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EXTENDED TRIGONOMETRIC CHEREDNIK ALGEBRAS AND NONSTATIONARY SCHRÖDINGER EQUATIONS WITH DELTA-POTENTIALS

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ABSTRACT. We realize an extended version of the trigonometric Cherednik algebra as affine Dunkl operators involving Heaviside functions. We use the quadratic Casimir element of the extended trigonometric Cherednik algebra to define an explicit nonstationary Schrödinger equation with delta-potential. We use coordinate Bethe ansatz methods to construct solutions of the nonstationary Schrödinger equation in terms of generalized Bethe wave functions. It is shown that the generalized Bethe wave functions satisfy affine difference Knizhnik-Zamolodchikov equations as functions of in the momenta. The relation to the vector valued root system analogs of the quantum Bose gas on the circle with delta-function interactions is indicated.

1. INTRODUCTION

The one dimensional quantum Bose gas with pairwise delta-function interactions [29] is among the first nontrivial quantum integrable systems that were successfully analyzed using the coordinate Bethe ansatz [29, 39, 40]. On the circle the coordinate Bethe ansatz leads to Bethe wave functions, given as explicit plane wave expansions with momenta subject to Bethe ansatz equations, which solve the associated stationary Schrödinger equation. In this paper we will construct Bethe wave type functions solving nonstationary extensions of Schrödinger equations associated to vector valued, root system analogs of the quantum Bose gas on the circle with pairwise delta-function interactions. We will show that the role of the Bethe ansatz equations is taken over by difference analogs of Knizhnik-Zamolodchikov equations.

It is well known that the quantum inverse scattering method can be applied to the one dimensional quantum Bose gas with pairwise delta-function interactions (see [28] and references therein). The key point is the fact that the pertinent quantum Bose gas arises as particle sector of the integrable quantum field theory in $1 + 1$ dimensions governed by the nonlinear Schrödinger equation. The resulting possibility to create and annihilate quantum particles in the quantum Bose gas has in particular led to the explicit evaluation of the norms of the Bethe wave functions (see [27, 28]) in terms of the Hessian of the Yang-Yang action [40].

Quantum Calogero-Moser systems [33, 2], which can be defined for any root system, form an important class of one dimensional integrable quantum many body systems with pairwise interactions of rational, trigonometric, hyperbolic or elliptic type. The quantum

trigonometric Calogero-Moser systems are naturally related to harmonic analysis on symmetric spaces [24]. Substantial progress has been made over the past decades in solving quantum Calogero-Moser systems, with main tool the explicit realization of (degenerate) Hecke algebra symmetries in terms of Dunkl type differential-reflection operators (see, e.g., [8, 22, 35, 7, 5, 2]). These systems do not arise though as particle sectors in particular integrable quantum field theories, hence the application of quantum inverse scattering techniques to such systems is limited. In particular there is no analog of particle creation and annihilation, which for instance explains the completely different techniques in deriving norm formulas; it is based on shift operators or intertwiners in the Hecke algebra context (see, e.g., [34, 7, 36]) and on quantum particle creation/annihilation in the quantum inverse scattering context [27].

The one dimensional quantum Bose gas with pairwise delta-function interactions, which is completely accessible to quantum inverse scattering techniques, can also be naturally viewed as member of the family of integrable quantum Calogero-Moser type systems. Firstly, the one dimensional quantum Bose gas with pairwise delta-function interactions has natural root system generalizations, going back to Gaudin, Gutkin and Sutherland [18, 21, 20]. For classical root systems it corresponds to imposing integrable reflecting boundary conditions on the quantum particles on the line. Secondly, Dunkl type operators and integral-reflection operators have been associated to the one dimensional quantum Bose gas with delta-function interactions and their root system generalizations, which opens the way to apply the Hecke algebra techniques (see, e.g., [20, 37, 23, 12, 13]).

The quantum Bose gas with pairwise delta-function interactions thus is accessible for an unusually large variety of techniques from integrable systems and representation theory. This feature places the one dimensional quantum Bose gas with delta-function interactions center stage of various new developments in mathematical physics and representation theory (see, e.g., [23] and [19] for two striking examples) and leads to the intriguing question how the quantum inverse scattering method and the Hecke algebra method can be united. For instance, conjectures [9] have been put forward extending Korepin's [27] norm formulas to Bethe wave functions of the root system analogs of the quantum Bose gas with pairwise delta-function interactions on the circle. Since Korepin's arguments are no longer applicable in the context of arbitrary root systems by the loss of quantum particle creation and annihilation techniques, it seems to require now Hecke algebra techniques instead to properly understand such norm formulas.

This formed an important motivation for the research leading up to the present work. Our starting point is the observation from [13] that the root system analogs of the quantum Bose gas on the circle with pairwise delta-function interactions arise from a representation of the trigonometric Cherednik algebra *at critical level* in terms of Dunkl type operators. This suggests the possibility to analyze the corresponding Bethe wave functions and norm formulas as limits of Bethe wave type functions associated to arbitrary level. Such an approach has been taken in recent years to successfully analyze the Bethe vectors for Gaudin models and their norms as critical level limit of integral solutions to Knizhnik-Zamolodchikov equations, see [31] and references therein, which in turn has interesting connections to the geometric Langlands correspondence [14].

Another important background for the current paper is the study of Knizhnik-Zamolodchikov-Bernard (KZB) heat equations, cf., e.g., cf. [1, 10, 15]. Correlation functions on the torus satisfy, besides KZB equations, the KZB heat equation. For one-point correlation functions the KZB heat equation is the only equation that remains. In that case it is natural to view the KZB heat equation as a nonstationary Schrödinger equation involving the modular parameter as time variable. At critical level it reduces to the quantum Hamiltonian of a vector valued version of the quantum elliptic Calogero-Moser-Sutherland model (see [10, §4] and [15]). Special solutions are the so called affine Jack polynomials; they can be defined algebraically [11, §6] or be expressed [11, Thm. 7.6] in terms of one-point correlation functions. Cherednik [5, Thm. 4.3] has derived such nonstationary Schrödinger equations using the action of the trigonometric Cherednik algebra by infinite trigonometric Dunkl operators. q-Extensions of various of these results have been obtained, see, e.g., [11, §11], [16, 17] and [6].

Let us describe now the results presented in this paper. In [12, 13] a suitable realization of the trigonometric Cherednik algebra at critical level in terms of affine Dunkl operators involving Heaviside functions enters the study of vector valued root system analogs of the quantum Bose gas on the circle with pairwise delta-function interactions (Dunkl type operators involving Heaviside functions appeared before in [37, 32, 26]). Following the idea of Cherednik [5], we generalize this realization to an extended version of the trigonometric Cherednik algebra at arbitrary level. It contains a quadratic Casimir element which we show to produce an explicit nonstationary Schrödinger equation involving delta-potentials (see Proposition 4.2).

We construct solutions of this nonstationary Schrödinger equation using an affine version of the coordinate Bethe ansatz. The associated generalized Bethe wave functions are defined in terms of an explicit cocycle of the extended affine Weyl group, which is obtained from the normalized intertwiners of the trigonometric Cherednik algebra.

We show that the generalized Bethe wave function satisfies a consistent system of equations as function of the momenta. These equations are expected to be degenerations of Cherednik's [6] affine difference-elliptic quantum Knizhnik-Zamolodchikov equations. These equations replace the requirement for the Bethe wave function of the (vector valued) root system version of the quantum Bose gas on the circle with pairwise delta-function interactions that the momenta satisfy Bethe ansatz type equations (see [12, Thm. 2.6] and [13, Thm. 5.10]).

With suitable restrictions on the momenta (see (3.11) for the explicit requirements) we show that the Bethe wave functions for the vector valued root system analog of the quantum Bose gas on the line (see [23]) are limits of the generalized Bethe wave functions. The limit to critical level is more subtle. We will only make some preliminary comments on it in this paper. A thorough understanding of this limit is expected to lead to new insights on the root system analogs of the quantum Bose gas on the circle with pairwise delta-function potentials, for instance on the quadratic norms of the scalar Bethe wave functions.

2. The extended trigonometric Cherednik algebra

2.1. **Notations.** Let $\mathfrak{g}$ be a complex finite dimensional reductive Lie algebra with $\mathfrak{g}_s := [\mathfrak{g}, \mathfrak{g}]$ simple. Write $\mathfrak{h} = \mathfrak{h}_s \oplus Z(\mathfrak{g})$ with $\mathfrak{h}_s$ a Cartan subalgebra of $\mathfrak{g}_s$ and with $Z(\mathfrak{g})$ the center of $\mathfrak{g}$.

Let $(\cdot, \cdot)_s : \mathfrak{h}_s \times \mathfrak{h}_s \to \mathbb{C}$ be the restriction of the Killing form of $\mathfrak{g}_s$ to $\mathfrak{h}_s \times \mathfrak{h}_s$. Let $R = R(\mathfrak{g}_s, \mathfrak{h}_s) \subset \mathfrak{h}_s$ be the set of roots of $\mathfrak{g}_s$ with respect to $\mathfrak{h}_s$. It is a finite, reduced, irreducible crystallographic root system, with the ambient Euclidean space V_s taken to be the real span of R and with scalar product the restriction of the bilinear form $(\cdot, \cdot)_s$ to $V_s \times V_s$. The root lattice Q (respectively the co-root lattice $Q^\vee$) is the rational integral span of all the roots $\alpha \in R$ (respectively the co-roots $\alpha^\vee$ ($\alpha \in R$)). These are lattices in $\mathfrak{h}$ satisfying $\mathbb{C} \otimes_{\mathbb{Z}} Q^\vee = \mathfrak{h}_s = \mathbb{C} \otimes_{\mathbb{Z}} Q$.

We fix a real form V of $\mathfrak{h}$ of the form $V_s \oplus V'$ with V' a real form of $Z(\mathfrak{g})$. We extend the scalar product $(\cdot, \cdot)_s$ on V_s to a scalar product $(\cdot, \cdot)$ on V such that $V' \perp V_s$. Its complex bilinear extension to a bilinear form on $\mathfrak{h}$ is also denoted by $(\cdot, \cdot)$. We use it to identify the linear dual $\mathfrak{h}^*$ with $\mathfrak{h}$. Set $\mathrm{O}(\mathfrak{h})$ for the group of invertible complex linear endomorphisms of $\mathfrak{h}$ preserving the bilinear form $(\cdot, \cdot)$.

Put $\widetilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c$, $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$, and extend the form $(\cdot, \cdot)$ to a non-degenerate symmetric bilinear form on $\widehat{\mathfrak{h}}$ by requiring

$$(c, d) = 1, \quad (c, c) = (d, d) = (c, v) = (d, v) = 0 \qquad \forall\, v \in \mathfrak{h}.$$

Write $\mathrm{O}(\widehat{\mathfrak{h}}) \subset \mathrm{GL}(\widehat{\mathfrak{h}})$ for the subgroup of invertible linear endomorphisms preserving the bilinear form $(\cdot, \cdot)$ on $\widehat{\mathfrak{h}}$. We identify $\mathrm{O}(\mathfrak{h})$ with the subgroup

$$\{\sigma \in \mathrm{O}(\widehat{\mathfrak{h}}) \mid \sigma(\mathfrak{h}) = \mathfrak{h} \ \& \ \sigma(c) = c, \, \sigma(d) = d\}$$

of $\mathrm{O}(\widehat{\mathfrak{h}})$. For $u \in \mathfrak{h}$ define $t_u \in \mathrm{O}(\widehat{\mathfrak{h}})$ by

(2.1)
$$t_u(v + \eta c + \xi d) = v + \xi u + \big(\eta - \frac{\xi}{2}(u, u) - (v, u)\big)c + \xi d,$$

where $v \in \mathfrak{h}$ and $\eta, \xi \in \mathbb{C}$. The map $\mathfrak{h} \to \mathrm{O}(\widehat{\mathfrak{h}})$ given by $u \mapsto t_u$ is a monomorphism of groups. If $\sigma \in \mathrm{O}(\mathfrak{h})$ then $\sigma \circ t_u = t_{\sigma(u)} \circ \sigma$. We conclude that $\mathrm{O}(\widehat{\mathfrak{h}})$ naturally contains the subgroup $\mathrm{O}(\mathfrak{h}) \ltimes \mathfrak{h}$ of affine linear transformations of $\mathfrak{h}$.

