MATHEMATICAL MATURITY FOR ENGINEERING STUDENTS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Electrical and Computer Engineering
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2018

Urbana, Illinois

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ABSTRACT

This dissertation presents four studies on the mathematical education of engineering students. The first study is a qualitative analysis of the beliefs of engineering faculty at a single institution regarding what constitutes "mathematical maturity" for engineering students. Faculty emphasized the need for mathematical modeling skills, fluent symbolic representation skills, and a combination of effortless algebraic fluency and ability to use computational tools. The second study is an analysis of the beliefs of engineering faculty at a variety of institutions. These faculty also emphasized modeling, representation, and computation, corroborating the results of the first study. The third study is an analysis of the mathematical content of engineering circuits and statics homework problems. Just 8% of statics problems and 20% of circuits problems use calculus, and in a much more limited way than what is taught in calculus. The fourth study presents a quantitative survey of engineering sophomores' perceptions of the relevance of mathematics to their engineering studies. The students have somewhat favorable views of the relevance of mathematics, but some high-performing students view mathematics as irrelevant.

To David Pierce, for starting me on the path to becoming an educator.
And to Caitlin McGuire, for helping me complete that journey.

ACKNOWLEDGMENTS

To David Pierce: Your friendship and mentoring are invaluable, and I would not be an educator if not for meeting you that day in the library.

To my advisor Geoffrey Herman: You are so unfailingly dutiful, supportive, generous with your time, and a thousand more adjectives. Every graduate student deserves to have an advisor like you, and I am lucky enough to have been one who actually did.

To my advisor Michael Loui: Your high standards keep me working to perform, and you support me when I'm stumbling and struggling. You are such a solid and ethical role model.

To my parents, Ed and Tammy Faulkner: I love both of you. Thank you for always encouraging my education and telling me I could do more and do better.

To Kelly Cross for your great patience with me and a your admirable courage.

To David Bahr for showing me what good mentoring was.

To Jennifer Cromley for being the most amazingly well read statistics nerd with a sharp critical eye and a willingness to help.

To my undergraduate professors, in no particular order: Ralph Neuhaus for keeping me under your wing, Thomas 'Doc' Bitterwolf for your inspiring enthusiasm I seek to always emulate, F. Marty Ytreberg for support with my project-turned-journal paper, Mark Nielsen for your commitment to improvement, Dave McIlroy for igniting my love of E&M, John Schneider for advice on graduate school selection, Christine Berven for the physics wisdom Bervenisms, Jason Barnes for your excitement both in class and at our SPS meetings.

To my sister Samantha: By being my student you taught me so much about being a teacher.

To Zak Estrada: If not for you, I wouldn't have found out about educational research and would not be writing this.

To Lara Waldrop and Paul Scott Carney for your service on my committee.

To the ECE110 team David Varodayan, Chris Schmitz, and Serge Minin for letting me into your classroom to do educational research.

To the staff at the UIUC writing center, for helping upgrade my terrible writing and your patience with my inside-out sentences, total lack of concision, and awful transitions.

To Nicole Johnson for help with my writing and bouncing ideas off each other.

To my partner Caitlin: You support and encourage me to be the best version of myself. You helped edit my work and encouraged me to go to the writing center. If not for you I wouldn't be finishing this now. You encourage me when I need encouragement. Thank you so much for all your love, dedication and help.

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LIST OF ABBREVIATIONS

ASEE	The American Society for Engineering Education
KOM	Dutch abbreviation, translated as "Competencies and Mathematical Learning"
MEA	Model Eliciting Activity
MWG	Mathematics Working Group, a subsection of the European Society for Engineering Education
SEFI	The European Society for Engineering Education

CHAPTER 1

INTRODUCTION

This work examines the mathematical needs of engineering students from multiple angles of approach and using multiple research methods. In this chapter, I summarize each of the four studies that compose this dissertation, the methods used, and the contribution to the existing literature.

1.1 Study A

In Study A, I present a qualitative thematic analysis of interviews with engineering faculty to determine their expectations of what mathematics their students need to succeed in their engineering courses. This study was motivated by asking what "mathematical maturity" for engineering students is. I conducted semistructured interviews with 27 engineering faculty in many disciplines from the University of Illinois at Urbana Champaign. Participants were generally most concerned about their students' abilities to apply mathematics to physical reasoning, to represent and communicate engineering ideas using mathematics as the language. Faculty expected that students have a balance of quick, fluent mathematical skills for very simple problems with an ability to prudently use computational tools for more complex problems. This work corroborates existing literature on the mathematical needs of engineering students by applying a more rigorous qualitative research methodology. This work was submitted to the International Journal of Research in Undergraduate Mathematics Education, and accepted with major revisions which have been included here. The three researchers who worked on this study were Brian Faulkner (research design, data collection, analysis, interpretation, writing), Katherine Earl (analysis), and Geoffrey Herman (research design, interpretation, writing).

1.2 Study B

In Study B, I present a second qualitative thematic analysis of interviews with engineering faculty. This replication study uses the same protocol as that of Study A, but the pool of subjects is drawn from many different types of institution to increase the generalizability of the results. The purpose of this study is to examine whether the results from Study A are still found in other institutional settings. The findings of this study are nearly identical to those of Study A, indicating that the themes likely generalize well to the nation at large. This work has been submitted to the International Journal of Engineering Education and is under review. The three researchers who worked on this study were Brian Faulkner (research design, data collection, analysis, interpretation, writing), Nicole Johnson-Glauch (analysis), and Geoffrey Herman (research design, interpretation, writing).

1.3 Study C

In Study C, I present a curricular matchup analysis of two engineering courses, Circuits and Statics, and compare the content of each against the calculus course which they list as a prerequisite. The full corpus of homework problems for one semester in each of these courses is taken as the sample, and analyzed with the mathematics-in-use technique [1]. Overall, very few of the assigned problems use any calculus at all, just 8% in statics and 20% in circuits. Of the problems that do use calculus, they apply just a tiny fraction of the preparatory mathematics taught in calculus. Applications of this finding to curriculum design and longitudinal reinforcement are presented. This work expands the existing literature by combining rigorous analysis of engineering assessments (avoiding the accuracy problems of self report) with a curriculum level scope (greater than that of previous assessment analyses). We plan to submit this study to the Journal of Engineering Education. The four researchers who contributed to this work were Brian Faulkner (research design, data collection, analysis, interpretation, writing), Nicole Johnson-Glauch (analysis), D.S. Choi (analysis), and Geoffrey Herman (research design, interpretation, writing).

1.4 Study D

In Study D, I present a preliminary quantitative analysis of engineering student attitudes towards the relevance of mathematics to their engineering studies. I used a previously applied survey instrument [2] to assess the attitudes of engineering students towards the relevance of their mathematics coursework. This replication study samples students at a different point in their mathematical education than previous studies; our subjects were sophomores in circuits or statics. Additionally, a conceptual evaluation of calculus skills was administered to evaluate whether attitudes toward mathematics were related to how much mathematics was remembered. Overall, students had mildly positive opinions of how relevant their mathematics coursework was to their engineering studies. This work has been submitted to the IEEE Frontiers in Education conference and is under review. The two researchers involved in this work were Brian Faulkner (research design, data collection, analysis, interpretation, writing) and Geoffrey Herman (interpretation, writing).

CHAPTER 2

STUDY A: INTERVIEWS WITH ILLINOIS ENGINEERING FACULTY

2.1 Introduction

> *It is to be hoped that these new [differential equations] courses will be taught by mathematicians rather than by engineers: the budget of any mathematics department is entirely dependent on the number of engineering students enrolled in our elementary courses. Were it not for these courses, which engineers generously defer to mathematicians, our mathematics departments would be doomed to extinction.*
>
> Gian-Carlo Rota, MAA invited address, 1997 [3]

Engineering departments are becoming increasingly concerned about retention and graduation rates as industry in the United States demands more engineering graduates to meet expected engineering job growth in coming decades [4]. However, since many students drop out of engineering, too few engineering students graduate to join industry [5]. More specifically, many students drop out of engineering not because they failed an engineering course, but because they failed a mathematics course [6, 7, 8]. Some programs blame mathematics courses for as many as a third of their dropouts [5, 9] . Most engineering programs require a standard "calculus sequence" of Calculus I, Calculus II, Calculus III, Linear Algebra, and Differential Equations. Students must pass prerequisite mathematics courses from the calculus sequence to continue into core engineering coursework [10, 11, 12]. The strictness of this prerequisite chain can particularly hamper students who are already disadvantaged due to disability or lack of access to high school calculus [13]. Students who do not start calculus-ready or fail a course in the calculus sequence may struggle to complete an engineering degree before financial aid runs out.

Because the calculus sequence has such a strong impact on engineering graduation, engineering departments are increasingly scrutinizing whether these high-failure courses are worth the investment. As Gian-Carlo Rota feared, engineering departments' dissatisfaction with the outcomes of calculus [14] has been used to justify drastic curricular change, such as that pursued by the Wright State program [15]. At Wright State, all engineering students take a special engineering mathematics course their first semester, which teaches all mathematics in the context of engineering problems and covers different topics from a typical first-semester calculus course. At other institutions like University of Louisville, the College of Engineering teaches all of the calculus sequence with its own faculty outside of a mathematics department. While it is not clear that the Mathematics Departments were actually at fault for poor student outcomes, the engineering faculty at these institutions clearly believed that to be the case and acted accordingly. Many hold the belief that engineering mathematics should be taught by engineers [16]. Having a better understanding of the mathematical outcomes expected by engineering faculty may help mathematics departments avoid these drastic options.

When asked why these courses are required, engineering faculty have responded that calculus is a prerequisite, in the words of one Operations Research instructor, *"for mathematical maturity more than just the actual calculus"* because *"the way [the engineering course] is taught, you can do it without calculus"* [17, p 49]). This idea begs the question, What is mathematical maturity according to engineering faculty?

Prior efforts have attempted to answer this question by promoting dialogue between mathematics and engineering faculty. For example, the Curriculum Renewal Across the First Two Years (CRAFTY) project led by the Mathematical Association of America [18] gathered 35 engineering faculty together with mathematics faculty members to generate a list of what engineering faculty expected students to know from their mathematics courses. Similarly, Ferguson [17] convened meetings between 12 engineering faculty members and 9 mathematicians to develop tasks that would indicate that a student had mastered calculus knowledge. These dialogues led to a consensus that more mathematical modeling, applied mathematics, and computational tools are needed than are in the current mathematics curriculum. In addition to these specific technical expectations, other subtler qualities like "life long learner, learning to think, mental discipline, and learning the mathematical thought process" [18] were proposed, though they were not well defined.

These prior efforts were selective and used time-intensive methods: multi-day or week-long workshops for the communities to build consensus. These prior efforts may bias findings toward over-representing the opinions of engineers who are willing to take such large amounts of time to discuss mathematics and are willing to work cooperatively with mathematics faculty. Additionally, these findings may over-represent opinions that both engineering and mathematics faculty found acceptable, downplaying expectations that engineering faculty might have that do not align with priorities of mathematicians. The process of dialogue changes the beliefs and knowledge of the participants as they learn to communicate across disciplinary boundaries [17]. Consequently, these opinions may not reflect the expectations of the engineering faculty who are most directly affected by students' preparation from the calculus sequence and may be vocal about their dissatisfaction. The opinions of these faculty may in part lead to the drastic curricular changes that mathematics departments seek to avoid.

To complement this prior work, we interviewed 27 engineering faculty from 11 disciplines to document their beliefs about what constitutes "mathematical maturity" for an engineering student. To fill the aforementioned gaps left by the previous efforts, we prioritized sampling faculty who taught courses that required courses in the calculus sequence as direct prerequisites. Additionally, while CRAFTY relied on faculty to articulate mathematical maturity in their own words, we seek in this study to situate and describe faculty members' beliefs within the context of prior mathematics education research to potentially reveal research-based avenues for addressing these faculty concerns. We analyzed these faculty interviews using thematic analysis in light of prior research on epistemic beliefs, symbol sense, and competencies to answer the following two research questions.

Research Question 1) What is "mathematical maturity" for engineering students, according to engineering faculty members?

Research Question 2) To what extent do engineering faculty perceive their students to possess mathematical maturity?

2.2 Background

Since *"Mathematical maturity is not a coherent, single entity, but an amorphous mix of diverse characteristics, each supported by special talent and special interests"* [19, p 107], we cannot expect that any single construct will provide a complete view of mathematical maturity. As we seek to define what mathematical maturity means to engineering faculty, we draw on three constructs from the mathematics education research literature as possible starting points: epistemic beliefs [20], competencies [21] and symbol sense [22].

Epistemic beliefs provide a framework to formalize the everyday notion of how students conceive of what mathematics is, what mathematics is for, and what activities count as mathematics. Epistemic beliefs were chosen as a construct for this study because they may capture the subtle qualities such as "learning to think" that have been left mostly undefined in previous efforts [18, 17]. From now on in this work, we will use "epistemic beliefs" to refer to the personal beliefs that students hold about the nature of mathematical knowledge and its relation to engineering, not to the students' philosophy of knowledge in general, nor the philosophy of knowledge held by the faculty [23].

Mathematical competencies from the Dutch KOM project [21] were chosen to capture skill-based aspects of mathematical maturity, and to see if engineers placed greater emphasis on one subset of competencies than others. Basic mechanical skills are needed, and the KOM competencies cast a wide net over many possible types of mathematical abilities. The KOM competencies are a general and broad framework of mathematical ability. Such a broad framework should capture any unexpected themes not well described by the other frameworks.

Symbol sense [22] was chosen to see if symbolic representation of mathematical ideas might be preventing students from being able to recognize the mathematical knowledge when it occurs in engineering courses. Since much of the mathematics done in engineering is in the symbolic domain, symbol sense was chosen rather than a more general construct like representational fluency. Symbol sense emphasizes recognizing common patterns and maintaining multiple simultaneous perspectives on an expression. This context-agnostic perspective may be part of the "learning to think" aspect mentioned in previous literature.

In summary, if a student has no mathematical capabilities (KOM competencies), one cannot hope to have the mathematical maturity to apply mathematics in engineering courses. If a student does not believe that mathematics applies (epistemic beliefs), they will not attempt to apply it when there is opportunity. If a student cannot recognize familiar mathematical structures (symbol sense), they cannot have the mathematical maturity to apply that knowledge in engineering. These three constructs provide adequate coverage to explore what engineering faculty believe constitutes mathematical maturity for their students, and their perception of the extent to which their students possess mathematical maturity.

2.2.1 Epistemic beliefs

When engineering faculty say things like "learning the mathematical thought process," it is reminiscent of epistemic beliefs: beliefs about knowledge and knowing [20]. For example, Gainsburg described the mathematical thought process of veteran structural engineers with the epistemic belief of "skeptical reverence," valuing mathematics while always double-checking the outcome of calculations [24, 25]. However, the connection between engineering faculty expectations and students' personal mathematical epistemic beliefs has not been explored in the research literature. Epistemic beliefs vary by discipline; for example, a student might believe mathematics knowledge is certain and fixed, but history knowledge is uncertain and contextual [26]. Epistemic beliefs also vary by context; a student might express one belief about knowledge when preparing for an exam and another belief when explaining knowledge to a friend [27]. At the same time, a student may have beliefs about knowledge that are contrary to an expert's beliefs. A student's possession of expert-like epistemic beliefs may compose part of 'mathematical maturity' for engineering students.

Epistemic beliefs about mathematics have been shown to impact engineering problem solving [28]. For example, in one study, a student's belief that he should trust mathematical calculation over physical intuition rendered him unable to resolve a sign error in his calculation [29]. Epistemic barriers separate formal mathematics from the informal, intuitive mathematics demanded in engineering. For example, students believe that intuitive, graphical, and informal reasoning is improper in engineering and avoid using such methods for fear of being marked wrong [28, 20, 27].

The typical epistemic beliefs of college freshmen toward mathematics are well documented (see Table 2.1). These beliefs include include the belief that mathematics is irrelevant outside the mathematics classroom (practical irrelevance), and that all mathematics problems have exactly one solution procedure (orderly process) [20, 26]. These epistemic beliefs can hinder students from learning mathematics or applying it to new contexts. For example, orderly process beliefs hinder students from solving the open-ended "realistic" problems that are important to engineering [29, 30]. These epistemic beliefs about mathematics, unlike other disciplines, are formed almost entirely by the student's experiences in school [20]. For example, these beliefs may form because the typical calculus curriculum emphasizes easy-to-test calculations [31]. Consequently, it is likely that the existence of these epistemic beliefs will manifest in engineering contexts. We used these epistemic beliefs to guide the construction of our interview protocol and included each of these beliefs in our coding scheme (see section 2.3).

2.2.2 Competencies

As described earlier, the CRAFTY project identified a set of mathematical skills (rather than beliefs) that engineering students needed. To parallel these findings, we incorporated the KOM competencies (translated from Dutch as "Competencies and Mathematical Learning") [21]. The KOM project sought to explore the many facets of mathematical competency at all levels of education, though it did not focus on university-level mathematics. In contrast with the focus on epistemologies, KOM lists students' abilities, not beliefs. A list of proposed competencies from KOM can be found in Table 2.2. We chose the KOM because it aligns with some findings from the CRAFTY project and Ferguson's [17] study, such as their finding about the importance of mathematical modeling.

The KOM competencies are also an appropriate framework to use in this study because they have been used in the context of engineering education. For example, Alpers documented the degree of coverage each competency

Table 2.1: The documented immature epistemic beliefs about mathematics and examples of negative associated student behaviors observed in previous literature [20, 32].

Belief	Definition	Associated Negative Behaviors
Innate Ability	Belief that mathematical ability is unchanging.	Poor study strategies [32].
Quick Learning	Belief that learning and problem solving are quick processes.	Giving up on problems taking more than 10 minutes. [32]
Orderly Process	Belief that mathematical knowledge and problem solving do not involve uncertainty or failure.	Greater difficulty solving open-ended "realistic" engineering problems [29, 30].
Simple Knowledge	Belief that mathematical knowledge is disconnected and isolated, and that information gained in one lesson has no bearing on knowledge in future or past lessons.	Greater difficulty with transfer, reduced comprehension and metacognition [32, 26]
Certain Knowledge	Belief that mathematical knowledge is perfect and certain.	Oversimplified conclusions about a problem [32].
Omniscient Authority	Belief that mathematical authority (textbook or instructor) forms the basis for truth.	Giving more incoherent and incorrect definitions of the limit in calculus [33].
Practical Irrelevance	Belief that school mathematics has no bearing outside the classroom and that formal mathematics is not connected to common sense.	Reduced attempts at sensemaking [34, 29].
Solitary Mathematics	Belief that mathematics is done by individuals alone and does not need to be communicated to others.	Reduced mathematical communication skill [20].

may require in engineering coursework [35]. Other researchers [36, 37, 38, 39] have examined how mathematical competencies developed in school may not align with those needed in industry.

Because mathematical modeling has been described as important, we briefly define this competency. The modeling cycle [40] involves mathematizing a verbal or pictorial description of the system, solving the resulting mathematical equations, then interpreting the solution in terms of the original situation.

Table 2.2: The KOM competencies and short definitions of them [21]. The competencies are overlapping, and some competencies might be more important to engineers than to mathematicians.

Competency	Definition
Thinking	The student can pose questions like "Is there a. . . ?" or "Is it possible that. . . ?" and has insight into answers. This student understands scope of statements, and statement types (definition, theorem, phenomenon).
Problem Tackling	The student can formulate and solve *non-routine* problems.
Modeling	The student can de-mathematize and interpret models, and actively create and criticize mathematical models.
Reasoning	The student can follow and assess formal reasoning and proof.
Representing	The student can apply algebraic, visual, graphical, tabular, verbal, geometric, diagrammatic and material objects as representations of mathematical truth, switching representation as needed.
Symbol & Formalism	The student can decode and translate symbols, use symbols, and has insight into rules for using symbols.
Communicating	The student is adept at reading and writing mathematics, with words, pictures, and equations.
Tools/Aids	The student knows powers and limitations of calculators, special paper, computer algebra systems, computational simulations, and physical props.

2.2.3 Symbol sense

One challenge for students as they transfer from mathematics to engineering contexts lies in differences in notational standards and conventions. For example, while the derivative of y with respect to x in mathematics is typically represented with $\frac{dy}{dx}$ or y', some engineering disciplines use the $\dot{y}$ notation. Similarly, mathematicians and physicists use the imaginary number convention for waves $e^{kx-i\omega t}$ where most engineers use the notation $e^{-kx+j\omega t}$.

To account for these disciplinary differences in notational standards and conventions, we used "symbol sense" (see Table 2.3) to inform the design of our interviews and included it in our analysis code book. Symbol sense broadly describes a student's tendencies when using symbols and notation [22]. As students progress from high school to college, the load on their symbolic reasoning ability rises sharply [41]. For example, new types of symbols appear such as vector and matrix valued quantities, complex numbers, and operators that act on functions. Many new symbols are introduced, and old symbols gain new meanings. Simultaneously, notation between and within disciplines begins to fragment [42]. Many students lack the ability to extract what type of object a symbol represents by looking at its place in the equation. Additionally, students have great difficulty with symbolic form answers [43] and struggle with the notational nuances of the object/process duality of derivatives [44].

Thus, the ability to use symbols flexibly, create and discard notation, and use symbols to reduce effort may be a part of mathematical maturity [19]. As mentioned by Arcavi [22], much of symbol sense may be epistemic in nature. For example, students do not see the point of problems with answers that are expressions of letters rather than single numbers [17]. Students do not know when to back out and try a new approach when the analytic representation is unproductive or reformulate the problem so that its symbolic form is more malleable [45].

2.3 Methods

Since the CRAFTY project and Ferguson's prior studies both relied on time-intensive, dialogue-based methods to encourage engineering and mathematics faculty to come to consensus about how to align the mathematics and engineering curricula, we emphasize the perspectives of engineering faculty who have not been as engaged in these inter-disciplinary discussions. To find faculty who were most directly impacted by the mathematics curriculum, we prioritized sampling faculty who taught courses that required courses in

Table 2.3: The elements of symbol sense and brief definitions [22].

