ALGEBRAICALLY CLOSED FIELDS WITH CHARACTERS; DIFFERENTIAL-HENSELIAN MONOTONE VALUED DIFFERENTIAL FIELDS

BY

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DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2018

Urbana, Illinois

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Abstract

This thesis consists of two unrelated research projects. In the first project we study the model theory of the 2-sorted structure $(\mathbb{F}, \mathbb{C}; \chi)$, where $\mathbb{F}$ is an algebraic closure of a finite field of characteristic p, $\mathbb{C}$ is the field of complex numbers and $\chi : \mathbb{F} \to \mathbb{C}$ is an injective, multiplication preserving map.

In the second project we study the model theory of the differential-henselian monotone valued differential fields. We also consider definability in differential-henselian monotone fields with c-map and angular component map.

To my family and my adviser Lou.

Acknowledgments

I cannot express enough thanks to my adviser Lou van den Dries for his constant help and support, for suggesting the problems and guidance throughout my graduate years. I also thank him for his generosity of his time to discuss ideas and read and make corrections to all my revisions.

I would also like to sincerely thank my committee members, Philipp Hieronymi, Anush Tserunyan, and Erik Walsberg, who offered guidance and support.

Special thanks to my wife Arpine, parents, sisters and brothers.

Many thanks to my mathematical and academic brothers Vahan Huroyan, Vahagn Aslanyan, Santiago Camacho, Allen Gehret, Nigel Pynn-Coates, and especially to Minh Chieu Tran for all the mathematical discussions and support.

I also thank all the other friends, especially Albert Tamazyan, Aram Grigoryan and Arsen Avagyan for making the life in Urbana-Champaign more interesting.

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Introduction

The current thesis consists of two unrelated research projects. Each project has its own introduction. Most of the first project presented in Chapter 1 is taken directly from [8] which is a joint work with Minh Chieu Tran. There, we study the model theory of the 2-sorted structure $(\mathbb{F}, \mathbb{C}; \chi)$, where $\mathbb{F}$ is an algebraic closure of a finite field of characteristic p, $\mathbb{C}$ is the field of complex numbers and $\chi : \mathbb{F} \to \mathbb{C}$ is an injective, multiplication preserving map. We obtain an axiomatization ACFC_p of $\mathrm{Th}(\mathbb{F}, \mathbb{C}; \chi)$ in a suitable language L, classify the models of ACFC_p up to isomorphism, prove a modified model companion result and give a description of definable sets inside a model of ACFC_p.

The second project is presented in Chapters 2, 3 and 4, most of which is taken directly from [6] and [7]. [6] is to appear in the Journal of Symbolic Logic. Scanlon [13] proves Ax-Kochen-Ershov type results for differential-henselian monotone valued differential fields with many constants. We show in Chapter 3 how to get rid of the condition *with many constants*. Chapter 4 considers definability in differential-henselian monotone fields with c-map and angular component map. We prove an Equivalence Theorem among whose consequences are a relative quantifier reduction and an NIP result.

Part I

Algebraically Closed Fields with a Generic Multiplicative Character

Chapter 1

Algebraically Closed Fields with a Generic Multiplicative Character

Most of this chapter is directly from [8], which is a joint work with Minh Chieu Tran.

1.1 Introduction

Fields with characters occur in many places; see for example Kowalski [9] for a case where also definability plays a role. This suggested it might be worth looking for model-theoretically tame pairs of fields with character maps between them.

Throughout in this chapter, $(F, K; \chi)$ is a structure where F and K are integral domains (usually fields), and $\chi : F \to K$ satisfies $\chi(ab) = \chi(a)\chi(b)$ for all $a, b \in F$, $\chi(0) = 0$ and $\chi(1) = 1$. Then $(F, K; \chi)$ is naturally a structure in the 2-sorted language L which consists of two disjoint copies of the language of rings, augmented by a unary function symbol χ. We call a function χ with the above properties a **character**.

We are particularly interested in the cases where F is an algebraic closure $\mathbb{F}$ of a finite field, K is the field $\mathbb{C}$ of complex numbers and $\chi : \mathbb{F} \to \mathbb{C}$ is *injective*. From now on, we let $(\mathbb{F}, \mathbb{C}; \chi)$ range over structures with these properties. Corollary 1.3.2 below says that for each prime p, there is up to L-isomorphism exactly one $(\mathbb{F}, \mathbb{C}; \chi)$ such that $\mathrm{char}(\mathbb{F}) = p$. In this Chapter we show that the L-theory $\mathrm{Th}(\mathbb{F}, \mathbb{C}; \chi)$ is tame in various ways. For precise statements we need some more terminology.

Let $(F, K; \chi)$ be given. For a tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in F^n$, $n \in \mathbb{N}^{\geq 1}$, and $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$ we set $\alpha^k := \alpha_1^{k_1} \cdots \alpha_n^{k_n}$. We call α **multiplicatively dependent** if $\alpha^k = \alpha^l$ for some distinct $k, l \in \mathbb{N}^n$, and **multiplicatively independent** otherwise. We say that $\chi : F \to K$ is **generic** if it is injective and for all multiplicatively independent $\alpha \in F^n$, $n \in \mathbb{N}^{\geq 1}$, the tuple $\chi(\alpha) := \big(\chi(\alpha_1), \ldots, \chi(\alpha_n)\big) \in K^n$ is algebraically independent in the fraction field of K over its prime field.

Theorem (Theorem 1.2.8)**.** *There is a recursive set* ACFC *of* $\forall\exists$*-axioms in* L *such that:*

(1) for all $(F, K; \chi)$, $(F, K; \chi) \models \mathrm{ACFC}$ *if and only if* F *and* K *are algebraically closed fields,* $\mathrm{char}(K) = 0$ *and* $\chi : F \to K$ *is generic;*

(2) for all p prime, if $\mathrm{char}(\mathbb{F}) = p$*, then* $(\mathbb{F}, \mathbb{C}; \chi) \models \mathrm{ACFC}$*.*

If p is either prime or zero, let ACFC_p be the set of $\forall\exists$-axioms in L obtained from ACFC by adding the statements expressing $\mathrm{char}(F) = p$ where $(F, K; \chi)$ is an L-structure.

Let κ, λ be (possibly finite) cardinals. In Section 3 we prove the following classification result. (If $p = 0$, set $\mathbb{F}_p := \mathbb{Q}$.)

Theorem (Theorem 1.3.4)**.** *For any* p, κ *and* λ*, there is up to isomorphism a unique model* $(F, K; \chi)$ *of* ACFC_p *such that* $\mathrm{trdeg}(F \mid \mathbb{F}_p) = \kappa, \mathrm{trdeg}\big(K \mid \mathbb{Q}(\chi(F))\big) = \lambda$*.*

By the wealth of results by Shelah in [15], we can get the following:

Corollary. ACFC_p *is superstable, shallow, without the dop, without the otop, without the fcp.*

By an analogue of Vaught Test, we have:

Theorem (Theorem 1.3.7)**.** ACFC_p *is complete.*

In section 1.4 we characterize the substructures of models of ACFC_p:

Proposition (Proposition 1.4.1)**.** *Given* $(F, K; \chi)$*, the following are equivalent:*

(1) $(F, K; \chi)$ *is a substructure of a model of* ACFC_p*;*

(2) χ *is generic and* $\mathrm{char}(F) = p$*.*

When is a substructure of a model of ACFC_p an elementary submodel? It is not enough that the substructure is a model of ACFC_p:

Proposition (Corollary 1.4.5)**.** ACFC_p *is not model complete.*

To deal with the above question we define a **regular submodel** of a model $(F', K'; \chi')$ of ACFC_p to be a substructure $(F, K; \chi) \models \mathrm{ACFC}_p$ of $(F', K'; \chi')$ such that $\mathbb{Q}\big(\chi(F')\big)$ is linearly disjoint with K over $\mathbb{Q}\big(\chi(F)\big)$ in K'. The more complicated notion of **regular L-substructure** will be defined in Section 1.4. Below we fix a set $\mathrm{ACFC}_p(\forall)$ of universal L-sentences whose models are the substructures of models of ACFC_p, as in Proposition 1.4.1.

Theorem (Theorem 1.4.8)**.** ACFC_p *is the* regular *model companion of* $\mathrm{ACFC}_p(\forall)$*. That is:*

(1) for models of ACFC_p*, the notions of* regular submodel *and* elementary submodel *are equivalent;*

(2) every model of $\mathrm{ACFC}_p(\forall)$ *is a regular substructure of a model of* ACFC_p*.*

In Section 1.5, we show that every definable set in a fixed model $(F, K; \chi)$ of ACFC_p has a simple description. This is comparable to the fact that every definable set in a model of ACF is a boolean combination of algebraic sets. A set $S \subseteq K^n$ is **algebraically presentable** if

$$S = \bigcup_{\alpha \in D} V_\alpha$$

for some definable $D \subseteq F^m$ and definable family $\{V_\alpha\}_{\alpha \in D}$ of K-algebraic subsets of K^n. Algebraically presentable sets should be thought of as geometrically simple. They are also existentially definable of a particular form. We also define in Section 1.5 the related notion of 0**-algebraically presentable** sets. The main result is:

Theorem (Theorem 1.5.12)**.** *If $X \subseteq K^n$ is definable, then X is a boolean combination of algebraically presentable subsets of K^n. Furthermore, if X is 0-definable, X is a boolean combination of 0-algebraically presentable subsets of K^n.*

The general case where a definable set is not a subset of K^n can be easily reduced to the above special case. We have better results for definable subsets of F^m:

Theorem (Theorem 1.5.15)**.** *Let $D \subseteq F^m$. If D is definable, then D is definable in the field F. If D is 0-definable, then D is 0-definable in the field F. Suppose $D = \chi^{-1}(V)$ with $V \subseteq K^m$ a K-algebraic set. Then D is F-algebraic. If V is defined over $\mathbb{Q}$ in the field sense, then D is defined over $\mathbb{F}_p$ in the field sense.*

The structure $(\mathbb{F}, \mathbb{C}; \chi)$ is similar to various known structures, for example $(\mathbb{C}, \mathbb{Q}^{ac})$ where $\mathbb{Q}^{ac}$ is the set of algebraic numbers regarded as an additional unary relation on $\mathbb{C}$. The study of the latter stretches back to Robinson (see [12]). Analogues of some of our results for $(\mathbb{C}, \mathbb{Q}^{ac})$ seem to be known as folklore; see for example [16]. However, our structure is mathematically even more closely related to $(\mathbb{C}, \mathbb{U})$ where $\mathbb{U} \subseteq \mathbb{C}$ is the group of all roots of unity regarded as an additional unary relation. In fact, we can almost view $(\mathbb{F}, \mathbb{C}; \chi)$ as $(\mathbb{C}, \mathbb{U})$ with some extra relations on $\mathbb{U}$. In consequence, several results of this Chapter are either directly implied or easy adaptations of results in [19] and [17]. These include axiomatization, ω-stability, quantifier reduction; the corresponding result of (κ, λ)-transcendental categoricity is known, according to Pillay, but not written down anywhere. There are also several results that hold in the above mentioned two structures and ought to have suitable analogues in our structure but we have not proven them yet. These include the study of imaginaries and definable groups; see [11] and [5].

On the other hand, some of our results are new, which also yield more information on the structures $(\mathbb{C}, \mathbb{Q}^{ac})$ and $(\mathbb{C}, \mathbb{U})$ as well. Through the notion of genericity, we obtain a more conceptual characterization of the class of models of ACFC_p other than using the axioms. From [19], we can already see that every model of ACFC_p satisfies the properties of this characterization. In this Chapter, we show the reverse direction. It is clear that one can obtain a characterization of the class of models of $\mathrm{Th}(\mathbb{C}, \mathbb{U})$ in the same way. Even though both use Mann's theorem in an essential way, our axiomatization strategy is slightly different from the strategy used in [19] and [17]. This modification, in particular, allows us to also axiomatize the class of substructures of the models and achieve the regular model companion result mentioned above. The regular model companion result should have analogues for $(\mathbb{C}, \mathbb{Q}^{ac})$ and $(\mathbb{C}, \mathbb{U})$ as well.

Notations and conventions

Throughout this chapter, let m, n range over the set of natural numbers (which includes zero), p either a prime number or zero, let $t = (t_1, \ldots, t_n)$ and $u = (u_1, \ldots, u_m)$ be tuples of variables of the first sort, and let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be tuples of variables of the second sort. For a field K, we denote by K^{ac} an algebraic closure of K. If a is in X^n, then $a = (a_1, \ldots, a_n)$ with $a_i \in X$ for $i \in \{1, \ldots, n\}$. For $A \subseteq K^{\neq} := K \setminus \{0\}$, set $\langle A \rangle_K$ to be the set of elements in $K^{\neq}$ which are in the subgroup generated by A in the fraction field of K; then $\langle A \rangle_K$ is a submonoid of $K^{\neq}$ and is a subgroup of $K^{\times}$ when K is a field; when the context is clear, we will write $\langle A \rangle$ instead of $\langle A \rangle_K$. If $P_1, \ldots, P_m$ are systems of polynomials in $K[x]$, we let $Z(P_1, \ldots, P_m) \subseteq K^n$ be the set of their common zeros.

1.2 Axiomatization

In this section, we also assume that $A, B \subseteq K^{\neq}$ and $C \subseteq K$. Let $\mathrm{acl}_C(A)$ denote the elements of K satisfying a nontrivial polynomial equation with coefficients in $\mathbb{Z}[A, C]$. We will give a definition of the notion of genericity which is slightly more general than what was given in the introduction. This is necessary for the purpose of axiomatization and will also play an important role in the next two sections.

The **multiplicative closure** of A **over** B, denoted by $\mathrm{mcl}_B(A)$, is the set

$$\{a \in K^{\neq} : a^n \in \langle A \cup B \rangle \text{ for some } n\}.$$

It can be easily checked that $(K^{\neq}, \mathrm{mcl}_B)$ is a pregeometry. Moreover, if K is a field, the notion of multiplicative closure over B coincides with the notion of divisible closure over B, viewing $K^{\times}$ as a $\mathbb{Z}$-module. We say A is **multiplicatively independent over** B if

$$a \notin \mathrm{mcl}_B(A \backslash \{a\}) \text{ for all } a \in A.$$

A **multiplicative basis** of A **over** B is an $A' \subseteq A$ such that A' is multiplicatively independent over B and $A \subseteq \mathrm{mcl}_B(A')$. General facts about pregeometry give us that there is a multiplicative basis of A over B; furthermore, any two such bases have the same cardinality. When $B = \varnothing$, we omit the phrase *over* B in the definition and the subscript B in the notation. We also note that $\mathrm{mcl}(\varnothing) = \mathbb{U} \cap K$.

We say A is **generic** if for all multiplicatively independent $a \in \langle A \rangle^n$ we also have a is algebraically independent. We say A is C-**generic over** B if for all B-multiplicatively independent $a \in \langle A \rangle^n$ we also have a is algebraically independent over $B \cup C$. The following follows easily from the exchange property of mcl:

Lemma 1.2.1. *Suppose A is C-generic over B. Then the preceding statement continues to hold as we:*

(1) replace K by an integral domain K' such that $A, B \subseteq {K'}^{\times}$ and $C \subseteq K'$,

(2) replace B with $B' \subseteq K^{\neq}$ such that $\mathrm{mcl}(B') = \mathrm{mcl}(B)$,

(3) replace C with $C' \subseteq K$ such that $\mathrm{acl}_B(C') = \mathrm{acl}_B(C)$,

(4) replace A with $A' \subseteq K^{\neq}$ such that $\mathrm{mcl}_B(A') = \mathrm{mcl}_B(A)$.

Corollary 1.2.2. *The following equivalence holds: A is C-generic over B if and only if there is a family $A' \subseteq A$ such that A' is algebraically independent over $B \cup C$ and $A \subseteq \mathrm{mcl}_B(A')$.*

The notions of *multiplicative closure* and *multiplicative independence* can also be understood using polynomials. A **monomial** in x is an element of $\mathbb{Q}[x]$ of the form x^k with $k \in \mathbb{N}^n$. Likewise, a B-**monomial** in x has the form $b^l x^k$ with $b \in B^m$, $l \in \mathbb{N}^m$ and $k \in \mathbb{N}^n$. In this section, let M and N range over the B-monomials. A B-**binomial** is a polynomial of the form $M - N$. If, moreover, M and N are monomials, we call $M - N$ a **binomial**. We call a B-binomial $M - N$ **nontrivial** if

$$M = b^{l_M} x^{k_M} \quad \text{and} \quad N = b^{l_N} x^{k_N} \quad \text{for some} \quad b \in B^m \quad \text{and distinct} \quad k_M, k_N \in \mathbb{N}^n.$$

It is easy to see that for $a \in K^{\neq}$, a is in $\mathrm{mcl}_B(A)$ if and only if a is a zero of a non-trivial $(A \cup B)$-binomial of one variable. Then A is multiplicatively independent over B if whenever $a_1, \ldots, a_n$ are pairwise distinct elements of A, then $a = (a_1, \ldots, a_n)$ is not in the zero-set of a nontrivial B-binomial of n variables.

Suppose K is a field, $H \subseteq G \subseteq K^{\times}$ are groups, C is a subfield of K, $g \in G^n$, and $a \in K^n$. The **multiplicative type** of g **over** H, denoted by $\mathrm{mtp}_H(g)$, is the quantifier free type of g in the language of groups with parameters from H. We can easily see that $\mathrm{mtp}_H(g)$ is completely characterized by the H-binomials vanishing on g. If $H = \{1\}$, we simply call this the **multiplicative type** of g, and denote this as $\mathrm{mtp}(g)$. Likewise, the **algebraic type** of a **over** C, denoted by $\mathrm{atp}_C(a)$, is the quantifier free type of a in the language of rings with parameters from C. Then $\operatorname{atp}_C(a)$ is completely characterized by the polynomials in $C[x]$ vanishing on a. If $C = \mathbb{Q}$, we call this the **algebraic type** of a, and denote this by $\operatorname{atp}(a)$. Suppose c is an n-tuple of elements in K and d is an element in K. A solution a of the equation $c \cdot x = d$ is called **non-degenerate** if we have $c_{i_1}a_{i_1} + \dots + c_{i_m}a_{i_m} \neq 0$ for all $\{i_1, \dots, i_m\} \subsetneq \{1, \dots, n\}$.

Proposition 1.2.3. *Suppose K is a field, $H \subseteq G \subseteq K^\times$ are groups, and C is a subfield of K. Moreover, suppose $\operatorname{mcl}(H) \cap G = H$. The following are equivalent:*

(1) G is C-generic over H;

(2) for all $g, g' \in G^n$, if $\operatorname{mtp}_H(g) = \operatorname{mtp}_H(g')$ then $\operatorname{atp}_{C(H)}(g) = \operatorname{atp}_{C(H)}(g')$;

(3) for all $g \in G^n$ and all $P \in C(H)[x]$, P vanishes on g if and only if P is in the ideal I_g of $C(H)[x]$ generated by H-binomials vanishing on g;

(4) if $c \in C^n$, and $g \in G^n$ is a non-degenerate solution of the equation $c \cdot x = 1$, then g is in H^n.

Without the condition $\operatorname{mcl}(H) \cap G = H$, we still have $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

Proof. Throughout the proof, we suppose K, C, G and H are as given. We first show that (4) implies (3). Suppose (4) holds, and P is in $C(H)[x]$ such that $P(g) = 0$. For our purpose, we can arrange that

$$P = \sum_{i=1}^{k} c_i M_i - M_{k+1}, \text{ where } c_1, \dots, c_k \in C^\times \text{ and } M_1, \dots, M_{k+1} \text{ are } H\text{-monomials.}$$

The cases where $k = 0, 1$ are immediate. Using induction, suppose $k > 1$ is the least case the statement has not been proven. Now suppose $\big(M_1(g), \dots, M_{k+1}(g)\big)$ is a non-degenerate solution of $c_1y_1 + \dots + c_ky_k - y_{k+1} = 0$. Hence, $M_{k+1}(g) \neq 0$ and

$$\big(M_1(g)M_{k+1}^{-1}(g), \dots, M_k(g)M_{k+1}^{-1}(g)\big)$$

is a non-degenerate solution of $c_1y_1 + \dots + c_ky_k = 1$. Hence, it follows from (4) that $M_i(g)M_{k+1}^{-1}(g) = h_i \in H$ for $i \in \{1, \dots, k\}$. As a consequence,

$$(c_1h_1 + \dots + c_kh_k - 1)M_{k+1} = P - \sum_{i=1}^{k} c_i(M_i - h_iM_{k+1})$$

vanishes on g. As $M_{k+1}(g) \neq 0$, the above implies $c_1h_1 + \dots + c_kh_k - 1 = 0$. Thus $P = \sum_{i=1}^k c_i(M_i - h_iM_{k+1})$ which is in I_g. The conclusion follows.

To show that (3) implies (2), let g and g' be as in (2). Then an H-binomial vanishes on g if and only if it vanishes on g', and so $I_g = I_{g'}$. The desired conclusion then follows from (3).

We now show that (2) implies (1). Suppose we have (2) and $g \in G^n$ is multiplicatively independent over H. We can arrange that K is algebraically closed by (1) of Lemma 1.2.1. The case where $n = 0$ is trivial. Using induction, suppose $n > 0$ is the least case the statement has not been proven. Then $g_1, \dots, g_{n-1}$ are algebraically independent over $C(H)$. Assume $P \in C(H)[x]$ is non-trivial. As $g_1, \dots, g_{n-1}$ are algebraically independent over $C(H)$, we get that

$$P(g_1, \dots, g_{n-1}, x_n) \neq 0 \text{ in } C(H, g_1, \dots, g_{n-1})[x_n],$$

and so it has at most finitely many roots. As a consequence, $P(g_1, \dots, g_{n-1}, g_n^m) \neq 0$ for some $m > 0$. Because $g = (g_1, \dots, g_n)$ is multiplicatively independent over H, for all m, $(g_1, \dots, g_{n-1}, g_n^m)$ has the same multiplicative type over H as $(g_1, \dots, g_n)$. By (2), for all m, $(g_1, \dots, g_{n-1}, g_n^m)$ has the same algebraic type over $C(H)$ as $(g_1, \dots, g_n)$. Therefore, $P(g_1, \dots, g_n) \neq 0$. Since P is chosen arbitrarily, g is algebraically independent over $C(H)$, and so we have (1).

We show that (1) implies (4). Suppose we have (1), $\mathrm{mcl}(H) \cap G = H$, and $g \in G^n$ is a non-degenerate solution of $c \cdot x = 1$. Let G' be the subgroup of G generated by g. As $\mathrm{mcl}(H) \cap G = H$, the group $G'/(H \cap G')$ is torsion-free of finite rank, and so we can choose $g'_1, \dots, g'_k$ in G' multiplicatively independent over H such that

$$g_i = M'_i(g'_1, \dots, g'_k) \text{ for some } H\text{-monomial } M'_i \text{ for } i \in \{1, \dots, n\}.$$

As $g'_1, \dots, g'_k$ are multiplicatively independent over H, they are algebraically independent over $C(H)$ by (1). As

$$g = \big(M'_1(g'_1, \dots, g'_k), \dots, M'_n(g'_1, \dots, g'_k)\big)$$

is a non-denegerate solution of the equation $c \cdot x = 1$, g'_j must appear with power 0 in all M'_i for all $i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$. Hence g is in H^n.

Finally, we observe that the condition $\mathrm{mcl}(H) \cap G = H$ is only used in showing (1) implies (4). Thus, the other implications still hold without this condition. □

Here, we present another property of genericity as a corollary of the previous proposition.

Corollary 1.2.4. *We have the following:*

(1) for $A \subseteq A' \subseteq A'' \subseteq K^\times$, A' is C-generic over A and A'' is C-generic over A' if and only if A'' is C-generic over A;

(2) suppose $\{A_\alpha\}_{\alpha < \kappa}$ is a sequence of subsets of $K^\times$ such that $A_\alpha \subseteq A_{\alpha+1}$ and $A_{\alpha+1}$ is C-generic over A_α

for all $\alpha < \kappa$, and $A_\beta = \bigcup_{\alpha<\beta} A_\alpha$ for all limit ordinals β. If $A = \bigcup_{\alpha<\kappa} A_\alpha$, then A is C-generic over A_α for all $\alpha < \kappa$.

Proof. By Lemma 1.2.1, we can arrange that C and K are fields and all the A_α's involved are multiplicatively closed in K. In particular, each A_α with the multiplication is a group. The conclusions follow easily from the equivalence of (1) and (4) of Proposition 1.2.3. □

We call a polynomial in $\mathbb{Q}[x]$ **special** if it has the form $\prod_\zeta (M - \zeta N)$ where ζ ranges over the set of k-th primitive roots of unity for some $k > 0$ and some monomials M and N.