For $a \in \widehat{\mathfrak{h}}$ such that $(a, a) \neq 0$ we set $a^\vee := 2a/(a, a) \in \widehat{\mathfrak{h}}$ and

$$s_a(\widehat{v}) := \widehat{v} - (\widehat{v}, a^\vee)a.$$

Note that $s_a \in \mathrm{O}(\widehat{\mathfrak{h}})$ is an involution. It fixes c if $a \in \widetilde{\mathfrak{h}}$ and also d if $a \in \mathfrak{h}$. Write $W \subset \mathrm{O}(\mathfrak{h})$ for the Weyl group of R generated by s_α ($\alpha \in R$). The lattices Q and $Q^\vee$ are W-invariant.

Set $\widehat{R} = R \oplus \mathbb{Z}c \subseteq \widetilde{\mathfrak{h}}$. It plays the role of the set of real roots of the untwisted affine Lie algebra associated to $\mathfrak{g}_s$, cf. [25]. If we identify $\widetilde{V} = V \oplus \mathbb{R}c$ with the linear functionals on V by interpreting $v + \xi c$ ($v \in V$, $\xi \in \mathbb{R}$) as the affine linear functional mapping v' to

$(v, v') + \xi$, then $\widehat{R}$ is a reduced irreducible affine root system in the sense of Macdonald [30].

The affine Weyl group $\widehat{W}$ of $\widehat{R}$ is defined to be the subgroup of $O(\widehat{\mathfrak{h}})$ generated by s_a $(a \in \widehat{R})$. The affine Weyl group is contained in the subgroup of affine linear transformations of $\mathfrak{h}$ since

$$s_{\alpha + mc} = s_\alpha t_{m\alpha^\vee}, \qquad \alpha \in R, \, m \in \mathbb{Z}.$$

In particular $\widehat{W} \simeq W \ltimes Q^\vee$.

Fix a choice of positive roots R^+ of R and let $F = \{a_1, \ldots, a_n\}$ be the associated set of simple roots. Let θ be the highest root in R^+ and set $a_0 := -\theta + c$. Then $\widehat{F} := \{a_0\} \cup F$ is a set of simple roots of $\widehat{R}$, i.e. every affine root $a \in \widehat{R}$ can uniquely be written as a nonnegative or nonpositive rational integral combination of the a_i's. Denote $\widehat{R}^+$ and $\widehat{R}^-$ for the associated sets of positive and negative affine roots, respectively. The affine Weyl group $\widehat{W}$ is a Coxeter group with Coxeter generators the simple reflections $s_i := s_{a_i}$ $(0 \le i \le n)$. The finite Weyl group $W \subset \widehat{W}$ is the standard parabolic subgroup generated by s_i $(1 \le i \le n)$. Note that $s_0 = s_\theta t_{-\theta^\vee} = t_{\theta^\vee} s_\theta$.

We fix a lattice Y in V containing $Q^\vee$ and satisfying $(Y, Q) \subseteq \mathbb{Z}$. Note that Y is automatically W-invariant. The associated extended affine Weyl group is defined by $\widehat{W}^Y := W \ltimes Y$. It contains the affine Weyl group $\widehat{W}$ as a normal subgroup, and $\widehat{W}^Y/\widehat{W} \simeq Y/Q^\vee$ is abelian.

The affine root system $\widehat{R} \subset \widetilde{\mathfrak{h}}$ and the level ξ hyperplanes

$$(2.2) \qquad \widehat{\mathfrak{h}}_\xi := \widetilde{\mathfrak{h}} + \xi d$$

are $\widehat{W}^Y$-invariant. Furthermore, $\widehat{W}^Y \simeq \Omega \ltimes \widehat{W}$ with $\Omega = \Omega^Y$ the subgroup of $\widehat{W}^Y$ consisting of elements ω such that $\omega(\widehat{R}^+) \subseteq \widehat{R}^+$. The group Ω permutes the simple affine roots.

2.2. **The definition.** The trigonometric Cherednik algebra, also known as the degenerate double affine Hecke algebra, was defined in [5, Def. 1.1]. We use an extended version, defined as follows.

Definition 2.1. *Let $k : \widehat{R} \to \mathbb{C}$ be a $\widehat{W}^Y$-invariant function, which we call a multiplicity function. The extended trigonometric Cherednik algebra $\widehat{H}^Y(k) = \widehat{H}^Y(k; V)$ is the associative unital $\mathbb{C}$-algebra satisfying*

(1) *$\widehat{H}^Y(k)$ contains the symmetric algebra $S(\widehat{\mathfrak{h}})$ and the group algebra $\mathbb{C}[\widehat{W}^Y]$ as subalgebras,*

(2) *the multiplication map defines a linear isomorphism $S(\widehat{\mathfrak{h}}) \otimes_{\mathbb{C}} \mathbb{C}[\widehat{W}^Y] \simeq \widehat{H}^Y(k)$,*

(3) *the following cross relations hold:*

$$(2.3) \qquad \begin{aligned} s_a \cdot \widehat{v} &= s_a(\widehat{v}) \cdot s_a - k_a(a, \widehat{v}), && \forall a \in \widehat{F}, \, \forall \widehat{v} \in \widehat{\mathfrak{h}}, \\ \omega \cdot \widehat{v} &= \omega(\widehat{v}) \cdot \omega, && \forall \omega \in \Omega, \, \forall \widehat{v} \in \widehat{\mathfrak{h}}. \end{aligned}$$

The subalgebra $\widetilde{H}^Y(k)$ of $\widehat{H}^Y(k)$ generated by $S(\widetilde{\mathfrak{h}})$ and $\mathbb{C}[\widehat{W}^Y]$ is the trigonometric Cherednik algebra.

Remark 2.2. **(i)** In the next subsection we verify that $\widehat{H}^Y(k)$ is well defined (see Proposition 2.5).

(ii) The trigonometric Cherednik algebra $\widetilde{H}^Y(k)$ admits a similar characterization (1)-(3) as $\widehat{H}^Y(k)$, with the role of the symmetric algebra $S(\widehat{\mathfrak{h}})$ taken over by $S(\widetilde{\mathfrak{h}})$.

(iii) Let $U \subseteq V$ be the real span of Y and $U_{\mathbb{C}}^{\perp} \subset \mathfrak{h}$ the complexification of the orthocomplement $U^{\perp}$ of U in V. Then $\widehat{H}^Y(k; V) \simeq \widehat{H}^Y(k; U) \otimes_{\mathbb{C}} S(U_{\mathbb{C}}^{\perp})$ as algebras.

We write $\widehat{H}(k) := \widehat{H}^{Q^{\vee}}(k)$ and $\widetilde{H}(k) := \widetilde{H}^{Q^{\vee}}(k)$. Note that

$$\widehat{H}^Y(k) \simeq \Omega \ltimes \widehat{H}(k), \qquad \widetilde{H}^Y(k) \simeq \Omega \ltimes \widetilde{H}(k).$$

Observe that $c \in Z(\widehat{H}^Y(k))$. We define the *extended trigonometric Cherednik algebra at level* κ to be $\widehat{H}_{\kappa}^Y(k) := \widehat{H}^Y(k)/(c - \kappa)$.

We call $\widetilde{H}_{\kappa}^Y(k) := \widetilde{H}^Y(k)/(c - \kappa)$ the trigonometric Cherednik algebra at level κ, and $\widetilde{H}_0^Y(k)$ the trigonometric Cherednik algebra at critical level. A detailed analysis of $\widetilde{H}_0^Y(k)$ in connection to the root system analogs of the one dimensional quantum Bose gas with pairwise delta-function interactions was undertaken in [12] and [13]. In this context the quantum Hamiltonians of the quantum integrable system arise from the center $Z(\widetilde{H}_0^Y(k))$ of $\widetilde{H}_0^Y(k)$, which contains $S(\mathfrak{h})^W$. In this paper we consider these structures away from critical level, replacing the role of $\widetilde{H}_0^Y(k)$ by the extended trigonometric Cherednik algebra $\widehat{H}^Y(k)$. Its center is described as follows.

Proposition 2.3. *If Y is a full lattice in V then $Z(\widehat{H}^Y(k)) = S(\widehat{\mathfrak{h}})^{\widehat{W}^Y} = \mathbb{C}[c, C]$, where $C \in S(\widehat{\mathfrak{h}})^{\widehat{W}^Y}$ is the Casimir element*

$$(2.4) \qquad\qquad C = \sum_{i=1}^{m} v_i^2 + 2cd,$$

and $\{v_i\}_{i=1}^{m}$ is an (arbitrary choice of) orthonormal basis of V with respect to $(\cdot, \cdot)$.

Proof. Let $\{b_j\}_{j=1}^{m+2}$ be a basis of $\widehat{V}$ and $\{b^j\}_{j=1}^{m+2}$ its dual basis with respect to $(\cdot, \cdot)$. The Casimir element can alternatively be defined by

$$(2.5) \qquad\qquad C = \sum_{j=1}^{m+2} b_j b^j.$$

From the $\widehat{W}^Y$-invariance of $(\cdot, \cdot)$ it then follows that $C \in S(\widehat{\mathfrak{h}})^{\widehat{W}^Y}$.

Now take an orthonormal basis $\{v_i\}_{i=1}^{n}$ of V_s. Set $\widehat{\mathfrak{h}}_s = \mathfrak{h}_s \oplus \mathbb{C}c \oplus \mathbb{C}d$ and write

$$C_s := \sum_{i=1}^{n} v_i^2 + 2cd \in S(\widehat{\mathfrak{h}}_s).$$

By [3, Prop. 4.1] we have $S(\widehat{\mathfrak{h}}_s)^{\widehat{W}} = \mathbb{C}[c, C_s]$. Considering $S(\widehat{\mathfrak{h}}_s)$ and $S(Z(\mathfrak{g}))$ as $\widehat{W}$-module subalgebras of $S(\widehat{\mathfrak{h}})$, we conclude that

$$S(\widehat{\mathfrak{h}})^{\widehat{W}} = S(\widehat{\mathfrak{h}}_s)^{\widehat{W}} S(Z(\mathfrak{g})) = \mathbb{C}[c, C_s] S(Z(\mathfrak{g})) = \mathbb{C}[C] S(Z(\mathfrak{g}) \oplus \mathbb{C}c),$$

where the last equality follows from the fact that $C - C_s \in S(Z(\mathfrak{g}))$. Set $Y_0 := Y \cap V'$ (recall that V' is the real form of $Z(\mathfrak{g})$ such that $V = V_s \oplus V'$). Since C is $\widehat{W}^Y$-invariant we conclude that

$$S(\widehat{\mathfrak{h}})^{\widehat{W}^Y} \subseteq \mathbb{C}[C] S(Z(\mathfrak{g}) \oplus \mathbb{C}c)^{Y_0}.$$

Furthermore,

$$S(Z(\mathfrak{g}) \oplus \mathbb{C}c)^{Y_0} = \mathbb{C}[c]$$

since Y_0 is a full lattice in V', cf. the proof of [3, Prop. 4.1]. We conclude that $S(\widehat{\mathfrak{h}})^{\widehat{W}^Y} = \mathbb{C}[c, C]$. Hence it remains to show that

$$Z(\widehat{H}^Y(k)) = S(\widehat{\mathfrak{h}})^{\widehat{W}^Y}.$$

This can be proved by a straightforward adjustment of the analogous statement (due to Lusztig) for the degenerate affine Hecke algebra, cf. [35, Prop. 1.1]. $\qquad\square$

2.3. Difference-reflection operators.

We construct now a representation of the extended trigonometric Cherednik algebra $\widehat{H}^Y(k)$ using Dunkl operators involving Heaviside functions. It generalizes the representation of the trigonometric Cherednik algebra $\widetilde{H}_0^Y(k)$ at critical level constructed in [13, §4.2].

