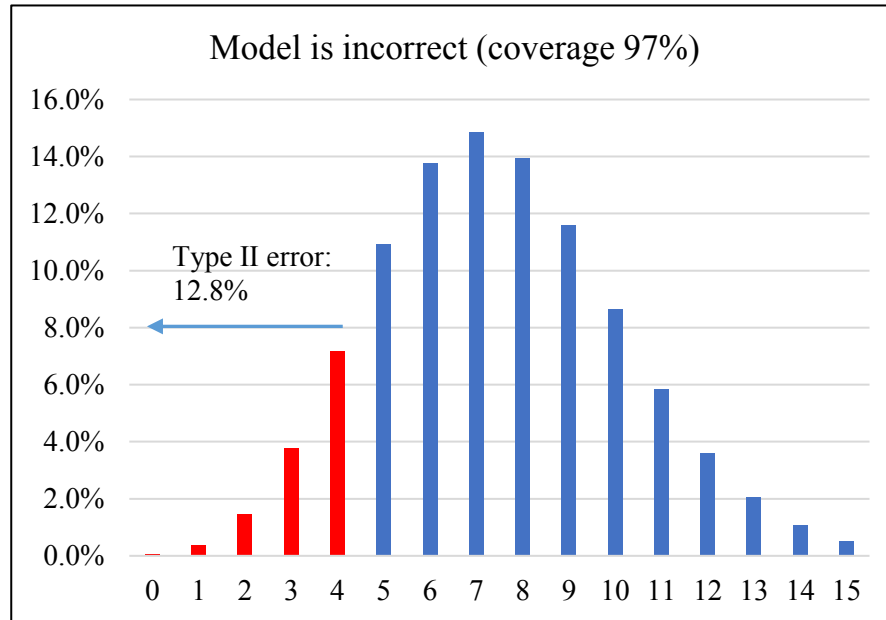


Figure 3.3 Distribution of exceptions when model is incorrect.



### 3.1 Kupiec's Unconditional Coverage Test

Some of the earliest proposed VaR backtests, e.g. Kupiec (1995), focused exclusively on the property of unconditional coverage. In short, these tests are concerned with whether the reported VaR is violated more or less than significance level or not. Kupiec (1995), for example, proposed a proportion of failures (POF) test that examines how many times a financial institution's VaR is violated over a given span of time. If we denote the number of exceptions by  $n_1$  and the total number of observations by  $n$ , we then define the rate of failure as  $n_1/n$ . Suppose a VaR number is reported at the confidence interval  $1 - \alpha$ , then an exception occurs if realized loss exceeds the VaR number. Therefore, the expected number of exceptions  $n_1$  in a total of  $n$  observations is  $\alpha \cdot n$ . Certainly, the number of exceptions will not be exactly  $\alpha \cdot n$ . Instead, it could swing within an acceptable range. In the backtesting method, the range for  $x$  will be calculated and thus the VaR model can be accepted or rejected, see Campbell (2005).

The null hypothesis for the POF-test is

$$H_0: \pi_{ex} = \pi_{obs} = \frac{n_1}{n} \quad (3.5)$$

We need to find out whether the observed failure rate  $\pi_{obs}$  is significantly different from  $\pi_{ex}$ , the failure rate suggested by the confidence level. According to Kupiec (1995), the POF-test



is best conducted as a likelihood-ratio (LR) test. The test statistic takes the form as follows,

$$LR_{Kupiec} = -2 \ln \left[ \frac{\pi_{ex}^{n_1} (1 - \pi_{ex})^{n_0}}{\pi_{obs}^{n_1} (1 - \pi_{obs})^{n_0}} \right], \quad (3.6)$$

where  $\pi_{ex}$  and  $\pi_{obs}$  are expected probability of exception occurring and observed probability of exception occurring, respectively,  $n$  is the length of the time series backtested,  $n_0$  is the number of zero and  $n_1$  is the number of ones, which are counted by formula (3.1) when the loss exceeds VaR. We know  $\pi_{ex}$  equals significant level  $\alpha$ ,  $\pi_{obs} = \frac{n_1}{n_0 + n_1}$  and  $n = n_0 + n_1$ . Therefore, the likelihood ratio can be expressed as follows:

$$LR_{Kupiec} = 2n_0 \cdot \ln \left( \frac{n_0}{n} \right) + 2n_1 \cdot \ln \left( \frac{n_1}{n} \right) - 2n_0 \cdot \ln(1 - \alpha) - 2n_1 \cdot \ln(\alpha). \quad (3.7)$$

Under the null hypothesis that the model is correct,  $LR_{Kupiec}$  is asymptotically chi-squared distributed with one degree of freedom. In probability theory and statistics, the chi-squared distribution with  $k$  degrees of freedom are the distribution of a sum of the squares of  $k$  independent standard normal random variables. The chi-squared distribution with  $k$  degrees of freedom is usually denoted as,

$$Q \sim X^2(k). \quad (3.8)$$

The chi-square distribution is a special case of the gamma distribution and is one of the most widely used probability distributions in inferential statistics, e.g. in hypothesis testing or in construction of confidence intervals. The chi-squared distribution is used in the common chi-squared tests for goodness of fit of an observed distribution to a theoretical one, the independence of two criteria of classification of qualitative data, and in confidence interval estimation for a population standard deviation of a normal distribution from a sample standard deviation, see Jonhson et al. (1994).

We know that there are two methods for a test of hypothesis whether the model accept or not, critical region and p-value. Critical values for a test of hypothesis depend upon a test statistic, which is specific to the type of test, and the significance level  $\alpha$ . Critical values are essentially cut-off values that define regions where the test statistic is unlikely to lie. For example, a region where the critical value is exceeded with probability  $\alpha$  if the null hypothesis is true. The null hypothesis

is rejected if the test statistic lies within this region which is often referred to as the rejection regions. Another quantitative measure for reporting the result of a test of hypothesis is the p-value. The p-value is the probability of the test statistic being at least as extreme as the one observed given that the null hypothesis is true. A small p-value is an indication that the null hypothesis is false.

In our cases, the null hypothesis is that the probability of exception occurring is equal to the expected value, that is what we want to prove. Therefore, we want to accept null hypothesis, which is contrary to the general statistical test that reject null hypothesis. Furthermore, considering critical values and p-value, we choose the p-value for testing hypothesis, because the higher p-value the more accurate the model.

Table 3.2 is a summary table of non-rejection regions for the Kupiec test for various observation period and confidence levels. The table shows that the interval, expressed as a proportion  $n_1/n$ , increases as the sample size increases.

Table 3.2 Non-rejection test confidence regions

Confidence level (1- $\alpha$ )	Significance level ( $\alpha$ )	Non-rejection region for number of failures $n_1$		
		$n = 251$ days	$n = 510$ days	$n = 1000$ days
99.0%	0.01	$n_1 < 7$	$1 < n_1 < 11$	$1 < n_1 < 17$
97.5%	0.025	$2 < n_1 < 12$	$6 < n_1 < 21$	$15 < n_1 < 36$
95.0%	0.05	$6 < n_1 < 20$	$16 < n_1 < 36$	$37 < n_1 < 65$
92.5%	0.075	$11 < n_1 < 28$	$27 < n_1 < 51$	$59 < n_1 < 92$
90.5%	0.1	$16 < n_1 < 36$	$38 < n_1 < 65$	$81 < n_1 < 120$

Source: adapted from Kupiec (1995)

For example, at 95% confidence level, the interval  $n_1/n$  for accepting the model with 251 observations are in the range

$$\left[ \frac{6}{251} = 0.0239; \frac{20}{251} = 0.0797 \right], \quad (3.9)$$

and compared to 1000 observations, the interval is

$$\left[ \frac{37}{1000} = 0.037; \frac{65}{1000} = 0.065 \right], \quad (3.10)$$

which is much tighter. With more data, we should be able to reject the model more easily if it is

false.

The Kupiec test has two main drawbacks. Firstly, the test is statistically weak with small sample sizes, for instance the Basel Committee’s regulatory framework sample (250 trading days), see Jorion (2006). Secondly, the Kupiec test examines only the failure rate (frequency of exceptions) and not the succession of occurrence. Therefore, it may fail to reject a model that produces serially dependent exceptions, a common weakness of unconditional coverage tests, see Campbell (2005).

### 3.2 Christoffersen’s Conditional Coverage Test

A separate issue from the number of exceptions is bunching. If daily assets change is independent, exceptions should be spread evenly throughout the period used for backtesting. In practice, they are often bunched together suggesting that losses on successive days are not independent. An early and influential test in this vein is Christoffersen’s (1998) Markov test. The Markov test examines whether or not the likelihood of a VaR violation depends on whether or not a VaR violation occurred on the previous day. If the VaR measure accurately reflects the underlying asset risk then the chance of violating today’s VaR should be independent of whether or not yesterday’s VaR was violated, see Campbell (2005). The testing procedure described below is explained, for example, in Jorion (2006), and in greater detail in Christoffersen (1998).

The test is then set out as follows. Suppose we have data of portfolio returns for  $T$  days. Each day we set a deviation indicator, and set the indicator value as follows:

$$I_t = \begin{cases} 1, & \text{if violation occurs} \\ 0, & \text{if no violation occurs} \end{cases} \quad (3.11)$$

Table 3.3 Contingency table for Markov independence test (Christoffersen, 1998)

	$I_{t-1} = 0$	$I_{t-1} = 1$	
$I_t = 0$	$n_{00}$	$n_{10}$	$n_{00} + n_{10}$
$I_t = 1$	$n_{01}$	$n_{11}$	$n_{01} + n_{11}$
	$n_{00} + n_{01}$	$n_{10} + n_{11}$	

The test is carried out by creating a 2×2 contingency table that records violations of the institution’s VaR on adjacent days as in Table 3.3. If the VaR measure accurately reflects the asset’s

risk then the proportion of violations that occur after a previous violation,  $I_{t-1} = 1$ , it should be the same as the proportion of violations that occur after a day in which no violation occurred,  $I_{t-1} = 0$ .

In addition, let  $\pi_i$  represent the probability of observing an exception conditional on state  $i$  on the previous day. It follows that,

$$\pi_0 = \frac{n_{01}}{n_{00} + n_{01}}, \pi_1 = \frac{n_{11}}{n_{10} + n_{11}}, \text{ and } \pi = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}}. \quad (3.12)$$

Under the null hypothesis that exceptions are independent across days, then the probabilities should be equal,  $\pi = \pi_0 = \pi_1$ . That is, the chance of an exception occurring after a day of no exception is the same as occurring after a day of an exception (Campbell, 2005). If these proportions differ greatly from each other, then this calls the validity of the VaR measure into question.

If the model is accurate, then a VaR violation today should not depend on whether or not a violation occurred on the previous day (Jorion, 2006). The relevant test statistic for independence of exceptions is a Likelihood-Ratio (LR) given by:

$$LR_{ind} = -2 \ln \left[ \frac{\pi^{n_{01}+n_{11}} \cdot (1-\pi)^{n_{00}+n_{10}}}{\pi_0^{n_{01}} (1-\pi_0)^{n_{00}} \cdot \pi_1^{n_{11}} (1-\pi_1)^{n_{10}}} \right]. \quad (3.13)$$

In order to simplify the calculation, the formula (3.13) is written as follows:

$$LR_{ind} = -2 \cdot \{[(n_{01} + n_{11}) \cdot \ln(\pi) + (n_{00} + n_{10}) \cdot \ln(1 - \pi)] - [n_{01} \cdot \ln(\pi_0) + n_{00} \cdot \ln(1 - \pi_0) + n_{11} \cdot \ln(\pi_1) + n_{10} \cdot \ln(1 - \pi_1)]\}. \quad (3.14)$$

Same as with the Kupiec test, the independence test follows the Chi-squared distribution with 1-degree of freedom. Similarly to previous chapter, if the value of the  $LR_{ind}$  falls below the critical value of Chi-squared distribution with 1-degree of freedom, the model passes the backtesting, otherwise the model is rejected.

The main defect of the Markov test is that its limited power against clustering. It mainly tests for independence of exceptions on two consecutive days. For instance, it might be the case that the likelihood of a VaR exception depends not on whether there was an exception the previous day, but whether there was an exception one week or two weeks ago. Clearly, if this is the way in which the

lack of the independence property manifests itself, then the Markov test won't have power to detect such violations of the independence property.

### 3.3 The Joint Test of Coverage and Independence

Given the shortcomings of the two tests presented above, it is clear that an accurate VaR measure should take into account both the unconditional coverage and independence properties. As such, tests that jointly examine the unconditional coverage and independence properties provide an opportunity to detect VaR measures which are deficient in one way or another (Campbell, 2005). One such test of conditional coverage which tests for both proportion of exceptions (unconditional coverage) and the clustering of exceptions (independence property) is the Christoffersens Interval Forecast test, again proposed by Christoffersen (1998).