Proposition 1.2.5. *Suppose K is a field, $G \subseteq K^\times$ is a group, and U is the set of all roots of unity in K. Moreover, suppose* $\mathrm{char}(K) = 0$. *Then the following are equivalent:*

(1) G is generic;

(2) for all $g, g' \in G^n$, if $\mathrm{mtp}(g) = \mathrm{mtp}(g')$ *then* $\mathrm{atp}(g) = \mathrm{atp}(g')$;

(3) for all $g \in G^n$ and $P \in \mathbb{Q}[x]$, P vanishes on g if and only if P is in $\sqrt{J_g}$ where $J_g \subseteq \mathbb{Q}[x]$ is the ideal generated by the special polynomials vanishing on g;

(4) if c is in $\mathbb{Q}^n$, and $g \in G^n$ is a non-degenerate solution of the equation $c \cdot x = 1$ then g is in U^n.

Proof. Throughout the proof, we suppose K, G and U are as stated. We first prove that (1) implies (3). As the statement is independent of the ambient field, we can arrange that K is algebraically closed. It is clear even without assuming (1) that the backward implication of (3) holds. Now we suppose (1) and prove the forward implication of (3). We reduce the problem to finding finitely many special polynomials $S_1, \ldots, S_l$ such that

$$Z(S_1, \ldots, S_l) \subseteq Z(P).$$

Indeed, suppose we managed to do so. Then, by the Nullstellensatz, this implies P^m is in the ideal generated by $S_1, \ldots, S_l$ in $K[x]$. Hence P^m is a K-linear combination of products $M_i S_j$ for $i \in \{1, \ldots, k\}$, $j \in \{1, \ldots, l\}$ and each M_i a monomial in x. By taking a linear basis of K over $\mathbb{Q}$ and taking into account the assumption that P is in $\mathbb{Q}(x)$, we get P^m is a $\mathbb{Q}$-linear combination of products of $M_i S_j$ as above. Therefore, P is in $\sqrt{J_g}$.

By equivalence of (1) and (3) in Proposition 1.2.3, we have that P lies in the ideal I_g of $\mathbb{Q}(U)[x]$ generated by polynomials of the form $M - \zeta N$ vanishing on g with M, N monomials in x and ζ a root of unity. As $\mathbb{Q}(U)[x]$ is Noetherian, there are binomials $M_1 - \zeta_1 N_1, \ldots, M_l - \zeta_l N_l$ generating I_g. Hence,

$$Z(M_1 - \zeta_1 N_1, \ldots, M_l - \zeta_l N_l) \subseteq Z(P).$$

Let ζ be a generator of the subgroup of U generated by $\zeta_1, \dots, \zeta_l$. Then there are natural numbers $s_1, \dots, s_l$ and $t_1, \dots, t_l$ such that $\zeta = \zeta_1^{s_1} \cdots \zeta_l^{s_l}$ and $\zeta_i = \zeta^{t_i}$ for all $i \in \{1, \dots, l\}$. Let

$$M' = \prod_{i=1}^{l} (M_i)^{s_i} \quad \text{and} \quad N' = \prod_{i=1}^{l} (N_i)^{s_i}.$$

We note that $Z(M_1 - \zeta_1 N_1, \dots, M_l - \zeta_l N_l)$ is equal to

$$Z\big(M' - \zeta N', (N')^{t_1} M_1 - (M')^{t_1} N_1, \dots, (N')^{t_l} M_l - (M')^{t_l} N_l\big).$$

Therefore, we might as well assume P vanishes on the zero set of polynomials $M_1(x) - \zeta N_1(x)$, $M_2(x) - N_2(x), \dots, M_l(x) - N_l(x)$.

With ζ, M_i, N_i as in the preceding statement, let ζ be a primitive k-th root of unity. Set

$$S_1 = \prod_{\varepsilon} (M_1 - \varepsilon N_1) \quad \text{where } \varepsilon \text{ ranges over the primitive } k\text{-th roots of unity}$$

and $S_2 = M_2(x) - N_2(x), \dots, S_l = M_l(x) - N_l(x)$. Note that each S_i is special. Suppose, $a \in K^n$ is in the zero set of the ideal of $\mathbb{Q}[x]$ generated by $S_1, \dots, S_l$. Then there is a primitive k-th root of unity ε such that $M_1(a) - \varepsilon N_1(a) = 0$. Since $\mathrm{char}(K) = 0$, there is an automorphism σ of K such that $\sigma(\varepsilon) = \zeta$. Hence,

$$M_1\big(\sigma(a)\big) - \zeta N_1\big(\sigma(a)\big) = S_2\big(\sigma(a)\big) = \cdots = S_l\big(\sigma(a)\big) = 0.$$

By the choice of ζ, M_i, N_i, we have $P\big(\sigma(a)\big) = 0$. As P is in $\mathbb{Q}[x]$, $P(a) = 0$. Thus, we have proven the reduction and hence (3).

Next, we prove that (3) implies (2). Suppose (3), and g, g' have the same multiplicative type. Let S be a special polynomial such that $S(g) = 0$ and $S = \prod_\zeta (M - \zeta N)$ where ζ ranges over all the primitive k-th roots of unity for some $k > 0$. Then $M(g)N^{-1}(g)$ is a primitive k-th k-th root of unity, so $M^k - N^k$ vanishes on g but $M^l - N^l$ does not vanish on g for $0 < l < k$. As g, g' have the same multiplicative type,

$$M^k(g') - N^k(g') = 0 \text{ but } M^l(g') - N^l(g') \neq 0 \text{ for } 0 < l < k.$$

So $M(g')N^{-1}(g')$ is a primitive k-th root of unity and $S(g') = 0$. Hence $J_g = J_{g'}$, and so $\mathrm{atp}(g) = \mathrm{atp}(g')$. Thus, we have (2).

The argument for (2) implying (1) is the same as the argument for (2) implying (1) in Proposition 1.2.3. Finally, by (2) of Lemma 1.2.1, G is generic if and only if G is generic over $G \cap U$. We note that $\mathrm{mcl}(G \cap U) \cap G =$

$G \cap U$, so the equivalence between (1) and (4) follows immediately from the equivalence between (1) and (4) in Proposition 1.2.3. $\square$

Proposition 1.2.6. *Suppose K is a field, $G \subseteq K^\times$ is a group, g is in G^n and H is a subgroup of G such that G is generic over H. Moreover, suppose $\operatorname{char}(K) = 0$, and $\operatorname{mcl}(H) \cap G = H$. Then $\mathbb{Q}(G)$ is a regular field extension of $\mathbb{Q}(H)$.*

Proof. As $\operatorname{char}(K) = 0$, $\mathbb{Q}(G)$ is a separable field extension of $\mathbb{Q}(H)$, so it suffices to check that $\mathbb{Q}(H)$ is algebraically closed in $\mathbb{Q}(G)$. Suppose $P, Q \in \mathbb{Q}[x]$, and $g \in G^n$ is such that $P(g)Q^{-1}(g)$ is algebraic over $\mathbb{Q}(H)$. Let G' be the subgroup of G generated by g. As $\operatorname{mcl}(H) \cap G = H$, $G'/(H \cap G')$ is torsion-free of finite rank, we can choose $g'_1, \dots, g'_k$ in G' multiplicatively independent over H such that

$$g_i \;=\; M'_i(g'_1, \dots, g'_k) \;\text{ where } M'_i \text{ is } H\text{-monomial for } i \in \{1, \dots, n\}$$

Hence we can find P', Q' coprime in $\mathbb{Q}(H)[y_1, \dots, y_k]$ such that $P'(g')Q'^{-1}(g')$ is equal to $P(g)Q^{-1}(g)$. As $g'_1, \dots, g'_k$ are multiplicatively independent over H, they are algebraically independent over $\mathbb{Q}(H)$. Therefore, in order to have $P'(g')Q'^{-1}(g')$ algebraic over $\mathbb{Q}(H)$, the polynomials P', Q' must have degree 0 and so $P'(g')Q'^{-1}(g')$ is in $\mathbb{Q}(H)$. The conclusion follows. $\square$

We recall the following version of a theorem of Mann from [10]:

Theorem (Mann). *Let U be the group of roots of unity in $\mathbb{Q}^{ac}$. There is a recursive function $d : \mathbb{N} \to \mathbb{N}$ such that if $a_1, \dots, a_n$ are in $\mathbb{Q}$ and $(y_1, \dots, y_n)$ in U^n is a tuple of non-degenerate solution of the equation $a_1y_1 + \cdots + a_ny_n = 1$, then $y_i^{d(n)} = 1$ for all i.*

For an L-structure $(F, K; \chi)$, it is easy to see that χ in generic if and only if $\chi(F^\times)$ is generic in the sense of this section. As a consequence we have:

Proposition 1.2.7. *There is a recursive set of universal statements in L whose models are (F, K, χ) with χ generic.*

Proof. Suppose F' and K' are respectively the fraction fields of F and K. Using only the conditions that χ is multiplication preserving, $\chi(0) = 0$ and χ is injective, we can extend χ to an injective character $\chi' : F' \to K'$; moreover, χ' maps multiplicatively independent elements to algebraically independent elements if and only if χ does so by Lemma 1.2.2. We also note that $(F', K'; \chi')$ is interpretable in $(F, K; \chi)$ in the obvious way. Hence we can reduce the problem to the case where F and K are fields.

Combining the equivalence between (1) and (4) of Proposition 1.2.5 and Mann's theorem, χ is generic if and only if for all n and all non-degenerate solutions of $a_1x_1 + \cdots + a_nx_n = 1$ in $\big(\chi(F)^\times\big)^n$ with a in $\mathbb{Q}^n$, we have

$x_i^{d(n)} = 1$ for $i \in \{1, \ldots, n\}$. It is clear that being a non-degenerate solution is definable by a quantifier-free formula. So we have the desired universal axiom scheme. □

Theorem 1.2.8. *There is a recursive set* ACFC *of* $\forall\exists$*-axioms in* L *such that:*

(1) for all $(F, K; \chi)$*,* $(F, K; \chi) \models$ ACFC *if and only if* F *and* K *are algebraically closed fields,* $\mathrm{char}(K) = 0$ *and* $\chi : F \to K$ *is generic;*

(2) if $\mathrm{char}(\mathbb{F}) = p$*, then* $(\mathbb{F}, \mathbb{C}; \chi) \models$ ACFC*.*

Proof. It follows easily from proposition 1.2.7 that we have the desired axiomatization. When $F = \mathbb{F}, K = \mathbb{C}$ and $\chi : F \to K$ is injective, we note that χ is automatically generic because there is no multiplicative independence between elements of $\chi(F^\times)$. □

Let $\mathcal{Q}$ be the set of prime powers. For each $q \in \mathcal{Q}$, let $\chi_q : \mathbb{F}_q \to \mathbb{C}$ be an injective map with $\chi_q(0) = 0$ and $\chi_q(ab) = \chi_q(a)\chi_q(b)$ for all $a, b \in \mathbb{F}_q$. With exactly the same method we get:

Proposition 1.2.9. *There is a recursive set of axioms* T *in* L *with the following properties:*

(1) for all $(F, K; \chi)$*,* $(F, K; \chi) \models T$ *if and only if* K *is an algebraically closed fields with* $\mathrm{char}(K) = 0$*,* F *is a pseudo-finite field and* $\chi : F \to K$ *is generic;*

(2) if $\mathcal{U}$ *is a non-principal ultrafilter on* $\mathcal{Q}$*, then* $\left(\prod_{q\in\mathcal{Q}}(\mathbb{F}_q, \mathbb{C}; \chi_q)\right)/\mathcal{U} \models T$*.*

This also allows us to conjecture that for every T-model $(F, K; \chi)$, there is an ultrafilter $\mathcal{U}$ on $\mathcal{Q}$ such that $(F, K; \chi) \equiv \left(\prod_{q\in\mathcal{Q}}(\mathbb{F}_q, \mathbb{C}; \chi_q)\right)/\mathcal{U}$.

1.3 Classification, completeness and decidability

We keep the notation conventions of the preceding section and moreover assume in this section that $(F, K; \chi) \models \mathrm{ACFC}_p$. We classify the models of ACFC_p up to isomorphism. From this we deduce that the theory ACFC_p is complete and decidable.

Proposition 1.3.1. *Suppose* $(F, K; \chi_1)$ *and* (F, K, χ_2) *are models of* ACFC_p *with* $\mathbb{Q}\big(\chi_1(F)\big) = \mathbb{Q}\big(\chi_2(F)\big)$*. Then there is an automorphism* σ *of* K *with* $\chi_2 = \sigma \circ \chi_1$*.*

Proof. Suppose F, K, χ_1 and χ_2 are as stated. Let $\alpha = (\alpha_i)_{i\in I}$ be a listing of the elements of $F^\times$. As χ_1, χ_2 are group homomorphisms, $\mathrm{mtp}\big(\chi_1(\alpha)\big) = \mathrm{mtp}\big(\chi_2(\alpha)\big)$. By Proposition 1.2.5, $\mathrm{atp}\big(\chi_1(\alpha)\big) = \mathrm{atp}\big(\chi_2(\alpha)\big)$, and so there is a field automorphism

$$\sigma : \mathbb{Q}\big(\chi_1(F)\big) \to \mathbb{Q}\big(\chi_2(F)\big)$$

such that $\chi_2 = \sigma \circ \chi_1$. We can further extend σ to a field automorphism of $\mathbb{Q}\big(\chi_1(F)\big)^{ac} = \mathbb{Q}\big(\chi_2(F)\big)^{ac}$ and then to an automorphism of K. $\square$

Corollary 1.3.2. *If p is prime, $F = \mathbb{F}_p^{ac}$ and $K = \mathbb{Q}^{ac}$, then there is a unique injective character from F to K up to isomorphism.*

Corollary 1.3.3. *If $\chi : F \to K$ is generic and if σ is an automorphism of F, then σ can be extended to an automorphism of $(F, K; \chi)$.*

We say $(F, K; \chi) \vDash \mathrm{ACFC}_p$ is (κ, λ)-**transcendental** if $\operatorname{trdeg}(F \mid \mathbb{F}_p) = \kappa$ and $\operatorname{trdeg}\big(K \mid \mathbb{Q}(G)\big) = \lambda$ with $G = \chi(F^\times)$.

Theorem 1.3.4. *For any p,κ and λ, there is a unique (κ, λ)-transcendental model of ACFC_p up to isomorphism .*

Proof. We first prove the uniqueness part of the lemma. Suppose $(F_1, K_1; \chi_1)$ and $(F_2, K_2; \chi_2)$ are (κ, λ)-transcendental models of ACFC_p. Let G_1 be $\chi(F_1^\times)$ and G_2 be $\chi(F_2^\times)$. As F_1 and F_2 are algebraically closed of the same characteristic and $\operatorname{trdeg}(F_1 \mid \mathbb{F}_p) = \operatorname{trdeg}(F_2 \mid \mathbb{F}_p)$, there is an isomorphism

$$\sigma : F_1 \to F_2.$$

Using Proposition 1.2.5 in a similar way as in the proof of Proposition 1.3.1, σ induces an isomorphism between $\mathbb{Q}(G_1)$ and $\mathbb{Q}(G_2)$; we will also call this σ. Finally, since $\operatorname{trdeg}\big(K_1 \mid \mathbb{Q}(G_1)\big)$ is equal to $\operatorname{trdeg}\big(K_2 \mid \mathbb{Q}(G_2)\big)$ we can extend σ to a field isomorphism from K_1 to K_2. It is easy to check that this is an isomorphism of L-structures.

We next prove the existence part of the lemma. For $p > 0$, ACFC_p clearly has a model. For $p = 0$, ACFC_p has a model by compactness. We can arrange to have for each p a model $(F, K; \chi)$ of ACFC_p such that $|F|, |K| > \max\{\kappa, \lambda, \aleph_0\}$. Choose an algebraically closed subfield F' of F with $\operatorname{trdeg}(F' \mid \mathbb{F}_p) = \kappa$. Then we have $\operatorname{trdeg}\big(K \mid \mathbb{Q}(\chi(F'))\big) > \lambda$. Choose an algebraically closed subfield K' of K containing $\chi(F')$ with $\operatorname{trdeg}\big(K' \mid \mathbb{Q}(\chi(F'))\big) = \lambda$. We can check that $(F', K'; \chi\restriction_{F'})$ is a (κ, λ)-transcendental model of ACFC_p. $\square$

Remark. *The fact that we can classify the models of ACFC_p has many consequences. For example, it follows from Shelah's main gap theorem [15, XII.6.1] that ACFC_p is superstable, shallow, without the dop, without the otop and from [15, VII.3.4] that ACFC_p is without the fcp. Direct simpler proof might also be feasible, but we leave this issue to the interested readers.*

Next we prove an analog of upward Löwenheim-Skolem theorem.

Lemma 1.3.5. *For $\chi : F \to K$ generic, K is an infinite extension of $\mathbb{Q}(\chi(F))$.*

Proof. Suppose F, K and χ are as stated. Let $G = \chi(F^\times)$. By Proposition 1.2.6, if U consists of the roots of unity in G, then $\mathbb{Q}(G)$ is a regular extension of $\mathbb{Q}(U)$. Hence,

$$\left[\mathbb{Q}^{ac} : \mathbb{Q}(U)\right] \le \left[\mathbb{Q}^{ac}\mathbb{Q}(G) : \mathbb{Q}(G)\right].$$

By Galois theory, $\left[\mathbb{Q}^{ac} : \mathbb{Q}(U)\right] = \infty$. Therefore, $\left[\mathbb{Q}^{ac}\mathbb{Q}(G) : \mathbb{Q}(G)\right] = \infty$ and so $\left[K : \mathbb{Q}(G)\right] = \infty$. □

Lemma 1.3.6. *Every model $(F, K; \chi)$ of ACFC_p has a (κ, κ)-transcendental elementary extension $(F', K'; \chi')$ for any cardinal $\kappa \ge \max(|F|, |K|)$.*

Proof. Let $(F, K; \chi)$ and κ be as stated. We construct an elementary extension $(F'', K''; \chi'')$ of $(F, K; \chi)$ with $\mathrm{trdeg}(F'' \mid \mathbb{F}_p) \ge \kappa$ and $\mathrm{trdeg}(K'' \mid G'') \ge \kappa$ with $G'' = \chi(F''^\times)$. For the later two conditions to hold, it suffices to ensure there are

$$\alpha \in (F'')^\kappa \text{ and } a \in (K'')^\kappa$$

such that components of α are all distinct and the components of a are algebraically independent over G''. Using compactness, we can reduce the problem to verifying the following: for arbitrary k, m, n, w of length m, x of length n and arbitrary polynomials $P_1, \ldots, P_l$ in $\mathbb{Q}[w, x]$, there are α in F^k and a in K^n such that components of α are pairwise different, and

$$P_i\big(\chi(\beta), a\big) \neq 0 \text{ for all } \beta \in F^m \text{ and } i \in \{1, \ldots, l\}.$$

It is easy to find α with the desired property. By the preceding lemma, $\left[K : \mathbb{Q}(G)\right]$ is infinite, so we can choose a so that $\left[\mathbb{Q}(G, a_1, \ldots, a_j) : \mathbb{Q}(G, a_1, \ldots, a_{j-1})\right] > N$ for $j \in \{1, \ldots, n\}$ where N is the maximum degree of P_i for $i \in \{1, \ldots, l\}$. We see that this choice of a works. We then get the desired $(F', K'; \chi')$ from $(F'', K''; \chi'')$ by taking the Skolem Hull of the suitable elements. □

Theorem 1.3.7. *For all p, ACFC_p is complete and decidable. When $p > 0$, ACFC_p axiomatizes $\mathrm{Th}(\mathbb{F}, \mathbb{C}; \chi)$ where $\mathrm{char}(\mathbb{F}) = p$.*

Proof. We first show that any two arbitrary models $(F_1, K_1; \chi_1)$ and $(F_2, K_2; \chi_2)$ of ACFC_p are elementarily equivalent. By the preceding lemma, we can arrange that $(F_1, K_1; \chi_1)$ and $(F_2, K_2; \chi_2)$ are both (κ, κ)-transcendental. It follows from Theorem 1.3.4 that for all p, ACFC_p is complete. The remaining conclusions are immediate. □

Corollary 1.3.8. *Let τ be an L-statement. The following are equivalent:*

(1) τ is true in some model of ACFC_0*;*

(2) there are arbitrarily large primes p such that τ is true in some model of ACFC_p*;*

(3) there is a number m such that for all primes $p > m$, τ is true in all models of ACFC_p*.*

1.4 Substructures and elementary substructures

From now on, let k, l range over the set of natural numbers, $s = (s_1, \dots, s_k)$, $v = (v_1, \dots, v_l)$ be tuples of variables of the first sort and $w = (w_1, \dots, w_k)$, $z = (z_1, \dots, z_l)$ be tuples of variables of the second sort. We also implicitly assume similar conventions for these letters with additional decorations.

In addition to the notation conventions in the first paragraph of section 1.2, we assume in this section that $(F, K; \chi)$ has $\mathrm{char}(K) = 0$. We use $\subseteq$ and $\preccurlyeq$ to denote the L-substructure and elementary L-substructure relations respectively. We will characterize the substructures and elementary substructures of a model of ACFC_p.

Proposition 1.4.1. *We have $(F, K; \chi)$ is an L-substructure of an ACFC_p-model if and only if χ is generic and* $\mathrm{char}(F) = p$*.*

Proof. The forward implication is clear. For the other direction, suppose χ is generic and $\mathrm{char}(F) = p$. We can embed $(F, K; \chi)$ into an L-structure $(F'', K''; \chi'')$ where F'', K'' are respectively the fraction fields of F, K and χ'' is the natural extension of χ to F''. We note that χ'' is still generic. Therefore, we can arrange that F and K are already fields.

Let G be $\chi(F^\times)$, F' be the algebraic closure of F, and K' be an algebraically closed field containing K such that $\mathrm{trdeg}(K' \mid K) > |F'|$. Let $\{\alpha_i\}_{i<\kappa}$ be a multiplicative basis of $F'^\times$ over $F^\times$. As $\mathrm{trdeg}(K' \mid K) > |F'|$, we can define a map

$$\chi' : \{\alpha_i\}_{i<\kappa} \to K'$$

such that the image is algebraically independent over K. Since $\mathrm{char}(K) = 0$, we have $\mathrm{mcl}_G\big(\{\chi'(\alpha_i)\}_{i<\kappa}\big)$ in $K'^\times$ is divisible. Hence we can extend χ' to an injective map $\chi' : F'^\times \to K'^\times$ extending χ. Let $G' = \chi(F'^\times)$. Then G' is K-generic over G by Corollary 1.2.2. Since G is generic, G' is also generic by Corollary 1.2.4. Thus the structure $(F', K'; \chi')$ is the desired model of ACFC_p. $\square$

Let $(F', K'; \chi')$ be an L-structure. We say that $(F, K; \chi)$ is a **regular substructure** of $(F', K'; \chi')$, denoted as $(F, K; \chi) \sqsubseteq (F', K'; \chi')$, if $(F, K; \chi) \subseteq (F', K'; \chi')$ and $\chi'(F'^\times)$ is K-generic over $\chi(F^\times)$. With the use of

Proposition 1.2.3, it can be seen that the above proof also gives us the following stronger statement:

Corollary 1.4.2. *If χ is generic then there is a model $(F', K'; \chi')$ of* ACFC_p *such that $(F, K; \chi) \sqsubseteq (F', K'; \chi')$.*

We now characterize the regular substructure relation for models of ACFC_p.

Proposition 1.4.3. *Suppose $(F, K; \chi) \subseteq (F', K'; \chi')$ are models of* ACFC_p. *Let $G = \chi(F^\times)$ and $G' = \chi'(F'^\times)$. Then the following are equivalent:*

(1) $(F, K; \chi) \sqsubseteq (F', K'; \chi')$;

(2) for all n, all $P_1, \ldots, P_n \in \mathbb{Q}[w]$ and all $a_1, \ldots, a_n \in K$, if there is a tuple $g' \in G'^k$ with $P_1(g'), \ldots, P_n(g')$ not all 0 and $a_1 P_1(g') + \cdots + a_n P_n(g') = 0$, then we can find such a tuple in G^k;

(3) $\mathbb{Q}(G')$ and K are linearly disjoint over $\mathbb{Q}(G)$ in K'.

Proof. Towards showing that (1) implies (2), suppose (1). Fix n, polynomials $P_1, \ldots, P_n \in \mathbb{Q}[w]$, K-elements $a_1, \ldots, a_n$ and g' as in (2). We want to find $g \in G^k$ with $P_1(g), \ldots, P_n(g)$ not all 0 and

$$a_1 P_1(g) + \cdots + a_n P_n(g) \;=\; 0.$$

Replacing $(F, K; \chi)$ and $(F', K'; \chi')$ concurrently with elementary extensions and noting that G' remains K-generic over G by the equivalence between (1) and (4) of Proposition 1.2.3, we can arrange that $(F, K; \chi)$ is $\aleph_0$-saturated. By the equivalence between (1) and (2) of Proposition 1.2.5,

$$\text{if } \operatorname{mtp}(g) \;=\; \operatorname{mtp}(g') \text{ then } P_1(g), \ldots, P_n(g) \text{ are not all } 0.$$

By the equivalence between (1) and (3) of Proposition 1.2.3, there are G-binomials $M_1 - N_1, \ldots, M_l - N_l$ vanishing on g' such that

$$M_1(g) - N_1(g) \;=\; \cdots \;=\; M_l(g) - N_l(g) \;=\; 0 \text{ implies } a_1 P_1(g) + \cdots + a_n P_n(g) \;=\; 0.$$

Let $\alpha = \chi^{-1}(g)$, $\alpha' = \chi^{-1}(g')$ and $\chi^{-1} M_i, \chi^{-1} N_i$ be the pullbacks of M_i and N_i under χ for $i \in \{1, \ldots, l\}$. It suffices to find $\alpha \in (F^\times)^k$ with $\operatorname{mtp}(\alpha) = \operatorname{mtp}(\alpha')$ and

$$\chi^{-1} M_1(\alpha) - \chi^{-1} N_1(\alpha) \;=\; \cdots \;=\; \chi^{-1} M_l(\alpha) - \chi^{-1} N_l(\alpha) \;=\; 0.$$

Such α can be found as F is an elementary substructure of F' in the language of fields and F is $\aleph_0$-saturated. Thus we have (2).