Set $\widetilde{V} = V \oplus \mathbb{R}c$ and $\widehat{V} := \widetilde{V} \oplus \mathbb{R}d$. The bilinear form $(\cdot, \cdot)$ on $\widehat{\mathfrak{h}}$ restricts to a real valued non-degenerate symmetric bilinear form on $\widehat{V}$. We also write $\widehat{V}^+ := \widetilde{V} \oplus \mathbb{R}_{>0}d$ and $\widehat{V}_\xi := \widetilde{V} + \xi d$. Note that $\widetilde{V}$, $\widehat{V}$, $\widehat{V}^+$ and $\widehat{V}_\xi$ are $\widehat{W}^Y$-invariant subsets of $\widehat{\mathfrak{h}}$.

The open subset

$$\widehat{V}_{reg}^+ := \{\widehat{v} = v + \eta c + \xi d \in \widehat{V}^+ \mid (a, \widehat{v}) \neq 0 \quad \forall a \in \widehat{R}^+\}$$

of regular elements in $\widehat{V}^+$ is $\widehat{W}^Y$-invariant. Denote by $\mathcal{C}$ the collection of connected components of $\widehat{V}_{reg}^+$. The affine Weyl group $\widehat{W}$ acts simply transitively on $\mathcal{C}$. The convex polytope

$$\widehat{C}_+ := \{\widehat{v} \in \widehat{V}^+ \mid (a, \widehat{v}) > 0 \quad \forall a \in \widehat{F}\}$$
$$= \{\widehat{v} \in V_s \oplus \mathbb{R}_{>0}d \mid (a, \widehat{v}) > 0 \quad \forall a \in \widehat{F}\} \oplus V' \oplus \mathbb{R}c$$

is a connected component of $\widehat{V}_{reg}^+$ which we call the fundamental chamber. Note that $\omega(\widehat{C}_+) = \widehat{C}_+$ for all $\omega \in \Omega$.

M will always stand for a finite dimensional, complex, left $\widehat{W}^Y$-module. Its representation map will be denoted by π_M.

We define the complex vector space

$$\mathcal{F}_M := \prod_{\widehat{C} \in \mathcal{C}} \left(C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M \right),$$

where $C^\omega(\widehat{V}^+)$ is the space of complex valued, real analytic functions on $\widehat{V}^+$. An element $f = (f_{\widehat{C}})_{\widehat{C} \in \mathcal{C}} \in \mathcal{F}_M$ should be thought of as a collection of real analytic M-valued functions $f_{\widehat{C}}$ on $\widehat{C}$ ($\widehat{C} \in \mathcal{C}$) with the additional requirement that each $f_{\widehat{C}}$ admits a real analytic

extension to $\widehat{V}^+$. We define the support $\mathrm{supp}(f)$ of $f = (f_{\widehat{C}})_{\widehat{C} \in \mathcal{C}} \in \mathcal{F}_M$ to be the collection of connected components $\widehat{C}$ for which $f_{\widehat{C}}$ is nonzero. Note that $\widehat{W}^Y$ acts on $f = (f_{\widehat{C}})_{\widehat{C} \in \mathcal{C}} \in \mathcal{F}_M$ by

$$(2.6) \qquad (wf)_{\widehat{C}}(\widehat{v}) := \pi_M(w)\big(f_{w^{-1}\widehat{C}}(w^{-1}\widehat{v})\big), \qquad w \in \widehat{W}^Y, \ \widehat{C} \in \mathcal{C}, \ \widehat{v} \in \widehat{V}^+.$$

For $\widehat{v} \in \widehat{V}$ we define the linear endomorphism $\partial_{\widehat{v}}$ of $\mathcal{F}_M$ as the componentwise directional derivative,

$$(\partial_{\widehat{v}} f)_{\widehat{C}}(\widehat{u}) := (\partial_{\widehat{v}} f_{\widehat{C}})(\widehat{u}) = \frac{d}{dt}\big|_{t=0} f_{\widehat{C}}(\widehat{u} + t\widehat{v}).$$

For $a \in \widehat{R}^+$ let $\chi_a : \widehat{V} \to \{0, 1\}$ be the characteristic function of the half-space $H_a^- := \{\widehat{v} \in \widehat{V} \mid (a, \widehat{v}) < 0\}$. For each chamber $\widehat{C} \in \mathcal{C}$ either $\chi_a|_{\widehat{C}} \equiv 1$ or $\chi_a|_{\widehat{C}} \equiv 0$. We also write χ_a for the linear endomorphism of $\mathcal{F}_M$ mapping $f = (f_{\widehat{C}})_{\widehat{C} \in \mathcal{C}}$ to $\chi_a f := \{\chi_a(\widehat{C}) f_{\widehat{C}}\}_{\widehat{C} \in \mathcal{C}}$.

Lemma 2.4. *Fix $\widehat{C} = w(\widehat{C}^+) \in \mathcal{C}$ ($w \in \widehat{W}$). If $a \in \widehat{R}^+$ then the following two statements are equivalent:*

   (1) $\widehat{C} \in \mathrm{supp}(\chi_a f)$ *for some $f \in \mathcal{F}_M$,*
   (2) $a \in \widehat{R}^+ \cap w(\widehat{R}^-)$ *(which is a finite set of positive affine roots).*

Proof. Both are easily seen to be equivalent to $\chi_a(\widehat{C}) = 1$. $\qquad \square$

The lemma allows us to define linear operators $\mathcal{D}_{\widehat{v}}^M$ ($\widehat{v} \in \widehat{V}$) on $\mathcal{F}_M$ by

$$\mathcal{D}_{\widehat{v}}^M f := \partial_{\widehat{v}} f - \sum_{a \in \widehat{R}^+} k_a (a, \widehat{v}) \chi_a s_a f.$$

The following proposition extends the results from [13, §4.2].

Proposition 2.5. *Let $k : \widehat{R} \to \mathbb{C}$ be a multiplicity function.*
   (1) *The extended trigonometric Cherednik algebra $\widehat{H}^Y(k)$ is well defined.*
   (2) *The assignments*

$$\widehat{v} \mapsto \mathcal{D}_{\widehat{v}}^M, \qquad \widehat{v} \in \widehat{V},$$
$$w \mapsto w, \qquad w \in \widehat{W}^Y,$$

   uniquely define an algebra morphism $\widehat{\pi} : \widehat{H}^Y(k) \to \mathrm{End}_{\mathbb{C}}(\mathcal{F}_M)$.

Proof. Repeating the arguments of the proof of [12, Thm. 4.1] gives that the operators $\mathcal{D}_{\widehat{v}}^M$ ($\widehat{v} \in \widehat{V}$), s_a ($a \in \widehat{F}$) and ω ($\omega \in \Omega$) on $\mathcal{F}_M$ satisfy the defining relations of $\widehat{H}^Y(k)$,

$$s_a \mathcal{D}_{\widehat{v}}^M = \mathcal{D}_{s_a \widehat{v}}^M s_a - k_a(a, \widehat{v}),$$
$$\omega \mathcal{D}_{\widehat{v}}^M = \mathcal{D}_{\omega \widehat{v}}^M \omega,$$
$$\mathcal{D}_{\widehat{v}}^M \mathcal{D}_{\widehat{v}'}^M = \mathcal{D}_{\widehat{v}'}^M \mathcal{D}_{\widehat{v}}^M.$$

For $Y = Q^\vee$ and $M = \mathrm{Triv}$ the trivial one-dimensional $\widehat{W}$-module the resulting linear map $S(\widehat{\mathfrak{h}}) \otimes_{\mathbb{C}} \mathbb{C}[\widehat{W}] \to \mathrm{End}(\mathcal{F}_{\mathrm{Triv}})$ is easily seen to be injective, hence the extended trigonometric Cherednik algebras $\widehat{H}(k)$ and $\widehat{H}^Y(k) = \Omega \ltimes \widehat{H}(k)$ are well defined. Part (2) of the proposition follows now immediately. $\qquad\square$

Remark 2.6. A trigonometric version of the representation $\widehat{\pi}|_{\widetilde{H}^Y(k)}$ was constructed by Cherednik in [5, Thm. 3.1] using infinite trigonometric Dunkl operators. He considers separately a trigonometric analog of the operator $\widehat{\pi}(d)$ (see [5, (4.12)]). He remarks that the cross relations (2.3) are respected (see [5, (4.13)]) but that, in contrast to our setup, it does not result in a representation of the extended trigonometric Cherednik algebra $\widehat{H}^Y(k)$.

Note that $\mathcal{D}_c^M = \partial_c$ since $(a, c) = 0$ for all $a \in \widehat{R}$. Hence for $\kappa \in \mathbb{C}$,
$$\mathcal{F}_M^\kappa := \{f \in \mathcal{F}_M \mid \partial_c f = \kappa f\}$$
is a $\widehat{H}^Y(k)$-submodule of $\mathcal{F}_M$, and the action of $\widehat{H}^Y(k)$ on $\mathcal{F}_M^\kappa$ descends to an action of the extended trigonometric Cherednik algebra $\widehat{H}_\kappa^Y(k)$ at level κ.

For $\widehat{\lambda} \in \widehat{\mathfrak{h}}$ let $e^{\widehat{\lambda}} \in \mathcal{O}(\widehat{\mathfrak{h}})$ be the holomorphic function $\widehat{\mu} \mapsto e^{(\widehat{\lambda}, \widehat{\mu})}$. Its restriction to $\widehat{V}^+$ defines a complex valued, real analytic function on $\widehat{V}^+$ which we also will denote by $e^{\widehat{\lambda}}$. We have
$$\mathcal{F}_M^\kappa = \prod_{\widehat{C} \in \mathcal{C}} \big(C_\kappa^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M\big)$$
where
$$C_\kappa^\omega(\widehat{V}^+) := \{f \in C^\omega(\widehat{V}^+) \mid \partial_c(f) = \kappa f\}.$$
Note that $C_\kappa^\omega(\widehat{V}^+) = e^{\kappa d} C_0^\omega(\widehat{V}^+)$.

2.4. **Integral-reflection operators.** In this subsection we give a second representation of the extended trigonometric Cherednik algebra $\widehat{H}^Y(k)$, now in terms of integral-reflection operators. The results in this subsection build on constructions from [20, 23, 12, 13].

For $a \in \widehat{R}$ define an integral-reflection operator $I(a)$ on $C^\omega(\widehat{V}^+)$ by

(2.7)
$$(I(a)f)(\widehat{v}) := \int_0^{(a, \widehat{v})} f(\widehat{v} - t a^\vee) dt.$$

The following is a version of [13, Thm. 4.11] at unspecified level.

Theorem 2.7. *Let M be a finite dimensional $\widehat{W}^Y$-module. There exists a unique algebra homomorphism $\widehat{Q} : \widehat{H}^Y(k) \to \mathrm{End}_{\mathbb{C}}\big(C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M\big)$ satisfying*

(2.8)
$$\begin{aligned}
\widehat{Q}(s_a) &= s_a \otimes \pi_M(s_a) - k_a I(a) \otimes \mathrm{Id}_M, & a &\in \widehat{F}, \\
\widehat{Q}(\omega) &= \omega \otimes \pi_M(\omega), & \omega &\in \Omega, \\
\widehat{Q}(\widehat{v}) &= \partial_{\widehat{v}} \otimes \mathrm{Id}_M, & \widehat{v} &\in \widehat{V}.
\end{aligned}$$

Proof. Consider $S(\widehat{\mathfrak{h}}) \otimes_{\mathbb{C}} M^*$ as left $\widehat{H}^Y(k)$-module by the canonical vector space identification

$$\operatorname{Ind}_{\mathbb{C}[\widehat{W}^Y]}^{\widehat{H}^Y(k)}(M^*) \simeq S(\widehat{\mathfrak{h}}) \otimes_{\mathbb{C}} M^*.$$

Using the complex linear antiinvolution $^\dagger : \widehat{H}^Y(k) \xrightarrow{\sim} \widehat{H}^Y(k)$ defined by $w^\dagger = w^{-1}$ ($w \in \widehat{W}^Y$) and $\widehat{v}^\dagger = \widehat{v}$ ($\widehat{v} \in \widehat{V}$), its linear dual $(S(\widehat{\mathfrak{h}}) \otimes_{\mathbb{C}} M^*)^*$ becomes a left $\widehat{H}^Y(k)$-module.