Symbol Sense	Definition
Quantitative Reasoning with Symbols	Scanning an algebraic expression to make rough estimates of the patterns that would emerge in numeric or graphic representation and can make informed comparisons of orders of magnitude for functions with rules of the form n, n^2, n^3 and n^k
Selecting a Symbol	Understanding how and when symbols can and should be used in order to display relationships, generalizations, and proofs which otherwise are hidden and invisible
Abandoning Symbols	Knowing when to abandon symbols in favor of alternative approaches in order to find an easier or more elegant solution or representation
Manipulating Symbols	Handling symbols quickly and efficiently, detaching oneself from their referents
Reading Meaning from Symbols	Being aware of the constant need to check symbol meanings while solving a problem, and comparing and contrasting those meanings with one's own intuitions or with the expected outcome of that problem
Symbols in Context	Sensing the different roles symbols can play in different contexts; One symbol may be bound to two different ideas. "Variables" and "parameters" permit different types of manipulations
Engineering (Designing) Symbolic Relationships	Creating an ad-hoc expression for a desired purpose

the calculus sequence as direct prerequisites. If students lack the desired "mathematical maturity" entering these follow-on courses, engineering faculty can easily construct an argument for drastic options such as teaching mathematics courses themselves. While the views of these faculty should not be taken as empirical fact, these perceptions do represent the constructed reality upon which these engineering faculty will act. Our study therefore takes a constructivist approach [46], seeking to document the opinions and beliefs of engineering faculty have constructed, apart from mathematicians. Mathematics faculty may actually be preparing engineering students better than engineering faculty realize or could do themselves, but perceptions of reality are the basis for decisions of individuals rather than objective reality. An understanding of these perceptions will help mathematics departments more clearly understand the wants and needs of engineering as a whole, to avoid undue influence from a single motivated person or department when negotiating and collaborating with their engineering college.

To document these perceptions, we chose to take a qualitative approach, conducting open-ended interviews that would encourage engineering faculty to fully explain their individual positions. The goal of our qualitative research is to richly document the perspectives of participants, presenting findings that represent the perspectives of our research population (internal validity) while acknowledging that our data does not provide evidence about the extent to which engineering faculty at large hold these views (external generalizability) [47]. We will discuss this consideration further in our limitations sections.

We chose to do interviews over surveys, because surveys do not easily permit follow-up questions to explore unexpected perceptions, limiting the richness of our descriptions. Since previous research [17, 18] showed it took a long time for faculty to uncover their expectations about what it really meant to understand calculus and there were numerous disagreements due to vocabulary and word use, we chose one-on-one interviews to allow for deeper conversations that can adapt to the vocabulary and expectations that are idiosyncratic to the individual or discipline.

To get sufficient breadth of observations, we chose to interview faculty from a variety of engineering disciplines. To allow sufficient time to explore each participant's perspective, we scheduled one-hour interviews but allowed them to go longer if the participants had more to say.

Consistent with our Constructivist approach, we chose to use thematic analysis [48] to document the expectations of the faculty interviewees. A thematic analysis interview method was not used in previous work [17]. Thematic analysis provides a robust and flexible way to analyze qualitative data that allows the perspectives of participants to emerge from the data. Thematic analysis is a data-driven approach that finds patterns and commonalities among participant's statements, enabling the researchers to describe participants' perspectives with their own words. This approach also lets the researchers build on prior theory without being constrained by it, thereby not forcing the researchers to conform their themes to existing theories so that prior work can enrich interpretation but not restrict it.

We chose a semi-open coding scheme with a priori codes in order to ground the research firmly in the existing mathematics education research, while remaining responsive to unexpected perceptions from the participants. We sought to ground our analysis in the existing research literature because these findings have been robust in understanding mathematical maturity in other contexts. However, it is possible that the constructs we selected may not span the entire range of the perceptions of engineering faculty. We therefore permitted additional codes to emerge during analysis.

2.3.1 Participant selection

We interviewed engineering faculty members about their experience teaching core engineering courses and the mathematical abilities of their students. We selected participants who had taught any engineering course that requires at least one course from the calculus sequence (Calculus I, Calculus II, Calculus III, Linear Algebra and Differential Equations) as either a direct prerequisite or corequisite. These faculty members had taught such an engineering course during the previous academic year, and 60 such faculty were identified. Faculty members who have taught these courses recently have their experiences fresh in their minds. Faculty members who teach immediate follow-up courses are the most strongly affected by mathematics preparation issues and are the group most likely to be involved in negotiations with mathematics departments during curricular decisions. We solicited participants by email and they were entered into a lottery for a $200 gift card as compensation.

We interviewed all 24 of these faculty members who volunteered. The participants varied from untenured assistant professors to senior faculty who had been teaching for 20 years.

The institutional context is a large, elite, American research-intensive institution with a student population of $\tilde{4}$0,000 students. The elite institution has very mathematically prepared students; the lower quartile of accepted students' ACT math standardized test score corresponds to the 95th percentile of the general population. The institution has a very high international undergraduate population of $\tilde{2}$0%, and programs within engineering generally have 10-30% female students.

If any participant personally recommended a peer as an excellent candidate for the study given their knowledge of students and mathematics, that peer was also sought out and interviewed. Three such additional faculty members were interviewed. These additional faculty were developing disciplinary mathematics courses within their departments. They possessed uniquely refined knowledge of mathematics needs inside their discipline and provided insight into their departments' dissatisfaction with mathematical preparation.

The sample of faculty included members from 12 of the 13 engineering departments on our campus: Industrial, Electrical, Mechanical, Civil, Nuclear, Agricultural, Chemical, Physics, Computer Science, Bioengineering, and Materials, but not Aerospace.

2.3.2 Interview procedure

We conducted semi-structured interviews with an initial interview protocol that allowed for asking off-script questions to explore the views of the participant when pertinent to the research question. The interview questions were designed to explore constructs we hypothesized were the roots of mathematical maturity (i.e., symbol sense, epistemic beliefs, and competencies). If the interviewed faculty member did not mention "mathematical maturity" or describe a similar concept with different words independently, then they were asked at the end of the interview if the term "mathematical maturity" meant anything to them. Interviews lasted for approximately one hour and participants were not sent the interview protocol in advance.

Example protocol questions included

- "What mathematical skills, abilities, or attitudes are essential to succeed in your course?" (general)
- "Do you encourage graphical/intuitive reasoning or analytic/formal reasoning?" (KOM competencies: representing competency)
- Do students perceive the "real life" applications of the math they have been taught? (Epistemic beliefs: practical irrelevance)
- How do your students manipulate and work with symbols abstractly? (Symbol sense: manipulating symbols)

2.3.3 Data analysis

Because the protocol was derived by two researchers who both had extensive backgrounds in mathematics and engineering, we added a third researcher who had minimal background with collegiate-level mathematics and no experience with engineering but with experience in qualitative research who could challenge the lead researcher's assumptions about the interplay between engineering and mathematics. This pair of researchers analyzed the interview data using thematic analysis. First, the team transcribed the interviews verbatim and agreed on a unit of analysis. The unit of analysis defines what statement or collection of statements can be assigned a code from the codebook. The team defined the unit of analysis as "an uninterrupted passage of participant speech between interviewer prompts." This unit of analysis is unambiguous and allows for full arguments on a topic by the faculty member to be considered as a single unit.

Second, the team constructed an a priori codebook to anchor the study to previous literature in mathematics education. The initial a priori codebook was constructed from documented immature epistemic beliefs, the KOM competencies, and symbol sense (Tables 2.1, 2.2 and 2.3).

Third, to ensure that both researchers agreed on the definition and interpretation of each code in the code book, the research team iteratively developed and refined the codebook. The team analyzed a random selection of three interviews, looking for perspectives not covered by the initial coding scheme. Each researcher coded these interviews independently, labeling each unit of analysis with a code (or multiple codes) that the researcher thought sufficiently described that participant's perspective. Interviews were coded independently to encourage the researcher with minimal mathematics background to inject their perspective into the analysis. After independent analysis, the two researchers conferred and resolved disagreements, refining the codebook by adding new codes and rewording definitions for each code. The team repeated this process two more times once with a second set of three randomly selected interviews and once more with another random selection of ten interviews.

Fourth, after analyzing these initial interviews, the team finalized the list of codes and their definitions. The team then independently analyzed the remaining eleven interviews using this finalized coding scheme (see Appendix B). The team compared which codes were applied to each unit of analysis, tallying agreements and disagreements. An inter-rater reliability of 81% was reached, indicating good agreement [49].

Fifth, we developed themes that connected the codes in the data together. The codes were sorted into themes, gathering codes together based on how often they co-occurred in the same unit of analysis or nearby in the same interview. These groups of codes suggest that the codes in the group are related to one another and have a sort of narrative structure. Each theme was held to the standard that it should be more than the sum of its component codes. We then reviewed these themes by generating a "thematic map" [48] of the analysis. These thematic maps (see Figure 2.1) relate multiple codes together into a more cohesive whole. After refining these themes through repeated discussion, we defined and named each theme.

After groups of related codes were assembled together, codes were arranged into an ordered structure that expresses containment, suspected causality, and centrality relationships. If the researchers believed such a narrative structure could be constructed, this code group was promoted to a candidate theme and a thematic map [48] for the candidate theme was generated. A thematic map forces the researchers to explicate the narrative structure of the theme. This explicit structure enables systematic and skeptical critique of the relationships between codes such as examination of co-occurrences or adjacent occurrences of codes in the data. This critique can lead to new codes being brought into the theme, or more often, codes with weaker supporting evidence being cut from the prospective theme. After refining the theme, a final thematic map is generated. This map was used to define and name the theme.

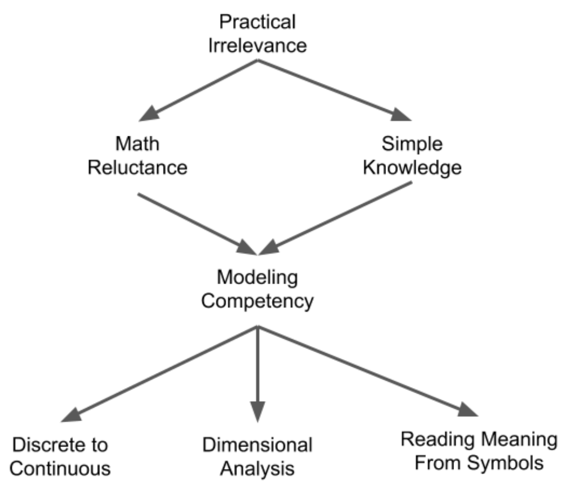


Figure 2.1: Thematic map. Codes are arranged and connected together from centralizing ideas to smaller, less central ideas. Nodes with many arrows leaving them are more important, more central ideas within the data, have suspected causal influence on those they point to, or contain in some way the objects they point to. The initial thematic map is condensed and refined to better display the narrative structure inherent in the data, by considering co-occurrence of codes within a particular passage and within interviews. Codes whose presence in the theme cannot be firmly justified are removed from the map, and connections between codes are arranged to align with the narrative of the theme.

2.3.4 Trustworthiness

The first coder (also the interviewer) had an undergraduate background in physics and mathematics, and graduate training in electrical engineering.

Participants were selected because they had taught an engineering course that used the calculus sequence as a prerequisite. The interviewer had never taught any of the engineering courses, and so had little disciplinary bias to favor one participant's expectations over another.

The second coder had no college-level mathematical experience beyond statistics for social scientists. The use of two coders helps reduce the likelihood that the results are a single researcher's opinion projected onto the data. The presence of a mathematical outsider keeps the preconceptions of the more mathematically entrenched first coder in check, since perceptions of the importance of ideas must be fully justified. We did not perform member checking on the results. When a faculty member said something very similar to previous participants and was informed of previous participants' ideas at the end of the interview, they did often comment that they felt good that they were not the only ones who felt that way about the topic. As will be seen later in this document, the results strongly corroborate other findings on engineering mathematics by other authors and by different methodologies.

2.3.5 Limitations

This study has a few limitations of this study that should be explicitly discussed, which are in addition to the limitations of all qualitative research.

- Our study focuses on describing engineering faculty's perceptions on what mathematical maturity students need to be successful. As we argue from our constructivist perspective, these perceptions may not accurately describe the reality of what mathematical skills and beliefs students actually need to be successful.

- The sample was drawn from a large, PhD-granting research institution with elite students with median ACT math scores of 31 as compared to the nationwide median of 24-26, who are more mathematically prepared than the typical undergraduate. The elite status of the institution may skew the findings, as these faculty may have higher expectations of their students than is typical nationwide. The faculty may also overemphasize mathematical skills that are useful for research rather than industrial practice, given the research-intensive nature of the institution. For example, the emphasis on computational tools or the dismissal of some skills as trivial may reflect the elite status of the institution but not the mathematical skills that engineering students at other institutions might need.

- The sample was drawn from a single institution. Campus colleagues are not independent and may develop a local culture regarding what engineering mathematics should be that varies from the population of engineering instructors at large. Further work will examine these beliefs with engineering faculty members from a variety of different institutions.

- The interviewer specifically discussed epistemic belief and symbol sense issues during the interviews. Participants' responses are influenced by prompting, and the symbol/representation theme may be over-represented. No similar theme emerged in previous studies by CRAFTY, Moore or Ferguson [18, 40, 17]. This bias is difficult to resolve and must be taken into consideration while examining the results.

- The interview questions included linguistic features that may trigger conformity to widespread views of the role of mathematics in engineering, such as "applicationism" [23].

- These interviews were single, isolated, one-hour events. The short time scale and lack of prior reflection may have led to shallow interview responses.

- The engineering faculty expectations were within the context of the typical lecture and exam assessment structure of most institutions of higher education in the United States. Participants gave their expectations within this framework.

- In addition, the first coder has his own biases as a researcher and as a practitioner of mathematics. The second coder has no college-level mathematics experience. This lack of experience helps mitigate this bias somewhat, since the first coder must explain views explicitly.

2.4 Results

In this section, we present the results of the thematic analysis. We claim that the following three themes are representative of the most common and important beliefs of the engineering faculty in our study.

- Theme 1: A mathematically mature student uses and interprets mathematical models.
- Theme 2: A mathematically mature student chooses and manipulates symbolic and graphical expressions.
- Theme 3: Computational tools reshape "what needs to be known" to be mathematically mature.

To demonstrate this generalizability within the sample, we present frequency counts [47] in Table 2.4 to illustrate how often each theme (and an associated representative code) appeared across all interviews and each participant.

Table 2.4: The three themes and each of their largest constituent codes. Modeling competency is the largest constituent code for theme 1. The total number of coded segments corresponding to each theme in the data are included to show presence in the data, and the percentage of participants mentioning that code or that theme is included to demonstrate representativeness.

	THEME 1	Modeling Competency	THEME 2	Symbol Competency	THEME 3	Tools/Aids Competency
Code/Theme Occurrences	310	74	74	23	260	68
% of Participants who Mentioned Code/Theme	100%	74%	81%	48%	100%	74%

2.4.1 Theme 1

"A mathematically mature student uses and interprets mathematical models."

This theme is supported by two beliefs expressed by the participants: Engineering faculty asserted that mathematical modeling skills are the most important skill for engineering students and these faculty believed that poor epistemic beliefs undermine students' ability to develop these modeling skills.

It is worth noting that the aspects of modeling that engineering faculty referred to did not span the entirety of the mathematical modeling cycle. Participants often referred to only two steps, the "mathematizing" step and the "interpreting" step. This finding is previously found in the literature [50], where engineering homework problems use only some of the mathematical modeling cycle, which could be called merely "application" rather than the more encompassing "modeling."

Strong modeling skills

Participants described that a mathematically mature engineering student is adept at modeling engineering problems and also believes mathematics is relevant to engineering. Of the KOM competencies, the modeling competency was the most important to participants: Participants' statements were coded as modeling competency 74 times, and 74% of participants had statements that were coded as modeling competency (the median KOM competency was coded 9 times). Similarly, all participants had at least one statement that was coded as part of this first theme. They described a mathematically mature student as able to translate from mathematical to physical representations of situations and to interpret the physical meaning of a mathematical result.

> *[I want students to] take a statement in plain English, describing how a system works, turn it into an algebraic equation, and then do the operations to solve the algebraic equation.*
>
> —Physics Faculty Member

According to participants, a mathematically mature student is expected to be able to set up integrals and derivatives to describe physical situations. For example, to participants' dismay, students struggle to set up control volumes for the flow of water or to sum up small differential elements of charge to find total charge, despite participants' expectations that these skills should be the primary outcome of finishing calculus. Faculty stated a mathematically mature student should be able to use units to solve problems and examine answers; however, they observed their students' ignorance of dimensional analysis. While students can manipulate simple algebra, they are unable to connect the mathematical meaning of those manipulations to any physical meaning. A faculty member describes the lack of mathematical maturity she observed in the classroom:

> *And sometimes they have problems understanding that the minimum in energy means that the force is zero. Again, because they may be able to do* $\frac{d}{dr}$*, but maybe they don't really know what it means physically. So many of them just set the energy to* 0*. They don't get it that there's a minimum in the energy curve, it means the force is zero.*
>
> – Materials Science Faculty Member

Participants stressed that mathematically mature students should already have the ability to extract mathematical meaning to set up a problem as well as perform algebraic/calculus manipulations. Participants highlighted their expectation for students to be able to *set up* an integral or derivative to describe a situation. A faculty member described his experience with his students struggling with these competencies.

> *The word problems are not saying solve the integral of this, it's here's a situation, translate which variables are known and which variables are unknown. Then set up the problem to solve it. It requires reasoning to set it up. I don't know how they don't know.*
>
> – Civil Engineering Faculty Member

As can be seen from these data, engineering faculty members expect their students to have solid modeling skills as a result of their previous mathematical experience. The expectations are mismatched with the realities of completing calculus. This mismatch between the expectations of outgoing students of mathematics and incoming students of engineering has been previously observed by Ferguson [17].

Unfortunately, faculty voiced their discontent with students' modeling competency. They noticed that students believe that mathematics is not relevant to engineering; without the mathematically mature belief that mathematics is relevant, students struggle with their modeling skills. Mathematical maturity entails not only the ability to use mathematics to make sense of the physical world, but also the belief that one should. This belief was captured with the code "Practical Irrelevance" 78 times whereas the median epistemological belief was coded only 19 times.

Not only can this irrelevance belief influence students' learning in pure mathematics courses, but it can extend to mathematics-heavy engineering courses. One materials engineering faculty member recalled her students commenting on her sophomore theory class: *"This isn't really [materials engineering]. This is just math or not really relevant to our major."* A civil engineering faculty member commented on his students' struggles with mathematics in the absence of relevant engineering examples, stating *"You're going to be a junior before you see any application of anything, so you better hold on tight!"* Although students' skills and abilities in mathematics may be admittedly poor, a deeper problem resides in their belief about mathematics. A physics faculty member said that students *"view calculus a, just equations and math in general as a calculating device,"* rather than the more mature perspective as a *"concise way of capturing the underlying physical principles."*

The belief that mathematics is irrelevant and painful shapes student choices. Many participants emphasized that this distaste for mathematics was a result of the lack of application in the early mathematics courses. A bioengineering faculty member stated, *"They take [bioengineering] because they thought it would be less math. The attitude of fear of math because it was disconnected [sic]."*

Faculty members claimed that the student attitude that mathematics is irrelevant results from the abstract, application-starved presentation of introductory mathematics courses. This abstract presentation cultivates student beliefs that mathematics is not relevant outside the mathematics classroom.

Faculty felt that they must convince students that mathematics they have learned in the past does provide useful techniques to understand the engineering systems they currently study. This bioengineering faculty member spoke about his struggle with convincing students of the relevance of previously learned mathematics.

> *As soon as you make the transition into the physiology domain, the problem is convincing students that what they did learn is applicable still, and that a plug and chug problem does provide useful information to physiological systems.*
>
> —Bioengineering Faculty Member

This bioengineering faculty member's challenge was not correcting their lack of mathematics skill, but correcting their beliefs about mathematics itself. His challenge was to convince students, that is, to change their beliefs. Students do not search for connections that they do not believe exist. For students to become skilled at finding connections, they must first be guided to believe that these connections are there to be found.

> *I think it's that they don't recognize that what we're talking about in my class is the same mathematical operation as what they learned to do in a methods of integration unit.*
>
> – Physics Faculty Member

Engineering faculty expect students to have attained these mathematically mature modeling skills as a result of their mathematics coursework. Due to this mismatch of expectations, students struggle in their subsequent engineering coursework. These findings align with previous literature, where modeling skills consistently dominate discussion of mathematical skills for engineers [17, 51, 18].

2.4.2 Theme 2

"A mathematically mature student chooses and manipulates symbolic and graphical expressions."

This theme was supported by the disproportionate application of two symbol sense codes, Reading Meaning from Symbols and Symbols in Context, applied 32 and 25 times respectively compared to a median application of 6 for symbol sense codes. The Symbol/Formalism competency was mentioned by 48% of participants, and 81% of participants mentioned something from the second theme.

The participants' definition of "reading meaning from symbols" was specifically concerned about students' ability to extract physical meaning from symbols as opposed to including ideas like searching for structural simplifications as in Arcavi's definition of reading meaning from symbols[22].

The participants' definition of "Symbols in Context" was more consonant with Arcavi's definition, describing a mathematically mature student as someone who recognizes that symbol meanings vary by context. For example, one agricultural engineering faculty member commented on how frequency is sometimes treated as a variable in the frequency domain, but is treated as a constant when performing the integrals that compute the Fourier transform into the frequency domain. A mathematically mature student must keep track of the changing roles symbols play in different contexts. A mechanical engineering faculty member commented:

> *I hammer it into them, what means what, what is varying, what is constant. When we do power spectral density, there the variable is actually frequency not time. I make sure that they understand what is integrated with respect to what. That notation is, needs to be explained and emphasized all the time.[sic]*
>
> -— Mechanical Engineering Faculty Member

Participants expressed dissatisfaction with many of their students' symbolic skills. Students could not transfer their mathematical knowledge between two phenomena with isomorphic symbolic form, such as inability to handle the notational difficulties arising from similar equations for shear forces and tensile forces.

Representational Fluency

Not surprisingly, participants stated that students' skills were weak at translating between different representations of a problem. For example, students struggle to, or did not attempt to transform an algebraic representation of an object to a graphical expression of that same object. According to participants, students do not emerge from their mathematics courses understanding that no one representation is the reality of the object; there are many representations that express different truths about the underlying object. The lack of mathematically mature representation skills becomes particularly worrisome in the context of linear algebra and the change of basis.

> *Objects that are doing things, linear in nature, but you can use numbers to represent them. But the numbers aren't the objects, they only represent them... And you can switch basis, but it's still the same object. I suspect that's true of the way they teach it here. That you emphasize that these are invariant objects that have a meaning in and of themselves, independently of the numbers that you chose to ascribe to them. And then you learn how to translate between different representations. I don't know whether the students get that or not, but that's an important aspect.*
>
> – Mechanical Engineering Faculty Member

Students tend to be attached to a single, canonical representation of a system; they struggle when they're presented with an alternate representation. Changing even a single letter can completely obscure their understanding of the system at hand. In particular, students are attached to using the letter x as the independent variable in all situations, as this physics faculty member points out:

> Part of the problem may be confronting symbolic expressions and generaliz[ing] from them. They may have derivative of $\ln(x)$ in calculus, but when they get the derivative of a bunch of symbolic constants in front of $\ln(r)$, it looks like a different problem to them.
>
> –Physics Faculty Member

Participants also complained about students' difficulties with the dummy variable of integration, saying that students frequently fail to recognize a dummy variable or when to use one. This lack of proper conceptual knowledge of dummy variables causes problems with later important concepts like convolution that depend strongly on dummy variables.