It is immediate that (2) implies (3). Towards showing that (3) implies (1), suppose (3) and $g' \in (G')^n$ is algebraically dependent over $K(G) = K$. We need to show that g' is multiplicatively dependent over G. Pick a non-trivial $P \in K[x]$ with $P(g') = 0$. Choose a linear basis $(b_i)_{i\in I}$ of K over $\mathbb{Q}(G)$. Then

$$P = \sum_{i\in I} P_i b_i \quad \text{with} \quad P_i \in \mathbb{Q}(G)[x] \quad \text{for } i \in I$$

and $P_i = 0$ for all but finitely many $i \in I$. Hence $\sum_{i\in I} P_i(g')b_i = 0$. By (3), $(b_i)_{i\in I}$ remains linearly independent over $\mathbb{Q}(G')$. Therefore, $\sum_{i\in I} P_i(g')b_i = 0$ implies that $P_i(g') = 0$ for all $i \in I$. Since P is non-trivial, at least one P_i is non-trivial, and hence g' is algebraically dependent over $\mathbb{Q}(G)$. Now, G' is generic so G' is generic over G by (1) of Corollary 1.2.4. By the definition of genericity, G' is $\mathbb{Q}(G)$-generic over G. Hence, g' is multiplicatively dependent over G which is the desired conclusion. $\square$

Corollary 1.4.4. *For* $(F, K; \chi), (F', K'; \chi') \models \mathrm{ACFC}_p$*, if* $(F, K; \chi) \preccurlyeq (F', K'; \chi')$*, then* $(F, K; \chi) \sqsubseteq (F', K'; \chi')$.

Proof. This follows from the equivalence between (1) and (2) of Proposition 1.4.3. $\square$

Corollary 1.4.5. *For all* p*,* ACFC_p *is not model complete, and has no model companion in* L*. The same conclusion applies to* ACFC.

Proof. We show that ACFC_p is not model complete. Let $(F, K; \chi)$ and $(F', K'; \chi)$ be models of ACFC_p such that the former is an L-substructure of the latter and the latter is κ-saturated with $\kappa > |F| + |K|$. Set $G = \chi(F^\times)$ and $G' = \chi(F'^\times)$. We get by saturation $a, b \in G'$ algebraically independent over K. We will show that

$$(F, K''; \chi) \not\sqsubseteq (F', K'; \chi) \quad \text{where} \quad K'' = K(a+b)^{ac},$$

which yields the desired conclusion by Corollary 1.4.4. Fix $g' \in G' \cap K''$. Then g' and $a + b$ are algebraically dependent over K and therefore so are g', a and b. Suppose G' is K''-generic over G. As a consequence, g', a and b are also multiplicatively dependent over G. By replacing g' with some power of it if needed, we arrange $g' = M(a, b)$ for some G-monomial M. Then $M(a, b)$ is in $K'' = K(a + b)^{ac}$ and so a and b are algebraically dependent over K, a contradiction. As a consequence, $(F, K''; \chi) \not\sqsubseteq (F', K'; \chi)$.

Suppose ACFC_p has a model companion T in L. Take any model $\mathcal{M}$ of T. Then $\mathcal{M}$ is an L-substructure of $(F, K; \chi) \models \mathrm{ACFC}_p$ which itself is an L-substructure of $\mathcal{N} \models T$. Let $\varphi(x)$ be an existential formula in L such that $\mathrm{ACFC}_p \models \forall x \varphi(x)$. Hence, for all a with components in $\mathcal{M}$ of suitable sorts, $(F, K; \chi) \models \varphi(a)$

and so $\mathcal{N} \models \varphi(a)$. As T is model complete, we also have $\mathcal{M} \models \varphi(a)$. Since ACFC_p is a set of $\forall\exists$-formulas, $\mathcal{M} \models \mathrm{ACFC}_p$. On the other hand, ACFC_p is complete. Hence, $T = \mathrm{Th}(\mathcal{M}) = \mathrm{ACFC}_p$, a contradiction as ACFC_p is not model complete.

It is easy to see that if a theory T has a model companion, then any of its extensions also has a model companion. The final conclusion thus follows. ☐

There are clearly some obstructions for one model of ACFC_p to be an elementary submodel of another model of ACFC_p that contains it. We will show that these are the only obstructions. For the main theorem of this section we need the following two technical lemmas:

Lemma 1.4.6. *The following statements hold:*

(1) If $(F, K; \chi) \sqsubseteq (F', K'; \chi')$ and $(F', K'; \chi') \sqsubseteq (F'', K''; \chi'')$, then we have $(F, K; \chi) \sqsubseteq (F'', K''; \chi'')$.

(2) Suppose $(F_0, K_0; \chi_0) = (F, K; \chi)$ and $(F_m, K_m; \chi_m) \sqsubseteq (F_{m+1}, K_{m+1}; \chi_{m+1})$ for every m. If $F' = \bigcup_m F_m, K' = \bigcup_m K_m$ and $\chi' = \bigcup_m \chi_m$, then we have $(F, K; \chi) \sqsubseteq (F', K'; \chi')$.

Proof. This follows from Corollary 1.2.4 and the definition of genericity. ☐

Lemma 1.4.7. *Let $(F_0, K_0; \chi_0) = (F, K; \chi)$, $(F_m, K_m; \chi_m) \sqsubseteq (F_{m+1}, K_{m+1}; \chi_{m+1})$ for each m, and $F' = \bigcup_m F_m, K' = \bigcup_m K_m$, $\chi' = \bigcup_m \chi_m$. If $(F_m, K_m; \chi_m)$ is a model of* ACFC_p *with $|K_m| = \kappa$ for each m and $(F, K; \chi)$ is (κ, κ)-transcendental, then $(F', K'; \chi')$ is a (κ, κ)-transcendental model of* ACFC_p.

Proof. In addition to the above notations, let $G = \chi(F^\times)$ and $G' = \chi(F'^\times)$. As ACFC_p is a set of $\forall\exists$-formulas, $(F', K'; \chi')$ is a model of ACFC_p. Since $(F, K; \chi)$ is (κ, κ)-transcendental, there is $a \in K^\kappa$ with all components algebraically independent over $\mathbb{Q}(G)$. By the preceding lemma,

$$(F, K; \chi) \ \sqsubseteq \ (F', K'; \chi').$$

By Proposition 1.4.3, K and $\mathbb{Q}(G')$ are linearly disjoint over $\mathbb{Q}(G)$ in K'. Hence the components of a remain algebraically independent over $\mathbb{Q}(G')$. Therefore, $\mathrm{trdeg}\big(K' \mid \mathbb{Q}(G')\big) \geq \kappa$. Also, $\mathrm{trdeg}(F' \mid \mathbb{F}_p) \geq \kappa$. Hence $|F'| = |K'| = \kappa$ by a cardinality argument. Thus, $(F', K'; \chi')$ must be (κ, κ)-transcendental. ☐

Theorem 1.4.8. ACFC_p *is the regular model companion of* $\mathrm{ACFC}_p(\forall)$*. That is:*

(1) every model of $\mathrm{ACFC}_p(\forall)$ *is a regular substructure of a model of* ACFC_p*;*

(2) when $(F, K; \chi) \sqsubseteq (F', K'; \chi')$ are models of ACFC_p*, then we have $(F, K; \chi) \preccurlyeq (F', K'; \chi')$.*

Proof. We have (1) follows from Corollary 1.4.2. The proof of (2) requires some preparation. We let L^+ be the language obtained by adding to L an n-ary relation $R_{P_1,\dots,P_n}$ for each n, and each choice of polynomials $P_1,\dots,P_n \in \mathbb{Q}[w]$. The theory ACFC_p^+ is obtained by adding to ACFC_p the following axioms for each choice of $n, P_1, \dots, P_n$:

$$R_{P_1,\dots,P_n}(x) \leftrightarrow \exists s\Bigg(\Big(\bigvee_{i=1}^{n} P_i\big(\chi(s)\big) \neq 0\Big) \wedge \Big(x_1P_1\big(\chi(s)\big) + \dots + x_nP_n\big(\chi(s)\big) = 0\Big)\Bigg).$$

We note that ACFC_p^+ is still a complete $\forall\exists$-theory. If $(F, K; \chi)$ is a model of ACFC_p, we will let $(F, K; \chi, R)$ be its natural expansion to a model of ACFC_p^+; here, R represents all the possible $R_{P_1,\dots,P_n}$ for simplicity of notation. Then, by equivalence of (1) and (2) of Proposition 1.4.3, (2) of this theorem is equivalent to saying the theory ACFC_p^+ is model complete in L^+.

It suffices to show that all models of ACFC_p^+ are existentially closed. Suppose we have a counterexample $(F, K; \chi, R)$. We first reduce to the case where $(F, K; \chi)$ is moreover (κ, κ)-transcendental for some infinite κ. By assumption, there is an ACFC_p^+-model $(F'', K''; \chi'', R'')$ extending $(F, K; \chi, R)$ as a L^+-substructure such that the latter is not existentially closed in the former. Consider the structure $(F'', K''; \chi'', R'', F, K, \chi, R)$ in the language where F, K, R, χ are regarded as relations on $(F'', K''; \chi'', R'')$. Note that if we replace this structure with an elementary extension we will still have $(F, K; \chi, R)$ a non-existentially closed ACFC_p^+-submodel of $(F'', K''; \chi'', R'')$. Using a similar trick as in Lemma 1.3.6, we can add the condition that $(F, K; \chi)$ is (κ, κ)-transcendental.

Next, we will construct $(F', K'; \chi', R')$ existentially closed such that $(F', K'; \chi')$ is (κ, κ)-transcendental. We start with $(F_0, K_0; \chi_0, R_0) = (F, K; \chi, R)$, the structure obtained at the end of the previous paragraph, and for each $m > 0$ construct the ACFC_p^+-model $(F_m, K_m; \chi_m, R_m)$ as follows. Choose $(F_{m+1}, K_{m+1}; \chi_{m+1}, R_{m+1})$ to be an ACFC_p^+-model extending $(F_m, K_m; \chi_m, R_m)$ realizing a maximal consistent set of existential formulas with parameters from $(F_m, K_m; \chi_m, R_m)$; concurrently, we use downward Löwenheim-Skolem theorem to arrange $|K_m| = \kappa$. Let

$$F' = \bigcup_m F_m,\ K' = \bigcup_m K_m,\ \chi' = \bigcup_m \chi_m,\ R' = \bigcup_m R_m.$$

By construction, $(F', K'; \chi')$ is an existentially closed model of ACFC_p^+. By the equivalence between (1) and (2) of Proposition 1.4.3, $(F_m, K_m; \chi_m)$ is a regular substructure of $(F_{m+1}, K_{m+1}; \chi_{m+1})$. It follows from the preceding lemma that $(F', K'; \chi')$ is (κ, κ)-transcendental.

Finally, by Theorem 1.3.4, $(F, K; \chi)$ and $(F', K'; \chi')$ are isomorphic. Hence, $(F, K; \chi, R)$ is also isomorphic to $(F', K'; \chi', R')$, a contradiction to the fact that the former is not existentially closed but the latter is. $\square$

Corollary 1.4.9. *Suppose $(F,K;\chi) \models \mathrm{ACFC}_p$ is (κ,λ)-transcendental and $(F',K';\chi') \models \mathrm{ACFC}_p$ is (κ',λ')-transcendental. Then $(F,K;\chi)$ can be elementarily embedded into $(F',K';\chi')$ if and only if $\kappa \le \kappa'$ and $\lambda \le \lambda'$.*

Proof. We prove the forward direction. Suppose $(F,K;\chi)$ and $(F',K';\chi')$ are as stated and $(F,K;\chi)$ is elementarily embeddable into $(F',K';\chi')$. We can arrange that $(F,K;\chi) \preccurlyeq (F',K';\chi')$. Clearly, $\kappa' \ge \kappa$. Furthermore, by Corollary 1.4.4 and $1 \Leftrightarrow (3)$ of Proposition 1.4.3, $\mathbb{Q}\big(\chi(F'^{\times})\big)$ and K are linearly disjoint over $\mathbb{Q}\big(\chi(F^{\times})\big)$ in K', and so $\lambda' \ge \lambda$.

For the backward direction, using Theorem 1.3.4 it suffices to show that a fixed (κ,λ)-transcendental model $(F,K;\chi)$ of ACFC_p has a (κ',λ')-transcendental elementary extension. Find F' extending F with $|F'| = \kappa'$, take K'' a sufficiently large algebraically closed field containing K and construct $K' \subseteq K''$ in the same fashion as in the proof of Proposition 1.4.1 to obtain $(F',K';\chi')$ such that $(F,K;\chi) \subseteq (F',K';\chi')$. This is the desired model by the preceding theorem. $\square$

1.5 Definable sets

We keep the notation conventions in the first paragraphs of sections 1.2 and 1.4. Moreover, we assume in this section that $(F,K;\chi) \models \mathrm{ACFC}_p$. A set $X \subseteq K^n$ is *definable in the field* K if it is definable in the language of rings. In this case, we use $\mathrm{r}_K(X), \mathrm{d}_K(X)$ to denote the corresponding Morley rank and degree. We equip K^n with the Zariski topology on K, also referred to as the *K-topology*. A *K-algebraic set* is a closed set in this topology. We define the corresponding notions for F in a similar fashion. In this section, we show that definable sets in a model of ACFC_p have a geometrically and syntactically simple description. The following observation is immediate:

Proposition 1.5.1. *Let $\chi : F^k \times K^n \to K^{k+n}, (\alpha, a) \mapsto \big(\chi(\alpha), a\big)$. If $X \subseteq F^k \times K^n$ is definable, then $\chi\restriction_X : X \to \chi(X)$ is a definable bijection. Moreover, $X \subseteq K^n$ is definable over $(\gamma, c) \in F^l \times K^m$ if and only if X is definable over $\big(\chi(\gamma), c\big) \in K^{l+m}$.*

Hence, we restrict our attention to definable subsets of K^n. For a similar reason, we only need to consider sets definable over $c \in K^m$.

Suppose $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$ are families of subsets of K^n. We say $(H_b)_{b\in Y}$ **contains** $(H'_{b'})_{b'\in Y'}$ if for each $b' \in Y'$, there is $b \in Y$ such that $H_b = H'_{b'}$; $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$ are **equivalent** if each contains the other. A **combination** of $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$ is any family of subsets of K^n containing both $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$, which is minimal with these properties in the obvious sense. A **fiberwise intersection** of

$(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$ is any family of subsets of K^n equivalent to $(H_b \cap H'_{b'})_{(b,b')\in Y\times Y'}$. A **fiberwise union** of $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$ is any family of subsets of K^n equivalent to $(H_b \cup H'_{b'})_{(b,b')\in Y\times Y'}$. A **fiberwise product** of $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$ is any family of subsets of K^{2n} equivalent to $(H_b \times H'_{b'})_{(b,b')\in Y\times Y'}$; this definition can be generalized in an obvious way for two families of subsets of different ambient spaces. The following is immediate from the above definitions:

Lemma 1.5.2. *Suppose $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$ are families of subsets of K^n. Let $X = \bigcup_{b\in Y} H_b$ and $X' = \bigcup_{b'\in Y'} H'_{b'}$. Then we have the following:*

(1) $X \cup X'$ is the union of any combination of $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$;

(2) $X \cap X'$ is the union of any fiberwise intersection of $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$;

(3) $X \cup X'$ is the union of any fiberwise union of $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$;

(4) $X \times X'$ is the union of any fiberwise product of $(H_b)_{b\in Y}$ and $(H'_{b'})_{b'\in Y'}$. This part of the lemma can be generalized in an obvious way for two families of subsets of different ambient spaces.

A family $(X_b)_{b\in Y}$ of subsets of K^n is **definable (over** c**)** if both Y and the set

$$\{(a,b) \in K^n \times Y : (a,b) \in X_b\}$$

are definable (over c). We note that if $(X_b)_{b\in Y}$ is definable over c, then for each $b \in Y$, X_b is definable over (b,c) but not necessarily over c.

For two families of subsets of K^n which are definable (over c), we can choose a combination, a fiberwise intersection, a fiberwise union and a fiberwise product of these two families to be definable (over c); the statement about fiberwise product can be generalized in an obvious way for two families of subsets of different ambient spaces. A **presentation** of $X \subseteq K^n$ is a definable family $(H_\alpha)_{\alpha\in D}$ such that

$$X = \bigcup_{\alpha\in D} H_\alpha \quad \text{and} \quad D \subseteq F^k \quad \text{for some } k.$$

An **algebraic presentation** $(V_\alpha)_{\alpha\in D}$ of $S \subseteq K^n$ is a presentation of S such that for each $\alpha \in D$, V_α is K-algebraic. If $S \subseteq K^n$ has an algebraic presentation (which is definable over c), we say S is **algebraically presentable (over** c**)**; if $S \subseteq K^n$ has an algebraic presentation which is 0-definable, we say S is **0-algebraically presentable**. If S plays no important role, we sometimes use the term *algebraic presentation* without mentioning S. For the rest of this section, S is an algebraically presentable subset of its ambient space. It is easy to observe that:

Lemma 1.5.3. *Suppose $(V_\alpha)_{\alpha\in D}$ and $(V'_{\alpha'})_{\alpha'\in D'}$ are algebraic presentations definable over $c \in K^m$. We can choose a combination (fiberwise intersection, fiberwise union, fiberwise product) of $(V_\alpha)_{\alpha\in D}$ and $(V'_{\alpha'})_{\alpha'\in D'}$ to also be an algebraic presentation definable over c.*

An algebraically presentable set can be considered geometrically simple, and next we show that 0-algebraically presentable sets are also syntactically simple.

Lemma 1.5.4. *Suppose $S \subseteq K^n$ is algebraically presentable over c. Then we can find an algebraic presentation $(V_\alpha)_{\alpha\in D}$ and a system of polynomials P in $\mathbb{Q}(c)[w,x]$ such that $V_\alpha = Z\Big(P\big(\chi(\alpha),x\big)\Big)$ for all $\alpha \in D$.*

Proof. Suppose S has an algebraic presentation $(W_\beta)_{\beta\in E}$ definable over c. For each choice $\mathcal{C}$ of k and a system P of polynomials in $\mathbb{Q}(c)[w,x]$, define $R_\mathcal{C} \subseteq F^k \times E$ by

$$(\alpha,\beta) \in R_\mathcal{C} \text{ if and only if } W_\beta = Z\Big(P\big(\chi(\alpha),x\big)\Big).$$

Then the relation $R_\mathcal{C}$ is definable and so are its projections $R^1_\mathcal{C}$ on F^k and $R^2_\mathcal{C}$ on E. For each $\beta \in E$, any automorphism of K fixing $\chi(F)$ and c will also fix W_β. Therefore, for each $\beta \in E$, W_β is definable in the field sense over $\mathbb{Q}\big(c,\chi(\alpha)\big)$ for some $\alpha \in F$. Hence, there is a choice $\mathcal{C}$ as above such that $\beta \in R^2_\mathcal{C}$. There are countably many such choices $\mathcal{C}$. By replacing $(F,K;\chi)$ by an elementary extension, if necessary, we can without loss of generality assume that the structure $(F,K;\chi)$ is $\aleph_0$-saturated. Hence, there are choices $\mathcal{C}_1,\dots,\mathcal{C}_l$ such that E is covered by $R^2_{\mathcal{C}_i}$ as i ranges over $\{1,\dots,l\}$.

For $i \in \{1,\dots,l\}$, obtain k_i and P_i from the choice $\mathcal{C}_i$ and let $D_i = R^1_{\mathcal{C}_i} \subseteq F^{k_i}$. Set $D = D_1 \times \cdots \times D_l$, $P = P_1 \cdots P_l$ and $V_\alpha = Z\Big(P\big(\chi(\alpha),x\big)\Big)$. It is easy to check that the family $(V_\alpha)_{\alpha\in D}$ satisfies the desired requirements. ☐

We have a slightly different version of the above lemma which will be used later.

Lemma 1.5.5. *Suppose $S \subseteq K^n$ has an algebraic presentation $(W_\beta)_{\beta\in E}$ definable over c. We can find an algebraic presentation $(V_\alpha)_{\alpha\in D}$ and systems $P_1,\dots,P_l$ of polynomials in $\mathbb{Q}(c)[w,x]$, such that $(V_\alpha)_{\alpha\in D}$ is equivalent to $(W_\beta)_{\beta\in E}$, $D \subseteq F^k$ is the disjoint union of $D_1,\dots,D_l$, each definable over c, and $V_\alpha = Z\big(P_i(\chi(\alpha),x)\big)$ for $i \in \{1,\dots,l\}$ and $\alpha \in D_i$.*

Proof. We get the choices $\mathcal{C}_1,\dots,\mathcal{C}_l$ in exactly the same way as in the first paragraph of the proof of the preceding lemma. By adding extra variables, if needed, we can arrange that $k_1 = \cdots = k_l = k$ where k_i is taken from the choice $\mathcal{C}_i$. We define D_i inductively. For each $i \in 1,\dots,l$, set

$$D_i = \Big\{\alpha \in F^k \backslash (\bigcup_{j<i} D_j) : \text{ there is } \beta \in E \text{ with } W_\beta = Z\Big(P_i\big(\chi(\alpha),x\big)\Big)\Big\}.$$

Let $D = \bigcup_{i=1}^{l} D_i$, and $(V_\alpha)_{\alpha\in D}$ be given by $V_\alpha = Z\big(P_i(\chi(\alpha), x)\big)$. It is easy to check that $(V_\alpha)_{\alpha\in D}$ is the desired algebraic presentation. □

Next, we prove that F is 0-stably embedded into $(F, K; \chi)$.

Lemma 1.5.6. *If $D \subseteq F^k$ is 0-definable, then it is 0-definable in the field F.*

Proof. By changing the model if needed, we can arrange that $(F, K; \chi)$ realizes all the 0-types. By Stone's representation theorem, it suffices to show that if α and α' are arbitrary elements in F^k with the same 0-type in the field F, then they have the same 0-type. Fix such α and α'. As F is a model of ACF, there is an automorphism of F sending α to α'. This automorphism can be extended to an automorphism of $(F, K; \chi)$ by Corollary 1.3.3, so α and α' have the same 0-type. □

Proposition 1.5.7. *If $S \subseteq K^n$ is 0-algebraically presentable, then we can find a formula $\varphi(s)$ in the language of rings and a system of polynomials $P \in \mathbb{Q}[w, x]$ such that S is defined by*

$$\exists s\Big(\varphi(s) \wedge P\big(\chi(s), x\big) = 0\Big).$$

Proof. This follows from Lemma 1.5.4 and Lemma 1.5.6. □

We next show that 0-definable sets are just boolean combinations of 0-algebraically presentable sets. Towards this, we need a number of lemmas.

Lemma 1.5.8. *The model $(F, K; \chi)$ has an elementary extension $(F', K'; \chi')$ such that $F' = F$ and K' is $|F'|^+$-saturated as a model of* ACF.

Proof. This follows from Corollary 1.4.9. □

The following lemma is well known about ACF. The proof is a consequence, for example, of the results in [18].

Lemma 1.5.9. *Let $(X_b)_{b\in Y}$ be a family of subsets of K^n definable (0-definable) in the field K. We have the following:*

(1) (Definability of dimension in families)

the set $Y_k = \big\{b \in Y : \mathrm{r}_K(X_b) = k\big\}$ is definable (0-definable) in the field K;

(2) (Definability of multiplicity in families)

the set $Y_{k,l} = \big\{b \in Y : \mathrm{r}_K(X_b) = k,\ \mathrm{d}_K(X_b) = l\big\}$ is definable (0-definable) in the field K;

(3) (Definability of irreducibility in algebraic families)

if X_b is K-algebraic for all $b \in Y$, then $Y_{\text{ired}} = \{b \in Y : X_b$ is irreducible$\}$ is definable (0-definable) in the field K.