View $C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M$ as linear subspace of $(S(\widehat{\mathfrak{h}}) \otimes_{\mathbb{C}} M^*)^*$ by interpreting $f \otimes m$ ($f \in C^\omega(\widehat{V}^*)$, $m \in M$) as the linear functional

$$p \otimes \psi \mapsto \psi(m)(p(\partial)f)(0), \qquad p \in S(\widehat{\mathfrak{h}}), \, \psi \in M^*$$

on $S(\widehat{\mathfrak{h}}) \otimes_{\mathbb{C}} M^*$, where $p(\partial)$ is the constant coefficient partial differential operator associated to $p \in S(\widehat{\mathfrak{h}})$. Then $C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M$ is a $\widehat{H}^Y(k)$-submodule of $(S(\widehat{\mathfrak{h}}) \otimes M^*)^*$. A direct computation establishes the explicit formulas (2.8) for the resulting action of $\widehat{H}^Y(k)$ on $C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M$. $\qquad \square$

Remark 2.8. The proof of Theorem 2.7 is simpler than the proof at critical level (see [13, Thm. 4.11]), since at critical level the arguments of the above proof lead to an explicit action by integral-reflection operators which is not yet of the desired form (see [13, Cor. 4.10]).

The integral-reflection operators $\widehat{Q}(w)$ ($w \in \widehat{W}$) gives rise to a linear map

$$T : C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M \to \mathcal{F}_M,$$

with $Tf = \{(Tf)_C\}_{C \in \mathcal{C}} \in \mathcal{F}_M$ for $f \in C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M$ defined by

$$(Tf)_{w\widehat{C}^+}(\widehat{v}) := \pi_M(w)\big((\widehat{Q}(w^{-1})f)(w^{-1}\widehat{v})\big), \qquad w \in \widehat{W}, \, \widehat{v} \in \widehat{V}^+.$$

It is the unique linear map satisfying $(Tf)_{\widehat{C}^+} = f$ and $T \circ \widehat{Q}(w) = \widehat{\pi}(w) \circ T$ for all $w \in \widehat{W}$, cf. [13, Lemma 4.14]. In fact, in analogy with [13, Prop. 4.15] we have

Proposition 2.9. *The linear map $T : C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M \to \mathcal{F}_M$ is $\widehat{H}^Y(k)$-linear,*

$$T \circ \widehat{Q}(h) = \widehat{\pi}(h) \circ T, \qquad \forall \, h \in \widehat{H}^Y(k).$$

Remark 2.10. For $\kappa \in \mathbb{C}$ the space $C^\omega_\kappa(\widehat{V}^+) \otimes M$ is a $\widehat{H}^Y(k)$-submodule of $C^\omega(\widehat{V}^+) \otimes_{\mathbb{C}} M$ with respect to the $\widehat{Q}$-action. The action on $C^\omega_\kappa(\widehat{V}^+) \otimes M$ descends to an action of $\widehat{H}^Y_\kappa(k)$ since $\widehat{Q}(c) = \partial_c$. The intertwiner T restricts to an intertwiner

$$T_\kappa : C^\omega_\kappa(\widehat{V}^+) \otimes_{\mathbb{C}} M \to \mathcal{F}^\kappa_M$$

of $\widehat{H}^Y_\kappa(k)$-modules.

3. Generalized Bethe wave functions

In this section we construct generalized Bethe wave functions, being $\widehat{Q}(\widehat{W}^Y)$-invariant eigenfunctions of the constant coefficient differential operator $\widehat{Q}(C)$, as infinite series expansions of plane waves with explicit cocycle values as coefficients. The relevant cocycle of $\widehat{W}^Y$ arises from the normalized intertwiners of the trigonometric Cherednik algebra $\widetilde{H}^Y(k)$. On the other hand, this cocycle can be used to define a consistent system of equations. These equations should be thought of as formal degenerations of Cherednik's [6] affine difference-elliptic quantum Knizhnik-Zamolodchikov equations. We show that the generalized Bethe wave functions satisfy this consistent system of equations as function of the momenta.

3.1. **The cocycle.** Let $k : \widehat{R} \to \mathbb{C}$ be a $\widehat{W}^Y$-invariant multiplicity function. Set

$$\widehat{\mathfrak{h}}_{reg} := \{\widehat{\lambda} \in \widehat{\mathfrak{h}} \mid (a^\vee, \widehat{\lambda}) \neq 0, k_a \quad \forall a \in \widehat{R}\},$$
$$\widehat{\mathfrak{h}}_{\kappa,reg} := \widehat{\mathfrak{h}}_{reg} \cap \widehat{\mathfrak{h}}_\kappa,$$

so that $\widehat{\mathfrak{h}}_{\kappa,reg} = \mathfrak{h}_{\kappa,reg} + \mathbb{C}c + \kappa d$ with

$$\mathfrak{h}_{\kappa,reg} = \{\lambda \in \mathfrak{h} \mid (\alpha^\vee, \lambda) + \frac{2m\kappa}{(\alpha,\alpha)} \neq 0, k_{\alpha+mc} \quad \forall \alpha \in R, \forall m \in \mathbb{Z}\}.$$

Let $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ be the open right half plane in $\mathbb{C}$. Set $\widehat{\mathfrak{h}}^+ := \widetilde{\mathfrak{h}} + \mathbb{C}_+ d$ and

$$\widehat{\mathfrak{h}}^+_{reg} := \widehat{\mathfrak{h}}_{reg} \cap \widehat{\mathfrak{h}}^+ = \bigcup_{\kappa \in \mathbb{C}_+} \widehat{\mathfrak{h}}_{\kappa,reg}.$$

Then $\widehat{\mathfrak{h}}^+_{reg} \subset \widehat{\mathfrak{h}}$ is open, connected, and the boundary of $\widehat{\mathfrak{h}}^+_{reg}$ contains $\widehat{\mathfrak{h}}_{0,reg}$.

Let $\mathbb{C}[\widehat{\mathfrak{h}}_{reg}]$ be the subalgebra of the field of rational functions on $\widehat{\mathfrak{h}}$ obtained by adjoining $(a^\vee - k_a)^{-1}$ and a^{-1} to $\mathbb{C}[\widehat{\mathfrak{h}}]$ for all $a \in \widehat{R}$. By [7] we have

Proposition 3.1. *There exist unique* $J_w \in \mathbb{C}[\widehat{\mathfrak{h}}_{reg}] \otimes \mathbb{C}[\widehat{W}^Y]$ *(*$w \in \widehat{W}^Y$*) satisfying, as rational* $\mathbb{C}[\widehat{W}^Y]$*-valued functions on* $\widehat{\mathfrak{h}}_{reg}$,

$$J_{uw}(\widehat{\lambda}) = J_u(w\widehat{\lambda})J_w(\widehat{\lambda}) \qquad \forall u, w \in \widehat{W}^Y$$

and satisfying

$$J_{s_a}(\widehat{\lambda}) = \frac{(a^\vee, \widehat{\lambda})s_a + k_a}{(a^\vee, \widehat{\lambda}) - k_a}, \qquad a \in \widehat{F},$$
$$J_\omega(\widehat{\lambda}) = \omega, \qquad\qquad \omega \in \Omega.$$

The proof of the proposition uses the normalized intertwiners of the trigonometric Cherednik algebra, cf. [7].

Remark 3.2. Viewing J_w as $\mathbb{C}[\widehat{W}^Y]$-valued rational function on $\widehat{\mathfrak{h}}$, we have $\partial_c(J_w) = 0$ for all $w \in \widehat{W}^Y$.

The ring $\mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+)$ of holomorphic functions on $\widehat{\mathfrak{h}}_{reg}^+$ is naturally a $\mathbb{C}[\widehat{\mathfrak{h}}_{reg}]$-module. If M is a $\widehat{W}^Y$-module, the cocycle $\{J_w\}_{w \in \widehat{W}^Y}$ thus canonically acts on $\mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$.

Corollary 3.3. *Let M be a finite dimensional $\widehat{W}^Y$-module. Set*

$$(3.1) \qquad (\Psi \cdot w)(\widehat{\lambda}) := \Psi(w\widehat{\lambda}) J_w(\widehat{\lambda}), \qquad w \in \widehat{W}^Y,\ \Psi \in \mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M),$$

where Ψ is viewed as $\mathrm{End}_{\mathbb{C}}(M)$-valued holomorphic function on $\widehat{\mathfrak{h}}_{reg}^+$. Then (3.1) defines a right $\widehat{W}^Y$-action on $\mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$.

We call the set of equations

$$(3.2) \qquad\qquad \Psi(t_y\widehat{\lambda}) J_{t_y}(\widehat{\lambda}) = \Psi(\widehat{\lambda}) \qquad \forall\, y \in Y$$

the affine difference Knizhnik-Zamolodchikov (adKZ) equations and

$$\mathrm{adKZ} := \left(\mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M) \right)^{\cdot Y}$$

the corresponding space of solutions. It is a $\cdot W$-submodule of $\mathrm{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$.

Remark 3.4. The affine difference KZ equations (3.2) are expected to be formal degenerations of Cherednik's [6] affine difference-elliptic quantum affine KZ equations, which are naturally associated to double affine Hecke algebras.

Explicitly, the affine difference KZ equations (3.2) read

$$(3.3)\ \ \Psi\big(\lambda + \kappa y + (\eta - \tfrac{\kappa}{2}(y,y) - (\lambda,y))c + \kappa d\big) J_{t_y}(\lambda + \kappa d) = \Psi(\lambda + \eta c + \kappa d) \qquad \forall\, y \in Y.$$

Set $\mathcal{O}_\xi(\widehat{\mathfrak{h}}_{reg}^+) = \{f \in \mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \mid \partial_c(f) = \xi f\}$, so that $\mathcal{O}_\xi(\widehat{\mathfrak{h}}_{reg}^+) = e^{\xi d} \mathcal{O}_0(\widehat{\mathfrak{h}}_{reg}^+)$. Observe that $\mathcal{O}_\xi(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$ is a $\cdot\widehat{W}^Y$-submodule of $\mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$. Set

$$\mathrm{adKZ}_\xi := \left(\mathcal{O}_\xi(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M) \right)^{\cdot Y}.$$

By (3.3), if $\Psi \in \mathcal{O}_\xi(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$ then $\Psi \in \mathrm{adKZ}_\xi$ iff

$$(3.4) \qquad \Psi(\lambda + \kappa y + \eta c + \kappa d) e^{-\frac{\kappa\xi}{2}(y,y) - \xi(\lambda,y)} J_{t_y}(\lambda + \kappa d) = \Psi(\lambda + \eta c + \kappa d) \qquad \forall\, y \in Y$$

as $\mathrm{End}_{\mathbb{C}}(M)$-valued holomorphic function in $\lambda + \eta c + \kappa d \in \widehat{\mathfrak{h}}_{reg}^+$. Note that (3.4) formally makes sense if $\kappa = 0$, in which case it gives

$$(3.5) \qquad\qquad \Phi(\lambda) e^{-\xi(\lambda,y)} J_{t_y}(\lambda) = \Phi(\lambda), \qquad \forall\, y \in Y$$

for $\Phi(\lambda) \in \mathrm{End}_{\mathbb{C}}(M)$ and $\lambda \in \widehat{\mathfrak{h}}_{0,reg}$. These equations are closely related to the Bethe ansatz equations for the vector valued root system analogs of the quantum Bose gas on the circle with pairwise delta-function interactions, see [13, Thm. 5.10].

For $\xi = 0$ the affine difference KZ equations (3.4) for fixed $\kappa \in \mathbb{C}_+$ take the form

$$(3.6) \qquad\qquad \Phi(\lambda + \kappa y) J_{t_y}(\lambda + \kappa d) = \Phi(\lambda), \qquad \forall\, y \in Y$$

for Φ a $\mathrm{End}_{\mathbb{C}}(M)$-valued holomorphic function on $\mathfrak{h}_{\kappa,reg}$. These are degenerations of Cherednik's [4] quantum affine KZ equations associated to double affine Hecke algebras.

They form a consistent system of difference equations naturally compatible to trigonometric KZ equations, see [38].