Participants said that students are attached not only to canonical symbolic representations of phenomena, but also to algebraic representations in general. Participants observed that their students are reluctant to use graphical representations to solve problems. A mathematically mature student uses simple graphical interpretations rather than sticking to the algebraic representation ill-suited to the problem at hand. One said:

> *[I wish students were good at] graphical solutions to problems. Being able to make a plot to solve a problem. Without me telling them plot these. Just being able to do graphical solutions to problems, being able to immediately resort to that technique.*
>
> – Materials Engineering Faculty Member

The above participant commented that it is her students' inclination toward algebraic representation that causes problems, not necessarily their ability to use graphs as a means of approaching a problem. Participants stated that students do not view notation and representation as a flexible tool for solving problems. Participants expressed a desire for students to show mathematical maturity by moving fluently between mathematical representations.

Symbols and Substitution

Within the algebraic representation, participants particularly wished that students understood substitution as a tool to gain insight into how a system works. Participants reported that students view substitution rules as meaningless tricks to reach the end of problems rather than flexible ways to change how one thinks about a problem. For example, when computing the integrals that describe the electric fields around a sphere, the arising trigonometric substitutions correspond to physical angles within the system. These correspondences can be used to help check and interpret the resulting integral. The participants teaching upper division courses (or computer science at any level) most emphasized the importance of flexible and incisive use of substitution compared to other participants.

> *Substitution rules are seen as just tricks. Maybe why they're not recalled as well I suspect. But I don't know I'm speculating. Some classes may very well put a lot of emphasis on this. The way I learned integration as a student was not by understanding invariance. It was by "Here's a bunch of rules you can apply to evaluate the integral." It was understood as* evaluating *the integral.* [emphasis original]
>
> – Mechanical Engineering Faculty Member

Not only are substitutions useful notational tools, but a mathematically mature student also recognizes that substitutions can give insight into how the system works. Symbols are powerful tools for solving problems and also for interpreting their results and checking for physical consistency. A mathematically mature student can use symbols and other representations to solve problems flexibly and understand answers.

2.5 Theme 3

"Computational tools reshape 'what needs to be known' to be mathematically mature."

Participants emphasized that the mathematics needed by engineers is changing. On one hand, they highlighted that mathematically mature students have fast, practiced fluency with basic mathematical operations; on the other hand, participants claimed that due to the increasing ubiquity of computational tools, students do not need fast, practiced fluency with advanced analytic techniques. Instead, participants stressed that an engineering student can be mathematically mature possessing merely an awareness of these techniques. The Tools/Aids competency was mentioned by 74% of participants, and all participants had at least one statement coded as part of the third theme.

> *Students should be able to work the simple problem on paper so they get some intuition. Most of the problems they're going to deal with are bigger than anything they could do on paper, so they need to understand how that translates to computer and visualization tools.*
>
> — Bioengineering Faculty Member

Fluent Basics

Participants emphasized that computational tools (e.g., ComSol, MatLab and Wolfram Alpha) change what mathematics students must know to be mathematically mature. According to participants, students must have fluent, effortless, fast skills with basic mathematics such as algebraic manipulations, derivatives of polynomials and logarithms, and first order differential equations. However, participants emphasized that students need this automaticity with only with the simplest functions: polynomials, roots, sinusoids, exponentials, and logarithms.

Participants observed that over-reliance on calculators for basic operations inhibits their students' understanding of larger problems. Specifically, students who display automaticity with basic operations rather than over-reliance on calculators have more mental space for the surrounding engineering problem.

> *What I want students to have is, on an exam, I don't want to be wasting my time with little diddly calculations... so I can allocate my mental resources to understanding the hard part of the problem.*
>
> – Industrial Engineering Faculty Member

Analytic Awareness

Participants emphasized that there is always more mathematics to know. They expressed that some mathematics should have practiced, honed fluency for quick accurate reproduction. By contrast, students merely need to have an awareness of techniques like integration by partial fractions. This partition of knowledge into these categories results from the ubiquity of computational tools. Ability to do integration by partial fractions quickly and accurately is no longer important, since the student will *"put it into Wolfram Alpha anyway"*. In the modern practice of engineering, practiced skill with complex paper-and-pencil calculations is unnecessary.

> *So I think students should be aware, but not be a technical expert in doing Gaussian elimination for instance. There are lots of things that we do in engineering that we don't need the details anymore.*
>
> – Electrical Engineering Faculty Member

These participants claimed that the computations students must do have changed, and that this change arises from computational resources now available. One civil engineering faculty member said that arduous analytic techniques like simplex optimization are now quick and easy with a computational tool, leaving him more room to discuss implications and bigger problems. He commented on how students are trained to execute computational algorithms by hand:

> *Math is a tool. Why are we thinking that our objective is to make [students] into a machine, rather than make them understand and use the machine. ... [People from theoretical mechanics] want to protect that domain and fight against people that are saying computers are adequate.*
>
> – Civil Engineering Faculty Member

In contrast to this faculty member's perception of the desires of other disciplines, in our data the sentiment that students need fewer fancy techniques and more computational skills was universal, regardless of engineering discipline. While disciplines did disagree on the emphasis to be put on complex numbers and transform methods, disciplines agreed that students need not memorize formulas for infrequently used techniques of integration.

2.6 Discussion

To answer **Research Question 1**: Through these interviews, we found that the engineering faculty in our study were primarily concerned about their students' mathematical modeling abilities and ability to apply mathematics to physical situations. We believe that this finding may generalize to engineering faculty at large since it aligns well with previous work in this area [17, 51, 18] despite the differences in methodology.

To answer **Research Question 2**: Engineering faculty members in our study were generally dissatisfied with the mathematical maturity of their students. Keep in mind that many participants teach early core engineering courses, so maturity may not have had time to fully develop. If the purpose of the calculus sequence is to create mathematical maturity, engineering faculty members do not appear to believe that the sequence is satisfactorily achieving that goal. While some faculty concerns are less common in juniors and seniors, even participants teaching students who had already taken differential equations as a prerequisite for their engineering course had serious concerns about their students' mathematical maturity, many no different from those teaching second-semester courses. We lack the data in this study to assert what causes this change, whether experience in the calculus sequence, unfit students dropping out, parallel experiences in engineering courses, or the maturing effect of years of life in general.

Our work expands on this finding by particularly exploring the epistemic underpinnings that threaten students' ability to develop modeling competencies. The belief that mathematics has practical relevance is necessary for extracting meaning from symbols or imbuing symbols with meaning. This belief may be a critical part of what engineering faculty mean by "the mathematics thought process."

The prevalence of the belief among students that mathematics is practically irrelevant should be disconcerting to both mathematicians and engineers because it is detrimental to students' desire to learn mathematics or engage in modeling competencies [29]. In mathematics, unlike other disciplines, students form the majority of their epistemic beliefs as a result of their mathematical experiences in school [20], so changes in mathematics instruction could have a significant effect on students' epistemic beliefs [27].

Concerns about students' modeling skills and beliefs about the practical irrelevance of mathematics are already prevalent in the mathematics education community, serving as the motivation for creating pedagogical techniques such as Model-Eliciting Activities (MEAs) [52]. An MEA is *"a problem that simulates an authentic, real-world situation that small groups of 3-5 students work to solve over one or two class periods. The crucial problem solving iteration of the MEA is to express, test, and revise models that will solve the problem"* [53]. These rich, open-ended problem-solving exercises allow students to generate their own models and engage in complex and challenging reasoning. MEAs have been suggested as a way to bridge the results of mathematics education research and engineering education research [53]. Ferguson's work [17] reached a similar conclusion as engineering and mathematics faculty co-generated their ideal assessments, which strongly resembled MEAs.

Consistent with our constructivist approach, we urge caution in using the expectations of engineering faculty members as the sole arbiter of the mathematics curriculum. The perceptions of engineering faculty may not represent students' actual needs and may even be misaligned with the skills that students need as practicing engineers [54]. As apocryphally attributed to automotive industrialist Henry Ford, *"If I had asked my customers what they wanted, they would have said faster horses."* Future data comparing student outcomes from different pedagogical approaches and content foci are needed to determine the accuracy of these perceptions. However, these findings do provide insight into the frustrations and desires that drive engineering faculty to change their curricula.

The alignment of the engineering faculty's perspectives with mathematics education research suggests that mathematics departments would be wise to lend some credence to these perspectives when designing their curricula, courses, and pedagogy. A core challenge may be learning how to develop students' modeling skills while also developing other core competencies such as following formal proofs. It is important note that the engineering faculty did not mention Reasoning Competency (ability to follow formal proof) or Thinking Competency (ability to carefully investigate definitions) even once. Mathematics courses are more than just what engineers need from them, and these competencies cannot be left behind in future discussions. There is still a duty to expose engineering students to these patterns of thinking as well as those they will apply.

In light of our findings and the potential challenges of balancing modeling competencies with other competencies within a limited number of credit hours, we provide some recommendations for how we might achieve this balance. These recommendations rely on the assumption that there is a degree of truth to the engineering faculty members' perceptions. We primarily focus on potential approaches that may shift which mathematics are taught when or identify content areas for reduced emphasis in order to increase emphasis elsewhere in the curriculum.

2.6.1 Recommendation 1: more modeling and more context

If a root problem for students is indeed their belief that mathematics is practically irrelevant, then the continued use of these contextless integrals that have no physical meaning may be exacerbating this problem. While it is not the mathematics department's job to teach physics or engineering, it can productively borrow examples from these disciplines to provide meaningful contexts without sacrificing mathematical goals.

To examine this claim, consider the following "great chain rule problem" from Stewart's Calculus text [55, p 204, exercise 3.4.39]:

$$\frac{d}{dx}\tan(\sec(\cos(x)))$$

This function does not describe any object in our universe, yet problems such as this one are typical exercises in introductory calculus [11]. This function is chosen solely because it requires careful navigation of manipulations through multiple algebraic steps. Unfortunately, upon reaching the answer to this problem, the student cannot examine the answer for its reasonableness as the question was not reasonable to begin with. This lack of context disrupts the last interpreting step of the mathematical modeling cycle [45].

Consider, by contrast, the following problem:

> The dispersion relation for surface water waves is $\Omega(k) = \sqrt{gk \cdot \tanh(kh)}$. Find the group velocity, $v_g = \frac{d\Omega}{dk}$.

Computing this derivative is much like that for the example from Stewart. It requires knowledge of special functions and multiple applications of the chain rule. Fortunately, after obtaining the derivative, the problem does not end. We can ask questions like "We know that wave packets with long central wavelength (small k) travel faster (have larger v_g) than packets with short central wavelength (large k). Explain whether your equation for group velocity correctly models this phenomenon." Using a physical context, we are able to give students practice in mathematical interpretation and sense-making *in addition to* practice with operations and algebra.

Using modeling examples from a variety of disciplines (e.g., mechanical, biological, economics, electrical, material, chemical, financial, abstract mathematical) may also help students better learn the fundamental underlying concepts [56, 38]. Not only can the variety of contexts provide motivation to students from different backgrounds, but it can also help students ignore idiosyncrasies of specific examples and identify the cross-cutting concepts that govern all of the examples.

2.6.2 Recommendation 2: Explicitly introduce and vary symbolic presentation

The goal of improving students' symbol sense and representational fluency may be easily achieved with a few small tweaks within the context of modeling exercises. For example, instructors could intentionally move away from always using the variable x for independent variables (and explicitly map context specific variables to those canonical forms), assign more problems with parameters left as letters, explicitly instruct which operations are permissible on variables but not on parameters, or require students to justify why they are choosing the representation they are using (e.g., graphical vs. algebraic).

Beyond, these changes, modeling will require explicit instruction of how to choose variable names that facilitate meaning making and modeling. Consider the following.

Alice's velocity is 1 m/s to the left. How do we represent this variable symbolically?

Should **A**lice's **V**elocity be written as $v_A = -1$, or $A_v = -1$? The choice for notation derives from the physical interpretation of the variables: the quantity type (velocity) takes priority and the object possessing that property (Alice) is put in the subscript. While this rule of mathematical communication is simple, it does require explicit attention. It is not uncommon for instructors to unintentionally forget to explicitly describe these automatized rules of communication [57, 58].

2.6.3 Recommendation 3: To make room, cut certain analytic techniques

Prior studies have highlighted that computational tools are becoming increasingly important for students to learn. Our study adds nuance to this observation, articulating that advances in computational tools are shifting the line between what students need to know well and what students can be merely aware of. Techniques that were once essential may now be vestigial components of our curricula.

In light of these expectations from engineering faculty members, we suggest cutting many analytic techniques from the core calculus sequence (e.g., integration by partial fractions, root test for convergence, the method of integrating factor) and allowing students to solve these by computer. We recognize that this suggestion is controversial. Does this claim not contradict what faculty said about students becoming over-reliant on calculators? We argue that removing techniques is not new to the mathematics curriculum and can be done without detriment in limited circumstances. To explore this claim, consider a simpler operation: computation of square roots.

Square roots are a fundamental operation in arithmetic. Despite their importance, high schools no longer teach the algorithm for manual computation of square roots because they are always done with a calculator. Students are still expected to know the square roots of the "simple problems" (i.e., the first few perfect squares). Students can then use this knowledge to judge a computation's reasonableness (i.e., that $\sqrt{17}$ is slightly greater than 4) but need not be able to calculate a root to three decimal places quickly and accurately. The emergence of cheap calculators lowers the threshold that separates "simple problems" that need to be known from "advanced" problems that do not. Systems like Wolfram Alpha lower the threshold for integration the same way hand calculators do for square roots. Students are still expected to be able to integrate polynomials, logarithms, exponentials, and trigonometric functions, and then use their experience with these simple integrals to be able to qualitatively evaluate whether Wolfram Alpha gave a reasonable result for more complex integrals such as $\frac{x^2+1}{x^3+2x^2+x}$.

Not teaching the square root algorithm is a good thing; the time that was once spent teaching that algorithm has been re-purposed to cover other content. Students still develop understanding of square roots but do not need to practice the algorithm to build that understanding. By the same argument, we would benefit from not teaching numerous integration techniques. Students can still possess strong conceptual understanding of how integrals work without being experts with advanced techniques of integration. This offloading to computational tools does not contradict professors' lamentations of over-reliance on hand calculators: they only want the very simple calculations to be done by hand/head and want students to use tools on the more cumbersome and complex cases.

Credit hours are a limited currency within a four-year curriculum. Each credit hour spent teaching advanced techniques is a credit hour not spent teaching something else such as modeling or reinforcing better epistemic beliefs. As educators, it is our duty to spend each credit hour as wisely as possible.

2.7 Conclusions

Since mathematics departments teach many service courses for majors in other departments, there will always be a tension between creating mathematics courses that teach mathematics and partner disciplines that want "mathematics-methods-for-engineers." Logistical difficulties such as alignment with AP calculus exams, course articulations for transfer students, and the expectations of graduate programs only serve to increase these tensions by making change difficult if possible at all. Although this tension has occasionally had catastrophic effects for some mathematics departments, we believe that our findings suggest productive pathways forward. We intend this research to help mathematics departments modify their courses and avoid losing them to engineering departments by providing a clear articulation of engineering departmental needs.

By making prudent cuts in the calculus sequence to create room for deeper instruction in modeling and computational skills, mathematics departments could meet the demands for mathematically mature students from partner disciplines. Such changes could also improve outcomes for students from a purely mathematical perspective by improving students' epistemic beliefs about mathematics. Further, the advances in computation tools may indeed be moving the line between what knowledge needs to be mastered and what knowledge can be offloaded onto tools, further decreasing the tension that departments of mathematics experience.

CHAPTER 3

STUDY B: INTERVIEWS WITH NON-ILLINOIS ENGINEERING FACULTY

3.1 Introduction

The study of the previous chapter of this dissertation asked the question, What is mathematical maturity according to engineering faculty? In our previous work in this topic, we interviewed 27 engineering faculty from 11 disciplines to get their raw thoughts on what constitutes "mathematical maturity" for an engineering student. In this work we extend our previous work to include faculty from a variety of different institutional sizes and types to increase the generalizability of the result.

3.2 Background

Prior efforts to define mathematical maturity for engineering students have relied on two primary approaches: consensus-building workshops led by professional societies and faculty interviews at individual institutions.

3.2.1 Consensus-building workshops

The Mathematical Association of America (MAA) has actively sought to understand the curricular needs of its "partner disciplines" (defined by the MAA as the physical sciences, engineering, and business). MAA's largest and most comprehensive project to document these mathematical expectations was the Curriculum Revision Across the First Two Years (CRAFTY) [18]. The CRAFTY workshops convened faculty members in partner disciplines and mathematics. They had several meetings, discussions, and revisions to develop consensus about the priorities of the introductory, post-secondary mathematics curriculum.

Similarly, the European Society for Engineering Education Mathematics Working Group (MWG) used the Competencies and Mathematical Learning (KOM), previously developed by the Dutch Ministry of Education [21], as a starting point for defining priorities for mathematics education for engineering students [35]. The MWG report divides all of the mathematical skills of an undergraduate education into four rings. The center (core 0) is material all engineers should be competent in. Progressing outward to outermost ring (elective 3) is material that only some disciplines need reliable access to.

Finally, the US Naval Academy [17] also explored what engineers and mathematicians both consider to be mathematical competence. During workshops lasting several weeks, engineering and mathematics faculty met to discuss what task would demonstrate that a student had mastered calculus.

All reports particularly emphasized that engineering students and engineers need to develop modeling competencies (translation between the physical and mathematical domains) during mathematics instruction. They also emphasized the increasing importance of competencies with tools and aids as computers become more powerful and ubiquitous. Both reports commented how computational tools can allow more realistic models to be explored more quickly and easily than is possible by hand, but that a loss of basic capabilities is a significant worry. Finally, these reports emphasized communication competency, urging mathematics faculty to require students to explain their results in words and incorporate assessments that require students to both read and write about mathematical ideas.

These efforts relied on consensus-building approaches to document the beliefs and attitudes of engineering faculty who were pre-selected for their prior engagement in this dialogue. This consensus-building approach reflects the consensus of highly engaged faculty who may not fully express their own personal opinions in hopes of achieving some consensus. Additionally, the reported competencies from these workshops were not critically analyzed or evaluated using any qualitative or quantitative research methods. Consequently, these reports may not accurately represent the expectations of rank-and-file faculty from a variety of institutions.

3.2.2 Single-institution interview studies

A complementary set of studies have investigated the beliefs of engineering students, faculty, and practicing engineers about the role of mathematics in the engineering curriculum.

Firouzian et al. [37] surveyed and interviewed engineering students, engineering faculty, and practicing engineers about the relative importance of the different KOM competencies. Their work showed that engineering faculty and practitioners weigh the importance of mathematical competencies differently from mathematicians. They found that modeling competency and tools/aids competency dominate far over the other competencies in terms of engineering need.

Gomes and Gonzalez-Martin [59] similarly found that an engineers emphasize the importance of checking the plausibility of a model against reality, particularly for models that come from computational tools. Many mathematical practices that are not rigorous by mathematicians' standards (such as the use of infinitesimals in derivations) are useful and standard mathematical techniques for engineers.

These prior studies relied on interviews with a small number of faculty members (four and one engineering faculty respectively). Our prior work sought to supplement these prior studies by interviewing 27 engineering faculty from 11 disciplines about what mathematical competencies their students need. In particular, we sought to better understand what faculty meant when they stated that students need to "learn how to think mathematically" or be "mathematically mature". The codebook (see Appendix B) we developed also used the KOM competencies as a starting point but also used other constructs from mathematics education research such as epistemology [20] and symbol sense [22].

Other researchers [60, 61] have sought to document what mathematical knowledge is actually used in engineering courses. These studies have found significant mismatches between what is taught in mathematics courses and what is used in engineering courses. These efforts, while related, are outside the scope of this study, as they focus more on specific content knowledge (e.g., ability to linearize a system) rather than broader competencies (e.g., modeling).

Critically, while these studies have documented the beliefs of individual faculty, they were all performed at individual institutions. Consequently, it is not clear whether any findings from these studies are generalizable.

3.3 Purpose of this study

The purpose of this study is to purposefully seek divergent opinions by interviewing individual faculty from a variety of institutions. This purpose complements the convergent approaches of the consensus-building workshops and supplements the institution-bound perspectives documented in the interview studies.

Promisingly, both the workshops and interviews have documented three common themes for defining mathematical maturity for engineering students:

- Advanced techniques (e.g., integration by parts) are increasingly automated by computer, but expertise with algebra is highly desired.
- Mathematical modeling skills are of supreme importance (units of measure and estimation in particular).
- Mathematical communication and representation skills are also desired.

Consequently, we pursued this study to observe whether these themes are sufficiently representative of faculty expectations. The engineering mathematics curriculum is a core backbone of almost all engineering programs and has been in place since the Grinter Report [62]. Hasty alteration to this backbone could have calamitous consequences, such as unintended effect on upper division courses or loss of skills from the industrial base. Therefore, seeking to document dissenting expectations or perspectives is essential before taking any action.

3.4 Methods

Since the CRAFTY project [18] and Ferguson's [17] prior studies both relied on time-intensive, dialogue-based methods to encourage engineering and mathematics faculty to come to consensus about how to align the mathematics and engineering curricula, we emphasize the perspectives of engineering faculty who have not been as engaged in these inter-disciplinary discussions. The goal of our approach is to document the views of engineering faculty who have not engaged in these discussions because these views may more accurately reflect the views of the typical engineering faculty member - the ones who will be making curricular decisions. While the views of these faculty should not be taken as empirical fact, these perceptions do represent the constructed reality upon which these engineering faculty will act. Our study therefore takes a constructivist approach [46], seeking to document the expectations and beliefs of engineering faculty have constructed, apart from mathematicians.

To document these perceptions, we chose to conduct open-ended interviews that would encourage engineering faculty to fully explain their individual positions. We chose to do interviews rather than surveys, because surveys do not easily permit follow-up questions to explore unexpected perceptions and because we could get richer observations from interviews than from surveys. Since previous research [17, 18] showed it took a long time for faculty to uncover their expectations about what it really meant to understand calculus and there were numerous disagreements due to vocabulary and word use, one-on-one interviews allow for deeper conversations that can adapt to the vocabulary and expectations that are idiosyncratic to the individual or discipline.

To get sufficient breadth of observations, we chose to interview faculty from a variety of engineering disciplines and from a variety of institutions. To allow sufficient time to explore each participant's perspective, we scheduled one hour interviews but allowed them to go longer if the participants had more to say.

Consistent with our constructivist approach, we chose to use thematic analysis [48] to document the perceptions of the faculty interviewees. A thematic analysis interview method was not used in previous work [17]. Thematic analysis provides a robust and flexible way to analyze qualitative data that allows the perspectives of participants to emerge from the data. The themes of the previous work were used as our starting point, open to possible refinement or nuance that might be different in other schools.