The method again applies the same Likelihood statistic testing framework as Kupiec, but extends the Kupiec test to jointly test for both unconditional coverage and independence. By combining the Kupiec test with the independence test, a joint test that examines both properties of an accurate VaR model is produced. The combined test statistic for conditional coverage is then given by as follows,

$$LR_{cc} = LR_{Kupiec} + LR_{ind}, \quad (3.15)$$

and the expansion formula can be written as follows,

$$LR_{cc} = -2 \log \left[ \frac{\pi_{ex}^{n_1} (1 - \pi_{ex})^{n_0}}{\pi_0^{n_{01}} (1 - \pi_0)^{n_{00}} \cdot \pi_1^{n_{11}} (1 - \pi_1)^{n_{10}}} \right], \quad (3.16)$$

which is a Chi-squared distribution with two degrees of freedom. Similarly to previous chapter, if the value of the  $LR_{cc}$  falls below the critical value, the model passes the backtesting, otherwise the model is rejected.

## 4 Empirical Results of Statistical Testing

In this chapter, we present the procedure of application of the selected VaR estimation approaches and the empirical result of them.

In the thesis, we utilize the single stocks for our calculation. We download the time series data from the finance.yahoo.com web site. We choose stocks of Apple Inc (AAPL) from American financial market, which is one of Dow Jones Industrial Average component. The prices are quoted in USD. The period we choose from December 31, 2007 to December 29, 2017. There are 2519 data in total. The frequency is daily. In the downloaded file, the first to fifth column are “Date”, “Open”, “High”, “Low”, “Close”, “Adj Close” and “Volume”, respectively. We just keep the first and fifth columns. The “Adj Close” means the prices are adjusted for splits and dividends paid, which is the better choice for historical prices. By means of checking, there is no missing data, so we can insert them into Excel directly, see the figure B in Annex B.

Because of the different historical windows, the length of the time series backtested is different. In order to unify the length of time series backtested ( $n$ ), we start VaR estimation for backtesting from the 501<sup>st</sup> day. Therefore, the backtesting is performed in period December 24, 2009 to December 29, 2017. The overall length of the time series backtested  $n = 2518 - 500 = 2018$ . The length of historical window ( $m$ ) is based on the period of trading days. For example, the  $m = 21$  is approximately one month,  $m = 63$  represents approximately three months,  $m = 250$  means approximately one year. For each VaR estimation method, we select significance level of 15%, 10%, 5%, 2.5%, 1% and 0.5%.

In order to represent the procedure of application, there are simplified examples in the beginning of each empirical study by means of different VaR estimation measures. For the simplified example, we select 31 days only, starting from December 31, 2007 to February 13, 2008. So, there are 31 historical prices. In terms of significance level, if we give 5% or 1% which are commonly utilized, the number of observed losses which exceeds VaR is too small to be accurately deducted, because of the small number of observations. So, we suppose the confidence level is 70%, and significance level  $\alpha$  is 30%. The historical window of 10 is given.

## 4.1 Historical Simulation

The VaR estimation measure of historical simulation is introduced in the chapter 2.6.1. Now, we introduce its application by means of Excel. This subchapter is divided into two parts. In the first part, the mathematical formulation and procedure applied on simplified example will be described. In the second part, the empirical results are interpreted.

Table 4.1 Mathematical formulations by means of historical simulation

Daily return

$$R_i = \frac{P_{i,t}}{P_{i,t-1}} - 1. \quad (E1)$$

VaR estimation

$$VaR_{R,\alpha} = \Delta V(k_{[T\alpha]+1}). \quad (E2)$$

Logical values that the observed loss exceed estimated loss

$$I_t = \begin{cases} 1, & \text{if } L_R(t; t + \Delta t) > VaR_{R,\alpha}(t; t + \Delta t) \\ 0, & \text{if } L_R(t; t + \Delta t) \leq VaR_{R,\alpha}(t; t + \Delta t) \end{cases} \quad (E3)$$

Likelihood ratio of Kupiec test

$$LR_{Kupiec} = 2n_0 \cdot \log\left(\frac{n_0}{n}\right) + 2n_1 \cdot \log\left(\frac{n_1}{n}\right) - 2n_0 \cdot \log(1 - \alpha) - 2n_1 \cdot \log(\alpha), \quad (E4)$$

where

$$n = n_0 + n_1. \quad (E5)$$

Likelihood ratio of Christoffersen test

$$LR_{ind} = -2 \cdot \{[(n_{01} + n_{11}) \cdot \ln(\pi) + (n_{00} + n_{10}) \cdot \ln(1 - \pi)] - [n_{01} \cdot \ln(\pi_0) + n_{00} \cdot \ln(1 - \pi_0) + n_{11} \cdot \ln(\pi_1) + n_{10} \cdot \ln(1 - \pi_1)]\}. \quad (E6)$$

where

$$\pi_0 = \frac{n_{01}}{n_{00} + n_{01}}, \pi_1 = \frac{n_{11}}{n_{10} + n_{11}}, \pi = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}}, \quad (E7)$$

Chi-squared distribution

$$LR \sim X^2(k). \quad (E8)$$

### 4.1.1 Simplified Example

We represent the simplified example by following three steps, mathematical formulations,

procedure by means of Excel, and solution and interpretation. In the simplified example, we mainly state the formulations we utilize, how to estimate VaR and its backtesting, and explain briefly the result.

### **A Mathematical formulation**

According to formulas (2.9), (2.20), (3.1), (3.7), (3.8), (3.12), (3.14) we can list the mathematical formulations in the table 4.1.

In the table 4.1,  $\Delta V$  means the return of asset which is ranked from smallest to biggest,  $[T_\alpha]$  means integral part of quantile  $T_\alpha$  on a given significance level  $\alpha$ . The formula (E8) is a notation, which means the chi-squared distribution with  $k$  degrees of freedom. In Kupiec test and Christoffersen test, we utilize one degree of freedom for calculating p-value.

### **B Procedure**

1. Calculating the daily return according to formula (E1).
2. Calculating the forecasts of VaR by means of PERCENTILE.EXC, input arguments of the historical window array and significance level.
3. Calculating whether the true loss is higher than VaR estimation or not according to (E3). We can utilize the Excel function IF.
4. Calculating likelihood ratio of Kupiec test and Christoffersen test according to (E4) and (E6).
5. Calculating the p-values of Kupiec test and Christoffersen test. They can be calculated using the CHISQ.DIST function in Excel. Input arguments are likelihood ratio and degree of freedom of 1. Cumulative distribution function is selected.

### **C Solution and interpretation**

Table 4.2 represents the procedure and result of VaR estimation and its backtesting by means of historical simulation. From this table, we can see there are five days that the observed loss exceeds VaR estimation and our expected value is 6. The exceptions occurred on January 16, 22, 23, February 6 and 12 in 2008, when the logical values  $I_t$  are equal to one. Based on the numbers of observed exceptions, we can calculate the p-values of Kupiec test and Christoffersen test, which are 62% and 94.6%, showing the accuracy of this model.



Table 4.2 VaR estimation and its backtesting using historical simulation

1	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
	Date	Adj Close	Return	VaR	Exceptions	n00	n01	n10	n11								
2	31/12/07	19.23404										Alpha	0.3				
3	02/01/08	18.91942	-0.01636	$=-(B3-B2)/B2$								Number of returns	30				
4	03/01/08	18.92817	0.00046	$=-PERCENTILE.EXC(D3:D12, \$M\$2)$								History window	10				
5	04/01/08	17.48328	-0.07634									n	20	$=M3-M4$			
6	07/01/08	17.24926	-0.01339									Exceeded	6	$=M5*M2$			
7	08/01/08	16.62878	-0.03597														
8	09/01/08	17.42016	0.04759									n1	5	$=SUM(F13:F32)$			
9	10/01/08	17.28617	-0.00769									n0	15	$=M5-M8$			
10	11/01/08	16.76861	-0.02994									Kupiec test	0.25				
11	14/01/08	17.35996	0.03527									p-value	62%	$=2*M9*LN(M9/M5)+2*M8*LN(M8/M5)-2*M9*LN(1-M2)-2*M8*LN(M2)$			
12	15/01/08	16.41419	-0.05448														
13	16/01/08	15.50142	-0.05561	0.03416	1												
14	17/01/08	15.62280	0.00783	0.04893	0	0	0	1	0					$=1-CHISQ.DIST(M10,1,1)$			
15	18/01/08	15.66844	0.00292	0.04893	0	1	0	0	0					$=SUM(G14:G32)$			
16	22/01/08	15.11301	-0.03545	0.03416	1	0	1	0	0			n00	11	$=SUM(H14:H32)$			
17	23/01/08	13.50403	-0.10646	0.03581	1	0	0	0	1			n01	3				
18	24/01/08	13.16708	-0.02495	0.04877	0	0	0	1	0			n10	4	$=SUM(I14:I32)$			
19	25/01/08	12.62428	-0.04122	0.04877	0	1	0	0	0			n11	1				
20	28/01/08	12.62428	0.00000	0.05050	0	1	0	0	0			$\pi_0$	0.2143	$=SUM(J14:J32)$			
21	29/01/08	12.77285	0.01177	0.05050	0	1	0	0	0			$\pi_1$	0.2				
22	30/01/08	12.83499	0.00487	0.05050	0	1	0	0	0			$\pi$	0.2105	$=M17/(M16+M17)$			
23	31/01/08	13.14378	0.02406	0.03949	0	1	0	0	0			Christoffersen test	0.0046	$=M19/(M18+M19)$			
24	01/02/08	12.98744	-0.01189	0.03230	0	1	0	0	0			p-value	94.62%	$=(M17+M19)/(M16+M17+M18+M19)$			
25	04/02/08	12.78353	-0.01570	0.03230	0	1	0	0	0								
26	05/02/08	12.56116	-0.01739	0.03230	0	1	0	0	0								
27	06/02/08	11.84649	-0.05690	0.02268	1	0	1	0	0					$=1-CHISQ.DIST(M23,1,1)$			
28	07/02/08	11.77269	-0.00623	0.02268	0	0	0	1	0					$=-2*(((M17+M19)*LN(M22)+(M16+M18)*LN(1-M22))-(M17*LN(M20)+M16*LN(1-M20)+M19*LN(M21)+M18*LN(1-M21)))$			
29	08/02/08	12.18440	0.03497	0.01689	0	1	0	0	0								
30	11/02/08	12.56990	0.03164	0.01456	0	1	0	0	0								
31	12/02/08	12.12420	-0.03546	0.01456	1	0	1	0	0								
32	13/02/08	12.56505	0.03636	0.01689	0	0	0	1	0					$=-PERCENTILE.EXC(D22:D31, \$M\$2)$			

### 4.1.2 Empirical Study

In the former subchapter, we describe the procedure of application on VaR estimation and its backtesting using historical simulation. In the empirical study, we mainly verify the accuracy of VaR estimation by means of historical simulation based on different length of historical windows with different significance levels. By means of calculation, the empirical result of VaR estimation and its backtesting for AAPL stock from December 3, 2007 to December 29, 2017 is shown in tables 4.3, 4.4 and 4.5. We show the VaR estimation in figure 4.1.

The table 4.3 shows the numbers of expected exceptions on different significance levels and the numbers of observed exceptions based on the different historical windows ( $m$ ). The first row shows the significance levels. The second row represents the numbers of expected exceptions. The rest rows show the number of observed exceptions based on different length of historical window with different significance levels.

Table 4.3 The numbers of exceptions by means of historical simulation.

Exceptions		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Expected		302.7	201.8	100.9	50.45	20.18	10.09
Observed	$m = 21$	301***	189***	96***	121	103	96
	$m = 42$	306***	208***	98***	55***	66	58
	$m = 63$	310***	199***	98***	50***	50	41
	$m = 125$	300***	204***	104***	45***	21***	24
	$m = 250$	291***	210***	101***	51***	19***	13***
	$m = 500$	261	189***	83*	40**	15***	10***

P-values of Kupiec test are higher than: 5%\*, 10%\*\* , 15%\*\*\*

From table 4.3, we prefer to see that the number of observed exceptions as close to expected value. For all significance levels, the longer the historical window, the less the observed exceptions. If the number of observed exceptions is lower than expected value, that means the current model is overestimating VaR. If the number of observed exceptions is higher than expected value, that means the current model is underestimating VaR. In table 4.3, the results of p-value are divided

into three different accuracy intervals based on Kupiec test. The p-value higher than 5% is recorded as \*, higher than 10% recorded \*\*, higher than 15% recorded as \*\*\*.