3.2. **Bethe wave functions at critical level.** Before constructing generalized Bethe wave functions and their relation to the affine difference KZ equations, we first recall the related results at critical level from [13]. Let $\xi > 0$ and write $Q_\xi : \widetilde{H}_0^Y(k) \to \mathrm{End}_{\mathbb{C}}(C^\omega(V) \otimes_{\mathbb{C}} M)$ for the analog of the integral-reflection action $\widehat{Q}$ (see [13, Thm. 4.11]), defined by $Q_\xi(v) = \partial_v$ $(v \in V)$ and

$$(Q_\xi(s_a)f)(v) = \pi_M(s_a)f(s_a \circ_\xi v) - k_a \int_0^{a^\xi(v)} f(v - tDa^\vee)dt, \qquad a \in \widehat{F},$$

$$(Q_\xi(\omega)f)(v) = \pi_M(\omega)f(\omega^{-1} \circ_\xi v), \qquad\qquad\qquad \omega \in \Omega,$$

where $a^\xi(v) = (\alpha, v) + m\xi$ and $Da = \alpha$ for $a = \alpha + mc$ and $v \in V$, and where the action $\widehat{W}^Y \times V \to V$, $(w, v) \mapsto w \circ_\xi v$ is defined by

$$w \circ_\xi v = w(v), \qquad w \in W,$$

$$t_y \circ_\xi v = v + \xi y, \qquad y \in Y.$$

Set $\widetilde{N}_\lambda = \{f \in C^\omega(V) \mid p(\partial)f = \chi_\lambda(p)f \ \forall p \in S(\mathfrak{h})^W\}$ for $\lambda \in \mathfrak{h}$. Here χ_λ is the linear character of $S(\mathfrak{h})$ satisfying $v \mapsto (\lambda, v)$ and $p(\partial)$ stands for the constant coefficient partial differential operator naturally associated to $p \in S(\mathfrak{h})$ by $v \mapsto \partial_v$ $(v \in V)$. Then $\widetilde{N}_\lambda$ is a $\#W$-dimensional vector space containing $e^{w\lambda}$ $(w \in W)$. In particular the $e^{w\lambda}$ $(w \in W)$ form a basis of $\widetilde{N}_\lambda$ if $\lambda \in \mathfrak{h}_{reg} := \{\lambda \in \mathfrak{h} \mid (\lambda, \alpha) \neq 0 \ \forall \alpha \in R\}$.

Since $S(\mathfrak{h})^W \subseteq Z(\widetilde{H}_0^Y(k))$,

$$\widetilde{S}_M(\lambda) := \widetilde{N}_\lambda \otimes_{\mathbb{C}} M$$

is a finite dimensional $Q_\xi(\widetilde{H}_0^Y(k))$-submodule of $C^\omega(V) \otimes_{\mathbb{C}} M$ for all $\lambda \in \mathfrak{h}^*$.

Define for $v \in V$ and $\lambda \in \mathfrak{h}_{0,reg}$ the Bethe wave function

$$\psi_\lambda(v) := \sum_{w \in W} e^{(w\lambda, v)} J_w(\lambda) \in \mathbb{C}[W].$$

Note that the $J_{t_y}(\lambda)$ $(y \in Y)$ pairwise commute. The following result from [13, Thm. 5.10] relates the coordinate Bethe ansatz for vector valued root system analogs of the quantum Bose gas on the circle with pairwise delta-function interactions to the study of the space of $Q_\xi(\widehat{W}^Y)$-invariants in $\widetilde{S}_M(\lambda)$.

Theorem 3.5. *Let M be a finite dimensional $\widehat{W}^Y$-module, $m \in M$ and $\lambda \in \mathfrak{h}_{0,reg}$. Then $\psi_\lambda(\cdot)m \in \widetilde{S}_M(\lambda)^{Q_\xi(\widehat{W}^Y)}$ if and only if*

$$J_{t_y}(\lambda)m = e^{\xi(\lambda, y)}m \qquad \forall y \in Y$$

(the Bethe ansatz equations).

3.3. **Generalized Bethe wave functions.** Now we consider unspecified level. Without loss of generality we assume that Y spans V (cf. Remark 2.2(**iii**)), in which case the center of $\widehat{H}^Y(k)$ is generated by c and C (see Proposition 2.3). The role of $\widetilde{N}_\lambda$ is taken over by

$$\widehat{N}_{\gamma,\Gamma} := \{f \in C^\omega(\widehat{V}^+) \mid \partial_c(f) = \gamma f \ \& \ \widehat{\Delta}(f) = \Gamma f\}$$

for $\gamma, \Gamma \in \mathbb{C}$, where $\widehat{\Delta}$ is the constant coefficient differential operator associated to C,

$$\widehat{\Delta} = \Delta + 2\partial_c\partial_d$$

and Δ is the Laplacean on V. If $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\kappa$ then

$$e^{w\widehat{\lambda}} \in \widehat{N}_{\kappa,(\widehat{\lambda},\widehat{\lambda})} \qquad \forall\, w \in \widehat{W}^Y.$$

Note that $\widehat{S}_M(\gamma,\Gamma) := \widehat{N}_{\gamma,\Gamma} \otimes_\mathbb{C} M$ is a $\widehat{Q}(\widehat{H}^Y(k))$-submodule of $C^\omega(\widehat{V}^+) \otimes M$. The generalized Bethe wave functions will be $\widehat{Q}(\widehat{W}^Y)$-invariant elements in the infinite dimensional vector space $\widehat{S}_M(\gamma,\Gamma)$, explicitly defined as a convergent series expansion in the $e^{w\widehat{\lambda}}$ ($w \in \widehat{W}^Y$) for some $\widehat{\lambda} \in \widehat{\mathfrak{h}}_\gamma$ (see (2.2)) satisfying $(\widehat{\lambda},\widehat{\lambda}) = \Gamma$. In addition we show that the role of the Bethe ansatz equations is taken over by the affine difference Knizhnik-Zamolodchikov equations.

If M is a finite dimensional $\widehat{W}^Y$-module then we regard $\mathrm{End}_\mathbb{C}(M)$ as $\widehat{W}^Y$-module by $w \cdot \psi := \pi_M(w) \circ \psi$. Note that evaluation at $m \in M$ defines a $\widehat{H}^Y(k)$-linear map $\widehat{S}_{\mathrm{End}_\mathbb{C}(M)}(\gamma,\Gamma) \to \widehat{S}_M(\gamma,\Gamma)$.

Theorem 3.6. *Let M be a unitarizable finite dimensional $\widehat{W}^Y$-module. For $\widehat{v} \in \widehat{V}^+$ and $\widehat{\lambda} \in \widehat{\mathfrak{h}}^+_{reg}$ the series*

$$E_M(\widehat{v};\widehat{\lambda}) := \sum_{y \in Y} e^{(t_y\widehat{\lambda},\widehat{v})} J_{t_y}(\widehat{\lambda}),$$

$$E_M^+(\widehat{v};\widehat{\lambda}) := \sum_{w \in \widehat{W}^Y} e^{(w\widehat{\lambda},\widehat{v})} J_w(\widehat{\lambda})$$

converge in $\mathrm{End}_\mathbb{C}(M)$ and satisfy
  (1) $E_M^+(\widehat{v};\widehat{\lambda}) = \sum_{w \in W} E_M(\widehat{v}; w\widehat{\lambda}) J_w(\widehat{\lambda}).$
  (2) *if $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{\kappa,reg}$ with $\kappa \in \mathbb{C}_+$ then*
      (a) $E_M(\cdot;\widehat{\lambda}) \in C^\omega_\kappa(\widehat{V}^+) \otimes_\mathbb{C} \mathrm{End}_\mathbb{C}(M),$
      (b) $E_M^+(\cdot;\widehat{\lambda}) \in \widehat{S}_{\mathrm{End}_\mathbb{C}(M)}(\kappa,(\widehat{\lambda},\widehat{\lambda}))^{\widehat{Q}(\widehat{W}^Y)}.$
  (3) *if $\widehat{v} \in \widehat{V}_\xi$ with $\xi > 0$ then $E_M(\widehat{v};\cdot) \in \mathrm{adKZ}_\xi$ and $E_M^+(\widehat{v};\cdot) \in \mathrm{adKZ}_\xi^W.$*

Proof. We first consider the convergence of the series, for which we use

Lemma 3.7. *Let $K \subset \widehat{\mathfrak{h}}^+_{reg}$ be a compact subset. There exists a positive constant $D = D(K)$ such that*

$$(3.7) \qquad \|J_w(\widehat{\lambda})\|_M \le D^{l(w)}, \qquad \forall\, w \in \widehat{W}^Y, \forall\, \widehat{\lambda} \in \bigcup_{w \in \widehat{W}^Y} wK,$$

where $\| \cdot \|_M$ is the norm on M turning M into a unitary $\widehat{W}^Y$-module.

Proof. Since $\widehat{\mathfrak{h}}_{reg}^+$ is $\widehat{W}^Y$-invariant, $\cup_{w \in \widehat{W}^Y} wK \subset \widehat{\mathfrak{h}}_{reg}^+$. Note that there exists a $s = s(K) > 0$ such that

$$|(\widehat{\lambda}, a^\vee) - k_a| \geq s \quad \forall a \in \widehat{R}, \forall \widehat{\lambda} \in K.$$

Then the estimate (3.7) is easily seen to be correct for $w \in \widehat{W}^Y$ with

$$l(w) := \#(\widehat{R}^+ \cap w^{-1}\widehat{R}^-) \leq 1$$

(i.e. for $w \in \widehat{W}^Y$ of the form $w = \omega$ or $w = s_a\omega$ with $a \in \widehat{F}$ and $\omega \in \Omega$) if the constant $D = D(K)$ is taken to be

$$D = 1 + \frac{2}{s} \max_{a \in \widehat{R}} |k_a|.$$

Note that D is well defined since k attains only finitely many values. The estimate (3.7) for arbitrary $w \in \widehat{W}^Y$ holds true with the same positive constant D. This follows by induction to $l(w)$ using the cocycle property of $J_w(\widehat{\lambda})$. $\qquad\square$

Write $\lambda = \text{Re}(\lambda) + \sqrt{-1}\text{Im}(\lambda)$ with $\text{Re}(\lambda), \text{Im}(\lambda) \in V$. Combining the lemma with the fact that

$$(3.8) \qquad l(t_y) = \sum_{\alpha \in R^+} |(y, \alpha)|$$

and

$$(3.9) \qquad e^{(t_y\widehat{\lambda}, \widehat{v})} = e^{u\kappa + \xi\eta + (\lambda, v) + (\kappa v - \xi\lambda, y) - \frac{\xi\kappa}{2}(y, y)}$$

for $y \in Y$, $\widehat{v} = v + uc + \xi d \in \widehat{V}_\xi$ ($\xi > 0$) and $\widehat{\lambda} = \lambda + \eta c + \kappa d \in \widehat{\mathfrak{h}}_{reg}^+$, we conclude that

$$\sum_{y \in Y} \|e^{(t_y\widehat{\lambda}, \widehat{v})} J_{t_y}(\widehat{\lambda})\|_M \leq e^{u\text{Re}(\kappa) + \xi\text{Re}(\eta) + \text{Re}((\lambda, v))} \sum_{y \in Y} e^{-\frac{\xi\text{Re}(\kappa)}{2}(y, y)} e^{(\text{Re}(\kappa)v - \xi\text{Re}(\lambda), y) + \log(D) \sum_{\alpha \in R^+} |(y, \alpha)|}$$

if $\widehat{v} \in \widehat{V}^+$ and $\widehat{\lambda} \in K$ (with $K \subset \widehat{\mathfrak{h}}_{reg}^+$ a fixed compact set and $D = D(K)$ the associated positive constant). The absolute convergence of the series $E_M(\widehat{v}; \widehat{\lambda})$ ($\widehat{v} \in \widehat{V}^+$ and $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{reg}^+$) follows. It also implies that $E_M(\cdot; \widehat{\lambda})$ is real analytic on $\widehat{V}^+$ for $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{reg}^+$ and $E(\widehat{v}; \cdot)$ is holomorphic on $\widehat{\mathfrak{h}}_{reg}^+$ for $\widehat{v} \in \widehat{V}^+$.