In our case the preceding lemma has the following consequence:

Corollary 1.5.10. *Let $(X_b)_{b \in Y}$ be a definable (0-definable) family of subsets of K^n with X_b definable in the field K for all $b \in Y$. Then we have the following:*

(1) (Definability of dimension in families)

the set $Y_k = \{b \in Y : \mathrm{r}_K(X_b) = k\}$ is definable (0-definable);

(2) (Definability of multiplicity in families)

the set $Y_{k,l} = \{b \in Y : \mathrm{r}_K(X_b) = k,\ \mathrm{d}_K(X_b) = l\}$ is definable (0-definable);

(3) (Definability of irreducibility in algebraic families)

if X_b is K-algebraic for all $b \in Y$, then $Y_{\text{ired}} = \{b \in Y : X_b$ is irreducible$\}$ is definable (0-definable).

Proof. We first prove (1) for the definable case. Let $(X_b)_{b \in Y}$ be a definable family as stated. For each $b \in Y$, there is a parameter free formula $\varphi(w, x)$ in the language of rings such that there is $c \in K^k$ with X_b defined by $\varphi(c, x)$. We note that there are only countably many parameter free formulas $\varphi(w, x)$ in the language of rings. By a standard compactness argument and a simple reduction we arrange that there is a formula $\varphi(w, x)$ such that for any b in Y, there is $c \in K^k$ such that X_b coincides with X'_c where $X'_d \subseteq K^n$ is defined by $\varphi(d, x)$ for $d \in K^k$. With Y_k as in the statement of the lemma, we have

$$Y_k = \{b \in Y : \text{ there is } c \in K^k \text{ such that } X_b = X'_c \text{ and } \mathrm{r}_K(X'_c) = k\}.$$

The definability of Y_k then follows from (1) of the preceding lemma.

For the 0-definable case, we can arrange that $(F, K; \chi)$ is $\aleph_0$-saturated and check that any automorphism of the structure fixing $(X_b)_{b \in Y}$ also fixes Y_k for all k. The statements (2) and (3) can be proven similarly. $\square$

Towards obtaining the main theorem, we need the following auxiliary lemma.

Lemma 1.5.11. *Let a be in K^n and $V \subseteq K^n$ be the K-algebraic set definable in the field sense over $\mathbb{Q}(\chi(F^\times))$ containing a such that $(\mathrm{r}_K(V), \mathrm{d}_K(V))$ is lexicographically minimized with respect to these conditions. Let a' also be in K^n and $V' \subseteq K^n$ be defined likewise but with a replaced by a'. Suppose there are $\alpha, \alpha' \in F^k$ of the same 0-type in F and a system P of polynomials in $\mathbb{Q}[w, x]$ with $V = Z(P(\chi(\alpha), x))$ and $V' = Z(P(\chi(\alpha'), x))$. Then a and a' have the same 0-type.*

Proof. Using Lemma 1.5.8, we can arrange that K is $|F|^+$-saturated as a model of ACF. Suppose $a, a', V, V', \alpha, \alpha'$ and P are as stated. Then we get an automorphism σ_F of F mapping α to α'. By Corollary 1.3.3, this can be extended to an automorphism (σ_F, σ_K) of $(F, K; \chi)$. In particular,

$$\sigma_K : \chi(\alpha) \mapsto \chi(\alpha') \text{ and } \sigma_K(V) = V'.$$

Then V' contains $\sigma_K(a)$, is defined over $\mathbb{Q}\big(\chi(F^\times)\big)$ and $\big(\mathrm{r}_K(V'), \mathrm{d}_K(V')\big)$ achieves the minimum value under these conditions. Hence, for an algebraic set $W \subseteq K^n$ definable in the field K over $\mathbb{Q}\big(\chi(F^\times)\big)$,

$$\sigma(a) \in W \text{ if and only if } \big(\mathrm{r}_K(V' \cap W), \mathrm{d}_K(V' \cap W)\big) = \big(\mathrm{r}_K(V'), \mathrm{d}_K(V')\big).$$

By the choice of V', exactly the same statement holds when $\sigma_K(a)$ is replaced with a'. By the quantifier elimination of ACF, $\sigma_K(a)$ and a' have the same type over $\mathbb{Q}\big(\chi(F^\times)\big)$ in the field K. As K is $|F|^+$-saturated, there is an automorphism τ_K of K fixing $\mathbb{Q}\big(\chi(F^\times)\big)$ pointwise and mapping $\sigma_K(a)$ to a'. It is easy to check that $(\sigma_F, \tau_K \circ \sigma_K)$ is an automorphism of $(F, K; \chi)$ mapping a to a'. Therefore, a and a' have the same 0-type. $\square$

Theorem 1.5.12. *If $X \subseteq K^n$ is 0-definable, then X is a boolean combination of 0-algebraically presentable subsets of K^n.*

Proof. We say $a, a' \in K^n$ have the same 0-ap-type if they belong to the same 0-algebraically presentable sets. By changing the model, if needed, we can arrange that $(F, K; \chi)$ realizes all the 0-types. By Stone's representation theorem, it suffices to show that if a and a' are arbitrary elements in K^n with the same 0-ap-type then they have the same 0-type.

Fix a and a' in K^n with the same 0-ap-type. Choose $V \subseteq K^n$ containing a and definable in the field K over $\mathbb{Q}\big(\chi(F^\times)\big)$ such that

$$\big(\mathrm{r}_K(V), \mathrm{d}_K(V)\big) \text{ is lexicographically minimized with respect to these conditions.}$$

Moreover, pick k, $D \subseteq F^k$ 0-definable in the field F, $\alpha \in D$ and a system P of polynomials in $\mathbb{Q}[w, x]$ such that $V = Z\big(P(\chi(\alpha), x)\big)$ and

$$\big(\mathrm{r}_F(D), \mathrm{d}_F(D)\big) \text{ is minimized under these conditions.}$$

We will find α' and V' in order to use Lemma 1.5.11. Set

$$E = \Big\{\beta \in D : \text{ if } V_\beta = Z\Big(P\big(\chi(\beta), x\big)\Big), \text{ then } \big(\mathrm{r}_K(V_\beta), \mathrm{d}_K(V_\beta)\big) = \big(\mathrm{r}_K(V), \mathrm{d}_K(V)\big)\Big\}.$$

We note that E is 0-definable by Corollary 1.5.10, and so by Lemma 1.5.6, E is also 0-definable in the field F. As α is in E,

$$\big(\mathrm{r}_F(E), \mathrm{d}_F(E)\big) = \big(\mathrm{r}_F(D), \mathrm{d}_F(D)\big)$$

by the choice of D. Let S be the definable subset of K^n given by the presentation $\big(Z\big(P(\chi(\beta), x)\big)\big)_{\beta\in E}$. Then we have $a \in S$, and so we also have $a' \in S$ since a and a' have the same 0-ap-type. Hence, there is $\alpha' \in E$ such that a' is an element of $V' = Z\big(P(\chi(\alpha'), x)\big)$.

We next verify that α' and V' satisfy the conditions of Lemma 1.5.11. It will then follow that a and a' have the same 0-type. We first check that

$$\big(\mathrm{r}_K(V'), \mathrm{d}_K(V')\big) = \min\Big\{\big(\mathrm{r}_K(W'), \mathrm{d}_K(W')\big) : W' \subseteq K^n \text{ is } K\text{-algebraic, } a' \in W'\Big\}.$$

As α' is in E,

$$\big(\mathrm{r}_K(V'), \mathrm{d}_K(V')\big) = \big(\mathrm{r}_K(V), \mathrm{d}_K(V)\big).$$

Suppose towards a contradiction that there is an irreducible algebraic set $W' \subseteq K^n$ containing a' with

$$\big(\mathrm{r}_K(W'), \mathrm{d}_K(W')\big) <_{\mathrm{lex}} \big(\mathrm{r}_K(V'), \mathrm{d}_K(V')\big).$$

We can do the same construction as above in the reverse direction to get W'' with

$$\big(\mathrm{r}_K(W''), \mathrm{d}_K(W'')\big) <_{\mathrm{lex}} \big(\mathrm{r}_K(V), \mathrm{d}_K(V)\big)$$

containing a, a contradiction to the choice of V. We next check that α and α' have the same 0-type in the field F. Suppose otherwise. Let D' be the smallest 0-definable F-algebraic set containing α'. Then

$$\big(\mathrm{r}_F(D'), \mathrm{d}_F(D')\big) <_{\mathrm{lex}} \big(\mathrm{r}_F(D), \mathrm{d}_F(D)\big).$$

Do the same construction in the reverse direction again to get $\alpha'' \in D'$ such that a satisfies $P\big(\chi(\alpha''), x\big) = 0$.

If D'' is the smallest 0-definable F-algebraic set containing α'', then

$$\big(\mathrm{r}_F(D''), \mathrm{d}_F(D'')\big) \ \leq_{\mathrm{lex}} \ \big(\mathrm{r}_F(D'), \mathrm{d}_F(D')\big) \ <_{\mathrm{lex}} \ \big(\mathrm{r}_F(D), \mathrm{d}_F(D)\big),$$

a contradiction to our choice of D, α and P. □

Suppose $D \subseteq F^k$ is definable. By Proposition 1.5.1, D can be identified with $\chi(D)$, which has a simple description by the preceding theorem. In the rest of the section, we give an improvement of the above result for this special case. For a system P in $K[w]$, we abuse the notation and let $Z\big(P(\chi(s))\big) \subseteq F^k$ be the set defined by $P\big(\chi(s)\big) = 0$.

Lemma 1.5.13. *For each k there is a system Q of polynomials in $F[s]$ such that the set defined by $\chi(s_1) + \dots + \chi(s_k) = 0$ is $Z(Q)$.*

Proof. For $I \subseteq \{1, \dots, k\}$, let $\sum_{i \in I} \chi(s_i) \overset{\mathrm{nd}}{=} 0$ denote the system which consists of $\sum_{i \in I} \chi(s_i) = 0$ and $\sum_{i \in I'} \chi(s_i) \neq 0$ for each non-empty proper subset I' of I. By Mann's theorem, there are $\alpha^{(1)}, \dots, \alpha^{(l)}$ in F^I, such that the set defined by $\sum_{i \in I} \chi(s_i) \overset{\mathrm{nd}}{=} 0$ precisely consists of $\beta\alpha^{(j)}$ with $\beta \in F^\times$ and $j \in \{1, \dots, l\}$. Hence, if $I \subseteq \{1, \dots, k\}$, then there is a system Q_I of polynomials in $F[s]$ such that the set defined by $\sum_{i \in I} \chi(s_i) \overset{\mathrm{nd}}{=} 0$ together with the tuple $(0, \dots, 0)$ is $Z(Q_I)$.

Consider all the partitions $\mathcal{P}$ of the set $\{1, \dots, k\}$ into non-empty subsets. Then we have the set defined by $\chi(s_1) + \dots + \chi(s_k) = 0$ is $\bigcup_{\mathcal{P}} \bigcap_{I \in \mathcal{P}} Z(Q_I)$. Note that finite unions and finite intersections of F-algebraic sets are again F-algebraic. Thus, we can find a system Q of polynomials in $F[s]$ as desired. □

Lemma 1.5.14. *The map $\chi : F^k \to K^k$ is continuous.*

Proof. For the statement of the lemma, we need to show that if $V \subseteq K^k$ is K-closed then $\chi^{-1}(V)$ is F-closed. It suffices to show that if P is in $\mathbb{Q}[w, x]$ and a is a tuple of elements in K, then $Z\big(P(\chi(s), a)\big)$ is F-algebraic. Choose a linear basis B of $\mathbb{Q}\big(\chi(F^\times), a\big)$ over $\mathbb{Q}\big(\chi(F^\times)\big)$. Then

$$P\big(\chi(s), a\big) \ = \ P_1\big(\chi(s)\big)b_1 + \dots + P_m\big(\chi(s)\big)b_m$$

where P_i has coefficients in $\mathbb{Q}(\chi(F^\times))$, $b_i \in B$ for $i \in \{1, \dots, m\}$, and $b_i \neq b_j$ for distinct $i, j \in \{1, \dots, m\}$. Therefore, $P\big(\chi(s), a\big) = 0$ is equivalent to $P_i\big(\chi(s)\big) = 0$ for all $i \in \{1, \dots, m\}$. Furthermore, for each $i \in \{1, \dots, m\}$, $P_i\big(\chi(s)\big) = 0$ is equivalent to an equation of the form

$$\chi\big(M_1(s, \alpha)\big) + \dots + \chi\big(M_{l_i}(s, \alpha)\big) \ = \ 0$$

where α is a tuple of elements in F, and M_j is a monomial for $j \in \{1,\ldots,l_i\}$. By the result of the preceding lemma, for each $i \in \{1,\ldots,m\}$, the polynomial equation $P_i\big(\chi(s)\big) = 0$ is equivalent to a system $Q_i\big(M_1(s,\alpha),\ldots,M_{l_i}(s,\alpha)\big) = 0$. Thus, $Z\big(P(\chi(s),a)\big) = \bigcap_{i=1}^k Z\big(P_i(\chi(s))\big)$ is F-algebraic. □

Theorem 1.5.15. *Let D be a subset of F^k. If D is definable, then D is definable in the field F. Moreover, when D is 0-definable, D is 0-definable in the field F. If $D = \chi^{-1}(V)$ with K-algebraic $V \subseteq K^n$, then D is an F-algebraic set. Moreover, when $V = Z(P)$ with P a system in $\mathbb{Z}[w]$, $D = Z(Q)$ with Q a system in $\mathbb{Z}[s]$.*

Proof. We prove the first assertion. It suffices to show that if $X \subseteq K^k$ is definable, then $\chi^{-1}(X)$ is definable in the field F. By Theorem 1.5.12, we only need to show that if $S \subseteq K^{k+m}$ is 0-algebraically presentable and $X = \big\{a : (a,b) \in S\big\}$ with $b \in K^m$ then $\chi^{-1}(X)$ is definable in the field F. It is easy to see that X is defined by a formula of the form

$$\exists t\Big(\varphi(t) \wedge P\big(w,\chi(t)\big) = 0\Big) \text{ where } P \text{ is a system of polynomials in } K[w,x].$$

Let V be $Z(P)$. Then by the preceding lemma, $\chi^{-1}(V)$ is $Z(Q)$ for some system Q in $F[s,t]$. Hence, $\chi^{-1}(X)$, which is defined by $\exists t\big(\varphi(t) \wedge P(\chi(s),\chi(t)) = 0\big)$, is also defined by $\exists t\big(\varphi(t) \wedge Q(s,t) = 0\big)$. Thus, $\chi^{-1}(X)$ is definable in the field F as desired. The second assertion is just Lemma 1.5.6. The third assertion is Lemma 1.5.14. The forth assertion follows from the second and third assertions. □

Part II

Differential-Henselian Monotone Fields

Chapter 2

Introduction and Preliminaries

2.1 Introduction

Let $\boldsymbol{k}$ be a differential field (always of characteristic 0 in what follows, with a single distinguished derivation). Let also an ordered abelian group Γ be given. This gives rise to the Hahn field $K = \boldsymbol{k}((t^\Gamma))$, to be considered in the usual way as a valued field. We extend the derivation ∂ of $\boldsymbol{k}$ to a derivation on K by

$$\partial(\sum_\gamma a_\gamma t^\gamma) \; := \; \sum_\gamma \partial(a_\gamma) t^\gamma.$$

Scanlon [13] extends the Ax-Kochen-Ershov theorem (see [2], [4]) to this differential setting. This includes requiring that $\boldsymbol{k}$ is linearly surjective in the sense that for each nonzero linear differential operator $A = a_0 + a_1\partial + \cdots + a_n\partial^n$ over $\boldsymbol{k}$ we have $A(\boldsymbol{k}) = \boldsymbol{k}$. Under this assumption, K is differential-henselian (see Section 2.2 for this notion), and the theory $\mathrm{Th}(K)$ of K as a valued differential field (see also Section 2.2 for this) is completely axiomatized by:

(1) the axiom that there are many constants;

(2) the theory $\mathrm{Th}(\boldsymbol{k})$ of the differential residue field $\boldsymbol{k}$;

(3) the theory $\mathrm{Th}(\Gamma)$ of the ordered abelian value group;

(4) the axioms for differential-henselian valued fields.

As to (1), having many constants means that every element of the differential field has the same valuation as some element of its constant field. This holds for K as above (whether or not $\boldsymbol{k}$ is linearly surjective) because the constant field of K is $C_K = C_{\boldsymbol{k}}((t^\Gamma))$. This axiom plays an important role in some proofs of [13]. In Chapter 3, most of which is directly from [6], we drop the "many constants" axiom and generalize the theorem above to a much larger class of differential-henselian valued fields. This involves a more general way of extending the derivation of $\boldsymbol{k}$ to K.

In more detail, let $c : \Gamma \to \boldsymbol{k}$ be an additive map. Then the derivation ∂ of $\boldsymbol{k}$ extends to a derivation ∂_c of K by setting

$$\partial_c(\sum_\gamma a_\gamma t^\gamma) \ := \ \sum_\gamma \big(\partial(a_\gamma) + c(\gamma) a_\gamma\big) t^\gamma.$$

Thus ∂_c is the unique derivation on K that extends ∂, respects infinite sums, and satisfies $\partial_c(t^\gamma) = c(\gamma)t^\gamma$ for all γ. The earlier case has $c(\gamma) = 0$ for all γ. Another case is where $\boldsymbol{k}$ contains $\mathbb{R}$ as a subfield, $\Gamma = \mathbb{R}$, and $c : \mathbb{R} \to \boldsymbol{k}$ is the inclusion map; then $\partial_c(t^r) = rt^r$ for $r \in \mathbb{R}$.

Let K_c be the valued differential field K with ∂_c as its distinguished derivation. Assume in addition that $\boldsymbol{k}$ is linearly surjective. Then K_c is differential-henselian, and Scanlon's theorem above generalizes as follows:

Theorem 2.1.1. *The theory* $\mathrm{Th}(K_c)$ *is completely determined by* $\mathrm{Th}(\boldsymbol{k}, \Gamma; c)$*, where* $(\boldsymbol{k}, \Gamma; c)$ *is the* 2*-sorted structure consisting of the differential field* $\boldsymbol{k}$*, the ordered abelian group* Γ*, and the additive map* $c : \Gamma \to \boldsymbol{k}$*.*

We actually prove in Section 3.1 a stronger version with the one-sorted structure K_c expanded to a 2-sorted one, with Γ as the underlying set for the second sort, and as extra primitives the cross-section $\gamma \mapsto t^\gamma : \Gamma \to K$, the set $\boldsymbol{k} \subseteq K$, and the map $c : \Gamma \to \boldsymbol{k}$.

The question arises: which complete theories of valued differential fields are covered by Theorem 2.1.1? The answer involves the notion of monotonicity: a valued differential field F with valuation v is said to be *monotone* if $v(f') \geq v(f)$ for all $f \in F$; as usual, f' denotes the derivative of $f \in F$ with respect to the distinguished derivation of F. The valued differential fields K_c are all clearly monotone. We show:

Theorem 2.1.2. *Every differential-henselian monotone valued field is elementarily equivalent to some* K_c *as in Theorem 2.1.1.*

This is proved in Section 3.2 and is analogous to the result from [13] that any differential-henselian valued field with many constants is elementarily equivalent to some K as in Scanlon's theorem stated in the beginning of this Introduction. (In fact, that result follows from the "complete axiomatization" given in that theorem.)

Theorem 2.1.2 has a nice algebraic consequence, generalizing [1, Corollary 8.0.2]:

Corollary 2.1.3. *If a valued differential field* F *is monotone and differential-henselian, then every valued differential field extension of* F *that is algebraic over* F *is also (monotone and) differential-henselian.*

See Section 3.3. To state further results it is convenient to introduce some notation. Let F be a differential field. For nonzero $f \in F$ we set $f^\dagger := f'/f$ and $F^\dagger \ := \ \{f^\dagger \ : \ f \ \in \ F^\times\}$, where $F^\times := F \setminus \{0\}$.

So far our only assumption on $c : \Gamma \to \boldsymbol{k}$ is that it is additive, but the case $c(\Gamma) \cap \boldsymbol{k}^\dagger = \{0\}$ is of particular interest: it is not hard to show that then the constant field of K_c is $C_{\boldsymbol{k}}((t^\Delta))$, where the value group Δ of the constant field equals $\ker(c)$ and is a pure subgroup of Γ. Conversely (see Section 3.2):

Theorem 2.1.4. *Every differential-henselian monotone valued field F such that $v(C_F^\times)$ is pure in $v(F^\times)$ is elementarily equivalent to some K_c as in Theorem 2.1.1 with $c(\Gamma) \cap \boldsymbol{k}^\dagger = \{0\}$.*

The referee showed us an example of a monotone henselian valued differential field F for which $v(C_F^\times)$ is not pure in $v(F^\times)$. In Section 3.3 we give an example of a differential-henselian monotone field F such that $v(C_F^\times)$ is not pure in $v(F^\times)$.

The hypothesis of Theorem 2.1.4 that $v(C_F^\times)$ is pure in $v(F^\times)$ holds if the residue field is algebraically closed or real closed (see Section 3.3). It includes also the case of main interest to us, where F has few constants, that is, the valuation is trivial on C_F. In that case any c as in Theorem 2.1.4 is injective by Corollary 3.2.2.

Section 3.2 contains examples of additive maps $c : \Gamma \to \boldsymbol{k}$ for which K_c has few constants, including a case where $\mathrm{Th}(K_c)$ is decidable. Two of those examples show that in Theorem 2.1.1, even when we have few constants, the traditional Ax-Kochen-Ershov principle without the map c does not hold. (It does hold in Scanlon's theorem where $c = 0$, but in general we do not expect to have a c that is definable in the valued differential field structure.)

Chapter 4, most of which is directly from [7], is a sequel to Chapter 3 and focuses on definability in differential-henselian monotone fields, similar to Scanlon [14].

More generally we consider there, as in Section 3.4, three-sorted structures

$$\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c)$$

where K and $\mathbf{k}$ are differential fields, Γ is an ordered abelian group, $v : K^\times \to \Gamma$ is a valuation which makes K into a valued differential field with a valuation ring $\mathcal{O} := \mathcal{O}_v$ such that K is monotone, $\pi : \mathcal{O} \to \mathbf{k}$ is a surjective differential ring morphism, $c : \Gamma \to \mathbf{k}$ is an additive map satisfying $\forall \gamma \exists x \neq 0 \big(v(x) = \gamma \ \&\ \pi(x^\dagger) = c(\gamma)\big)$. To make the maps π and v total, we add a formal symbol ∞ to the sorts of the residue field and the value group as default values, i.e. $\pi(x) = \infty$ if and only if $x \in K \smallsetminus \mathcal{O}$, and $v(x) = \infty$ if and only if $x = 0$. We construe these $\mathcal{K}$ as L_3-structures for a natural 3-sorted language L_3 (with unary function symbols for π, v and c). We have an obvious set $\mathrm{Mo}(c)$ of L_3-sentences whose models are exactly these $\mathcal{K}$. See Section 3.4 for details.

To study definability we consider, as in Scanlon [14], expansions of such $\mathcal{K}$ by an angular component map $\mathrm{ac} : K \to \mathbf{k}$ on $\mathcal{K}$. In Section 4.1 we introduce a suitable notion of *angular component map on* $\mathcal{K}$ (the definition is not completely obvious) and show its existence if $\mathcal{K}$ is $\aleph_1$-saturated.

To state our results precisely, we need to be more specific about the language. The language L_3 has three sorts: f (the main field sort), r (the residue field sort), and v (the value group sort); it contains an f-copy $L_{\mathrm{f}} = \{0, 1, -, +, \cdot, \partial\}$ of the language of differential rings, an r-copy $L_{\mathrm{r}} = \{0, 1, -, +, \cdot, \bar{\partial}\}$ of this language, and a v-copy $L_{\mathrm{v}} = \{\leq, 0, -, +\}$ of the language of ordered abelian groups, with disjoint L_{f}, L_{r}, L_{v}. It also has the unary function symbols π, v, c of (mixed) sorts fr, fv, vr, respectively. This completes the description of L_3. Let L_{rv} be the 2-sorted sublanguage of L_3 consisting of L_{r} and L_{v}, and the function symbol c. By $L_3(\mathrm{ac})$ we mean L_3 augmented by a new unary function symbol ac of sort fr. Let T be the $L_3(\mathrm{ac})$-theory of d-henselian monotone valued differential fields with angular component map. Let the $L_3(\mathrm{ac})$-structures

$$\mathcal{K}_1 \ = \ (K_1, \mathbf{k}_1, \Gamma_1;\ \pi_1, v_1, c_1, \mathrm{ac}_1), \qquad \mathcal{K}_2 \ = \ (K_2, \mathbf{k}_2, \Gamma_2;\ \pi_2, v_2, c_2, \mathrm{ac}_2)$$

be models of T. The main result of Chapter 4 is the Equivalence Theorem 4.2.1 among whose consequences are the following (see Section 4.3):

Theorem 2.1.5. *If $\mathcal{K}_1 \subseteq \mathcal{K}_2$ and $(\mathbf{k}_1, \Gamma_1; c_1) \preccurlyeq_{L_{\mathrm{rv}}} (\mathbf{k}_2, \Gamma_2; c_2)$, then $\mathcal{K}_1 \preccurlyeq \mathcal{K}_2$.*

Here "$\subseteq$" means "substructure of". We also derive a relative quantifier reduction result. This uses a technical notion of *special formula* whose definition can be found in Section 4.3.