Table 4.4 P-values of Kupiec test

P-value (K-test)		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Observed	$m = 21$	91.6%	33.8%	61.4%	0.0%	0.0%	0.0%
	$m = 42$	83.7%	64.7%	76.6%	52.2%	0.0%	0.0%
	$m = 63$	65.0%	83.5%	76.6%	94.9%	0.0%	0.0%
	$m = 125$	86.6%	87.1%	75.3%	42.9%	85.5%	0.0%
	$m = 250$	46.3%	54.5%	99.2%	93.8%	79.0%	37.9%
	$m = 500$	0.8%	33.8%	6.0%	12.2%	22.5%	97.7%

Table 4.5 P-values of Christoffersen test

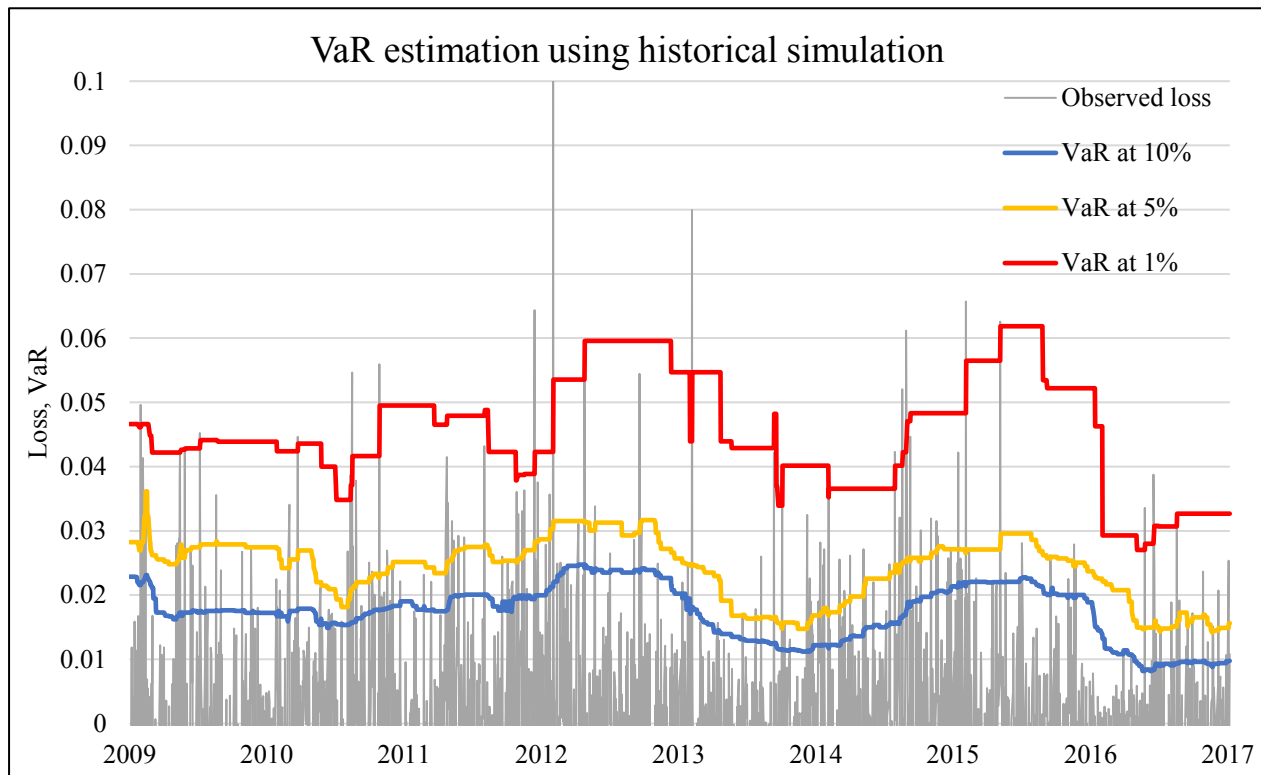
P-value (C-test)		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Observed	$m = 21$	15.6%	6.8%	12.4%	16.6%	5.0%	12.4%
	$m = 42$	0.8%	0.5%	6.6%	68.9%	57.5%	79.7%
	$m = 63$	0.1%	0.4%	2.6%	16.7%	16.7%	26.6%
	$m = 125$	0.0%	0.0%	5.6%	9.7%	50.6%	44.7%
	$m = 250$	0.0%	0.0%	3.9%	78.6%	54.8%	68.1%
	$m = 500$	0.1%	0.0%	7.3%	20.3%	63.5%	75.2%

In the table 4.4, the accuracy of this model is represented. The higher the p-value, the more accurate the model. Note that there are several p-values of Kupiec test equal to zero. Because as the VaR estimated, the function PERCENTILE.EXC only works if  $\alpha$  is between  $1/m$  and  $1 - 1/m$ . For the smaller significance levels, we apply the function PERCENTILE.INC for smaller  $m$ , which is less accurate, but it works for any value of  $\alpha$  between 0 and 1. Therefore, the p-values by means of PERCENTILE.INC is zero. Because of the unprecise function, we cannot distinguish if the model is accurate or not with p-value of zero in this case. According to the table 4.4, we can draw a conclusion that the most results of the numbers of observed exceptions can be statistically accepted, except only overestimating VaR with significance level of 15% for historical window of

500. The numbers of observed exceptions are rejected of which the VaR estimated by PERCENTILE.INC.

The table 4.5 shows the p-value of Christoffersen test. As we can see with significance level of 15% and 10%, only the empirical result can be accepted from 21 days. For significance level of 5%, the acceptable empirical results are historical windows of 21, 42, 125 and 500. The p-value of them shows poor accuracy of the model. However, for the significance levels of 2.5%, 1% and 0.5%, all the empirical results are accepted, although some of them the VaR is underestimated. Overall, the acceptable empirical results of Kupiec test are more than Christoffersen test, and more resealable.

Figure 4.1 VaR estimation using historical simulation



In order to represent more visually, we make figure 4.1. The exceptions shown in the figure is the part of observed loss above the VaR at a given significance level. According to our statistical result, the historical window of 250 is more accuracy for all selected significance levels than others. So, we choose the historical window of 250 days, VaR estimation with significance levels of 10%, 5%, and 1%.

According to the shape of VaR curves, we can see they maintain a fixed level over a period of time. That is the unique characteristic of historical simulation to the other VaR estimation methods. Because when we calculated the VaR by means of historical simulation, we utilize the given quantile of historical returns as a negative VaR estimation. The given quantile is determined by significance level. So, the VaR might be unchanged for a while. From the figure we can observe that the huge volatility happened at August and October in 2012, January in 2014, August in 2015 and January in 2016. Especially in January 24, 2013, the stock price from \$53.14 decreased to \$46.57, which fall more than 12%.

## **4.2 Filtered Historical Simulation**

The VaR estimation measure of filtered historical simulation is introduced in the chapter 2.6.2. In this chapter, we explain the application procedure by means of Excel. Similarly to previous chapter, we represent the simplified example first and then empirical study. For filtered historical simulation, the AR (1)-GARCH (1,1) is applied.

### **4.2.1 Simplified Example**

Compared with procedure of historical simulation, the FHS method is more complex. The selected history prices, the significance level of 30% and historical window of 10 are the same as historical simulation. The difference of procedure is that we need to find out several parameters for calculating one day ahead VaR estimation. In our case, we suppose that the expected mean value of return over the whole period is zero, and there is no autocorrelation of returns. That means all random variables follow normal distribution. In the simplified example, we mainly state the formulations that we need to utilize, what the procedure is, and explain the result briefly.

#### **A Mathematical formulation**

According to formulas (2.16), (2.19), (2.20), (3.1), (3.7), (3.8), (3.12), (3.14) we can list the mathematical formulation in the table 4.6. First, we need to set up solver and calculate the parameters of GARCH (1,1) model over one period. Then, we calculate the VaR estimation and its backtesting.

Table 4.6 Mathematical formulation by means of filtered historical simulation

Daily return

$$R_i = \frac{P_{i,t}}{P_{i,t-1}} - 1. \quad (E1)$$

Objective function

$$L(\omega, \alpha, \beta) = \sum_t z_t \rightarrow \max.$$

Constraints

$$\alpha + \beta < 1, \quad (C1)$$

$$\omega, \alpha, \beta \geq 0, \quad (C2)$$

where

$$z_t = -\ln \sigma_{t,t-1}^2 - \frac{R_t^2}{\sigma_{t,t-1}^2}, \quad (R1)$$

$$\sigma_{t+1,t}^2 = \omega + \alpha \cdot R_t^2 + \beta \cdot \sigma_{t,t-1}^2 \quad (R2)$$

Standardized residual returns

$$e_t = \frac{R_t}{\sigma_t}. \quad (E2)$$

VaR estimation

$$VaR_{t+1} = \sigma_{t+1} q_\alpha, \quad (E3)$$

Logical values that the observed loss exceed estimated loss

$$I_t = \begin{cases} 1, & \text{if } L_R(t; t + \Delta t) > VaR_{R,\alpha}(t; t + \Delta t) \\ 0, & \text{if } L_R(t; t + \Delta t) \leq VaR_{R,\alpha}(t; t + \Delta t) \end{cases} \quad (E4)$$

Likelihood ratio of Kupiec test

$$LR_{Kupiec} = 2n_0 \cdot \log\left(\frac{n_0}{n}\right) + 2n_1 \cdot \log\left(\frac{n_1}{n}\right) - 2n_0 \cdot \log(1 - \alpha) - 2n_1 \cdot \log(\alpha), \quad (E5)$$

where

$$n = n_0 + n_1. \quad (E6)$$

Likelihood ratio of Christoffersen test

$$LR_{ind} = -2 \cdot \{[(n_{01} + n_{11}) \cdot \ln(\pi) + (n_{00} + n_{10}) \cdot \ln(1 - \pi)] - [n_{01} \cdot \ln(\pi_0) + n_{00} \cdot \ln(1 - \pi_0) + n_{11} \cdot \ln(\pi_1) + n_{10} \cdot \ln(1 - \pi_1)]\}. \quad (E7)$$

where

$$\pi_0 = \frac{n_{01}}{n_{00} + n_{01}}, \pi_1 = \frac{n_{11}}{n_{10} + n_{11}}, \pi = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}}, \quad (E8)$$

Chi-squared distribution

$$LR \sim X^2(k). \quad (E9)$$

In the table 4.6, the  $\omega, \alpha, \beta$  are parameters,  $e_t$  is standardized residual returns,  $R_t$  is daily return,  $\sigma_t$  is daily volatility,  $\sigma_{t+1}$  is volatility forecast.

### **B procedure**

1. Calculating the daily return according to formula (E1).
2. Calculating the forecasted conditional variance  $\sigma_{t+1,t}^2$  according to (R2). For the unconditional variance, we utilize Excel function AVERAGE.
3. Setting up the *Solver* and calculating the parameters of GARCH (1,1) model over one period, see figure 4.2.
4. Calculating the forecast VaR according to (E3). We utilize Excel function PERCENTILE.EXC to determine the quantile of standardized residual returns  $e_t$ . The standard deviation  $\sigma_{t+1}$  which is square root of variance  $\sigma_{t+1,t}^2$  is calculated by Excel function SQRT.
5. Calculating whether the true loss is higher than VaR estimation or not according to (E4). We can utilize the Excel function IF.
6. Calculating likelihood ratio of Kupiec test and Christoffersen test according to (E5), and (E7).
7. Calculating the p-value of Kupiec test and Christoffersen test using the CHISQ.DIST function.

### **C Solution and interpretation**

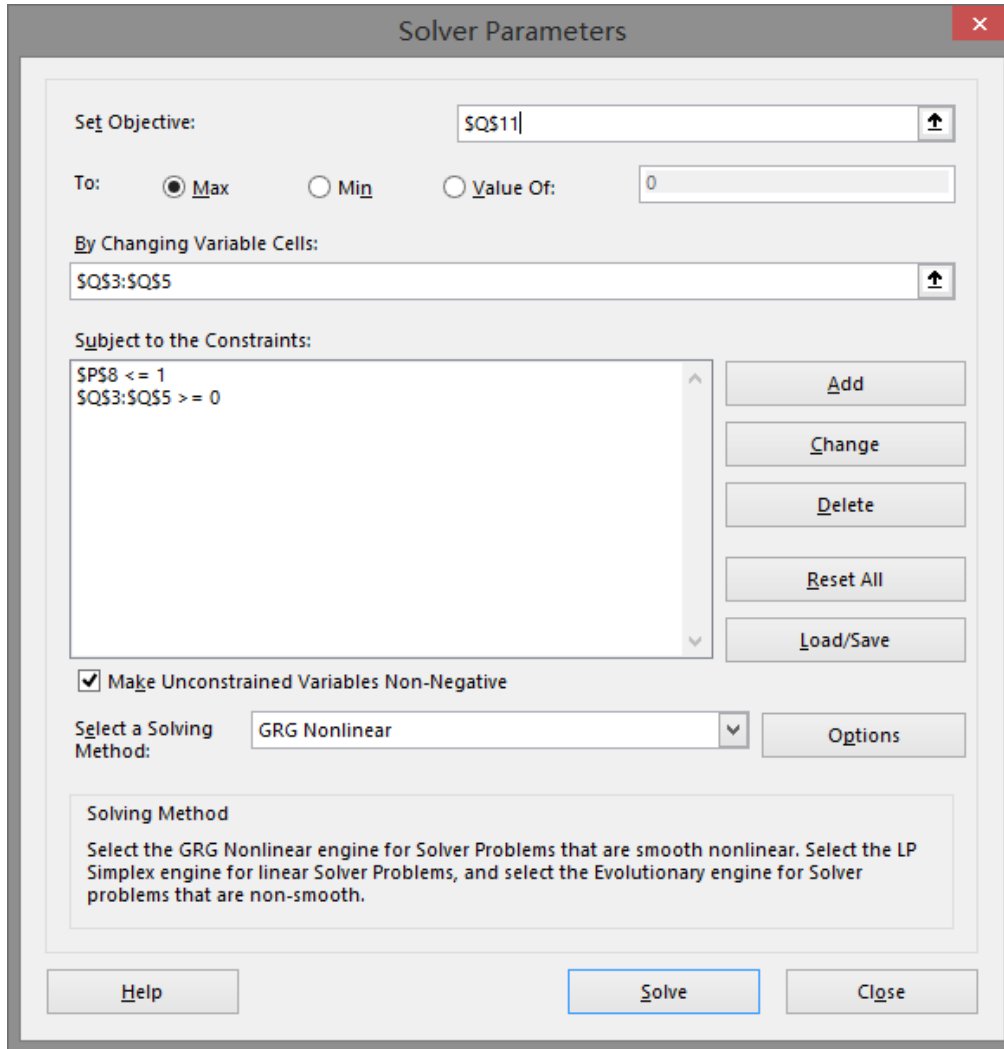
Table 4.7 shows the procedure and result of VaR estimation and its backtesting by means of filtered historical simulation. Figure 4.2 shows the setting procedure of GARCH model by means of *Solver*. From table 4.7, we find out the parameters of  $\omega = 0.0000, \alpha = 0.0000, \beta = 0.99423$ . As we can see there are 5 days that the observed loss exceeds VaR estimation. For backtesting, the number of observed exceptions are 5, and our expected value is 6. The p-value of Kupiec test and Christoffersen test are 62% and 94.62%.