Since the purpose of our study is to see if our findings from the original research site (with "internal faculty") still adequately describe perspectives of faculty at other institutions ("external faculty"), we re-used the codebook from the previous research. If the codebook developed for one institution could fully describe faculty perceptions from external institutions, this similarity is strong evidence for saturation of observations.

3.4.1 Research question

Is the perception of “mathematical maturity” at a variety of institutions consistent with the perception found at a single institution?

3.4.2 Participant selection: Saturation sampling

To explore whether perceptions are consistent across institutions, our sampling must aim to find dissenting opinions. Consequently, we intended to sample seeking to observe saturation of observations even when faculty were recruited from a variety of institution types (size, selectivity, research output) [63] and engineering disciplines. Highly selective schools (which we proxy by the lower quartile of ACT mathematics score) may have higher mathematical expectations of their students, since they accept only students of higher demonstrated mathematical ability. Research-intensive schools may have different expectations than teaching-focused schools, and might prioritize mathematical skills that are necessary for research over those for industry. Schools with large student bodies and attendant large class sizes could have different priorities for mathematical skills that are time-consuming to grade at large scale (e.g. mathematical communication).

The previous research [64] was only at a single large, research intensive school with very high admission requirements. It is reasonable to suspect that faculty in this institution may have higher expectations about students’ level of preparation and may downplay the importance of some mathematical skills or may value skills more important for preparing students for research rather than for industry. Sampling across institution types may reveal different perspectives on “mathematical maturity” as students’ preparedness or career expectations vary.

To recruit faculty from other institutions who would be willing to talk about the intersection between engineering and mathematics, we first identified faculty who had participated in an engineering mathematics consortium at the American Society for Engineering Education conference in 2016. Of the 18 people solicited for interview, 7 (39%) people answered and were interviewed. We also used snowball sampling, following up on recommendations of participants and colleagues. In our prior work, saturation of observation [64] was found after 7-10 of the 27 interviews were completed. So an initial sample of seven external faculty were selected. If new observations were found in this initial sample, more data would be collected, but no new observations relevant to the research question were found.

Keeping with the goal of sampling from a variety of perspectives (see Table 3.1), we sampled faculty from four-year, masters-granting, R2, and R1 institutions. We also sampled from a variety of geographical locations. The participants varied from pre-tenure assistant professors to senior faculty who had been teaching for 20 years. The sample of faculty included members of electrical engineering, civil engineering, mechanical engineering, bioengineering, and computer science departments. We operationalized selectivity of institutions as the lower quartile of ACT math scores, and sampled from schools with a variety of ranges.

3.4.3 Interview procedure

We conducted semi-structured interviews with an initial interview protocol that allowed for asking off-script questions to explore the views of the participant when pertinent to the research question. The interview questions were designed to explore constructs we hypothesized were the roots of mathematical maturity (i.e., symbol sense, epistemology, and competencies). If the interviewed faculty member did not mention a concept like "mathematical maturity," "mathematical sophistication," or "mathematical flexibility" independently, participants were asked at the end of the interview if the term "mathematical maturity" meant anything to them. Interviews lasted for approximately one hour and participants were not sent the interview protocol in advance.

Example protocol questions included

- What courses do you teach and what are the prerequisites?
- What mathematical skills, abilities, or attitudes are essential to succeed in your course?

Table 3.1: Institutional demographics of sample.

Institution Description	Carnegie Classification	Lower Quartile ACT Math for Incoming Engineering Students	Student Population (rounded to nearest thousand)
Small Southwestern Masters University	Masters-Granting	20	7,000
Midwestern Undergraduate College (2 faculty members)	Four-year	28	2,000
Large Midwestern Research University	R1	29	66,000
Medium Midwestern Research University	R2	25	20,000
Medium Pacific Northwest University	R1	23	30,000
Medium South East University	R1	25	23,000
Original Research Site Midwestern Research University	R1	31	44,000

- What is your perception of the mathematical abilities of your incoming students?
- What mathematical behaviors inhibit your students' abilities in your course?
- If you could guarantee that 100% of your incoming students possessed one mathematical ability, what would you choose?
- Do you assign problems that contain uncertainty or imprecision?

3.4.4 Data analysis

Interviews were transcribed verbatim. To analyze the data from faculty interviews, two researchers conducted a thematic analysis. An uninterrupted passage of participant speech between interviewer prompts constituted one unit of analysis. This unit of analysis is unambiguous and allows for full arguments on a topic by the faculty member to be considered as a single unit.

The codebook from the previous study [64] with only faculty from the primary research site was used to code the data. The first coder was the author of that previous paper. The second coder was a graduate student in materials science with experience in qualitative research.

The coding scheme was open to having additional categories for observations emerge, given that they were relevant to the research question and were found in moderate numbers. Some interesting miscellaneous statements were found, but only in small numbers. These few novel observations did not constitute a new code, let alone a new theme, so were left as a miscellaneous category. There were some novel observations, but they did not have anything to do with the research question.

The first round of coding resulted in only 60% inter-rater reliability. A second round after discussing disagreements only reached 72%. We then reviewed the codebook and merged small codes consistently with the thematic analysis from the previous work. For example the "reading meaning from symbols" in symbol sense was so connected to reading physical meaning from symbols that it was merged into the modeling competency code. We also instituted double-checking protocols to prevent careless miscoding errors. After all that, we achieved 84% reliability, which is good agreement [49].

The new data did not have any substantial new observations from which to build new codes. The observations naturally arranged themselves into themes, and these themes were the same themes as in our original research, strongly echoing other previous literature.

3.4.5 Trustworthiness

The first coder had an undergraduate background in physics and mathematics, and graduate training in electrical engineering. The second coder had an undergraduate background in physics and graduate training in materials science. The use of two coders helps reduce the likelihood that the results are a single researcher's opinion projected onto the data. Since definitions of existing codes were static, variety in researcher experience would not strongly impact results. We did not perform member checking on the results. When a faculty member said something very similar to previous participants and was informed of this similarity to others, they often commented that they felt good that they were not alone in feeling this way. As will be seen later in this document, the results strongly corroborate other findings on engineering mathematics by other authors and by different methodologies.

3.5 Results

Results from these interviews corroborated results from our prior study with faculty at our institution. No new themes or even new codes resulted, despite the variety of different institutions. During analysis of the interview data, we noted that the proportion of time external faculty spent on the various themes was similar to the time spent by faculty at our institution, providing strong evidence for saturation of observation. We illustrate this qualitative observation by showing that the percentage of coded statements falling into each code group is qualitatively similar (see Table 3.2).

Table 3.2: Codes were sorted into three themes. The percentage of coded segments that were in each theme (code group) is shown. Some coded segments aligned with multiple themes or with none of them, so percentages may not total to 100%.

	External	Participants	Internal	Participants
THEMES	Number of statements	Proportion of coded segments	Number of statements	Proportion of all coded statements
Modeling	54	39%	338	46%
Computation	43	31%	240	33%
Representation	22	16%	91	12%

The same themes from the original research site remain nearly unchanged in proportion of the total coding. Though the finding of how often each code varied within a theme often varied considerably (particularly given the smaller number of the external faculty), the fact that the themes have nearly identical relative frequency in both cases indicates a surprisingly high level of consistency. While one could suspect that the original research site, as a highly selective large research university, might have different or higher expectations of what constitutes mathematical proficiency for engineers, this does not appear to be the case.

Not only were the participants' numerical concentrations of commentary similar, but the contents of faculty comments were also similar. In the following sections, we pair quotations from the original research site with quotations from external faculty to illustrate this similarity.

3.5.1 Theme 1: Modeling

Participants emphasized that mathematical modeling of physical systems was an essential part of mathematical maturity of engineers. Participants described mathematical modeling as being able to turn a physical description into a solvable mathematical system, and being able to translate mathematical solutions into interpreted physical meaning. The modeling theme occupied the most coded segments of the three themes, corroborating findings from our prior research. Faculty wanted their students to connect the math they had learned to the engineering they were learning, but they believed many students viewed math and engineering as separate, distinct worlds of knowledge (Table 3.3).

Table 3.3: Students need to see mathematics as related to the real world.

Internal Bioengineer	External Mechanical Engineer
"As soon as you make the transition into the physiology domain, the problem is convincing students that what they did learn is applicable still, and that a plug and chug problem does provide useful information to physiological systems."	"Getting them to not view math as numbers on a page, but how everything in the world can be represented through math."

External faculty continue to emphasize that sense-making and identifying physically implausible results is an important aspect of mathematics (Table 3.4).

The persistent feeling that mathematics is not relevant to their engineering studies is particularly a problem. Students feel unmotivated and do not believe math they are learning will ultimately reach fruition (Table 3.5). Students are entering college with this defeating and demotivating attitude (Table 3.6).

Table 3.4: Students need to engage in sense-making with mathematics.

Internal Materials Engineer	External Bioengineer
"Sometimes they have problems understanding that the minimum in energy means that the force is zero. Again, because they may be able to do $\frac{d}{dr}$, but maybe they don't really know what it means physically. So many of them just set the energy to 0. They don't get it that there's a minimum in the energy curve, it means the force is zero."	"I actually keep pounding that into them. If you come up with a probability that is less than zero or greater than one, you know you did something wrong. Same thing with understanding that standard deviations and certain statistics can only have positive values. I keep harping on them for that. Some students get that, others do not."

Table 3.5: Applications of mathematics are distant in time from when students learn mathematical content.

Internal Civil Engineer	External Electrical Engineer
"You're going to be a junior before you see any application of anything, so you better hold on tight!"	"By the time they get to the applications they've probably forgotten it."

3.5.2 Theme 2: Computation

Participants emphasized that the mathematics needed by engineers is changing. On one hand, they highlighted that mathematically mature students have fast, practiced fluency with basic mathematical operations; on the other hand, participants claimed that due to the increasing ubiquity of computational tools, students do not need fast, practiced fluency with advanced analytic techniques. Instead, participants stressed that an engineering student can be mathematically mature possessing merely an awareness of these techniques. This result is consonant with previous literature [65, 66].

Faculty in both studies expressed similar beliefs about which computational tools students must master. For example, faculty mentioned spreadsheets as a primary computational tool over other tools such Matlab, citing their presence on the Fundamentals of Engineering exam.

External faculty also expressed similar attitudes about the proper place of computational tools, combining caution and excitement. Computational tools provide great possible educational opportunities, but they stressed that

Table 3.6: Perceived lack of application demotivates students.

Internal Bioengineer	External Electrical Engineer
“They take [bioengineering] because they thought it would be less math. The attitude of fear of math because it was disconnected.” [sic]	“I’ll ask when they’re interviewing: have you ever had a class in math where someone in the class asks ‘How is this going to apply?’ And they already know what I’m going to say, before I can even get it out, their face contorts and everything. And they’re already ready to say ‘You just need to learn it.’ And so when I’m interviewing it’s the survivors of that.”

ability to interpret the result of a computation was of utmost importance. Without a firm understanding of “garbage in, garbage out,” computational tools are risky (See Table 3.7).

Table 3.7: Students need mathematical intuition to interpret the output of computational tools.

Internal Bioengineer	External Electrical Engineer
“Students should be able to work the simple problem on paper so they get some intuition. Most of the problems they’re going to deal with are bigger than anything they could do on paper, so they need to understand how that translates to computer and visualization tools.”	“We want to have some balance between manual manipulation vs using computing power. We want them to see the concept behind those things. There’s always time for them to pick up the more complex tool like integrals.com and plug it in, but you don’t understand what you’ve got when you get it out. You don’t even understand if you made a mistake if it kicked out a mistake if you don’t know what you’re doing.”

Just like the internal faculty, the external faculty stressed the importance of having fast, fluent fundamentals. They argued that automaticity with basic skills leaves surplus cognitive capacity for thinking about the engineering situation surrounding the math (Table 3.8).

In the same manner as at the original research site, external faculty stated that long, complex problems usually required a lot of algebra, but only a small amount of calculus (see Table 3.9), despite the volume of calculus required to enter their courses.

Table 3.8: Students need fast fundamentals to help them focus on understanding problems.

Internal Industrial Engineer	External Mechanical Engineer
"What I want students to have is, on an exam, I don't want to be wasting my time with little diddly calculations... so I can allocate my mental resources to understanding the hard part of the problem"	"The math skill to where you're not struggling to solve a derivative, you're not struggling to solve this trig problem. Then when the problem is put in front of you, your brain energy can be put towards looking at it from a conceptual point of view and not 'Oh my gosh what formula can I use to solve this.' "

Table 3.9: Engineering courses need algebra more than calculus.

Internal Nuclear Engineer	External Bioengineer
"The vast majority of it is algebra. A little bit of linear algebra, they need to pick that up themselves. The differential equations I give the solution."	"Our students really are struggling with just algebra. Solving long, complex problems usually involves a lot of algebra. It involves a little calculus or whatever."

3.5.3 Theme 3: Representation

Participants emphasized that a fluency with symbolic, graphical, and verbal mathematical representations is an important aspect of engineering students' mathematical maturity [40]. Our participants stressed that students who understand the material will be able to translate information from one representation to another. In doing so, students demonstrate the ability to identify and manipulate the most important information within a problem. Many participants shared the belief that their engineering students do not understand why it is useful to represent a physical system or data describing a physical system using different types of visual representations. Similarly, students do not recognize the mathematical forms they have seen when those same forms are presented using variables from an engineering context.

The external participants again confirmed that students are highly attached to the canonical (often algebraic) representation of the system. Students are unable to connect the meaning in one symbolic representation that they learned in mathematics to a representation in an applied course (Table 3.10). This inability to connect knowledge of already learned things to the new weakens student ability to do transfer, which is required for a prerequisite to serve its purpose: allowing the engineering instructor to pick up where the mathematics instructor left off.

Table 3.10: Students need to be able to generalize symbolic knowledge away from canonical representations.

Internal Physicist	External Electrical Engineer
"Part of the problem may be confronting symbolic expressions and generaliz[ing] from them. They may have derivative of $ln(x)$ in calculus, but when they get the derivative of a bunch of symbolic constants in front of $ln(r)$, it looks like a different problem to them."	"$y = mx + b$, you mean that can be $v_{final} = at+v_0$? You mean that can be the same thing? And because the shift of seeing those variables represented in multiple ways."

External faculty had similar complaints about student attachment to algebraic expressions as those at the original research site. The reluctance of students to move between representations to simplify or understand problems is previously documented. Faculty comment on the reluctance of students to use graphical representations (Table 3.11).

Table 3.11: Students need to know when to move between representations.

Internal Materials Engineer	External Electrical Engineer
"[I wish students were good at] graphical solutions to problems. Being able to make a plot to solve a problem. Without me telling them plot these. Just being able to do graphical solutions to problems, being able to immediately resort to that technique.	"I think they just don't like to draw pictures. I think because it takes time that they feel isn't really productive. I don't know. It's a weird mindset and I can't get into it so I don't really. I don't think they draw as many pictures as they should. They want to get to a number as quickly as possible. And for some reason it doesn't feel to them that a picture gets them to that desired outcome."

The emphasis on communication skills was slightly different between the external faculty and internal faculty. External faculty emphasized communication *with the general public*, in addition to communicating with other technical professionals.

"We have a focus, the freshman year, and it's tied into this math. We have a focus on communicating the things they're learning to the general public, which you assume a 8th grade reading level."

- External Electrical Engineer

3.5.4 New Finding

One observation that was much more common among the external participants was the verbalized expression of the likely endpoint of their students. More of them explicitly stated things about how their students would likely become bench engineers, and that a lot of mathematical techniques are useful only to those bound for graduate school. For example, one civil engineering faculty member said:

" Like 95% of our students will become practicing engineers."– External Civil Engineer

3.6 Discussion and Limitations

This study has limitations that should be explicitly discussed. These are limitations of this study in particular, which are in addition to the limitations of qualitative research in general.

- This study did not capture any community college level pre-engineering. Community colleges do play a large role in the dynamics of engineering math since so many engineering students take their mathematics coursework at community colleges before transferring to universities to complete degrees in engineering. Since community colleges teach few, if any, engineering courses beyond prerequisite math and science courses, they are further distanced from what mathematics engineering students need to be successful in industry or academia. Consequently, we chose not to recruit faculty from these institutions.

- The interviewer specifically discussed epistemological and symbol sense issues during the interviews. Participants' responses are influenced by prompting, and the symbol/representation theme may be over-represented. This bias is difficult to resolve and must be taken into consideration while examining the results.

- These interviews were single, isolated, one-hour events. The short time scale and lack of prior reflection may have led to shallow interview responses.

- In addition, there are the first coder's biases as a researcher and as a practitioner of mathematics, having an undergraduate degree in physics and mathematics and a master's in electrical engineering. The second coder shared the same undergraduate training in physics.

- The selection of faculty who had already participated in a mathematics reform seminar may be biased to be more dissatisfied with the mathematics curriculum than the average engineering faculty member.

- The participants were all white and male, potentially limiting the diversity of perspectives on the importance of mathematics.

Given these limitations, the uniformity in the data across sample populations is an indicator that we have reached saturation. Despite many studies by many authors in many disciplines and in many institutions [17, 18, 35, 66], the core themes in this study are consonant with prior efforts. This level of saturation of observation means that further qualitative sampling is likely unnecessary. We feel confident that our core three themes are representative of the majority opinion of engineering instructors in the United States and Europe.

We did see one nuance in the perspectives from the external faculty at smaller institutions: their students would be working as practicing engineers and would need to be able to communicate with the general public. This observation does appear to be related to the institution type, as the original research site is a research-dominated environment, but this comment came from the two smaller institutions in this sample. Finding these nuances underscores the importance of seeking out dissenting and diverse perspectives. Future work should explore how widely these perspectives are held using broader survey measures that particularly aim to sample among demographics that were not adequately represented in this study.

3.7 Conclusions

Even across varied types of institution, the general themes of engineering mathematical proficiency hold to a surprisingly similar degree. Engineering faculty place mathematical modeling as the principal mathematical competency for their engineering students. The mathematics that students must know is changing as a result of the ubiquity of computers, but computational power must be tempered with solid fundamentals and sense-making. Students must be able to represent and communicate mathematical ideas, both to other engineers and to the world at large. This result corroborates previous work [17, 18].

Mathematics filters engineering students from their degree program, with many students not even reaching classes with engineering faculty. That may not be the best use for mathematics within engineering, but it is the consequence of the current curriculum and prerequisites. The consistency of findings across these different methodological approaches suggests that we can recommend potential changes to why or how we require mathematics prerequisites that would be broadly accepted by engineering faculty.

CHAPTER 4

STUDY C: ANALYSIS OF ENGINEERING HOMEWORK PROBLEMS

> *To be effective and useful the design of mathematics courses for engineering students must involve a continuous and informed dialogue between engineering and mathematics departments to which each must contribute fully. The process of dialogue is essential since neither must be the dominant partner. The difficulties usually arise not in deciding what is to be taught but how and at what level. This is where the engineering department must have a* clear understanding of what is needed *and be able to communicate this effectively to the mathematicians.*
>
> J.O. Scanlan [67, Emphasis mine]

Because of the negative impact of mathematics on graduation rates from engineering [6, 7, 8], we are striving to rigorously document what mathematics knowledge and skills students need to successfully enter their engineering degree programs and inform the dialogue that Scanlan [67] calls for.

Most engineering programs require a prerequisite "calculus sequence" of Calculus I, Calculus II, Calculus III, Linear Algebra, and Differential Equations [66]. Students must pass some combination of prerequisite courses from this sequence to continue into core engineering coursework such as statics, dynamics, circuits, and thermodynamics [10, 11, 12, 68, 66]. Due to the length of these prerequisite chains in the "math-science death march" [69] (see Figure 4.1), engineering students may not take their first course with engineering faculty until their sophomore or junior year [70, 71]. These prerequisite mathematics courses often have high failure/withdrawal rates [72, 73, 74, 75], and a failure in one of these courses pushes back a student's graduation by a semester or more. The strictness of this prerequisite chain can particularly hamper female and minority students [71] and students who are already disadvantaged due to disability or lack of access to high school calculus [13]. Students who do not start calculus-ready or fail a course in the calculus sequence may struggle to complete an engineering degree before financial aid runs out. Consequently, there is a need to explore whether there are ways to shorten these prerequisite chains or bypass them [71] while still supplying students with the foundation in mathematics that they need to be successful in engineering. Can prerequisite structures be modified to include fewer stumbling blocks that may delay graduation in order to improve engineering retention and graduation rates?

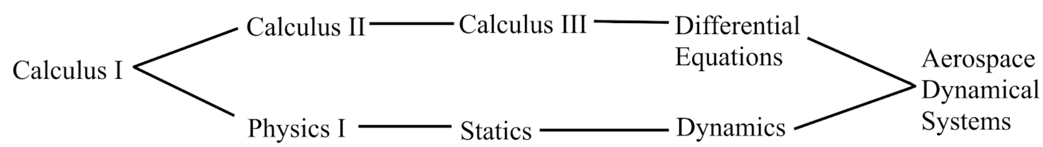


Figure 4.1: The prerequisite relationships at our institution leading to one particular required junior-year course in aerospace engineering. Failing any of the prerequisite courses delays entry into the required course.

In this study, we apply the mathematics-in-use technique to map the knowledge learned in the Calculus I to when that knowledge is used in core engineering courses Statics and Circuits. This mapping can provide evidence that can guide curricular decision making and dialogue between engineering and mathematics faculty regarding prerequisites to reduce the number of stumbling blocks for students. The purpose of this study is to document a clear understanding of what is needed by engineering departments, which mathematics departments can use to make informed decisions about their courses, as well as to assist engineering departments in revising their own curricula and prerequisite requirements. This study maps out explicitly *which* mathematical techniques taught in Calculus are applied in the follow-on engineering courses, to show *when* and *how often* mathematical techniques are applied. This study also examines *how* the application of those techniques differs between prerequisite mathematics courses and follow-on engineering courses.

4.1 Background

Prerequisite mathematics course performance usually has a moderate Pearson correlation ($r = 0.4$ to 0.7) with subsequent engineering course performance [76, 77, 78, 79, 80]. Given prerequisite structures, we might assume that the content of these prerequisite mathematics courses is strongly linked to that of subsequent engineering courses.

Ideally, the preparation from prerequisite courses allows an engineering professor to pick up where the mathematics professor left off, applying the knowledge of the previous course. A successful hand-off requires that students will both be able to recall the mathematical knowledge from the prerequisite course and transfer that knowledge to engineering. Unfortunately, students often do not remember content from previous courses. About 85% of knowledge that has not been refreshed will be forgotten within a year [81, 82, 83, 84, 85, 86]. To be effective, prerequisite courses must be located temporally close to their follow-on material.