Theorem 2.1.6. *Every $L_3(\mathrm{ac})$-formula is T-equivalent to a special $L_3(\mathrm{ac})$-formula.*

We use this to prove the following for models $\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c, \mathrm{ac})$ of T:

Corollary 2.1.7. *If a set $X \subseteq \mathbf{k}^m \times \Gamma^n$ is definable in $\mathcal{K}$, then X is already definable in the L_{rv}-structure $(\mathbf{k}, \Gamma; c)$.*

Corollary 2.1.8. *$\mathcal{K}$ has* NIP *if and only if the 2-sorted structure $(\mathbf{k}, \Gamma; c)$ has* NIP*.*

We also show how to eliminate the angular component maps from Theorem 2.1.5 and Corollaries 2.1.7 and 2.1.8.

2.2 Preliminaries

Adopting terminology from [1], a *valued differential field* is a differential field K together with a (Krull) valuation $v : K^\times \to \Gamma$ whose residue field $\mathbf{k} := \mathcal{O}/\mathcal{o}$ has characteristic zero; here $\Gamma = v(K^\times)$ is the value group, and we also let $\mathcal{O} = \mathcal{O}_K$ denote the valuation ring of v with maximal ideal $\mathcal{o}$, and let

$$C \ = \ C_K \ := \ \{f \in K :\ f' = 0\}$$

denote the constant field of the differential field K. We use notation from [1]: for elements a, b of a valued field with valuation v we set

$$a \asymp b :\Leftrightarrow va = vb, \quad a \preceq b \Leftrightarrow b \succeq a :\Leftrightarrow va \geq vb, \quad a \prec b \Leftrightarrow b \succ a :\Leftrightarrow va > vb.$$

Let K be a valued differential field as above, and let ∂ be its derivation. We say that K has *many constants* if $v(C^\times) = \Gamma$. We say that the derivation of K is *small* if $\partial(\smallo) \subseteq \smallo$. If K, with a small derivation, has many constants, then K is *monotone* in the sense of [3], that is, $v(f) \leq v(f')$ for all $f \in K$. We say that K has *few constants* if $v(C^\times) = \{0\}$. Note: if K is monotone, then its derivation is small; if the derivation of K is small, then ∂ is continuous with respect to the valuation topology on K. Note also that if K is monotone, then so is any valued differential field extension with small derivation and the same value group as K.

From now on we assume that the derivation of K is small. This has the effect (see [3] or [1, Lemma 4.4.2]) that also $\partial(\mathcal{O}) \subseteq \mathcal{O}$, and so ∂ induces a derivation on the residue field; we view $\mathbf{k}$ below as equipped with this induced derivation, and refer to it as the *differential residue field of K*.

We say that K is *differential-henselian* (for short: d-*henselian*) if every differential polynomial $P \in \mathcal{O}\{Y\} = \mathcal{O}[Y, Y', Y'', \dots]$ whose reduction $\overline{P} \in \mathbf{k}\{Y\}$ has total degree 1 has a zero in $\mathcal{O}$. (Note that for ordinary polynomials $P \in \mathcal{O}[Y]$ this requirement defines the usual notion of a henselian valued field, that is, a valued field whose valuation ring is henselian as a local ring.)

If K is d-henselian, then its differential residue field is clearly *linearly surjective*: any linear differential equation $y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_0 y = b$ with coefficients $a_i, b \in \mathbf{k}$ has a solution in $\mathbf{k}$. This is a key constraint on our notion of d-henselianity. If K is d-henselian, then $\mathbf{k}$ has a *lift to K*, meaning, a differential subfield of K contained in $\mathcal{O}$ that maps isomorphically onto $\mathbf{k}$ under the canonical map from $\mathcal{O}$ onto $\mathbf{k}$; see [1, 7.1.3]. Other items from [1] that are relevant in this chapter are the following differential analogues of Hensel's Lemma and of results due to Ostrowski/Krull/Kaplansky on valued fields:

(DV1) If the derivation of $\mathbf{k}$ is nontrivial, then K has a spherically complete immediate valued differential field extension with small derivation; [1, 6.9.5].

(DV2) If $\mathbf{k}$ is linearly surjective and K is spherically complete, then K is d-henselian; [1, 7.0.2].

(DV3) If $\mathbf{k}$ is linearly surjective and K is monotone, then any two spherically complete immediate monotone valued differential field extensions of K are isomorphic over K; [1, 7.4.3].

We also need a model-theoretic variant of (DV3):

(DV4) Suppose $\mathbf{k}$ is linearly surjective and K is monotone. Let $K^{\bullet}$ be a spherically complete immediate valued differential field extension of K with small derivation. Then $K^{\bullet}$ can be embedded over K into any $|v(K^{\times})|^{+}$-saturated d-henselian monotone valued differential field extension of K; [1, 7.4.5].

In Chapter 4 we will need an auxiliary result presented below. Consider 3-sorted structures

$$\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c)$$

where K and $\mathbf{k}$ are differential fields, Γ is an ordered abelian group, $v : K^{\times} \to \Gamma$ is a valuation which makes K into a monotone valued differential field, $\pi : \mathcal{O} \to \mathbf{k}$ with $\mathcal{O} := \mathcal{O}_v$ is a surjective differential ring morphism, $c : \Gamma \to \mathbf{k}$ is an additive map satisfying $\forall \gamma \exists x \neq 0 \big(v(x) = \gamma \;\&\; \pi(x^{\dagger}) = c(\gamma)\big)$. We construe these $\mathcal{K}$ as L_3-structures for the 3-sorted language L_3 described in the previous section. We also have an obvious set $\mathrm{Mo}(c)$ (see also Section 3.4) of L_3-sentences whose models are exactly these $\mathcal{K}$.

Lemma 2.2.1. *Let $\mathcal{K} \vDash \mathrm{Mo}(c)$ be as above and $b \in K^{\times}$. Then $\pi(b^{\dagger}) = \pi(a^{\dagger}) + c(v(b))$ for some $a \asymp 1$ in K.*

Proof. Take $x \in K^{\times}$ with $x \asymp b$ and $\pi(x^{\dagger}) = c(v(b))$. Then $a := b/x$ works. $\square$

Chapter 3

An Ax-Kochen-Ershov Theorem for Differential-Henselian Monotone Fields

3.1 Elementary equivalence of differential-henselian monotone fields

In this section we obtain Theorem 2.1.1 from the introduction as a consequence of a more precise result in a 2-sorted setting. We consider 2-sorted structures

$$\mathcal{K} \;=\; (K, \Gamma; v, s, c),$$

where K is a differential field equipped with a differential subfield $\boldsymbol{k}$ (singled out by a unary predicate symbol), Γ is an ordered abelian group, $v : K^\times \to \Gamma = v(K^\times)$ is a valuation that makes K into a monotone valued differential field such that $\boldsymbol{k} \subseteq K$ is a lift of the differential residue field, $s : \Gamma \to K^\times$ is a cross-section of v (that is, s is a group morphism and $v \circ s = \mathrm{id}_\Gamma$), and $c : \Gamma \to \boldsymbol{k}$ satisfies $c(\gamma) = s(\gamma)^\dagger$ for all $\gamma \in \Gamma$ (so c is additive). We construe these $\mathcal{K}$ as L_2-structures for a natural 2-sorted language L_2 (with unary function symbols for v, s, and c). We have an obvious set $\mathrm{Mo}(\ell, s, c)$ of L_2-sentences whose models are exactly these $\mathcal{K}$; the "ℓ" is to indicate the presence of a lift.

For example, for $K = \boldsymbol{k}((t^\Gamma))$ as in the introduction and additive $c : \Gamma \to \boldsymbol{k}$ we consider K_c as a model of $\mathrm{Mo}(\ell, s, c)$ in the obvious way by taking $\boldsymbol{k} \subseteq K$ as lift, and $\gamma \mapsto t^\gamma$ as cross-section.

Theorem 3.1.1. *If $\mathcal{K}$ is* d*-henselian, then* $\mathrm{Th}(\mathcal{K})$ *is axiomatized by:*

(1) $\mathrm{Mo}(\ell, s, c)$*;*

(2) *the axioms for* d*-henselianity;*

(3) *$Th(\boldsymbol{k}, \Gamma; c)$ with $\boldsymbol{k}$ as differential field and Γ as ordered abelian group.*

We first develop the required technical material, and give the proof of this theorem at the end of this section. Until further notice, $\mathcal{K} = (K, \Gamma; \boldsymbol{k}, v, s, c) \models \mathrm{Mo}(\ell, s, c)$. For any subfield E of K we set $\Gamma_E := v(E^\times)$.

We define a *good subfield of* $\mathcal{K}$ to be a differential subfield of K such that (i) $\boldsymbol{k} \subseteq E$, (ii) $s(\Gamma_E) \subseteq E$, and (iii) $|\Gamma_E| \leq \aleph_0$. Thus $\boldsymbol{k}$ is a good subfield of $\mathcal{K}$.

Lemma 3.1.2. *Let E be a good subfield of $\mathcal{K}$ and $x \in K \smallsetminus E$. Then $|\Gamma_{E(x)}| \leq \aleph_0$.*

This is well-known; see for example [1, Lemma 3.1.10].

Lemma 3.1.3. *Let $E \subseteq K$ be a good subfield of $\mathcal{K}$ and $\gamma \in \Gamma \smallsetminus \Gamma_E$, that is, $s(\gamma) \notin E$. Then $E(s(\gamma))$ is also a good subfield of $\mathcal{K}$.*

Proof. From $c(\gamma) \in \boldsymbol{k} \subseteq E$ and $s(\gamma)' = c(\gamma)s(\gamma)$ we get that $E(s(\gamma))$ is a differential subfield of K and that condition (i) for being a good subfield is satisfied by $E(s(\gamma))$. For condition (ii) we distinguish two cases:

(1) $n\gamma \in \Gamma_E$ for some $n \in \mathbb{N}^{\geq 1}$. Take $n \geq 1$ minimal with $n\gamma \in \Gamma_E$. Then $0, \gamma, 2\gamma, \ldots, (n-1)\gamma$ are in different cosets of Γ_E, so for every $q(X) \in E[X]^{\neq}$ of degree $< n$ we get $q(s(\gamma)) \neq 0$. Hence the minimum polynomial of $s(\gamma)$ over E is $X^n - s(n\gamma)$. Thus, given any $x \in E(s(\gamma))^{\times}$, we have

$$x \;=\; q_0 + q_1 s(\gamma) + \cdots + q_{n-1} s(\gamma)^{n-1}$$

with $q_0, \ldots, q_{n-1} \in E$, not all 0, so $v(x) = \min\limits_{i=0,\ldots,n-1}\{v(q_i) + i\gamma\}$. Therefore, $\Gamma_{E(s(\gamma))} = \Gamma_E + \mathbb{Z}\gamma$ and hence $s(\Gamma_{E(s(\gamma))}) \subseteq s(\Gamma_E) \cdot s(\gamma)^{\mathbb{Z}} \subseteq E(s(\gamma))$.

(2) $n\gamma \notin \Gamma_E$ for all $n \in \mathbb{N}^{\geq 1}$. Then $0, \gamma, 2\gamma, \ldots$ are in different cosets of Γ_E, so $s(\gamma)$ is transcendental over E and for any polynomial $q(X) = q_0 + q_1 X + \cdots + q_n X^n \in E[X]$, we have $v(q(s(\gamma))) = \min\limits_{i=0,\ldots,n}\{v(q_i) + i\gamma\}$. As in case (1) this yields $\Gamma_{E(s(\gamma))} = \Gamma_E + \mathbb{Z}\gamma$ and so $s(\Gamma_{E(s(\gamma))}) \subseteq s(\Gamma_E) \cdot s(\gamma)^{\mathbb{Z}} \subseteq E(s(\gamma))$.

Thus condition (ii) of good subfields holds for $E(s(\gamma))$. Condition (iii) is satisfied by Lemma 3.1.2. $\square$

In the rest of this section we fix a d*-henselian* $\mathcal{K}$*.* Let $T_{\mathcal{K}}$ be the L_2-theory given by (1)–(3) in Theorem 3.1.1. Assume CH (the Continuum Hypothesis), and let

$$\mathcal{K}_1 \;=\; (K_1, \Gamma_1; v_1, s_1, c_1), \qquad \mathcal{K}_2 \;=\; (K_2, \Gamma_2; v_2, s_2, c_2)$$

be saturated models of $T_{\mathcal{K}}$ of cardinality $\aleph_1$; remarks following Corollary 3.1.6 explain why we can assume CH. Then the structures $(\boldsymbol{k}_1, \Gamma_1; c_1)$ and $(\boldsymbol{k}_2, \Gamma_2; c_2)$ are also saturated of cardinality $\aleph_1$, where $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ are the lifts of the differential residue fields of K_1 and K_2 respectively. Since $(\boldsymbol{k}_1, \Gamma_1; c_1)$ and $(\boldsymbol{k}_2, \Gamma_2; c_2)$ are elementarily equivalent to $(\boldsymbol{k}, \Gamma; c)$, we have an isomorphism $f = (f_r, f_v)$ from $(\boldsymbol{k}_1, \Gamma_1; c_1)$ onto $(\boldsymbol{k}_2, \Gamma_2; c_2)$ with $f_r : \boldsymbol{k}_1 \to \boldsymbol{k}_2$ and $f_v : \Gamma_1 \to \Gamma_2$.

A map $g : E_1 \to E_2$ between good subfields E_1 and E_2 of $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, will be called *good* if

(1) $g : E_1 \to E_2$ is a differential field isomorphism,

(2) g extends f_r,

(3) $f_v \circ v_1 = v_2 \circ g$,

(4) $g \circ s_1 = s_2 \circ f_v$.

Note that then g is also an isomorphism of the valued subfield E_1 of K_1 onto the valued subfield E_2 of K_2. The map $f_r : \boldsymbol{k}_1 \to \boldsymbol{k}_2$ is clearly a good map.

Proposition 3.1.4. $\mathcal{K}_1 \cong \mathcal{K}_2$.

Proof. We claim that the collection of good maps is a back-and-forth system between K_1 and K_2. (By the saturation assumption this yields the desired result.) This claim holds trivially if $\Gamma_1 = \{0\}$, so assume $\Gamma_1 \neq \{0\}$, and thus $\Gamma_2 \neq \{0\}$.

Let $g : E_1 \to E_2$ be a good map and $\gamma \in \Gamma_1 \smallsetminus \Gamma_{E_1}$. By Lemma 3.1.3 we have good subfields $E_1\big(s_1(\gamma)\big)$ of $\mathcal{K}_1$ and $E_2\big(s_2(f_v(\gamma))\big)$ of $\mathcal{K}_2$. The proof of that lemma then yields easily a good map

$$g_\gamma : E_1\big(s_1(\gamma)\big) \to E_2\big(s_2(f_v(\gamma))\big)$$

that extends g with $g_\gamma\big(s_1(\gamma)\big) = s_2\big(f_v(\gamma)\big)$.

Let $g : E_1 \to E_2$ be a good map and $x \in K_1 \smallsetminus E_1$. We show how to extend g to a good map with x in its domain.

By condition (i) of being a good subfield, $E_1 \supseteq \boldsymbol{k}_1$ and $E_2 \supseteq \boldsymbol{k}_2$. The group $\Gamma_{E_1\langle x\rangle}$ is countable by Lemma 3.1.2. Thus by applying iteratively the construction above to elements $\gamma \in \Gamma_{E_1\langle x\rangle}$, we can extend g to a good map $g^1 : E_1^1 \to E_2^1$ with $\Gamma_{E_1^1} = \Gamma_{E_1\langle x\rangle}$. Likewise we can extend g^1 to a good map $g^2 : E_1^2 \to E_2^2$ with $\Gamma_{E_1^2} = \Gamma_{E_1^1\langle x\rangle}$. Iterating this process and taking the union $E_i^\infty = \bigcup_n E_i^n$, for $i = 1, 2$, we get a good map $g^\infty : E_1^\infty \to E_2^\infty$ extending g such that $\Gamma_{E_1^\infty} = \Gamma_{E_1^\infty\langle x\rangle}$, so the valued differential field extension $E_1^\infty\langle x\rangle$ of E_1^∞ is immediate. By (DV1) and (DV4) we have a spherically complete immediate valued differential field extension $E_1^\bullet \subseteq K_1$ of $E_1^\infty\langle x\rangle$. Note that then $E_1^\bullet$ is also a spherically complete immediate valued differential field extension of E_1^∞. Likewise we have a spherically complete immediate valued differential field extension $E_2^\bullet \subseteq K_2$ of E_2^∞. By (DV3) we can extend g^∞ to a valued differential field isomorphism $g^\bullet : E_1^\bullet \to E_2^\bullet$. It is clear that then $g^\bullet$ is a good map extending g with x in its domain.

This finishes the proof of the *forth* part. The *back* part is done likewise. $\square$

Proof of Theorem 3.1.1. We can assume the Continuum Hypothesis (CH) for this argument. (This is explained further in the remarks following Corollary 3.1.6.) Our job is to show that the theory $T_{\mathcal{K}}$ is complete. In other words, given any two models of $T_{\mathcal{K}}$ we need to show they are elementarily equivalent. Using CH we can assume that these models are saturated of cardinality $\aleph_1$, and so they are indeed isomorphic by Proposition 3.1.4. □

Note that Theorem 2.1.1 is a consequence of Theorem 3.1.1.

Corollary 3.1.5. *Suppose* $\mathcal{K}_1 = (K_1, \Gamma_1; v_1, s_1, c_1)$ *and* $\mathcal{K}_2 = (K_2, \Gamma_2; v_2, s_2, c_2)$ *are* d*-henselian models of* $\mathrm{Mo}(\ell, c, s)$*. Then:* $\mathcal{K}_1 \equiv \mathcal{K}_2 \Longleftrightarrow (\boldsymbol{k}_1, \Gamma_1; c_1) \equiv (\boldsymbol{k}_2, \Gamma_2; c_2)$.

In connection with eliminating the use of CH we introduce the L_2-theory T whose models are the d-henselian models of $\mathrm{Mo}(\ell, s, c)$. The structures $(\boldsymbol{k}, \Gamma; c)$ where $\boldsymbol{k}$ is a differential field, Γ is an ordered abelian group, and $c : \Gamma \to \boldsymbol{k}$, are L_c-structures for a certain sublanguage L_c of L_2. Now Corollary 3.1.5 yields:

Corollary 3.1.6. *Every* L_2*-sentence is* T*-equivalent to some* L_c*-sentence.*

The above proof of Corollary 3.1.6 depends on CH, but T has an explicit axiomatization and so the statement of this corollary is "arithmetic". Therefore this proof can be converted to one using just ZFC (without CH). Thus as an obvious consequence of Corollary 3.1.6, Theorem 3.1.1 also holds without assuming CH.

3.2 Existence of $\boldsymbol{k}$, s, c

In this section we construct under certain conditions a lift $\boldsymbol{k}$, a cross-section s, and a map c as in the previous section.

Proposition 3.2.1. *Assume* $\mathcal{K} = (K, \Gamma; v, s, c) \models \mathrm{Mo}(\ell, c, s)$*. Then*

$$s(\ker(c)) = C^\times \cap s(\Gamma) \quad (\textit{so } \ker(c) \subseteq v(C^\times)), \qquad c\big(v(C^\times)\big) \subseteq \boldsymbol{k}^\dagger,$$
$$c(\Gamma) \cap \boldsymbol{k}^\dagger = \{0\} \iff \ker(c) = v(C^\times).$$

Proof. Let $\gamma \in \Gamma$. If $c(\gamma) = 0$, then $s(\gamma)^\dagger = 0$, so $s(\gamma) \in C^\times \cap s(\Gamma)$. If $s(\gamma) \in C^\times$, then $c(\gamma) = s(\gamma)^\dagger = 0$, so $\gamma \in \ker(c)$. This proves the first equality. Next, for the inclusion $c\big(v(C^\times)\big) \subseteq \boldsymbol{k}^\dagger$, suppose $\gamma = va$ with $a \in C^\times$. Then $s(\gamma) = ua$ with $u \asymp 1$ in K, so $u = d(1+\epsilon)$ with $d \in \boldsymbol{k}^\times$ and $\epsilon \prec 1$. Hence

$$c(\gamma) = s(\gamma)^\dagger = u^\dagger = d^\dagger + (1+\epsilon)^\dagger = d^\dagger + \frac{\epsilon'}{1+\epsilon}.$$

Since $c(\gamma), d^\dagger \in \boldsymbol{k}$ and $\epsilon' \prec 1$, this gives $\epsilon' = 0$, so $c(\gamma) \in \boldsymbol{k}^\dagger$, as claimed. As to the equivalence, suppose $c(\Gamma) \cap \boldsymbol{k}^\dagger = \{0\}$. Then $c\big(v(C^\times)\big) = \{0\}$ by the inclusion that we just proved, so $v(C^\times) \subseteq \ker(c)$. We already have the reverse inclusion, so $\ker(c) = v(C^\times)$. For the converse, assume $\ker(c) = v(C^\times)$. Let $\gamma \in \Gamma$ be such that $c(\gamma) = d^\dagger$ with $d \in \boldsymbol{k}^\times$. Then $s(\gamma)^\dagger = d^\dagger$, so $s(\gamma)/d \in C^\times$, hence $\gamma = v\big(s(\gamma)/d\big) \in v(C^\times)$, and thus $c(\gamma) = 0$, as claimed. □

Examples where $c(\Gamma) \cap \boldsymbol{k}^\dagger \neq \{0\}$: Take any differential field $\boldsymbol{k}$ with $\boldsymbol{k} \neq C_{\boldsymbol{k}}$, and take $\Gamma = \mathbb{Z}$. Then $\boldsymbol{k}^\dagger \neq \{0\}$; take any nonzero element $u \in \boldsymbol{k}^\dagger$. Then for the additive map $c : \Gamma \to \boldsymbol{k}$ given by $c(1) = u$ we have $c(\Gamma) = \mathbb{Z}u \subseteq \boldsymbol{k}^\dagger$, and so $\boldsymbol{k}((t^\Gamma))_c$ is a model of $\mathrm{Mo}(\ell, c, s)$ with $c(\Gamma) \cap \boldsymbol{k}^\dagger \neq \{0\}$. By taking $\boldsymbol{k}$ to be linearly surjective, this model is d-henselian.

An example where $c(\Gamma) \cap \boldsymbol{k}^\dagger = \{0\}$: Take $\boldsymbol{k} = \mathbb{T}_{\log}$, the differential field of logarithmic transseries; see [1, Chapter 15 and Appendix A] about $\mathbb{T}_{\log}$, especially the fact that $\mathbb{T}_{\log}$ is linearly surjective. Also $\mathbb{T}_{\log}$ contains $\mathbb{R}$ as a subfield, and $f^\dagger \notin \mathbb{R}$ for all nonzero $f \in \mathbb{T}_{\log}$. Next, take $\Gamma = \mathbb{R}$ and define $c : \Gamma \to \boldsymbol{k}$ by $c(r) = r$. Then $K := \boldsymbol{k}((t^\Gamma))$ yields a d-henselian model K_c of $\mathrm{Mo}(\ell, c, s)$ with $c(\Gamma) \cap \boldsymbol{k}^\dagger = \{0\}$. Allen Gehret conjectured an axiomatization of $\mathrm{Th}(\mathbb{T}_{\log})$ that would imply its decidability, and thus the decidability of the theory of K_c. This K_c has few constants by the following obvious consequence of Proposition 3.2.1:

Corollary 3.2.2. *Suppose $\mathcal{K} = (K, \Gamma; v, s, c) \models \mathrm{Mo}(\ell, c, s)$. Then:*

$$c \text{ is injective and } c(\Gamma) \cap \boldsymbol{k}^\dagger = \{0\} \iff \mathcal{K} \text{ has few constants.}$$

We now provide an example to show that in Theorem 2.1.1 we cannot drop the map c in the case of few constants. Take $\boldsymbol{k} = \mathbb{T}_{\log}$ and $\Gamma = \mathbb{Z}$. Define the additive maps $c_1 : \Gamma \to \boldsymbol{k}$ by $c_1(1) = 1$ and $c_2 : \Gamma \to \boldsymbol{k}$ by $c_2(1) = \sqrt{2}$; instead of $\sqrt{2}$, any irrational real number will do. Let $K_1 := \boldsymbol{k}((t^\Gamma))$ and $K_2 := \boldsymbol{k}((t^\Gamma))$ be the differential Hahn fields with derivations defined as in the introduction using the maps c_1 and c_2, respectively. They are d-henselian monotone valued differential fields. As in the previous example they have few constants by Corollary 3.2.2. We claim that K_1 and K_2 are not elementarily equivalent as valued differential fields (without c_1 and c_2 as primitives), so the traditional Ax-Kochen-Ershov principle does not hold. In K_1, we have $t^\dagger = c(1) = 1$ and so $K_1 \models \exists a \neq 0 (a^\dagger = 1)$. We now show that $K_2 \not\models \exists a \neq 0 (a^\dagger = 1)$. Towards a contradiction, assume $a \in K_2^\times$ is such that $a^\dagger = 1$. Then $a = t^k d(1 + \epsilon)$ with $k \in \mathbb{Z}$, $d \in \boldsymbol{k}^\times$ and $\epsilon \in K_2$ with $\epsilon \prec 1$. Hence $a^\dagger = c_2(k) + d^\dagger + (1 + \epsilon)^\dagger$, so

$$k\sqrt{2} + d^\dagger + \frac{\epsilon'}{1 + \epsilon} = 1.$$

Since $\epsilon' \prec 1$ we get $k\sqrt{2} + d^\dagger = 1$ and $\epsilon' = 0$. Thus $d^\dagger = 1 - k\sqrt{2} \in \mathbb{R}$. Since $1 - k\sqrt{2} \neq 0$, this contradicts $\mathbb{T}^\dagger_{\log} \cap \mathbb{R} = \{0\}$.