Table 4.7 VaR estimation and its backtesting using filtered historical simulation

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U
1	Date	Adj Close	Return	R^2	Volatility	z_t	e_t	VaR	Exceptions	n00	n01	n10	n11								
2	31/12/07	19.2340	= $(B3-B2)/B2$	= $D3^2$	= $AVERAGE(E3:E32)$	= $LN(F3)-E3/F3$	= $D3/SQRT(F3)$									Parameters					
3	02/01/08	18.9194	-0.0164	0.00027	0.00142	6.36894	-0.43415									$\omega$	0.00000				
4	03/01/08	18.9282	0.0005	0.00000	0.00141	6.56906	-0.01230	= $SQ53+SQ54*E3+SQ55*F3$								$\alpha$	0.00000				
5	04/01/08	17.4833	-0.0763	0.00583	0.00140	2.41631	-2.03781									$\beta$	0.99423				
6	07/01/08	17.2493	-0.0134	0.00018	0.00140	6.44636	-0.35837	= $PERCENTILE.EXC(H3:H12,SQ514)*SQRT(F13)$								Constraints					
7	08/01/08	16.6288	-0.0360	0.00129	0.00139	5.64769	-0.96586									0.994227	<=	1			
8	09/01/08	17.4202	0.0476	0.00226	0.00138	4.94397	1.28156									$\omega, \alpha, \beta$	>=	0			
9	10/01/08	17.2862	-0.0077	0.00006	0.00137	6.54901	-0.20774									Objective function	166.85034				
10	11/01/08	16.7686	-0.0299	0.00090	0.00136	5.94035	-0.81093														
11	14/01/08	17.3600	0.0353	0.00124	0.00136	5.68612	0.95793														
12	15/01/08	16.4142	-0.0545	0.00297	0.00135	4.40678	-1.48417														
13	16/01/08	15.5014	-0.0556	0.00309	0.00134	4.30711	-1.51928	0.03365	1												= $SUM(G3:G32)$
14	17/01/08	15.6228	0.0078	0.00006	0.00133	6.57508	0.21454	0.04849	0	0	0	1	0			Alpha	0.3				
15	18/01/08	15.6684	0.0029	0.00001	0.00132	6.62046	0.08028	0.04835	0	1	0	0	0			Number of returns	30				
16	22/01/08	15.1130	-0.0354	0.00126	0.00132	5.67827	-0.97694	0.03336	1	0	1	0	0			History window	10				
17	23/01/08	13.5040	-0.1065	0.01133	0.00131	-2.02030	-2.94258	0.03523	1	0	0	0	1			n	20				= $Q15-Q16$
18	24/01/08	13.1671	-0.0250	0.00062	0.00130	6.16590	-0.69164	0.04805	0	0	0	1	0			Expected	6				= $Q17*Q14$
19	25/01/08	12.6243	-0.0412	0.00170	0.00129	5.33668	-1.14603	0.04701	0	1	0	0	0			n1	5				= $SUM(J13:J32)$
20	28/01/08	12.6243	0.0000	0.00000	0.00129	6.65585	0.00000	0.04959	0	1	0	0	0			n0	15				= $Q17-Q20$
21	29/01/08	12.7728	0.0118	0.00014	0.00128	6.55336	0.32906	0.04945	0	1	0	0	0			Kupiec test	0.25				= $2*Q21*LN(Q21/Q17)+2*Q20*LN(Q20/Q17)-2*Q21*LN(1-Q14)-2*Q20*LN(Q14)$
22	30/01/08	12.8350	0.0049	0.00002	0.00127	6.64882	0.13643	0.04931	0	1	0	0	0			P-value	62.0%				= $1-CHISQ.DIST(Q22,1,1)$
23	31/01/08	13.1438	0.0241	0.00058	0.00126	6.21542	0.67661	0.03895	0	1	0	0	0								
24	01/02/08	12.9874	-0.0119	0.00014	0.00126	6.56646	-0.33548	0.03160	0	1	0	0	0								
25	04/02/08	12.7835	-0.0157	0.00025	0.00125	6.48755	-0.44413	0.03151	0	1	0	0	0								
26	05/02/08	12.5612	-0.0174	0.00030	0.00124	6.44708	-0.49347	0.03142	0	1	0	0	0								
27	06/02/08	11.8465	-0.0569	0.00324	0.00124	4.07605	-1.61874	0.02222	1	0	1	0	0								
28	07/02/08	11.7727	-0.0062	0.00004	0.00123	6.67057	-0.17775	0.02216	0	0	0	1	0								
29	08/02/08	12.1844	0.0350	0.00122	0.00122	5.70642	1.00077	0.01673	0	1	0	0	0			n00	11				= $SUM(L14:L32)$
30	11/02/08	12.5699	0.0316	0.00100	0.00121	5.88927	0.90801	0.01434	0	1	0	0	0			n01	3				
31	12/02/08	12.1242	-0.0355	0.00126	0.00121	5.67799	-1.02056	0.01430	1	0	1	0	0			n10	4				
32	13/02/08	12.5650	0.0364	0.00132	0.00120	5.62369	1.04959	0.01658	0	0	0	1	0			n11	1				
33																$\pi_0$	0.2143				= $SUM(M14:M32)$
34																$\pi_1$	0.2				= $SUM(N14:N32)$
35																$\pi$	0.2105				= $Q29/(Q28+Q29)$
36																Christoffersen test	0.0046				= $Q31/(Q30+Q31)$
37																p-value	94.62%				= $(Q29+Q31)/(Q28+Q29+Q30+Q31)$
38																					
39																					
40																					
41																					
42																					



Figure 4.2 Solver – the GARCH model



### 4.2.2 Empirical Study

Following the procedure of simplified example, we can do the statistical testing by means of filtered historical simulation for AAPL from December 3, 2007 to December 29, 2017. In the empirical study, we mainly state the value of parameters by means of *Solver*, and verify the accuracy of VaR estimation utilizing filtered historical simulation based on different length of historical windows at different significance levels. Then, we show VaR estimation in figure 4.3. Focusing on estimated parameters, we calculate them by means of *Solver*. The results are shown as table 4.8. The numbers of obtained exception are shown in table 4.9 and p-values of Kupiec test and Christoffersen test are shown in table 4.10 and table 4.11 respectively.

Table 4.8 Parameters estimation

Parameters	Value
$\omega$	0.000012
$\alpha$	0.090008
$\beta$	0.896438

Table 4.9 The numbers of exceptions for filtered historical simulation

Exceptions		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Expected		302.7	201.8	100.9	50.45	20.18	10.09
Observed	$m = 21$	302***	197***	100***	125	107	100
	$m = 42$	301***	207***	94***	59***	63	61
	$m = 63$	302***	209***	101***	47***	49	39
	$m = 125$	311***	196***	101***	50***	20***	25
	$m = 250$	297***	196***	103***	49***	21***	9***
	$m = 500$	280***	186***	101***	50***	19***	10***

P-values of Kupiec test are higher than: 5%\*, 10%\*\* , 15%\*\*\*

Table 4.10 P-values of Kupiec test.

P-value (K-test)		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Observed	$m = 21$	96.5%	72.1%	92.7%	0.0%	0.0%	0.0%
	$m = 42$	91.6%	70.1%	47.6%	23.5%	0.0%	0.0%
	$m = 63$	96.5%	59.5%	99.2%	61.9%	0.0%	0.0%
	$m = 125$	60.6%	66.6%	99.2%	94.9%	96.8%	0.0%
	$m = 250$	72.2%	66.6%	83.1%	83.5%	85.5%	72.6%
	$m = 500$	15.3%	23.5%	99.2%	94.9%	79.0%	97.7%

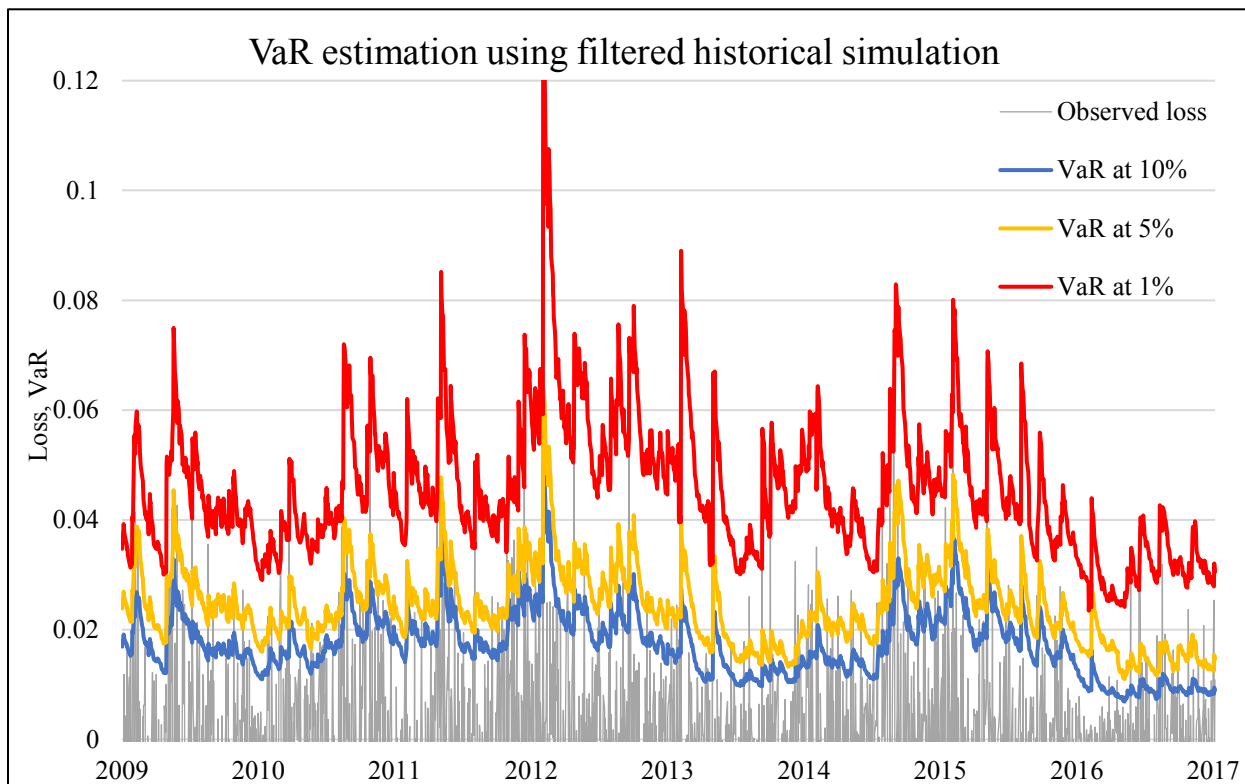
As can be seen from the table 4.9, most of the numbers of exceptions can be statistically accepted based on Kupiec test. Note that the rejected numbers of exceptions are calculated by function PERCENTILE.INC, which is not working as we needs. This point we already discussed in chapter 4.1.2. If we ignore the results which are calculated by PERCENTILE.INC, all the

numbers of exceptions can be statistically accepted. Moreover, the number of exceptions do not rely on the chosen length of historical window.