Additionally, students often fail to transfer mathematical knowledge to other disciplines without special prompting [34, 9, 87, 77, 88]. Transfer is a complex process that allows students to apply the knowledge learned in one domain to another domain. This transfer process is complex and proceeds by four classical mechanisms (identical rules, analogy, knowledge compilation, and constraint violation) that mediate different modes of transfer. These mechanisms are activated by the applied task itself as well as environmental, social, and personal situated cues [89]. Mathematics knowledge rarely transfers to engineering if students believe mathematics to be unrelated or irrelevant, as many engineering students do [20, 21, 29, 90]. When transfer fails, successful students engage in reduplicated learning, constructing an isomorphic, compartmentalized version of previously learned knowledge that is activated by different context clues [91, 28]. For example, a student may have one schema for determining the approximation accuracy of a truncated Taylor series for $y = \arctan(x)$ in a mathematics context, and has constructed a completely separated mental structure associated with determining the approximation accuracy for an approximation of the period of a pendulum in a physics context [28]. Such students do not associate the two mental structures with each other and thus prerequisites may not effectively prepare students. Engineering faculty must reteach the content that has been forgotten or compartmentalized. Given these issues with recall and transfer, one questions whether engineering students are getting the right mathematical content at the right time [66].

4.1.1 Curricular analysis

Previous literature has examined the connections between mathematical content in the engineering curriculum [60, 92, 35, 61, 93]. Overall, these studies agree that only a small portion of the mathematical content in the calculus sequence is actually applied in engineering courses; application may be separated from the prerequisite course by a year or more, and some applications are taught before the underlying mathematics in sequence. However, one key weakness of this part of the literature is a dependence on faculty self-reporting to document curricular connection. Most engineering faculty largely do not know the particular content taught in a given mathematics course [60, 61]. Furthermore, faculty self-reporting is often inaccurate because a professor's perception often fails to match their practice. For example, one study found that while 68% of calculus teachers claimed in a survey to require explanations frequently on their exams, analysis of the exams themselves revealed that only 3% of all problems involved explanation [94, 33]. Because faculty self-report may not reflect student experience, we argue that research into curricular alignment necessitates an alternate means of inquiry to fill this gap. In this study, we chose to examine homework as a course artifact, due to homework's role as the assessment students engage with most often, and with the most diversity in content coverage.

4.1.2 Assessment analysis

Previous research on application of calculus in engineering homework problems reveals that the way that calculus is applied in engineering may differ greatly from how that same calculus was covered in calculus courses. One study [95] found that in statics, students may not recognize that the integration process relating shear force to bending moment is even the same object as the integration they learned in calculus (a failure of transfer) [95]. Analysis of tasks in engineering courses has shown that an introductory electrical engineering course uses primarily the "mathematics of physical quantities" (management of units and orders of magnitude) which is taught neither in high school mathematics courses nor in university-level mathematics courses [96]. These previous works exhibit a gap: analyzing single problems or first-semester courses lacks the curriculum-wide scope to compare calculus as a whole course against the curriculum of courses that follow it.

4.1.3 Research questions

While there have been studies that have examined the alignment of the engineering and mathematics curriculum through faculty self-reporting, these suffer from accuracy problems. And studies focused on homeworks examined insufficiently large and comprehensive scope of problems to examine questions at the curricular level. Because of the accuracy problems with faculty self-reporting, and scope problems with homework problem analysis, results from previous work may be ineffective at enacting the curricular change that research hoped to promote. This evidence presented in these previous works may be less effective when engineering and mathematics departments negotiate over curricular revision. In the spirit of creating "*clear understanding of what is needed*", this study seeks to provide more accurate, more comprehensively scoped articulation of engineering's mathematical needs and expands the previous literature by analyzing the ways that calculus is applied in homework problems in two core engineering courses.

Research Question 1) Which concepts and skills from Calculus I are applied in engineering statics and circuits homework?

Research Question 2) How are calculus skills applied in statics and circuits homework?

4.2 Methods

4.2.1 Mathematics-in-use

We employed the "mathematics-in-use" technique for the analysis of course artifacts [1]. This technique involves solving a problem completely without skipping a single step, including steps that might be obvious to an expert. It also requires exploring many alternate solution paths. Mathematics-in-use analyzes each problem for the concepts and skills in calculus that are required to solve the problem. Concepts are low-level ideas about mathematics (e.g. "derivatives express a rate of change"). Skills are the procedural sequences of steps used to solve a particular type of problem (e.g., "how to compute the derivative of a polynomial"). The resulting data is a narrative description of the problem solution (see Figure 4.2), a list of the concepts and skills that are needed to solve that problem, and a summary of which concepts and skills were applied (see Table 4.1). The concepts and skills for calculus were taken from the list generated by Czocher [1], derived from interviews from veteran calculus instructors [97]. An example of the analysis of one problem is shown in the following section.

4.2.2 Example of mathematics-in-use

Many problems have multiple possible solutions to be documented. Concepts and skills from calculus are **bolded** when they appear in the analysis. The bolded items for each problem are summarized in Table 4.1.

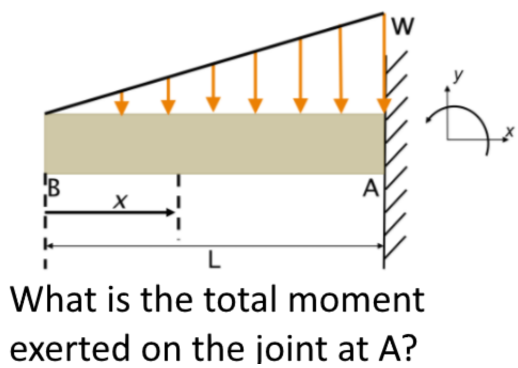


Figure 4.2: An example problem prompt from statics for the mathematics-in-use analysis. Analysis of this problem can proceed via two common paths, "the calculus way", and "the algebra way" which makes use of centroids as the instructor solution presents.

The Calculus Way

Since the load is distributed, every little chunk of moment comes at a different location. The total moment is made of a **bunch of small pieces of moment (concept: integral)**.

$$\Delta M = \Delta F \times r$$

We must find a way to make our variable of integration x since that is the part of the geometry we can vary. The small chunk of force will be related to the loading-per-unit-length at that its location.

$$\Delta F(x) = w(x)\Delta x$$

We must construct an algebraic expression for the loading $w(x)$ from the diagram using point-slope form for lines. If we let the maximum value of $w(x)$ be P, we obtain a linear expression.

$$w(x) = \frac{P}{L}x + 0$$

Since more advanced x's have shorter lever arm, the lever arm r is related to the coordinate x by $r = L - x$. The combination of all these **algebraic expressions** yields

$$\Delta M = +(\frac{P}{L}x)(L - x)\Delta x$$

The sign is chosen to be positive since the moment is counterclockwise. We then use **concept of integration** to add up all the little pieces to get a continuous whole

$$\sum \Delta M = \sum +(\frac{P}{L}x)(L - x)\Delta x$$

The sum becomes an integral and the differential element Δx becomes dx. The first piece is located at $x = 0$ and the last piece at $x = L$

$$M = \int_{x=0}^{x=L} +(\frac{P}{L}x)(L - x)dx$$

Distribute to make the expression easier to integrate

$$M = \int_{x=0}^{x=L} +(-\frac{P}{L}x^2 + Px)dx$$

We must apply the **technique of integration: polynomial**.

$$M = -\frac{1}{3}\frac{P}{L}x^3 + \frac{1}{2}Px^2\Big|_{x=0}^{x=L}$$

$$M = -\frac{1}{3}\frac{P}{L}L^3 + \frac{1}{2}PL^2 - (-\frac{1}{3}\frac{P}{L}0^3 + \frac{1}{2}P0^2)$$

$$\boxed{M = \frac{1}{6}PL^2}$$

The Algebra Way

The total force from a distributed load is the **area of the triangle (area/volume)** with width L and height equal to the maximum value of $w(x)$ being called P.

$$F = \frac{1}{2} \times base \times height = \frac{1}{2}PL$$

.

A distributed load exerts its moment at the location of the centroid, which according to the given formula for right triangles is $\frac{1}{3}$ of the way from the right angle, so

$$r = \frac{1}{3}L$$

Combining these **algebraic expressions** yields

$$M = r \times F = +|r||F| = (\frac{1}{3}L)(\frac{1}{2}PL)$$

The sign is positive since the moment is counter-clockwise.

$$\boxed{M = \frac{1}{6}PL^2}$$

4.2.3 Course content analysis

All homework problems in one semester for a course under consideration are analyzed this way using mathematics-in-use. Groups of related problems on a given topic from the engineering course have their tables combined. Topics are broad ideas found in tables of contents, syllabi, or at the top of lecture slides (e.g., "Bending Moments"). When combining the tables for each topic in the course, one constructs a visual chart that documents the usage of calculus concepts and skills over the entirety of the course content (see Figure 4.3 for the chart for the Differential Equations course). This format lays bare

Table 4.1: The resultant table contribution for the example mathematics-in-use problem. An • indicates the skill is used along the instructor suggested problem pathway, and a ◦ that the skill could be used on an alternative pathway. In this case, the problem does not use calculus along the instructor-selected pathway.

CONCEPTS		SKILLS	
Derivative		Derivative Computations	
Integral	◦	Integration Techniques	◦
Fundamental Theorem		Limit calculations	
Limit		sequences/Series	
Approximation		Algebraic Expressions	•
Riemann sums		Area/Volume	•
Parametric/Polar		Parametric Equations	
Continuity		Polar Coordinates	
Optimization		Trigonometric Manipulations	
		Logs & Exponentials	
		$\epsilon - \delta$	
		Listening & Reading Comprehension	
		Definitions & Notation	
		Limit Calculations	

the most immediately recognizable content from a course (the topics), but shows the base elements from the prerequisite Calculus course required to solve that content.

4.2.4 Data selection

We have chosen to study how calculus skills/concepts are applied in two high-enrollment core engineering courses: Statics and Circuits. These are often the first engineering courses that students take following the calculus sequence and are considered the gateway to upper-level engineering courses. Many departments require Statics, Circuits or both. The content of these Statics courses is consistent between institutions [98] so the analysis should generalize well. At the original research site, the median incoming engineering student has an unusually high ACT math of 34 (99th percentile nationally) and most students have AP calculus credit. We expect that the calculus content here to form a conservative upper bound on the amount of calculus used elsewhere due to the high level of calculus-readiness among our freshmen.

Unlike other work that analyzes interviews or workshops with faculty [18,

	topics	classification	FOL, variable coeffs	separable eqns	applications with FO	linear vs. nonlinear	autonomous eqns	popluation dynamics	exact eqns	integrating factor	existence / uniqueness	SO linear	fundamental solns	wronskian	complex/ repeated roots	undetermined coeffs	applications (vibrations)	higher order eqn	two-point BVP's	PDE's	Fourier series	separation of variables	heat, wave, laplace eqns	phase planes	systems of linear eqns
calculus concepts	derivative				●			●	●			●				●	●			●		●	●	●	●
	integral			●	●				●	●		●		●		●					●				●
	fundamental thm			●						●															
	limit																●				●		●	●	
	approximation																								
	sequences & series																			●	●		●		
	riemann sums																								
	paramteric & polar eqns																								●
	continuity																								
	optimization																								
calculus skills	derivative computations				●		●	●				●				●	●		●	●	●		●	●	
	integration techniques		●	●	●		●	●	●	●		●		●			●		●	●	●		●		
	limit calculations				●			●									●								
	sequences/series																				●	●	●		
	algebraic expressions		●	●	●				●			●	●	●	●	●	●		●	●	●	●	●	●	
	area/volume				●																				
	parametric eqns																								●
	polar coord.																								
	trig manipulations		●	●	●		●		●			●			●		●		●	●		●	●		●
	logarithms and exponentials		●	●	●				●	●		●		●	●	●			●	●			●		●
	epislon-delta																								
	listening & reading comprehension							●			●														
	definitions & notations	●				●					●							●							●

Figure 4.3: An example matrix for the course Differential Equations [1]. Topics (columns) come from the follow-on course, Differential Equations. Concepts and skills (rows) come from the prerequisite Calculus course(s). A filled in square in the table indicates that the postrequisite course topic in its column applies the calculus concept/skill from its row. Columns with many dots are topics that require many techniques of calculus to understand and solve. Columns with no dots are topics that do not make use of calculus. Rows with many dots are concepts/skills from calculus that are frequently required to solve problems in the course. Rows with no dots (pink) are concepts/skills from calculus that are not applied in this course.

17, 64], we have chosen to analyze homework problems. Faculty testimony may include knowledge of the subject that is not explicitly incorporated into the course in the same way as homework. These homework assessments are concrete representations of values, standards, and expectations that students are expected to achieve [99]. We analyzed homeworks, rather than exams, because homework is the primary formative assessment. Homework problems are also more numerous than exam questions and more likely to contain lengthy calculations.

Statics

Statics (TAM 211) is required by 7 of the 13 engineering majors on our campus, and leads to required mechanics classes like Mechanics of Materials, Fluid Mechanics, and Dynamics. Hundreds of students take the three-credit TAM 211 course each semester at our institution. Statics typically has a prerequisite of first-semester calculus, and is often taken in the sophomore year, since a calculus-ready student must take Calculus I, first-semester physics, and is then permitted access to Statics. The analysis was carried out by Brian Faulkner and another graduate student with experience in materials science. The homework problems and instructor solutions were obtained from the instructor at our institution. The sample had 12 homework assignments with a total of 84 problems. The topics list was taken from the homework titles from the syllabus publicly available on the course website.

Circuits

Circuits (ECE 205) is required by 5 of 13 engineering majors on our campus. The ECE 205 course is a service course for non-electrical and non-computer engineers. Hundreds of students take the three-credit ECE 205 course each semester at our institution. We chose circuits-for-nonmajors because it is required by more departments and likely more representative of institutions that offer only one circuits course. Circuits requires two semesters of calculus as a prerequisite, and is often taken in the sophomore year. Analysis of the Circuits problems was conducted by me and another graduate student in electrical engineering. The topics for mathematics-in-use were taken from the course syllabus publicly available on the course website. The sample has 12 homework assignments with a total of 70 problems. The homework problems and instructor solutions were obtained from the instructor.

4.2.5 Trustworthiness

I am an engineering-mathematics education researcher with a background in physics, mathematics, and electrical engineering. The data were primarily analyzed by me with the mathematics-in-use technique. Two random problems in each lesson in statics were re-coded by a second researcher (with a materials science background). The circuits problems were re-analyzed by a third researcher (with an electrical engineering background) in the same fashion. An inter-rater reliability of 87% was achieved.

Member checking [100] was performed by showing the resulting topic-concept/skill chart to the course instructors and asking if the evaluation seemed correct. The instructor of the Circuits course indicated that the topic-skill/concept table matched their perception of the mathematical content course. The instructor for Statics similarly confirmed that the table matched their perception of the course.

4.3 Results

The outcomes of the analysis for Statics (see Table 4.2) and Circuits (see Table 4.3) are summarized in the following section. Given the status of calculus as a prerequisite to these core engineering courses, the quantity and variety of calculus applied in these courses is strangely small and limited.

4.3.1 Statics

As can be clearly seen in the Table 4.2, the majority of the mathematics applied in Statics is algebra, geometry, and trigonometry ($\bullet$'s in Table 4.2). Very little calculus is used. Of all the problems in an entire semester of statics (84 problems), just 7 (8%) of these problems require calculus. Five of those 7 are from a single lesson (internal forces).

In our Statics course, energy functions and unstable equilibrium were not covered, leading to the lack of derivatives and optimization as observed in statics by others [50].

4.3.2 Circuits

Of the 70 homework problems in one semester of Circuits (see Table 4.3), 14 of them (20%) require calculus concepts or skills to solve. Most calculus (10 of the 14) is concentrated in two lessons, the first on LR and RC circuits, and the second on LRC circuits. Although Calculus II is an indirect prerequisite for Circuits, and sequences & series and parametric/polar are two primary topics in Calculus II, neither of these two topics occurs in Circuits.

4.3.3 Comparison of the use of calculus ideas

In addition to the limited application of calculus ideas in both Circuits and Statics, the way those concepts are used also varies between the mathematical prerequisite course and the engineering courses that follow it. These differences are summarized in Tables 4.4, 4.5, 4.6, 4.7 and 4.8. These ideas are further elaborated upon in Section 4.4.

4.4 Discussion

4.4.1 Research question 1

Which concepts and skills from Calculus are applied in engineering statics and circuits?

The skills from Calculus that are applied in Statics are low in both abundance and diversity. Only 8% of problems in statics apply calculus in any way, and the portion of calculus that is applied is very limited. Only a tiny fraction of the content taught in calculus is used in Statics.

The situation in Circuits is similar. Only 20% of problems in Circuits use calculus, but the diversity of concepts that are called upon in Circuits is moderately diverse. Integration, differentiation, and limits all occur in at least one lesson as a real tool. Reiterating, this circuits course is a service course offered to non-electrical engineering students. The circuits course for

Table 4.2: Calculus concepts and skills in Statics. A concept or skill from calculus (row) in a statics lesson (column) is indicated by a ● on the instructor path or a ○ on an alternative path.

		Vector Math	Force Vectors	Equilibrium	3D equilibrium	Distributed loads	Rigid Bodies	Rigid Equilibrium	Trusses and Frames	Internal Forces	Friction	Moment of Inertia	Fluids & Virt. Work
Calc I Concepts	Derivative									●			●
	Integral						○			●		○	○
	Fundamental Thm.									○			●
	Limit Approximation												
	Riemann sums												
	Continuity									●			
	Optimization												
Calc I Skills	Derivative Comp.												
	Integration Tech.						●			●		○	○
	ϵ-δ Definitions												
	Limit Calculations									○			
Calc II Concepts	Sequences & Series												
	Polar & Parametric												
Calc II Skills	Sequences & Series												
	Parametric Eqns.												
	Polar Coordinates												
Precalc Skills	Algebraic Expr.	●	●	●	●	●	●	●	●	●	●	●	●
	Area & Volume						●			●		●	●
	Trigonometry		●	●	●	●	●	●	●	●	●		●
	Exponentials												
	Reading Comp.		●	●	●	●		●	●	●			●

Table 4.3: Calculus concepts and skills in Circuits. A concept or skill from calculus (row) in a statics lesson (column) is indicated by a • on the instructor path or a ○ on an alternative path.

		Voltage & Current	Series & Parallel	Nodes & Loops	Superposition	RC & LR	Phasors	Power	LRC Circuits	ω Resp. & OpAmps	Diodes	BJT's	Logic Gates
Calc I Concepts	Derivative					•	•	○	•				
	Integral					•			•				
	Fundamental Thm.					•			•				
	Limit					•				•	○		
	Approximation												
	Riemann sums												
	Continuity					•							
	Optimization							○					
Calc I Skills	Derivative Comp.					•	•	○	•				
	Integration Tech.					•			•				
	ϵ-δ Definitions	•	•	•		•		•					•
	Limit Calculations					○				•			
Calc II Concepts	Sequences & Series												
	Polar & Parametric												
Calc II Skills	Sequences & Series												
	Parametric Eqns.												
	Polar Coordinates												
Precalc Skills	Algebraic Expr.	•	•	•		•	•	•	•	•	•	•	•
	Area & Volume												
	Trigonometry				•		•	•		•			
	Exponentials					•	•		•				
	Reading Comp.												

Table 4.4: Mismatches in how the Derivative concept is taught in Calculus and applied in Circuits and Statics.

Derivatives in Calculus [1]	Derivatives in Circuits	Derivatives in Statics	Synthesis
Computation; rate of change.	Derive current to find voltage or vice versa in inductors and capacitors; match to an initial condition for I'(t).	Infinitesimal displacements in virtual work; check integral calculations.	Derivatives are pre-defined relationships between physical quantities.

Table 4.5: Mismatches in how the Integral concept is taught in Calculus and applied in Circuits and Statics.

Integrals in Calculus [1]	Integrals in Circuits	Integrals in Statics	Synthesis
Antiderivative; measurement of area; volume and accumulation.	Integrate current or voltage to find the other in capacitors and inductors.	Integrate load density to get shear force; integrate shear force to get bending moment in beams.	Integrals are pre-defined relationships between physical quantities.

Table 4.6: Mismatches in how the Fundamental Theorem of Calculus concept is taught in Calculus and applied in Circuits and Statics. The usages of the concept overlap very little in the two courses.

Fundamental Theorem in Calculus [1]	Fundamental Theorem in Circuits	Fundamental Theorem in Statics	Synthesis
Formal justification for using antiderivatives instead of definite integrals; shortcut for computing certain derivatives and definite integrals.	Produces parameter for initial currents in LRC circuits; inverts derivative relationships into integral relationships when desired quantity is inside either operator.	Virtual work justification.	N/A

Table 4.7: Mismatches in how the Limit concept is taught in Calculus and applied in Circuits and Statics.

Limits in Calculus [1]	Limits in Circuits	Limits in Statics	Synthesis
Algebraic computation; exposure to formal ($\epsilon - \delta$) definitions; basis for derivative definition	Evaluate high frequency response of circuits.	Evaluate right and left side values around point loads.	Formalism of limits is not applied, only very simple limits are evaluated.

Table 4.8: Mismatches in how the Continuity concept is taught in Calculus and applied in Circuits and Statics. The fact that different physical quantities have different continuity constraints is a key feature in both courses.

Continuity in Calculus[1]	Continuity in Circuits	Continuity in Statics	Synthesis
Property to be checked.	Current in capacitors and inductors cannot change instantaneously.	Shear force and bending moment only have jumps at locations of point loads, applied moments and joints.	Guarantee desirable physical properties.

electrical engineers may use much more calculus. But this course is likely representative of service circuits courses, and at institutions that offer only one circuits course.

The low percentage of problems that require calculus sheds light on the likely source of the correlation between performance in engineering courses and their calculus prerequisites. Even optimistically assuming perfect transfer of knowledge, calculus can account for a share of the correlation at most equal to its percentage of the assessments. So in cases where the correlation coefficient (often $r = 0.4$ to 0.7 in the literature [76, 77, 78, 79, 80]) is higher than the percentage of problems that use calculus (0.08 and 0.2 in this work), we can conclude that much of this correlation comes from a confounding variable. A likely candidate would be algebra skills, since calculus courses and engineering courses both require large amounts of algebraic calculation.

This result does not have the power to suggest curricular change to prerequisites. However, it does suggest a question: **Given the high engineering student attrition in calculus, what is the minimum amount of calculus content in an engineering course that justifies requiring calculus as a prerequisite?**

Answering this question is beyond the scope of this work, as it depends on many contextual factors. Course grades are often more determined by exams, which may not have a similar concept/skill distribution to that on the homework. Other sources of assessment such as projects or participation may not depend on calculus at all. However, the assertion "calculus must be a strict prerequisite for statics because statics uses calculus" might be overly simplistic. Perhaps a leaner calculus-like course might provide adequate preparation while consuming less time in the curriculum. Some programs, such as the Wright State Model, have explored modifying the first-semester mathematics curriculum to increase alignment between engineering and mathematical preparation [15].

Recall and Transfer

Most engineering students will forget approximately 85% of mathematical concepts they learned in calculus after one year [81, 86, 85, 82, 83, 84]. Due to this forgetting, students might not be able to recall knowledge from calculus by the time it is applied in engineering courses. In Table 4.9, we elaborate on some of the possibilities of mathematical concepts from Calculus 1 and how they are applied in statics and/or dynamics.