Next we give an example of a decidable d-henselian monotone valued differential field with few constants. The valued differential field $\mathbb{T}$ of transseries is linearly surjective by [1, Corollary 15.0.2] and [1, Corollary 14.2.2]. As $\mathbb{T}[\mathrm{i}]$ with $\mathrm{i}^2 = -1$ is algebraic over $\mathbb{T}$, it is also linearly surjective by [1, Corollary 5.4.3]. The proof of [1, Proposition 10.7.10] gives $(\mathbb{T}[\mathrm{i}]^\times)^\dagger = \mathbb{T} + \mathrm{i}\partial\smallo$, where $\smallo$ is the maximal ideal of the valuation ring of $\mathbb{T}$. Thus taking $\boldsymbol{k} = \mathbb{T}[\mathrm{i}]$, $\Gamma = \mathbb{R}$ and the additive map $c : \Gamma \to \boldsymbol{k}$ given by $c(r) = \mathrm{i}r$, we have $c(\Gamma) \cap \boldsymbol{k}^\dagger = \mathrm{i}\mathbb{R} \cap (\mathbb{T} + \mathrm{i}\partial\smallo) = \{0\}$ and therefore $K := \mathbb{T}[\mathrm{i}]((t^{\mathbb{R}}))_c$ will be a d-henselian monotone valued differential field with few constants by Corollary 3.2.2. Moreover, $\mathrm{Th}(K)$ is decidable by Theorem 2.1.1, since the 2-sorted structure $(\mathbb{T}[\mathrm{i}], \mathbb{R}; c)$ is interpretable in the valued differential field $\mathbb{T}$ and the latter has decidable theory by [1, Corollary 16.6.3].

In what follows in this chapter, we fix a differential field K with a valuation $v : K^\times \to \Gamma = v(K^\times)$ such that $(K, \Gamma; v)$ is a monotone valued differential field.

Lemma 3.2.3. *Suppose $(K, \Gamma; v)$ is* d*-henselian and $\boldsymbol{k}$ is a lift of its differential residue field. Then $G := \{a \in K^\times : a^\dagger \in \boldsymbol{k}\}$ is a subgroup of $K^\times$ with $v(G) = \Gamma$.*

Proof. Using $(a/b)^\dagger = a^\dagger - b^\dagger$ for $a, b \in K^\times$ we see that G is a subgroup of $K^\times$. Let $\gamma \in \Gamma$; our goal is to find a $g \in G$ with $vg = \gamma$. Take $f \in K^\times$ with $vf = \gamma$. If $f' \prec f$, then [1, 7.1.10] gives $g \in C^\times$ such that $f \asymp g$, so $g \in G$ and $vg = \gamma$. Next, suppose $f' \asymp f$. Then $f^\dagger \asymp 1$, so $f^\dagger = a + \epsilon$ with $a \in \boldsymbol{k}$ and $\epsilon \in \smallo$. By [1, Corollary 7.1.9] we have $\smallo = (1 + \smallo)^\dagger$, so $\epsilon = (1 + \delta)^\dagger$ with $\delta \in \smallo$. Then $(\frac{f}{1+\delta})^\dagger = a \in \boldsymbol{k}$, so $\frac{f}{1+\delta} \in G$ and $v(\frac{f}{1+\delta}) = \gamma$. □

Recall that if $(K, \Gamma; v)$ is d-henselian, then a lift of the differential residue field exists. Below we assume a lift $\boldsymbol{k}$ of the differential residue field is given, and we consider the 2-sorted structure $\big((K, \boldsymbol{k}), \Gamma; v\big)$ (so $\boldsymbol{k}$ is a distinguished subset of K).

Lemma 3.2.4. *Suppose $\big((K, \boldsymbol{k}), \Gamma; v\big)$ is $\aleph_1$-saturated and G is a definable subgroup of $K^\times$ such that $v(G) = \Gamma$. Then there exists a cross-section $s : \Gamma \to K^\times$ such that $s(\Gamma) \subseteq G$.*

Proof. First note that $H := \mathcal{O}^\times \cap G$ is a pure subgroup of G. The inclusion $H \to G$ and the restriction of the valuation v to G yield an exact sequence

$$1 \to H \to G \to \Gamma \to 0$$

of abelian groups. Since H is $\aleph_1$-saturated as an abelian group, this exact sequence splits; see [1, Corollary 3.3.38]. This yields a cross-section $s : \Gamma \to K^\times$ with $s(\Gamma) \subseteq G$. □

Combining the previous two lemmas gives us the main result of this section:

Theorem 3.2.5. *Suppose* $\big((K,\boldsymbol{k}),\Gamma;v\big)$ *is* d*-henselian and* $\aleph_1$*-saturated. Then there is a cross-section* $s:\Gamma\to K^\times$ *and an additive map* $c:\Gamma\to\boldsymbol{k}$ *with* $s(\gamma)^\dagger=c(\gamma)$ *for all* $\gamma\in\Gamma$.

Proof. Since $\boldsymbol{k}$ is now part of the structure, the subgroup G of $K^\times$ from Lemma 3.2.3 is definable. Now apply Lemma 3.2.4 and get a cross-section $s:\Gamma\to K^\times$ such that $s(\Gamma)^\dagger\subseteq\boldsymbol{k}$. Take the additive map $c:\Gamma\to\boldsymbol{k}$ to be given by $c(\gamma)=s(\gamma)^\dagger$. □

Proof of Theorem 2.1.2. Let a d-henselian monotone valued field be given. Then it has a lift of its differential residue field, and fixing such a lift $\boldsymbol{k}$, it is a structure $\big((K,\boldsymbol{k}),\Gamma;v\big)$ as above. Passing to an elementary extension, we can assume $\big((K,\boldsymbol{k}),\Gamma;v\big)$ is $\aleph_1$-saturated. Then Theorem 3.2.5 yields a cross-section $s:\Gamma\to K^\times$ and an additive map $c:\Gamma\to\boldsymbol{k}$ with $s(\gamma)^\dagger=c(\gamma)$ for all $\gamma\in\Gamma$. This in turn yields a Hahn field $\boldsymbol{k}((t^\Gamma))_c$ that is elementarily equivalent to $\big((K,\boldsymbol{k}),\Gamma;v,s,c\big)$. □

We can now prove Theorem 2.1.4:

Proof of Theorem 2.1.4. Let F be a d-henselian monotone valued field such that $v_F(C_F^\times)$ is pure in $\Gamma_F=v_F(F^\times)$. The valued differential field F has a lift of its differential residue field, and fixing such a lift $\boldsymbol{k}_F$ we get the structure $\big((F,\boldsymbol{k}_F),\Gamma_F;v_F\big)$. Take an elementary extension $\big((K,\boldsymbol{k}),\Gamma;v\big)$ of it that is $\aleph_1$-saturated. Then $\Delta:=v(C_K^\times)$ is pure in $v(K^\times)$. Since Δ is also $\aleph_1$-saturated (as an abelian group), we have a direct sum decomposition $\Gamma=\Delta\oplus\Gamma^*$ by [1, Corollary 3.3.38]. Since the valued subfield $C:=C_K$ of K is $\aleph_1$-saturated, it has a cross-section $s_C:\Delta\to C^\times$. Theorem 3.2.5 yields a cross-section $\tilde{s}:\Gamma\to K^\times$ of the valued field K such that $\tilde{s}(\Gamma)^\dagger\subseteq\boldsymbol{k}$. By the definition of Δ we have $\tilde{s}(\gamma)\notin C$ for all $\gamma\in\Gamma\setminus\Delta$.

Let s be the cross-section of the valued field K that agrees with s_C on Δ and with $\tilde{s}$ on Γ^*. Then $s(\gamma)^\dagger\in\boldsymbol{k}$ for all $\gamma\in\Gamma$, so we have an additive map $c:\Gamma\to\boldsymbol{k}$ given by $c(\gamma)=s(\gamma)^\dagger$. Moreover, for $\gamma\in\Gamma$,

$$c(\gamma)=0\ \Leftrightarrow\ s(\gamma)'=0\ \Leftrightarrow\ s(\gamma)\ \in C\ \Leftrightarrow\ \gamma\in\Delta.$$

This gives $\ker(c)=v(C^\times)$, and thus $c(\Gamma)\cap\boldsymbol{k}^\dagger=\{0\}$ by Proposition 3.2.1. Since $\ker(c)$ is a pure subgroup of Γ then so is Δ. This in turn yields a Hahn field $\boldsymbol{k}((t^\Gamma))_c$ with the required properties that is elementarily equivalent to $\big((K,\boldsymbol{k}),\Gamma;v,s,c\big)$. □

3.3 Eliminating the cross-section

Note that every $\mathcal{K}\models\mathrm{Mo}(\ell,s,c)$ satisfies the sentences

(1) $\forall\gamma\forall\delta \quad c(\gamma+\delta) = c(\gamma) + c(\delta)$,

(2) $\forall\gamma\exists x \neq 0 \quad v(x) = \gamma \;\&\; x^{\dagger} = c(\gamma)$.

These sentences don't mention the cross-section s. Below we derive the analogue of Theorem 3.1.1 in the setting without a cross-section. Let L_2^- be the language L_2 with the symbol s for the cross-section removed. Let $\mathrm{Mo}(\ell, c)$ be the L_2^--theory whose models are the L_2^--structures

$$\mathcal{K} \;=\; (K, \Gamma; v, c),$$

where K is a differential field equipped with a differential subfield $\boldsymbol{k}$ (singled out by a unary predicate symbol), Γ is an ordered abelian group, $v : K^{\times} \to \Gamma = v(K^{\times})$ is a valuation that makes K into a monotone valued differential field such that $\boldsymbol{k} \subseteq K$ is a lift of the differential residue field, and $c : \Gamma \to \boldsymbol{k}$ is such that the sentences (1) and (2) above are satisfied.

Lemma 3.3.1. *Suppose $\mathcal{K} = (K, \Gamma; v, c) \models \mathrm{Mo}(\ell, c)$ is* d*-henselian and $\aleph_1$-saturated. Then there is a cross-section $s : \Gamma \to K^{\times}$ such that $s(\gamma)^{\dagger} = c(\gamma)$ for all $\gamma \in \Gamma$.*

Proof. By (1) and (2) we have a definable subgroup $G := \{x \in K^{\times} : x^{\dagger} = c(v(x))\}$ of $K^{\times}$ with $v(G) = \Gamma$. Now, use Lemma 3.2.4 to get a cross section $s : \Gamma \to K^{\times}$ with $s(\Gamma) \subseteq G$. This s has the desired property. $\square$

Theorem 3.3.2. *Suppose $\mathcal{K} = (K, \Gamma; v, c) \models \mathrm{Mo}(\ell, c)$ is* d*-henselian. Then $\mathrm{Th}(\mathcal{K})$ is axiomatized by the following axiom schemes:*

(1) $\mathrm{Mo}(\ell, c)$*;*

(2) *the axioms for* d*-henselianity;*

(3) $\mathrm{Th}(\boldsymbol{k}, \Gamma; c)$ *with $\boldsymbol{k}$ as differential field and Γ as ordered abelian group.*

Proof. Let any two $\aleph_1$-saturated models of the axioms in the theorem be given. By Lemma 3.3.1 we have in both models a cross-section that make these into models of $\mathrm{Mo}(\ell, s, c)$. It remains to appeal to Theorem 3.1.1 to conclude that these two models are elementarily equivalent. $\square$

Before giving the proof of Corollary 2.1.3 from the introduction we note that any algebraic valued differential field extension of a monotone valued differential field is again monotone; see [1, Corollary 6.3.10].

Proof of Corollary 2.1.3. Let K range over d-henselian monotone valued differential fields. As in [1, Proof of Corollary 8.0.2] we have a set Σ_n of sentences in the language of valued differential fields, independent of K, such that $K \models \Sigma_n$ if and only if every valued differential field extension L of K with $[L : K] = n$ is d-henselian. Now by Theorem 2.1.2 we have $K \equiv \boldsymbol{k}((t^\Gamma))_c$ for a suitable differential field $\boldsymbol{k}$, ordered abelian group Γ, and additive map $c : \Gamma \to \boldsymbol{k}$. Every valued differential field extension L of $\boldsymbol{k}((t^\Gamma))_c$ of finite degree is spherically complete as a valued field and so d-henselian by [1, Corollary 5.4.3 and Theorem 7.2.6]. Hence $\boldsymbol{k}((t^\Gamma))_c \models \Sigma_n$ and thus $K \models \Sigma_n$, for all $n \geq 1$. □

We now give an example of a d-henselian monotone field F such that $v(C_F^\times)$ is not pure in $v(F^\times)$. This elaborates on an example by the referee of a monotone henselian valued differential field F for which $v(C_F^\times)$ is not pure in $v(F^\times)$.

Let the additive map $c : \mathbb{Z} \to \mathbb{T}_{\log}$ be given by $c(1) = 1$. With the usual derivation on $\mathbb{T}_{\log}$, this yields the (discretely) valued differential field $\boldsymbol{k} = \mathbb{T}_{\log}((s^{\mathbb{Z}}))_c$, with $s' = s$. Since $\mathbb{T}_{\log}$ is linearly surjective, $\boldsymbol{k}$ is d-henselian field and thus linearly surjective. We now forget about the valuation of $\boldsymbol{k}$, consider it just as a differential field, and introduce $K := \boldsymbol{k}((t^{\mathbb{Z}}))_d$ with the additive map $d : \mathbb{Z} \to \boldsymbol{k}$ given by $d(1) = 0$, so $t' = 0$. Then K is a d-henselian monotone field with $v(K^\times) = \mathbb{Z}$. Finally, let $F := K(\sqrt{st})$, which is naturally a valued differential field extension of K. Since F is algebraic over K, it is monotone and d-henselian too, by Corollary 2.1.3. Clearly, $v(F^\times) = \frac{1}{2}\mathbb{Z}$. We claim that $v(C_F^\times) = \mathbb{Z}$ and so it is not pure in $v(F^\times)$. From $t^{\mathbb{Z}} \subseteq C_F$ we get $\mathbb{Z} \subseteq v(C_F^\times)$. For the reverse inclusion, let any element $a + b\sqrt{st} \in C_F^\times$ be given with $a, b \in K$, not both zero. Now,

$$(a + b\sqrt{st})' = a' + b'\sqrt{st} + b(\sqrt{st})' = a' + b'\sqrt{st} + b(\sqrt{st}/2) = a' + (b' + b/2)\sqrt{st},$$

so $a' = 0$ and $b' + b/2 = 0$. From $b' = -b/2$ we now derive $b = 0$. (Then $a + b\sqrt{st} = a \in C_{\boldsymbol{k}}((t^{\mathbb{Z}}))$, and thus $v(a + b\sqrt{st}) \in \mathbb{Z}$, as claimed.) Let k, l range over $\mathbb{Z}$. Towards a contradiction, suppose $b = \sum_{l \geq l_0} b_l t^l$ with all $b_l \in \boldsymbol{k}$, $l_0 \in \mathbb{Z}$, $b_{l_0} \neq 0$. Then $b' = \sum_{l \geq l_0} b_l' t^l$ and so the equality $b' = -b/2$ takes the form

$$\sum_{l \geq l_0} b_l' t^l \;=\; -\frac{1}{2} \sum_{l \geq l_0} b_l t^l \;=\; \sum_{l \geq l_0} -\frac{1}{2} b_l t^l.$$

Therefore $b_l' = -b_l/2$ for all $l \geq l_0$, in particular for $l = l_0$. Assume $b_{l_0} = \sum_{k \geq k_0} u_k s^k$, with all $u_k \in \mathbb{T}_{\log}$, and $k_0 \in \mathbb{Z}$, $u_{k_0} \neq 0$. We have $b_{l_0}' = \sum_{k \geq k_0} (u_k' + k u_k) s^k$ and $-\frac{1}{2} b_{l_0} = \sum_{k \geq k_0} -\frac{1}{2} u_k s^k$. Thus $u_k' + k u_k = -u_k/2$ for all $k \geq k_0$. For $k = k_0$ we have $u_{k_0} \neq 0$, and so this gives $u_{k_0}^\dagger = -k_0 - 1/2$. However, this contradicts $\mathbb{T}_{\log}^\dagger \cap \mathbb{R} = \{0\}$ and hence the claim is proved.

On the other hand:

Proposition 3.3.3. *Let F be a henselian valued differential field with algebraically closed or real closed residue field. Then $v(C_F^\times)$ is pure in $v(F^\times)$.*

Proof. Let $n\alpha = \beta$ with $\alpha \in v(F^\times), \beta \in v(C_F^\times), n \geq 1$; our job is to show that then $\alpha \in v(C_F^\times)$. Take $a \in F^\times$ with $v(a) = \alpha$ and $b \in C_F^\times$ with $v(b) = \beta$, so $v(b/a^n) = 0$; if the residue field is real closed we also arrange that the residue class of b/a^n is positive. Considering the polynomial $P(Y) = Y^n - (b/a^n) \in \mathcal{O}_F[Y]$, the henselianity of F and the assumption on the residue field gives a zero $y \asymp 1$ in F of P. Then $(ay)^n = b \in C_F^\times$, hence $ay \in C_F^\times$ with $v(ay) = \alpha$. □

A valued differential field with small derivation is said to be d-*algebraically maximal* if it has no proper immediate d-algebraic valued differential field extension. For monotone valued differential fields with linearly surjective differential residue field,

$$\text{d-algebraically maximal} \implies \text{d-henselian}$$

by [1, Theorem 7.0.1]. By [1, Theorem 7.0.3], the converse holds in the case of few constants, but an example at the end of Section 7.4 of [1] shows that this converse fails for some d-henselian monotone valued differential field with many constants. Below we generalize this example as follows:

Corollary 3.3.4. *Let K be a* d-*henselian, monotone, valued differential field with $v(C^\times) \neq \{0\}$. Then some $L \equiv K$ is not* d-*algebraically maximal.*

Proof. By Theorems 2.1.1 and 2.1.2 and Löwenheim-Skolem we can arrange $K = \boldsymbol{k}((t^\Gamma))_c$ where the differential field $\boldsymbol{k}$ and the ordered abelian group Γ are countable and $c : \Gamma \to \boldsymbol{k}$ is additive. With $C := C_K$, take $a \in C^\times$ with $va = \gamma_0 > 0$. Then $a = \sum_{\gamma \geq \gamma_0} a_\gamma t^\gamma$, with $\partial(a_\gamma) + c(\gamma)a_\gamma = 0$ for all γ, in particular for $\gamma = \gamma_0$. Hence $\mathfrak{m} := a_{\gamma_0} t^{\gamma_0} \in C$, and so all infinite sums $\sum_n q_n \mathfrak{m}^n$ with rational q_n lie in C as well. Thus C is uncountable.

On the other hand, $\boldsymbol{k}(t^\Gamma)$ is countable and so by Löwenheim-Skolem we have a countable $L \prec K$ that contains $\boldsymbol{k}(t^\Gamma)$. Thus K is an immediate extension of L and we can take $a \in C \setminus L$. Then $L\langle a \rangle = L(a)$ is a proper immediate d-algebraic extension of L and therefore L is not d-algebraically maximal. □

3.4 Eliminating the lift of the differential residue field

In this section we drop the requirement of having a *lift* of the differential residue field in our structure and instead use a copy of the differential residue field. For this purpose we consider 3-sorted structures

$$\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c)$$

where K and $\mathbf{k}$ are differential fields, Γ is an ordered abelian group, $v : K^\times \to \Gamma$ is a valuation which makes K into a monotone valued differential field, $\pi : \mathcal{O} \to \mathbf{k}$ with $\mathcal{O} := \mathcal{O}_v$ is a surjective differential ring morphism, $c : \Gamma \to \mathbf{k}$ is an additive map satisfying $\forall \gamma \exists x \neq 0 \ \left[v(x) = \gamma \ \& \ \pi(x^\dagger) = c(\gamma)\right]$. We construe these $\mathcal{K}$ as L_3-structures for a natural 3-sorted language L_3 (with unary function symbols for π, v and c). We have an obvious set $\mathrm{Mo}(c)$ of L_3-sentences whose models are exactly these $\mathcal{K}$.

Lemma 3.4.1. *Suppose* $(K, \mathbf{k}, \Gamma; \pi, v, c) \vDash \mathrm{Mo}(c)$ *is* d-*henselian, and* $\iota : \mathbf{k} \to K$ *is a lifting of the differential residue field* $\mathbf{k}$ *to* K*, that is, a differential field embedding with image in* $\mathcal{O}$ *such that* $\pi(\iota(x)) = x$ *for all* $x \in \mathbf{k}$*. Then* $\Big(\big(K, \iota(\mathbf{k})\big), \Gamma; v, \iota \circ c\Big) \vDash \mathrm{Mo}(\ell, c)$.

Proof. We need to check the two conditions from the previous section. First of all $\iota \circ c$ is obviously additive. Fix $\gamma \in \Gamma$. There is an element $x \in K^\times$ with $v(x) = \gamma$ and $\pi(x^\dagger) = c(\gamma)$. Let $a = (\iota \circ \pi)(x^\dagger) = (\iota \circ c)(\gamma)$. As $a \in \iota(\mathbf{k})$ and $\pi(a) = \pi(x^\dagger)$, we get $x^\dagger = a + \epsilon$ for some $\epsilon \prec 1$. By [1, Corollary 7.1.9] we have $\epsilon = (1+\delta)^\dagger$ for some $\delta \prec 1$ and thus

$$(\iota \circ c)(\gamma) = a = x^\dagger - (1+\delta)^\dagger = \Big(\frac{x}{1+\delta}\Big)^\dagger, \text{ and } v\Big(\frac{x}{1+\delta}\Big) = v(x) = \gamma.$$

This completes the proof of the lemma. □

Theorem 3.4.2. *Suppose* $\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c) \vDash \mathrm{Mo}(c)$ *is* d-*henselian. Then* $\mathrm{Th}(\mathcal{K})$ *is axiomatized by the following axiom schemes:*

(1) $\mathrm{Mo}(c)$*;*

(2) *the axioms for* d-*henselianity;*

(3) $\mathrm{Th}(\mathbf{k}, \Gamma; c)$ *with* $\mathbf{k}$ *as differential field and* Γ *as ordered abelian group.*

Proof. Let any two $\aleph_1$-saturated models $\mathcal{K}_1 = (K_1, \mathbf{k}_1, \Gamma_1; \pi_1, v_1, c_1)$ and $\mathcal{K}_2 = (K_2, \mathbf{k}_2, \Gamma_2; \pi_2, v_2, c_2)$ of the axioms in the theorem be given. By Lemma 3.4.1 we have in both models lifts of the differential residue fields that make these into models of $\mathrm{Mo}(\ell, c)$. So $\mathrm{Th}(\mathbf{k}_i, \Gamma_i; c_i) = \mathrm{Th}(\boldsymbol{k}_i, \Gamma_i; \iota_i \circ c_i)$ where ι_i is the lifting of the differential residue field $\mathbf{k}_i$ to K_i for $i = 1, 2$. It remains to appeal to Theorem 3.3.2 to conclude that these two models are elementarily equivalent. $\square$

Chapter 4

Definability in Differential-Henselian Monotone Fields

4.1 Angular components

Let $\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c) \models \mathrm{Mo}(c)$.

An **angular component map** on $\mathcal{K}$ is a map $\mathrm{ac} : K^\times \to \mathbf{k}^\times$ such that

(1) ac is a multiplicative group morphism and $\mathrm{ac}(a) = \pi(a)$ for all $a \asymp 1$,

(2) $\mathrm{ac}(a^\dagger) = \mathrm{ac}(a)^\dagger + c(v(a))$ for all $a \in K^\times$ with $a' \asymp a$,

(3) $\mathrm{ac}(d)^\dagger = -c(v(d))$ for all $d \in C^\times$.

We extend such a map to all of K by $\mathrm{ac}(0) = 0 \in \mathbf{k}$, so (3) then becomes the equality in (2) for $a' = 0$ instead of $a' \asymp a$. Note that if $\mathcal{K}$ has few constants, then (1) and (2) together imply (3). If $\mathcal{K}$ has many constants, then this notion of angular component map is easily seen to agree with that in [1, Section 8.1].