We continue now with the proof of (1)-(3).
(1) is immediate from the cocycle property of $\{J_w\}_{w \in \widehat{W}^Y}$.
(2) We already established part (a).
(b) For $a \in \widehat{F}$,

$$\widehat{Q}(s_a)E_M^+(\cdot; \widehat{\lambda}) = \sum_{w \in \widehat{W}^Y} e^{s_a w\widehat{\lambda}} s_a J_w(\widehat{\lambda}) - k_a \sum_{w \in \widehat{W}^Y} \left(I(a)e^{w\widehat{\lambda}}\right) J_w(\widehat{\lambda}).$$

Now use that for $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{reg}^+$,

$$I(a)\left(e^{\widehat{\lambda}}\right) = \frac{e^{\widehat{\lambda}} - e^{s_a\widehat{\lambda}}}{(a^\vee, \widehat{\lambda})},$$

hence

$$(3.10) \qquad \widehat{Q}(s_a)E_M^+(\cdot;\widehat{\lambda}) = \sum_{w\in\widehat{W}^Y} e^{s_a w\widehat{\lambda}} s_a J_w(\widehat{\lambda}) - k_a \sum_{w\in\widehat{W}^Y} \frac{e^{w\widehat{\lambda}} - e^{s_a w\widehat{\lambda}}}{(a^\vee, w\widehat{\lambda})} J_w(\widehat{\lambda}).$$

Since $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{reg}^+$, the explicit expression of the cocycle value $J_{s_a}(w\widehat{\lambda})$ allows us to write

$$s_a = \left(1 - \frac{k_a}{(a^\vee, w\widehat{\lambda})}\right) J_{s_a}(w\widehat{\lambda}) - \frac{k_a}{(a^\vee, w\widehat{\lambda})}.$$

Substituting in (3.10) and using the cocycle condition we get

$$\widehat{Q}(s_a)E_M^+(\cdot;\widehat{\lambda}) = \sum_{w\in\widehat{W}^Y} \left(1 - \frac{k_a}{(a^\vee, w\widehat{\lambda})}\right) e^{s_a w\widehat{\lambda}} J_{s_a w}(\widehat{\lambda}) - \sum_{w\in\widehat{W}^Y} \frac{k_a}{(a^\vee, w\widehat{\lambda})} e^{w\widehat{\lambda}} J_w(\widehat{\lambda}).$$

In the first sum replace the summation variable w by $s_a w$. It follows that $\widehat{Q}(s_a)E_M^+(\cdot;\widehat{\lambda}) = E^+(\cdot;\widehat{\lambda})$. If $\omega \in \Omega$ then, using $\widehat{Q}(\omega) = \omega \otimes \pi_M(\omega)$ and $J_\omega(w\widehat{\lambda}) = \omega$ for all $w \in \widehat{W}^Y$,

$$\widehat{Q}(\omega)E_M^+(\cdot;\widehat{\lambda}) = \sum_{w\in\widehat{W}^Y} e^{\omega w\widehat{\lambda}} J_{\omega w}(\widehat{\lambda}) = E_M^+(\cdot;\widehat{\lambda}).$$

Hence $E_M^+(\cdot;\widehat{\lambda})$ is $\widehat{Q}(\widehat{W}^Y)$-invariant.
(3) The fact that $E_M(\widehat{v};\cdot)$ solves the affine difference KZ equations is direct by the cocycle condition: if $y' \in Y$ then

$$E_M(\widehat{v}; t_{y'}\widehat{\lambda}) J_{t_{y'}}(\widehat{\lambda}) = \sum_{y\in Y} e^{(t_{y+y'}\widehat{\lambda}, \widehat{v})} J_{t_{y+y'}}(\widehat{\lambda}) = E(\widehat{v}; \widehat{\lambda}).$$

Similarly one establishes $E_M^+(\widehat{v}; \cdot) \in \mathrm{adKZ}^{\cdot W}$. $\qquad\qquad\qquad\qquad\square$

We call $E_M^+(\cdot, \widehat{\lambda}) \in \widehat{S}_{\mathrm{End}_\mathbb{C}(M)}(\kappa, (\widehat{\lambda}, \widehat{\lambda}))$ for $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{\kappa,reg}$ ($\kappa \in \mathbb{C}_+$) the *generalized Bethe wave functions of level κ*.

Remark 3.8. Formally one can think of the affine difference KZ equations as defining a difference connection on the bundle over $\widehat{\mathfrak{h}}_{reg}^+$ with fiber at $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{\kappa,reg}$ ($\kappa \in \mathbb{C}_+$) given by $\widehat{S}_M(\kappa, (\widehat{\lambda}, \widehat{\lambda}))$. The difference connection commutes with the action of $\widehat{Q}(\widehat{H}^Y(k))$ on the fibers of the bundle. The generalized Bethe wave function $\widehat{\lambda} \mapsto E_M^+(\cdot, \widehat{\lambda})$ then defines a flat $\widehat{Q}(\widehat{W}^Y)$-invariant section.

Remark 3.9. Fix $\kappa \in \mathbb{C}_+$. If $\lambda \in \mathfrak{h}_{\kappa,reg}$ satisfies

$$(3.11) \qquad\qquad (\mathrm{Re}(\lambda), y) > -\frac{\mathrm{Re}(\kappa)}{2}(y, y) \quad \forall y \in Y \setminus \{0\},$$

then

$$\lim_{\xi\to\infty} E_M(v + \xi d, \lambda + \kappa d) = e^{(\lambda, v)} \mathrm{Id}_M,$$

$$\lim_{\xi\to\infty} E_M^+(v + \xi d, \lambda + \kappa d) = \sum_{w\in W} e^{(w\lambda, v)} J_w(\lambda) = \psi_\lambda(v)$$

uniformly for v in compacta of V. Note that $v \mapsto \psi_\lambda(v)$ coincides with the Bethe wave function of the vector valued root system analog of the quantum Bose gas on the line with pairwise delta-function interactions, cf., e.g., [23], [13] and references therein. The fundamental property of $\psi_\lambda \in C^\omega(V) \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(M)$ is the fact that it is a $Q(W)$-invariant real analytic solution to the differential equations $p(\partial)f = p(\lambda)f$ for all $p \in S(\mathfrak{h})^W$, where Q is the W-action on $C^\omega(V) \otimes_{\mathbb{C}} \operatorname{End}_{\mathbb{C}}(M)$ given by integral-reflection operators, cf. [23, 12, 13].

Example 3.10. *The simplest example corresponds to the Steinberg module $M = \operatorname{St}$, which is the one-dimensional $\widehat{W}^Y$-module with associated linear character $w \mapsto (-1)^{l(w)}$ ($w \in \widehat{W}^Y$), since $J_w(\widehat{\lambda})|_{\operatorname{St}} = (-1)^{l(w)}$ ($w \in \widehat{W}^Y$). We regard E_{St} and E_{St}^+ as scalar valued functions. Assume for simplicity that $(Y, P) \subseteq \mathbb{Z}$, where P is the weight lattice of R. Then $l(t_y)$ is even for all $y \in Y$ (indeed, by W-invariance it suffices to prove it when $(y, \alpha) \geq 0$ for all $\alpha \in R$, in which case $l(t_y) = 2(\rho, y)$ with $\rho \in P$ the half sum of positive roots by (3.8)). Then*

$$E_{\operatorname{St}}(\widehat{v}, \widehat{\lambda}) = \sum_{y \in Y} e^{(t_y \widehat{\lambda}, \widehat{v})}$$
$$= e^{u\kappa + \xi\eta + (\lambda, v)} \sum_{y \in Y} e^{\kappa(v, y) - \xi(\lambda, y) - \frac{\xi\kappa}{2}(y, y)}$$

if $\widehat{v} = v + uc + \xi d \in \widehat{V}_\xi$ ($\xi > 0$) and $\widehat{\lambda} = \lambda + \eta c + \kappa d \in \widehat{\mathfrak{h}}_{\kappa, reg}$ ($\kappa \in \mathbb{C}_+$). This is essentially a classical theta function (cf. [25, Chpt. 13] and references therein). Furthermore,

$$E_{\operatorname{St}}^+(\widehat{v}, \widehat{\lambda}) = \sum_{w \in W} (-1)^{l(w)} E_{\operatorname{St}}(\widehat{v}, w\widehat{\lambda}).$$

Example 3.11. *Consider the trivial $\widehat{W}^Y$-module $M = \operatorname{Triv}$, which is the one-dimensional $\widehat{W}^Y$-module with associated linear character $w \mapsto 1$ ($w \in \widehat{W}^Y$). Let $j_w \in \mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+)$ ($w \in \widehat{W}^Y$) such that $J_w(\widehat{\lambda})$ acts as multiplication by $j_w(\widehat{\lambda})$ on Triv. Viewing E_{Triv} and $E_{\operatorname{Triv}}^+$ as scalar valued functions we thus get*

$$E_{\operatorname{Triv}}(\widehat{v}, \widehat{\lambda}) = \sum_{y \in Y} j_{t_y}(\widehat{\lambda}) e^{(t_y \widehat{\lambda}, \widehat{v})},$$
$$E_{\operatorname{Triv}}^+(\widehat{v}, \widehat{\lambda}) = \sum_{w \in W} E_{\operatorname{Triv}}(\widehat{v}, w\widehat{\lambda}) j_w(\widehat{\lambda}).$$

The cocycle values are explicitly given by

$$j_w(\widehat{\lambda}) = \prod_{a \in \widehat{R}^+ \cap w^{-1}\widehat{R}^-} \frac{(a^\vee, \widehat{\lambda}) + k_a}{(a^\vee, \widehat{\lambda}) - k_a}, \qquad w \in \widehat{W}^Y.$$

In particular, if $k_{\alpha+mc} = k_\alpha$ for all $\alpha \in R$ and $m \in \mathbb{Z}$ (this is automatically true if $P^\vee \subseteq Y$, with $P^\vee$ the coweight lattice of R) then

$$j_{t_y}(\widehat{\lambda}) = \prod_{\alpha \in R^+:(\alpha,y)>0} \prod_{m=0}^{(\alpha,y)-1} \frac{m\kappa_\alpha^\vee + (\alpha^\vee,\lambda) + k_\alpha}{m\kappa_\alpha^\vee + (\alpha^\vee,\lambda) - k_\alpha} \prod_{\beta \in R^+:(\beta,y)<0} \prod_{m=1}^{-(\beta,y)} \frac{m\kappa_\beta^\vee - (\beta^\vee,\lambda) + k_\beta}{m\kappa_\beta^\vee - (\beta^\vee,\lambda) - k_\beta},$$

where $\widehat{\lambda} = \lambda + \eta c + \kappa d$ and $\kappa_\alpha^\vee := 2\kappa/(\alpha,\alpha)$.

We present now a straightforward generalization of some of the statements of Theorem 3.6. Call $f \in \mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$ $\widehat{W}^Y$-invariant if $f \lhd w = f$ for $w \in \widehat{W}^Y$, where $\lhd$ is the right $\widehat{W}^Y$-action on $\mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$ defined by

$$(f \lhd w)(\widehat{\lambda}) := J_w(\widehat{\lambda})^{-1} f(w\widehat{\lambda}) J_w(\widehat{\lambda}).$$

Corollary 3.12. *Let M be a finite dimensional unitarizable $\widehat{W}^Y$-module and take $f \in \mathcal{O}(\widehat{\mathfrak{h}}_{reg}^+) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(M)$. Set*

$$E_{M,f}^+(\widehat{v}, \widehat{\lambda}) := E_M^+(\widehat{v}, \widehat{\lambda}) f(\widehat{\lambda})$$

for $\widehat{v} \in \widehat{V}^+$ and $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{reg}^+$.