Table 4.11 P-values of Christoffersen test.

P-value (C-test)		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Observed	$m = 21$	1.1%	3.6%	18.3%	23.7%	17.1%	18.3%
	$m = 42$	0.4%	0.7%	1.4%	12.4%	5.9%	15.3%
	$m = 63$	0.4%	0.5%	20.0%	12.1%	15.0%	22.2%
	$m = 125$	0.0%	0.1%	20.0%	4.1%	52.7%	42.8%
	$m = 250$	0.0%	0.0%	11.5%	85.4%	50.6%	77.6%
	$m = 500$	0.1%	0.0%	9.3%	82.0%	54.8%	75.2%

Figure 4.3 VaR estimation using filtered historical simulation



Compared with the results of historical simulation, the results of filtered historical simulation indicate the greater accuracy, because the numbers of observed exceptions are closer to the expected values and the most of p-values are higher in general. The Kupiec test is more reflected point than Christoffersen test. For visual representation, we choose the historical window of 250

days to make a figure. The figure 4.2 shows the visual representation of the VaR estimated for AR(1)-GARCH(1,1) historical simulation with 250 days of historical window at significance level of 10%, 5% and 1%.

According to figure 4.3, the trend of VaR curves are almost the same as trend of observed loss. The degree of fitting is high. That means the model reacts to the volatility quickly. So, in our case, the AR(1)-GARCH(1,1) historical simulation is a great accurate model for VaR estimation.

### **4.3 Normal Distribution**

The theoretical part of analytical solution for normal distribution was described in chapter 2.6.3. In this chapter, we introduce the application of normal distribution on VaR estimation and its backtesting result. Similarly to previous chapter, we utilize the same period historical data of AAPL to represent procedure on simplified example and empirical study.

#### **4.3.1 Simplified Example**

Similarly to previous chapter, the historical data of simplified example is AAPL form December 31, 2007 to February 13, 2008. The historical window is 10 days, and significance level we choose 30%. In the simplified example, we mainly state the formulations that we need to utilize, the procedure and Excel functions, and explain briefly the result.

##### **A Mathematical formulation**

According to formulas (2.20), (2.34), (3.1), (3.7), (3.8), (3.12), (3.14) we can list the mathematical formulations in the table 4.12.

In the table 4.12, the  $E(R)$  is expected returns,  $\sigma(R)$  is standard deviation of asset return,  $\Phi_{\alpha}^{-1}$  is an inverse function of normal cumulative distribution function on a given significance level  $\alpha$ .

##### **C Procedure**

1. Calculating the daily returns according to formula (E1).
2. Calculating the forecast VaR according to formula (E2). The inverse function of standard normal distribution on given significance level  $\alpha$ ,  $\Phi_{\alpha}^{-1}$ , by means of the Excel function NORM.S.INV. The mean value  $E(R)$  is calculated by Excel function AVERAGE. The

standard deviation  $\sigma(R)$  is calculated by Excel function STDEV.S.

3. Calculating whether the true loss is higher than VaR estimation or not according to (E3). We can utilize the Excel function IF.
4. Calculating likelihood ratio of Kupiec test and Christoffersen test, according to (E4) and (E6).
5. Calculating the p-values of Kupiec test and Christoffersen test, respectively. They can be calculated using the CHISQ.DIST function in Excel. Input arguments are likelihood ratio and degree freedom of 1. The cumulative distribution function is selected.

Table 4.12 Mathematical formulation by means of normal distribution

Daily return	$R_i = \frac{P_{i,t}}{P_{i,t-1}} - 1.$	(E1)
VaR estimation	$VaR_{R,\alpha} = -E(R) - \Phi_{\alpha}^{-1} \cdot \sigma(R)$	(E2)
Logical values that the observed loss exceed estimated loss	$I_t = \begin{cases} 1, & \text{if } L_R(t; t + \Delta t) > VaR_{R,\alpha}(t; t + \Delta t) \\ 0, & \text{if } L_R(t; t + \Delta t) \leq VaR_{R,\alpha}(t; t + \Delta t) \end{cases}$	(E3)
Likelihood ratio of Kupiec test	$LR_{Kupiec} = 2n_0 \cdot \log\left(\frac{n_0}{n}\right) + 2n_1 \cdot \log\left(\frac{n_1}{n}\right) - 2n_0 \cdot \log(1 - \alpha) - 2n_1 \cdot \log(\alpha),$	(E4)
where	$n = n_0 + n_1.$	(E5)
Likelihood ratio of Christoffersen test	$LR_{ind} = -2 \cdot \{[(n_{01} + n_{11}) \cdot \ln(\pi) + (n_{00} + n_{10}) \cdot \ln(1 - \pi)] - [n_{01} \cdot \ln(\pi_0) + n_{00} \cdot \ln(1 - \pi_0) + n_{11} \cdot \ln(\pi_1) + n_{10} \cdot \ln(1 - \pi_1)]\}.$	(E6)
where	$\pi_0 = \frac{n_{01}}{n_{00} + n_{01}}, \pi_1 = \frac{n_{11}}{n_{10} + n_{11}}, \pi = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}},$	(E7)
Chi-squared distribution	$LR \sim X^2(k).$	(E8)

Table 4.13 VaR estimation and its backtesting using normal distribution

1	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
1	Date	Adj Close	Return	VaR	Exceptions	n00	n01	n10	n11								
2	31/12/07	19.23404										Alpha	0.3				
3	02/01/08	18.91942	-0.01636									Number of returns	30				
4	03/01/08	18.92817	0.000462									History window	10				
5	04/01/08	17.48328	-0.07634									n	20				
6	07/01/08	17.24926	-0.01339									Expected	6				
7	08/01/08	16.62878	-0.03597									n1	5				
8	09/01/08	17.42016	0.047591									n0	15				
9	10/01/08	17.28617	-0.00769									Kupiec test	0.25				
10	11/01/08	16.76861	-0.02994									p-value	62%				
11	14/01/08	17.35996	0.035265														
12	15/01/08	16.41419	-0.05448														
13	16/01/08	15.50142	-0.05561	0.034803	1												
14	17/01/08	15.6228	0.00783	0.039848	0	0	0	1	0								
15	18/01/08	15.66844	0.002921	0.039356	0	1	0	0	0								
16	22/01/08	15.11301	-0.03545	0.028678	1	0	1	0	0								
17	23/01/08	13.50403	-0.10646	0.031355	1	0	0	0	1								
18	24/01/08	13.16708	-0.02495	0.043913	0	0	0	1	0								
19	25/01/08	12.62428	-0.04122	0.047782	0	1	0	0	0								
20	28/01/08	12.62428	0	0.050935	0	1	0	0	0								
21	29/01/08	12.77285	0.011768	0.048539	0	1	0	0	0								
22	30/01/08	12.83499	0.004865	0.049062	0	1	0	0	0								
23	31/01/08	13.14378	0.024058	0.043294	0	1	0	0	0								
24	01/02/08	12.98744	-0.01189	0.035801	0	1	0	0	0								
25	04/02/08	12.78353	-0.0157	0.037331	0	1	0	0	0								
26	05/02/08	12.56116	-0.01739	0.038838	0	1	0	0	0								
27	06/02/08	11.84649	-0.0569	0.036808	1	0	1	0	0								
28	07/02/08	11.77269	-0.00623	0.025544	0	0	0	1	0								
29	08/02/08	12.1844	0.034972	0.023502	0	1	0	0	0								
30	11/02/08	12.5699	0.031639	0.016586	0	1	0	0	0								
31	12/02/08	12.1242	-0.03546	0.014634	1	0	1	0	0								
32	13/02/08	12.56505	0.036361	0.020261	0	0	0	1	0								

### C Solution and interpretation

Table 4.13 shows the procedure and result of VaR estimation and its backtesting by means of normal distribution. From this table, we can see there are 5 days that the observed loss exceeds VaR estimation. For backtesting, the number of observed exceptions are 5, and our expected value is 6. The p-value of Kupiec test and Christoffersen test are 62% and 94.62%. Although the number of observed exceptions is the same as in previous two simplified examples, the values of VaR estimation are different.

#### 4.3.2 Empirical Study

Following the procedure of simplified example, we can do the statistical testing by means of normal distribution for AAPL from December 3, 2007 to December 29, 2017. The obtained numbers of exceptions are shown in table 4.14 and p-values calculated by Kupiec test and Christoffersen test are shown in table 4.15 and 4.16, respectively. In the empirical study, we mainly verify the accuracy of VaR estimation by means of normal distribution based on different length of historical windows at different significance levels. Then, we show the different VaR estimation in figure 4.4 and 4.5, according to the different length of historical windows.

Table 4.14 The numbers of exceptions for normal distribution

Exceptions		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Expected		302.7	201.8	100.9	50.45	20.18	10.09
Observed	$m = 21$	264	196***	120*	78	54	41
	$m = 42$	254	180**	107***	71	40	31
	$m = 63$	232	170	100***	71	35	25
	$m = 125$	238	153	92***	57***	34	26
	$m = 250$	238	159	91***	56***	33	25
	$m = 500$	212	153	75	43***	22***	18

P-values of Kupiec test are higher than: 5%\*, 10%\*\* , 15%\*\*\*

From table 4.14 and 4.15, we find that the normal distribution is a poor accuracy model for VaR estimation in our case. For all length of historical windows, all the VaR estimation are

overestimating with 15% and underestimating with 0.5%. Only the following VaR estimation can be statistically accepted:

- VaR estimation from 21 and 42 days with significance level of 10%.
- VaR estimation from 21 days or longer historical windows, except 500 days with significance level of 5%.
- VaR estimation from 125 days or longer historical windows with significance level of 2.5%.
- VaR estimation from 500 days with significance level of 1%.

Therefore, we can draw a conclusion that the VaR estimation by means of normal distribution is only statistically accepted with some specific conditions and the results indicate a poor accuracy of the model.

Table 4.15 P-values of Kupiec test

P-value (K-test)		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Observed	$m = 21$	1.4%	66.6%	5.8%	0.0%	0.0%	0.0%
	$m = 42$	0.2%	10.0%	53.7%	0.6%	0.0%	0.0%
	$m = 63$	0.0%	1.6%	92.7%	0.6%	0.3%	0.0%
	$m = 125$	0.0%	0.0%	35.6%	36.0%	0.5%	0.0%
	$m = 250$	0.0%	0.1%	30.4%	43.7%	0.9%	0.0%
	$m = 500$	0.0%	0.0%	0.6%	27.6%	68.8%	2.5%

Table 4.16 P-values of Christofferson test

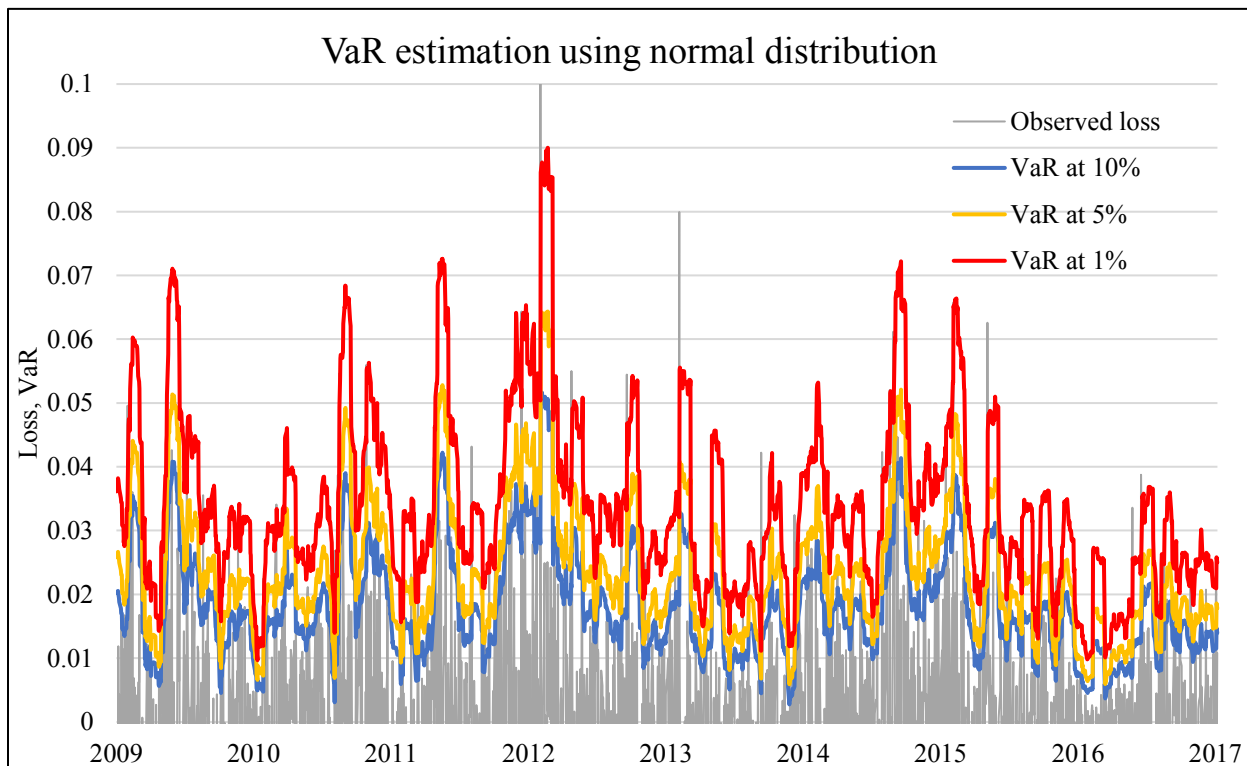
P-value (C-test)		$\alpha = 0.15$	$\alpha = 0.1$	$\alpha = 0.05$	$\alpha = 0.025$	$\alpha = 0.01$	$\alpha = 0.005$
Observed	$m = 21$	0.4%	0.5%	7.5%	57.3%	68.7%	85.7%
	$m = 42$	0.0%	0.0%	0.2%	1.4%	24.3%	50.0%
	$m = 63$	0.0%	0.1%	0.1%	1.4%	14.8%	31.7%
	$m = 125$	0.1%	0.1%	0.3%	2.6%	13.2%	34.5%
	$m = 250$	0.0%	1.6%	0.9%	8.9%	56.8%	42.8%
	$m = 500$	0.0%	0.0%	8.0%	93.0%	48.6%	56.9%