Table 4.9: A mock-up of the consequences of several curricular cases, using the prerequisite chain from Calculus I to Statics to Dynamics as an example. (Dynamics was not analyzed in this study; this table is merely a discussion aid.)

	Example Skill/Concept	Calculus I	Statics	Dynamics	IMPLICATION
A	Integration of polynomials	•	•	•	Longitudinal reinforcement
B	Computation of derivatives	•		•	85% Forgotten [81]
C	$\epsilon - \delta$ limit	•			Pure math?
D	Quotient Rule	•			Obsolete?

The result is three types of cases with their own implications for students' ability to recall mathematics concepts and thus performance in engineering problems.

Case A

Content in row A is repeatedly refreshed and re-learned through new contexts. The prerequisite system functions as intended, assuming adequate transfer.

Case B

Content in row B does not occur in Statics, but does occur in the following course, Dynamics. Techniques taught in Calculus that are not refreshed may be forgotten by the time they must be recalled in Dynamics. However, structured review of content can provide longitudinal reinforcement in off-semesters and reduce how much is forgotten [101, 102], and is not as drastic a revision as full curricular reform. For case B, the skill of computation of derivatives is not practiced, and may be forgotten by the time students reach dynamics. To remedy this lapse in reinforcement, instructors might alter exercises to take the derivative from bending moment to shear force, rather than exclusively integrating from shear force to bending moment.

Case C

Mathematical content in row C may not have application anywhere in the sequence. It may be targeted at mathematics majors. For example, the formalism of the $\epsilon - \delta$ definition of the limit may not be seen again.

Case D

Content that is not applied in subsequent courses might also be an obsolete topic relegated to computers and no longer in active use in engineering, such as exact equations and integrating factors commonly taught in Ordinary Differential Equations courses [3]. The obsolescence explanation is suggested by interviews with engineering faculty [17, 18, 61, 77, 64].

4.4.2 Research question 2

How are the skills that are applied in Statics and Circuits used differently than in calculus?

Of the content from calculus that is applied in Circuits and Statics, the usage and nature of that content knowledge vary considerably. The following section illuminates a few of these differences. For the following section, it is important to note that the topic coverage of calculus courses around the nation is very standardized [103]. This high degree of uniformity, reinforced by market leader textbooks such as Stewart's Calculus [55] allows us to make conclusions about calculus teaching as a whole. There are certainly some variants such as the Wright State program [15] but these are a small minority.

Complexity of Functions

In Calculus, students learn a set of rules powerful enough to take the derivative of any combination of functions. Calculus students learn how to differentiate and integrate the six standard trigonometric functions, exponentials, logarithms, and roots. Standard techniques include the chain rule for function compositions like $\sqrt{\sin(x^2)}$, the product rule for multiplied functions like $e^x \cdot \sqrt{x}$, and the quotient rule for division of functions like $\frac{\sin(x)}{x}$[55]. Second semester calculus inverts these techniques for integration, covering u-substitution, trigonometric integrals, and integration by parts. Only a subset of these practiced skills find application in Circuits or Statics.

Statics coursework does not require students to use any type of derivatives; students do not use the chain rule, product rule, or quotient rule at all. In contrast, students do use integration in statics. However, students typically only integrate piecewise polynomial functions and completely ignore functions such as exponentials, logs and roots.

Circuits uses mostly polynomial functions, along with exponential functions. The chain rule is limited to functions of the form e^{ax}. The product rule appears just once in Circuits (for the function $x \cdot e^{ax}$). The quotient rule is entirely absent in Circuits. Surprisingly, Circuits (for non-electrical majors) never takes the derivative of a sinusoid to see the voltage-leads-current phase relationship in inductors, using frequency domain instead. Circuits uses none of the advanced techniques of integration taught in its prerequisite Calculus II.

This result corroborated evidence with interviews with engineering faculty, who stated that only a small fraction of the rules learned in calculus are used regularly in engineering [17, 61, 64, 91, 18]. Only the simplest rules of calculus are applied in Circuits and Statics. While previous results have also made this point, it is important to stress just how simple “simple” is. The most complex function encountered would be a simple homework exercise in a calculus class. Much class time in Calculus is dedicated to techniques that do not reach application, at least not immediately.

Continuity

The continuity concept has interesting epistemic mismatches between calculus and the applied courses. Continuity appears as a necessary concept in 4/7 of the statics problems that use calculus, and in 7/14 of the circuits problems that use calculus. In Calculus, continuity is just a "property to be checked" [1]; students are tested on evaluating whether a given function is continuous.

In Statics, continuity determines what is allowed. Students must manage which quantities are allowed to be discontinuous, and under what conditions. In Statics, the shear force on a beam can jump only (be discontinuous) at the location of a point load, and the magnitude of that discontinuity must be equal to the size of the point load. Continuity constraints must be applied to solve many problems in statics, particularly with piecewise functions.

In Circuits, the current in a resistor may change instantaneously (can be discontinuous) but the current in an inductor cannot change instantaneously (may not be discontinuous). Management of discontinuity is a key that enables the solution of several problems, providing information that makes a system solvable.

In both Circuits and Statics, most integration and differentiation acts on piecewise (often piecewise linear) functions. This use contrasts with instruction in calculus, where piecewise functions are covered, but are not a focus of instruction.

Management of continuity concerns is particularly important for interacting with these piecewise functions in Circuits and Statics (see Figure 4.4). When integrating current with time to obtain charge transferred, for instance, the total charge must be continuous. This use of continuity is most evident if students have an algebraic expression for each piece of the piecewise function. The result of one integration, combined with continuity constraints, becomes the $+C$ used for the next piece. The continuity relationships between segments are always related to physical events or locations, such as the presence of a point load in Statics or the moment a switch is thrown in Circuits.

This use of continuity contrasts with the types of integration and differentiation practice of much of calculus, where most functions are single algebraic expressions rather than piecewise functions. Also in Statics and Circuits, the explicit algebraic expression for these piecewise functions is rarely given. An

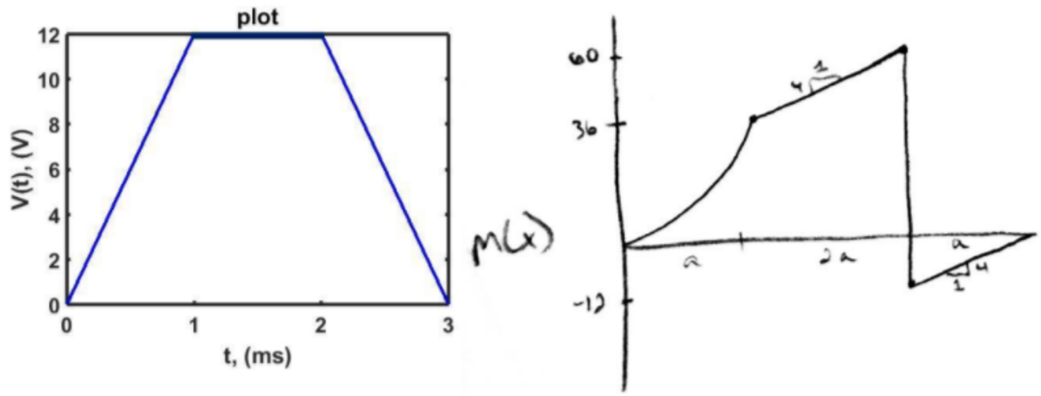


Figure 4.4: Typical functions from Circuits (left) and Statics (right) from the instructor solutions. Piecewise functions containing quadratic or linear segments are the norm.

expression must be obtained from a given graph, or the function must be integrated graphically rather than symbolically. The solution is expected in graph form, not analytic/algebraic form.

When doing these piecewise integrations, more complicated kinds of continuity are sometimes encountered. At the location of an applied moment (i.e., a point-like twisting force) or a reaction moment at a fixed joint in Statics, there is a kind of discontinuity beyond first-semester calculus (Dirac δ function) that is not represented in the shear diagram. This lack of representation is a key source of error when applying integration from shear force to bending moment [104].

This application of continuity and its importance is consistent between these statics and circuits. This consistency may indicate that increasing the emphasis on discontinuous and piecewise functions in the teaching of Calculus (though at the expense of other content) may better prepare students for subsequent coursework.

Usage of Limits

In calculus, limits form the first few weeks of instruction. The definition of the limit may be presented. Students learn various rules for evaluating limits for many sums and products of limits, as well as rules for handling ill-behaved indeterminate forms.

In Statics, the breadth of limits encountered is much more narrow than that learned in calculus. Limits only appear once in Statics. One homework problem asks the student to obtain a numeric value for the (discontinuous) shear force just to the right/left of the point load at point a. This problem uses the $v(a^+)$ notation from calculus, though any "limit" is simply evaluation of the right side of the piecewise function.

Circuits use limits to examine the response of filters at high frequency. However, all limits in this course use only the rules of very large numbers: If $A \gg b$, then $A + b \approx A$. This number system is commonly called the Extended Real Line. Only the limit at infinity appears in Circuits. Advanced techniques like L'Hospital's rule for ill-behaved functions such as $Sinc(x) = \frac{sin(x)}{x}$ (an important function in signal processing) are absent. The most complex limit students must evaluate is

$$|H(\omega = \infty)| = \lim_{\omega \to \infty} \frac{\omega}{\sqrt{\omega^4 + A\omega^2 + B}} \approx \frac{\omega}{\sqrt{\omega^4 + A\omega^2}} \approx \frac{\omega}{\sqrt{\omega^4}} \approx \frac{1}{\infty} \approx 0$$

The application of limits in Circuits and Statics is much simpler and less mathematically rigorous than the limits in Calculus courses. Historically, this limited application makes sense: Calculus had been used to develop beam theory [105] nearly a century before the first epsilon-delta limit proof was published [106]. Stronger, more rigorous formulations of limits were a response to strange, paradoxical corner cases that defy the intuitive notions of limiting behavior. However, the simple, well-behaved physical systems under consideration in these early courses are described by simple, well-behaved functions. The ill-behaved functions that make mathematically rigorous definitions worthwhile are not encountered.

Reduction to Algebra

Our analysis technique, mathematics-in-use, follows all the solution paths that a student might have access to by this point in their education. Many problems in both Statics and Circuits could be solved using the methods taught in calculus (in the charts, the ∘'s). However, the instructor solution to these problems does not use calculus and is a purely algebraic solution. Upon reflection, this preference for algebraic solutions makes sense. Calculus is more difficult than algebra, so engineers who must do many calculations are incentivized to reduce a calculus-based problem to an algebraic one whenever possible. This phenomenon has been seen before; students often reduce to algebra even when the pre-computed algebraic expression is inappropriate in Physics II [77].

In Statics, many problems involving bending moment are in principle problems of integration, but are not in practice from the students' perspective. Most of the "integrations" are over simple rectangular or triangular functions (piecewise linear functions); the "integrals" for shear forces use basic geometry, not calculus. The integral may be the area under the curve, but that interpretation of calculus only matters when the functions of interest are curved. Computation of bending moments also avoids use of calculus through centroids. Students use a provided table of centroids combined with algebra, rather than calculus, to compute bending moments. One can compare directly the length and complexity of these problems as seen in the example mathematics-in-use (see Section 4.2.2).

When discussing loads and power in the Circuits course, students must calculate which load would receive maximum power from a given source. One could construct an expression for the power delivered as a function of the source load resistances, take its derivative with respect to the load resistance, set this expression equal to zero, and solve for the load resistance, just as optimization problems are taught in calculus. However, the instructor solution to the homework simply states as a fact that matching impedances produce the maximum power. The student is not expected to perform any optimization.

4.4.3 Limitations

We must interpret these results conservatively.

- This analysis covers only two courses. Other core courses in engineering, science, or business may apply more or different calculus. For example, though sequences and series do not occur in either of these courses, they are essential tools in signal processing. These results cannot suggest any curricular change. A similar study of greatly increased scope could make credible suggestions.

- This study's data makes no claims about how much of this required calculus knowledge is remembered by these students. We also cannot speculate about the quality of successful transfer of concepts/skills from Calculus to Circuits/Statics.

- This study describes only these classes as they are. We cannot infer from this data why they are that way, or what they should be like. Engineering faculty may wish to use more estimation/approximation [107], but feel they cannot, or may have decreased the quantity of calculus applied in the course in response to falling mathematical competence of the typical student [66]. A study of an entirely different methodology could probe these questions.

- This analysis does not permit examining topics that are not covered due to issues of topic sequencing. For example, in this analysis, no linear system bigger than 3×3 was encountered. This absence may be because techniques for larger systems are taught in linear algebra, which often is taught after Circuits or Statics.

- This study is limited to a single offering at a single institution. Other instructors may place more or less emphasis on a topic, or explore it more mathematically. However, due to the high level of calculus-readiness at our institution, on average other institutions will probably use even less calculus in their Circuits and Statics courses.

- This study's data does not let us infer why a particular course is a prerequisite. It may be a prerequisite for reasons not having to do with content, such as gate keeping or practice with more fundamental skills. A completely different methodology would be necessary to investigate this aspect.

- This analysis does not account for successive re-learning. After all, isn't forgetting and re-learning essential to long-term learning? True, but the timescale for relearning effects is typically on the order of days or weeks [108]. This contrasts with the 6-18 month time spans between learning and re-learning for mathematical content in the engineering mathematics sequence.

- This analysis investigates only homework problems, but exams usually make up more of the student's grade. In courses where exams differ greatly from the homework, this kind of analysis is inaccurate.

4.5 Conclusions

From these data, we cannot be too hasty to make changes or suggest reform. There are many factors that influence prerequisite structures, the content of courses, and how students progress through it all. The multi-faceted needs of departments and the multiple objectives of introductory mathematics (to serve engineers, scientists, mathematicians, and general education) further cloud any attempt to make policy suggestions. However, these results can suggest a shape for future research and discussions. The relatively narrow span of applications may be a boon. Many teachers of calculus feel they are "in a rush to cover everything" [109, 110], even lecturing during recitation times to cover all the content in the calculus syllabus [111]. Perhaps these teachers could relax, slow down, and deepen some content at the expense of less-vital advanced techniques [112, 113].

Specific, detailed mapping of the mathematical needs of the engineering curriculum could allow engineering departments to more clearly communicate what is needed to mathematics departments. Mathematics faculty are aware that engineers are dissatisfied with calculus outcomes and want to change to please these client disciplines, but are themselves not well-versed enough in the applications of calculus to do so alone [77, 91]. Previous studies have found that both mathematics and engineering faculty are ignorant of what goes on in the others' classrooms [61, 60, 91, 114]. Future studies could repeat this analysis for more courses such as introductory physics, chemistry, dynamics, and thermodynamics. A sufficiently large collection of such analyses could not only provide powerful evidence for reform, but also assist engineers in arranging longitudinal reinforcement of mathematical topics within their own courses. We can work towards a future where students in mathematics courses need not ask the ever-present question *"When am I ever going to use this?"*

CHAPTER 5

STUDY D: SURVEY OF ENGINEERING STUDENTS BELIEFS

5.1 Introduction

Due to the length of these prerequisite chains in the "math-science death march" [69], engineering students may not take their first course with engineering faculty until their sophomore or junior year [70, 71]. Since students do not interact with engineering faculty early on, they often wonder if their math courses are actually relevant or are just a barrier to get through. As a consequence, students often form deleterious beliefs about mathematics and its relationship with engineering [20]. These beliefs may be contributing to the problems of dropout, as students who do not believe mathematics is relevant will be less interested, less motivated, and exert less effort, exacerbating other problems. Due to the difficult nature of prerequisite mathematics courses, and their position at the root of the prerequisite tree, any lowering of motivation in these courses as a result of low perceived relevance could have great consequence for students' ability to graduate in four years.

5.2 Background

5.2.1 Previous work on why students' attitudes about the relevance of mathematics to engineering matter

Engineering students hold varying beliefs about the relevance of their mathematics coursework to engineering. Belief that mathematics is not relevant to engineering correlates with increased rates of dropout from engineering [115]. Many students believe that mathematics is not connected enough to engineering, and having high relevance beliefs is an important motivational factor to encourage diligent study and learning [116]. Higher relevance beliefs are correlated with higher rates of transferring mathematical knowledge to engineering and being able to successfully apply it in that new context [117].

5.2.2 Previous work on what attitudes are in what stage of college

Generally, engineering students' attitudes towards mathematics (particularly their belief in its relevance to engineering) change as they progress through the engineering curriculum from freshmen to seniors.

In general, incoming freshmen engineering students do not believe their mathematics coursework is relevant to engineering [118, 119, 120]. Students may not even realize there is any relationship between mathematics and engineering until after their first year of study [84]. Unsurprisingly, students rate math for which they have not yet seen any applications as being irrelevant to engineering [121]. Students perceive mathematics as unconnected to engineering design [70]. Over the course of the freshmen year, many engineering students' relevance beliefs actually get lower than when they entered college [122, 123]. This phenomenon is not entirely uniform, as freshmen in an engineering-math course had relatively high relevance beliefs towards mathematics [2].

Upperclassmen have more mixed opinions of the relevance of mathematics. According to some studies, juniors and seniors have low relevance beliefs [93, 124] and believe that their mathematics coursework had too few applications relevant to engineering [111]. However, other studies [125] indicate that they believe mathematics to be very relevant to engineering. In further interviews, students reported that the content of math courses is mainly procedural, but that conceptual math is more relevant to engineering careers, and to their engineering studies [126].

The previous literature undersamples the sophomore year, focusing on freshmen and seniors. However, students begin taking core engineering courses in the sophomore year. It may be exposure to these applied courses that shapes students' attitudes towards mathematics. Furthermore, the sophomore year directly follows the high-attrition freshman year. Examination of the sophomore year can shed light on whether the difference between freshman and senior surveys is a result of student attrition.

5.2.3 Research questions

Research Question 1) How relevant do sophomore engineering students believe their mathematics coursework is?

Research Question 2) Is this answer related to how much calculus they remember?

5.3 Methods

To answer these research questions, we surveyed engineering students about their beliefs regarding the relevance of their mathematics coursework to their engineering studies. The survey instrument was taken from Flegg et al.'s work [2]. To expand on this instrument, we added items to the survey about what "being good at math in engineering" means to students. To answer the second research question, we added a conceptual calculus assessment [127, 128]. This instrument requires no calculations, only graphical evaluations of functions and derivatives and concept knowledge of integrals and limits.

The structure of the survey instrument was as follows:

- Survey items [2] (10 five-point Likert scale items)
- Additional survey items created by the authors (7 five-point Likert scale items)
- Conceptual measure of calculus knowledge (11 true/false items, 14 multiple choice items) [128]

5.3.1 Institutional context

This study was carried out in a large, elite, American, research-intensive university with a student population of about 44,000 students. This study compares to Flegg et al.'s work [2] with a large, elite, Australian, research-intensive university with about 48,000 students. With similar institutional context (other than country), we can expect to be able to compare the results reasonably.

5.3.2 Participant selection

Participants were students currently taking Engineering Statics or Engineering Circuits during the Spring semester of 2018. Six hundred students were solicited by email through the course instructor. A total of 66 students responded (response rate 11%). Students were incentivized to participate by entry into a raffle for one of four $100 gift cards. Demographic characteristics of the respondents were not collected. Two incomplete responses were discarded.

Sophomore courses are often students' first exposure to engineering [70, 71]. If students are indeed revising their beliefs in response to seeing applications of mathematics, it is reasonable that it would happen during the sophomore year in these introductory core engineering courses. Circuits and Statics have large enrollments and maximized the chances of getting a large sample.

5.4 Results

This work-in-progress presents some initial results of this survey. Unfortunately, the small achieved sample size precludes more advanced analyses such as exploratory factor analysis. However, basic statistics and comparison to previous work can be presented. See Tables 5.1 and 5.2 for Cronbach's α analysis of the internal consistency of the Likert scale items. The responses to the survey (for items that were not eliminated) are presented in Figures 5.1 and 5.2. Overall, students have moderate views of how relevant their mathematics coursework is to their engineering studies.

The conceptual calculus instrument's basic statistics are presented in Figure 5.3. "Difficulty" is the percentage of students who answered correctly (A problem with difficulty below 0.2 is hard, a problem with difficulty above 0.8 is easy). Discrimination (Pearson point-biserial correlation) measures how well the problem correlates with overall score; items with discrimination below 0.2 are nearly as likely to be answered right by low-performing as by high-performing students.

Combining the conceptual instrument data and the items from the relevance survey, we can see how relevance beliefs and performance on the calculus instrument are related.

Table 5.1: Cronbach α data for items from Flegg et al. [2]. Cronbach α with that item removed is shown in the right column. Eliminating items that did not contribute to internal validity (~~struckthrough~~) yields $\alpha = 0.91$, a quite high level of internal consistency [129]. Items 10 and 11 indeed do not seem to be asking about perceptions of relevance, and it makes sense that they do not load with the rest of the items.

Whole Instrument after removing rejected items	**0.91**
3. I can see how the mathematics skills that I am currently developing will be useful in an engineering career.	0.66
5. In my current mathematics course, I am being taught ways of thinking that will remain with me long after I graduate.	0.67
6. I believe that my current mathematics course teaches me how to formulate and solve problems that are directly related to engineering.	0.66
7. My current mathematics course exposes me to ideas which I know I will need later on in my engineering degree.	0.66
8. I believe that being able to communicate effectively using mathematical arguments is an important skill to have.	0.67
9. The formal and rigorous aspects of mathematics that I have seen in my current mathematics course are important for my future engineering career.	0.67
4. Being integrated with mathematics majors in my math courses helps me to get a better understanding of the uses of mathematics as a whole rather than just in the engineering fields.	0.69
10. For me, I only want to learn what I feel is likely to be graded.	~~0.79~~
11. At some stage during my degree program I have been so overwhelmed by mathematical content that I have considered withdrawing from my engineering degree.	~~0.85~~

Table 5.2: Cronbach's α data for the additional survey items. Eliminating items that did not contribute to reliability (~~struckthrough~~) yields reliability of $\alpha = 0.76$, indicating good agreement for those items. The derogatory wording of "plug and chug" likely explains the elimination of item 16.