For $\widehat{\lambda} \in \widehat{\mathfrak{h}}_{\kappa,reg}$ and $\widehat{v} \in \widehat{V}_\xi^+$ ($\kappa \in \mathbb{C}_+$ and $\xi \in \mathbb{R}_{>0}$) we then have

(1) $E_{M,f}^+(\cdot, \widehat{\lambda}) \in \widehat{S}_{\mathrm{End}_{\mathbb{C}}(M)}(\kappa, (\widehat{\lambda}, \widehat{\lambda}))^{\widehat{Q}(\widehat{W}^Y)}$,
(2) *if f is $\widehat{W}^Y$-invariant then $E_{M,f}^+(\widehat{v}, \cdot) \in \mathrm{adKZ}_\xi^{\cdot W}$.*

Proof. (1) is clear from the theorem.
(2) Since f is $\widehat{W}^Y$-invariant,

$$E_{M,f}^+(\widehat{v}, \widehat{\lambda}) = \sum_{w \in \widehat{W}^Y} e^{(w\widehat{\lambda},\widehat{v})} f(w\widehat{\lambda}) J_w(\widehat{\lambda}),$$

which is $\cdot\widehat{W}^Y$-invariant. $\square$

Let $\widehat{\lambda} : (0,1) \to \widehat{\mathfrak{h}}_{reg}^+$ be a path such that $\widehat{\lambda}(t) \to \lambda \in \mathfrak{h}_{0,reg}$ if $t \downarrow 0$. Then for all $w \in \widehat{W}^Y$,

$$\lim_{t \downarrow 0} e^{(\widehat{\lambda}(t),w(v+\xi d))} = e^{(\lambda,w\circ_\xi v)}$$

and for all $a \in \widehat{R}$,

$$\lim_{t \downarrow 0} (I(a)e^{\widehat{\lambda}(t)})(v + \xi d) = \int_0^{a^\xi(v)} e^{(\lambda,v-tDa^\vee)} dt.$$

Consequently

$$\lim_{t \downarrow 0} (\widehat{Q}(w)e^{\widehat{\lambda}(t)})(v + \xi d) = (Q_\xi(w)e^\lambda)(v), \qquad w \in \widehat{W}^Y.$$

This suggests that if $E_M^+(v+\xi d, \widehat{\lambda}(t))$ converges as $t \downarrow 0$, then it converges to the Bethe wave function $\psi_\lambda(v)$ associated to the vector valued root system analog of the quantum Bose gas on the circle with pairwise delta-function interactions. More concretely we expect that for

$\widehat{v} \in \widehat{V}_\xi$ and for $(\lambda, m) \in \mathfrak{h}_{0,reg} \times M$ satisfying the Bethe ansatz equations $J_{t_y}(\lambda)m = e^{\xi(\lambda,y)}m$ for all $y \in Y$, a renormalization of $E_M(\widehat{v}, \widehat{\lambda})m$ converges to $e^{\xi(\lambda,v)}m$ when $\widehat{\lambda}$ tends to λ along specific paths.

Here we will only consider the simplest case that $M = \mathrm{St}$. For the sake of simplicity we assume that $(Y, P) \subseteq \mathbb{Z}$ (cf. Example 3.10). Note that the k-dependence drops out for $M = \mathrm{St}$ (it relates to the limit $k \to \infty$ of the theory for the trivial representation $M = \mathrm{Triv}$, cf. [12, §3]). The $J_{t_y}(\widehat{\lambda})$ $(y \in Y)$ act trivially on St. Hence the Bethe ansatz equations simplify to the requirement that $\lambda \in \mathfrak{h}_{reg}$ satisfies

$$e^{\xi(\lambda,y)} = 1 \qquad \forall y \in Y.$$

Equivalently $\lambda \in \frac{2\pi\sqrt{-1}}{\xi} X_{reg}$, where $X_{reg} = X \cap \mathfrak{h}_{reg}$ and $X \subset V$ is the lattice dual to Y with respect to $(\cdot, \cdot)$. Fix $\lambda \in \frac{2\pi\sqrt{-1}}{\xi} X_{reg}$ and $\widehat{v} \in \widehat{V}_\xi$, say $\widehat{v} = v + \eta c + \xi d$. Then

$$E_{St}(\widehat{v}, \lambda + \kappa d) = e^{(\lambda,v)} E_{St}(\widehat{v}, \kappa d),$$

hence for all $y \in Y$,

$$\begin{aligned}
E_{St}(\widehat{v}, \kappa d)^{-1} E_{St}(\widehat{v}, \lambda + \kappa y + \kappa d) &= E_{St}(\widehat{v}, \kappa)^{-1} E_{St}(\widehat{v}, t_y(\lambda + \kappa d)) e^{\frac{\kappa\xi}{2}(y,y)+\xi(y,\lambda)} \\
&= E_{St}(\widehat{v}, \kappa)^{-1} E_{St}(\widehat{v}, \lambda + \kappa d) e^{\frac{\kappa\xi}{2}(y,y)+\xi(y,\lambda)} \\
&= e^{\frac{\kappa\xi}{2}(y,y)+(\lambda,v)}
\end{aligned}$$

which converges to $e^{(\lambda,v)}$ as $\kappa \to 0$. In particular, for $\widehat{v} = v + \eta c + \xi d \in \widehat{V}_\xi$ and $\lambda \in \frac{2\pi\sqrt{-1}}{\xi} X_{reg}$,

$$\lim_{\mathbb{C}_+ \ni \kappa \to 0} E_{St}(\widehat{v}, \kappa d)^{-1} E_{St}^+(\widehat{v}, \lambda + \kappa d) = \sum_{w \in W} (-1)^{l(w)} e^{(w\lambda, v)},$$

in accordance to [12, Prop. 3.3].

4. Time dependent Schrödinger equations with delta potentials

In this section we construct the nonstationary Schrödinger equation with delta potentials and its solutions. We first need to describe the image of the propagation operator T in more detail. For $b \in \widehat{R}$ set $H_b := \{\widehat{v} \in \widehat{V}^+ \mid (\widehat{v}, b) = 0\}$.

Proposition 4.1. *Let M be a $\widehat{W}^Y$-module and let $w \in \widehat{W}^Y$, $b \in w\widehat{F}$, $\widehat{v} \in H_b$, $p \in S(\widehat{\mathfrak{h}})$ and $f \in T\big(C^\omega(\widehat{V}) \otimes_{\mathbb{C}} M\big)$. Then*

$$(4.1) \qquad \big(p(\partial)f_{w\widehat{C}^+}\big)(\widehat{v}) - \big(p(\partial)f_{s_b w\widehat{C}^+}\big)(\widehat{v}) = k_b \pi_M(s_b)\Big(\big(\Delta_b(p)(\partial)f_{w\widehat{C}^+}\big)(\widehat{v})\Big),$$

where $\Delta_b(p) := \frac{s_b p - p}{b^\vee} \in S(\widehat{\mathfrak{h}})$. In particular $f_{w\widehat{C}^+}(\widehat{v}) = f_{s_b w\widehat{C}^+}(\widehat{v})$ and if $\widehat{u} \in \widehat{V}$, regarded as an element in $S(\widehat{\mathfrak{h}})$ of degree one, then

$$(\partial_{\widehat{u}} f_{w\widehat{C}^+})(\widehat{v}) - (\partial_{\widehat{u}} f_{s_b w\widehat{C}^+})(\widehat{v}) = -k_b(\widehat{u}, b)\pi_M(s_b) f_{w\widehat{C}^+}(\widehat{v}).$$

Proof. (compare with the proof of [12, Thm 5.3(ii)] and [13, Prop. 4.16]).

By $\widehat{W}^Y$-equivariance of T and (2.6), it suffices to prove (4.1) for $w = 1$.

Let $a \in \widehat{F}$, $\widehat{v} \in H_a$, $p \in S(\widehat{\mathfrak{h}})$ and $f = Tg$ with $g \in C^\omega(\widehat{V}) \otimes_\mathbb{C} M$. In $\widehat{H}^Y(k)$ we have the cross relation

$$s_a \cdot p - s_a(p) \cdot s_a = k_a \Delta_a(p).$$

Applying the representation map $\widehat{Q}$, acting on g and evaluating at $\widehat{v}$ we get

$$\pi_M(s_a)(p(\partial)g)(\widehat{v}) - \big((s_a p)(\partial)\widehat{Q}(s_a)g\big)(\widehat{v}) = k_a\big(\Delta_a(p)(\partial)g\big)(\widehat{v})$$

because $(\widehat{Q}(s_a)p(\partial)g)(\widehat{v}) = \pi_M(s_a)(p(\partial)g)(\widehat{v})$. Acting on both sides by $\pi_M(s_a)$ we get

$$(p(\partial)g)(\widehat{v}) - \big(p(\partial)\pi_M(s_a)(\widehat{Q}(s_a)g)(s_a\cdot)\big)(\widehat{v}) = k_a\pi_M(s_a)\big(\Delta_a(p)(\partial)g\big)(\widehat{v}).$$

Since $f_{\widehat{C}^+} = g$ and $f_{s_a\widehat{C}^+} = \pi_M(s_a)(\widehat{Q}(s_a)g)(s_a\cdot)$, we get the desired identity

$$\big(p(\partial)f_{\widehat{C}^+}\big)(\widehat{v}) - \big(p(\partial)f_{s_a\widehat{C}^+}\big)(\widehat{v}) = k_a\pi_M(s_a)\big(\Delta_a(p)(\partial)f_{\widehat{C}^+}\big)(\widehat{v}).$$

$\square$

For $f \in \mathcal{F}_M$ write $f^\dagger \in L^1_{loc}(\widehat{V}^+) \otimes_\mathbb{C} M$ such that $f^\dagger(\widehat{v}) = f_{\widehat{C}}(\widehat{v})$ if $\widehat{v} \in \widehat{C}$ ($\widehat{C} \in \mathcal{C}$). Write

$$C_M := \{f \in \mathcal{F}_M \mid f_{w\widehat{C}^+}(\widehat{v}) = f_{s_a w\widehat{C}^+}(\widehat{v}) \quad \forall\, w \in \widehat{W}^Y,\, a \in w\widehat{F} \text{ and } \widehat{v} \in H_a\}.$$

If $f \in C_M$ then the local L^1-function $f^\dagger$ can be represented by a continuous function on $\widehat{V}^+$, which we will also denote by $f^\dagger$. It is explicitly given by $f^\dagger(\widehat{v}) = f_{\widehat{C}}(\widehat{v})$ if $\widehat{v} \in \widehat{V}^+ \cap \overline{\widehat{C}}$. Note that $T(C^\omega(\widehat{V}^+) \otimes_\mathbb{C} M) \subseteq C_M$ in view of Proposition 4.1.

We consider $\widehat{V}$ as Euclidean space with scalar product

$$\langle \widehat{u}, \widehat{v} \rangle := \big(E(\widehat{u}), \widehat{v}\big),$$

where E is the linear involution of $\widehat{V}$ satisfying $E|_V = \mathrm{Id}_V$, $E(c) = d$ and $E(d) = c$. If $\{v_1, \ldots, v_m\}$ is an orthonormal basis of V with respect to $(\cdot, \cdot)$ then $\{v_1, \ldots, v_m, c, d\}$ is an orthonormal basis of $\widehat{V}$ with respect to $\langle\cdot, \cdot\rangle$. Let $\mathrm{d}\widehat{v}$ denote the Lebesgue measure on $\widehat{V}$ and $\mathrm{d}\sigma(\widehat{v})$ the induced measure on a hypersurface.

Define

$$\iota : L^1_{\mathrm{loc}}(\widehat{V}^+) \otimes_\mathbb{C} M \to \mathrm{Hom}_\mathbb{C}\big(C^\infty_c(\widehat{V}^+), M\big)$$

by

$$(\iota g)(\varphi) = \int_{\widehat{V}^+} g(\widehat{v})\varphi(\widehat{v})\, \mathrm{d}\widehat{v}.$$

Write

$$\widehat{\mathcal{H}}^M_k = \widehat{\Delta} + \sum_{a \in \widehat{R}^+} k_a \sqrt{\langle a, a \rangle} \delta\big((a, \cdot)\big) \pi_M(s_a)$$

for the linear map $\widehat{\mathcal{H}}^M_k : C(\widehat{V}^+) \otimes_\mathbb{C} M \to \mathrm{Hom}_\mathbb{C}\big(C^\infty_c(\widehat{V}^+), M\big)$ defined by

$$(\widehat{\mathcal{H}}^M_k g)(\varphi) = \int_{\widehat{V}^+} g(\widehat{v})(\widehat{\Delta}\varphi)(\widehat{v})\, \mathrm{d}\widehat{v} + \sum_{a \in \widehat{R}^+} k_a \sqrt{\langle a, a \rangle} \int_{H_a} \pi_M(s_a)g(\widehat{v})\varphi(\widehat{v})\, \mathrm{d}\sigma(\widehat{v}).$$

Proposition 4.2. *Let M be a finite dimensional $\widehat{W}^Y$-module. Assume $f \in C_M$ satisfies the derivative jump conditions*

$$(4.2) \qquad (\partial_{\widehat{u}} f_C)(\widehat{v}) - (\partial_{\widehat{u}} f_{s_b C})(\widehat{v}) = -k_b(b, \widehat{u}) \pi_M(s_b) f_C(\widehat{v})$$

for all $b \in w\widehat{F}$, $C = w\widehat{C}_+$, $w \in \widehat{W}$, $\widehat{u} \in \widehat{V}$ and $\widehat{v} \in H_b$. Then

$$\iota\big((\widehat{\Delta} f)^\dagger\big) = \widehat{\mathcal{H}}_k^M(f^\dagger).$$

Remark 4.3. By Proposition 4.1, the jump conditions (4.2) are satisfied if f lies in the image of the propagation operator T.