Examples are Hahn differential fields $\mathbf{k}((t^\Gamma))_c$ (the differential residue field being identified with $\mathbf{k}$ in the usual way), with the angular component map given by $\mathrm{ac}(a) = a_{\gamma_0}$ for non-zero $a = \sum a_\gamma t^\gamma$ and $\gamma_0 = v(a)$.

More generally, let $s : \Gamma \to K^\times$ be a cross-section with $\pi(s(\gamma)^\dagger) = c(\gamma)$ for all γ. We claim that this yields an angular component map ac on $\mathcal{K}$ by

$$\mathrm{ac}(a) \;:=\; \pi\Big(\frac{a}{s(v(a))}\Big) \text{ for } a \in K^\times.$$

Condition (1) is obviously satisfied. As to condition (2), first note that $\mathrm{ac}(s(\gamma)) = 1$ for $\gamma \in \Gamma$. Next, let $a \in K^\times$ and $a' \asymp a$. Then $a = s(v(a))u$ where $u \asymp 1$ and hence $\mathrm{ac}(a) = \pi(u)$. Also,

$$\pi(a^\dagger) \;=\; \pi\Big(s(v(a))^\dagger + u^\dagger\Big) \;=\; \pi\Big(s(v(a))^\dagger\Big) + \pi(u^\dagger) \;=\; c(v(a)) + \pi(u^\dagger).$$

Using $a^\dagger \asymp 1$ and $u \asymp 1$, this yields $\mathrm{ac}(a^\dagger) = c(v(a)) + \pi(u)^\dagger = \mathrm{ac}(a)^\dagger + c(v(a))$. As to (3), let $d \in C^\times$, so

$d = u \cdot s(v(d))$ with $u \asymp 1$, hence $\mathrm{ac}(d) = \mathrm{ac}(u) \cdot \mathrm{ac}(s(v(d)))$ with $\mathrm{ac}(s(v(d)) = 1$, and thus

$$\mathrm{ac}(d)^\dagger \;=\; \mathrm{ac}(u)^\dagger \;=\; \pi(u)^\dagger \;=\; \pi(u^\dagger).$$

Morever, $0 = d^\dagger = u^\dagger + s(v(d))^\dagger$, so $u^\dagger = -s(v(d))^\dagger$, and thus

$$\pi(u^\dagger) \;=\; -\pi\big(s(v(d))^\dagger\big) \;=\; -c(v(d)).$$

Therefore, $\mathrm{ac}(d)^\dagger = -c(v(d))$, as claimed. This leads to the following:

Corollary 4.1.1. *Suppose the model $\mathcal{K}$ of* $\mathrm{Mo}(c)$ *is $\aleph_1$-saturated. Then there exists an angular component map on $\mathcal{K}$.*

Proof. This is close to the proof of Theorem 3.2.5 and so we shall be brief. Let $G := \{a \in K^\times : \pi(a^\dagger) = c(va)\}$, a subgroup of $K^\times$, definable in $\mathcal{K}$, with $v(G) = \Gamma$. Then $H := \mathcal{O}^\times \cap G$ is a pure subgroup of G, and so we get a cross-section $s : \Gamma \to K^\times$ with $s(\Gamma) \subseteq G$ as in the proof of Lemma 3.2.4. Then $\pi(s(\gamma)^\dagger) = c(\gamma)$ for all $\gamma \in \Gamma$. This gives an angular component map ac on $\mathcal{K}$ by $\mathrm{ac}(a) := \pi\Big(a/s(v(a))\Big)$ for $a \in K^\times$. $\square$

4.2 Equivalence over substructures

In this section we consider 3-sorted structures

$$\mathcal{K} \;=\; (K, \mathbf{k}, \Gamma; \pi, v, c, \mathrm{ac})$$

where $(K, \mathbf{k}, \Gamma; \pi, v, c) \models \mathrm{Mo}(c)$ and $\mathrm{ac} : K \to \mathbf{k}$ is an angular component map on $(K, \mathbf{k}, \Gamma; \pi, v, c)$. These 3-sorted structures are naturally $L_3(\mathrm{ac})$-structures where the language $L_3(\mathrm{ac})$ is L_3 augmented by a unary function symbol ac of sort fr. Let $\mathrm{Mo}(c, \mathrm{ac})$ be the set of $L_3(\mathrm{ac})$-sentences consisting of $\mathrm{Mo}(c)$ and a sentence expressing that ac is an angular component map as defined in the previous section. Then these 3-sorted structures are exactly the models of $\mathrm{Mo}(c, \mathrm{ac})$. Given $\mathcal{K}$ as above we regard any subfield E of K as a valued subfield of K, so the valuation ring of such E is $\mathcal{O}_E = E \cap \mathcal{O}_v$. We say that a differential subfield E of K *satisfies the c-condition* if for all $\gamma \in v(E^\times)$ there is an $x \in E^\times$ such that $v(x) = \gamma$ and $\pi(x^\dagger) = c(\gamma)$. For example, K satisfies the c-condition.

Define a **good substructure** of $\mathcal{K}$ to be a triple $\mathbf{E} = (E, \mathbf{k}_{\mathbf{E}}, \Gamma_{\mathbf{E}})$ such that

(1) E is a differential subfield of K;

(2) $\mathbf{k}_{\mathbf{E}}$ is a differential subfield of $\mathbf{k}$ with $\mathrm{ac}(E) \subseteq \mathbf{k}_{\mathbf{E}}$ (and thus $\pi(\mathcal{O}_E) \subseteq \mathbf{k}_{\mathbf{E}}$);

(3) $\Gamma_{\mathbf{E}}$ is an ordered abelian subgroup of Γ, with $v(E^\times) \subseteq \Gamma_{\mathbf{E}}$ and $c(\Gamma_{\mathbf{E}}) \subseteq \mathbf{k}_{\mathbf{E}}$.

Note that we do *not* demand here that $\pi(\mathcal{O}_E) = \mathbf{k}_{\mathbf{E}}$ or $v(E^\times) = \Gamma_{\mathbf{E}}$. For good substructures $\mathbf{E} = (E, \mathbf{k}_{\mathbf{E}}, \Gamma_{\mathbf{E}})$ and $\mathbf{F} = (F, \mathbf{k}_{\mathbf{F}}, \Gamma_{\mathbf{F}})$ of $\mathcal{K}$ we define $\mathbf{E} \subseteq \mathbf{F}$ to mean that $E \subseteq F$, $\mathbf{k}_{\mathbf{E}} \subseteq \mathbf{k}_{\mathbf{F}}$ and $\Gamma_{\mathbf{E}} \subseteq \Gamma_{\mathbf{F}}$. Now let

$$\mathcal{K}_1 = (K_1, \mathbf{k}_1, \Gamma_1; \pi_1, v_1, c_1, \mathrm{ac}_1), \quad \mathcal{K}_2 = (K_2, \mathbf{k}_2, \Gamma_2; \pi_2, v_2, c_2, \mathrm{ac}_2)$$

be models of $\mathrm{Mo}(c, \mathrm{ac})$, set $\mathcal{O}_1 := \mathcal{O}_{v_1}$ and $\mathcal{O}_2 := \mathcal{O}_{v_2}$, and let

$$\mathbf{E}_1 = (E_1, \mathbf{k}_{\mathbf{E}_1}, \Gamma_{\mathbf{E}_1}), \qquad \mathbf{E}_2 = (E_2, \mathbf{k}_{\mathbf{E}_2}, \Gamma_{\mathbf{E}_2})$$

be good substructures of $\mathcal{K}_1, \mathcal{K}_2$, respectively. A **good map** $\mathbf{f} : \mathbf{E}_1 \to \mathbf{E}_2$ is a triple $\mathbf{f} = (f, f_r, f_v)$ consisting of a differential field isomorphism $f : E_1 \to E_2$, a differential field isomorphism $f_r : \mathbf{k}_{\mathbf{E}_1} \to \mathbf{k}_{\mathbf{E}_2}$, and an ordered group isomorphism $f_v : \Gamma_{\mathbf{E}_1} \to \Gamma_{\mathbf{E}_2}$, such that

(4) $f_r(\mathrm{ac}_1(a)) = \mathrm{ac}_2(f(a))$ for all $a \in E_1$;

(5) $f_v(v_1(a)) = v_2(f(a))$ for all $a \in E_1^\times$;

(6) (f_r, f_v) is elementary as a partial map between $(\mathbf{k}_1, \Gamma_1; c_1)$ and $(\mathbf{k}_2, \Gamma_2; c_2)$, in particular, $f_r(c_1(\gamma)) = c_2(f_v(\gamma))$ for all $\gamma \in \Gamma_{\mathbf{E}_1}$.

Let $\mathbf{f} = (f, f_r, f_v) : \mathbf{E}_1 \to \mathbf{E}_2$ be a good map. Then

$$\mathbf{f}^{-1} := (f^{-1}, f_r^{-1}, f_v^{-1}) : \mathbf{E}_2 \to \mathbf{E}_1$$

is a good map as well, and $f(\mathcal{O}_{E_1}) = \mathcal{O}_{E_2}$ by (5), so f is an isomorphism of valued fields. Using also (4) we obtain $f_r(\pi_1(a)) = \pi_2(f(a))$ for all $a \in \mathcal{O}_{E_1}$.

We say that a good map $\mathbf{g} = (g, g_r, g_v) : \mathbf{F}_1 \to \mathbf{F}_2$ **extends f** if $\mathbf{E}_1 \subseteq \mathbf{F}_1$, $\mathbf{E}_2 \subseteq \mathbf{F}_2$, and g, g_r, g_v extend f, f_r, f_v, respectively. Note that if a good map $\mathbf{E}_1 \to \mathbf{E}_2$ exists, then $(\mathbf{k}_1, \Gamma_1; c_1) \equiv (\mathbf{k}_2, \Gamma_2; c_2)$ by (6). Our goal is:

Theorem 4.2.1. *If $\mathcal{K}_1$ and $\mathcal{K}_2$ are* d*-henselian, then any good map $\mathbf{E}_1 \to \mathbf{E}_2$ is a partial elementary map between $\mathcal{K}_1$ and $\mathcal{K}_2$.*

Towards the proof we establish some lemmas; these do not assume d-henselianity.

Lemma 4.2.2. *Let $\mathbf{f} : \mathbf{E}_1 \to \mathbf{E}_2$ be a good map and suppose $F_1 \supseteq E_1$ and $F_2 \supseteq E_2$ are differential subfields of K_1 and K_2, respectively, such that $\pi_1(\mathcal{O}_{F_1}) \subseteq \mathbf{k}_{\mathbf{E}_1}$ and $v_1(F_1^\times) = v_1(E_1^\times)$. Let $g : F_1 \to F_2$ be a valued differential field isomorphism such that g extends f and $f_r(\pi_1(u)) = \pi_2(g(u))$ for all $u \asymp 1$ in F_1. Then* $\mathrm{ac}_1(F_1) \subseteq \mathbf{k}_{\mathbf{E}_1}$ *and $f_r(\mathrm{ac}_1(a)) = \mathrm{ac}_2(g(a))$ for all $a \in F_1$, and thus also* $\mathrm{ac}_2(F_2) \subseteq \mathbf{k}_{\mathbf{E}_2}$.

Proof. Let $a \in F_1$. Then $a = bu$ where $b \in E_1$ and $u \asymp 1$ in F_1, so $\mathrm{ac}_1(a) = \mathrm{ac}_1(b)\pi_1(u) \in \mathbf{k}_{\mathbf{E}_1}$. Thus $f_r(\mathrm{ac}_1(a)) = f_r(\mathrm{ac}_1(b))f_r(\pi_1(u)) = \mathrm{ac}_2(f(b))\pi_2(g(u)) = \mathrm{ac}_2(g(a))$. □

Lemma 4.2.3. *Suppose $\mathbf{f} : \mathbf{E}_1 \to \mathbf{E}_2$ is a good map, $\pi_1(\mathcal{O}_{E_1}) = \mathbf{k}_{\mathbf{E}_1}$, and $F_1 \supseteq E_1$, $F_2 \supseteq E_2$ are differential subfields of K_1 and K_2, respectively, such that $v_1(F_1^\times) = v_1(E_1^\times)$. Let $g : F_1 \to F_2$ be a valued differential field isomorphism extending f, and $g_r : \pi_1(\mathcal{O}_{F_1}) \to \pi_2(\mathcal{O}_{F_2})$ the differential field isomorphism induced by g. Then $\mathrm{ac}_1(F_1) = \pi_1(\mathcal{O}_{F_1})$ and $g_r(\mathrm{ac}_1(a)) = \mathrm{ac}_2(g(a))$ for all $a \in F_1$. Moreover, $\mathrm{ac}_2(F_2) = \pi_2(\mathcal{O}_{F_2})$ and $v_2(F_2^\times) = v_2(E_2^\times)$.*

Proof. Let $a \in F_1^\times$. Then $a = bu$ where $a \asymp b \in E_1^\times$ and $1 \asymp u \in \mathcal{O}_{F_1}$, so $\mathrm{ac}_1(a) = \mathrm{ac}_1(b)\pi_1(u) \in \pi_1(\mathcal{O}_{F_1})$. It is clear that g_r extends f_r. Thus

$$\begin{aligned} g_r(\mathrm{ac}_1(a)) &= g_r(\mathrm{ac}_1(b))g_r(\pi_1(u)) = f_r(\mathrm{ac}_1(b))\pi_2(g(u)) = \mathrm{ac}_2(f(b))\mathrm{ac}_2(g(u)) \\ &= \mathrm{ac}_2(g(b))\mathrm{ac}_2(g(u)) = \mathrm{ac}_2(g(bu)) = \mathrm{ac}_2(g(a)), \end{aligned}$$

as claimed. □

With the assumptions of the lemma above, g_r extends f_r, $(F_1, \pi_1(\mathcal{O}_{F_1}), \Gamma_{\mathbf{E}_1})$ and $(F_2, \pi_2(\mathcal{O}_{F_2}), \Gamma_{\mathbf{E}_2})$ are good substructures of $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively, and

$$\mathbf{g} = (g, g_r, f_v) : (F_1, \pi_1(\mathcal{O}_{F_1}), \Gamma_{\mathbf{E}_1}) \to (F_2, \pi_2(\mathcal{O}_{F_2}), \Gamma_{\mathbf{E}_2})$$

extends $\mathbf{f}$ and satisfies conditions (4) and (5) for good maps.

Corollary 4.2.4. *Suppose $\mathbf{f} : \mathbf{E}_1 \to \mathbf{E}_2$ is a good map, $\pi_1(\mathcal{O}_{E_1}) = \mathbf{k}_{\mathbf{E}_1}$, and $F_1 \supseteq E_1$ and $F_2 \supseteq E_2$ are differential subfields of K_1 and K_2, respectively, and are immediate extensions of E_1 and E_2, respectively. Let $g : F_1 \to F_2$ be a valued differential field isomorphism extending f. Then $\mathbf{g} = (g, f_r, f_v)$ is a good map that extends $\mathbf{f}$.*

This follows by verifying the hypotheses of Lemma 4.2.3.

Proof of Theorem 4.2.1

Assume $\mathcal{K}_1$ and $\mathcal{K}_2$ are d-henselian, and let

$$\mathbf{f} = (f, f_r, f_v) : \mathbf{E}_1 \to \mathbf{E}_2$$

be a good map. We have to show that this is a partial elementary map between $\mathcal{K}_1$ and $\mathcal{K}_2$. The case $\Gamma_1 = \{0\}$ is a routine exercise, so assume $\Gamma_1 \neq \{0\}$.

By passing to suitable elementary extensions of $\mathcal{K}_1$ and $\mathcal{K}_2$ we arrange that both are κ-saturated, where κ is an uncountable cardinal such that $|\mathbf{k}_{\mathbf{E}_1}|, |\Gamma_{\mathbf{E}_1}| < \kappa$. We call a good substructure $\mathbf{E} = (E, \mathbf{k}_{\mathbf{E}}, \Gamma_{\mathbf{E}})$ of $\mathcal{K}_1$ **small** if $|\mathbf{k}_{\mathbf{E}}|, |\Gamma_{\mathbf{E}}| < \kappa$. We prove that the good maps whose domain $\mathbf{E}$ is small (such as the above $\mathbf{f}$) form a back-and-forth system from $\mathcal{K}_1$ to $\mathcal{K}_2$. This is enough to establish the theorem. We now present 8 procedures to extend a good map $\mathbf{f}$ as above:

(1) *Given $d \in \mathbf{k}_1$, arranging that $d \in \mathbf{k}_{\mathbf{E}_1}$.* This can be done by saturation without changing $f, f_v, E_1, \Gamma_{\mathbf{E}_1}$ by extending f_r to a map with domain $\mathbf{k}_{\mathbf{E}_1}\langle d\rangle$ which together with f_v gives a partial elementary map between $(\mathbf{k}_1, \Gamma_1; c_1)$ and $(\mathbf{k}_2, \Gamma_2; c_2)$.

(2) *Given $\gamma \in \Gamma_1$, arranging that $\gamma \in \Gamma_{\mathbf{E}_1}$.* We can assume $c(\gamma) \in \mathbf{k}_{\mathbf{E}_1}$ by (1). Without changing f, f_r, E_1, $\mathbf{k}_{\mathbf{E}_1}$ we then use saturation as in (1) to extend f_v to a map with domain $\Gamma_{\mathbf{E}_1} + \mathbb{Z}\gamma$ which together with f_r is a partial elementary map between $(\mathbf{k}_1, \Gamma_1; c_1)$ and $(\mathbf{k}_2, \Gamma_2; c_2)$.

(3) *Arranging* $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{E_1})$. Let $d \in \mathbf{k}_{\mathbf{E}_1}$ and $d \notin \pi_1(\mathcal{O}_{E_1})$. Set $e = f_r(d)$. There are two possibilities:

(i) d is d-transcendental over $\pi_1(\mathcal{O}_{E_1})$. Then e is d-transcendental over $\pi_2(\mathcal{O}_{E_2})$. Take $a \in \mathcal{O}_1$ and $b \in \mathcal{O}_2$ with $\pi_1(a) = d$ and $\pi_2(b) = e$. Then by [1, Lemma 6.3.1] we have $v_1(E_1^\times) = v_1(E_1\langle a\rangle^\times)$ and we get an isomorphism $g : E_1\langle a\rangle \to E_2\langle b\rangle$ of valued differential fields which extends f and sends a to b. Therefore by Lemma 4.2.2, $\mathbf{g} = (g, f_r, f_v)$ is a good map between $(E_1\langle a\rangle, \mathbf{k}_{\mathbf{E}_1}, \Gamma_{\mathbf{E}_1})$ and $(E_2\langle b\rangle, \mathbf{k}_{\mathbf{E}_2}, \Gamma_{\mathbf{E}_2})$ and it extends $\mathbf{f}$.

(ii) d is d-algebraic over $\pi_1(\mathcal{O}_{E_1})$. Take a minimal annihilator $\bar{P} \in \pi_1(\mathcal{O}_{E_1})\{Y\}$ of d over $\pi_1(\mathcal{O}_{E_1})$. Note that applying f_r to the coefficients of $\bar{P}$ yields a minimal annihilator of e over $\pi_2(\mathcal{O}_{E_2})$. Next, take $P \in \mathcal{O}_{E_1}\{Y\}$ such that applying π_1 to the coefficients of P yields $\bar{P}$ and such that P has the same complexity as $\bar{P}$. Now $\mathcal{K}_1$ is d-henselian, so we obtain $a \in \mathcal{O}_1$ with $\pi_1(a) = d$ and $P(a) = 0$. As in the proof of [1, Lemma 7.1.4] one shows that P is then a minimal annihilator of a over E_1. Applying f to the coefficients of P yields $fP \in \mathcal{O}_{E_2}\{Y\}$, and as $\mathcal{K}_2$ is d-henselian we obtain likewise an element $b \in \mathcal{O}_2$ with $\pi_2(b) = e$ and $fP(b) = 0$; then fP is again a minimal annihilator of b over E_2. An argument in the beginning of the proof of [1, Theorem 6.3.2] yields $v_1(E_1\langle a\rangle^\times) = v_1(E_1^\times)$, $v_2(E_2\langle b\rangle^\times) = v_2(E_2^\times)$, and the uniqueness part of that theorem then gives us a valued differential field isomorphism $g : E_1\langle a\rangle \to E_2\langle b\rangle$ that extends f and sends a to b; the same theorem also gives $\pi_1(\mathcal{O}_{E_1\langle a\rangle}) = \pi_1(\mathcal{O}_{E_1})\langle d\rangle$ and $\pi_2(\mathcal{O}_{E_2\langle b\rangle}) = \pi_2(\mathcal{O}_{E_2})\langle e\rangle$. Using Lemma 4.2.2 this yields a good map $\mathbf{g} = (g, f_r, f_v)$ extending $\mathbf{f}$ with small domain $(E_1\langle a\rangle, \mathbf{k}_{E_1}, \Gamma_{E_1})$.

By iterating the extension procedures in (i) and (ii) we complete step (3), that is, arrange $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{E_1})$.

(4) *Arranging that* $\mathbf{k}_{\mathbf{E}_1}$ *is linearly surjective and* $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{E_1})$. This is done by first iterating (1) and then applying (3).

(5) *Arranging that* E_1 *is* d-*henselian as a valued differential field and* $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{E_1})$. First apply (4) to arrange that $\mathbf{k}_{\mathbf{E}_1}$ is linearly surjective and $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{E_1})$. Next, use (DV1)–(DV4) from Chapter 2 to pass to spherically complete immediate extensions of E_1 and E_2 inside K_1 and K_2, and use Corollary 4.2.4.

(6) *Arranging that* E_1 *satisfies the c-condition.* Let $\gamma \in v_1(E_1^\times)$. Take $b \in E_1^\times$ with $v_1(b) = \gamma$. Then Lemma 2.2.1 gives $a \asymp 1$ in $K_1^\times$ with $\pi_1(b^\dagger) = \pi_1(a^\dagger) + c_1(\gamma)$. Using (1) we arrange $\pi_1(a) \in \mathbf{k}_{\mathbf{E}_1}$. Next, use (3) to arrange $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{E_1})$. Now choose $a^\star \in \mathcal{O}_{E_1}$ with $\pi_1(a^\star) = \pi_1(a)$. Then $\pi_1(a^\dagger) = \pi_1(a)^\dagger = \pi_1(a^\star)^\dagger$, so $\pi_1(b^\dagger) = \pi_1(a^\star)^\dagger + c_1(\gamma)$. Thus $x := b/a^\star$ satisfies $v_1(x) = \gamma$ and $\pi_1(x^\dagger) = c_1(\gamma)$. This takes care of a single γ, and doing the above iteratively we can deal with all $\gamma \in v_1(E_1^\times)$, preserving $|\mathbf{k}_{\mathbf{E}_1}| < \kappa$; this process does not change $\Gamma_{\mathbf{E}_1}$.

In steps (1)–(6) the value group $v_1(E_1^\times)$ does not change, so if the first field E_1 of the domain $\mathbf{E}_1$ of our good map $\mathbf{f}$ satisfies the c-condition, then so does the first field of the domain of the extension of $\mathbf{f}$ constructed in each of (1)–(6).

In steps (7) and (8) below we assume that the domain $\mathbf{E}_1$ of our good map $\mathbf{f}$ has the following properties: E_1 is d-henselian, E_1 satisfies the c-condition, and $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{E_1})$. Note that then the codomain $\mathbf{E}_2$ has the corresponding properties. In view of these properties of $\mathbf{E}_i$ we have

$$\big(E_i, \mathbf{k}_{\mathbf{E}_i}, v_i(E_i^\times);\ \pi_i|_{\mathcal{O}_{E_i}}, v_i|_{E_i^\times}, c_i|_{v_i(E_i^\times)}\big) \models \mathrm{Mo}(c) \qquad (i = 1, 2).$$

In (7) and (8) below we construct models of the theory $\mathrm{Mo}(\ell, c)$ defined in the beginning of Section 3.4.

(7) *Towards* $v_1(E_1^\times) = \Gamma_{\mathbf{E}_1}$*; the case of no torsion modulo* $v_1(E_1^\times)$. Suppose $\gamma \in \Gamma_{\mathbf{E}_1}$ has no torsion modulo $v_1(E_1^\times)$, that is, $n\gamma \notin v_1(E_1^\times)$ for all $n \geq 1$. Take a lifting ι_1 of the differential residue field $\mathbf{k}_{\mathbf{E}_1}$ to E_1. Then

$$\iota_2 \ :=\ f \circ \iota_1 \circ f_r^{-1}\ :\ \mathbf{k}_{\mathbf{E}_2} \to E_2$$

is a lifting of the differential residue field $\mathbf{k}_{\mathbf{E}_2}$ to E_2. The proof of [1, Proposition 7.1.3] shows how to extend

ι_i to a lifting, to be denoted also by ι_i, of $\mathbf{k}_i$ to K_i for $i = 1, 2$. Then by Lemma 3.4.1,

$$(*) \qquad \Big(\big(K_i, \iota_i(\mathbf{k}_i)\big), \Gamma_i; v_i, \iota_i \circ c_i\Big) \models \mathrm{Mo}(\ell, c) \qquad (i = 1, 2).$$

It follows that we can take $a \in K_1^\times$ such that $v_1(a) = \gamma$ and $a^\dagger = (\iota_1 \circ c_1)(\gamma)$. We distinguish two cases:

(i) $c_1(\gamma) = 0$. Then $a \in C_{K_1}$ and so $\mathrm{ac}_1(a)^\dagger = -c_1(\gamma) = 0$. Replace a with $a/\iota_1(\mathrm{ac}_1(a))$.