Whole instrument excluding rejected items	**0.76**
13. Being good at math in engineering means describing real world situations with math equations.	0.69
14. Being good at math in engineering means knowing if the equation makes physical sense.	0.67
17. Being good at math in engineering means being good at manipulating equations.	0.67
15. Being good at math in engineering means being good at word problems.	0.67
12. Being good at math in engineering means being able to solve math problems quickly.	0.69
18. Being able to solve integrals by hand doesn't matter because Wolfram Alpha can do them.	0.72
16. Being good at math in engineering means being able to plug and chug equations.	~~0.76~~

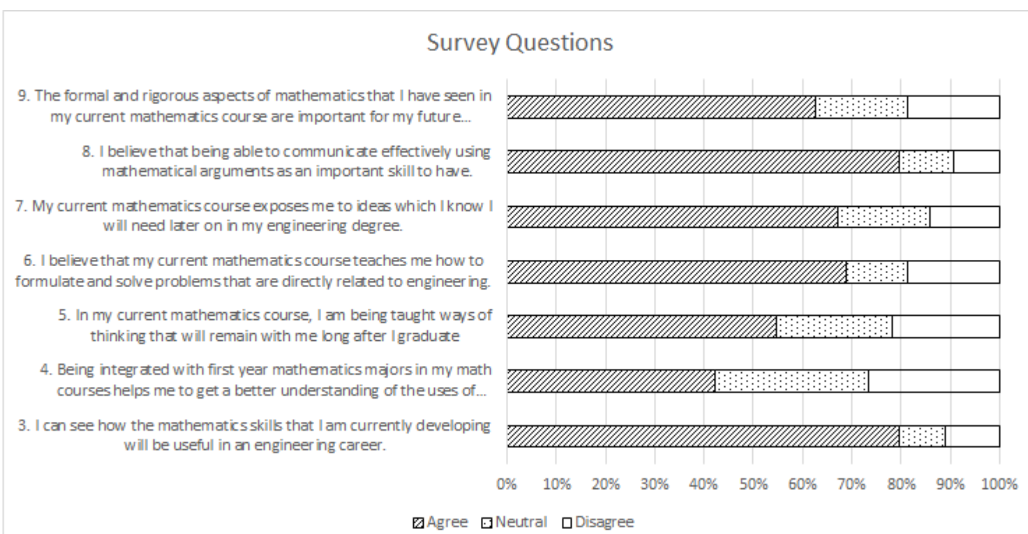


Figure 5.1: Student survey responses to Likert scale items copied from Flegg et al.(after inconsistent items removed).

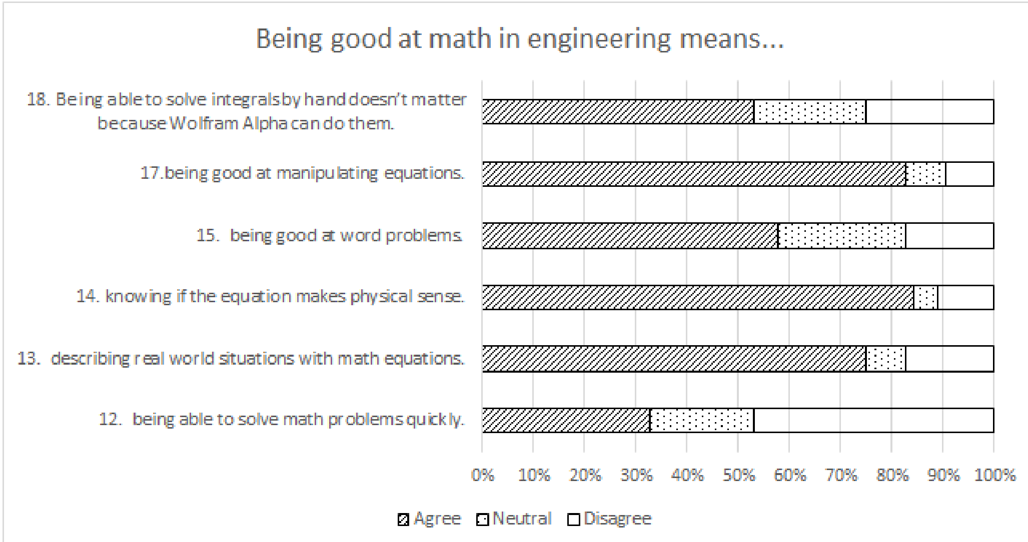


Figure 5.2: Student survey responses to new Likert scale items (after inconsistent items removed).

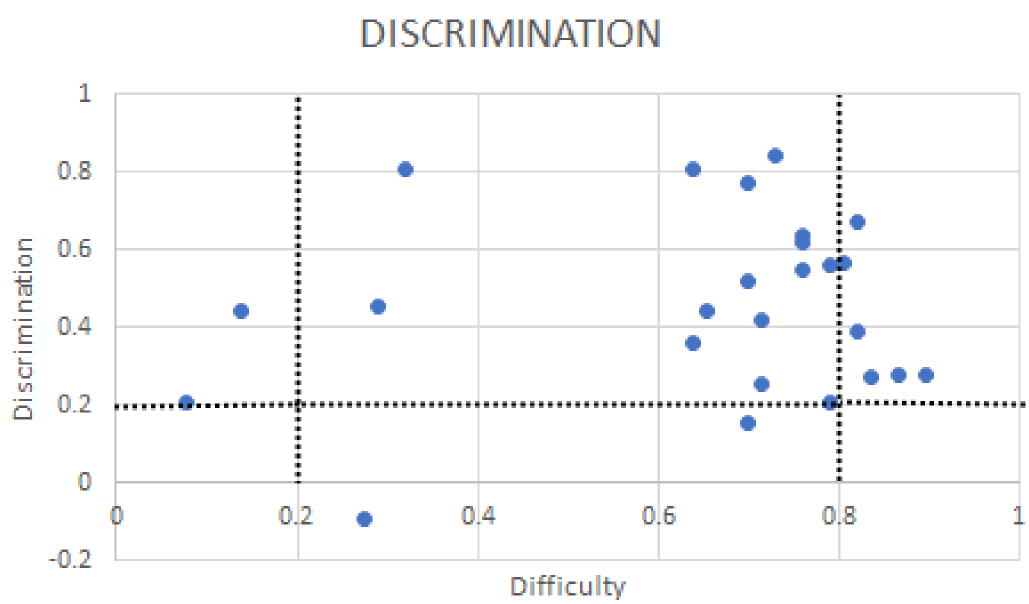


Figure 5.3: Descriptive statistics (difficulty and discrimination) for items in the conceptual measure of calculus knowledge. Most items had acceptable discrimination. There were multiple items with very high difficulty but also high discrimination above 0.4. Two of these items are mathematical theory questions about fundamental theorems, which only very few high-scoring students answered correctly.

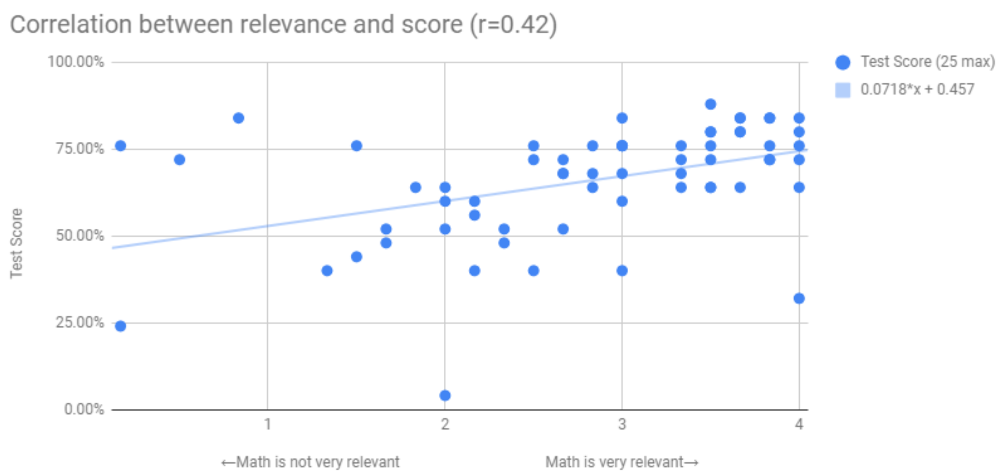


Figure 5.4: Plot of the average of the Likert scale items (after removal of low consistency items). The median score on the conceptual measure of calculus knowledge is 72%, and the median relevance rating is 3 (slightly agree that math is relevant). Belief in relevance and score are loosely correlated (Pearson correlation 0.42). The slope of the trendline is 0.07. This slope means a student answering items with one more point of relevance on average does about 7% better on the concept test. So a student answering 2 (neutral on whether mathematics is relevant) scores an expected 62% on the assessment, but one answering a 3 (slightly agree that mathematics is relevant) scores a 69%. Neither the relevance beliefs nor the relevance beliefs are homoscedastic (variance on the left is visibly higher); the linear regression is presented only as a rough magnitude of effect. A Spearman's rank order correlation results in $\rho_S = 0.48$.

5.5 Discussion

Given the moderate correlation between performance on the calculus assessment and the relevance beliefs, we can add evidence for some interpretations of why relevance beliefs seem to change over time. Students who remember more, and are thus probably earning higher grades, also have higher relevance beliefs. This result is consistent with previous literature, that high relevance beliefs can influence students' study strategies and have higher transfer to engineering. Very concerning are the students in the upper left quadrant of Figure 5.4, who despite high performance on the calculus assessment, believe mathematics is mostly irrelevant. This data contradicts the claim that only poor or ill-prepared students think math is irrelevant. These students are performing well, but even high performers fail to see the utility of the content they have been learning. Poor beliefs cannot be blamed on ill-prepared and unmotivated students in this case.

Consistent with results from Flegg et al. [2], about 30% of students agreed with the statement "At some stage during my degree program I have been so overwhelmed by mathematical content that I have considered withdrawing from my engineering degree." Since most students who drop out of engineering do so before beginning their sophomore year, these are the students who nearly dropped out recently.

Overall, students had moderate to slightly positive views of the relevance of mathematics to engineering. This result appears much like a dampened version of the results from Flegg et al.'s work. There is a majority of students that believe math is relevant, but this majority is not as overwhelmingly large as in Flegg et al.'s work [2].

The jump between the high irrelevance beliefs (and steadily falling relevance beliefs) in freshman year to the mildly positive beliefs in sophomore suggests the dropout hypothesis: students who did not believe mathematics was relevant also had low performance in their mathematics classes, and dropped out of engineering. The remainder of students have higher beliefs in the relevance of mathematics on average, because the bottom has dropped out of the student distribution.

While there is some moderate positive correlation between calculus test score and relevance beliefs, these data are not strong enough to answer Research Question 2. Most students with high test scores who remembered more calculus also had higher relevance beliefs. The presence of some high-scoring but low-believing students in the sample lends credence to the hypothesis that students of high mathematical ability need to have seen engineering applications to achieve the higher relevance beliefs observed in 3rd and 4th year students. This result matches previous literature for engineering-mathematics classes [2].

5.5.1 Limitations

The results of this work should be interpreted in the light of the following limitations:

- The limited sample size and response rate reduce confidence in the results.

- Data come from a single elite research institution and may not generalize to typical engineering institutions around the nation. Students at less selective schools may have less mathematical preparation and stronger beliefs that mathematics is irrelevant, and faculty at less research-intensive schools may not stress the mathematical aspects of engineering as heavily.

- Students normally take Statics in their 3rd semester (Fall of sophomore year). Spring statics courses are largely composed of advanced students who enter college with calculus credit (a semester ahead), students who failed Statics and are retaking it, and students who were unable to take Calculus their first semester (a semester behind). This data likely samples the extremes of the population more than a fall offering.

- The sample is largely white or Asian male students; these results may not generalize to other demographics.

- When interviewed, even practicing engineers do not recognize when they are doing math in their work, despite observers seeing the mathematics in the engineers' practice. The practicing engineers also thought their math coursework was not relevant [38]. Student perceptions may be similarly biased, not recognizing the mathematics they did as relevant or present in their work.

5.6 Conclusions

Overall, the sophomore students we surveyed have somewhat positive opinions of the relevance of mathematics to their careers as engineering students and as engineers. Consonant with previous work, engineering students have varying beliefs regarding the relevance of mathematics to their engineering studies. The sophomore year is filled with important core engineering courses like Statics and Circuits, which shape students' opinions of what engineering is and how it relates to other disciplines. The task remains to investigate how we can best encourage engineering students to develop productive expert-like beliefs towards mathematics, particularly during the freshman year when such beliefs are most connected to persistence and retention.

CHAPTER 6

SCHOLARLY AMBITIONS AND FUTURE WORK

Building on these studies, going forward I want to pursue a path of scholarship that enables more effective communication of needs between engineering departments and the mathematics departments that support them [67]. Most mathematics departments do care about their service mission, and sincerely want to help their students going forward. Unfortunately, many suffer from change fatigue [130]. After a small group of motivated engineers demand changes, the mathematics department makes the demanded changes, and just as many engineers are unsatisfied next year. When facing the choice between doing the hard work of curricular change and having everyone be mad, or doing nothing and have everyone be mad anyway, the choice is obvious.

I want more students to get more out of the first years of engineering education, much of which is from mathematics. Partially, I am a critic of the dominant epistemology of "applicationism" [23], which holds that mathematics and application are separated, that they do not change each other, and can perfectly well be learned independently of each other. This perspective is at odds with history [105]. Much mathematics was invented to solve physics problems. The cost of this "applicationism" epistemology is dire, and I want students passing their mathematics coursework, entering engineering coursework, and graduating with degrees in engineering.

6.1 Expansion Studies (Next Two Years)

6.1.1 National survey (extends Studies A and B)

One extension to this study could be a large-scale national quantitative survey of engineering faculty. Such a study would get a much larger, more representative sample of faculty and build on the results of the interviews in

Studies A and B. This approach could make a more powerful case for the generality and uniformity of the assertions in this thesis, and perhaps make a more convincing case for reform. However, 20 years of educational research in this direction has made so little difference in the way that mathematics is taught to engineering students, perhaps other tactics might be more effective in using research to enact revisions.

6.1.2 Full curriculum artifact analysis (extends Study C)

An extension of the course artifacts analysis presented in this dissertation to more (six to ten) core engineering courses would dramatically expand the power of the result. Such a set of studies could examine whole course sequences and take a critical look at the engineering side of the curriculum, pointing out when different techniques lapse in reinforcement and give students the opportunity to forget the mathematics they have learned. A larger set of courses would also be much clearer, more trustworthy information to give to mathematics departments. Many mathematics departments are beset by many requests for changes, and cannot be expected know how many of those changes are universally popular among the client disciplines and how many are unique demands. An analysis of the entire engineering core could be very clear communication from engineering instructors to their colleagues in mathematics. Furthermore, artifact analysis of this nature could help make difficult decisions regarding cuts in the curriculum, since this analysis would provide precise information about how many majors use content and how often they do so.

6.1.3 Multi-institution perceptions of relevance (extends Study D)

I am currently collaborating with Jaqi McNeil at the University of Louisville. We plan to corroborate the results of Study D, examining the students' perceptions of relevance at multiple institutions. Since introductory mathematics courses at that institution are taught within the engineering college (by the Engineering Fundamentals department), it may be that students there will have higher perceptions of the relevance of mathematics to their engineering studies. This pattern was the case in Flegg et al.'s work in an applied engineering mathematics courses [2].

6.2 Medium Term Studies (Next 5 Years)

6.2.1 Transfer from Mathematics to Engineering

This dissertation does not study transfer, but transfer is essential to the very idea of prerequisites' existence. Existing literature clearly states that transfer is difficult without explicit transfer-oriented support, but our curricula are often structured around the implicit assumption that transfer is easy and automatic. I will study the rate of successful transfer from mathematics to engineering, what can be done to maximize the chances, and help set reasonable expectations. I am particularly interested in how matters of notational convention and representation affect transfer from mathematics to engineering. Can we modify tasks to make this "handoff" from mathematics to engineering more efficient?

In the next 5 years, I hope to establish close enough relationships with my institution's mathematics department that we could engage in collaborative investigations of transfer from mathematics coursework to engineering coursework. I would like to study modified homework problems that incorporate some of the recommendations of this thesis. Such problems could be fit into a standard mathematics course without substantially altering the curriculum. Even if the list of topics and techniques stays fixed, there is much room within homework problems to introduce symbolic variety or sense-making opportunities. Derivatives need not always be taken with respect to x, homeworks could have $\frac{d}{dt}[h + v_0 \cdot t - \frac{1}{2}g \cdot t^2]$ as more of the standard drill exercises. One study could examine transfer success after exposure to such problems within their mathematics courses. Additional types of assessment activities could be incorporated in this phase of research.

6.2.2 Symbolic domain modeling

One modeling and symbolic facet that I may want to pursue is the notion of different ranks of variables: Constants, parameters, variables, and others such as temporary variables summoned for notational convenience. This distinction is essential to many applied mathematics tasks, particularly when displaying and communicating data, but is not brought up in mathematical courses. I would be interested to study how this conception influences students applied math and math modeling skills. Dr. Kevin Hadley from the South Dakota School of Mines has expressed interest in collaborating on this topic, since it closely aligns with his area of study of engineering numeracy.

6.2.3 Infinitesimals and engineering modeling

Typical calculus classes are taught with limit-based methods, as opposed to the infinitesimal-based methods employed by Newton and Leibniz. However, these infinitesimal-based methods are still common practice in physics and engineering. I would be interested in studying whether being taught with infinitesimal methods in mathematics courses leads to better modeling outcomes when describing physical systems using calculus. Mathematics-education researcher Dr. Robert Ely from the University of Idaho would be a potential collaborator.

6.3 Long Term Studies (Next 10 years)

6.3.1 Model eliciting activities

MEAs have been successfully used as a pedagogical tool at Purdue and other institutions [131]. Much good mathematics education research, engineering education research, and education research on the boundary of those two domains [17] indicates that these activities may be a part of the solution to the current mismatch between engineering's expectations and mathematics's preparation. By five years into my career, I hope to have established strong, productive collaborations between engineering and mathematics at my institution, such that efforts like incorporating and studying MEAs would be a possibility. I would like to study the impact of MEAs on both engineering outcomes (ill-posed problems, communication) and mathematical outcomes (abstracted conceptual knowledge, proof/derivation).

6.3.2 Skeptical reverence

Gainsburg's model of "Skeptical Reverence"[25] for mathematics (appreciating the power of mathematics while also keeping in mind its limitations) seems like a noble goal toward which to bend mathematical education for engineers. As more of the particular skills are subsumed by computers, it becomes that much more essential to develop in our students the skills that computers do not yet have. Perhaps such a high level of epistemic maturity cannot be reached in only 4 years of undergraduate education. However, this viewpoint should be consistently presented to our students so that they emerge from undergrad with the most mature and useful of mathematical attitudes possible. This change would make their first few years as practitioners more productive, and help align education with practice. Epistemic beliefs *can* be changed through targeted instruction [27]. A very long-term agenda would be interventions in engineering math (and core engineering) courses that push students' epistemic beliefs towards those of mature veteran practitioners.

APPENDIX A

INTERVIEW PROTOCOL

The interview protocol is a semi-structured interview with a set of main questions and sample follow-up questions that might be asked depending on how the interviewee responds. This protocol is intended to take between 30 and 60 minutes.

1. What engineering courses do you teach?
    (a) How often do you teach these courses?
    (b) Why do you teach these particular courses?
2. Have you been an engineer in industry before your present career in academia?
    (a) What did you do? What mathematics did you use on the job?
3. What are the prerequisite math courses for the courses you teach?
    (a) In what ways does students' performance in each course you teach depend on their performance in the mathematics prerequisites?
    (b) How do you perceive the content of the courses you teach build on the content of the prerequisite courses?
    (c) In the courses you teach, how would you describe the importance of mathematics?
4. What is your general perception of the mathematical preparedness of the students rising up to your course?
    (a) How many lectures do you spend on purely mathematical review for your course?
    (b) Does this vary by semester?

(c) How do you interpret students' lack of preparation?

(d) What part do the instructors of the prerequisite courses play?

(e) What part does the curriculum itself play?

5. What types of attitudes toward mathematics in Engineering do you perceive in your students? How do those attitudes impede students' learning?

    (a) Do students perceive the "real life" applications of the math they have been taught? Can they connect math to application? The engineering uses?

    (b) What do you do to alter this attitude?

    (c) Do you consider attitudes about learning to be key goals?

6. Is the math students use in your class genuine to the experience of engineering (in your experience in academia/industry)?

    (a) Do you assign problems that have multiple correct solutions? Solutions that aren't immediately apparent?

    (b) How many minutes do the longest problems take an average students to solve?

    (c) How do your students think about models, critically or hegemonically? Does this improve after taking your course?

7. What do you think should be the relationship between the engineering curriculum and mathematics curriculum? How well are the current curricula meeting your expectations and needs?

    (a) What do you think about the choice of content and order of content presentation?

    (b) Do students remember/transfer what they have learned in math?

    (c) Are math courses a form of mental exercise for engineers but not directly applicable? Just a form of general education?

8. Do you believe that some mathematics topics or skills are taught better by mathematics faculty and some are taught better by engineering faculty? Why or Why not?

(a) For example, electrical engineering makes much more extensive use of Laplace transforms, but they are still covered in the standard Differential Equations class.

9. (3rd year course faculty only) What emphasis do you put on derivations in your course? Do you think they are important for future practicing engineers? For future engineering researchers?

10. What specific mathematical knowledge must students have mastered to do well in your course?

    (a) Is it things like integration by parts, or how to define variables, or "what is a scalar?"

11. What mathematical knowledge do you think students should have mastered but is not taught in prerequisite calculus sequence course (dimensional analysis, name that object game, extreme case analysis, etc)?

12. Describe your students' skills at manipulating notation or working with symbols abstractly.

    (a) Do you encourage graphical/intuitive methods over analytic/formal methods?

    (b) Do you encourage estimation (over calculation) in your class?

    (c) Do they think they can create their own methods or are they beholden to the formula sheet?

APPENDIX B

CODEBOOK

This appendix shows the codes and definitions for the qualitative analysis used in Study A and Study B.

Discrete to continuous: Use this code when the faculty member mentions a transition between discrete and continuous phenomena. Examples: Setting up a discrete equation for conservation. Running $\frac{\Delta M}{\Delta t}$ into $\frac{dM}{dt}$. Constructing control volumes or infinitesimal elements. Setting up discretization, or chopping a domain into tiny pieces.

Derivations: This code is for discussions of what derivations are for in a class. These are formal proofs for equations that are used in class, or perhaps of special results. Examples: What is the role of derivations in this course. Do not use this code for casual mentions of derivations without comment on their role in the course.

Analytic Awareness: Use this code when the participant mentions the proper place of "fancy" techniques, advanced manipulation skill, techniques of integration or similar topics. Examples: Students should have an awareness that advanced techniques exist, but should not be expected to apply and recall them on their own without access to reference material. They need to be aware that they can look up the specific fancy technique for their own discipline. They need to be aware that there are all these techniques out there. We don't use and apply most of the complicated integration techniques. I think students should be aware, but not be a technical expert in doing Gaussian elimination for instance.

Fluent Basics: Use this code for references to algebra skills being fast, accurate, and second nature. This includes elementary geometry such as circles, spheres and cuboids. Examples: $\frac{d}{dx}(x^2 + 2x + 1)$ should fast and second nature. Their algebra needs to be fast. They need to know basics.