Proof. Let $\varphi \in C_c^\infty(\widehat{V}^+)$ be a test function, then

$$\iota\big((\widehat{\Delta} f)^\dagger\big)(\varphi) = \sum_{\widehat{C} \in \mathcal{C}} \int_{\widehat{C}} (\widehat{\Delta} f_{\widehat{C}})(\widehat{v}) \varphi(\widehat{v}) d\widehat{v}.$$

Substituting the definition of $\widehat{\Delta}$ and applying repeatedly the divergence theorem one obtains

$$
\begin{aligned}
(4.3) \quad \iota\big((\widehat{\Delta} f)^\dagger\big)(\varphi) &= \int_{\widehat{V}^+} f^\dagger(\widehat{v})(\widehat{\Delta} \varphi)(\widehat{v}) d\widehat{v} \\
&+ \sum_{\widehat{C} \in \mathcal{C}} \int_{\partial \widehat{C}} \Big(\langle (\mathrm{grad} f_{\widehat{C}})(\widehat{v}), E(N^{\widehat{C}}(\widehat{v})) \rangle \varphi(\widehat{v}) - f_{\widehat{C}}(\widehat{v}) \langle (\mathrm{grad} \varphi)(\widehat{v}), E(N^{\widehat{C}}(\widehat{v})) \rangle \Big) d\sigma(\widehat{v}),
\end{aligned}
$$

where $N^{\widehat{C}} : \partial \widehat{C} \to \widehat{V}$ is the unit outward normal vector field on the boundary $\partial \widehat{C}$ of $\widehat{C}$.

We simplify now the boundary terms. Write

$$
\sum_{\widehat{C} \in \mathcal{C}} \int_{\partial \widehat{C}} f_{\widehat{C}}(\widehat{v}) \langle (\mathrm{grad} \varphi)(\widehat{v}), E(N^{\widehat{C}}(\widehat{v})) \rangle d\sigma(\widehat{v}) =
$$

$$
= \sum_{a \in \widehat{R}^+} \sum_{\substack{\widehat{C} \in \mathcal{C}: \\ (a, \widehat{C}) \subseteq \mathbb{R}_{>0}}} \int_{\partial \widehat{C}^+ \cap H_a} \Big(f_{\widehat{C}}(\widehat{v}) \langle (\mathrm{grad} \varphi)(\widehat{v}), E(N^{\widehat{C}}(\widehat{v})) \rangle
$$

$$
+ f_{s_a \widehat{C}}(\widehat{v}) \langle (\mathrm{grad} \varphi)(\widehat{v}), E(N^{s_a \widehat{C}}(\widehat{v})) \rangle \Big) d\sigma(\widehat{v}).
$$

The contribution of the integral over $\widehat{C} = w\widehat{C}^+$ ($w \in \widehat{W}$) will be zero unless $a \in w\widehat{F}$, in which case $N^{\widehat{C}}(\widehat{v}) = -\frac{E(a)}{\sqrt{\langle a, a \rangle}}$ and $N^{s_a \widehat{C}}(\widehat{v}) = \frac{E(a)}{\sqrt{\langle a, a \rangle}}$ for $\widehat{v} \in \partial \widehat{C}^+ \cap H_a$. Since $f \in C_M$ we have in addition $f_{\widehat{C}}(\widehat{v}) = f_{s_a \widehat{C}}(\widehat{V})$ for $\widehat{v} \in H_a$, hence

$$\sum_{\widehat{C} \in \mathcal{C}} \int_{\partial \widehat{C}} f_{\widehat{C}}(\widehat{v}) \langle (\mathrm{grad} \varphi)(\widehat{v}), E(N^{\widehat{C}}(\widehat{v})) \rangle d\sigma(\widehat{v}) = 0.$$

Similarly

$$\sum_{\widehat{C}\in\mathcal{C}}\int_{\partial\widehat{C}}\langle(\mathrm{grad}f_{\widehat{C}})(\widehat{v}), E(N^{\widehat{C}}(\widehat{v}))\rangle\varphi(\widehat{v})d\sigma(\widehat{v})$$

$$=-\sum_{a\in\widehat{R}^+}\frac{1}{\sqrt{\langle a,a\rangle}}\sum_{\substack{\widehat{C}\in\mathcal{C}:\\(a,\widehat{C})\subseteq\mathbb{R}_{>0}}}\int_{\partial\widehat{C}^+\cap H_a}\Big(\frac{\partial f_{\widehat{C}}}{\partial E(a)}(\widehat{v})-\frac{\partial f_{s_a\widehat{C}}}{\partial E(a)}(\widehat{v})\Big)\varphi(\widehat{v})d\sigma(\widehat{v}).$$

By (4.2) it follows that

$$\sum_{\widehat{C}\in\mathcal{C}}\int_{\partial\widehat{C}}\langle(\mathrm{grad}f_{\widehat{C}})(\widehat{v}), E(N^{\widehat{C}}(\widehat{v}))\rangle\varphi(\widehat{v})d\sigma(\widehat{v}) = \sum_{a\in\widehat{R}^+}k_b\sqrt{\langle a,a\rangle}\pi_M(s_a)\int_{H_a}f^\dagger(\widehat{v})\varphi(\widehat{v})d\sigma(\widehat{v}).$$

Substitution in (4.3) shows that $\iota((\widehat{\Delta}f)^\dagger) = \widehat{\mathcal{H}}_k^M(f^\dagger)$. $\qquad\square$

For $\widehat{\lambda}\in\widehat{\mathfrak{h}}_{reg}^+$ and a unitarizable finite dimensional $\widehat{W}^Y$-module M set

$$\phi_{\widehat{\lambda}} := T\big(E_M^+(\cdot;\widehat{\lambda})\big) \in C_{\mathrm{End}_\mathbb{C}(M)}.$$

Note that $\phi_{\widehat{\lambda}}$ is the unique $\widehat{W}^Y$-invariant element in $\mathcal{F}_{\mathrm{End}_\mathbb{C}(M)}$ satisfying

$$\phi_{\widehat{\lambda},\widehat{C}^+} = E_M^+(\cdot;\widehat{\lambda}).$$

Note that $\widehat{\lambda}\mapsto\phi_{\widehat{\lambda}}(\widehat{v})$ is in adKZ_ξ^W if $\widehat{v}\in\widehat{V}_\xi$ ($\xi>0$).

Proposition 4.4. *With the above assumptions,*

$$\widehat{\mathcal{H}}_k^{\mathrm{End}_\mathbb{C}(M)}(\phi_{\widehat{\lambda}}^\dagger) = (\widehat{\lambda},\widehat{\lambda})\iota(\phi_{\widehat{\lambda}}^\dagger).$$

In particular, if $\widehat{\lambda}\in\widehat{\mathfrak{h}}_{reg,\kappa}$ ($\kappa\in\mathbb{C}_+$) then

$$(4.4)\qquad\big(2\kappa\partial_d + \Delta + \sum_{a\in\widehat{R}^+}k_a\sqrt{\langle a,a\rangle}\delta((a,\cdot))\pi_M(s_a)\big)\phi_{\widehat{\lambda}}^\dagger = (\widehat{\lambda},\widehat{\lambda})\iota(\phi_{\widehat{\lambda}}^\dagger)$$

in the weak sense, meaning that the left hand side tested against $\varphi\in C_c^\infty(\widehat{V}^+)$ is

$$\int_{\widehat{V}^+}\phi_{\widehat{\lambda}}^\dagger(\widehat{v})((-2\kappa\partial_d + \Delta)\varphi)(\widehat{v})d\widehat{v} + \sum_{a\in\widehat{R}^+}k_a\sqrt{\langle a,a\rangle}\int_{H_a}\pi_M(s_a)\phi_{\widehat{\lambda}}^\dagger(\widehat{v})\varphi(\widehat{v})d\sigma(\widehat{v}).$$

Proof. $\phi_{\widehat{\lambda}}\in C_{\mathrm{End}_\mathbb{C}(M)}$ satisfies the jump conditions (4.2) by Proposition 4.1. Hence

$$\widehat{\mathcal{H}}_k^M(\phi_{\widehat{\lambda}}^\dagger) = \iota((\widehat{\Delta}\phi_{\widehat{\lambda}})^\dagger) = (\widehat{\lambda},\widehat{\lambda})\iota(\phi_{\widehat{\lambda}})$$

by Proposition 4.2. The second statement follows from the first since $\partial_c(E_M^+(\cdot;\widehat{\lambda})) = \kappa E_M^+(\cdot;\widehat{\lambda})$. $\qquad\square$

Formula (4.4) can be interpreted as $\phi_{\widehat{\lambda}}$, with $\widehat{\lambda} = \lambda + \eta c + \kappa d \in \widehat{\mathfrak{h}}_{reg,\kappa}$ and $\kappa \in \mathbb{C}_+$, solving (weakly) the time dependent Schrödinger equation

$$\partial_d f = H_\kappa f$$

with quantum Hamiltonian

$$H_\kappa = \frac{1}{2\kappa}\left(-\Delta - \sum_{a \in \widehat{R}^+} k_a \sqrt{\langle a, a \rangle} \delta((a, \cdot)) \pi_M(s_a) + (\lambda, \lambda)\right) + \eta.$$

Remark 4.5. At $\kappa = 0$, (4.4) formally reduces to a weak eigenvalue equation for

$$\Delta + \sum_{a \in \widehat{R}^+} k_a \sqrt{\langle a, a \rangle} \delta((a, \cdot)) \pi_M(s_a).$$

This is *not* the weak quantum Hamiltonian of the vector valued root system analog of the quantum Bose gas on the circle with pairwise delta-function interactions as analyzed in [13, §4.5] with the help of the trigonometric Cherednik algebra $\widetilde{H}_0^Y(k)$, which is given by

$$\Delta + \sum_{a \in \widehat{R}^+} k_a \sqrt{(a, a)} \delta((a, \cdot)) \pi_M(s_a),$$

see [13, (4.18)].

Remark 4.6. Take $\widehat{\lambda} = \lambda + \kappa d \in \widehat{\mathfrak{h}}_{reg,\kappa}$ with λ satisfying (3.11). Then

$$\psi_\lambda^0(v) := \lim_{\xi \to \infty} \phi_{\lambda+\kappa d}^\dagger(v + \xi d), \qquad v \in V$$

is well defined and independent of $\kappa \in \mathbb{C}_+$. An explicit expression is given as follows. Let

$$K^+ := \{v \in V \mid (v, \alpha) > 0 \quad \forall \alpha \in F\}$$

be the fundamental Weyl chamber of V. Choose $w \in W$ such that $v \in \overline{wK^+}$. Then $v + \xi d \in \overline{w\widehat{C}^+}$ if $\xi \gg 0$ and

$$\psi_\lambda^0(v) = \pi_M(w)\big(Q(w^{-1})\psi_\lambda\big)(w^{-1}v),$$

cf. Remark 3.9. This shows that ψ_λ^0 is the Bethe wave function for the vector valued root system analog of the quantum Bose gas on the line with pairwise delta-function interactions. It is continuous and satisfies

$$\big(\Delta + \sum_{\alpha \in R^+} k_\alpha \sqrt{(\alpha, \alpha)} \delta((\alpha, \cdot)) \pi_M(s_\alpha)\big)\psi_\lambda = (\lambda, \lambda)\psi_\lambda$$

in the weak sense. These observations should be compared to the analysis at the trigonometric level in [10, §5].

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