Similarly to previous chapter, table 4.16 shows the p-value of Christofferson test. For all length of historical windows, the empirical results are rejected with significance level of 15% and 10%. Only the following VaR estimation can be statistically accepted:

- VaR estimation from 21 and 500 days with significance level of 5%, but the accuracy of them are poor.
- VaR estimation form 21 days, 250 and 500 days with significance level of 2.5%.
- For significance level of 1% and 0.5%, the VaR estimation from all length of historical windows.

Figure 4.4 VaR estimation using normal distribution with historical window of 21 days

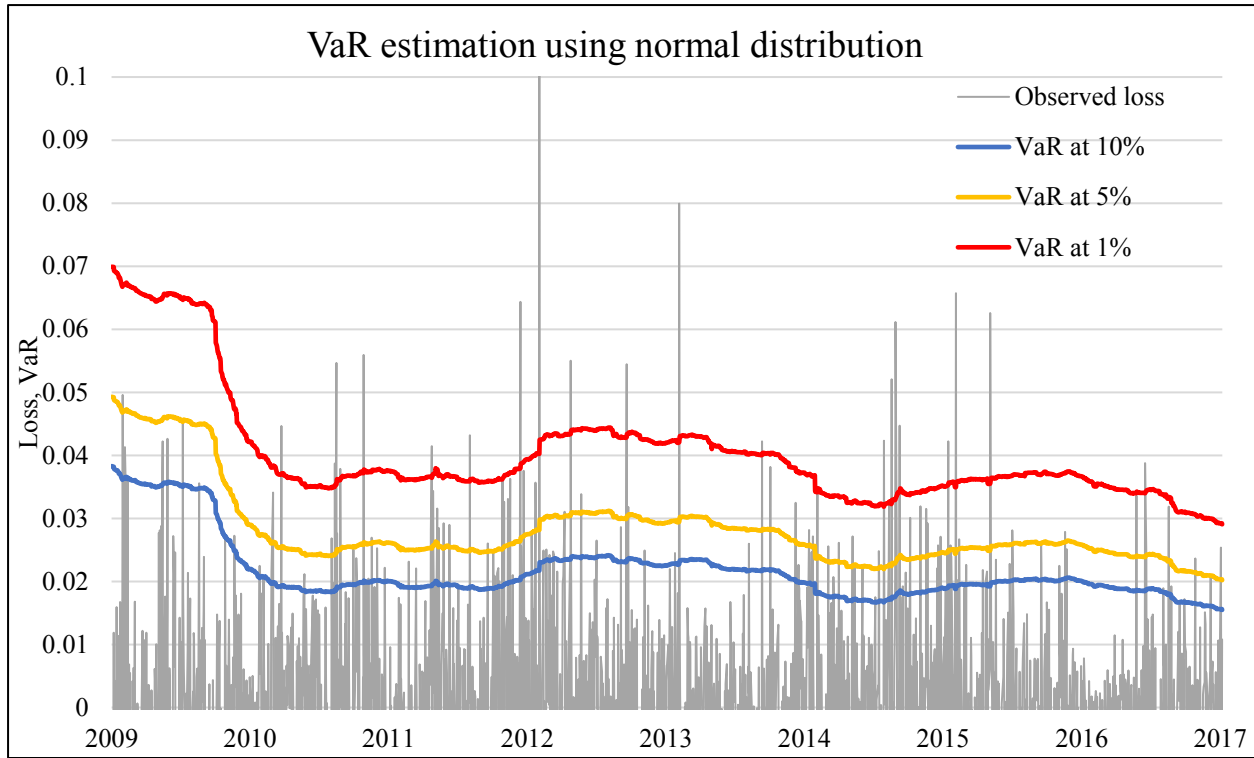


Moreover, the degree of accuracy of this model depends on the length of historical window by means of normal distribution. So, we make the figure 4.4 and 4.5 to comparing the VaR estimation at given significance level with different length of historical window.

Compared figure 4.4 and 4.5, we can see that the VaR estimation is more volatile with historical window of 21 days. The curve of VaR estimation with historical window of 500 days is more stable than 21 days. Therefore, we can draw a conclusion that in general, the longer the

historical windows, the more stable VaR estimation by means of normal distribution.

Figure 4.5 VaR estimation using normal distribution with historical window of 500 days



#### 4.4 Student's $t$ -distribution

The theoretical part of analytical solution for Student distribution is described in chapter 2.6.3. For the small significance level, we need Student's  $t$ -distribution, which can reflect the situation of heavier tail. According to the empirical results of normal distribution, we can find the most VaR estimation are rejected at the smaller significance level (i.e. 1% and 0.5%). Therefore, for empirical study, Student's  $t$ -distribution is applied with 0.5% significance level.

##### 4.4.1 Simplified Example

Similarly to previous chapter, the historical data of simplified example is AAPL from December 31, 2007 to February 13, 2008. The historical window is 10 days, and significance level we choose 30%, and we suppose that the degree of freedom is one in our simplified example.

##### A Mathematical formulation

According to formulas (2.20), (2.34), (3.1), (3.7), (3.8), (3.12), (3.14) we can list the

mathematical formulation in the table 4.17.

Table 4.17 Mathematical formulation by means of Student  $t$ -distribution

Daily return	$R_i = \frac{P_{i,t}}{P_{i,t-1}} - 1.$	(E1)
VaR estimation	$VaR_{R,\alpha} = -E(R) - F_{\alpha}^{-1}(v) \cdot \sigma(R)$	(E2)
Logical values that the observed loss exceed estimated loss	$I_t = \begin{cases} 1, & \text{if } L_R(t; t + \Delta t) > VaR_{R,\alpha}(t; t + \Delta t) \\ 0, & \text{if } L_R(t; t + \Delta t) \leq VaR_{R,\alpha}(t; t + \Delta t) \end{cases}$	(E3)
Likelihood ratio of Kupiec test	$LR_{Kupiec} = 2n_0 \cdot \log\left(\frac{n_0}{n}\right) + 2n_1 \cdot \log\left(\frac{n_1}{n}\right) - 2n_0 \cdot \log(1 - \alpha) - 2n_1 \cdot \log(\alpha),$	(E4)
where	$n = n_0 + n_1.$	(E5)
Likelihood ratio of Christoffersen test	$LR_{ind} = -2 \cdot \{[(n_{01} + n_{11}) \cdot \ln(\pi) + (n_{00} + n_{10}) \cdot \ln(1 - \pi)] - [n_{01} \cdot \ln(\pi_0) + n_{00} \cdot \ln(1 - \pi_0) + n_{11} \cdot \ln(\pi_1) + n_{10} \cdot \ln(1 - \pi_1)]\}.$	(E6)
where	$\pi_0 = \frac{n_{01}}{n_{00} + n_{01}}, \pi_1 = \frac{n_{11}}{n_{10} + n_{11}}, \pi = \frac{n_{01} + n_{11}}{n_{00} + n_{01} + n_{10} + n_{11}},$	(E7)
Chi-squared distribution	$LR \sim X^2(k).$	(E8)

In the table 4.17,  $E(R)$  is expected returns,  $\sigma(R)$  is standard deviations of asset return,  $F_{\alpha}^{-1}(v)$  is an inverse function of cumulative Student  $t$ -distribution function on a given significance level  $\alpha$  with freedom degree of  $v$ .

### C Procedure

1. Calculating the daily return according to formula (E1).
2. Calculating the forecast VaR according to formula (E2).  $F_{\alpha}^{-1}(v)$  is an inverse function of cumulative Student  $t$ -distribution on given significance level  $\alpha$ , by means of the Excel function

Table 4.17 VaR estimation and its backtesting using Student *t*-distribution

1	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q
	Date	Adj Close		Return	VaR	Exceptions	n00	n01	n10	n11							
2	31/12/07	19.23404			$= (B3-B2)/B2$							v	1				
3	02/01/08	18.91942		-0.01636								Alpha	0.3				
4	03/01/08	18.92817		0.000462	$= -\text{AVERAGE}(D3:D12) - \text{T.INV}(\$M\$3, \$M\$2) * \text{STDEV.S}(D3:D12)$							Number of returns	30				
5	04/01/08	17.48328		-0.07634								History window	10	$= M4 - M5$			
6	07/01/08	17.24926		-0.01339								n	20	$= M6 * M3$			
7	08/01/08	16.62878		-0.03597								Excepted	6				
8	09/01/08	17.42016		0.047591								n1	4	$= \text{SUM}(F13:F32)$			
9	10/01/08	17.28617		-0.00769		$= \text{IF}(-D13 > E13, 1, 0)$						n0	16	$= M6 - M9$			
10	11/01/08	16.76861		-0.02994								Kupiec test	1.03				
11	14/01/08	17.35996		0.035265								p-value	31%	$= 2 * M10 * \text{LN}(M10/M6) + 2 * M9 * \text{LN}(M9/M6) - 2 * M10 * \text{LN}(1-M3) - 2 * M9 * \text{LN}(M3)$			
12	15/01/08	16.41419		-0.05448										$= 1 - \text{CHISQ.DIST}(M11, 1, 1)$			
13	16/01/08	15.50142		-0.05561	0.042403	1											
14	17/01/08	15.6228		0.00783	0.04788	0	0	0	1	0							
15	18/01/08	15.66844		0.002921	0.047482	0	1	0	0	0							
16	22/01/08	15.11301		-0.03545	0.035744	0	1	0	0	0							
17	23/01/08	13.50403		-0.10646	0.038602	1	0	1	0	0							
18	24/01/08	13.16708		-0.02495	0.053285	0	0	0	1	0							
19	25/01/08	12.62428		-0.04122	0.055848	0	1	0	0	0							
20	28/01/08	12.62428		0	0.058923	0	1	0	0	0							
21	29/01/08	12.77285		0.011768	0.056758	0	1	0	0	0							
22	30/01/08	12.83499		0.004865	0.056578	0	1	0	0	0							
23	31/01/08	13.14378		0.024058	0.050873	0	1	0	0	0							
24	01/02/08	12.98744		-0.01189	0.043563	0	1	0	0	0							
25	04/02/08	12.78353		-0.0157	0.044922	0	1	0	0	0							
26	05/02/08	12.56116		-0.01739	0.046293	0	1	0	0	0							
27	06/02/08	11.84649		-0.0569	0.044176	1	0	1	0	0							
28	07/02/08	11.77269		-0.00623	0.030481	0	0	0	1	0							
29	08/02/08	12.1844		0.034972	0.028373	0	1	0	0	0							
30	11/02/08	12.5699		0.031639	0.021729	0	1	0	0	0							
31	12/02/08	12.1242		-0.03546	0.020244	1	0	1	0	0							
32	13/02/08	12.56505		0.036361	0.026219	0	0	0	1	0							

3. T.INV with degree of freedom of one. The mean value  $E(R)$  is calculated by Excel function AVERAGE. The standard deviation  $\sigma(R)$  is calculated by Excel function STDEV.S.
4. Calculating whether the true loss is higher than VaR estimation or not according to (E3). We can utilize the function IF.
5. Calculating likelihood ratio of Kupiec test and Christoffersen test according to (E4) and (E6).
6. Calculating the p-value of Kupiec test and Christoffersen test. It can be calculated using the CHISQ.DIST function in Excel. Input arguments are likelihood ratio and degree freedom of 1, and cumulative distribution function is selected.