Fundamentals: The math that is applied in engineering courses, and needs to be remembered by engineering students, is just the simplest concepts from math courses. Examples: Derivatives of polynomials and logs, but not derivatives of complicated product rule chain rule compositions. First order $y' = ky$ equations, not the Frobenius method. 2×2 matrices, not finding eigenspaces of 4×4 systems by hand. Vector addition and multiplication, not gradients and curls. We don't teach them all the nitty-gritty detail, we have a chance to teach students to become better engineering students.

Forgetting College Math: Use this code when the subject mentions that students have forgotten mathematics from their previous college courses. The participant regrets having to review, expects that they should know this already, or knows that it has been a few semesters since it has been reviewed and reinforced.

Forgetting High school math: Use this for mentions of students who have forgotten the rules for manipulating simple high school level constructions like factoring, square roots, division of functions, rules for exponents, etc. If they mention the student having taken calculus in high school, apply this code, but generally this will be for pre-calculus courses and below.

Varied Student Ability: Use this code when the faculty member mentions that they have a wide range of abilities in math in their class. Example: There is a top crust that get everything, and a bottom dregs that always struggle. Do not use this code if they compare across universities or compare college students to general population.

Solving Symbolically: Use this when faculty mention that students hsould not plug in numerical values immediately. In the problem "Find the time it takes for a ball to fall 20m if released from rest," One would use the equation $y = y_0 + v_0 t + \frac{1}{2}at^2$. Solving symbolically manipulates this expression without substituting any non-zero numerical values (the $y_0 = 20m$ and $a = -9.9m/s^2$ in this problem) to obtain $0 = h + \frac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2h}{a}} = \sqrt{\frac{2 \cdot 20}{9.9}}$. Solving numerically in this context means solving the equation $0 = 20 + 0 + \frac{1}{2}(-9.8)t^2$. Examples: You do the same manipulations, but on cumbersome, error-prone numbers rather than sleek letters. You are given numerical values, but you delay choosing to use them. It's easier to solve a problem with letters than to put the numbers in immediately.

Ignorance of Prereqs: Use this code when the subject does not know the prerequisites are for their course, or for ignorance of the content taught in that prerequisite course. Examples: I'm not sure what they cover in differential equations. I'm surprised they don't cover that in calculus. I don't know what course PHYS212 even is. Do not use this code if they are just saying that students can ignore the prerequisite system at their own peril.

Dimensional Analysis: Use this code for any mentions of using the units of physical quantities to assist in problem solving. Examples: Knowing that acceleration is $meters \cdot seconds^2$. Catching mistakes by finding things are dimensionally inconsistent. Adding apples and oranges. Do not use this code for merely mentioning that things have units. They need to mention using the units of things to aid in solving problems or understanding the equations in some way.

Math reluctance: Use this code for mentions that students are reluctant to do math, engage with math, learn math, etc. This code captures the sentiment of I hate math but I can't articulate why. Also use this code if the faculty member expresses a desire to reverse this and generate excitement.

Letters can be answers: Use for mentions that algebraic expressions with letters are a valuable type of answer, in addition to just numerical answers. Examples: Expressions with integrals or derivatives can also be answers. You are not solving for an answer, solving for a behavior.

Thinking Competency: A student with strong Thinking Competency can pose questions and has insight into answers. This student understands scope, and statement types (definition, theorem, phenomenon). Questions like "Is there a. . . ?" or "Is it possible that. . . ?" indicate strong Thinking competency, they question the boundaries of the mathematics that they learn. Examples: I want them to anticipate the breakdown cases. They should know what counterexamples to suggest. They should generalize results to a larger class of objects. They should know the scope of a given concept. For example taking the average of everyone's favorite color has gone beyond the scope of what an average means.

Problem tackling competency: A student with strong Problem-Tackling competency can formulate and solve problems. However, non-routine problems are what count; not all questions are problems. The ability to solve routine exercises is not included in problem-tackling competency. Examples: They just can't solve multistep problems. Anything that they haven't seen that exact one before, they're flummoxed. Do not use this code for: They can't do the simple things right. Even standard problems they just don't know how to do.

Modeling: A student with Modeling Competency can de-mathematize and interpret models, and actively create models. Furthermore, he or she can criticize models, and make non-evident assumptions/decisions. This involves all and any of the steps in the Lesh modeling cycle: Identify the real-world phenomenon, simplify or idealize the phenomenon, express the idealized phenomenon mathematically (i.e., mathematize), perform the mathematical manipulations (i.e., solve the model), interpret the mathematical solution in real-world terms, test the interpretation against reality. Examples: Building it up from basic parts. They need to do word problems. They need to take the result they get and interpret what it means in terms of the original problem statement. They need to turn the situation into a simple object. Examining assumptions such as no friction or perfectly circular. Validating an equation with real data. Translating a real world problem into differential equations. Interpreting, but not actually solving, a problem. Do not use this code when: The emphasis is just on the mathematical manipulations (solve the model) step. There is no physical context. When the professor wants to make sure all the problems are physically reasonable. When they discuss the student having a reluctance to look for applications (that should be coded with Practical Irrelevance).

Reasoning competency: A student with Reasoning Competency can follow and asses reasoning. This competency is not coming up with ideas, but confirming that proof/justification is solid, assessing formal argument and proof. This student can dismantle bad reasoning and spot shaky proof. Examples: Assessing if a proof has been done correctly. Spotting errors in reasoning. Do not use this code for: Checking your work and making sure there are no sign errors. Knowing if an assumption is physically reasonable.

Representing competency: A student with Representing Competency can apply algebraic, visual, graphical, tabular, verbal, geometric, diagrammatic and material objects as representations of mathematical truth. This student can choose and switch representation as needed. This competency contains the aspects of representational fluency. Examples: Knowing when to switch to a different representation. Ability to correctly switch representations. Observe corresponding features between representations. Plots and graphs. Do not use this code for: Mentions of types of graphs being used (the subject must have a mention of how the students should use graphs, how they need to gain skill in graphs, how they are doing graphs wrong or how they are unable to interpret the graphs). This code overlaps with the symbol sense code: Abandoning Symbols.

Symbol/formalism competency: A student with Symbol/Formalism competency can decode and translate symbols, use symbols, and has insight into rules for using symbols. This code overlaps with Symbol sense codes. Use this category to sort miscellaneous notation statements that don't fall readily into the symbol sense categories, or the Solve Symbolically code. Examples: Knowing the difference between σ and τ (stresses). Name and identify a symbol.

Communicating competency: A student with strong Communicating Competency is adept at reading and writing mathematics, with words, pictures, and equations. Mathematics is the language of technical communication. Examples: They just do not know how to label things correctly. They choose conventions that do not let me know what is going on. Writing/ justifying answers within math. The culture of engineering or culture of physics. This code overlaps with the aspects of the Solitary Mathematics epistemology. Communicating Competency is an ability to communicate mathematics effectively, the Solitary Mathematics is believing that one should communicate mathematics, or that communication factors should determine some mathematical choices. Use Communicating competency if the faculty member mentions poor skills at communicating (such as bad presentations or reports), rather than student inclination to communicate mathematically.

Tools/aids competency: A student with strong Tools and Aids competency knows possibilities and limitations associated with mathematical tools such as calculators, special paper, computer algebra systems, computational simulations, and physical props. Examples: Using computers to do mathematics in engineering. Using MatLab, Mathematica, Python, or other scripting computation. Using COMSOL, Fluent or other computational simulation software. Using Wolfram Alpha, computer algebra systems, and symbolic manipulation systems. Do not use this code for: When subject mentions general programming ability.

Quantitative reasoning with symbols: This code captures the ability to scan an algebraic expression to make rough estimates of the patterns that would emerge in numeric or graphic representation. Examples: Ability to scan a table of function values or a graph or to interpret verbally stated conditions. Ability to identify the likely form of an algebraic rule that expresses the appropriate pattern. Ability to make informed comparisons of orders of magnitude for functions with rules of the form n, n^2, n^3, and n^k. If you change this variable, what happens to that variable.

Selecting a symbol: An understanding of and an aesthetic feel for the power of symbols. Understanding how and when symbols can and should be used in order to display relationships, generalizations, and proofs which otherwise are hidden and invisible. Seeing equivalent expressions for non-equivalent meanings. Different symbolic representations can display more information for example how $4(n)(n-1)$ is a multiple of 8). Information can be built into a symbol, and students should try to maximize the information that can be gleaned from the expression. How much of the meaning do we pre-pack into the expression. We have the freedom to express and represent how we want. The initial binding of symbols is not binding, we can re-represent when it is convenient. Students have a feel for an optimal choice of symbols. Ability to determine which of several equivalent forms might be most appropriate for answering particular questions. The ability to select a possible symbolic representation of a problem, and, if necessary, to have the courage, first, to recognize and heed one's dissatisfaction with that choice, and second, to be resourceful in searching for a better one as replacement. Examples: Choosing y' over $\frac{dy}{dx}$ over $\dot{y}(x)$. Choosing $-\alpha^2$ over λ in an eigenvalue problem. Choosing $n(n-1)$ instead of n^2-n (to make evenness obvious). Creating your own symbol or notation. Do we represent the acceleration due to gravity as g or as $-g$?

Abandoning Symbols: Use this code when the subject mentions a feeling for when to abandon symbols in favor of other approaches in order to make progress with a problem, or in order to find an easier or more elegant solution or representation. Having symbol sense should include the intuitive feel for when to call on symbols in the process of solving a problem, and conversely, when to abandon a symbolic treatment for better tools.

Manipulating symbols: An ability to manipulate and to "read" symbolic expressions as two complementary aspects of solving algebraic problems. On the one hand, the detachment of meaning necessary for manipulation coupled with a global "gestalt" view of symbolic expressions makes symbol-handling relatively quick and efficient. Extending the number of important operations we can perform without thinking about them. The student knows when they can reasonably forget the referents of the symbol. Examples: Anticipate deriving a tautology (like in a system of degenerate equations). Solve $v\sqrt{u} = 1 + 2v\sqrt{1+u}$ for v is easy, the u's don't matter. The student can prudently detach the meaning of the symbols and their referents.

Reading meaning from symbols: An ability to "read" symbolic expressions. The reading of the symbolic expressions towards meaning can add layers of connections and reasonableness to the results. Examples: A priori inspection before manipulation. Reading as a goal for manipulations. Read and check for reasonableness (the six times as many students as professors $6s = p$ problem). Ability to inspect algebraic operations and predict the form of the result or, as in arithmetic estimation, to inspect the result and judge the likelihood that it has been performed correctly. Reduction to a previous result (this is the choosing sine or cosine for the ramp problem, reduction of the jacketed cable to a solid cable, or reduction of a disk to a point charge very far away. This also has some serious linkage to modeling competency. Taking appropriate limits, examining whether it makes sense in a particular limit. Does this function have this property? Using sanity checks (e.g., some things should not go to infinity, things are forbidden to be negative).

Symbols in context: Sensing the different roles symbols can play in different contexts. Examples: In $y = mx + b$, there are 'variables' and 'parameters'. Both are letters that represent numbers, but have very different meanings. The student can navigate conventional meanings of letters (such as 'a' meaning an offset of a circle and finding a circle that goes through $(0, a)$).Choosing i or j for an imaginary unit. Understanding notation in different contexts; the same symbol can have different meanings in different places or classes. Understanding the role of a dummy variable.

Engineer symbolic relationships: Use this code when the subject mentions the awareness that one can successfully engineer symbolic relationships which express the verbal or graphical information needed to make progress in a problem, and the ability to engineer those expressions. Examples: Creating an ad-hoc expression for a desired purpose. Developing an expression with a desired property. Creating an algebraic expression that matches a graph. Do not use this code for: Creating your own symbol or choosing a symbol.

These epistemic belief codes contain both the "negative" and "positive" ends of that attribute. A quote about a growth mindset would be put in the Innate Ability code, since it is the opposite of Innate Ability, the mature epistemic belief along the same axis.

Innate Ability: Students believe that mathematical ability is fixed and unchanging. Students then interpret mistakes as personal inadequacy. Furthermore, they believe that only geniuses can truly understand or create mathematics. Students are quick to give up when faced with a challenging problem and causes students to devalue education itself.

Quick Learning: Students believe that learning is quick process, and furthermore so is problem solving. Examples: Students believe problems should be solvable within five minutes, and will give up on a problem as impossible after just 10 minutes. Students focus on speed when studying rather than on understanding. They believe they don't need to practice. The student claims they can learn it as they go.

Orderly Process: Students believe that mathematical knowledge and knowing does not involve uncertainty or failure. Examples: Student is reluctant to change tactics, and pursues their first strategy "come hell or high water." They will not just try something random and see if it works.

Simple Knowledge: Students believe mathematical knowledge is disconnected, and that information gained in one lesson has no bearing on knowledge in future or past lessons. Examples: Students prefer that there be only one method to solve a problem, as it reduces the amount of memorizing they have to do. The students plug and chug equations. Engage in memorization without meaning. The student does not see ow it all fits together or connections between concepts. They do not understand the fundamentals at a deep level. Student should new methods because they are connected to other things, not for their own sake.

Certain Knowledge: Students believe mathematical knowledge is highly certain, particularly material that is presented in class. The idea that "all models are wrong, some are useful" conflicts directly with this epistemic belief held by many students. Examples: Wanting excessively many significant figures. Needing 99% accuracy.

Omniscient Authority: The belief that mathematical authority (usually a textbook or the instructor) forms the basis for truth. Students believe that mathematical problems have one and only one correct answer, and that answers become correct when ratified by the teacher.

Practical Irrelevance: Students believe that the mathematics learned in school has no bearing on their out-of-school lives. Students rarely try to connect formal mathematical knowledge to everyday common sense. Examples: Connecting derivations to practical problems. Connecting mathematics to physics. Math isn't just something you have to get through, you will use it later. Learning math in engineering context vs learning math purely in the abstract. The professor trying to make connection between coursework and practical applications. Use this code to capture students not seeing applications, not transferring knowledge from math to engineering, mathematics courses being too abstract and not having applications. This code is for a belief/tendency to look for applications, not an ability to do them correctly (that is modeling competency).

Solitary Mathematics: Students believe that mathematics is done by individuals alone. This belief undermines the nature of mathematics as a form of communication. Examples: Math is a language of technical communication or "mathematics is the language of engineering." Students are disinclined to write down explanations. This code overlaps with Communicating Competency. Code statements that have to do with students not believing that they should be concerned with communicative issues in mathematics in Solitary Mathematics and code commentary on quality of communication into Communicating Competency.

APPENDIX C

SURVEY INSTRUMENT

This instrument is adapted from Flegg et al. [2].

Please indicate how much you agree with the following statements: 1 - strongly disagree, 2 - disagree, 3 - neutral 4 - agree, 5 - strongly agree

I can see how the mathematics skills that I am currently developing will be useful in an engineering career.	1 2 3 4 5
Being integrated with first year mathematics majors helps me to get a better understanding of the uses of mathematics as a whole rather than just in the engineering fields.	1 2 3 4 5
In my current mathematics course, I am being taught ways of thinking that will remain with me long after I graduate	1 2 3 4 5
I believe that my current mathematics course teaches me how to formulate and solve problems that are directly related to engineering.	1 2 3 4 5
My current mathematics course exposes me to ideas which I know I will need later on in my engineering degree.	1 2 3 4 5
I believe that being able to communicate effectively using mathematical arguments as an important skill to have. The formal and rigorous aspects of mathematics that I have seen in my current mathematics course are important for my future engineering career.	1 2 3 4 5 1 2 3 4 5
For me, I only want to learn what I feel is likely to be graded.	1 2 3 4 5
At some stage during my degree program I have been so overwhelmed by mathematical content that I have considered withdrawing from my engineering degree.	1 2 3 4 5

Please indicate how much you agree with the following statements: 1 - strongly disagree, 2 - disagree, 3 - neutral 4 - agree, 5 - strongly agree

Being good at math in engineering means being able to solve math problems quickly.	1 2 3 4 5
Being good at math in engineering means describing real world situations with math equations	1 2 3 4 5
Being good at math in engineering means knowing if the equation makes physical sense.	1 2 3 4 5
Being good at math in engineering means being good at word problems	1 2 3 4 5
Being good at math in engineering means being able to plug and chug equations	1 2 3 4 5
Being good at math in engineering means being good at manipulating equations.	1 2 3 4 5
Being able to solve integrals by hand doesn't matter because Wolfram Alpha can do them	1 2 3 4 5

APPENDIX D

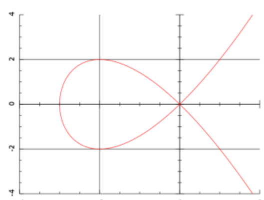
CONCEPTUAL MEASURE OF CALCULUS KNOWLEDGE

This instrument is taken from Cromley et al. [128].

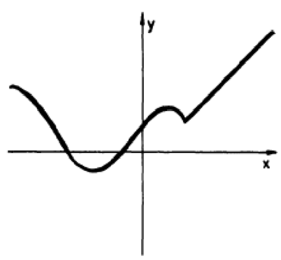
Directions: Solve each of the following problems, using any available space for scratch work.

$\boxed{\text{Problem 1}}$

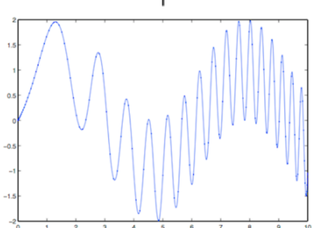
Is each of the following a graph of a function in the form of $y = f(x)$? Circle YES or NO.



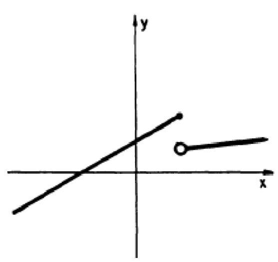
YES NO



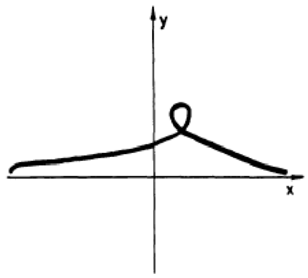
YES NO



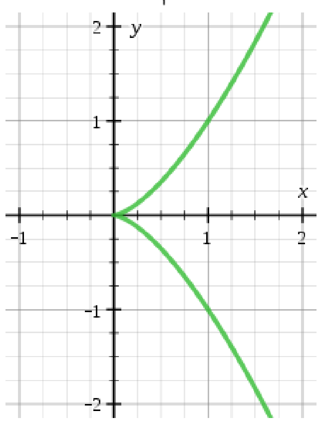
YES NO



YES NO



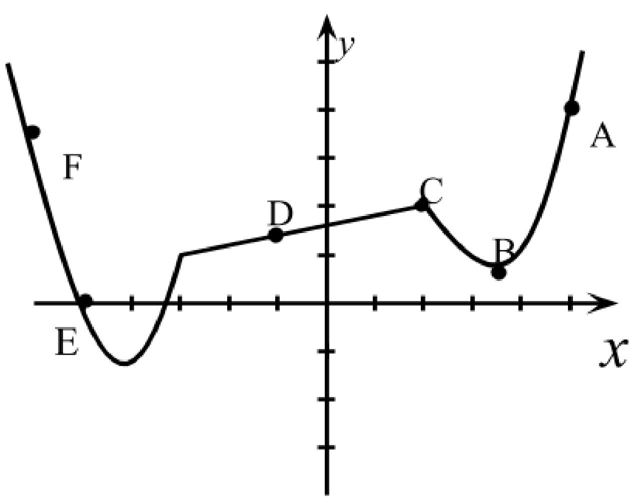
YES NO



YES NO

Problem 2

For the function f whose graph is shown below, circle each labeled point(s) that satisfies the following conditions.



Circle all that apply.

a.	$f'(x) = 0$	A	B	C	D	E	F
b.	$0 < f'(x) < 1$	A	B	C	D	E	F
c.	$f'(x) > 1$	A	B	C	D	E	F
d.	$f(x) = 0$	A	B	C	D	E	F
e.	$f'(x) < 0$	A	B	C	D	E	F
f.	$f'(x) < 1$	A	B	C	D	E	F
g.	$f'(x)$ is not defined	A	B	C	D	E	F

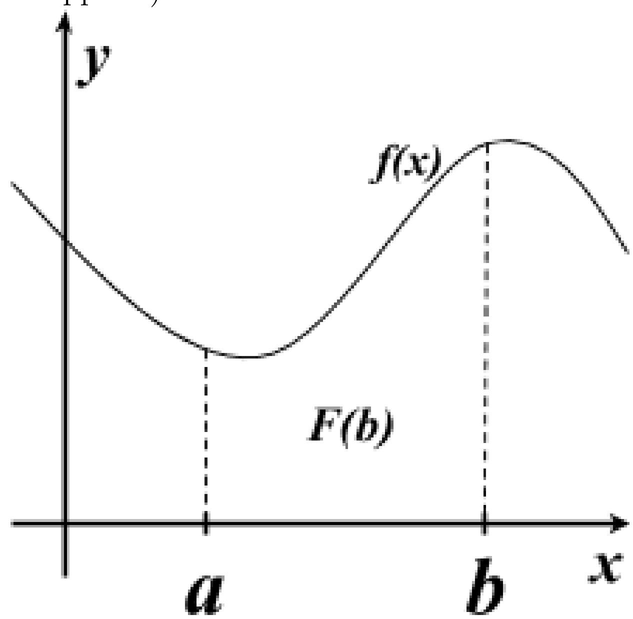
Problem 3

Which information would you use to determine each of the following? (Circle all that apply.)

If f has a critical point at $x = 3$	f	f'	f''
The zeros off	f	f'	f''
If the graph of f has an inflection point at $x = -1$	f	f'	f''
Intervals on which f is decreasing	f	f'	f''

Problem 4

What information would you need to know about the function, f(x) (shown below), in order to determine each of the following? (Mark an X for each that applies.)



	An antiderivative g exists	Whether $f(x)$ is even or odd	$f(x)$ is defined for all real values	$f(x)$ is continuous over a closed interval	$f(x)$ is differentiable on an open interval	Inflection points of $f(x)$
If there exists a point c on $f(x)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$(If the Mean Value Theorem for derivatives applies)						
If the definite integral of the derivative of $f(x)$ is given by $\int_a^b f(x)dx = g(b) - g(a)$,where g is the antiderivative (If the First Fundamental Theorem of Calculus applies)						
If the area $F(b)$is given by $F(b) = \int_a^b f(x)dt$(If the Second Fundamental Theorem of Calculus applies)						

$\boxed{\text{Problem 5}}$

State whether each of the following is an accurate statement about limits. (Circle YES or NO.)

a.	The limit of a function $f(x)$at a given value of x may not exist, even though $f(x)$is defined at x	YES NO
b.	In order to determine the limit of $f(x)$,the formula for $f(x)$ must be given	YES NO
c.	The limit of a function $f(x)$ at a given value of x may be infinite	YES NO
d.	If $f(x)$ is continuous at a given value of x, it has a limit for that value of x	YES NO
e.	The value of the limit of $f(x)$ at the point $f(a)$ is the same as the value of $f(a)$	YES NO

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