(ii) $c_1(\gamma) \neq 0$. Then $a' \asymp a$, so $\mathrm{ac}_1(a^\dagger) = \mathrm{ac}_1(a)^\dagger + c_1(\gamma)$. On the other hand, $\mathrm{ac}_1(a^\dagger) = \pi_1(a^\dagger) = c_1(\gamma)$. So $\mathrm{ac}_1(a)^\dagger = 0$ and we replace a with $a/\iota_1(\mathrm{ac}_1(a))$.

In both cases the updated a still satisfies $v_1(a) = \gamma$ and $a^\dagger = (\iota_1 \circ c_1)(\gamma)$; in addition, we have $\mathrm{ac}_1(a) = 1$. Note that a is transcendental over E_1 and $P(a) = 0$ where

$$P(Y) \;:=\; Y' - (\iota_1 \circ c_1)(\gamma) Y \;\in\; \mathcal{O}_{E_1}\{Y\}.$$

In the same way, we get $b \in K_2^\times$ with $v_2(b) = f_v(\gamma)$, $b^\dagger = (\iota_2 \circ c_2)(f_v(\gamma))$ and $\mathrm{ac}_2(b) = 1$. Then b is transcendental over E_2 and $P^f(b) = 0$ where the differential polynomial $P^f(Y) \in \mathcal{O}_{E_2}\{Y\}$ is given by

$$P^f(Y) \;:=\; Y' - (f \circ \iota_1 \circ c_1)(\gamma) Y \;=\; Y' - (\iota_2 \circ f_r \circ c_1)(\gamma) Y = \; Y' - (\iota_2 \circ c_2)(f_v(\gamma)) Y.$$

Then [1, Lemma 3.1.30] gives

$$v_1(E_1(a)^\times) \;=\; v_1(E_1^\times) + \mathbb{Z}\gamma \;\subseteq\; \Gamma_{\mathbf{E}_1}, \qquad \pi_1(\mathcal{O}_{E_1(a)}) \;=\; \pi_1(\mathcal{O}_{E_1}).$$

It also yields a valued field isomorphism $g : E_1(a) \to E_2(b)$ extending f with $g(a) = b$. Note that g is in addition a differential field isomorphism. It is now routine to verify that (g, f_r, f_v) is a good map with domain $(E_1(a), \mathbf{k}_{\mathbf{E}_1}, \Gamma_{\mathbf{E}_1})$. Using that E_1 satisfies the c-condition, it follows easily that $E_1(a)$ does as well.

Next we pass to immediate extensions of $E_1(a)$ and $E_2(b)$ using (DV1)–(DV4) from Chapter 2 and appeal to Corollary 4.2.4 to obtain a good map (h, f_r, f_v) extending (g, f_r, f_v) whose domain $(F_1, \mathbf{k}_{\mathbf{E}_1}, \Gamma_{\mathbf{E}_1})$ is such that F_1 is d-henselian, F_1 satisfies the c-condition, and $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{F_1})$. What we have gained is that $\gamma \in v_1(F_1^\times)$.

(8) *Towards $v_1(E_1^\times) = \Gamma_{\mathbf{E}_1}$; the case of prime torsion modulo $v_1(E_1^\times)$.* Suppose $\gamma \in \Gamma_{\mathbf{E}_1} \setminus v_1(E_1^\times)$ and $l\gamma \in v_1(E_1^\times)$ where l is a prime number. Let ι_1 and ι_2 be as in (7). Using $(*)$ as in (7) we obtain $a \in K_1^\times$ such

that $v_1(a) = \gamma$, $a^\dagger = (\iota_1 \circ c_1)(\gamma)$, and $\mathrm{ac}_1(a) = 1$. From Lemma 3.4.1 we also get

$$(**) \qquad \Big(\big(E_i, \iota_i(\mathbf{k}_{\mathbf{E}_i})\big), v_i(E_i^\times);\ v_i|_{E_i^\times}, \iota_i \circ c_i|_{v_i(E_i^\times)}\Big) \vDash \mathrm{Mo}(\ell, c) \qquad (i = 1, 2).$$

This yields an element $b \in E_1^\times$ such that $v_1(b) = l\gamma$ and $b^\dagger = l(\iota_1 \circ c_1)(\gamma)$, and as in (7) we can arrange in addition that $\mathrm{ac}_1(b) = 1$. Our next aim is to find $d \in K_1^\times$ such that $d^l = b$ and $\mathrm{ac}_1(d) = 1$. To this end we consider $P(Y) := Y^l - b/a^l \in \mathcal{O}_1[Y]$. From $\mathrm{ac}_1(b/a^l) = 1$ and $b/a^l \asymp 1$ we get $v_1(1 - b/a^l) > 0$, that is, $v_1(P(1)) > 0$. Moreover, $P'(1) = l$. By henselianity we get $u \in K_1$ such that $P(u) = 0$ and $v_1(u-1) > 0$. Setting $d := au \in K_1^\times$, we have

$$d^l \ = \ b, \qquad \pi_1(d^\dagger) \ = \ c_1(\gamma), \qquad v_1(d) \ = \ \gamma, \qquad \mathrm{ac}_1(d) \ = \ \mathrm{ac}_1(a)\mathrm{ac}_1(u) \ = \ 1.$$

Now $v_2(f(b)) = lf_v(\gamma)$, $f(b)^\dagger = l(\iota_2 \circ c_2)(f_r(\gamma))$, and $ac_2(f(b)) = 1$, so in the same way we constructed d, we find $e \in K_2^\times$ such that

$$e^l \ = \ f(b), \qquad \pi_2(e^\dagger) \ = \ c_2(f_r(\gamma)), \qquad v_2(e) \ = \ f_r(\gamma), \qquad \mathrm{ac}_2(e) \ = \ 1.$$

Then [1, Lemma 3.1.28] gives us an isomorphism $g : E_1(d) \to E_2(e)$ of valued fields extending f and sending d to e. This isomorphism is also a differential field isomorphism. Using that same lemma it is routine to check that $\mathbf{g} := (g, f_r, f_v)$ is a good map with domain $(E_1(d), \mathbf{k}_{\mathbf{E}_1}, \Gamma_{\mathbf{E}_1})$. Using that E_1 satisfies the c-condition and $\pi(d^\dagger) = c_1(\gamma)$, it follows that $E_1(d)$ satisfies the c-condition.

Unlike in (7) we do not need to extend further to immediate extensions of $E_1(d)$ and $E_2(e)$ to regain d-henselianity: By Corollary 2.1.3, $E_1(d)$ and $E_2(e)$ are d-henselian. What we have gained is that $\gamma \in v_1(E_1(d)^\times)$.

Now let any $a \in K_1$ be given; we need to extend $\mathbf{f}$ to a good map with small domain and with a in its domain. Using (1)–(8) we arrange that E_1 is d-henselian, E_1 satisfies the c-condition, $\mathbf{k}_{\mathbf{E}_1} = \pi_1(\mathcal{O}_{E_1})$, and $v_1(E_1^\times) = \Gamma_{\mathbf{E}_1}$. Using that $E_1\langle a\rangle$ has countable transcendence degree over E_1 it follows from [1, Lemma 3.1.10] that $|\pi_1(\mathcal{O}_{E_1\langle a\rangle})| < \kappa$ and $|v_1(E_1\langle a\rangle^\times)| < \kappa$. Using again (1)–(8) we extend $\mathbf{f}$ to a good map $\mathbf{f}_1 = (f_1, f_{1,r}, f_{1,v})$ with small domain $\mathbf{E}_1^1 = (E_1^1, \mathbf{k}_{\mathbf{E}_1^1}, \Gamma_{\mathbf{E}_1^1})$ such that E_1^1 is d-henselian, satisfies the c-condition, and

$$\pi_1(\mathcal{O}_{E_1\langle a\rangle}) \ \subseteq \ \mathbf{k}_{\mathbf{E}_1^1} \ = \ \pi_1(\mathcal{O}_{E_1^1}), \qquad v_1(E_1\langle a\rangle^\times) \ \subseteq \ \Gamma_{\mathbf{E}_1^1} \ = \ v_1((E_1^1)^\times).$$

Next we extend $\mathbf{f}_1$ in the same way to $\mathbf{f}_2 = (f_2, f_{2,r}, f_{2,v})$ with small domain $\mathbf{E}_1^2 = (E_1^2, \mathbf{k}_{\mathbf{E}_1^2}, \Gamma_{\mathbf{E}_1^2})$ such that

E_1^2 is d-henselian, E_1^2 satisfies the c-condition, and

$$\pi_1(\mathcal{O}_{E_1^1\langle a\rangle}) \subseteq \mathbf{k}_{\mathbf{E}_1^2} = \pi_1(\mathcal{O}_{E_1^2}), \qquad v_1(E_1^1\langle a\rangle^\times) \subseteq \Gamma_{\mathbf{E}_1^2} = v_1((E_1^2)^\times).$$

Continuing in this manner and taking the union of the resulting good maps and small domains, we get a good map $\mathbf{f}_\infty = (f_\infty, f_{\infty,r}, f_{\infty,v})$ with small domain $\mathbf{E}_1^\infty = (E_1^\infty, \mathbf{k}_{\mathbf{E}_1^\infty}, \Gamma_{\mathbf{E}_1^\infty})$ and codomain $\mathbf{E}_2^\infty = (E_2^\infty, \mathbf{k}_{\mathbf{E}_2^\infty}, \Gamma_{\mathbf{E}_2^\infty})$ such that E_1^∞ is d-henselian, E_1^∞ satisfies the c-condition, and

$$\pi_1(\mathcal{O}_{E_1^\infty\langle a\rangle}) = \mathbf{k}_{\mathbf{E}_1^\infty} = \pi_1(\mathcal{O}_{E_1^\infty}), \qquad v_1(E_1^\infty\langle a\rangle^\times) = \Gamma_{\mathbf{E}_1^\infty} = v_1((E_1^\infty)^\times).$$

Therefore, the differential valued field extension $E_1^\infty\langle a\rangle$ of E_1^∞ is immediate. By (DV1) and (DV4) we have a spherically complete immediate valued differential field extension $E_i^\bullet \subseteq K_i$ of $E_i^\infty\langle a\rangle$ (and thus of E_i) for $i = 1, 2$. Then by (DV3) and Corollary 4.2.4 we can extend $\mathbf{f}_\infty$ to a good map $\mathbf{f}_\bullet$ with small domain $(E_1^\bullet, \mathbf{k}_{\mathbf{E}_1^\infty}, \Gamma_{\mathbf{E}_1^\infty})$ and codomain $(E_2^\bullet, \mathbf{k}_{\mathbf{E}_2^\infty}, \Gamma_{\mathbf{E}_2^\infty})$. It remains to note that $a \in E_1\langle a\rangle \subseteq E_1^\bullet$.

This finishes the proof of the *forth* part. The *back* part is done likewise.

4.3 Relative quantifier elimination

In this section we derive various consequences of Theorem 4.2.1. Recall from the introduction the 3-sorted languages L_3 and $L_3(ac)$ and the 2-sorted language L_{rv}. We also defined there T to be the $L_3(\mathrm{ac})$-theory of d-henselian monotone valued differential fields with angular component map as defined in Section 3. Let

$$\mathcal{K}_1 = (K_1, \mathbf{k}_1, \Gamma_1;\ \pi_1, v_1, c_1, \mathrm{ac}_1), \qquad \mathcal{K}_2 = (K_2, \mathbf{k}_2, \Gamma_2;\ \pi_2, v_2, c_2, \mathrm{ac}_2)$$

be models of T considered as $L_3(\mathrm{ac})$-structures.

Corollary 4.3.1. *$\mathcal{K}_1 \equiv \mathcal{K}_2$ if and only if $(\mathbf{k}_1, \Gamma_1; c_1) \equiv (\mathbf{k}_2, \Gamma_2; c_2)$ as L_{rv}-structures.*

Proof. Suppose $(\mathbf{k}_1, \Gamma_1; c_1) \equiv (\mathbf{k}_2, \Gamma_2; c_2)$. Then we have good substructures $\mathbf{E}_1 = (\mathbb{Q}, \mathbb{Q}; \{0\})$ of $\mathcal{K}_1$, $\mathbf{E}_2 = (\mathbb{Q}, \mathbb{Q}; \{0\})$ of $\mathcal{K}_2$, and an obviously good map $\mathbf{E}_1 \to \mathbf{E}_2$, so Theorem 4.2.1 applies. The other direction of the corollary is trivial. □

Corollary 4.3.2. *Let $\mathcal{K}_1$ be a substructure of $\mathcal{K}_2$, with $(\mathbf{k}_1, \Gamma_1; c_1) \preccurlyeq (\mathbf{k}_2, \Gamma_2; c_2)$ as L_{rv}-structures. Then $\mathcal{K}_1 \preccurlyeq \mathcal{K}_2$.*

Proof. With $(K_1, \mathbf{k}_1, \Gamma_1)$ in the role of a good substructure of $\mathcal{K}_1$ as well as of $\mathcal{K}_2$, the identity on $(K_1, \mathbf{k}_1, \Gamma_1)$

is a good map. Hence $\mathcal{K}_1 \preccurlyeq \mathcal{K}_2$ by Theorem 4.2.1. $\square$

To eliminate angular components in Corollary 4.3.2, consider L_3-structures

$$\mathcal{E} = (E, \mathbf{k}_E, \Gamma_E;\ \pi_E, v_E, c_E), \qquad \mathcal{F} = (F, \mathbf{k}_F, \Gamma_F;\ \pi_F, v_F, c_F)$$

that are d-henselian models of $\mathrm{Mo}(c)$.

Corollary 4.3.3. *Suppose $\mathcal{E}$ is a substructure of $\mathcal{F}$ and $(\mathbf{k}_E, \Gamma_E; c_E) \preccurlyeq (\mathbf{k}_F, \Gamma_F; c_F)$ as L_{rv}-structures. Then $\mathcal{E} \preccurlyeq \mathcal{F}$.*

Proof. By passing to suitable elementary extensions we arrange that $\mathcal{E}$ and $\mathcal{F}$ are $\aleph_1$-saturated. Then the proof of Corollary 4.1.1 yields a cross-section $s_E : \Gamma_E \to E^\times$ such that $\pi_E(s_E(\gamma)^\dagger) = c_E(\gamma)$ for all $\gamma \in \Gamma_E$, and also a cross-section $s_F : \Gamma_F \to F^\times$ such that $\pi_F(s_F(\gamma)^\dagger) = c_F(\gamma)$ for all $\gamma \in \Gamma_F$. Now Γ_E is an $\aleph_1$-saturated pure subgroup of Γ_F and thus we have an internal direct sum decomposition $\Gamma_F = \Gamma_E \oplus \Delta$ by [1, Corollary 3.3.38]. This gives a cross-section $s : \Gamma_F \to F^\times$ that agrees with s_E on Γ_E and with s_F on Δ. Moreover, $\pi_F(s(\gamma)^\dagger) = c_F(\gamma)$ for all $\gamma \in \Gamma_F$. This yields an angular component map ac_F on $\mathcal{F}$ by $\mathrm{ac}(x) = \pi_F\big(x/s(v_F(x))\big)$. Its restriction to Γ_E is angular component map on $\mathcal{E}$. Now use Corollary 4.3.2. $\square$

Let x be an l-tuple of distinct f-variables, y an m-tuple of distinct r-variables, and z an n-tuple of distinct v-variables. Call an $L_3(\mathrm{ac})$-formula $\phi(x,y,z)$ *special* if

$$\phi(x,y,z) = \psi\big(\mathrm{ac}(P_1(x)), \dots, \mathrm{ac}(P_p(x)), v(Q_1(x)), \dots, v(Q_q(x)), y, z\big),$$

for some L_{rv}-formula $\psi\big(u_1, \dots, u_p, w_1, \dots, w_q, y, z\big)$ where $u_1, \dots, u_p$ are extra r-variables, $w_1, \dots, w_q$ are extra v-variables, and where the differential polynomials $P_1, \dots, P_p, Q_1, \dots, Q_q \in \mathbb{Q}\{x\}$ have all their coefficients in $\mathbb{Z}$. Note that a special formula contains no quantified f-variables.

We can now prove the Theorem 2.1.6.

Proof of the Theorem 2.1.6. For a model $\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c, \mathrm{ac})$ of T and for a tuple (a,d,γ) where $a \in K^l$, $d \in \mathbf{k}^m$ and $\gamma \in \Gamma^n$, define the *special type of* (a,d,γ) (in $\mathcal{K}$), denoted by $\mathrm{sptp}(a,d,\gamma)$, to be the following set of special formulas:

$$\mathrm{sptp}(a,d,\gamma) = \{\phi(x,y,z) : \phi \text{ is a special } L_3(\mathrm{ac})\text{-formula and } \mathcal{K} \models \phi(a,d,\gamma)\}.$$

Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be models of T and let $a_i \in K_i^l$, $d_i \in \mathbf{k}_i^m$ and $\gamma_i \in \Gamma_i^n$ for $i = 1, 2$ be such that (a_1, d_1, γ_1) and (a_2, d_2, γ_2) have the same special type, in $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively. By Stone's Representation Theorem, it is enough to show that then (a_1, d_1, γ_1) and (a_2, d_2, γ_2) realize the same type in $\mathcal{K}_1$ and $\mathcal{K}_2$, respectively. Consider the differential subfield $E_i := \mathbb{Q}\langle a_i \rangle$ of K_i, the ordered subgroup $\Gamma_{\mathbf{E}_i}$ of Γ_i generated by $v_i(E_i^\times)$ and γ_i, and the differential subfield

$$\mathbf{k}_{\mathbf{E}_i} := \mathbb{Q}\langle \mathrm{ac}_i(E_i), c_i(\Gamma_{\mathbf{E}_i}), d_i \rangle$$

of $\mathbf{k}_i$, for $i = 1, 2$. Then $\mathbf{E}_i := (E_i, \mathbf{k}_{\mathbf{E}_i}, \Gamma_{\mathbf{E}_i})$ is a good substructure of $\mathcal{K}_i$, for $i = 1, 2$. Note that for all $P \in \mathbb{Q}\{x\}$ and for $i = 1, 2$ we have $P(a_i) = 0$ iff $\mathrm{ac}_i(P(a_i)) = 0$. Since a_1 and a_2 have the same special type, this yields a differential field isomorphism $f : E_1 \to E_2$ with $f(a_1) = a_2$. It is also routine to show that we have an ordered group isomorphism $f_v : \Gamma_{\mathbf{E}_1} \to \Gamma_{\mathbf{E}_2}$ such that $f_v(\gamma_1) = \gamma_2$ and $f_v(v_1(a)) = v_2(f(a))$ for all $a \in E_1^\times$, and a differential field isomorphism $f_r : \mathbf{k}_{\mathbf{E}_1} \to \mathbf{k}_{\mathbf{E}_2}$ with $f_r(d_1) = d_2$, $f_r(c_1(\gamma)) = c_2(f_v(\gamma))$ for all $\gamma \in \Gamma_{\mathbf{E}_1}$ and $f_r(\mathrm{ac}_1(a)) = \mathrm{ac}_2(f(a))$ for all $a \in E_1$. These properties of the maps f, f_r, f_v only use that the tuples (a_1, d_1, γ_1) and (a_2, d_2, γ_2) realize the same *quantifier-free* special formulas $\phi(x, y, z)$ in $\mathcal{K}_1$ and $\mathcal{K}_2$ respectively, but the full assumption on these two tuples guarantees that (f_r, f_v) is a partial elementary map between $(\mathbf{k}_1, \Gamma_1; c_1)$ and $(\mathbf{k}_2, \Gamma_2; c_2)$. Thus we have a good map $\mathbf{f} = (f, f_r, f_v)$, and it remains to apply Theorem 4.2.1. □

Corollary 4.3.4. *Let $\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c, \mathrm{ac}) \models T$. Then any set $X \subseteq \mathbf{k}^m \times \Gamma^n$ that is definable in $\mathcal{K}$ is definable in the L_{rv}-structure $(\mathbf{k}, \Gamma; c)$. In particular, $(\mathbf{k}, \Gamma; c)$ is stably embedded in $\mathcal{K}$.*

The angular component map does not occur in the above reduct $(\mathbf{k}, \Gamma; c)$, so we can eliminate it in the result above in view of Corollary 4.1.1:

Corollary 4.3.5. *Let $\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c)$ be a* d*-henselian model of* $\mathrm{Mo}(c)$*. Then any set $X \subseteq \mathbf{k}^m \times \Gamma^n$ that is definable in $\mathcal{K}$ is definable in the L_{rv}-structure $(\mathbf{k}, \Gamma; c)$.*

4.4 NIP

Let $\mathcal{K} = (K, \mathbf{k}, \Gamma; \pi, v, c, \mathrm{ac})$ be a model of T. When does $\mathcal{K}$ have NIP (the Non-Independence Property)? This can be reduced to the same question for $(\mathbf{k}, \Gamma; c)$:

Corollary 4.4.1. *$\mathcal{K}$ has* NIP *if and only if the L_{rv}-structure $(\mathbf{k}, \Gamma; c)$ has* NIP*.*

Proof. The forward direction is clear. To prove the contrapositive of the other direction, assume $\varphi(x, y, z; \tilde{x}, \tilde{y}, \tilde{z})$ is an $L_3(\mathrm{ac})$-formula having IP in $\mathcal{K}$, with $|x| = k$, $|y| = l$, $|z| = m$, $|\tilde{x}| = \tilde{k}$, $|\tilde{y}| = \tilde{l}$, $|\tilde{z}| = \tilde{m}$.

Moreover, without loss of generality, we can assume $\mathcal{K}$ is $2^{\aleph_0}$-saturated. This means we have a sequence $\{(a_i, u_i, \gamma_i)\}_{i\in\mathbb{N}}$ with $a_i \in K^k$, $u_i \in \mathbf{k}^l$ and $\gamma_i \in \Gamma^m$ and for every $I \subseteq \mathbb{N}$, tuples $\tilde{a}_I \in K^{\tilde{k}}$, $\tilde{u}_I \in \mathbf{k}^{\tilde{l}}$ and $\tilde{\gamma}_I \in \Gamma^{\tilde{m}}$, such that for all $i \in \mathbb{N}$ and $I \subseteq \mathbb{N}$,

$$\mathcal{K} \models \varphi(a_i, u_i, \gamma_i; \tilde{a}_I, \tilde{u}_I, \tilde{\gamma}_I) \Leftrightarrow i \in I.$$

By Theorem 2.1.6, $\varphi(x, y, z; \tilde{x}, \tilde{y}, \tilde{z})$ is T-equivalent to a special formula

$$\psi\Big(\mathrm{ac}\big(\vec{P}(x)\big), v\big(\vec{Q}(x)\big), y, z; \mathrm{ac}\big(\vec{R}(\tilde{x})\big), v\big(\vec{S}(\tilde{x})\big), \tilde{y}, \tilde{z}\Big),$$

where ψ is an L_{rv}-formula and $\vec{P}$, $\vec{Q}$, $\vec{R}$, $\vec{S}$ are finite tuples of differential polynomials in $\mathbb{Q}\{\tilde{x}\}$. This yields tuples witnessing that ψ has IP in $(\mathbf{k}, \Gamma; c)$:

$$(\mathbf{k}, \Gamma; c) \models \psi\Big(\mathrm{ac}\big(\vec{P}(a_i)\big), v\big(\vec{Q}(a_i)\big), u_i, \gamma_i; \mathrm{ac}\big(\vec{R}(\tilde{a}_I)\big), v\big(\vec{S}(\tilde{a}_I)\big), \tilde{u}_I, \tilde{\gamma}_I\Big) \Leftrightarrow i \in I,$$

for all $i \in \mathbb{N}$ and $I \subseteq \mathbb{N}$. $\square$

By Corollary 4.1.1 the result just proved goes through for d-henselian model of $\mathrm{Mo}(c)$.

Example of a d-*henselian model of* $\mathrm{Mo}(c)$ *with few constants that has NIP.* Let $K := \mathbb{T}[\mathrm{i}]((t^{\mathbb{R}}))_c$ be the d-henselian monotone valued differential field considered in Section 3.2; here $\mathbb{T}$ is the valued differential field of transseries, $\mathrm{i}^2 = -1$, and $c : \mathbb{R} \to \mathbb{T}[\mathrm{i}]$ is the additive map given by $c(r) = \mathrm{i}r$. By [1, Proposition 16.6.6], $\mathbb{T}$ has NIP. Then the L_{rv}-structure $(\mathbb{T}[\mathrm{i}], \mathbb{R}; c)$ has NIP, since it is interpretable in the valued differential field $\mathbb{T}$. Therefore, $\mathcal{K} = (K, \mathbb{T}[\mathrm{i}], \mathbb{R}; \pi, v, c)$, where π and v are the obvious maps, also has NIP.

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