### **C Solution and interpretation**

Table 4.17 shows the procedure and result of VaR estimation and its backtesting by means of Student  $t$ -distribution. From this table, we can see there are four days that the observed loss exceeds VaR estimation. They occurred on January 16, 23, and February 6 and 12, 2008. For backtesting, the numbers of observed exception are 4, and our expected value is 6. The p-values of Kupiec test and Christoffersen test are 31% and 21.14%.

#### **4.4.2 Empirical Study**

Following the procedure of simplified example, we can do the statistically testing by means of Student  $t$ -distribution for AAPL from December 3, 2007 to December 29, 2017. The obtained numbers of exception are shown in table 4.18. The p-values of Kupiec test and Christoffersen test are shown in table 4.19 and table 4.20. From the empirical result of normal distribution, the numbers of observed exceptions are totally rejected with significance level of 0.5%. So, we choose the significance level of 0.5% for empirical study utilizing Student  $t$ -distribution. In this way, we can observe the small numbers of exceptions. As we mentioned in chapter 2.6.3, as the degree of freedom  $\nu$  increase, the Student  $t$ -distribution becomes closer to normal distribution. In this empirical study, we mainly verify which value of degree of freedom is best suited for our case by means of Student  $t$ -distribution.

According to the table 4.18, we can see that for the same significance level, there are the different number of observed exceptions based on different degrees of freedom. Due to the same

significance level of 0.5%, the number of expected exceptions are the same, approximately 10 days. As we can see, when degree of freedom is 3, the VaR is overestimated. For degree of freedom of 5 and 9, the most numbers of observed exceptions are accepted. The most accurate model is with degree of freedom of 7. We can see the table 4.19 that all p-values are higher than 15% with degree of freedom of 7 at 0.5% significance level.

Table 4.18 The numbers of exception for Student *t*-distribution

Exceptions		$\nu = 3$	$\nu = 5$	$\nu = 7$	$\nu = 9$
Expected		10.09	10.09	10.09	10.09
Observed	$m = 21$	2	7***	13***	16*
	$m = 42$	1	6***	10***	16*
	$m = 63$	1	5*	12***	15**
	$m = 125$	1	5*	11***	15**
	$m = 250$	1	4	10***	10***
	$m = 500$	1	5*	8***	9***

P-values of Kupiec test are higher than: 5%\*, 10%\*\* , 15%\*\*\*

Table 4.19 P-values of Kupiec test.

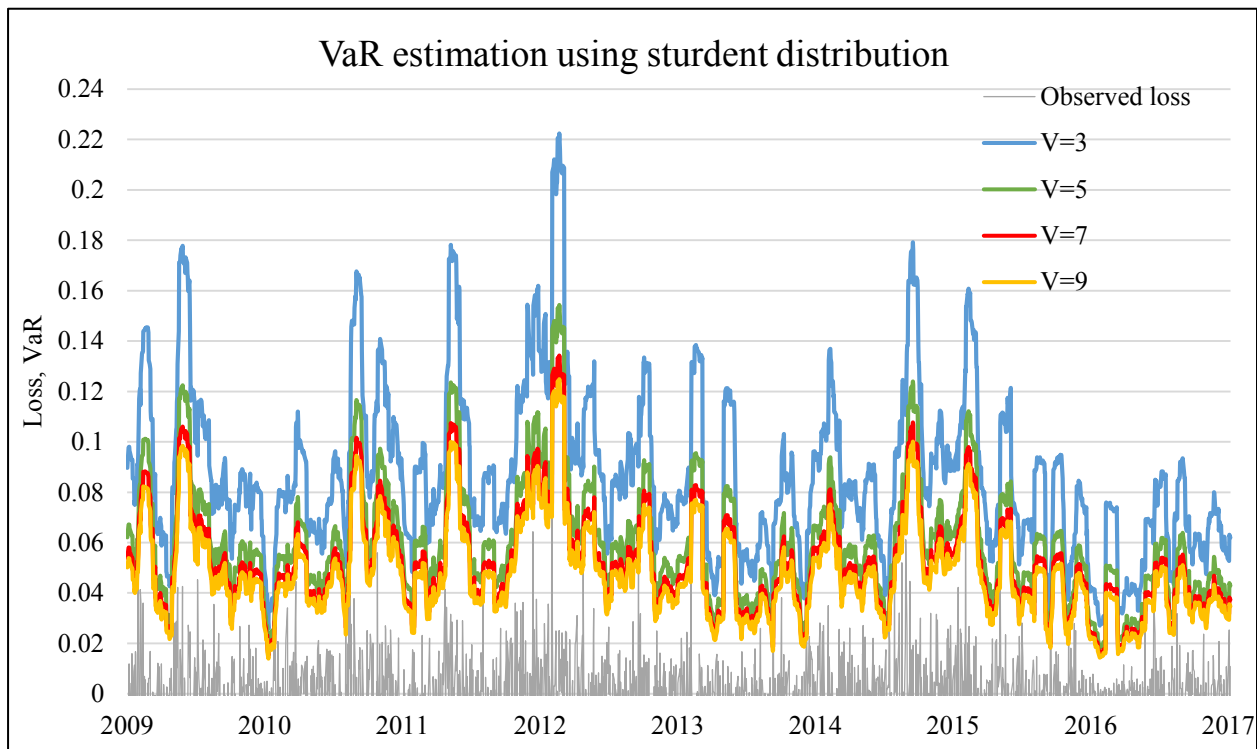
P-value (K-test)		$\nu = 3$	$\nu = 5$	$\nu = 7$	$\nu = 9$
Observed	$m = 21$	0.2%	30.2%	37.9%	8.6%
	$m = 42$	0.0%	16.2%	97.7%	8.6%
	$m = 63$	0.0%	7.5%	55.8%	14.9%
	$m = 125$	0.0%	7.5%	77.7%	14.9%
	$m = 250$	0.0%	2.9%	97.7%	97.7%
	$m = 500$	0.0%	7.5%	49.4%	72.6%

From the table 4.20, we can see that all the numbers of observed exceptions are statistically accepted by means of Christoffersen test, and all the p-values are higher than 15%, which shows a great accuracy. Moreover, the number of exceptions do not rely on the chosen length of historical window.

Table 4.20 P-values of Christoffersen test.

P-value (C-test)		$\nu = 3$	$\nu = 5$	$\nu = 7$	$\nu = 9$
Observed	$m = 21$	95.0%	82.5%	68.1%	61.3%
	$m = 42$	97.5%	85.0%	75.2%	61.3%
	$m = 63$	97.5%	87.5%	70.5%	63.5%
	$m = 125$	97.5%	87.5%	72.8%	63.5%
	$m = 250$	97.5%	90.0%	75.2%	75.2%
	$m = 500$	97.5%	87.5%	80.1%	77.6%

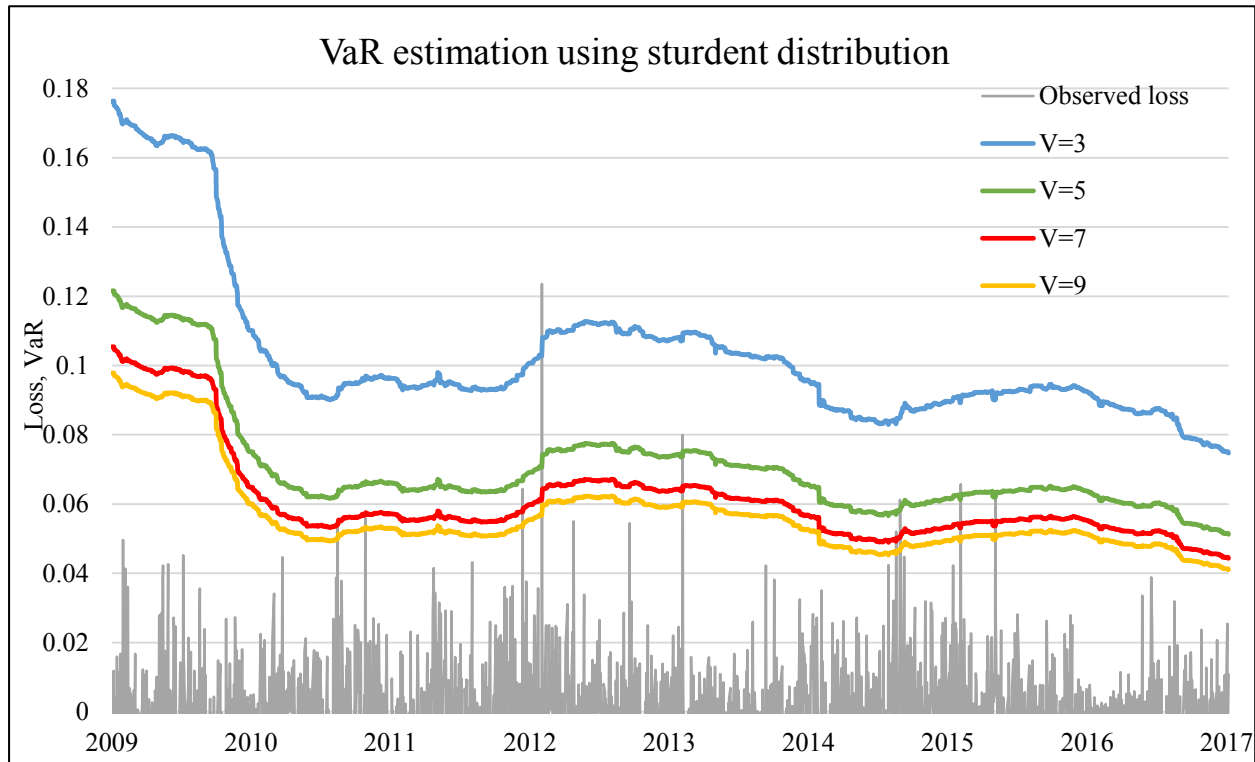
Figure 4.6 VaR estimation using Student  $t$ -distribution ( $\alpha = 0.5\%$ ,  $n = 21$ )



The figure 4.6 shows the VaR estimation with different degrees of freedom from historical window if 21 days. The figure 4.7 shows the VaR estimation with different degrees of freedom from historical window of 500 days. As we can see that for empirical study based on Kupiec test, with significance level of 0.5% the most accurate degree of freedom for VaR estimation by means of Student's  $t$ -distribution is 7. In the figure 4.6 and 4.7, the red curve shows the degree of freedom of 7, which is most accurate one. Based on Christoffersen test, the VaR estimation at all the four

degrees of freedom is accepted and they have great accuracy. Comparing figure 4.6 and 4.7, we can see that the VaR estimation is more fluctuate from historical window of 21 than historical of 500. Therefore, we can draw a conclusion that similarly to normal distribution, the longer the historical window the more stable VaR estimation.

Figure 4.7 VaR estimation using Student  $t$ -distribution ( $\alpha = 0.5\%$ ,  $n = 500$ )





## 5 Conclusion

The objective of the thesis is to verify different VaR estimation approaches by means of backtesting on chosen time series.

In the second chapter, we mainly discuss the historical simulation, filtered historical simulation, analytical solutions with normal distribution and Student  $t$ -distribution and Monte Carlo simulation. The historical simulation is a non-parametrical method for VaR estimation based on historical observed returns, which the procedure is repeated many times using all past returns. The filtered historical simulation overcome the defect of historical simulation that the volatility clustering does not considered. Therefore, the filtered historical simulation method is to utilize these standardized residuals rescaled by the actual volatility. The analytical solution computes an asset's maximum loss based on the price movements of asset over a past period within a probability level. It is based on assumption that the asset returns follow a given probability distribution. In the thesis, we introduced two types probability distributions, normal distribution and Student  $t$ -distribution. Monte Carlo simulation is another method of VaR estimation method. The basis of the Monte Carlo simulation method of calculating VaR that generates random movements in the risk factors by using past events to model future return distribution of an asset. But we do not utilize this method for VaR estimation in empirical study, because of its complexity.

VaR has become one of the most popular methods for measuring market risk. But the worrying issue is the accuracy of VaR estimations. In the third chapter, we described the fundamental properties of an accurate VaR model, unconditional coverage, the independence property and joint properties. The Kupiec's unconditional coverage test and Christoffersen's conditional coverage test are applied in empirical studies.

From the presented backtesting results of particular VaR estimation methods, we can summarize following findings. The most accurate method of VaR estimation is filtered historical simulation. The filtered historical simulation is a semi-parametrical method which is a combination of volatility model and non-parametrical model. The second-best model is historical simulation, which is non-parametrical method. The analytical solution is only suitable for several specific

situations in terms of normal distribution method. The most appropriate situations we already mentioned before. Considering the Student  $t$ -distribution, the most accurate degree of freedom is founded by means of backtesting procedure. The longer the historical window, the more stable VaR estimation.

## Bibliography

### a) Books

- 1) APOSTOLIK, R., C. DONOHUE and P. WENT. *Foundations of Banking Risk: An Overview of Banking, Banking Risks, and Risk-Based Banking Regulation*. Chichester: John Wiley & Sons, 2009. ISBN 978-0470442197
- 2) COX, D.R. and HINKLEY, D.V. *Theoretical Statistics*. London: Chapman & Hall, 1979. ISBN 978-0412161605.
- 3) DOWD, Kevin. *Beyond Value at Risk*. Chichester: John Wiley & Sons, 1999. ISBN 978-0471976226.
- 4) DOWD, Kevin. *Measuring Market Risk*. 2nd ed. Chichester: John Wiley & Sons, 2005. ISBN 978-0470013038.
- 5) GLYN, A. Holton. *Value at Risk: Theory and Practice*, 2nd ed. London: Chapman & Hall, 2009. ISBN 978-1420092523.
- 6) HULL, J. C. *Risk Management and Financial Institutions*. 4th ed. Chichester: John Wiley & Sons, 2012. ISBN 978-1118955949.
- 7) JONHSON, N. L., KOTZ, S. and BALAKRISHNAN, N. *Continuous Univariate Distributions*. 2nd ed. Chichester: John Wiley & Sons, 1994. ISBN 0-471-58495-9.
- 8) JORION, Philippe. *Value at Risk: The New Benchmark for Managing Financial Risk*. 3rd ed. New York: McGraw-Hill, 2006. ISBN 978-0-07-146495-6.
- 9) KRESTA, Aleš. *Financial Engineering in Matlab: Selected Approaches and Algorithms*. Ostrava: VŠB-TU Ostrava, 2015. ISBN 978-80-248-3702-4.
- 10) ZMEŠKAL, Z., D. DLUHOŠOVÁ and T. TICHÝ. *Financial Models*. Ostrava: VŠB-TU Ostrava, 2004. ISBN 978-80-248-0754-8.

### b) Articles

- 1) ARTZNER, P., DELBAEN, F., EBER, J.-M. and HEATH, D. Thinking coherently. *Risk*. 1997, vol. 10, no. 11, pp 68-71. ISSN 0952-8776.

- 2) BARONE-ADESI Giovanni and Kostas GIANNOPOULOS. Non parametric VaR Techniques. Myths and Realities. *Economic Notes*. 2003, vol. 30, no. 2, pp. 167-181. ISSN 1468-0300
- 3) BARONE-ADESI, G., ROBERT E. and LORIANO M. Option Pricing Model with Filtered Historical Simulation. *Review of Financial Studies*. 2008, vol. 21, no. 3, pp. 1223 – 1258. ISSN 0893-9454.
- 4) BOLLERSLEV, Tim. Generalized Autoregressive Conditional Heteroskedasticity. *Journal of Econometrics*. 1986, vol. 31, no. 3, pp. 307–327. ISSN 0304-4076.
- 5) CHRISTOFFERSON, Peter. F. Evaluating interval forecasts. *International Economic Review*. 1998, vol. 39, issue 4, 841-862. ISSN 1468-2354.
- 6) ENGLE, Robert F. Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation. *Econometrica*. 1982, vol. 50, no. 4, pp. 987–1007. ISSN 0012-9682.
- 7) GOSSET, W. (Originally published under the pseudonym "Student") The probable error of a mean. *Biometrika*. 1908, vol. 6, no. 1, pp. 1–25. ISSN 0006-3444.
- 8) KUPIEC, Paul. Techniques for Verifying the Accuracy of Risk Management Models. *Journal of Derivatives*. 1995, vol. 3 no. 2, pp. 73-84. ISSN 10741240.
- 9) LYON, A. Why are Normal Distributions Normal? *The British Journal for the Philosophy of Science*. 2014, vol. 65, no. 3, pp. 621-649. ISSN 0007-0882.
- 10) ROCKAFELLAR, R. T. and URYASEV, S. Conditional Value-at-Risk for general loss distributions. *Journal of Banking & Finance*. 2002, vol. 26, no. 7, pp. 1443-1471. ISSN 0378-4266.
- 11) ROCKAFELLAR, R. T. and URYASEV, S. Optimization of conditional Value-at-Risk. *Journal of Risk*. 2000, vol. 2, no. 3, pp. 21-41. ISSN 1465-1211.
- 12) SENN, S. and RICHARDSON, W. The first t-test. *Statistics in Medicine*. 1994, vol. 13, no. 8, pp. 785–803. ISSN 1097-0258.
- 13) ZOLA, L., C. FERNANDES, A. PIZZINGA and R. RIERA. The log-periodic-AR(1)-

GARCH(1,1) model for financial crashes. *The European Physical Journal B*. 2008, vol. 61, no. 3, pp. 355-362. ISSN 1434-6036.

**c) Diploma thesis**

- 1) Gerald Z. Katsenga. *Value at Risk (VaR) Backtesting 'Evidence from a South African Market Portfolio'*, 2013. University of Witwatersrand Business School.
- 2) KISIALA, Jakob. *Conditional Value-at-Risk: Theory and Applications*, 2015. The University of EDINBURGH.
- 3) LINSMEIER, J. and PEARSON, N.D. *Risk Measurement: An Introduction to Value at Risk*. Illinois, 1996, University of Illinois at Urbana-Champaign, Department of Accountancy and Department of Finance, respectively.
- 4) NIEPPOLA, Olli. *Backtesting value-at-risk models*, 2009. Helsinki School of Economics, Department of Economics.

**d) Electronic documents**

- 1) Basle Committee of Banking Supervision. *Supervisory framework for the use of "backtesting" in conjunction with the internal models approach to market risk capital requirements*. 1996. [Online]. Available on: <https://www.bis.org/>.
- 2) BROWN, A. *Private Profits and Socialized Risk – Counterpoint: Capital Inadequacy*. Global Association of Risk Professionals, June/July 2008 issue. [Online]. Available on: <http://www.garp.org>
- 3) CAMPBELL, Sean D. *A Review of Backtesting and Backtesting Procedure*. Finance and Economics Discussion Series, 2005-21. Board of Governors of the Federal Reserve System (U.S.). [Online]. Available on: <http://www.federalreserve.gov/>
- 4) *Value at Risk – Monte Carlo simulation*, 2002. [Online]. Available on: <http://treasurytoday.com/2002/02/value-at-risk-monte-carlo-simulation>.
- 5) Yahoo Finance. [Online]. Available on: <https://finance.yahoo.com/>.

## List of abbreviations

ARCH	Autoregressive conditional heteroskedasticity
C-test	Christoffersen's conditional coverage test
CVAR	Conditional Value at Risk
FHS	Filtered historical simulation
GARCH	Generalized autoregressive conditional heteroskedasticity
K-test	Kupiec's unconditional coverage test
POF	Proportion of failures
P-value	Probability value
VaR	Value at Risk

## Declaration of Utilisation of Results from the Diploma Thesis

Herewith I declare that

- I am informed that Act No. 121/2000 Coll. – the Copyright Act, in particular, Section 35 Utilisation of the Work as a Part of Civil and Religious Ceremonies, as a Part of School Performances and the Utilisation of a School Work – and Section 60 – School Work, fully applies to my diploma (bachelor) thesis;
- I take account of the VSB – Technical University of Ostrava (hereinafter as VSB-TUO) having the right to utilize the diploma (bachelor) thesis (under Section 35(3)) unprofitably and for own use;
- I agree that the diploma (bachelor) thesis shall be archived in the electronic form in VSB-TUO's Central Library and one copy shall be kept by the supervisor of the diploma (bachelor) thesis. I agree that the bibliographic information about the diploma (bachelor) thesis shall be published in VSB-TUO's information system;
- It was agreed that, in case of VSB-TUO's interest, I shall enter into a license agreement with VSB-TUO, granting the authorization to utilize the work in the scope of Section 12(4) of the Copyright Act;
- It was agreed that I may utilize my work, the diploma (bachelor) thesis or provide a license to utilize it only with the consent of VSB-TUO, which is entitled, in such a case, to claim an adequate contribution from me to cover the cost expended by VSB-TUO for producing the work (up to its real amount).

Ostrava dated 24. 04. 2018

Yuling Li, 李钰灵

Student's name and surname

## **List of Annexes**

- Annex A      Tables of binomial distribution
- Annex B      Figure of stock price of AAPL



## Annex A

Table A-1 Distribution of numbers of exceptions by binomial distribution ( $\pi_{ex} = 0.01, n = 250$ )

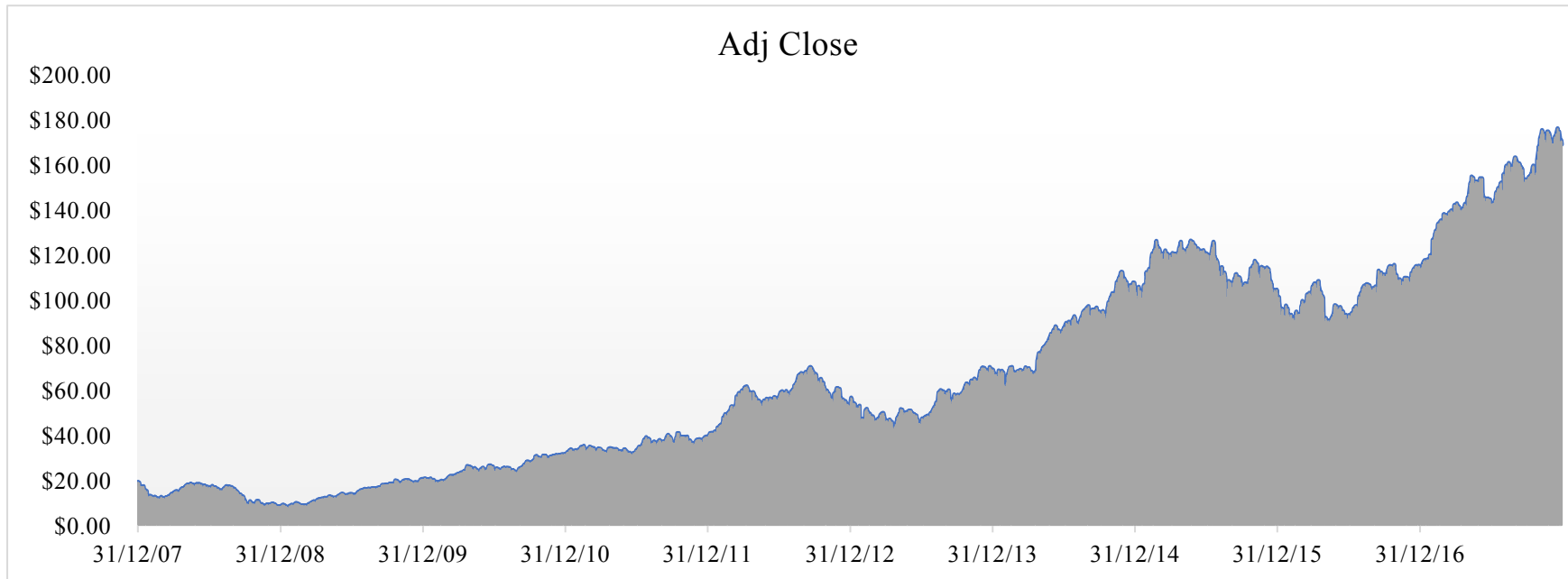
$n_1$	$\Pr(n_1 \pi_{ex}, n)$	$F(n_1 \pi_{ex}, n)$	$1 - F(n_1 \pi_{ex}, n)$
0	8.1%	8.1%	91.9%
1	20.5%	28.6%	71.4%
2	25.7%	54.3%	45.7%
3	21.5%	75.8%	24.2%
4	13.4%	89.2%	10.8%
5	6.7%	95.9%	4.1%
6	2.7%	98.6%	1.4%
7	1.0%	99.6%	0.4%
8	0.3%	99.9%	0.1%
9	0.1%	100.0%	0.0%
10	0.0%	100.0%	0.0%
11	0.0%	100.0%	0.0%
12	0.0%	100.0%	0.0%
13	0.0%	100.0%	0.0%
14	0.0%	100.0%	0.0%
15	0.0%	100.0%	0.0%

Table A-2 Distribution of numbers of exceptions by binomial distribution ( $\pi_{ex} = 0.03, n = 250$ )

$n_1$	$\Pr(n_1 \pi_{ex}, n)$	$F(n_1 \pi_{ex}, n)$	$1 - F(n_1 \pi_{ex}, n)$
0	0.0%	0.0%	100.0%
1	0.4%	0.4%	99.6%
2	1.5%	1.9%	98.1%
3	3.8%	5.7%	94.3%
4	7.2%	12.8%	87.2%
5	10.9%	23.7%	76.3%
6	13.8%	37.5%	62.5%
7	14.9%	52.4%	47.6%
8	14.0%	66.3%	33.7%
9	11.6%	77.9%	22.1%
10	8.6%	86.6%	13.4%
11	5.8%	92.4%	7.6%
12	3.6%	96.0%	4.0%
13	2.0%	98.0%	2.0%
14	1.1%	99.1%	0.9%
15	0.5%	99.6%	0.4%

## Annex B

Figure B. Stock price of AAPL from December 31, 2007 to December 29, 2017



Source: <https://finance.yahoo.com>.